Gerstenhaber brackets on Hochschild cohomology of twisted tensor products

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Abstract. We construct the Gerstenhaber bracket on Hochschild cohomology of a twisted tensor product of algebras, and, as examples, compute Gerstenhaber brackets for some quantum complete intersections arising in the work of Buchweitz, Green, Madsen, and Solberg. We prove that a subalgebra of the Hochschild cohomology ring of a twisted tensor product, on which the twisting is trivial, is isomorphic, as a Gerstenhaber algebra, to the tensor product of the respective subalgebras of the Hochschild cohomology rings of the factors.

1. Introduction

The Hochschild cohomology $HH^*(\Lambda)$ of an associative algebra Λ has a cup product under which it is a graded commutative ring. In 1963, Gerstenhaber [4] introduced the bracket product $[\cdot,\cdot]$ (or Gerstenhaber bracket) of degree -1, to give a second multiplicative structure on the Hochschild cohomology ring. Thus one combines the structures of a graded commutative algebra and a graded Lie algebra, to form what is generally called a Gerstenhaber algebra, of which the Hochschild cohomology ring is an example. Gerstenhaber [5] showed that the bracket plays a role in the deformation theory of algebras.

Recently, Le and Zhou [7] defined the tensor product of two Gerstenhaber algebras. They proved that, given algebras R and S over a field k, at least one of which is finite dimensional, the Hochschild cohomology of the tensor product algebra $R \otimes_k S$ is isomorphic to the tensor product of the respective Hochschild cohomologies of R and of S, as Gerstenhaber algebras.

In this paper, we work more generally in the twisted tensor product setting of Bergh and Oppermann [1]. Let R and S be k-algebras graded by abelian groups A

and B respectively, and consider $R \otimes_k^t S$, where a twist t is defined using the gradings of R and of S (see Section 3 below). In the succeeding sections, we show the following main results:

(1) We construct the Gerstenhaber bracket on the Hochschild cohomology of $R \otimes_k^t S$ in Section 3 by employing and augmenting techniques of Negron and the third author [9]. In Section 5, we apply this construction to compute brackets for the quantum complete intersection

$$\Lambda_q := k \langle x, y \rangle / (x^2, y^2, xy + qyx), \quad q \in k^\times,$$

which can be considered as a twisted tensor product $k[x]/(x^2) \otimes_k^t k[y]/(y^2)$. We take advantage of the known algebra structure of $\mathrm{HH}^*(\Lambda_q)$, for various values of q, as given by Buchweitz, Green, Madsen, and Solberg [2]. Our computations give information about the structures of the Lie algebra $\mathrm{HH}^1(\Lambda_q)$ and its module $\mathrm{HH}^*(\Lambda_q)$.

(2) In Section 6, we let A' and B' be subgroups of A and B, respectively, on which the twisting t is trivial (see (6.2)), and show that the graded algebra isomorphism given by Bergh and Oppermann [1, Theorem 4.7], namely

$$\mathrm{HH}^{*, A' \oplus B'}(R \otimes_k^t S) \cong \mathrm{HH}^{*, A'}(R) \otimes \mathrm{HH}^{*, B'}(S),$$

is in fact an isomorphism of Gerstenhaber algebras. This generalizes the result of Le and Zhou [7] to the twisted setting. Our proof relies on twisted versions of the Alexander–Whitney and Eilenberg–Zilber chain maps, and uses techniques from [9]. Examples are in Section 5.

Gerstenhaber brackets are in general difficult to compute. Our results described in (1) above include a new class of examples which moreover illustrate the techniques of [9], showing that bracket computations can be simplified by defining brackets directly on a resolution other than the bar resolution. An advantage of these techniques is in eliminating the necessity of using explicit formulas for chain maps between resolutions, which traditional approaches typically require. Our main theorem described in (2) above gives a way to compute brackets on a subalgebra of the Hochschild cohomology of a twisted tensor product, saving time for some classes of examples. The statement and proof are quite general, showing that while the techniques of [9] were primarily developed for Koszul algebras, they can in fact be helpful for other algebras as well. Many algebras of interest may be described as twisted tensor products or deformations of twisted tensor products, including our examples of Section 5 and generalizations, skew polynomial rings, many quantum groups and many Nichols algebras arising in results on classification of finite dimensional Hopf algebras.

Throughout the article, k is a field. All tensor products of modules are taken over k unless otherwise indicated. Additionally, when writing elements in tensor product modules, we often will omit the subscripts on the tensor product symbols when they are clear from context.

2. Preliminaries

In this section, we summarize and augment the results of [9] that we will need. Let Λ be a k-algebra and $\Lambda^e := \Lambda \otimes \Lambda^{op}$ be its enveloping algebra, that is it has the tensor product algebra structure, where Λ^{op} is Λ with the opposite multiplication. Then a left Λ^e-module is a Λ-bimodule, and vice versa.

The *Hochschild cohomology* of Λ is

$$\mathrm{HH}^*(\Lambda) := \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda).$$

It is a *Gerstenhaber algebra*, that is, it is a graded commutative algebra via the cup product $\smile$, it is a graded Lie algebra via the Lie bracket (or Gerstenhaber bracket) $[\cdot, \cdot]$, and it satisfies various conditions. See, for example, [4]. We will not need the standard definition here. Instead we will recall a construction of these operations that will suit our purposes. For this we will need the bar resolution $\mathbb{B}$ and a resolution $\mathbb{K}$ satisfying some properties ($\mathbb{K} = \mathbb{B}$ is one choice), which we introduce next.

Let $\mathbb{B} = \mathbb{B}(\Lambda)$ denote the *bar resolution* of Λ,

$$\cdots \xrightarrow{\delta_2} \Lambda^{\otimes 3} \xrightarrow{\delta_1} \Lambda^{\otimes 2} \xrightarrow{m} \Lambda \to 0, \tag{2.1}$$

where m denotes multiplication, and for each i, δ_i is the Λ^e-module map determined by its values on monomials,

$$\delta_i(\lambda_0 \otimes \cdots \otimes \lambda_{i+1}) = \sum_{j=0}^{i} (-1)^j \lambda_0 \otimes \cdots \otimes \lambda_j \lambda_{j+1} \otimes \cdots \otimes \lambda_{i+1},$$

for $\lambda_0, \ldots, \lambda_{i+1} \in \Lambda$. We will also use the *normalized bar resolution* $\overline{\mathbb{B}} = \overline{\mathbb{B}}(\Lambda)$, whose ith component is $\Lambda \otimes \overline{\Lambda}^{\otimes i} \otimes \Lambda$, where $\overline{\Lambda} = \Lambda/(k \cdot 1)$ as a k-vector space. One checks that each differential δ_i defined above factors through $\Lambda \otimes \overline{\Lambda}^{\otimes i} \otimes \Lambda$ by employing a choice of section of the quotient map $\Lambda \to \overline{\Lambda}$. Abusing notation, we will not always distinguish between elements of Λ and those of $\overline{\Lambda}$, making use of our choice of section as needed.

There is a chain map $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B} \otimes_\Lambda \mathbb{B}$, called a *diagonal map*, given on monomials by

$$\Delta_{\mathbb{B}}(\lambda_0 \otimes \cdots \otimes \lambda_{i+1}) = \sum_{j=0}^{i} (\lambda_0 \otimes \cdots \otimes \lambda_j \otimes 1) \otimes_\Lambda (1 \otimes \lambda_{j+1} \otimes \cdots \otimes \lambda_{i+1}), \tag{2.2}$$

for all $\lambda_0, \ldots, \lambda_{i+1} \in \Lambda$.

The *cup product* on Hochschild cohomology may be defined at the chain level as follows. Let $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes(i+2)}, \Lambda)$ and $g \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes(j+2)}, \Lambda)$. Then $f \smile g \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes(i+j+2)}, \Lambda)$ is defined on monomials by

$$(f \smile g)(\lambda_0 \otimes \cdots \otimes \lambda_{i+j+1}) = f(\lambda_0 \otimes \cdots \otimes \lambda_i \otimes 1)g(1 \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_{i+j+1}),$$

for all $\lambda_0, \ldots, \lambda_{i+j+1} \in \Lambda$. This can be viewed as a composition of maps

$$\mathbb{B} \xrightarrow{\Delta_{\mathbb{B}}} \mathbb{B} \otimes_{\Lambda} \mathbb{B} \xrightarrow{f \otimes g} \Lambda \otimes_{\Lambda} \Lambda \xrightarrow{\sim} \Lambda. \tag{2.3}$$

The cup product may be defined similarly on the normalized bar resolution, or indeed on any resolution.

Let $\mathbb{K} \to \Lambda$ be any resolution of Λ by free Λ^e-modules. We define chain maps $F_{\mathbb{K}}^{\ell}, F_{\mathbb{K}}^{r} : \mathbb{K} \otimes_{\Lambda} \mathbb{K} \to \mathbb{K}$ as follows. Letting $\mu : \mathbb{K} \to \Lambda$ be the natural quasi-isomorphism,

$$F_{\mathbb{K}}^{\ell} = \mu \otimes \mathbf{1}_{\mathbb{K}} \quad \text{and} \quad F_{\mathbb{K}}^{r} = \mathbf{1}_{\mathbb{K}} \otimes \mu, \tag{2.4}$$

where $\mathbf{1}_{\mathbb{K}}$ is the identity map on $\mathbb{K}$. Note that the maps $F_{\mathbb{K}}^{\ell}, F_{\mathbb{K}}^{r} : \mathbb{K} \otimes_{\Lambda} \mathbb{K} \to \mathbb{K}$ are chain maps by their definitions.

For our formulation of the Gerstenhaber bracket, we will assume that $\mathbb{K}$ satisfies the following conditions from [9, 3.1].

Conditions 2.5. We assume:

(a) There is an embedding $\iota : \mathbb{K} \to \mathbb{B}$ lifting the identity map on Λ.

(b) There is a chain map $\pi : \mathbb{B} \to \mathbb{K}$ for which $\pi\iota = \mathbf{1}_{\mathbb{K}}$.

(c) There is a chain map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_{\Lambda} \mathbb{K}$ for which $\Delta_{\mathbb{B}}\iota = (\iota \otimes_{\Lambda} \iota)\Delta_{\mathbb{K}}$.

Clearly if we set $\mathbb{K} = \mathbb{B}$, it will satisfy these conditions. It is explained in [9] that if Λ is a Koszul algebra and $\mathbb{K}$ is its Koszul resolution, then $\mathbb{K}$ satisfies these conditions; in particular, the needed diagonal maps $\Delta_{\mathbb{K}}$ are given in [3]. We will use this fact to compute brackets for some quantum complete intersections in Section 5 below. An advantage of this method over traditional methods is that we do not need to use or even know the often cumbersome map π explicitly. For our theorem in Section 6, giving an isomorphism of Gerstenhaber algebras in the context of a twisted tensor product, we will take $\mathbb{K}$ to be the total complex of the twisted tensor product of two normalized bar resolutions.

Now let

$$F_{\mathbb{K}} = F_{\mathbb{K}}^{\ell} - F_{\mathbb{K}}^{r}, \tag{2.6}$$

where $F_{\mathbb{K}}^{\ell}, F_{\mathbb{K}}^{r}$ are defined in equation (2.4). This is the chain map $F_{\mathbb{K}}$ as defined in [9]. It is shown there that $F_{\mathbb{K}}$ is a boundary in $\mathrm{Hom}_{\Lambda^e}(\mathbb{K} \otimes_{\Lambda} \mathbb{K}, \mathbb{K})$, and so there is a map $\phi : \mathbb{K} \otimes_{\Lambda} \mathbb{K} \to \mathbb{K}$ for which

$$d(\phi) := d_{\mathbb{K}}\phi + \phi d_{\mathbb{K} \otimes_{\Lambda} \mathbb{K}} = F_{\mathbb{K}}, \tag{2.7}$$

that is ϕ is a *contracting homotopy* for $F_{\mathbb{K}}$. Let $f \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_i, \Lambda)$ and let $g \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_j, \Lambda)$ represent elements of Hochschild cohomology in degrees i and j, respectively. By [9, Theorem 3.2.5], their *Gerstenhaber bracket* on Hochschild cohomology is given at the chain level by

$$[f, g] = f \circ g - (-1)^{(i-1)(j-1)} g \circ f \tag{2.8}$$

where the *circle product* $f \circ g$ is the composition

$$\mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}^{(2)}} \mathbb{K} \otimes_\Lambda \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\mathbf{1}_{\mathbb{K}} \otimes g \otimes \mathbf{1}_{\mathbb{K}}} \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\phi} \mathbb{K} \xrightarrow{f} \Lambda, \tag{2.9}$$

and $g \circ f$ is defined similarly. The definition of the map $\mathbf{1}_{\mathbb{K}} \otimes g \otimes \mathbf{1}_{\mathbb{K}}$ above includes "Koszul signs," that is, on elements the map is given by

$$x \otimes y \otimes z \mapsto (-1)^{lj} x \otimes g(y) \otimes z \tag{2.10}$$

for all $x \in \mathbb{K}_l$, $y \in \mathbb{K}_m$, $z \in \mathbb{K}_n$. The map $\Delta_{\mathbb{K}}^{(2)}$ is given by $(\Delta_{\mathbb{K}} \otimes \mathbf{1}_{\mathbb{K}})\Delta_{\mathbb{K}}$ (which is equal to $(\mathbf{1}_{\mathbb{K}} \otimes \Delta_{\mathbb{K}})\Delta_{\mathbb{K}}$ by a calculation using Condition 2.5(c)). In [9], this circle product is denoted by $\circ_\phi$ and bracket by $[\cdot, \cdot]_\phi$ in order to distinguish these maps at the chain level. By [9, Theorem 3.2.5], the operations induced by $\circ_\phi$ and $[\cdot, \cdot]_\phi$ on cohomology do not depend on ϕ, and so we choose not to make such a distinction here.

In case $\mathbb{K} = \mathbb{B}$, as in [9], we may set $\phi = G_{\mathbb{B}}$ where

$$
\begin{aligned}
G_{\mathbb{B}}((\lambda_0 \otimes &\cdots \otimes \lambda_{p-1}) \otimes (\lambda_p) \otimes (\lambda_{p+1} \otimes \cdots \otimes \lambda_{n+1})) \\
&= (-1)^{p-1} \lambda_0 \otimes \cdots \otimes \lambda_{p-1} \otimes \lambda_p \otimes \lambda_{p+1} \otimes \cdots \otimes \lambda_{n+1}
\end{aligned} \tag{2.11}
$$

for all $\lambda_0, \ldots, \lambda_{n+1} \in \Lambda$. If $\mathbb{K} = \overline{\mathbb{B}}$, the normalized bar resolution, we may set $\phi = G_{\overline{\mathbb{B}}}$ where $G_{\overline{\mathbb{B}}}$ is defined similarly, by replacing λ_j by its image in $\overline{\Lambda}$ in the formula; the proof of [9, Proposition 2.0.8] may be adapted to show that $G_{\overline{\mathbb{B}}}$ is indeed a contracting homotopy for $F_{\overline{\mathbb{B}}}$.

One of the properties of the Gerstenhaber bracket is a compatibility relation with the cup product: On Hochschild cohomology,

$$[f \smile g, h] = [f, h] \smile g + (-1)^{i(u-1)} f \smile [g, h], \tag{2.12}$$

where u is the homological degree of h.

3. Gerstenhaber brackets for twisted tensor products

Let R and S be k-algebras, graded by abelian groups A and B, respectively. Let

$$t : A \otimes_{\mathbb{Z}} B \to k^{\times}$$

be a *twisting*, that is a homomorphism of abelian groups, denoted $t(a \otimes_{\mathbb{Z}} b) = t^{\langle a | b \rangle}$ for all $a \in A$, $b \in B$. Let $R \otimes^t S$ denote the twisted tensor product of algebras as in Bergh and Oppermann [1]. That is, $R \otimes^t S = R \otimes S$ as a vector space, and

$$(r \otimes s) \cdot^t (r' \otimes s') = t^{\langle |r'| \| |s| \rangle} r r' \otimes s s'$$

for all homogeneous $r, r' \in R$ and $s, s' \in S$, where $|r'|$, $|s|$ are the degrees of r', s in A, B, respectively. We will often write $t^{\langle r' | s \rangle}$ in place of $t^{\langle |r'| \| |s| \rangle}$. Note that $R \otimes^t S$ is $(A \oplus B)$-graded.

If X is an A-graded R^e-module and Y a B-graded S^e-module, denote by $X \otimes^t Y$ the tensor product $X \otimes Y$ as a vector space, with $(R \otimes^t S)^e$-module structure given by

$$(r \otimes s)(x \otimes y)(r' \otimes s') = t^{\langle x | s \rangle} t^{\langle r' | y \rangle} t^{\langle r' | s \rangle} r x r' \otimes s y s' \tag{3.1}$$

for all homogeneous $r, r' \in R$, $s, s' \in S$, $x \in X$, and $y \in Y$ (see [1, Definition/Construction 4.1]). By [1, Lemma 4.3], if X and Y are projective modules, then $X \otimes^t Y$ is an $(A \oplus B)$-graded projective $(R \otimes^t S)^e$-module.

Let

$$P: \quad \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} R \to 0$$

be an A-graded R^e-projective resolution of R and let

$$Q: \quad \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{d_0^Q} S \to 0$$

be a B-graded S^e-projective resolution of S. In particular, the differentials are graded maps (i.e. preserve degree). By [1, Lemmas 4.3, 4.4, 4.5], the total complex of $P \otimes^t Q$ is an $(A \oplus B)$-graded $(R \otimes^t S)^e$-projective resolution of $R \otimes^t S$. The differentials are given as usual by $d_{i,j}^{P \otimes^t Q} := d_i^P \otimes \mathbf{1} + (-1)^i \mathbf{1} \otimes d_j^Q$.

Now assume that P is a free resolution of R as an R^e-module, and that Q is a free resolution of S as an S^e-module. Assume $P_0 = R \otimes R$ and $Q_0 = S \otimes S$, and d_0^P and d_0^Q are multiplication maps. Then $P_0 \otimes^t Q_0 = (R \otimes R) \otimes^t (S \otimes S)$, which is isomorphic to $(R \otimes^t S)^e$ by the proof of [1, Lemma 4.3] (see also Lemma 3.2 below). We will identify P_n with $R \otimes W_n \otimes R$ for a vector space W_n, for each n, and similarly Q_n.

Assume that ϕ_P and ϕ_Q are contracting homotopies for F_P and F_Q (see (2.6) and (2.7)), respectively, that is, $d(\phi_P) = F_P$ and $d(\phi_Q) = F_Q$. We will construct from these a contracting homotopy $\phi = \phi_{P \otimes^t Q}$ for $F_{P \otimes^t Q}$.

By its definition in (2.6), $F_{P \otimes^t Q}$ is a map from $(P \otimes^t Q) \otimes_{R \otimes^t S} (P \otimes^t Q)$ to $P \otimes^t Q$. We will want to compare it with maps from $(P \otimes_R P) \otimes^t (Q \otimes_S Q)$ to $P \otimes^t Q$. We will need the following isomorphism of $(R \otimes^t S)^e$-modules, similar to that found in the proof of [1, Lemma 4.3].

Lemma 3.2. *Let X, X' be A-graded R^e-modules and Y, Y' be B-graded S^e-modules. There is an isomorphism of $(R \otimes^t S)^e$-modules,*

$$\sigma : (X \otimes^t Y) \otimes_{R \otimes^t S} (X' \otimes^t Y') \xrightarrow{\sim} (X \otimes_R X') \otimes^t (Y \otimes_S Y'),$$

given by $\sigma((x \otimes y) \otimes (x' \otimes y')) = t^{\langle x'|y \rangle}(x \otimes x') \otimes (y \otimes y')$ for all homogeneous $x \in X$, $x' \in X'$, $y \in Y$, and $y' \in Y'$.

Proof. It may be checked that this yields a well-defined map on the tensor product in each degree. We check that this is an $(R \otimes^t S)^e$-module homomorphism. Choose homogeneous elements $r \in R$ and $s \in S$. We check the left action:

$$\sigma\big((r \otimes s) \cdot ((x \otimes y) \otimes (x' \otimes y'))\big) = \sigma\big(t^{\langle x|s \rangle}(rx \otimes sy) \otimes (x' \otimes y')\big)$$
$$= t^{\langle x|s \rangle}t^{\langle x'|sy \rangle}(rx \otimes x') \otimes (sy \otimes y'),$$
$$(r \otimes s) \cdot \sigma\big((x \otimes y) \otimes (x' \otimes y')\big) = (r \otimes s) \cdot \big(t^{\langle x'|y \rangle}(x \otimes x') \otimes (y \otimes y')\big)$$
$$= t^{\langle x'|y \rangle}t^{\langle x \otimes x'|s \rangle}(rx \otimes x') \otimes (sy \otimes y').$$

Now $t^{\langle x'|sy \rangle} = t^{\langle x'|s \rangle}t^{\langle x'|y \rangle}$ and $t^{\langle x \otimes x'|s \rangle} = t^{\langle x|s \rangle}t^{\langle x'|s \rangle}$ so the above expressions are the same. Similarly, the right action commutes with σ. Clearly this $(R \otimes^t S)^e$-module map has an inverse given by $(x \otimes x') \otimes (y \otimes y') \mapsto t^{-\langle x'|y \rangle}(x \otimes y) \otimes (x' \otimes y')$. $\qquad\square$

We next modify σ by a sign to define a chain map from $(P \otimes^t Q) \otimes_{R \otimes^t S} (P \otimes^t Q)$ to $(P \otimes_R P) \otimes^t (Q \otimes_S Q)$ (cf. the map τ of [7, p. 1471]).

Lemma 3.3. *There is a chain map*

$$\sigma : (P \otimes^t Q) \otimes_{R \otimes^t S} (P \otimes^t Q) \to (P \otimes_R P) \otimes^t (Q \otimes_S Q)$$

that is an isomorphism of $(R \otimes^t S)^e$-modules in each degree, given by

$$\sigma((x \otimes y) \otimes (x' \otimes y')) = (-1)^{jp}t^{\langle x'|y \rangle}(x \otimes x') \otimes (y \otimes y')$$

on $(P_i \otimes^t Q_j) \otimes_{R \otimes^t S} (P_p \otimes^t Q_q)$.

Proof. That σ is an isomorphism of $(R \otimes^t S)^e$-modules follows from Lemma 3.2: The extra sign in the definition still yields an $(R \otimes^t S)^e$-module map, since action by elements of $R \otimes^t S$ does not change the homological degree. A calculation shows that this map σ commutes with the differentials. $\qquad\square$

We will need to switch notation back and forth, using the isomorphism of Lemma 3.3, in our computations. The following lemma may be proven directly by comparing the values of the two given maps on the chain complex. One checks that the effect of twisting is as expected.

Lemma 3.4. *The map $F = F_{P \otimes^t Q}$ on $(P \otimes^t Q) \otimes_{R \otimes^t S} (P \otimes^t Q)$, as defined in (2.6), is precisely*

$$F = (F_P^{\ell} \otimes F_Q^{\ell} - F_P^r \otimes F_Q^r)\sigma,$$

where σ is defined in Lemma 3.3.

We next use Lemma 3.4 to construct a contracting homotopy for $F_{P \otimes^t Q}$.

Lemma 3.5. *Let ϕ_P, ϕ_Q be contracting homotopies for F_P, F_Q, respectively. Let $\phi = \phi_{P \otimes^t Q} : (P \otimes^t Q) \otimes_{R \otimes^t S} (P \otimes^t Q) \to P \otimes^t Q$ be defined by*

$$\phi := (\phi_P \otimes F_Q^{\ell} + (-1)^{i+p} F_P^r \otimes \phi_Q)\sigma$$

on $(P_i \otimes^t Q_j) \otimes_{R \otimes^t S} (P_p \otimes^t Q_q)$, where σ is the isomorphism of Lemma 3.3. Then ϕ is a contracting homotopy for $F = F_{P \otimes^t Q}$, that is, $d(\phi) = F$.

Proof. In the following calculation, the exponent $*$ in $(-1)^*$ varies and is determined when needed. As $F_P^{\ell}, F_Q^{\ell}, F_P^r, F_Q^r$ are chain maps, they commute with the differentials. By the definition of ϕ on $(P_i \otimes^t Q_j) \otimes_{R \otimes^t S} (P_p \otimes^t Q_q)$,

$$
\begin{aligned}
d(\phi) &:= d\phi + \phi d \\
&= (d \otimes \mathbf{1} + (-1)^* \otimes d)(\phi_P \otimes F_Q^{\ell} + (-1)^{i+p} F_P^r \otimes \phi_Q)\sigma \\
&\quad + (\phi_P \otimes F_Q^{\ell} + (-1)^* F_P^r \otimes \phi_Q)\sigma(d \otimes \mathbf{1} + (-1)^{i+p} \otimes d) \\
&= \big(d\phi_P \otimes F_Q^{\ell} + (-1)^{i+p} dF_P^r \otimes \phi_Q + (-1)^{i+p+1}\phi_P \otimes dF_Q^{\ell} \\
&\qquad + F_P^r \otimes d\phi_Q + \phi_P d \otimes F_Q^{\ell} + (-1)^{i+p}\phi_P \otimes F_Q^{\ell} d \\
&\qquad + (-1)^{i+p-1} F_P^r d \otimes \phi_Q + F_P^r \otimes \phi_Q d\big)\sigma \\
&= \big((d\phi_P + \phi_P d) \otimes F_Q^{\ell} + F_P^r \otimes (d\phi_Q + \phi_Q d)\big)\sigma \\
&= \big(F_P \otimes F_Q^{\ell} + F_P^r \otimes F_Q\big)\sigma \\
&= \big((F_P^{\ell} - F_P^r) \otimes F_Q^{\ell} + F_P^r \otimes (F_Q^{\ell} - F_Q^r)\big)\sigma \\
&= \big(F_P^{\ell} \otimes F_Q^{\ell} - F_P^r \otimes F_Q^r\big)\sigma.
\end{aligned}
$$

Now apply Lemma 3.4. $\square$

As a consequence, the map ϕ given in Lemma 3.5 may be used to compute Gerstenhaber brackets on Hochschild cohomology of $R \otimes^t S$, under the isomorphism of complexes given by Lemma 3.3, provided Conditions 2.5(a)–(c) hold for

$$\mathbb{K} := \mathrm{Tot}(P \otimes^t Q).$$

That is, under those conditions, $[f, g] = f \circ g - (-1)^{(i-1)(j-1)} g \circ f$, where i, j are the homological degrees of f, g, respectively, and the circle product is given as in (2.9) by

$$f \circ g = f\phi(\mathbf{1}_{\mathbb{K}} \otimes g \otimes \mathbf{1}_{\mathbb{K}})\Delta_{\mathbb{K}}^{(2)},$$

where the definition of the map $\mathbf{1}_{\mathbb{K}} \otimes g \otimes \mathbf{1}_{\mathbb{K}}$ involves Koszul signs, and similarly $g \circ f$. We will use these formulas in the remainder of the paper.

4. Maps for some quantum complete intersections

Let $q \in k^\times$, and let

$$\Lambda = \Lambda_q := k \langle x, y \rangle / (x^2, y^2, xy + qyx),$$

which is a Koszul algebra whose Hochschild cohomology was computed (as an algebra) by Buchweitz, Green, Madsen, and Solberg [2]. In the next section, we compute its Gerstenhaber brackets, after defining all the needed maps in this section. We can identify Λ as the twisted tensor product $R \otimes_k^t S$, where $R := k[x]/(x^2)$, $S := k[y]/(y^2)$, $A = B = \mathbb{Z}$, and $t : \mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z} \to k^\times$ is the homomorphism of abelian groups defined by $t(1 \otimes_\mathbb{Z} 1) = -q^{-1}$. (We take $|x| = 1$, $|y| = 1$.)

We will use the techniques in [9], in combination with our results in Section 3, to compute the Gerstenhaber brackets for $\Lambda = R \otimes^t S$. Let

$$\mathbb{K}^x : \quad \cdots \xrightarrow{\cdot u} R^e \xrightarrow{\cdot v} R^e \xrightarrow{\cdot u} R^e \xrightarrow{m} R \to 0$$

be the R^e-projective resolution of R where $u = x \otimes 1 - 1 \otimes x$, $v = x \otimes 1 + 1 \otimes x$, and m is the multiplication map. Letting ϵ_i denote the element $1 \otimes 1$ of R^e in homological degree i, we see that we must give ϵ_i the graded degree i as an element of $\mathbb{Z}$ as well, in order for the differentials to be of graded degree 0. We may thus view the resolution $\mathbb{K}^x$ more precisely as a resolution of graded modules:

$$\mathbb{K}^x : \quad \cdots \xrightarrow{\cdot u} R^e \langle 2 \rangle \xrightarrow{\cdot v} R^e \langle 1 \rangle \xrightarrow{\cdot u} R^e \xrightarrow{m} R \to 0, \tag{4.1}$$

the (standard) notation for the degree shift as in [1]. Similarly, we have the S^e-projective resolution $\mathbb{K}^y$ of S. Take the total complex of $\mathbb{K}^x \otimes^t \mathbb{K}^y$ and call it $\mathbb{K}$. As explained in Section 3 (setting $P = \mathbb{K}^x$, $Q = \mathbb{K}^y$), the complex $\mathbb{K} := \mathrm{Tot}(\mathbb{K}^x \otimes^t \mathbb{K}^y)$ is a graded projective resolution of Λ as a Λ^e-module.

Denote the generators of $\mathbb{K}_n$ as a Λ^e-module by $\{\epsilon_{i,j}\}_{i+j=n}$, where we define $\epsilon_{i,j} := \epsilon_i \otimes \epsilon_j$, that is, $\epsilon_{i,j}$ is the copy of $1 \otimes 1$ with homological degree i in x and degree j in y. One can check that after appropriate identifications, in all degrees $n = i + j$, the differentials are given by:

$$d_{i,j}^{\mathbb{K}} : \epsilon_{i,j} \mapsto x\epsilon_{i-1,j} + (-1)^n q^j \epsilon_{i-1,j} x + q^i y \epsilon_{i,j-1} + (-1)^n \epsilon_{i,j-1} y.$$

Let $\mathbb{B}$ be the bar resolution of Λ as defined in (2.1). Since Λ is Koszul and $\mathbb{K}$ is a Koszul resolution, Conditions 2.5(a)–(c) hold (see [3,9]). Therefore we may indeed compute Gerstenhaber brackets using the techniques in [9], in combination with our results in Section 3. We will need the following explicit formulas for some of the relevant maps:

The embedding chain map $\iota : \mathbb{K} \to \mathbb{B}$. We have in low degrees, from [2],

$$\iota_0 : \epsilon_{0,0} \mapsto 1 \otimes 1,$$
$$\iota_1 : \epsilon_{0,1} \mapsto 1 \otimes y \otimes 1,$$
$$\epsilon_{1,0} \mapsto 1 \otimes x \otimes 1,$$
$$\iota_2 : \epsilon_{0,2} \mapsto 1 \otimes y \otimes y \otimes 1,$$
$$\epsilon_{1,1} \mapsto 1 \otimes x \otimes y \otimes 1 + q \otimes y \otimes x \otimes 1,$$
$$\epsilon_{2,0} \mapsto 1 \otimes x \otimes x \otimes 1,$$
$$\iota_3 : \epsilon_{0,3} \mapsto 1 \otimes y \otimes y \otimes y \otimes 1,$$
$$\epsilon_{1,2} \mapsto 1 \otimes x \otimes y \otimes y \otimes 1 + q \otimes y \otimes x \otimes y \otimes 1 + q^2 \otimes y \otimes y \otimes x \otimes 1,$$
$$\epsilon_{2,1} \mapsto 1 \otimes x \otimes x \otimes y \otimes 1 + q \otimes x \otimes y \otimes x \otimes 1 + q^2 \otimes y \otimes x \otimes x \otimes 1,$$
$$\epsilon_{3,0} \mapsto 1 \otimes x \otimes x \otimes x \otimes 1.$$

In general, $\iota_n(\epsilon_{i,l}) = \tilde{f}_l^{i+l}$ in the notation of [2], where $n = i + l$, and this identifies our complex $\mathbb{K}$ with $\mathbb{P}$ of [2], at the same time verifying Condition 2.5(a).

We will not need an explicit formula for a chain map $\pi : \mathbb{B} \to \mathbb{K}$ satisfying Condition 2.5(b). This is an advantage of our approach, in comparison with traditional methods. Existence of π is guaranteed by the observation that, for each n, the image of $\{\epsilon_{i,l} \mid i + l = n\}$ under ι_n in $\mathbb{B}$ may be extended to a free Λ^e-basis of $\mathbb{B}_n$.

The diagonal map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_{\Lambda} \mathbb{K}$. Condition 2.5(c) states that this map must satisfy the relation $\Delta_{\mathbb{B}} \circ \iota = (\iota \otimes_{\Lambda} \iota) \circ \Delta_{\mathbb{K}}$, where $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B} \otimes_{\Lambda} \mathbb{B}$ is given by (2.2). By [2, p. 810], via the identification $\tilde{f}_l^n \leftrightarrow \epsilon_{i,l}$ ($i + l = n$), such a diagonal map is given by

$$\Delta_{\mathbb{K}}(\epsilon_{i,l}) = \sum_{w=0}^{i+l} \sum_{j=\max\{0,-i+w\}}^{\min\{w,l\}} q^{j(i+j-w)} \epsilon_{w-j,j} \otimes_{\Lambda} \epsilon_{i+j-w,l-j}.$$

We will write out lower degree terms that are needed for some of our calculations:

$$(\Delta_{\mathbb{K}})_0 : \epsilon_{0,0} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{0,0},$$
$$(\Delta_{\mathbb{K}})_1 : \epsilon_{0,1} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{0,1} + \epsilon_{0,1} \otimes_{\Lambda} \epsilon_{0,0},$$
$$\epsilon_{1,0} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{1,0} + \epsilon_{1,0} \otimes_{\Lambda} \epsilon_{0,0},$$
$$(\Delta_{\mathbb{K}})_2 : \epsilon_{0,2} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{0,2} + \epsilon_{0,1} \otimes_{\Lambda} \epsilon_{0,1} + \epsilon_{0,2} \otimes_{\Lambda} \epsilon_{0,0},$$
$$\epsilon_{1,1} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{1,1} + \epsilon_{1,0} \otimes_{\Lambda} \epsilon_{0,1} + q\epsilon_{0,1} \otimes_{\Lambda} \epsilon_{1,0} + \epsilon_{1,1} \otimes_{\Lambda} \epsilon_{0,0},$$
$$\epsilon_{2,0} \mapsto \epsilon_{0,0} \otimes_{\Lambda} \epsilon_{2,0} + \epsilon_{1,0} \otimes_{\Lambda} \epsilon_{1,0} + \epsilon_{2,0} \otimes_{\Lambda} \epsilon_{0,0},$$

$$(\Delta_{\mathbb{K}})_3 : \epsilon_{0,3} \mapsto \epsilon_{0,0} \otimes_\Lambda \epsilon_{0,3} + \epsilon_{0,1} \otimes_\Lambda \epsilon_{0,2} + \epsilon_{0,2} \otimes_\Lambda \epsilon_{0,1} + \epsilon_{0,3} \otimes_\Lambda \epsilon_{0,0},$$

$$\epsilon_{1,2} \mapsto \epsilon_{0,0} \otimes_\Lambda \epsilon_{1,2} + \epsilon_{1,0} \otimes_\Lambda \epsilon_{0,2} + q\epsilon_{0,1} \otimes_\Lambda \epsilon_{1,1} + \epsilon_{1,1} \otimes_\Lambda \epsilon_{0,1}$$
$$+ q^2 \epsilon_{0,2} \otimes_\Lambda \epsilon_{1,0} + \epsilon_{1,2} \otimes_\Lambda \epsilon_{0,0},$$

$$\epsilon_{2,1} \mapsto \epsilon_{0,0} \otimes_\Lambda \epsilon_{2,1} + \epsilon_{1,0} \otimes_\Lambda \epsilon_{1,1} + q^2 \epsilon_{0,1} \otimes_\Lambda \epsilon_{2,0} + \epsilon_{2,0} \otimes_\Lambda \epsilon_{0,1}$$
$$+ q\epsilon_{1,1} \otimes_\Lambda \epsilon_{1,0} + \epsilon_{2,1} \otimes_\Lambda \epsilon_{0,0},$$

$$\epsilon_{3,0} \mapsto \epsilon_{0,0} \otimes_\Lambda \epsilon_{3,0} + \epsilon_{1,0} \otimes_\Lambda \epsilon_{2,0} + \epsilon_{2,0} \otimes_\Lambda \epsilon_{1,0} + \epsilon_{3,0} \otimes_\Lambda \epsilon_{0,0}.$$

Next we construct maps $\phi : \mathbb{K} \otimes_{R \otimes^t S} \mathbb{K} \to \mathbb{K}$, using Lemma 3.5, that we will need to compute brackets via the method in [9]. We will first need such maps for each of the factor algebras R and S. The following lemma is straightforward to check.

Lemma 4.2. *Letting* $R = k[x]/(x^2)$ *and* $\mathbb{K}^x$ *as defined in (4.1), the following map is a contracting homotopy for* $F_{\mathbb{K}^x}$:

$$\phi_{i+j}(\epsilon_i \otimes x^m \epsilon_j) = \delta_{m,1}(-1)^i \epsilon_{i+j+1}.$$

Letting $S = k[y]/(y^2)$, similarly we obtain a contracting homotopy for $F_{\mathbb{K}^y}$. As a consequence of Lemmas 3.5 and 4.2, a contracting homotopy $\phi := \phi_{R \otimes^t S}$ of $F_{\mathbb{K}}$ is as follows: To evaluate ϕ on $\epsilon_{i,j} \otimes_{R \otimes^t S} x^l y^m \epsilon_{p,r}$, we first apply the isomorphism

$$\sigma : (\mathbb{K}^x \otimes^t \mathbb{K}^y) \otimes_{R \otimes^t S} (\mathbb{K}^x \otimes^t \mathbb{K}^y) \xrightarrow{\sim} (\mathbb{K}^x \otimes_R \mathbb{K}^x) \otimes^t (\mathbb{K}^y \otimes_S \mathbb{K}^y)$$

of Lemma 3.3. Then

$$\phi(\epsilon_{i,j} \otimes_{R \otimes^t S} x^l y^m \epsilon_{p,r})$$
$$= \phi\big((\epsilon_i \otimes \epsilon_j) \otimes_{R \otimes^t S} ((x^l \otimes y^m)(\epsilon_p \otimes \epsilon_r))\big)$$
$$= \phi\big((\epsilon_i \otimes \epsilon_j) \otimes t^{\langle \epsilon_p | y^m \rangle}(x^l \epsilon_p \otimes y^m \epsilon_r)\big)$$
$$= (-1)^{jp}\big(\phi_{\mathbb{K}^x} \otimes F_{\mathbb{K}^y}^l + (-1)^{i+p} F_{\mathbb{K}^x}^r \otimes \phi_{\mathbb{K}^y}\big)$$
$$\qquad\qquad \big(t^{\langle \epsilon_p | y^m \rangle} t^{\langle x^l \epsilon_p | \epsilon_j \rangle}(\epsilon_i \otimes x^l \epsilon_p) \otimes (\epsilon_j \otimes y^m \epsilon_r)\big)$$
$$= (-1)^{jp} t^{\langle \epsilon_p | y^m \rangle} t^{\langle x^l \epsilon_p | \epsilon_j \rangle}$$
$$\qquad \cdot \big(\delta_{l,1}(-1)^i \epsilon_{i+p+1} \otimes \delta_{j,0} y^m \epsilon_r + (-1)^{i+p}\delta_{p,0}\epsilon_i x^l \otimes \delta_{m,1}(-1)^j \epsilon_{j+r+1}\big)$$
$$= (-1)^{jp}(-q^{-1})^{pm+(l+p)j}$$
$$\qquad \cdot \big(\delta_{j,0}\delta_{l,1}(-1)^i \epsilon_{i+p+1} \otimes y^m \epsilon_r + \delta_{p,0}\delta_{m,1}(-1)^{i+p+j}\epsilon_i x^l \otimes \epsilon_{j+r+1}\big).$$

If $j = 0$ and $p > 0$, by recalling the bimodule action of $R \otimes^t S$ on $\mathbb{K}^x \otimes^t \mathbb{K}^y$, this is

$$(-q^{-1})^{pm}\delta_{l,1}(-1)^i \epsilon_{i+p+1} \otimes y^m \epsilon_r$$
$$= (-q^{-1})^{pm}\delta_{l,1}(-1)^i (-q)^{(i+p+1)m} y^m \epsilon_{i+p+1,r}$$
$$= (-q)^{(i+1)m}(-1)^i \delta_{l,1} y^m \epsilon_{i+p+1,r}.$$

Similarly, if $j = 0$, $p = 0$, then we have

$$\delta_{l,1}(-1)^i \epsilon_{i+1} \otimes y^m \epsilon_r + \delta_{m,1}(-1)^i \epsilon_i x^l \otimes \epsilon_{r+1}$$
$$= (-q)^{(i+1)m} \delta_{l,1}(-1)^i y^m \epsilon_{i+1,r} + (-q)^{l(r+1)} \delta_{m,1}(-1)^i \epsilon_{i,r+1} x^l.$$

If $j > 0$, $p = 0$, we have

$$(-q^{-1})^{lj} \delta_{m,1}(-1)^{i+j} \epsilon_i x^l \otimes \epsilon_{j+r+1}$$
$$= (-q^{-1})^{lj}(-q)^{l(j+r+1)} \delta_{m,1}(-1)^{i+j} \epsilon_{i,j+r+1} x^l$$
$$= (-q)^{l(r+1)} \delta_{m,1}(-1)^{i+j} \epsilon_{i,j+r+1} x^l.$$

If $j > 0$, $p > 0$, we have 0. To summarize, the contracting homotopy ϕ is

$$\phi(\epsilon_{i,j} \otimes_\Lambda x^l y^m \epsilon_{p,r}) = \begin{cases} (-q)^{mi+m} \delta_{l,1}(-1)^i y^m \epsilon_{i+p+1,r}, & \text{if } j = 0, p > 0, \\ (-q)^{mi+m} \delta_{l,1}(-1)^i y^m \epsilon_{i+1,r} \\ \quad + (-q)^{lr+l} \delta_{m,1}(-1)^i \epsilon_{i,r+1} x^l, & \text{if } j = 0, p = 0, \\ (-q)^{lr+l} \delta_{m,1}(-1)^{i+j} \epsilon_{i,j+r+1} x^l, & \text{if } j > 0, p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

5. Brackets for some quantum complete intersections

In this section, we will compute the Gerstenhaber brackets on the Hochschild cohomology of $\Lambda = \Lambda_q := k \langle x, y \rangle / (x^2, y^2, xy + qyx)$, using the technique and maps described in previous sections. We consider various cases of $q \in k^\times$. We will find that in many cases, the Hochschild cohomology in degree 1 (that is, $\mathrm{HH}^1(\Lambda)$) is a finite dimensional abelian Lie algebra over which $\mathrm{HH}^*(\Lambda)$ is a module, the chosen generators being common eigenvectors. Exceptions occur when $q = \pm 1$. In particular, if $q = 1$ and $\mathrm{char}(k) \neq 2$, we find that $\mathrm{HH}^1(\Lambda)$ is isomorphic to the Lie algebra $\mathfrak{gl}_2(k)$ with a rather more complicated action on $\mathrm{HH}^*(\Lambda)$. Generally these brackets are however sufficient to determine the rest of the graded Lie structure, as we will see that brackets among higher degree generators are zero.

5.1. q **is not a root of unity ([2, 2.1]).** As computed in [2, 2.1],

$$\mathrm{HH}^*(\Lambda) \cong k[xy]/((xy)^2) \times_k \bigwedge^*(u_0, u_1),$$

the fiber product of rings, where $u_0 = (x, 0)$, $u_1 = (0, y)$ are of (homological) degree 1, and $k[xy]/((xy)^2)$ is the center of Λ (homological degree 0). That is, $\mathrm{HH}^*(\Lambda)$ is the subring of $k[xy]/((xy)^2) \oplus \bigwedge^*(u_0, u_1)$ consisting of pairs (ζ, ξ) such that the images of ζ and ξ under the respective augmentation maps are equal.

(Here, xy, u_0, and u_1 are in the kernels of their respective augmentation maps.) After translating the notation of [2] to that of our Section 4, we may identify $u_0 = x\epsilon_{1,0}^*$ and $u_1 = y\epsilon_{0,1}^*$, where $\epsilon_{i,l}^*(\epsilon_{j,m}) = \delta_{i,j}\delta_{l,m}$. Hence, we need to compute the circle products for pairs of elements from the set of algebra generators $\{xy, x\epsilon_{1,0}^*, y\epsilon_{0,1}^*\}$. The rest will follow using (2.12). Applying the formula (2.9), we have the following:

$$(x\epsilon_{1,0}^* \circ x\epsilon_{1,0}^*)(\epsilon_{1,0}) = x,$$
$$(x\epsilon_{1,0}^* \circ y\epsilon_{0,1}^*)(\epsilon_{0,1}) = 0,$$
$$(y\epsilon_{0,1}^* \circ x\epsilon_{1,0}^*)(\epsilon_{1,0}) = 0,$$
$$(y\epsilon_{0,1}^* \circ y\epsilon_{0,1}^*)(\epsilon_{0,1}) = y.$$

The corresponding Gerstenhaber brackets are thus all 0. Non-zero brackets arising when pairing generators with the degree 0 element xy are:

$$[x\epsilon_{1,0}^*, xy] = xy \quad \text{and} \quad [y\epsilon_{0,1}^*, xy] = xy.$$

In particular, we see that the Hochschild cohomology in degree 1 is a 2-dimensional abelian Lie algebra whose action on $\mathrm{HH}^*(\Lambda)$ is diagonal on the chosen basis, with eigenvalues 0, 1.

5.2. char$(k) \neq 2$ and $q = -1$ ([2, 3.4]). In this case, $\Lambda \cong R \otimes S$ is just the usual tensor product, and $\mathrm{HH}^*(\Lambda) \cong \mathrm{HH}^*(R) \otimes \mathrm{HH}^*(S)$, a graded tensor product of algebras. This isomorphism preserves the Gerstenhaber structure, as expected from [7, Theorem 3.3]. We give details next.

By [2, 3.4], after translating the notation to ours,

$$\mathrm{HH}^*(\Lambda) \cong (\Lambda \otimes \textstyle\bigwedge^*(x\epsilon_{1,0}^*, y\epsilon_{0,1}^*))[\epsilon_{2,0}^*, \epsilon_{0,2}^*]/(x(x\epsilon_{1,0}^*), y(y\epsilon_{0,1}^*), x\epsilon_{2,0}^*, y\epsilon_{0,2}^*).$$

We will compute circle products of pairs of elements from the set of generators $\{x, y, x\epsilon_{1,0}^*, y\epsilon_{0,1}^*, \epsilon_{2,0}^*, \epsilon_{0,2}^*\}$. The rest will follow by applying (2.12). By (2.9), direct computation yields non-zero circle products among these pairs of generators:

$$(x\epsilon_{1,0}^* \circ x)(\epsilon_{0,0}) = x,$$
$$(y\epsilon_{0,1}^* \circ y)(\epsilon_{0,0}) = y,$$
$$(x\epsilon_{1,0}^* \circ x\epsilon_{1,0}^*)(\epsilon_{1,0}) = x,$$
$$(y\epsilon_{0,1}^* \circ y\epsilon_{0,1}^*)(\epsilon_{0,1}) = y,$$
$$(\epsilon_{2,0}^* \circ x\epsilon_{1,0}^*)(\epsilon_{2,0}) = 2,$$
$$(\epsilon_{0,2}^* \circ y\epsilon_{0,1}^*)(\epsilon_{0,2}) = 2.$$

Therefore the non-zero Gerstenhaber brackets on generators of $\mathrm{HH}^*(\Lambda)$ are

$$[x\epsilon_{1,0}^*, x] = x,$$
$$[y\epsilon_{0,1}^*, y] = y,$$
$$[x\epsilon_{1,0}^*, \epsilon_{2,0}^*] = -2\epsilon_{2,0}^*,$$
$$[y\epsilon_{0,1}^*, \epsilon_{0,2}^*] = -2\epsilon_{0,2}^*.$$

By [7, Proposition-Definition 2.2] as summarized in (6.1) below, since $\Lambda \cong R \otimes S$, we can alternatively use the Gerstenhaber bracket structure on $\mathrm{HH}^*(R) \cong \mathrm{HH}^*(S)$ to compute the brackets on $\mathrm{HH}^*(\Lambda)$. We outline this approach next, for comparison.

For brevity, we will suppress the steps and show only the results. By [2, 3.4], we know $\mathrm{HH}^*(R) \cong R \times_k \bigwedge^*(x\epsilon_1^*)[\epsilon_2^*]$. Of the brackets we need to confirm our computations, the non-zero brackets among generators are

$$[x\epsilon_1^*, x] = x \quad \text{and} \quad [\epsilon_2^*, x\epsilon_1^*] = 2\epsilon_2^*,$$

as may be computed using (2.9) and Lemma 4.2. The Hochschild cohomology $\mathrm{HH}^*(S)$ will have the same Gerstenhaber bracket structure. By direct computation using (6.1) below and this bracket structure, we again find that the non-zero Gerstenhaber brackets on our choice of generators in $\mathrm{HH}^*(\Lambda)$ are

$$[x\epsilon_1^* \otimes 1, x \otimes 1] = x \otimes 1,$$
$$[1 \otimes y\epsilon_1^*, 1 \otimes y] = 1 \otimes y,$$
$$[x\epsilon_1^* \otimes 1, \epsilon_2^* \otimes 1] = -2\epsilon_2^* \otimes 1,$$
$$[1 \otimes y\epsilon_1^*, 1 \otimes \epsilon_2^*] = -2 \otimes \epsilon_2^*,$$

confirming our earlier calculations. In particular, we see that Hochschild cohomology in degree 1 is a 4-dimensional Lie algebra with basis $x\epsilon_{1,0}^*$, $y\epsilon_{0,1}^*$, $xy\epsilon_{1,0}^*$, $xy\epsilon_{0,1}^*$. The non-zero brackets among these basis elements are

$$[x\epsilon_{1,0}^*, xy\epsilon_{0,1}^*] = xy\epsilon_{0,1}^* \quad \text{and} \quad [y\epsilon_{0,1}^*, xy\epsilon_{1,0}^*] = xy\epsilon_{1,0}^*.$$

The action of $\mathrm{HH}^1(\Lambda)$ on $\mathrm{HH}^*(\Lambda)$ is nondiagonal on the chosen generating set, for example, applying (2.12) and the above equations, we find that

$$[xy\epsilon_{1,0}^*, \epsilon_{2,0}^*] = -2y\epsilon_{2,0}^*.$$

5.3. char(k) $\neq$ 2 and q is an odd root of unity ([2, 3.1]). Let q be a primitive rth root of unity, r odd. By [2, 3.1], translated into our notation,

$$\mathrm{HH}^*(\Lambda) \cong k[xy]/((xy)^2)$$
$$\times_k (\bigwedge^*(x\epsilon_{1,0}^*, y\epsilon_{0,1}^*)[\epsilon_{2r,0}^*, \epsilon_{r,r}^*, \epsilon_{0,2r}^*]/(\epsilon_{2r,0}^*\epsilon_{0,2r}^* - (\epsilon_{r,r}^*)^2)).$$

Thus we need to calculate the brackets on pairs of elements from the set

$$\{xy, x\epsilon^*_{1,0}, y\epsilon^*_{0,1}, \epsilon^*_{2r,0}, \epsilon^*_{r,r}, \epsilon^*_{0,2r}\}.$$

The rest will follow by applying (2.12). Of these pairs, the non-zero Gerstenhaber brackets are

$$[x\epsilon^*_{1,0}, xy] = xy,$$
$$[y\epsilon^*_{0,1}, xy] = xy,$$
$$[\epsilon^*_{2r,0}, x\epsilon^*_{1,0}] = 2r\epsilon^*_{2r,0},$$
$$[\epsilon^*_{r,r}, x\epsilon^*_{1,0}] = r\epsilon^*_{r,r},$$
$$[\epsilon^*_{r,r}, y\epsilon^*_{0,1}] = r\epsilon^*_{r,r},$$
$$[\epsilon^*_{0,2r}, y\epsilon^*_{0,1}] = 2r\epsilon^*_{0,2r}.$$

Other brackets may be computed using (2.12), for example,

$$[(\epsilon^*_{2r,0})^2, x\epsilon^*_{1,0}] = [\epsilon^*_{2r,0}, x\epsilon^*_{1,0}] \smile \epsilon^*_{2r,0} + \epsilon^*_{2r,0} \smile [\epsilon^*_{2r,0}, x\epsilon^*_{1,0}] = 4r(\epsilon^*_{2r,0})^2.$$

In this case, the degree 1 elements of Hochschild cohomology form a 2-dimensional abelian Lie algebra acting on Hochschild cohomology $HH^*(\Lambda)$, diagonally on the chosen generating set, with eigenvalues $1, -r, -2r$.

5.4. char$(k) \neq 2$, q is an even root of unity, and $q \neq 1$; or char$(k) = 2$, q is a root of unity, and $q \neq 1$ ([2, 3.2]). Let r be the order of q as a root of unity. As computed in [2, 3.2],

$$HH^*(\Lambda) \cong k[xy]/((xy)^2) \times_k (\textstyle\bigwedge^*(x\epsilon^*_{1,0}, y\epsilon^*_{0,1})[\epsilon^*_{r,0}, \epsilon^*_{0,r}].$$

Hence we need to compute brackets on pairs of elements from the generating set

$$\{xy, x\epsilon^*_{1,0}, y\epsilon^*_{0,1}, \epsilon^*_{r,0}, \epsilon^*_{0,r}\}.$$

Of these pairs, the non-zero Gerstenhaber brackets are

$$[x\epsilon^*_{1,0}, xy] = xy,$$
$$[y\epsilon^*_{0,1}, xy] = xy,$$
$$[\epsilon^*_{r,0}, x\epsilon^*_{1,0}] = r\epsilon^*_{r,0},$$
$$[\epsilon^*_{0,r}, y\epsilon^*_{0,1}] = r\epsilon^*_{0,r}.$$

Thus in degree 1, the Hochschild cohomology forms a 2-dimensional abelian Lie algebra acting diagonally on the chosen generating set for Hochschild cohomology, with eigenvalues $1, -r$.

5.5. char$(k) = 2$ and $q = 1$ ([2, 3.3]). As computed in [2, 3.3],

$$HH^*(\Lambda) \cong \Lambda[\epsilon_{1,0}^*, \epsilon_{0,1}^*].$$

We will compute brackets on pairs of elements from the set

$$\{x, y, \epsilon_{1,0}^*, \epsilon_{0,1}^*\}.$$

The non-zero Gerstenhaber brackets on generators are

$$[x, \epsilon_{1,0}^*] = 1,$$
$$[y, \epsilon_{0,1}^*] = 1.$$

Again, Λ is a tensor product of algebras, and the above brackets may be found alternatively by using formula (6.1) below, due to Le and Zhou [7]. Note that even though many brackets on pairs of generators are 0, there are many non-zero brackets, for example, using (2.12) we find that $[x\epsilon_{1,0}^*, \epsilon_{1,0}^*] = \epsilon_{1,0}^*$. Thus the degree 1 elements of Hochschild cohomology form an 8-dimensional nonabelian Lie algebra, with basis $\epsilon_{1,0}^*, \epsilon_{0,1}^*, x\epsilon_{1,0}^*, x\epsilon_{0,1}^*, y\epsilon_{1,0}^*, y\epsilon_{0,1}^*, xy\epsilon_{1,0}^*, xy\epsilon_{0,1}^*$. This Lie algebra acts nondiagonally on the chosen set of generators for $HH^*(\Lambda)$. For example, $[\epsilon_{1,0}^*, x\epsilon_{1,0}^*\epsilon_{0,1}^*] = \epsilon_{1,0}^*\epsilon_{0,1}^*$.

5.6. char$(k) \neq 2$ and $q = 1$ ([2, 3.5]). As computed in [2, 3.5],

$$HH^*(\Lambda) \cong \left(k[xy]/((xy)^2) \times_k \bigwedge{}^*(x\epsilon_{1,0}^*, y\epsilon_{1,0}^*, x\epsilon_{0,1}^*, y\epsilon_{0,1}^*)\right)[\epsilon_{2,0}^*, \epsilon_{1,1}^*, \epsilon_{0,2}^*]/I$$

where I is generated by

$$x\epsilon_{1,0}^*x\epsilon_{0,1}^*, \ y\epsilon_{1,0}^*y\epsilon_{0,1}^*, \ x\epsilon_{1,0}^*y\epsilon_{1,0}^* - xy\epsilon_{2,0}^*,$$
$$x\epsilon_{1,0}^*y\epsilon_{0,1}^* - xy\epsilon_{1,1}^*, \ x\epsilon_{0,1}^*y\epsilon_{0,1}^* - xy\epsilon_{0,2}^*, \ y\epsilon_{1,0}^*x\epsilon_{0,1}^* + xy\epsilon_{1,1}^*,$$
$$x\epsilon_{1,0}^*\epsilon_{1,1}^* - x\epsilon_{0,1}^*\epsilon_{2,0}^*, \ y\epsilon_{1,0}^*\epsilon_{1,1}^* - y\epsilon_{0,1}^*\epsilon_{2,0}^*, \ x\epsilon_{1,0}^*\epsilon_{0,2}^* - x\epsilon_{0,1}^*\epsilon_{1,1}^*,$$
$$y\epsilon_{1,0}^*\epsilon_{0,2}^* - y\epsilon_{0,1}^*\epsilon_{1,1}^*, \ \epsilon_{2,0}^*\epsilon_{0,2}^* - (\epsilon_{1,1}^*)^2.$$

We will compute brackets on pairs of elements from the set

$$\{xy, x\epsilon_{1,0}^*, y\epsilon_{1,0}^*, x\epsilon_{0,1}^*, y\epsilon_{0,1}^*, \epsilon_{2,0}^*, \epsilon_{1,1}^*, \epsilon_{0,2}^*\}.$$

Of these pairs, the non-zero Gerstenhaber brackets are

$$[xy, x\epsilon_{1,0}^*] = -xy,$$
$$[xy, y\epsilon_{0,1}^*] = -xy,$$
$$[xy, \epsilon_{2,0}^*] = -2y\epsilon_{1,0}^*,$$
$$[xy, \epsilon_{1,1}^*] = -y\epsilon_{0,1}^* + x\epsilon_{1,0}^*,$$

$$[xy, \epsilon_{0,2}^*] = 2x\epsilon_{0,1}^*,$$
$$[x\epsilon_{1,0}^*, y\epsilon_{1,0}^*] = -y\epsilon_{1,0}^*,$$
$$[x\epsilon_{1,0}^*, x\epsilon_{0,1}^*] = x\epsilon_{0,1}^*,$$
$$[y\epsilon_{1,0}^*, x\epsilon_{0,1}^*] = y\epsilon_{0,1}^* - x\epsilon_{1,0}^*,$$
$$[y\epsilon_{1,0}^*, y\epsilon_{0,1}^*] = -y\epsilon_{1,0}^*,$$
$$[x\epsilon_{0,1}^*, y\epsilon_{0,1}^*] = x\epsilon_{0,1}^*,$$
$$[x\epsilon_{1,0}^*, \epsilon_{2,0}^*] = -2\epsilon_{2,0}^*,$$
$$[x\epsilon_{1,0}^*, \epsilon_{1,1}^*] = -\epsilon_{1,1}^*,$$
$$[y\epsilon_{1,0}^*, \epsilon_{1,1}^*] = -\epsilon_{2,0}^*,$$
$$[y\epsilon_{1,0}^*, \epsilon_{0,2}^*] = -2\epsilon_{1,1}^*,$$
$$[x\epsilon_{0,1}^*, \epsilon_{2,0}^*] = -2\epsilon_{1,1}^*,$$
$$[x\epsilon_{0,1}^*, \epsilon_{1,1}^*] = -\epsilon_{0,2}^*,$$
$$[y\epsilon_{0,1}^*, \epsilon_{1,1}^*] = -\epsilon_{1,1}^*,$$
$$[y\epsilon_{0,1}^*, \epsilon_{0,2}^*] = -2\epsilon_{0,2}^*.$$

Again, we may use (2.12) to compute other brackets, e.g., $[\epsilon_{2,0}^*, xy\epsilon_{2,0}^*] = -2y\epsilon_{1,2}^*$. We find that the degree 1 elements of Hochschild cohomology form a Lie algebra isomorphic to $\mathfrak{gl}_2(k)$ via the following map:

$$x\epsilon_{1,0}^* \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x\epsilon_{0,1}^* \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$y\epsilon_{1,0}^* \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad y\epsilon_{0,1}^* \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Its action on $HH^*(\Lambda)$ is nondiagonal on the chosen set of generators.

In the more complicated cases where $q = \pm 1$, it would be interesting to obtain a more detailed description of the action of the Lie algebra $HH^1(\Lambda)$ on $HH^*(\Lambda)$.

6. A Gerstenhaber algebra isomorphism

We return to the general setting of a twisted tensor product $\Lambda = R \otimes^t S$, where R and S are graded by abelian groups A and B respectively, as defined in Section 3. Our main result is Theorem 6.3 below, which generalizes the main theorem of Le and Zhou [7] to the twisted setting of Bergh and Oppermann [1]. The result of [7] involves the known algebra isomorphism of the Hochschild cohomology of a tensor product of algebras with the graded tensor product of their Hochschild cohomologies, which is valid under some finiteness assumptions (see Mac Lane [8, Theorem X.7.4],

who cites Rose [10] with the first proof). Le and Zhou show that this isomorphism of algebras preserves Gerstenhaber brackets, so that it is in fact an isomorphism of Gerstenhaber algebras. Our result more generally takes Bergh and Oppermann's algebra isomorphism from a subalgebra of Hochschild cohomology of a *twisted* tensor product of algebras to a tensor product of subalgebras of their Hochschild cohomology rings, and shows that it preserves Gerstenhaber brackets, so that it is in fact an isomorphism of Gerstenhaber algebras. In order to do this, we will use twisted versions of the Alexander–Whitney and Eilenberg–Zilber maps. Our proof then diverges from that of Le and Zhou to take advantage of the general construction of Gerstenhaber brackets in [9] as applied to twisted tensor products specifically via our techniques from Section 3.

In this section, all algebras and modules will be graded, and HH will denote graded Hochschild cohomology. That is, if X, Y are A-graded R-modules, we let $\mathrm{Hom}(X, Y) := \oplus_{a \in A} \mathrm{Hom}(X, Y)_a$ where $\mathrm{Hom}(X, Y)_a$ consists of all R-homomorphisms from X to Y such that $|f(x)| = |x| - a$ for all homogeneous $x \in X$. Graded Hochschild cohomology arises by applying Hom to the appropriate resolution and taking homology.

Tensor product of Gerstenhaber algebras ([7, Proposition-Definition 2.2]). Let H_1 and H_2 be two Gerstenhaber algebras over k. Let $f, f' \in H_1$ be elements of degrees m, m', and let $g, g' \in H_2$ be of degrees n, n', respectively. Then $H_1 \otimes H_2$ is a Gerstenhaber algebra with product

$$(f \otimes g) \smile (f' \otimes g') := (-1)^{m'n}(f \smile f') \otimes (g \smile g')$$

and bracket

$$[f \otimes g, f' \otimes g'] := (-1)^{(m+n-1)n'}[f, f'] \otimes (g \smile g')$$
$$+ (-1)^{m(m'+n'-1)}(f \smile f') \otimes [g, g']. \quad (6.1)$$

Returning to our graded algebras R and S, the grading by groups A and B passes to cohomology (e.g. via the grading on the bar resolutions of R and S, respectively), so that the Hochschild cohomologies of R and S are bigraded. Specifically, letting $n \in \mathbb{N}$ and $a \in A$, an element of $\mathrm{HH}^{n,a}(R)$ is represented by an R^e-homomorphism $f : R^{\otimes(n+2)} \to R$ with $|f(r_0 \otimes \cdots \otimes r_{n+1})| = |r_0| + \cdots + |r_{n+1}| - a$ for all homogeneous $r_0, \ldots, r_{n+1} \in R$. Similarly the Hochschild cohomology of S is bigraded by $\mathbb{N}$ and B. Let

$$A' = \bigcap_{b \in B} \mathrm{Ker}\, t^{\langle -|b \rangle} \quad \text{and} \quad B' = \bigcap_{a \in A} \mathrm{Ker}\, t^{\langle a|- \rangle}, \quad (6.2)$$

which are subgroups of A and B, respectively. Let $H_1 = \mathrm{HH}^{*,A'}(R)$ and $H_2 = \mathrm{HH}^{*,B'}(S)$. These are Gerstenhaber subalgebras of $\mathrm{HH}^*(R)$ and of $\mathrm{HH}^*(S)$,

respectively, as may be seen from formulas (2.3), (2.8), (2.9), and (2.11) with $\mathbb{K} = \mathbb{B}$. Thus $H_1 \otimes H_2$ is a Gerstenhaber algebra with bracket defined by formula (6.1). The following theorem states that the algebra isomorphism of [1, Theorem 4.7] is in fact a Gerstenhaber algebra isomorphism.

Theorem 6.3. *Let R and S be k-algebras graded by abelian groups A and B, respectively, at least one of which is finite dimensional, and let t be a twisting. There is an isomorphism of Gerstenhaber algebras*

$$\mathrm{HH}^{*,A'}(R) \otimes \mathrm{HH}^{*,B'}(S) \cong \mathrm{HH}^{*,A'\oplus B'}(R \otimes^t S),$$

where the Gerstenhaber bracket on the left side is given by (6.1).

Remarks 6.4. (i) In the statement of the theorem, the tensor product of Gerstenhaber algebras is understood to restrict to the usual tensor product of graded algebras, that is the twisting sends $((i, a'), (j, b'))$ to $(-1)^{ij}$. In [1], this is given explicitly in the notation, while in [7] it is not. We use the notation of [7].

(ii) The reason this isomorphism is restricted to subalgebras corresponding to A' and B' is that the Hom, $\otimes$ interchange does not behave well with respect to graded bimodules and degree shifts. In particular, if $\alpha \in \mathrm{Hom}(X, R)_a$ and $\beta \in \mathrm{Hom}(Y, S)_b$ for some R^e-module X and S^e-module Y, then $\alpha \otimes \beta$ is generally *not* an $(R \otimes^t S)^e$-module homomorphism from $X \otimes^t Y$ to $R \otimes^t S$, unless $a \in A'$, $b \in B'$, since the module structure of $X \otimes^t Y$ involves the twist. See Remark 4.2 and the proof of Theorem 4.7 in [1] for more details.

Example 6.5. Many of the algebras Λ_q of Section 5 provide nontrivial illustrations of Theorem 6.3. For example, if q is a primitive rth root of unity, r odd (as in Section 5.3 above), then $\mathrm{HH}^{*,A'\oplus B'}(\Lambda_q)$ is a significant part of $\mathrm{HH}^*(\Lambda_q)$. The generators that are in this subalgebra are $x\epsilon_{1,0}^*$, $y\epsilon_{0,1}^*$, $\epsilon_{2r,0}^*$, and $\epsilon_{0,2r}^*$ (since $(-q^{-1})^{2r} = 1$). Brackets of pairs of these elements may be computed via formula (6.1), once brackets in $\mathrm{HH}^*(k[x]/(x^2))$ have been computed, for example, by the techniques of [9] or otherwise. Such computations yield the same results as in Section 5.3 above with less work. Some of the other choices of values of q in Section 5 similarly yield nontrivial illustrations of Theorem 6.3.

In order to prove Theorem 6.3, we will need to construct twisted versions of the Alexander–Whitney and Eilenberg–Zilber maps. This we do next (cf. [6]).

Choose a section of the quotient map of vector spaces from R to $\overline{R} := R/k \cdot 1$ (respectively, from S to $\overline{S}$), by which to identify $\overline{R}$ with a vector subspace of R (respectively, $\overline{S}$ of S). Choose a compatible section of the map from $R \otimes^t S$ to $\overline{R \otimes^t S}$, that is, identify $R \otimes^t S$ with the direct sum of its four subspaces $\overline{R} \otimes^t \overline{S}$, $\overline{R} \otimes^t k$, $k \otimes^t \overline{S}$, and $k \otimes^t k$, the sum of the first three of which is a subspace of $R \otimes^t S$ that we identify with $\overline{R \otimes^t S}$. Let $\overline{\mathbb{B}} = \overline{\mathbb{B}}(R \otimes^t S)$ be the normalized bar resolution of $R \otimes^t S$ and let

$$\mathbb{K} := \mathrm{Tot}(\overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S))$$

be the total complex of the twisted tensor product of the normalized bar resolutions of R and of S.

We define a twisted Alexander–Whitney map $\mathrm{AW}_*^t : \overline{\mathbb{B}}(R \otimes^t S) \to \overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S)$, generalizing that used in [7] to the twisted tensor product. In degree 0, let

$$\mathrm{AW}_0^t : (R \otimes^t S) \otimes (R \otimes^t S) \to (R \otimes R) \otimes^t (S \otimes S)$$
$$r \otimes^t s \otimes r' \otimes^t s' \mapsto t^{\langle r'|s \rangle} r \otimes r' \otimes^t s \otimes s',$$

for all homogeneous $r, r' \in R$ and $s, s' \in S$.

It is straightforward to check that AW_0^t is an $(R \otimes^t S)^e$-module homomorphism with module action on $(R \otimes R) \otimes^t (S \otimes S)$ as given by (3.1) and module action on $(R \otimes^t S) \otimes (R \otimes^t S)$ given by multiplication on the left and right.

To define AW_n^t for $n > 0$, we use the identification of $\overline{R}$ as a subspace of R (respectively, $\overline{S}$ of S, $\overline{R \otimes^t S}$ of $R \otimes^t S$) as discussed at the beginning of this section, keeping in mind that if one of the r_i or s_i in the expression below is in the field k, then the only possibly non-zero summands in the expression are those for which it is in the first or last tensor factor. Define the $(R \otimes^t S)^e$-module homomorphism as follows:

$$\mathrm{AW}_n^t : (R \otimes^t S) \otimes \overline{R \otimes^t S}^{\otimes n} \otimes (R \otimes^t S)$$
$$\to \bigoplus_{d=0}^{n} (R \otimes \overline{R}^{\otimes n-d} \otimes R) \otimes^t (S \otimes \overline{S}^{\otimes d} \otimes S)$$
$$1 \otimes^t 1 \otimes r_1 \otimes^t s_1 \otimes \cdots \otimes r_n \otimes^t s_n \otimes 1 \otimes^t 1$$
$$\mapsto \sum_{d=0}^{n} (-1)^{d(n-d)} t^* r_1 r_2 \cdots r_d \otimes r_{d+1} \otimes \cdots$$
$$\cdots \otimes r_n \otimes 1 \otimes^t 1 \otimes s_1 \otimes \cdots \otimes s_d \otimes s_{d+1} \cdots s_n,$$

where

$$t^* = t^{\langle r_1|1 \rangle} t^{\langle r_2|s_1 \otimes 1 \rangle} \cdots t^{\langle r_n|s_{n-1} \otimes \cdots \otimes s_1 \otimes 1 \rangle} t^{\langle 1|s_n \otimes \cdots \otimes s_1 \otimes 1 \rangle},$$

for all homogeneous $r_i \in R, s_j \in S$. It may be checked that AW_n^t does indeed define an $(R \otimes^t S)^e$-module homomorphism. Moreover, by a lengthy calculation, it can be seen that this choice of AW_n^t commutes with the differentials.

Similarly, we generalize the Eilenberg–Zilber chain map

$$\mathrm{EZ}_*^t : \overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S) \to \overline{\mathbb{B}}(R \otimes^t S)$$

as in [7] to the twisted case. Let

$$\mathrm{EZ}_0^t : (R \otimes R) \otimes^t (S \otimes S) \to (R \otimes^t S) \otimes (R \otimes^t S)$$
$$r \otimes r' \otimes^t s \otimes s' \mapsto t^{-\langle r'|s \rangle} r \otimes^t s \otimes r' \otimes^t s'.$$

To define EZ_n^t for $n > 0$, we need the following notation from [7]: $S_{n-d,d}$ is the set of $(n-d, d)$-shuffles, that is the permutations ξ in the symmetric group S_n for which $\xi(1) < \xi(2) < \cdots < \xi(n-d)$ and $\xi(n-d+1) < \xi(n-d+2) < \cdots < \xi(n)$.

For all $\xi \in S_{n-d,d}$, all $r_1, \ldots, r_{n-d} \in R$ and $s_1, \ldots, s_d \in S$, let

$$F_\xi(r_1 \otimes \cdots \otimes r_{n-d} \otimes^t s_1 \otimes \cdots \otimes s_d) = F(x_{\xi^{-1}(1)}) \otimes \cdots \otimes F(x_{\xi^{-1}(n)})$$

where $x_1 = r_1, \ldots, x_{n-d} = r_{n-d}, x_{n-d+1} = s_1, \ldots, x_n = s_d$ and $F(r) = r \otimes 1$, $F(s) = 1 \otimes s$ for $r \in R$, $s \in S$. We will also use the notation

$$\mathrm{inv}(\xi) = \{(i, j) \mid 1 \leq i < j \leq n \text{ and } \xi(i) > \xi(j)\},$$
$$|\xi| = |\mathrm{inv}(\xi)|,$$
$$t^{-\mathrm{inv}(\xi)} = \prod_{(i,j) \in \mathrm{inv}(\xi)} t^{-\langle r_i | s_{j-n+d} \rangle}.$$

Now define the $(R \otimes^t S)^e$-module homomorphism:

$$\mathrm{EZ}_n^t : \bigoplus_{d=0}^n (R \otimes \overline{R}^{\otimes n-d} \otimes R) \otimes^t (S \otimes \overline{S}^{\otimes d} \otimes S)$$
$$\to (R \otimes^t S) \otimes \overline{R \otimes^t S}^{\otimes n} \otimes (R \otimes^t S)$$

$$1 \otimes r_1 \otimes \cdots \otimes r_{n-d} \otimes 1 \otimes^t 1 \otimes s_1 \otimes \cdots \otimes s_d \otimes 1$$
$$\mapsto 1 \otimes^t 1 \otimes \left(\sum_{\xi \in S_{n-d,d}} (-1)^{|\xi|} t^{-\mathrm{inv}(\xi)} F_\xi(r_1 \otimes \cdots \right.$$
$$\left. \cdots \otimes r_{n-d} \otimes^t s_1 \otimes \cdots \otimes s_d) \right) \otimes 1 \otimes^t 1.$$

As with AW_*^t, it can be checked that EZ_*^t is in fact a chain map.

Now, in order to use the methods of [9] to describe the Gerstenhaber brackets on the Hochschild cohomology of $\Lambda = R \otimes^t S$, we must check Conditions 2.5(a)–(c) on $\mathbb{K} = \mathrm{Tot}(\overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S))$:

(a) Let $\iota = \iota_{\mathbb{B}} \mathrm{EZ}_*^t$, where $\iota_{\mathbb{B}} : \overline{\mathbb{B}}(R \otimes^t S) \to \mathbb{B}(R \otimes^t S)$ is a choice of embedding compatible with our identifications of $\overline{R}$ and $\overline{S}$ as subspaces of R and S.

(b) Let $\pi = \mathrm{AW}_*^t \pi_{\mathbb{B}}$, where $\pi_{\mathbb{B}} : \mathbb{B}(R \otimes^t S) \to \overline{\mathbb{B}}(R \otimes^t S)$ is the quotient map. We want to show that $\pi \iota := \mathrm{AW}_*^t \circ \mathrm{EZ}_*^t = \mathbf{1}_{\mathbb{K}}$. By their definitions, $\pi_{\mathbb{B}} \iota_{\mathbb{B}} = \mathbf{1}_{\mathbb{B}}$, and as in [7], we know that for the maps without the twist, $\mathrm{AW}_* \circ \mathrm{EZ}_* = \mathbf{1}_{\mathbb{K}}$. Therefore, we need only check that the coefficients included in relation to the twist cancel:

$$\mathrm{AW}_n^t \circ \mathrm{EZ}_n^t \left((1 \otimes r_1 \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_d \otimes 1) \right)$$
$$= \mathrm{AW}_n^t \left(1 \otimes^t 1 \otimes \left(\sum_{\xi \in S_{n-d,d}} (-1)^{|\xi|} t^{-\mathrm{inv}(\xi)} F(x_{\xi^{-1}(1)}) \otimes \cdots \right. \right.$$
$$\left. \left. \cdots \otimes F(x_{\xi^{-1}(n)}) \right) \otimes 1 \otimes^t 1 \right),$$

where for each i, $x_{\xi^{-1}(i)}$ is either $r_{\xi^{-1}(i)}$ or $s_{\xi^{-1}(i)-(n-d)}$ depending on the value of $\xi^{-1}(i)$. Then $F(x_{\xi^{-1}(i)})$ is either $r_{\xi^{-1}(i)} \otimes 1$ or $1 \otimes s_{\xi^{-1}(i)-(n-d)}$. After applying AW_n^t, the twisting coefficient for the term corresponding to ξ is $t^{\mathrm{inv}(\xi)-\mathrm{inv}(\xi)} = 1$. Therefore $\pi \iota = \mathbf{1}_{\mathbb{K}}$.

(c) Consider $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_{R \otimes^t S} \mathbb{K}$ defined by

$$(\Delta_{\mathbb{K}})_n \big((1 \otimes r_1 \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_d \otimes 1) \big)$$

$$= \sum_{j=0}^{n-d} \sum_{i=0}^{d} (-1)^{i(n-d-j)} t^{-\langle r_{j+1} \otimes \cdots \otimes r_{n-d} | s_1 \otimes \cdots \otimes s_i \rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_j \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_i \otimes 1)] \otimes_{R \otimes^t S}$$
$$[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_d \otimes 1)]$$

for all homogeneous $r_l \in R$ and $s_m \in S$.

Then

$$(\iota_{\mathbb{B}} EZ_*^t \otimes_{R \otimes^t S} \iota_{\mathbb{B}} EZ_*^t) \Delta_{\mathbb{K}} \big((1 \otimes r_1 \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_d \otimes 1) \big)$$

$$= (\iota_{\mathbb{B}} EZ_*^t \otimes_{R \otimes^t S} \iota_{\mathbb{B}} EZ_*^t) \bigg(\sum_{j=0}^{n-d} \sum_{i=0}^{d} (-1)^{i(n-d-j)} t^{-\langle r_{j+1} \otimes \cdots \otimes r_{n-d} | s_1 \otimes \cdots \otimes s_i \rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_j \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_i \otimes 1)] \otimes_{R \otimes^t S}$$
$$[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_d \otimes 1)] \bigg)$$

$$= \sum_{j=0}^{n-d} \sum_{i=0}^{d} (-1)^{i(n-d-j)} t^{-\langle r_{j+1} \otimes \cdots \otimes r_{n-d} | s_1 \otimes \cdots \otimes s_i \rangle}$$
$$[1 \otimes^t 1 \otimes \bigg(\sum_{\xi' \in S_{j,i}} (-1)^{|\xi'|} t^{-\operatorname{inv}(\xi')} \cdot F_{\xi'}(r_1 \otimes \cdots \otimes r_j \otimes^t s_1 \otimes \cdots \otimes s_i) \bigg) \otimes 1 \otimes^t 1] \otimes_{R \otimes^t S}$$
$$[1 \otimes^t 1 \otimes \bigg(\sum_{\xi'' \in S_{n-d-j,d-i}} (-1)^{|\xi''|} t^{-\operatorname{inv}(\xi'')} \cdot F_{\xi''}(r_{j+1} \otimes \cdots \otimes r_{n-d} \otimes^t s_{i+1} \otimes \cdots \otimes s_d) \bigg) 1 \otimes^t 1]$$

and

$$\Delta_{\mathbb{B}} (\iota_{\mathbb{B}} EZ_*^t) \big((1 \otimes r_1 \otimes \cdots \otimes r_{n-d} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_d \otimes 1) \big)$$

$$= \Delta_{\mathbb{B}} \bigg(1 \otimes^t 1 \otimes \bigg(\sum_{\xi \in S_{n-d,d}} (-1)^{|\xi|} t^{-\operatorname{inv}(\xi)} \cdot F_\xi(r_1 \otimes \cdots \otimes r_{n-d} \otimes^t s_1 \otimes \cdots \otimes s_d) \bigg) \otimes 1 \otimes^t 1 \bigg)$$

$$= \Delta_{\mathbb{B}} \bigg(1 \otimes^t 1 \otimes \bigg(\sum_{\xi \in S_{n-d,d}} (-1)^{|\xi|} t^{-\operatorname{inv}(\xi)} \cdot F(x_{\xi^{-1}(1)}) \otimes \cdots \otimes F(x_{\xi^{-1}(n)}) \bigg) \otimes 1 \otimes^t 1 \bigg)$$

$$= \sum_{i=0}^{n} \sum_{\xi \in S_{n-d,d}} (-1)^{|\xi|} t^{-\operatorname{inv}(\xi)} [1 \otimes^t 1 \otimes F(x_{\xi^{-1}(1)}) \otimes \cdots \otimes F(x_{\xi^{-1}(i)}) \otimes 1 \otimes^t 1]$$
$$\otimes_{R \otimes^t S} [1 \otimes^t 1 \otimes F(x_{\xi^{-1}(i+1)}) \otimes \cdots \otimes F(x_{\xi^{-1}(n)}) \otimes 1 \otimes^t 1],$$

where for each i, $x_{\xi^{-1}(i)}$ is either $r_{\xi^{-1}(i)}$ or $s_{\xi^{-1}(i)-(n-d)}$, depending on the value of $\xi^{-1}(i)$. Now notice in both expressions, we are allowing all possible arrangements of r's and s's, thus, we need only check that the corresponding coefficients agree. Given a fixed arrangement of the r's and s's determined by $\xi \in S_{n-d,d}$, we see that $(-1)^{|\xi|}t^{-\mathrm{inv}(\xi)}$ is uniquely determined by the s's and r's that are moved past each other. The corresponding term in the first expression has coefficient

$$(-1)^{i(n-d-j)+|\xi'|+|\xi''|}t^{-\langle r_{j+1}\otimes\cdots\otimes r_{n-d}|s_1\otimes\cdots\otimes s_i\rangle-\mathrm{inv}(\xi')-\mathrm{inv}(\xi'')},$$

for some i and j, and $\xi' \in S_{j,i}$, and $\xi'' \in S_{n-d-j,d-i}$, which is again uniquely determined by the s's and r's that are moved past each other. Thus, because we are assuming we have the same arrangement of r's and s's,

$$(-1)^{i(n-d-j)+|\xi'|+|\xi''|}t^{-\langle r_{j+1}\otimes\cdots\otimes r_{n-d}|s_1\otimes\cdots\otimes s_i\rangle-\mathrm{inv}(\xi')-\mathrm{inv}(\xi'')} = (-1)^{|\xi|}t^{-\mathrm{inv}(\xi)}$$

when we view the term as coming from $\xi \in S_{n-d,d}$. Therefore,

$$(\iota_{\mathbb{B}}\,\mathrm{EZ}_*^t \otimes_{R\otimes^t S}\iota_{\mathbb{B}}\,\mathrm{EZ}_*^t)\Delta_{\mathbb{K}} = \Delta_{\mathbb{B}}(\iota_{\mathbb{B}}\,\mathrm{EZ}_*^t).$$

We now have chain maps π, ι, and $\Delta_{\mathbb{K}}$ satisfying Conditions 2.5(a)–(c). Therefore, we may use the formulas (2.8) and (2.9) to describe Gerstenhaber brackets, via a contracting homotopy ϕ of $F_{\mathbb{K}}$. By Lemma 3.5, we may choose

$$\phi = (G_{\overline{\mathbb{B}}(R)} \otimes F_{\overline{\mathbb{B}}(S)}^{\ell} + (-1)^* F_{\overline{\mathbb{B}}(R)}^{r} \otimes G_{\overline{\mathbb{B}}(S)})\sigma,$$

where $G_{\overline{\mathbb{B}}(R)}$, $G_{\overline{\mathbb{B}}(S)}$ are defined in equation (2.11), $F_{\overline{\mathbb{B}}(S)}^{\ell}$, $F_{\overline{\mathbb{B}}(R)}^{r}$ are defined in equation (2.4), and σ is the map from Lemma 3.3. (See Lemma 3.5 for the precise value of $(-1)^*$.)

Proof of Theorem 6.3. Bergh and Oppermann [1, Theorem 4.7] proved that there is such an isomorphism of associative algebras. Their isomorphism may be realized explicitly at the chain level by using $\mathbb{K} = \mathrm{Tot}(\overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S))$ to express elements on the right-hand side, via the Hom, $\otimes$ interchange, as elements on the left-hand side. Our diagonal map $\Delta_{\mathbb{K}}$ may be used to describe cup products. We need only show that this isomorphism also preserves Gerstenhaber brackets. One approach would be to use the known algebra isomorphism combined with (2.12), showing that some mixed terms are 0. Another approach would be to generalize the proof of Le and Zhou, which is an explicit computation using the chain maps AW_* and EZ_*. We take yet another approach, using the theory we have developed for twisted tensor products in Section 3 and in the first part of this section, which has the advantage of avoiding explicit computations with the cumbersome chain maps AW_* and EZ_* themselves.

Brackets on the right-hand side of the isomorphism will be described by using $\mathbb{K} = \mathrm{Tot}(\overline{\mathbb{B}}(R) \otimes^t \overline{\mathbb{B}}(S))$. We will use the chain maps ι and π which are comparison morphisms between $\mathbb{K}$ and $\mathbb{B} = \mathbb{B}(R \otimes^t S)$, and the diagonal map $\Delta_{\mathbb{K}}$ which allows a construction of the bracket operation on $\mathbb{K}$ via formulas (2.8) and (2.9).

Let $\alpha \in \mathrm{HH}^{m,A'}(R)$, $\alpha' \in \mathrm{HH}^{m',A'}(R)$, $\beta \in \mathrm{HH}^{n,B'}(S)$, and $\beta' \in \mathrm{HH}^{n',B'}(S)$. By abuse of notation, we also denote by $\alpha, \alpha', \beta, \beta'$ the morphisms representing the corresponding cohomology elements. We will write $\alpha \otimes \beta$ and $\alpha' \otimes \beta'$ to represent elements in $\mathrm{HH}^{*,A'\oplus B'}(R \otimes^t S)$ via its algebra isomorphism to $\mathrm{HH}^{*,A'}(R) \otimes \mathrm{HH}^{*,B'}(S)$. We will need the finite dimension hypothesis in interchanging Hom and $\otimes$ in the tensor product of chain complexes, as we are working with bar resolutions. We will compute $[\alpha \otimes \beta, \alpha' \otimes \beta']$ as an element of $\mathrm{HH}^{*,A'\oplus B'}(R \otimes^t S)$ using (2.8) and (2.9), and we will show that it agrees with the Gerstenhaber bracket on a tensor product of two Gerstenhaber algebras as defined in (6.1).

We will want to apply $[\alpha \otimes \beta, \alpha' \otimes \beta']$ to elements of the form

$$(1 \otimes r_1 \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_{n''} \otimes 1)$$

where $m'' + n'' = m + m' + n + n' - 1$ and $r_1, \ldots, r_{m''} \in \overline{R}$, $s_1, \ldots, s_{n''} \in \overline{S}$. In the calculation below, we will see that $(-1)^*$ is $(-1)^{m''-m'}$, partway through the calculation, as by that point we will already have applied α' to some of the input, thus lowering its homological degree. There are signs associated to application of each of the maps $(\mathbf{1} \otimes (\alpha' \otimes \beta') \otimes \mathbf{1})$ (the "Koszul signs" in (2.10)), and σ, $G_{\overline{\mathbb{B}}(R)}$ and $G_{\overline{\mathbb{B}}(S)}$ (in their definitions in Lemma 3.3 and in (2.11)). We start by computing a circle product:

$$(\alpha \otimes \beta) \circ (\alpha' \otimes \beta')\big((1 \otimes r_1 \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_{n''} \otimes 1)\big)$$

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^{\ell}_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)\sigma\big(\mathbf{1} \otimes (\alpha' \otimes \beta') \otimes \mathbf{1}\big)\Delta^{(2)}_{\mathbb{K}}$$
$$\big((1 \otimes r_1 \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_{n''} \otimes 1)\big)$$

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^{\ell}_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)\sigma\big(\mathbf{1} \otimes (\alpha' \otimes \beta') \otimes \mathbf{1}\big)(\Delta_{\mathbb{K}} \otimes \mathbf{1})$$
$$\left(\sum_{j=0}^{m''}\sum_{i=0}^{n''}(-1)^{i(m''-j)}t^{-\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_i\rangle}\right.$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_j \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_i \otimes 1)]\otimes_{R\otimes^t S}$$
$$\left.[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)]\right)$$

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^{\ell}_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)\sigma\big(\mathbf{1} \otimes (\alpha' \otimes \beta') \otimes \mathbf{1}\big)$$
$$\left(\sum_{j=0}^{m''}\sum_{i=0}^{n''}\sum_{p=0}^{i}\sum_{l=0}^{j}(-1)^{i(m''-j)}(-1)^{p(j-l)}\right.$$
$$t^{-\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_i\rangle}t^{-\langle r_{l+1}\otimes\cdots\otimes r_j|s_1\otimes\cdots\otimes s_p\rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_l \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_p \otimes 1)]\otimes_{R\otimes^t S}$$
$$[(1 \otimes r_{l+1} \otimes \cdots \otimes r_j \otimes 1) \otimes^t (1 \otimes s_{p+1} \otimes \cdots \otimes s_i \otimes 1)]\otimes_{R\otimes^t S}$$
$$\left.[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)]\right).$$

Now, in order to apply $(\mathbf{1} \otimes (\alpha' \otimes \beta') \otimes \mathbf{1})$, we must have $m' = j - l$, $n' = i - p$. The Koszul sign from (2.10) is thus

$$(-1)^{(p+l)(m'+n')} = (-1)^{(m'+n')(j-m'+i-n')},$$

and the above becomes

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^{\ell}_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)\sigma$$
$$\bigg(\sum_{j=m'}^{m''} \sum_{i=n'}^{n''} (-1)^{i(m''-j)}(-1)^{(i-n')m'}(-1)^{(m'+n')(j-m'+i-n')}$$
$$t^{-\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_i\rangle - \langle r_{j-m'+1}\otimes\cdots\otimes r_j|s_1\otimes\cdots\otimes s_{i-n'}\rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_{j-m'} \otimes 1) \otimes^t (1 \otimes s_1 \otimes \cdots \otimes s_{i-n'} \otimes 1)]\otimes_{R\otimes^t S}$$
$$[\alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1) \otimes^t \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1)]\otimes_{R\otimes^t S}$$
$$[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)] \bigg).$$

After applying the definition (3.1) of the module action, and applying σ (which comes with a sign of $(-1)^{(i-n')(m''-j)}$), the above becomes

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^r_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)\sigma$$
$$\bigg(\sum_{j=m'}^{m''} \sum_{i=n'}^{n''} (-1)^{i(m''-j)}(-1)^{(i-n')m'}(-1)^{(m'+n')(j-m'+i-n')}$$
$$t^{-\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_i\rangle - \langle r_{j-m'+1}\otimes\cdots\otimes r_j|s_1\otimes\cdots\otimes s_{i-n'}\rangle}$$
$$t^{\langle \alpha'(1\otimes r_{j-m'+1}\otimes\cdots\otimes r_j \otimes 1)|s_1\otimes\cdots\otimes s_{i-n'}\rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_{j-m'} \otimes \alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1))\otimes^t$$
$$(1 \otimes s_1 \otimes \cdots \otimes s_{i-n'} \otimes \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1))]\otimes_{R\otimes^t S}$$
$$[(1 \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1) \otimes^t (1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)] \bigg)$$

$$= (\alpha \otimes \beta)\big(G_{\overline{\mathbb{B}}(R)} \otimes F^{\ell}_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}\big)$$
$$\bigg(\sum_{j=m'}^{m''} \sum_{i=n'}^{n''} (-1)^{-n'(m''-j)}(-1)^{(i-n')m'}(-1)^{(m'+n')(j-m'+i-n')}$$
$$t^{-\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_i\rangle - \langle r_{j-m'+1}\otimes\cdots\otimes r_j|s_1\otimes\cdots\otimes s_{i-n'}\rangle}$$
$$t^{\langle \alpha'(1\otimes r_{j-m'+1}\otimes\cdots\otimes r_j \otimes 1)|s_1\otimes\cdots\otimes s_{i-n'}\rangle}$$
$$t^{\langle r_{j+1}\otimes\cdots\otimes r_{m''}|s_1\otimes\cdots\otimes s_{i-n'}\otimes\beta'(1\otimes s_{i-n'+1}\otimes\cdots\otimes s_i \otimes 1)\rangle}$$
$$[(1 \otimes r_1 \otimes \cdots \otimes r_{j-m'} \otimes \alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1))$$
$$\otimes_R (1 \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1)]\otimes^t$$
$$[(1 \otimes s_1 \otimes \cdots \otimes s_{i-n'} \otimes \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1))$$
$$\otimes_S (1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)] \bigg).$$

For brevity, we denote by t^* the twisting coefficient in the above equation. Now $\alpha' \in \mathrm{HH}^{m',A'}(R)$ and $\beta' \in \mathrm{HH}^{n',B'}(S)$, that is, α' and β' have graded degrees in the kernel of the twist homomorphism, and it follows that

$$t^{\langle \alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1) | s_1 \otimes \cdots \otimes s_{i-n'} \rangle} = t^{\langle r_{j-m'+1} \otimes \cdots \otimes r_j | s_1 \otimes \cdots \otimes s_{i-n'} \rangle},$$

$$t^{\langle r_{j+1} \otimes \cdots \otimes r_{m''} | \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1) \rangle} = t^{\langle r_{j+1} \otimes \cdots \otimes r_{m''} | s_{i-n'+1} \otimes \cdots \otimes s_i \rangle}.$$

Thus, $t^* = 1$. Now we are ready to apply $G_{\overline{\mathbb{B}}(R)} \otimes F^\ell_{\overline{\mathbb{B}}(S)} + (-1)^* F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}$, and there are signs associated to each term. In order to apply $G_{\overline{\mathbb{B}}(R)} \otimes F^\ell_{\overline{\mathbb{B}}(S)}$, we must have $i = n'$ for the map to be non-zero, and the sign incurred is $(-1)^{j-m'}$. In order to apply $F^r_{\overline{\mathbb{B}}(R)} \otimes G_{\overline{\mathbb{B}}(S)}$, we must have $j = m''$ for the map to be non-zero, and the sign incurred is $(-1)^{i-n'}$; in addition, for this application, we find that $(-1)^* = (-1)^{j-m'+m''-j} = (-1)^{m''-m'} = (-1)^m$ (as for this term, $m'' = m + m'$). The above expression becomes

$$= (\alpha \otimes \beta)\bigg(\sum_{j=m'}^{m''} (-1)^{-n'(m''-j)}(-1)^{(m'+n')(j-m')}(-1)^{j-m'}$$

$$[1 \otimes r_1 \otimes \cdots \otimes r_{j-m'} \otimes \alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1) \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1]$$

$$\otimes^t [\beta'(1 \otimes s_1 \otimes \cdots \otimes s_i \otimes 1) \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1]$$

$$+ \sum_{i=n'}^{n''} (-1)^{(i-n')m'}(-1)^{(m'+n')(m+i-n')}(-1)^{i-n'}(-1)^m$$

$$[1 \otimes r_1 \otimes \cdots \otimes r_m \otimes \alpha'(1 \otimes r_{m+1} \otimes \cdots \otimes r_{m''})] \otimes^t$$

$$[1 \otimes s_1 \otimes \cdots \otimes s_{i-n'} \otimes \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1) \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1]\bigg)$$

$$= \sum_{j=m'}^{m''} (-1)^{-n'(m''-j)}(-1)^{(m'+n')(j-m')}(-1)^{j-m'}$$

$$\alpha(1 \otimes r_1 \otimes \cdots \otimes r_{j-m'} \otimes \alpha'(1 \otimes r_{j-m'+1} \otimes \cdots \otimes r_j \otimes 1) \otimes r_{j+1} \otimes \cdots \otimes r_{m''} \otimes 1)$$

$$\otimes^t \beta'(1 \otimes s_1 \otimes \cdots \otimes s_i \otimes 1)\beta(1 \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1)$$

$$+ \sum_{i=n'}^{n''} (-1)^{(i-n')m'}(-1)^{(m'+n')(m+i-n')}(-1)^{i-n'}(-1)^m$$

$$\alpha(1 \otimes r_1 \otimes \cdots \otimes r_m \otimes 1)\alpha'(1 \otimes r_{m+1} \otimes \cdots \otimes r_{m''}) \otimes^t$$

$$\beta(1 \otimes s_1 \otimes \cdots \otimes s_{i-n'} \otimes \beta'(1 \otimes s_{i-n'+1} \otimes \cdots \otimes s_i \otimes 1) \otimes s_{i+1} \otimes \cdots \otimes s_{n''} \otimes 1).$$

We wish to rewrite the sums. The first sum involves $\alpha \circ \alpha'$, in which the term indexed by j has a sign $(-1)^{(m'-1)(j-m')}$. The second sum involves $\beta \circ \beta'$, in which the term indexed by i has a sign $(-1)^{(n'-1)(i-n')}$. Accommodating these signs and rewriting, the above is equal to

$$(-1)^{n'(m-1)}(\alpha \circ \alpha') \otimes (\beta' \smile \beta) + (-1)^{m(m'+n'-1)}(\alpha \smile \alpha') \otimes (\beta \circ \beta')$$

applied to the input. Similarly,

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (-1)^{n(m'-1)}(\alpha' \circ \alpha) \otimes (\beta \smile \beta')$$
$$+ (-1)^{m'(m+n-1)}(\alpha' \smile \alpha) \otimes (\beta' \circ \beta).$$

We will use the following relation from [4, Theorem 7.3] to reverse the order of the cup product $\alpha' \smile \alpha$ in the above expression (and a similar relation for $\beta' \smile \beta$):

$$\alpha \circ (d^*\alpha') - d^*(\alpha \circ \alpha') + (-1)^{m'-1}(d^*\alpha) \circ \alpha' = (-1)^{m'-1}\big(\alpha' \smile \alpha - (-1)^{mm'}\alpha \smile \alpha'\big).$$

Now, α and α' are cocycles, so the two outermost terms on the left-hand side of the above equation are 0. Putting it all together, using this relation and formula (2.8), we obtain the Gerstenhaber bracket:

$$[\alpha \otimes \beta, \alpha' \otimes \beta']$$
$$= (\alpha \otimes \beta) \circ (\alpha' \otimes \beta') - (-1)^{(m+n-1)(m'+n'-1)}(\alpha' \otimes \beta') \circ (\alpha \otimes \beta)$$
$$= (-1)^{n'(m-1)}(\alpha \circ \alpha') \otimes (\beta' \smile \beta)$$
$$+ (-1)^{m(m'+n'-1)}(\alpha \smile \alpha') \otimes (\beta \circ \beta')$$
$$- (-1)^{(m+n-1)(m'+n'-1)+n(m'-1)}(\alpha' \circ \alpha) \otimes (\beta \smile \beta')$$
$$- (-1)^{(m+n-1)(m'+n'-1)+m'(m+n-1)}(\alpha' \smile \alpha) \otimes (\beta' \circ \beta)$$
$$= (-1)^{n'(m+n-1)}(\alpha \circ \alpha') \otimes (\beta \smile \beta') + (-1)^{mn'}(\alpha \circ \alpha') \otimes d^*(\beta \circ \beta')$$
$$+ (-1)^{m(m'+n'-1)}(\alpha \smile \alpha') \otimes (\beta \circ \beta')$$
$$+ (-1)^{m(m'+n'-1)+nn'-m'-n'}(\alpha' \circ \alpha) \otimes (\beta \smile \beta')$$
$$+ (-1)^{(m+n-1)(n'-1)+mm'+1}(\alpha \smile \alpha') \otimes (\beta' \circ \beta)$$
$$- (-1)^{(m+n-1)(n'-1)+m'}d^*(\alpha \circ \alpha') \otimes (\beta' \circ \beta).$$

We claim that the terms involving $d^*(\beta \circ \beta')$ and $d^*(\alpha \circ \alpha')$ sum to a boundary:

$$d^*((\alpha \circ \alpha') \otimes (\beta' \circ \beta)) = d^*(\alpha \circ \alpha') \otimes (\beta' \circ \beta)$$
$$+ (-1)^{m+m'-1}(\alpha \circ \alpha') \otimes d^*(\beta' \circ \beta).$$

Since β, β' are cocycles, $d^*([\beta, \beta']) = 0$, that is,

$$d^*(\beta' \circ \beta) = (-1)^{(n-1)(n'-1)}d^*(\beta \circ \beta'),$$

which implies

$$d^*((\alpha \circ \alpha') \otimes (\beta' \circ \beta)) = d^*(\alpha \circ \alpha') \otimes (\beta' \circ \beta)$$
$$+ (-1)^{m+m'-1+(n-1)(n'-1)}(\alpha \circ \alpha') \otimes d^*(\beta \circ \beta'),$$

and this is $(-1)^{(m+n-1)(n'-1)+m'-1}$ times the sum of the two terms in our previous expression involving $d^*(\beta \circ \beta')$, $d^*(\alpha \circ \alpha')$. We now see that as elements in cohomology,

$$[\alpha \otimes \beta, \alpha' \otimes \beta']$$
$$= (-1)^{n'(m+n-1)}(\alpha \circ \alpha' - (-1)^{(m+n-1)(m'-1)+n(m'-1)}\alpha' \circ \alpha) \otimes (\beta \smile \beta')$$
$$+ (-1)^{m(m'+n'-1)}(\alpha \smile \alpha') \otimes (\beta \circ \beta' - (-1)^{(n-1)(m'+n'-1)+m'(n-1)}\beta' \circ \beta)$$
$$= (-1)^{(m+n-1)n'}[\alpha, \alpha'] \otimes (\beta \smile \beta') + (-1)^{m(m'+n'-1)}(\alpha \smile \alpha') \otimes [\beta, \beta'],$$

which agrees with formula (6.1). Thus we have proved that the algebra isomorphism

$$\mathrm{HH}^{*,A'}(R) \otimes \mathrm{HH}^{*,B'}(S) \cong \mathrm{HH}^{*,A' \oplus B'}(R \otimes^t S)$$

of Bergh and Oppermann [1] also preserves Gerstenhaber brackets. Therefore, it is an isomorphism of Gerstenhaber algebras, as claimed. □

In conclusion, the results of this section and of this paper may be applied effectively, to many algebras of interest, to obtain information about the Gerstenhaber algebra structure of Hochschild cohomology and to answer questions about deformations of algebras that involve this structure. The quantum complete intersections of Section 5 are of interest in their own right, and may also be generalized to include more generators and more general relations. The techniques in this paper may be applied to skew polynomial rings (cf. [11]). They may also be used to gain information about deformations of all of these algebras and their extensions by group actions, which include a wide variety of algebras of interest such as some quantum groups, some Nichols algebras arising in results on classification of finite dimensional Hopf algebras, and quantum versions of Drinfeld Hecke algebras or rational Cherednik algebras and related quotient algebras.

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