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The noncommutative Kalton–Peck spaces

Félix Cabello Sánchez*

Abstract. For every von Neumann algebra $\mathcal{M}$ and $0 < p < \infty$ we construct a nontrivial exact sequence of $\mathcal{M}$-bimodules and homomorphisms $0 \to L_p \to Z_p(\mathcal{M}) \to L_p \to 0$, where L_p is the Haagerup L_p space built over $\mathcal{M}$. The middle space $Z_p(\mathcal{M})$ can be seen as a noncommutative version of the Kalton–Peck space Z_p.

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1. Introduction and preliminaries

The aim of this short note is to construct a noncommutative version of the Kalton–Peck spaces in [23] for L_p spaces associated to a general von Neumann algebra, semifinite or not.

1.1. Background. Before going further let us recall some highlights on the "three-space problem" that may help the reader to understand how the content of this paper is related to the existing constructions. The reader can find other points of view in [8,14,22,24].

In 1975, Enflo, Lindenstrauss, and Pisier [11] proved that there exist "twisted Hilbert spaces", that is, Banach spaces Z with a closed subspace Y such that both Y and the quotient Z/Y are (isomorphic to) Hilbert spaces but Z is not. This can be rephrased by saying that there are nontrivial exact sequences of Banach spaces

$$0 \longrightarrow \ell_2 \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } \ell_2 \longrightarrow 0.$$

Here, "nontrivial" means that the image of $\imath$ is not complemented in Z.

Later on Kalton and Peck gave very explicit examples of nontrivial sequences

$$0 \longrightarrow \ell_p \longrightarrow Z \longrightarrow \ell_p \longrightarrow 0 \tag{1.1}$$

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for $0 < p < \infty$. Their most popular examples are the so-called Kalton–Peck spaces Z_p, which are "extreme" in many respects. From a "modern" perspective the space Z_p arises from the quasilinear map $\Omega : \ell_p \to \mathbb{C}^{\mathbb{N}}$ defined coordinatewise by $\Omega(f) = f \log(|f|/\|f\|_p)$. Indeed,

$$Z_p = \left\{(g, f) \in \mathbb{C}^{\mathbb{N}} \times \ell_p : \|g - \Omega(f)\|_p + \|f\|_p < \infty\right\}$$

with the quasinorm you can imagine. Soon afterwards it was realized that Ω is, not merely quasilinear, but also a centralizer: one has

$$\|\Omega(af) - a\Omega(f)\|_p \leq C\|a\|_\infty \|f\|_p \quad (a \in \ell_\infty, f \in \ell_p),$$

which makes Z_p into an ℓ_∞-module in such a way that the arrows in (1.1) become homomorphisms. Actually the quasilinear maps constructed in [23] are all centralizers.

This was claiming for generalization, and Kalton undertook the task in two different directions: he considered the function space version in [19] (regarding the Lebesgue spaces L_p as L_∞-modules) and the Schatten classes in [20] (regarding the Schatten classes S_p as $\mathcal{B}(H)$-modules). Kalton shows by sheer force that the mapping $\Omega : S_p \to \mathcal{B}(H)$ defined by $\Omega(f) = f \log(|f|/\|f\|_p)$ is a bicentralizer: it obeys an estimate of the form

$$\|\Omega(afb) - a\Omega(f)b\|_{S_p} \leq C\|a\|_\infty \|f\|_{S_p} \|b\|_\infty \quad (a, b \in \mathcal{B}(H), f \in S_p),$$

where $|f| = (f^* f)^{1/2}$ and the logarithm is defined by the functional calculus.

Although this point is deliberately neglected in [20], this gives rise to an extension of bimodules over $\mathcal{B}(H)$ that was studied in [33].

In the meantime Rochberg and Weiss had discovered that self-extensions arise naturally in interpolation theory. The basic idea is that whenever one finds a given Banach space X inside an "interpolation scale" one can construct a self-extension of X by "differentiating" the scale. The resulting extension may depend on the interpolation method that defines the scale, of course. Thus, for instance, when $p > 1$, the genuine Kalton–Peck space Z_p can be obtained by "interpolating" ℓ_∞ and ℓ_1 by the complex method. Replacing the couple (ℓ_∞, ℓ_1) by $(\mathcal{B}(H), S_1)$ one obtains the Schatten classes S_p and the corresponding "noncommutative" Kalton–Peck spaces.

Later on, Kalton proved the astonishing fact that every centralizer on $L_p(\mu)$, with $1 < p < \infty$, arises by interpolation of two or three Köthe function spaces; see the paper [21] which in a sense can be seen both as a synthesis and a culmination of [19] and [20].

In [5] these ideas are applied to obtain bicentralizers on (and the corresponding self-extensions of) all noncommutative L_p spaces built on semifinite von Neumann algebras when $p > 1$.

However, for type III algebras, the interpolation mechanism of [5] produces two (rather than one) self-extensions of L_p of one sided modules, one of left modules and other of right modules. This is due to the "asymmetry" of the embedding $\mathcal{M} \to \mathcal{M}_*$ used to obtain the (Kosaki) L_p spaces by interpolation. Section 5.2 explains why this attempt was doomed to fail.

1.2. Results. In this paper we complete the results of [5] by using the Haagerup L_p spaces from the very beginning. The key point is to slightly change the perspective focusing on the spaces of analytic functions rather than on the associated analytic families of Banach spaces. This is done in Section 2 which treats the case $p > 1$. In Section 3 we adapt a factorization argument which goes back to Kalton's memoir [19] to cover the case $0 < p \le 1$. The proof of "non-triviality" is achieved in Section 4 which also contains some duality issues.

In the end it turns out that the mapping $f \mapsto \Omega(f) = f \log(|f|/\|f\|)$ still defines a (two-sided) centralizer on $L_p(\mathcal{M})$ but taking values in $L^0(\mathcal{R})$, where $\mathcal{R}$ is the crossed product of $\mathcal{M}$ and $\mathbb{R}$ induced by the modular group of a weight on $\mathcal{M}$. Thus, the space $Z_p(\mathcal{M}) = \{(g, f) \in L^0(\mathcal{R}) \times L^0(\mathcal{R}) : f, g - \Omega(f) \in L_p(\mathcal{M})\}$ quasinormed by $\|(g, f)\|_\Omega = \|g - \Omega f\|_{L_p(\mathcal{M})} + \|f\|_{L_p(\mathcal{M})}$ fits into a nontrivial sequence of $\mathcal{M}$-bimodules and homomorphisms

$$0 \longrightarrow L_p(\mathcal{M}) \longrightarrow Z_p(\mathcal{M}) \longrightarrow L_p(\mathcal{M}) \longrightarrow 0$$

and can be seen as a rather natural noncommutative version of Kalton–Peck's Z_p.

1.3. Haagerup L_p spaces. We assume the reader is acquainted with noncommutative Banach function spaces associated to semifinite von Neumann algebras and Haagerup construction of L_p for general von Neumann algebras (cf. [15]). Here we will only recall the facts we need to fix the notation and we refer the reader to the classical sources [25,34] for full details.

Let $\mathcal{M}$ be a von Neumann algebra with a distinguished normal, faithful, semifinite weight φ. We denote by $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ the "modular" group of φ. The crossed product $\mathcal{R} = \mathbb{R} \ltimes_\sigma \mathcal{M}$ has a trace τ such that $\tau \circ \hat{\sigma}_t = e^{-t}\tau$ for every $t \in \mathbb{R}$, where $\hat{\sigma}_t$ is the dual action of σ_t on $\mathcal{R}$. Let $L^0(\mathcal{R}, \tau)$ be the space of all τ-measurable operators affiliated to $\mathcal{R}$. Then, for $0 < p < \infty$, the Haagerup noncommutative L_p-space associated to $(\mathcal{M}, \varphi)$ is

$$L_p(\mathcal{M}, \varphi) = \left\{ f \in L^0(\mathcal{R}, \tau) : \hat{\sigma}_t(f) = e^{-t/p} f \text{ for every } t \in \mathbb{R} \right\},$$

while

$$L_\infty(\mathcal{M}, \varphi) = \left\{ f \in L^0(\mathcal{R}, \tau) : \hat{\sigma}_t(f) = f \text{ for every } t \in \mathbb{R} \right\}$$

is a subalgebra of $\mathcal{R}$ isomorphic to $\mathcal{M}$.

There is an order preserving, linear isomorphism $\phi \in \mathcal{M}_* \mapsto h_\phi \in L_1(\mathcal{M}, \varphi)$. Let $\mathrm{Tr} : L_1(\mathcal{M}, \varphi) \to \mathbb{C}$ be the Haagerup "trace", that is, $\mathrm{Tr}(h_\phi) = \langle 1_{\mathcal{M}}, \phi\rangle$. One has

$$\mathrm{Tr}\,|h_\phi| = \|\phi\|_{\mathcal{M}_*} \quad (\phi \in \mathcal{M}_*).$$

This allows one to define a quasinorm (actually a norm when $1 \le p < \infty$) on the spaces $L_p(\mathcal{M}, \varphi)$ by letting

$$\|f\|_{L_p(\mathcal{M},\varphi)} = (\mathrm{Tr}\,|f|^p)^{1/p}\,.$$

From now on, we will omit the underlying algebra $\mathcal{M}$ and the reference weight φ (but not $\mathcal{R}$ nor τ) in the notations. Accordingly, we write L_p instead of $L_p(\mathcal{M}, \varphi)$ and denote its quasinorm by $\|\cdot\|_p$. We use subscripts for Haagerup spaces and superscripts for "tracial" spaces. The distinction is pertinent because if the weight φ happens to be a trace, then the Haagerup space $L_p = L_p(\mathcal{M}, \varphi)$ is not the same as the "tracial" $L^p(\mathcal{M}, \varphi) = \{f \in L^0(\mathcal{M}, \varphi) : (\varphi|f|^p)^{1/p} < \infty\}$, even if they are isometrically isomorphic bimodules over $\mathcal{M}$.

One has the following version of Hölder's inequality: if $f \in L_p$ and $g \in L_q$, then $fg \in L_r$, where $r^{-1} = p^{-1} + q^{-1}$, and $\|fg\|_r \le \|f\|_p \|g\|_q$. Since L_∞ "agrees" with $\mathcal{M}$, this provides the $\mathcal{M}$-module structures of the spaces L_p.

Actually, it also provides the dual of L_p for $p \in [1, \infty)$. Indeed, the dual of L_p is isometric to L_q, where $p^{-1} + q^{-1} = 1$ under the pairing $\langle f, g\rangle = \mathrm{Tr}(fg)$ for $f \in L_p$, $g \in L_q$.

1.4. Lorentz spaces over $\mathcal{R}$. Recalling that $\mathcal{R}$ is semifinite, with trace τ, we may consider the noncommutative Lorentz space

$$L^{p,\infty}(\mathcal{R}, \tau) = \big\{f \in L^0(\mathcal{R}, \tau) : \mu_f \in L^{p,\infty}(\mathbb{R}_+)\big\}.$$

Here, $\mu_f : \mathbb{R}_+ \to \mathbb{R}$ is the "generalized singular value" function of f and we put $\|f\|_{p,\infty} = \|\mu_f\|_{L^{p,\infty}(\mathbb{R}_+)}$. Our interest in these spaces relies on the fact, due to Kosaki, that each $f \in L_p$ belongs to $L^{p,\infty}(\mathcal{R}, \tau)$ and $\|f\|_p = \|f\|_{p,\infty}$; see [12, Lemma 4.8 and the comment closing Section 4 on p. 296] and [26]. Note that the "tracial" $L^p(\mathcal{R}, \tau)$ does not contain L_p since Tr is not the restriction of τ to L_1.

1.5. Extensions. Let X and Y be quasi-Banach modules over a Banach algebra $\mathcal{M}$. An extension of X by Y is an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 0 \qquad (1.2)$$

where Z is another quasi-Banach module over $\mathcal{M}$ and the arrows represent (continuous) homomorphisms. By the open mapping theorem $\imath$ embeds Y as a closed submodule of Z and the quotient is isomorphic to X. The extension (1.2) is said to be trivial, or to split, if there is a homomorphism $\varpi : Z \to Y$ such that $\imath \circ \varpi = \mathbf{I}_Y$ or, equivalently, if there is a a homomorphism $\jmath : X \to Z$ such that $\jmath \circ \pi = \mathbf{I}_X$. In this case Z is isomorphic to $Y \times X$ through the mapping $z \mapsto (\varpi(z), \pi(z))$. In particular Z is linearly isomorphic to $Y \times X$.

1.6. Centralizers and extensions. Let X and Y be a quasi-Banach modules over a Banach algebra $\mathcal{M}$ and let W be another $\mathcal{M}$ module containing Y the the purely algebraic sense. Let further $\Phi : X \to W$ be a homogeneous mapping, that is, $\Phi(\lambda f) = \lambda \Phi f$ for every $\lambda \in \mathbb{C}$ and $f \in X$.

(a) We say that Φ is quasilinear from X to Y if, for every $f, g \in X$, the difference $\Phi(f + g) - \Phi f - \Phi g$ belongs to Y and

$$\|\Phi(f + g) - \Phi(f) - \Phi(g)\|_Y \leq Q\big(\|f\|_X + \|g\|_X\big)$$

for some constant Q independent on f, g.

(b) We say that Φ is a left-centralizer from X to Y if there is a constant C such that for every $a \in \mathcal{M}$ and every $f \in X$ the difference $\Phi(af) - a\Phi(f)$ belongs to Y and

$$\|\Phi(af) - a\Phi(f)\|_Y \leq C\|a\|_{\mathcal{M}}\|f\|_X.$$

Right-centralizers are defined analogously, using right-module structures.

(c) Finally, Φ is said to be a bicentralizer over $\mathcal{M}$ if it is both a left-centralizer and a right-centralizer. A bicentralizer obeys an estimate of the form

$$\|\Phi(afb) - a\Phi(f)b\|_Y \leq C\|a\|_{\mathcal{M}}\|f\|_X\|b\|_{\mathcal{M}}.$$

The least constant for which the preceding inequality holds is denoted by $C[\Phi]$.

In this paper we will always have $X = Y = L_p$ for some finite p and $W = L^0(\mathcal{R}, \tau)$ in whose case we say that Φ is a (bi-) centralizer on L_p. A centralizer on L_p is said to be "real" if it takes self-adjoint operators (of L_p) to self-adjoint operators (of $L^0(\mathcal{R}, \tau)$).

Let is briefly describe the connection between centralizers and extensions. Suppose $\Phi : X \to W$ is quasilinear from X to Y. Then the set

$$Y \oplus_\Phi X = \big\{(g, f) \in W \times X : g - \Phi f \in Y\big\}.$$

is a linear subspace of $W \times X$ and the functional

$$\|(g, f)\|_\Phi = \|g - \Phi f\|_Y + \|f\|_X$$

is a quasinorm on it. We define maps $\imath : Y \to Y \oplus_\Phi X$ and $\pi : Y \oplus_\Phi X \to X$ by $\imath(g) = (g, 0)$ and $\pi(g, f) = f$, respectively. Both maps are easily seen to be relatively open continuous operators and moreover $\imath(Y) = \ker \pi$, so that

$Y \oplus_\Phi X/\imath(Y)$ is isomorphic to X. As completeness is a "three-space" property (cf. [31, Theorem 12.1]) this implies that $Y \oplus_\Phi X$ is complete and

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y \oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0 \tag{1.3}$$

is an extension of X by Y, which is called the extension induced by Φ with good reason.

If Φ is a centralizer, then $Y \oplus_\Phi X$ is a quasi-Banach module with outer product $a \cdot (g, f) = (ag, af)$ and the arrows in (1.3) become homomorphisms.

And, if Φ is a bicentralizer, then $Y \oplus_\Phi X$ is a bimodule under $a \cdot (g, f) \cdot b = (agb, afb)$ and (1.3) is an extension of bimodules.

1.7. Two simplifications. We pause a moment to record the following observation, which generalizes [20, Proposition 4.1]:

Lemma 1.1. (a) *Every (left) centralizer from L_p to a quasi-Banach module over $\mathcal{M}$ is quasilinear.*

(b) *Every real centralizer on L_p is a bicentralizer.*

Proof. (a) The key point is that if $f, g \in L_p$, then $h = (f^*f + g^*g)^{1/2}$ belongs to L_p and one has $f = ah, g = bh$ for certain contractive $a, b \in \mathcal{M}$ –whose initial projections agree with the final projection of h, if you want. Indeed one may take $a = f(f^*f + g^*g)^{-1/2}$ which is contractive by Schmitt's [32, Lemma 2.2(c)]: just set $T = f^*f, S = h$ and follow Schmitt's notations.

As for the quasinorm of h we have

$$\begin{aligned}\|(f^*f + g^*g)^{1/2}\|_p &= \|f^*f + g^*g\|_{p/2}^{1/2} \le \Delta_{p/2}^{1/2}\big(\|f^*f\|_{p/2} + \|g^*g\|_{p/2}\big)^{1/2} \\ &\le \Delta_{p/2}^{1/2}\big(\|f^*f\|_{p/2}^{1/2} + \|g^*g\|_{p/2}^{1/2}\big) = \Delta_{p/2}^{1/2}\big(\|f\|_p + \|g\|_p\big),\end{aligned}$$

where Δ_r denotes the "modulus of concavity" of L_r, that is, $\Delta_r = 2^{1/r-1}$ for $r < 1$ and $\Delta_r = 1$ for $r \ge 1$. Now, if $\Phi : L_p \to W$ is a centralizer from L_p to Y, and $f, g \in L_p$, then

$$\begin{aligned}&\|\Phi(f+g) - \Phi f - \Phi g\|_Y \\ &= \big\|\Phi\big((a+b)h\big) - \Phi(ah) - \Phi(bh)\big\|_Y \\ &\le C\Big(\big\|\Phi\big((a+b)h\big) - (a+b)\Phi(h)\big\|_Y + \|a\Phi h - \Phi(ah)\|_Y + \|b\Phi h - \Phi(bh)\|_Y\Big) \\ &\le 4C\|h\|_p \le C'\big(\|f\|_p + \|g\|_p\big).\end{aligned}$$

(b) It is obvious that if Φ is a left-centralizer on L_p such that $\Phi(f^*) = (\Phi f)^*$, then Φ is a bicentralizer (cf. [20, p. 51]). If Φ is any centralizer on L_p and $f \in L_p$, then writing $f = f_1 + if_2$, with f_i self-adjoint, and using (a) we see that

$$\|\Phi f - \Phi f_1 - i\Phi f_2\|_p \le C\big(\|f_1\| + \|f_2\|\big) \le C'\|f\|_p.$$

But the map $f \in L_p \longmapsto \Phi f_1 + i\Phi f_2 \in L^0(\mathcal{R})$ is a bicentralizer on L_p and so Φ is. □

2. The case $p > 1$

2.1. Admissible spaces of analytic functions. The following definition is taken from Kalton and Montgomery-Smith's [24, Section 10]. Let U be an open set of $\mathbb{C}$ conformally equivalent to the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and W a complex Banach space. A Banach space $\mathscr{F}$ of analytic functions $F : U \to W$ is said to be admissible provided:

(a) For each $z \in U$, the evaluation map $\delta_z : \mathscr{F} \to W$ is bounded.

(b) If $h : U \to \mathbb{D}$ is a conformal equivalence and $F : U \to W$ is analytic, then $F \in \mathscr{F}$ if and only if $h \cdot F \in \mathscr{F}$ and $\|h \cdot F\|_{\mathscr{F}} = \|F\|_{\mathscr{F}}$.

For each $z \in U$ we define $\mathscr{F}(z) = \{x \in W : x = F(z) \text{ for some } F \in \mathscr{F}\}$ with the norm $\|x\| = \inf\{\|F\|_{\mathscr{F}} : x = F(z)\}$ so that $\mathscr{F}(z)$ is isometric to $\mathscr{F} / \ker \delta_z$ with the quotient norm. One often says that the spaces $\mathscr{F}(z)$ with z varying in U form an analytic family of Banach spaces.

The following result is classical. We write the easy proof for the convenience of the reader.

Lemma 2.1. *The map* $\delta'_z : \mathscr{F} \to W$ *is bounded from* $\ker \delta_z$ *to* $\mathscr{F}(z)$.

Proof. Let $h : U \to \mathbb{D}$ be a fixed conformal equivalence having a (single) zero at z. Suppose $F \in \mathscr{F}$ vanishes at z. Then one can write $F = h \cdot G$, where $G : U \to W$ is analytic. By (b) one has $G \in \mathscr{F}$, and $\|F\| = \|G\|$. As $F'(z) = h'(z)G(z)$ we have that $F(z)$ belongs to $\mathscr{F}(z)$ and

$$\|\delta'_z F\|_{\mathscr{F}(z)} = \|F'(z)\|_{\mathscr{F}(z)} = \|h'(z)G(z)\|_{\mathscr{F}(z)} \leq |h'(z)|\|G\|_{\mathscr{F}} = |h'(z)|\|F\|_{\mathscr{F}},$$

so $\|\delta'_z : \ker \delta_z \to \mathscr{F}(z)\| \leq |h'(z)|$. □

Although we shall not need it explicitly, let us explain how admissible spaces are used to construct self-extensions of Banach spaces (cf. [24, § 10]). Suppose $X = \mathscr{F}(z)$ for some $z \in U$. Put $Z = \{(F'(z), F(z)) : F \in \mathscr{F}\}$ and furnish it with the norm $\|(y, x)\|_Z = \inf\{\|F\|_{\mathscr{F}} : F \in \mathscr{F} \text{ is such that } x = F(z) \text{ and } y = F'(z)\}$. Then one has an exact sequence

$$0 \longrightarrow X \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 0,$$

in which $\imath(y) = (y, 0)$ and $\pi(y, x) = x$. In fact the only point that is not completely obvious is that $\imath$ is correctly defined and bounded. Pick $y \in X$. Take $F \in \mathscr{F}$ such that $y = F(z)$, with $\|F\|_{\mathscr{F}} \leq (1 + \varepsilon)\|y\|_X$ and let $h : U \to \mathbb{D}$ be a conformal equivalence with a single zero at z. Taking $G = h'(z)^{-1} \cdot h \cdot F$ we see that $\|G\|_{\mathscr{F}} = |h'(z)^{-1}|\|F\|_{\mathscr{F}}$ and, besides, $G'(z) = y, G(z) = 0$. This shows that $(y, 0) \in Z$ and $\|(y, 0)\|_Z \leq (1 + \varepsilon) \cdot |h'(z)^{-1}| \cdot \|y\|_X$.

2.2. The basic construction. In this section we define the space of analytic functions needed to “twist” L_p when $1 < p < \infty$. We use the Lorentz “norm” of $L^{p,\infty}(\mathcal{R},\tau)$ to control the norm in L_p and the group $(\hat{\sigma}_t)$ to keep the values in the right place. We also introduce the “delimiters” $1 < q < r < \infty$ to stay in the locally convex setting and to avoid any difficulty on the boundary.

Consider the strip $U = \{z \in \mathbb{C} : r^{-1} < \Re(z) < q^{-1}\}$ and denote by $\overline{U}$ its closure in $\mathbb{C}$. Let $\mathcal{F} = \mathcal{F}(q,r)$ be the space (of restrictions to U) of those bounded, continuous functions $F : \overline{U} \to L^{q,\infty}(\mathcal{R},\tau) + L^{r,\infty}(\mathcal{R},\tau)$ that are analytic on U and satisfy the following “boundary” conditions:

- For $s = q, r$ one has $F(z) \in L^{s,\infty}(\mathcal{R},\tau)$, where $\Re(z) = s^{-1}$ and

$$\sup\left\{\|F(z)\|_{s,\infty} : \Re(z) = s^{-1}\right\} < \infty.$$

We equip $\mathcal{F}$ with the (quasi) norm

$$\|F\|_{\mathcal{F}} = \sup\left\{\|F(1/q+iy)\|_{q,\infty}, \|F(1/r+iy)\|_{r,\infty} : y \in \mathbb{R}\right\},$$

which is equivalent to a norm rendering $\mathcal{F}$ complete.

Now, consider the following subspace of $\mathcal{F}$:

$$\begin{aligned}\mathcal{G} &= \left\{F \in \mathcal{F}(q,r) : F(z) \in L_p \text{ for every } z \in U \cap \mathbb{R}, \text{ where } 1/p = z\right\}\\ &= \left\{F \in \mathcal{F}(q,r) : \hat{\sigma}_t(F(z)) = e^{-tz}F(z) \text{ for every real } z \in (r^{-1}, q^{-1})\right\}.\end{aligned}$$

Observe that $U \cap \mathbb{R} = (r^{-1}, q^{-1})$ and that the definition of $\mathcal{G}$ involves only those $z \in U$ that are real.

Lemma 2.2. *The space $\mathcal{G}$ is admissible and, for $r^{-1} < z < q^{-1}$, one has $\mathcal{G}(z) = L_p$, where $1/p = z$.*

Proof. That $\mathcal{F}(q,r)$ is admissible is clear. And that its subspace $\mathcal{G}$ is admissible again is then both obvious and trivial. As for the second part, first note that if $z \in U$ and $F \in \mathcal{F}$, then $F(z) \in L^{p,\infty}(\mathcal{R},\tau)$, with $\|F(z)\|_{p,\infty} \le \|F\|_{\mathcal{F}}$, where $1/p = \Re z$. This follows from the commutative case, proved by Calderón in [7, Section 13.5, p. 125] and from general results about “noncommutative Banach function spaces”; see [10, Theorem 3.2] for a precise reference.

Now, by the very definition of $\mathcal{G}$, we see that, for $F \in \mathcal{G}$ and $z \in U$ real, one has $F(z) \in L_p$ and $\|F(z)\|_p \le \|F\|_{\mathcal{F}}$, where $p = 1/z$.

It only remains to check that for every $f \in L_p$ there is $F \in \mathcal{G}$ such that $F(z) = f$ at $z = 1/p$, with $\|F\|_{\mathcal{F}} = \|f\|_p$. This is obvious if f is normalized in L_p since one may use the “extremal” $F(z) = u|f|^{pz}$, where $f = u|f|$ is the “polar decomposition”. For arbitrary $f \in L_p$ just take $F(z) = u\|f\|_p(|f|/\|f\|_p)^{pz}$. $\square$

Corollary 2.3. *For $1 < p < \infty$, the mapping $\Omega_p : L_p \to L^0(\mathcal{R})$ given by $\Omega_p(f) = pf\log(|f|/\|f\|_p)$ is a bicentralizer on L_p.*

Proof. Fix $q, r \in (1, \infty)$ so that $q < p < r$ and let $\mathcal{F}(q, r)$ and $\mathcal{G}$ be as before. Set $\theta = 1/p$. We observe that for $f \in L_p$,

$$\frac{d}{dz}\left(u\|f\|_p\left(\frac{|f|}{\|f\|_p}\right)^{pz}\right)\Big|_{z=\theta} = pf\log\left(\frac{|f|}{\|f\|_p}\right).$$

Now, we identify $\mathcal{M}$ with L_∞. Pick $f \in L_p$ and $a, b \in L_\infty$. Consider the functions F and G defined by

$$F(z) = au\|f\|_p\big(|f|/\|f\|_p\big)^{pz}b \quad \text{and} \quad G(z) = v\|g\|_p\big(|g|/\|g\|_p\big)^{pz}$$

for $z \in \overline{U}$, where $g = afb$ and v is the phase of g. Then $F, G \in \mathcal{G}$, and $\|F\|_{\mathcal{G}}, \|G\|_{\mathcal{G}} \le \|a\|_\infty\|f\|_p\|b\|_\infty$. Besides, $F(\theta) = G(\theta) = afb$, so $G - F \in \ker\delta_\theta$. We know from Lemma 2.1 that δ'_θ is bounded from $\ker\delta_\theta$ to $\mathcal{G}(\theta) = L_p$, hence

$$\|\delta'_\theta(G - F)\|_p \le M\|G - F\|_{\mathcal{G}} \le 2M\|a\|_\infty\|f\|_p\|b\|_\infty.$$

But

$$\begin{aligned}\delta'_\theta(G - F) &= pg\log\left(\frac{|g|}{\|g\|_p}\right) - a\left(pf\log\left(\frac{|f|}{\|f\|_p}\right)\right)b\\ &= \Omega_p(afb) - a(\Omega_p f)b,\end{aligned}$$

as required. □

3. The case $0 < p \le 1$

3.1. Transfer. What if $0 < p \le 1$? Well, in this case $L^{p,\infty}(\mathcal{R})$ is not locally convex and we cannot guarantee that Lemma 2.2 and the proof of Corollary 2.3 work. Anyway we can still use the following generalization of [2, Lemma 5] (which in turn is a generalization of [19, Theorem 5.1]) to transfer arbitrary centralizers on L_p to any L_q with $0 < q < p$.

Proposition 3.1. *Let $\Phi : L_p \to L^0(\mathcal{R}, \tau)$ be a left centralizer on L_p and $0 < r < \infty$. Define q by letting $q^{-1} = p^{-1} + r^{-1}$. Then there is a centralizer $\Phi^{(r)} : L_q \to L^0(\mathcal{R}, \tau)$ such that*

$$\|\Phi^{(r)}(fg) - f\Phi g\|_q \le C\|f\|_r\|g\|_p \quad (f \in L_r, g \in L_p). \tag{3.1}$$

Moreover, if Γ is another centralizer satisfying the corresponding estimate, then $\Gamma \approx \Phi^{(r)}$.

Proof. Let us first prove that if $\Phi : L_p \to L^0(\mathcal{R}, \tau)$ is a centralizer and $f_1 g_1 = f_2 g_2$, with $f_i \in L_r$ and $g_i \in L_p$ the difference $f_1\Phi(g_1) - f_2\Phi(g_2)$ belongs to L_q, where $q^{-1} = r^{-1} + p^{-1}$ and

$$\|f_1\Phi(g_1) - f_2\Phi(g_2)\|_q \le C\big(\|f_1\|_r\|g_1\|_p + \|f_2\|_r\|g_2\|_p\big), \tag{3.2}$$

for some C independent on f_i and g_i.

Recall from the proof of Lemma 1.1 that there is $g \in L_p$ such that $g_i = a_i g$ for certain contractive $a_i \in \mathcal{M}$ whose initial projections agree with the final projection of g. Now, since $f_1 a_1 g = f_2 a_2 g$ we have $f_1 a_1 = f_2 a_2$. For $i = 1, 2$, one has

$$\| f_i \Phi(g_i) - f_i a_i \Phi(g)\| \le C[\Phi]\| f_i\|_r \|g\|_p \le C\| f_i\|_r\big(\|g_1\|_p + \|g_2\|_p\big),$$

and combining we arrive to

$$\| f_1 \Phi(g_1) - f_2 \Phi(g_2)\|_q \le C\big(\| f_1\|_r + \| f_2\|_r\big)\big(\|g_1\|_p + \|g_2\|_p\big).$$

But Φ is homogeneous and since $f_1 g_1 = \alpha f_1 \alpha^{-1} g_1$ and $f_2 g_2 = \beta f_2 \beta^{-1} g_2$, for $\alpha, \beta > 0$, we obtain

$$\| f_1 \Phi(g_1) - f_2 \Phi(g_2)\|_q \le C\big(\alpha\| f_1\|_r + \beta\| f_2\|_r\big)\big(\alpha^{-1}\|g_1\|_p + \beta^{-1}\|g_2\|_p\big).$$

Minimizing the right-hand side over $\alpha, \beta > 0$ we obtain

$$\begin{aligned}\| f_1 \Phi(g_1) - f_2 \Phi(g_2)\|_q &\le C\big(\| f_1\|_r^{1/2}\|g_1\|_p^{1/2} + \| f_2\|_r^{1/2}\|g_2\|_p^{1/2}\big)^2\\ &\le C\Delta_{1/2}\big(\| f_1\|_r\|g_1\|_p + \| f_2\|_r\|g_2\|_p\big),\end{aligned}$$

which proves (3.2).

Now, we define $\Phi^{(r)} : L_q \to L^0(\mathcal{R}, \tau)$ by $\Phi^{(r)}(f) = u|f|^{q/r}\Phi|f|^{q/p}$, where q is given by $q^{-1} = p^{-1} + r^{-1}$ and $f = u|f|$ is the polar decomposition. To check that $\Phi^{(r)}$ it is indeed a centralizer on L_q, take $a \in \mathcal{M}$ and $f \in L_q$ and let v be the phase of af, so that $af = v|af|$. One has

$$a\Phi^{(r)}(f) = au|f|^{q/r}\Phi|f|^{q/p}, \quad \text{while } \Phi^{(r)}(af) = v|af|^{q/r}\Phi|af|^{q/p}.$$

And since $au|f|^{q/r}|f|^{q/p} = v|af|^{q/r}|af|^{q/p} = af$ we may apply (3.2) to get

$$\begin{aligned}\|\Phi^{(r)}(af) - a\Phi^{(r)}(f)\|_q &\le C\|au|f|^{q/r}\|_r\||f|^{q/p}\|_p + \|v|af|^{q/r}\|_r\||af|^{q/p}\|\\ &\le C\big(\|a\|_\infty\| f\|_q + \|af\|_q\big).\end{aligned}$$

The estimate (3.1) is clear from (3.2). Finally, if $\Gamma : L_q \to L^0(\mathcal{R}, \tau)$ is any mapping such that $\|\Gamma(fg) - f\Phi g\|_q \le C\| f\|_r\|g\|_p$, then given $h \in L_q$ we may take $f = u|h|^{q/r}$ and $g = |h|^{q/p}$ to obtain

$$\|\Gamma(h) - \Phi^{(r)}(h)\|_q = \|\Gamma(fg) - f\Phi g\|_q \le C\| f\|_r\|g\|_p = C\|h\|_q.$$

This completes the proof. □

Corollary 3.2. *The map* $\Omega_p : L_p \to L^0(\mathcal{R}, \tau)$ *given by* $\Omega_p(f) = pf\log(|f|/\| f\|_p)$ *is a bicentralizer on* L_p *for each* $0 < p < \infty$.

Proof. We already know that Ω_p is a centralizer when $p > 1$ and in particular if $p = 2$. Fix any $0 < q < 2$ and take r so that $q^{-1} = r^{-1} + 2^{-1}$. According to Proposition 3.1 $\Omega_2^{(r)}$ is a centralizer on L_q. But for $f \in L_q$ one has

$$\Omega_2^{(r)}(f) = u|f|^{q/r}\Omega_2|f|^{q/2} = u|f|^{q/r} \cdot 2 \cdot |f|^{q/2} \log \frac{|f|^{q/2}}{\||f|^{q/2}\|_2} = \Omega_q(f)$$

since $\||f|^{q/2}\|_2 = \|f\|_q^{q/2}$. Hence Ω_q is a centralizer and since it is real, it is a bicentralizer by Lemma 1.1(b). □

Now we can consider the whole range of noncommutative Kalton–Peck spaces by letting

$$Z_p(\mathcal{M}) = L_p \oplus_{\Omega_p} L_p = \{(g, f) \in L^0(\mathcal{R}, \tau) \times L_p : g - \Omega_p(f) \in L_p\}$$

with the quasinorm $\|(g, f)\|_{\Omega_p} = \|g - \Omega_p(f)\|_p + \|f\|_p$ and the corresponding self-extension

$$0 \longrightarrow L_p \xrightarrow{\ \imath\ } Z_p(\mathcal{M}) \xrightarrow{\ \pi\ } L_p \longrightarrow 0 \tag{3.3}$$

for $0 < p < \infty$.

4. Non triviality

4.1. Duality. The following result is a noncommutative version of [23, Theorem 5.1]. We present it right now to ease the proof of the forthcoming Theorem 4.2.

Corollary 4.1. *If $1 < p, q < \infty$ are conjugate exponents, that is, $p^{-1} + q^{-1} = 1$, then the dual of Z_p is isomorphic to Z_q under the pairing*

$$\langle (g', g); (f', f)\rangle = \mathrm{Tr}(f'g - fg') \quad ((g', g) \in Z_q,\ (f', f) \in Z_p). \tag{4.1}$$

Proof. This can be proved "by interpolation"; see [30, Proposition 2.11] to get the basic idea. However, at this point we have a simpler proof based on Proposition 3.1 at hand. Indeed, applying Proposition 3.1 to Ω_q we get a (left) centralizer $\Omega_q^{(p)}$ on L_1 such that

$$\|\Omega_q^{(p)}(fg) - f\Omega_q(g)\|_1 \le C\|f\|_p\|g\|_q \quad (f \in L_p,\ g \in L_q).$$

An elementary computation shows that $\Omega_q^{(p)}$ agrees with Ω_1, so

$$\|\Omega_1(fg) - f\Omega_q(g)\|_1 \le C\|f\|_p\|g\|_q \quad (f \in L_p,\ g \in L_q).$$

By symmetry one also has

$$\|\Omega_1(fg) - \Omega_p(f) \cdot g\|_1 \le C\|f\|_p\|g\|_q \quad (f \in L_p,\ g \in L_q).$$

Combining we obtain the estimate

$$\|(\Omega_p(f))g - f\Omega_q(g)\|_1 \le C\|f\|_p\|g\|_q \quad (f \in L_p,\ g \in L_q).$$

This implies that the pairing (4.1) is continuous since

$$\begin{aligned}
&|\operatorname{Tr}(f'g - fg')| \\
&\le \|f'g - fg'\|_1 = \|f'g - \Omega_p(f)g + \Omega_p(f)g - f\Omega_q(g) + f\Omega_q(g) - fg'\|_1 \\
&\le \|f' - \Omega_p(f)\|_p\|g\|_q + \|\Omega_p(f)g - f\Omega_q(g)\|_1 + \|f\|_p\|\Omega_q(g) - g'\|_q \\
&\le C\|(f', f)\|_{Z_p}\|(g', g)\|_{Z_q}.
\end{aligned}$$

Therefore, the mapping $\varkappa : Z_q \to (Z_p)'$ given by $(\varkappa(g', g))(f', f) = \operatorname{Tr}(f'g - fg')$ is a bounded operator fitting in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_q & \xrightarrow{\ \imath\ } & Z_q & \xrightarrow{\ \pi\ } & L_q & \longrightarrow & 0 \\
 & & {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle \varkappa} & & \| & & \\
0 & \longrightarrow & L_p' & \xrightarrow{\ \pi'\ } & Z_q & \xrightarrow{\ \imath'\ } & L_p' & \longrightarrow & 0
\end{array}$$

and the 5-lemma shows that $\varkappa$ is an isomorphism. □

4.2. True twist. And, of course:

Theorem 4.2. *If $\mathcal{M}$ is infinite-dimensional, then the exact sequence (3.3) is not trivial, even in the category of quasi-Banach spaces.*

Proof. We will prove that $Z_p(\mathcal{M})$ is not linearly isomorphic to the direct sum $L_p \oplus L_p$. Let $u_1, \dots, u_n$ be disjoint projections in $\mathcal{M}$ and for each $1 \le i \le n$ take a positive, normalized $f_i \in L_p$ such that $f_i = u_i f_i u_i$. Set $x_i = (f_i \log f_i, f_i)$. Suppose $r_i = \pm 1$ for $1 \le i \le n$. Then $\|x_i\|_{\Omega_p} = 1$ for all i, while

$$\begin{aligned}
\left\|\sum_{i=1}^n r_i x_i\right\|_{\Omega_p} &= \left\|\left(\sum_{i=1}^n r_i f_i \log f_i, \sum_{i=1}^n r_i f_i\right)\right\|_{\Omega_p} \\
&= \left\|\sum_{i=1}^n r_i f_i \log f_i - \Omega_p\left(\sum_{i=1}^n r_i f_i\right)\right\|_p + \left\|\sum_{i=1}^n r_i f_i\right\|_p \\
&= \left\|\sum_{i=1}^n r_i f_i \log f_i - \sum_{i=1}^n r_i f_i \log\left(\frac{f_i}{n^{1/p}}\right)\right\|_p + n^{1/p} \\
&= \left\|\sum_{i=1}^n r_i f_i \log n^{1/p}\right\|_{L_p} + n^{1/p} = \left(1 + \frac{\log n}{p}\right)n^{1/p}.
\end{aligned}$$

The end of the proof depends on the values of p:

(1) If $0 < p \leq 1$, then taking $r_i = 1$ for every i and letting $n \to \infty$ we conclude that Z_p is not p-normable, while $L_p \oplus L_p$ is.

(2) If $1 < p \leq 2$ we see that Z_p does not have type p, while, as it is well known, the spaces L_p have type $\min(p, 2)$ for $0 < p < \infty$. Actually this argument works for every $0 < p \leq 2$, but (1) is simpler.

(3) Finally, if $2 < p < \infty$ we apply Corollary 4.1 and (2).

This completes the proof. □

5. Concluding remarks

5.1. More centralizers, please. It would be interesting to know if all symmetric centralizers on (the commutative) $L^p(\mathbb{R}_+)$ can be transferred to $L_p = L_p(\mathcal{M})$ for a general von Neumann algebra $\mathcal{M}$ as it is the case when $\mathcal{M}$ is semifinite (cf. [5, Theorem 2]).

The main obstruction to proceed as in Section 1 is that $L^{p,\infty}(\mathbb{R}_+)$ is a bad space for interpolation (a nonseparable space with no lower estimate) and most probably $L^{p,\infty}(\mathbb{R}_+)$ has "less" centralizers than $L^p(\mathbb{R}_+)$ (see, however, [4, Section 4.1]).

One might consider $L_q(\mathcal{M})$ as a quotient of $L^{q,1}(\mathcal{R})$ when $q > 1$ together with the fact that $L^{q,1}(\mathbb{R}_+)$ has "the same" centralizers as $L^q(\mathbb{R}_+)$, which follows by juxtaposition of results in [2] and [3]. But in this case, even if every centralizer on $L^{q,1}(\mathbb{R}_+)$ can be obtained by interpolation of a family of quasi-Banach function spaces, we don't know if the basic result by the Dodds and de Pagter we used in the proof of Lemma 2.2 would survive to the lack of local convexity.

More precisely, and without any reference to interpolation theory, we may ask the following.

Problem 5.1. Let E and F be symmetric (or fully symmetric) quasi-Banach function spaces on $\mathbb{R}_+$ and let G denote the product space $E \cdot F$, with the product quasinorm (cf. [3, Section 2.2]). Is $E(\tau) \cdot F(\tau) = G(\tau)$ for every trace τ?

5.2. The role of the ambient space. A curious by-product of Corollary 3.2 is that if $f, g \in L_p$ and $a, b \in \mathcal{M}$, then the differences

$$(f + g)\log|f + g| - f\log|f| + g\log|g| \quad \text{and} \quad af\log|f|b - afb\log|afb|$$

are in L_p. On the other hand it is clear that $f\log|f|$ cannot fall in L_p unless $f = 0$. We observe that in most typical situations where Lemma 2.1 applies, for instance for centralizers $\Phi : L^p(\mathcal{M}, \tau) \to L^0(\mathcal{M}, \tau)$ the so-called domain of Φ

$$\mathrm{Dom}(\Phi) = \{f \in L^p(\mathcal{M}, \tau) : \Phi(f) \in L^p(\mathcal{M}, \tau)\}$$

is a dense submodule of $L^p(\mathcal{M},\tau)$. One may wonder if the fact that $\mathrm{Dom}(\Omega_p)=0$ for the Kalton–Peck centralizers on Haagerup L_p spaces merely reflects some peculiarity of Haagerup's construction or if there is a "real" obstruction to have bicentralizers with nontrivial domain. The following result shows that this is indeed the case.

Example 5.2. A von Neumann algebra $\mathcal{N}$ such that if Φ is a centralizer from $L_p(\mathcal{N})$ to any quasi-Banach bimodule over $\mathcal{N}$, then either $\mathrm{Dom}(\Phi)\neq 0$ or Φ is bounded: $\mathrm{Dom}(\Phi)=L_p(\mathcal{N})$ and $\sup\{\|\Phi f\|:\|f\|_{L_p(\mathcal{N})}\leq 1\}<\infty$.

Proof. By a celebrated result of Connes and Størmer [9, Theorem 4], if $\mathcal{M}$ is a factor of type III_1, then, given states $\phi,\psi\in\mathcal{M}_*$ and $\varepsilon>0$, there is a unitary $u\in\mathcal{M}$ such that $\|u^*\phi u-\psi\|_{\mathcal{M}_*}<\varepsilon$, where $u^*\phi u$ is defined by $\langle u^*\phi u,x\rangle=\langle\phi,uxu^*\rangle$ for $x\in\mathcal{M}$.

It follows from the generalized Powers–Størmer inequality (see the Appendix of [17]) that L_p has a similar almost transitivity property: given positive $f,g\in L_p$ with $\|f\|_p=\|g\|_p=1$ and $\varepsilon>0$ there is a unitary $u\in\mathcal{M}$ such that $\|u^*fu-g\|_p<\varepsilon$. It readily follows that for arbitrary $f,g\in L_p$ with $\|f\|_p=\|g\|_p=1$ and $\varepsilon>0$ there are unitaries $u,v\in\mathcal{M}$ such that $\|vfu-g\|_p<\varepsilon$.

Let $\mathcal{U}$ be a countably incomplete ultrafilter on the integers. According to Raynaud, there is a von Neumann algebra $\mathcal{N}$ containing the ultrapower C*-algebra $\mathcal{M}_\mathcal{U}$ such that $(L_p)_\mathcal{U}=L_p(\mathcal{N})$ for every $0<p<\infty$; see [29]. Hence, if f and g have the same norm in $L_p(\mathcal{N})$, then there are unitaries $u,v\in\mathcal{N}$ such that $g=vfu$.

Now, the statement follows from the very definition of a bicentralizer. □

5.3. "Higher order" extensions. Although this paper treats the same problems as [5], our approach here is more akin to [6]. Actually, once one has established Lemma 2.2, the abstract results in [6] apply and one can associate to the admissible space $\mathcal{G}$ (for every fixed $1<p<\infty$) its sequence of Rochberg spaces $\mathcal{X}^{(n)}$. These can be arranged to form nontrivial (bimodule) extensions

$$0\longrightarrow\mathcal{X}^{(n)}\longrightarrow\mathcal{X}^{(n+k)}\longrightarrow\mathcal{X}^{(k)}\longrightarrow 0.$$

Here, one has $\mathcal{X}^{(1)}=L_p$, $\mathcal{X}^{(2)}=Z_p(\mathcal{M})$ and so on: in particular $\mathcal{X}^{(3)}$ can be seen both as an extension of L_p by $Z_p(\mathcal{M})$ and as an extension of $Z_p(\mathcal{M})$ by L_p. See [6, Section 5] for details.

5.4. Transfer in semifinite L_p spaces. Proposition 3.1 applies in the tracial case just considering $L^p(\mathcal{M},\tau)$ as a subspace of $L^0(\mathcal{M},\tau)$. This allows one to extend the results proved in [5] for $1<p<\infty$ (in particular Theorem 2) to any $0<p<\infty$. A sample: if $\phi:\mathbb{R}\to\mathbb{C}$ is a Lipschitz function vanishing at zero, then the map $\Phi:L^p(\mathcal{M},\tau)\to L^0(\mathcal{M},\tau)$ given by

$$\Phi(f)=f\phi\left(\log\frac{|f|}{\|f\|}\right)$$

is a bicentralizer on $L^p(\mathcal{M},\tau)$ for every $0<p<\infty$; cf. [5, Corollary 1].

5.5. Independence on the weight. Although Haagerup construction of L_p starts with the choice of a weight φ, the resulting "structures" are basically independent on φ, and the same happens to "our" $Z_p(\mathcal{M})$. Let us explain why. Suppose φ' is another weight on $\mathcal{M}$ whose associated objects we decorate with "primes". Recall that if $\mathcal{M}$ acts on H, then $\mathcal{R}$ and $\mathcal{R}'$ act on $L_2(\mathbb{R}, H)$. It turns out that there is a unitary u on $L_2(\mathbb{R}, H)$ such that $f \mapsto ufu^*$ maps $\mathcal{R}$ onto $\mathcal{R}'$. Besides, one has $\tau'(ufu^*) = \tau(f)$ for every $f \in \mathcal{R}$ and therefore the mapping $\varkappa : L^0(\mathcal{R}, \tau) \to L^0(\mathcal{R}', \tau')$ defined by $\varkappa(f) = ufu^*$ is a $*$-isomorphism. On the other hand, $\varkappa$ intertwines the dual actions: for every real t one has $\varkappa \circ \hat{\sigma}_t = \hat{\sigma}'_t \circ \varkappa$, that is, $u(\hat{\sigma}_t(f))u^* = \hat{\sigma}'_t(ufu^*)$ for every $f \in L^0(\mathcal{R}, \tau)$.

It follows that $\varkappa$ maps $L^{p,\infty}(\mathcal{R}, \tau)$ isometrically onto $L^{p,\infty}(\mathcal{R}', \tau')$ and $L_p(\mathcal{M}, \varphi)$ onto $L_p(\mathcal{M}, \varphi')$ and also that $\varkappa$ intertwines $\Omega_p : L_p(\mathcal{M}, \varphi) \to L^0(\mathcal{R}, \tau)$ and $\Omega'_p : L_p(\mathcal{M}, \varphi') \to L^0(\mathcal{R}', \tau')$ in the sense that $\Omega'_p \circ \varkappa = \varkappa \circ \Omega_p$ so that $\Omega'_p(f') = u(\Omega_p(u^* f' u))u^*$ for every $f' \in L_p(\mathcal{M}, \varphi')$.

Thus, the mapping $(g, f) \in Z_p(\mathcal{M}, \varphi) \mapsto (ugu^*, ufu^*) \in Z_p(\mathcal{M}, \varphi')$ is a surjective isometry witnessing that the "new" extension

$$0 \longrightarrow L_p(\mathcal{M}, \varphi') \stackrel{\imath}{\longrightarrow} Z_p(\mathcal{M}, \varphi') \stackrel{\pi}{\longrightarrow} L_p(\mathcal{M}, \varphi') \longrightarrow 0$$

is "isometrically equivalent" to the "old one" in (3.3).

5.6. Yamagami spaces. The family of Haagerup spaces L_p extends quite naturally to a larger family studied by Yamagami in [35]; see also [27]. Given $z \in \mathbb{C}$, consider

$$\mathcal{R}(z) = \{f \in L^0(\mathcal{R}) : \hat{\sigma}_t(f) = e^{-tz} f \text{ for every } t \in \mathbb{R}\}.$$

When $x = \Re(z) > 0$ the space $\mathcal{R}(z)$ is a quasi-Banach space with quasinorm

$$\|f\|_{\mathcal{R}(z)} = \left(\operatorname{Tr}\left((f^* f)^{1/(2x)}\right)\right)^x.$$

In particular $L_p = \mathcal{R}(1/p)$ for $0 < p < \infty$. When $\Re(z) = 0$ one has $\mathcal{R}(z) \subset \mathcal{R}$ and $\mathcal{R}(z)$ is equipped with the restriction of the norm of $\mathcal{R}$. Finally, if $\Re(z) < 0$, then $\mathcal{R}(z) = 0$. For what this paper is concerned, the most important feature of Yamagami's spaces is that they form a "graded algebra": if $f \in \mathcal{R}(a)$ and $g \in \mathcal{R}(b)$, then $fg \in \mathcal{R}(a+b)$ and $\|fg\|_{\mathcal{R}(a+b)} \le \|f\|_{\mathcal{R}(a)} \|g\|_{\mathcal{R}(b)}$. From this it is relatively easy to see that $\mathcal{R}(z) \subset L^{p,\infty}(\mathcal{R})$ "isometrically" provided $p^{-1} = \Re(z)$ and also that if $\Re(z) = \Re(\zeta)$, then $\mathcal{R}(z)$ and $\mathcal{R}(\zeta)$ are isometric as left (or right) $\mathcal{M}$-modules, though not as bimodules in general.

It turns out that, with the notations of Lemma 2.2, one has $\mathcal{G}(z) = \mathcal{R}(z)$ for every $z \in U$, where $U = \{z \in \mathbb{C} : r^{-1} < \Re(z) < q^{-1}\}$. Observe that, if F belongs to $\mathcal{G}$, then since $\hat{\sigma}_t(F(z)) = e^{-tz} F(z)$ for every z in the real interval $U \cap \mathbb{R}$, one also has $\hat{\sigma}_t(F(z)) = e^{-tz} F(z)$ for every $z \in U$, by analytic continuation, and therefore, $\mathscr{G}(z) \subset \mathscr{R}(z)$ for every $z \in U$. Besides, $\|F(z)\|_{\mathscr{R}(z)} = \|F(z)\|_{L^{p,\infty}(\mathscr{R})} \leq \|F\|_{\mathscr{F}} = \|F\|_{\mathscr{G}}$, which shows that $\|f\|_{\mathscr{R}(z)} \leq \|f\|_{\mathscr{G}(z)}$ for every $f \in \mathscr{G}(z)$. To stablish the reversed relations one has to construct, for a given normalized $f \in \mathscr{R}(z)$, an "extremal" $F : U \to L^0(\mathscr{R})$.

Let $f = u|f|$ be the polar decomposition of f in $L^0(\mathscr{R})$. Note that this time $|f|$ does not belong to $\mathscr{R}(z)$ unless z is real and in fact $|f|$ is in $\mathscr{R}(\Re(z)) = L_p$, where $\Re(z) = 1/p$. Also $u \in \mathscr{R}(iy)$, where $y = \Im(z)$. Set

$$F(w) = u|f|^{\frac{w-yi}{x}} \quad (w \in U),$$

where we are treating w as a variable and $z = x + iy$ as a constant. Clearly, $F \in \mathscr{F}(q,r)$, and $F(z) = f$. Let us check that $F \in \mathscr{G}$. If $s \in U$ is real, then

$$\begin{aligned}\hat{\sigma}_t(F(s)) = \hat{\sigma}_t\big(u|f|^{\frac{s-yi}{x}}\big) &= \hat{\sigma}_t(u)\hat{\sigma}_t\big(|f|^{\frac{s-yi}{x}}\big)\\ &= e^{-yti}ue^{-t(s-yi)}|f|^{\frac{s-yi}{x}} = e^{-st}F(s),\end{aligned}$$

so $F \in \mathscr{G}$. Evaluating the derivative at $w = z$ we obtain

$$F'(z) = x^{-1}u|f|^{\frac{-yi}{x}}|f|\log|f| = x^{-1}f|f|^{-yi/x}\log|f|.$$

From where it follows the following extension of Corollary 3.2:

Corollary 5.3. *For every fixed $z = x + yi$, with $x > 0$, the map $\Phi : \mathscr{R}(z) \to L^0(\mathscr{R},\tau)$ given by $\Phi f = f|f|^{-yi/x}\log(|f|/\|f\|_{\mathscr{R}(z)})$ is a bicentralizer on $\mathscr{R}(z)$.*

To be true, the preceding argument applies for $0 < x < 1$ only. If $x \geq 1$ one has to use a variation of Proposition 3.1 to get the result. Observe that the "spinning" part of the centralizer is hidden when z is real.

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F. Cabello Sánchez, Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain

E-mail: fcabello@unex.es