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Ring-theoretic blowing down. I

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Abstract. One of the major open problems in noncommutative algebraic geometry is the classification of noncommutative projective surfaces (or, slightly more generally, of noetherian connected graded domains of Gelfand–Kirillov dimension 3). Earlier work of the authors classified the connected graded noetherian subalgebras of Sklyanin algebras using a noncommutative analogue of blowing up. In order to understand other algebras birational to a Sklyanin algebra, one also needs a notion of blowing down. This is achieved in this paper, where we give a noncommutative analogue of Castelnuovo's classic theorem that (-1)-lines on a smooth surface can be contracted. The resulting noncommutative blown-down algebra has pleasant properties; in particular it is always noetherian and is smooth if the original noncommutative surface is smooth.

In a companion paper we will use this technique to construct explicit birational transformations between various noncommutative surfaces which contain an elliptic curve.

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1. Introduction

Throughout the paper, $\Bbbk$ will denote an algebraically closed field and all rings will be $\Bbbk$-algebras. A $\Bbbk$-algebra R is *connected graded* or *cg* if $R = \bigoplus_{n\geq 0} R_n$ is a finitely generated, $\mathbb{N}$-graded algebra with $R_0 = \Bbbk$. For such a ring R, the category of graded noetherian right R-modules will be denoted gr-R with quotient category qgr-R obtained by quotienting out the Serre subcategory of finite dimensional modules. An effective intuition is to regard qgr-R as the category of coherent sheaves on the (nonexistent) space $\operatorname{Proj}(R)$.

The classification of noetherian, connected graded domains R of Gelfand–Kirillov dimension 3 (or the corresponding noncommutative surfaces qgr-R) is one of the

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major open problems in noncommutative algebraic geometry. This has been solved in many particular cases and those solutions have lead to some fundamental advances in the subject; see, for example, [3,10,18,22–24] and the references therein. In [1], Artin conjectured that, birationally at least, there is a short list of such surfaces, with the generic case being a Sklyanin algebra. Here, the graded quotient ring $Q_{gr}(R)$ of R is obtained by inverting the non-zero homogeneous elements and two such domains R, S are *birational* if $Q_{gr}(R)_0 \cong Q_{gr}(S)_0$. Sklyanin algebras are defined in Example 4.2.

In earlier work of the authors the connected graded noetherian subalgebras of any Sklyanin algebra were classified [16,18] and this was achieved through a noncommutative variant of blowing up. However, if one wishes to classify more general algebras birational to a Sklyanin algebra one certainly also needs an analogue of blowing down (contracting) exceptional lines. This is achieved in this paper. Before describing these results in detail we set the stage by reviewing key classical results from commutative algebraic geometry. Thus, let x be a closed point on a smooth projective surface X over $\mathbb{k}$, and let $\pi : Bl_x(X) \to X$ be the blowup of X at x. These maps, also known as monoidal transformations, are of course fundamental to the birational geometry of surfaces.

It is well known that:

Proposition 1.1. *$Bl_x(X)$ is also a smooth projective surface. If $L = \pi^{-1}(x)$ is the exceptional locus of π, then $L \cong \mathbb{P}^1$ with self-intersection $L{\bullet}L = -1$.*

A celebrated theorem of Castelnuovo says that the properties of L given in the proposition also characterise curves that can be contracted to smooth points.

Theorem 1.2 (Castelnuovo). *Let Y be a smooth projective surface, and let L be a curve on Y with $L \cong \mathbb{P}^1$ and $L{\bullet}L = -1$. Then there are a smooth projective surface X and a birational morphism $\pi : Y \to X$ so that L is the exceptional locus of π; in fact $Y \cong Bl_x(X)$, where $x = \pi(L)$.*

The main aim of this paper is to give noncommutative versions of Proposition 1.1 and Theorem 1.2. These results apply to a class of algebras known as elliptic algebras that occur naturally among algebras birational to the Sklyanin algebras and are defined as follows. An *elliptic algebra* is a connected graded domain R containing a central $g \in R_1$ so that $R/(g)$ is isomorphic to a *twisted homogeneous coordinate ring* $B(E, \mathcal{M}, \tau)$, where E is an elliptic curve, with an ample invertible sheaf $\mathcal{M}$ and infinite order automorphism τ. We say that E is the *elliptic curve associated* to R and define the *degree* of R to be the degree of the line bundle $\mathcal{M}$. (See Section 4 for more details.) For example, the third Veronese ring $T = S^{(3)}$ of a Sklyanin algebra S is elliptic; the Veronese ring is needed to ensure that the central element has degree one, but this is a fairly harmless change since qgr-$S \cong$ qgr-T. Likewise, if S' is a 3-dimensional cubic Sklyanin algebra, as discussed in Example 4.2, then $T' = (S')^{(4)}$ is elliptic. The space qgr-S should be thought of as a noncommutative $\mathbb{P}^2$, while qgr-$S' \cong$ qgr-T' should be thought of as a noncommutative version of $\mathbb{P}^1 \times \mathbb{P}^1$.

An appropriate noncommutative analogue of a monoidal transformation of an elliptic algebra R is known. In more detail, $R_1/g\Bbbk \cong B(E,\mathcal{M},\tau)_1$ may be identified with global sections of the invertible sheaf $\mathcal{M}$. If $p \in E$ and $\deg \mathcal{M} \geq 3$, the *blowup of R at p* is defined to be the subalgebra $P = Bl_p(R)$ of R generated by those elements of R_1 whose images mod g vanish at p. By [16, Theorem 1.1], P is again an elliptic algebra and, moreover, has properties analogous to those of a (commutative) blowup. In particular, it has an analogue of an exceptional line. To be precise, a graded P-module $L = \bigoplus_{n\in\mathbb{Z}} L_n$ is a *line module* if L is cyclic with Hilbert series $\mathsf{hilb}\, L = \bigoplus_{n\in\mathbb{Z}}(\dim_\Bbbk L_n)s^n = 1/(1-s)^2$. Then $P = Bl_p(R)$ does indeed have a distinguished line module L, called the *exceptional line module* and characterised by the fact that $R/P \cong \bigoplus_{i\geq 1} L[-i]$ as P-modules. Inducting on this procedure one can blow up as many as seven points on the noncommutative projective plane qgr-T (one can even blow up eight points although the definition is more subtle since the ring is no longer generated in degree one [17]).

We would like to reverse this procedure. For a noncommutative version of Castelnuovo's Theorem 1.2, we need not only an analogue of a line but also an analogue of self-intersection. If qgr-P is a *smooth* noncommutative scheme in the sense that the category qgr-P has finite homological dimension, then an appropriate notion of intersection number is

$$(M \bullet_{MS} N) = \sum_{n\geq 0}(-1)^{n+1} \dim_\Bbbk \operatorname{Ext}^n_{\text{qgr-}P}(M,N),$$

for line modules M and N (see [14]). Unfortunately even if qgr-R is smooth, if $P = Bl_p(R)$ then the blowup qgr-P need not be smooth, in which case the self-intersection $(L \bullet_{MS} L)$ of a line module L can be undefined. (See Section 10, where an example is constructed by blowing up an elliptic algebra twice at the same point.) So, we use a weaker notion of intersection number, defined as follows. Let grk denote the torsion-free rank of a finitely generated $\Bbbk[g]$-module. Then, it is not hard to show that $(M \bullet_{MS} N) = \sum_{n\geq 0}(-1)^{n+1} \operatorname{grk} \operatorname{Ext}^n_P(M,N)$ (combine Proposition 6.4 and Lemma 6.2, in the notation from the beginning of Section 2). Moreover, as is discussed in Sections 6 and 7, the simpler sum

$$(M \bullet N) = -\operatorname{grk} \operatorname{Hom}_P(M,N) + \operatorname{grk} \operatorname{Ext}^1_P(M,N)$$

is a satisfactory alternative to $(M \bullet_{MS} N)$.

It will actually be convenient to use the following, still weaker concept. Assume that L is a line module over an elliptic algebra P and write $L = P/J$ for the *line ideal* J. We say two graded, locally finite dimensional vector spaces M and N are *numerically equivalent* if they have the same Hilbert series: $\mathsf{hilb}\, M = \mathsf{hilb}\, N$. Then the relevant condition is:

For a line module $L = P/J$, the rings P and $\operatorname{End}_P(J)$ are numerically equivalent. $(*)$

This notion is appropriate, as the next result shows.

Proposition 1.3 (Theorems 7.1 and 8.6). (1) *Let P be an elliptic algebra such that* qgr-P *has finite homological dimension, and let L be a line module over P. Then*
$$(L \bullet_{MS} L) = -1 \iff (L \bullet L) = -1 \iff (*)$$
holds.

(2) *If $P = Bl_p(R)$ is the blowup of an elliptic algebra R, then $(*)$ holds for the exceptional line module L.*

Our definition of self-intersection leads to a noncommutative version of Castelnuovo's Theorem 1.2, as we next show.

Theorem 1.4 (Theorems 8.3 and 8.6). (1) *Let P be an elliptic algebra with associated elliptic curve E and let L be a line module with $(L \bullet L) = -1$ or, more generally, one that satisfies $(*)$. Then one can blow down the line L.*

More precisely, there are an elliptic algebra $R = \widetilde{P} \supseteq P$, again associated to E, and a point $p \in E$ so that $P \cong Bl_p(R)$, with exceptional line L.

(2) *Conversely, if Q is an elliptic algebra of degree ≥ 4, then blowing Q up at a point p of the associated elliptic curve E and blowing down the exceptional line of $Bl_p(Q)$ returns the algebra Q.*

Definition. The ring $\widetilde{P}$ from part (1) of the theorem is called the *blowdown* of P at L.

The key step in the proof of part (1) of the theorem is to show that there exists a right P-module M with $Q_{gr}(P) \supset M \supset P$ for which $M/P \cong \bigoplus_{i \geq 1} L[-i]$. One then shows that M is actually a ring with the properties specified by the theorem.

Elliptic algebras have a number of pleasant properties; for example they are automatically noetherian and satisfy the Artin–Schelter, Gorenstein, and Cohen–Macaulay conditions (see Proposition 4.3). Thus, in particular, these conditions hold for the blowdown of an elliptic algebra. More subtly we have an analogue of the smoothness part of Castelnuovo's Theorem 1.2.

Theorem 1.5 (Corollary 9.2). *Let P be an elliptic algebra and suppose that L is a line module satisfying $(*)$, with blowdown $\widetilde{P}$. Assume, moreover, that $L[g^{-1}]_0$ has finite projective dimension over $P[g^{-1}]_0$.*

Then the noncommutative scheme qgr-$\widetilde{P}$ *is smooth if and only if* qgr-P *is smooth.*

Our eventual goal is to classify graded algebras birational to a Sklyanin algebra. Using the commutative geometry of surfaces as a guide, one would presumably need to classify "minimal models" (in the appropriate sense) and to show that any reasonable algebra in this class can be blown down to a minimal model. Clearly, the noncommutative versions of $\mathbb{P}^2$ and Van den Bergh's quadrics [24] should be minimal, and in forthcoming work we show that this is true [20]. We do not yet know whether these are the only minimal models.

We also do not know how to show that any algebra birational to a Sklyanin algebra can be blown down to give a minimal model. In the birational theory of commutative surfaces, this is proved using the following consequence of Zariski's Main Theorem:

Theorem 1.6 (Zariski). *Let $X \dashrightarrow Y$ be a birational map of smooth projective surfaces. Then there are a smooth projective surface Z and compositions of monoidal transformations $Z \to X$, $Z \to Y$ so that*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

commutes.

As yet, there is no noncommutative analogue of Theorem 1.6 although in the companion paper [19], we do prove:

Theorem 1.7. *Let E be the elliptic curve associated to the cubic Sklyanin algebra S', as defined above, and let $r \in E$ be generic. Then there is a Sklyanin algebra S associated to E and points $p, q \in E$ so that*

$$Bl_r((S')^{(4)}) \cong Bl_{p,q}(S^{(3)}).$$

In fact this theorem also holds when S' is replaced by any generic noncommutative quadric surface in the sense of [24]. This theorem is a noncommutative version of the isomorphism $Bl_{p,q}(\mathbb{P}^2) \cong Bl_r(\mathbb{P}^1 \times \mathbb{P}^1)$ arising from Theorem 1.6. The birationality of S and S' was first proved by Van den Bergh, with a detailed proof given in [15].

The paper is organised as follows. In Section 2 we review background on twisted homogeneous coordinate rings of elliptic curves. In Section 3 we study point modules over such a ring. In Section 4 we define elliptic algebras and give their basic properties, and in section 5 we define and study line modules over elliptic algebras. In Sections 6 and 7 we develop noncommutative intersection theory and prove part (1) of Proposition 1.3. In Section 8 we prove our main blowing down Theorem 1.4, and in Section 9 we prove Theorem 1.5. Finally, in Section 10 we study the effect of blowing up the same point twice.

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2. Basic concepts

In this section we review some basic material, including twisted homogeneous coordinate rings, that will be used frequently and without particular comment throughout the paper.

Throughout we work over an algebraically closed field $\Bbbk$, and rings will be $\Bbbk$-algebras unless otherwise noted. Given a noetherian $\mathbb{N}$-graded $\Bbbk$-algebra A, let Gr-A be the category of $\mathbb{Z}$-graded right A-modules, with morphisms $\mathrm{Hom}_{\text{Gr-}A}$ preserving degree. Write gr-A for the full subcategory of noetherian modules. Let $[1] : \text{Gr-}A \to \text{Gr-}A$ be the *shift functor*: the autoequivalence sending

$$M = \bigoplus M_i \to M[1] = \bigoplus M[1]_i,$$

where $M[1]_n = M_{n+1}$. For $M, N \in \text{Gr-}A$ the *graded Hom* groups are

$$\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\text{Gr-}A}(M, N[n]),$$

with derived functors

$$\underline{\mathrm{Ext}}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\text{Gr-}A}(M, N[n]).$$

If A and M are noetherian, then $\underline{\mathrm{Hom}}_A(M, N)$ equals the usual ungraded Hom which will always be written $\mathrm{Hom}_A(M, N)$. Similarly, $\underline{\mathrm{Ext}}^i_A(M, N) = \mathrm{Ext}^i_A(M, N)$, see [5, Proposition 3.1]. Finally, set $\underline{\mathrm{End}}_A(M) = \underline{\mathrm{Hom}}_A(M, M)$. We will frequently use the fact that

$$\mathrm{Ext}^r_{\text{Gr-}A}(M[-n], N) = \mathrm{Ext}^r_{\text{Gr-}A}(M, N[n]) = \mathrm{Ext}^r_{\text{Gr-}A}(M, N)[n], \tag{2.1}$$

for any M, N, n and r.

Let $A = \bigoplus_{n\geq 0} A_n$ be a cg noetherian algebra, and note that A is necessarily *locally finite* in the sense that $\dim_\Bbbk A_n < \infty$ for all n. Let tors-A be the category of modules in gr-A which are finite-dimensional over $\Bbbk$, and let Tors-A be the subcategory of Gr-A consisting of direct limits of finite-dimensional modules. Write Qgr-A for the quotient category Qgr-$A/$ Tors-A, with quotient functor $\pi : \text{Gr-}A \to$ Qgr-A. Then qgr-A = gr-$A/$ tors-A is identified with the noetherian objects in Qgr-A. Following [5], the pair (Qgr-$A, \pi(A)$) is called the *noncommutative projective scheme* associated to A. The autoequivalence [1] of Gr-A induces an autoequivalence, again written [1], of Qgr-A. We again have graded Hom groups

$$\underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}A}(\mathcal{M}, \mathcal{N}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Qgr}\text{-}A}(\mathcal{M}, \mathcal{N}[n]).$$

We emphasise here that a graded module is called *torsionfree* provided it has no finite dimensional submodules. In contrast a module M (graded or not) over a prime ring R is called *Goldie torsionfree* if no element of the module is killed by a regular element of the ring.

Next we review some important homological conditions.

Definition 2.1. A ring A is called *Auslander–Gorenstein* if

(i) $\mathrm{injdim}(A) < \infty$, in the sense that A has finite injective dimension on both left and right;

(ii) if $0 \le p < q$ and M is a finitely generated A-module, then $\mathrm{Ext}^p_A(N, A) = 0$ for every submodule N of $\mathrm{Ext}^q_A(M, A)$.

Write $\mathrm{GKdim}(M)$ for the Gelfand–Kirillov dimension of an A-module M, as in [11]. An R-module M is called d*-pure* if $\mathrm{GKdim}\, N = d = \mathrm{GKdim}\, M$ for all nonzero submodules N of M, and is d*-critical* if $\mathrm{GKdim}\, M/N < d$ for all all nonzero submodules N of M. Let A be a noetherian Auslander–Gorenstein $\Bbbk$-algebra with $\mathrm{GKdim}(A) < \infty$. For an A-module M, write

$$j(M) = \min\left\{r : \mathrm{Ext}^r_A(M, A) \neq 0\right\}$$

for the *homological grade* of M. The algebra A is called *Cohen–Macaulay* (or CM), provided that $j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A)$ holds for every finitely generated A-module M. The module M is then called *Cohen–Macaulay* (or CM) if $\mathrm{Ext}^r_A(M, A) = 0$ for all $r \neq j(M)$.

Finally, a cg noetherian $\Bbbk$-algebra A is called *Artin–Schelter (AS) Gorenstein* if $d = \mathrm{injdim}(A) < \infty$ and $\underline{\mathrm{Ext}}^j_A(\Bbbk, A) \cong \delta_{j,d}\Bbbk[\ell]$, where $\Bbbk = A/A_{\ge 1}$ is the *trivial module*, and ℓ is some shift of grading.

Let X, Y be $\Bbbk$-schemes. Write $\mathrm{Qcoh}\, X$ for the category of quasi-coherent sheaves on X, with $\mathrm{coh}\, X$ the subcategory of coherent sheaves. Given a morphism of $\Bbbk$-schemes $\phi : X \to Y$ and $\mathcal{F} \in \mathrm{Qcoh}\, Y$, we write $\mathcal{F}^\phi$ for the pullback $\phi^*(\mathcal{F})$. If $\mathcal{L}$ is an invertible sheaf on X, and $\tau \in \mathrm{Aut}_\Bbbk(X)$ is a $\Bbbk$-automorphism, one defines the *TCR or twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \tau) = \bigoplus_{n\ge 0} H^0(X, \mathcal{L}_n)$, where $\mathcal{L}_n = \mathcal{L}\otimes\mathcal{L}^\tau\otimes\cdots\otimes\mathcal{L}^{\tau^{n-1}}$. This is a $\mathbb{N}$-graded $\Bbbk$-algebra with multiplication defined as follows: for $x \in B_m$, $y \in B_n$, then $x \star y = \mu(x \otimes y^{\tau^m})$, where

$$\mu : H^0(E, \mathcal{L}_m) \otimes H^0(E, \mathcal{L}_n^{\tau^m}) \to B_{n+m} = H^0(E, \mathcal{L}_{n+m})$$

is the obvious map.

In this paper we are primarily concerned with the TCR $B(E, \mathcal{L}, \tau)$ of an elliptic curve E. In this case, the following result is well known:

Lemma 2.2. *Let E be an elliptic curve over $\Bbbk$ and $\mathcal{L}, \mathcal{M}$ be invertible sheaves on E of degree ≥ 2.*

(1) *The natural map*

$$\mu : H^0(E, \mathcal{L}) \otimes H^0(E, \mathcal{M}) \to H^0(E, \mathcal{L} \otimes \mathcal{M})$$

is surjective unless $\mathcal{L} \cong \mathcal{M}$ has degree 2, in which case $\dim \operatorname{Coker} \mu = 1$.

(2) *Let $\tau \in \operatorname{Aut}_{\Bbbk}(X)$ have infinite order. Then $B = B(E, \mathcal{L}, \tau)$ is generated as an algebra in degree* 1.

(3) *If B is as in (2), then B is a noetherian domain which is Auslander–Gorenstein, CM, and AS-Gorenstein, with* $\operatorname{injdim} B_B = 2$ *and* $\underline{\operatorname{Ext}}^j_B(\Bbbk, B) = \delta_{2j}\Bbbk$.

(4) *If B is as in (2), the map $\mathcal{F} \mapsto \pi(\bigoplus_{n \geq 0} H^0(E, \mathcal{F} \otimes \mathcal{L}_n))$ defines an equivalence of categories* $\operatorname{Qcoh} E \to \operatorname{Qgr-}B$.

Proof. Parts (1) and (2) are [16, Lemma 3.1]. That B is a domain follows immediately from the definition, and the noetherian property is [4, Theorem 1.4], while part (4) is [4, Theorem 1.3]. The remaining homological properties follow from [12, Theorems 6.3 and 6.6]. (Levasseur assumes that $\deg \mathcal{L} \geq 3$, but the proof only uses that $\mathcal{L}$ is ample). □

Notation 2.2. The quotient functor $\pi : \operatorname{Gr-}B \to \operatorname{Qgr-}B$ has a right adjoint $\omega : \operatorname{Qgr-}B \to \operatorname{Gr-}B$ called the section functor, which may be described more explicitly as follows. If

$$M = \bigoplus_{n \geq 0} H^0(E, \mathcal{F} \otimes \mathcal{L}_n)$$

for a coherent sheaf $\mathcal{F}$, then

$$\omega\pi(M) = \bigoplus_{n \in \mathbb{Z}} H^0(E, \mathcal{F} \otimes \mathcal{L}_n),$$

where we define $\mathcal{L}_n$ for $n < 0$ by

$$\mathcal{L}_n = (\mathcal{L}^{\tau^n} \otimes \cdots \otimes \mathcal{L}^{\tau^{-1}})^{-1}.$$

We say that a graded B-module M is *saturated* if it is in the image of the section functor ω. By[5, (2.2.3)], this is equivalent to $\underline{\operatorname{Ext}}^1_B(\Bbbk, M) = 0$.

Given an $\mathbb{N}$-graded noetherian domain A, the localisation of A at the set of nonzero homogeneous elements exists and is called the *graded quotient ring* $Q = Q_{\mathrm{gr}}(A)$ of A. Given noetherian graded A-submodules M, N of Q, we identify $\underline{\operatorname{Hom}}_A(M, N)$ with $\{x \in Q : xM \subseteq N\}$. In particular, $\underline{\operatorname{Hom}}_A(M, A)$ is identified with $M^* = \{x \in Q : xM \subseteq A\}$ and M is *reflexive* if $M = M^{**}$. If $A = B = B(E, \mathcal{L}, \tau)$ as in Lemma 2.2, then $Q_{gr}(B) \cong \Bbbk(E)[t, t^{-1}; \tau]$ where $\Bbbk(E)$ is the function field of E

with the induced action of τ. We sometimes fix an isomorphism $Q_{\mathrm{gr}}(B) \cong Q = \Bbbk(E)[t, t^{-1}; \tau]$, and write B as the explicit subalgebra $B = \bigoplus_{n\geq 0} H^0(E, \mathcal{L}_n)t^n$ of Q, where each $H^0(E, \mathcal{L}_n)$ is then given a fixed embedding into $\Bbbk(E)$. The following result will be useful in calculating Homs between B-submodules of Q.

Lemma 2.3. *Let $B = B(E, \mathcal{L}, \tau) = \bigoplus_{n\geq 0} H^0(E, \mathcal{L}_n)t^n \subseteq Q = \Bbbk(E)[t, t^{-1}; \tau]$ for an elliptic curve E over $\Bbbk$, with invertible sheaf $\mathcal{L}$ of degree ≥ 2 and infinite order automorphism $\tau \in \mathrm{Aut}_{\Bbbk}(E)$. Let $\mathcal{F}$ and $\mathcal{G}$ be invertible $\mathcal{O}_E$-subsheaves of the constant sheaf $\Bbbk(E)$, and let $M = \bigoplus_{n\in\mathbb{Z}} H^0(E, \mathcal{F} \otimes \mathcal{L}_n)t^n$, $N = \bigoplus_{n\in\mathbb{Z}} H^0(E, \mathcal{G} \otimes \mathcal{L}_n)t^n$ be saturated B-submodules of Q. Then*

$$\underline{\mathrm{Hom}}_B(M, N) = \bigoplus_{n\in\mathbb{Z}} H^0\big(E, (\mathcal{F}^{\tau^n})^{-1} \otimes \mathcal{G} \otimes \mathcal{L}_n\big)t^n \subseteq Q.$$

Proof. This is similar to the proof of [18, Lemma 6.14(i)], but since we use the result frequently we give the details.

Write $H = \underline{\mathrm{Hom}}_B(M, N)$, and $X = \bigoplus_{n\in\mathbb{Z}} H^0(E, (\mathcal{F}^{\tau^n})^{-1} \otimes \mathcal{G} \otimes \mathcal{L}_n)t^n$ both of which can be identified with subspaces of Q. For each n, let $\mathcal{H}_n$ be the subsheaf of the constant sheaf $\Bbbk(E)$ generated by $H_n t^{-n} \subseteq \Bbbk(E)$. Let $\mathcal{M}_n = \mathcal{F} \otimes \mathcal{L}_n$; thus $M_n = H^0(E, \mathcal{M}_n)$, and M_n generates the sheaf $\mathcal{M}_n$, for $n \gg 0$, say for $n \geq n_0$, because $\mathcal{L}$ is τ-ample by [4, Proposition 1.5]. Similarly, write $\mathcal{N}_n = \mathcal{G} \otimes \mathcal{L}_n$ for all n.

For $n \geq n_0$ and $r \geq 0$, the equation $H_r M_n \subseteq N_{n+r}$ forces $\mathcal{H}_r \mathcal{M}_n^{\tau^r} \subseteq \mathcal{N}_{n+r}$ and so

$$\mathcal{H}_r \otimes \big(\mathcal{F} \otimes \mathcal{L}_n\big)^{\tau^r} \subseteq \mathcal{G} \otimes \mathcal{L}_{n+r}.$$

Equivalently $\mathcal{H}_r \subseteq (\mathcal{F}^{\tau^r})^{-1} \otimes \mathcal{G} \otimes \mathcal{L}_r$ and so $H \subseteq \bigoplus H^0(E, \mathcal{H}_r)t^r \subseteq X$. Another calculation shows that $\big((\mathcal{F}^{\tau^r})^{-1} \otimes \mathcal{G} \otimes \mathcal{L}_r\big)\,\mathcal{M}_n^{\tau^r} = \mathcal{N}_{n+r}$ for $r, n \geq 0$ and taking sections for $n \geq n_0$ shows that $X \subseteq \underline{\mathrm{Hom}}_B(M_{\geq n_0}, N)$.

To complete the proof we need to prove that $H = \underline{\mathrm{Hom}}_B(M_{\geq n_0}, N)$. Clearly, $H \subseteq \underline{\mathrm{Hom}}_B(M_{\geq n_0}, N)$. However, if $\theta \in \underline{\mathrm{Hom}}_B(M_{\geq n_0}, N)_t$ for some t, then we may consider θ as an element of Q. We see that $Z = (\theta M + N)$ is a B-submodule of Q such that $ZB_{\geq n_0} \subseteq N$. Since N is saturated, this forces $Z \subseteq N$ and $\theta \in H$, as desired. □

Notation 2.3. Let A be connected graded. A $\mathbb{Z}$-graded A-module M is *left* respectively *right bounded* if $M_n = 0$ for $n \ll 0$, respectively for $n \gg 0$. Obviously right bounded modules are in Tors-A, and finitely generated graded modules M are left bounded. Importantly, if A is noetherian and M and N are finitely generated graded A-modules, then by considering a free resolution of M, each $\underline{\mathrm{Ext}}^i_A(M, N)$ is left bounded and locally finite. If M is locally finite, the *Hilbert series* of M is the formal Laurent series $\mathsf{hilb}\, M = h_M(s) = \sum_{n\in\mathbb{Z}} \dim_{\Bbbk} M_n s^n \in \mathbb{Z}((s))$. Note that we use the notations $\mathsf{hilb}\, M$ and h_M interchangeably. Given two Hilbert series $g(s) = \sum a_n s^n$ and $h(s) = \sum b_n s^n$ we write $g(s) \leq h(s)$ if $a_n \leq b_n$ for all $n \in \mathbb{Z}$.

3. Point modules

In this section we study some homological properties of point modules over a twisted homogeneous coordinate ring. Throughout the section, we fix an elliptic curve E, an automorphism $\tau \in \mathrm{Aut}_{\Bbbk}(E)$ of infinite order and an invertible sheaf $\mathcal{M}$ with $\deg \mathcal{M} \geq 3$, although many of the results hold more generally. Corresponding to this data, set $B = B(E, \mathcal{M}, \tau)$. Points of E will always mean closed points.

Definition 3.1. Let A be a cg $\Bbbk$-algebra that is generated in degree one. Then a *point module* over A is a graded cyclic module M with Hilbert series $h_M(s) = 1/(1-s)$. If M is an A-point module, then there is a graded isomorphism $M \cong A/I$ for a unique right ideal I, called a (right) *point ideal*. Now let $B = B(E, \mathcal{M}, \tau)$ be a TCR as defined in the last section; thus B is generated in degree 1 by Lemma 2.2. By Lemma 2.2(4), the isomorphism classes of B-point modules are in one-to-one correspondence with the closed points of E; explicitly, if $p \in E$ with skyscraper sheaf $\mathcal{O}_p$ and ideal sheaf $\mathcal{I}_p$, then $p \in E$ corresponds to the point module

$$M_p = \bigoplus_{n \geq 0} H^0\big(E, \mathcal{O}_p \otimes \mathcal{M}_n\big),$$

with point ideal

$$I_p = \bigoplus_{n \geq 0} H^0\big(E, \mathcal{I}_p \otimes \mathcal{M}_n\big) \subseteq B.$$

It is easy to see that I_p is a saturated right ideal in the sense of Notation 2.2.

When considering shifts of point modules, the following formula will be useful.

$$M_p[n]_{\geq 0} \cong M_{\tau^n p} \qquad \text{for any } p \in E \text{ and } n \geq 0 \tag{3.1}$$

(see, for example, [17, Lemma 4.8(1)]). In particular, $(M_p)_{\geq n} \cong (M_{\tau^n p})[-n]$ for all $p \in E$ and $n \geq 0$.

Remark 3.2. We occasionally work with left point modules over $B = B(E, \mathcal{M}, \tau)$. Of course there are left-sided versions of all of the results above; in particular, the left B-point modules are again in bijection with the points of E. In this case the equivalence of categories Qcoh $E \to B$-Qgr is induced by the functor

$$\mathcal{F} \mapsto \bigoplus_{n \geq 0} H^0\big(E, \mathcal{F}^{\tau^{n-1}} \otimes \mathcal{M}_n\big).$$

In particular, the left point module corresponding to $p \in E$ is

$$M_p^{\ell} = \bigoplus_{n \geq 0} H^0\big(E, (\mathcal{O}_p)^{\tau^{n-1}} \otimes \mathcal{M}_n\big).$$

Moreover, $M_p^{\ell} \cong B/J_p$ for the *left point ideal*

$$J_p = \bigoplus_{n \geq 0} H^0\big(E, (\mathcal{I}_p)^{\tau^{n-1}} \otimes \mathcal{M}_n\big).$$

Note that the correspondence is set up so that if I_p is the right point ideal corresponding to p, and J_p the left point ideal, then

$$(I_p)_1 = (J_p)_1 = H^0\big(E, \mathcal{I}_p \otimes \mathcal{M}\big).$$

We also have the following analogue of (3.1):

$$\big(M_q^{\ell}\big)[n]_{\geq 0} \cong M^{\ell}_{\tau^{-n}q} \qquad \text{for any } q \in E \text{ and } n \geq 0. \tag{3.2}$$

Lemma 3.3. *Let $B = B(E, \mathcal{M}, \tau)$ as above with a right point module $M = M_p$. Then M is CM and $\underline{\mathrm{Ext}}^1_B(M_p, B)[1] \cong M^{\ell}_{\tau^{-2}(p)}$ is a left point module. The analogous result holds for left point modules.*

Proof. We freely use the properties of B given by Lemma 2.2. Set $E^{pq}(M) = \underline{\mathrm{Ext}}^p_B(\underline{\mathrm{Ext}}^q_B(M, B), B)$.

We first show that M is CM, which will be a routine consequence of the spectral sequence

$$E_2^{p,-q} = E^{pq}(M) \Rightarrow \mathbb{H}^{p-q}(M) = \begin{cases} M & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

as described in [12, Theorem 2.2(a)] or [6, §I.1]. Note that, as B is generated in degree one, every proper factor module of M is finite dimensional, and so M is 1-critical. Now B is Auslander–Gorenstein of injective dimension 2. Therefore, by the 1-criticality of M and [12, Theorem 2.4(b)], and in the notation of that result, $F^j(M) = 0$ for $j > 1$. Hence $M = E^{11}(M)$ but $E^{22}(M) = 0$ by [12, Theorem 2.2(b)]. In particular, $\underline{\mathrm{Ext}}^2_B(M, B) = 0$ by the Gorenstein property while $\underline{\mathrm{Ext}}^0_B(M, B) = 0$ since B is a domain. Thus M is indeed CM.

It is almost immediate from (3.3) that $\underline{\mathrm{Ext}}^1_B(M, B)[1]$ is a point module, but as we need to identify the corresponding point, we take a different approach. Write $M = M_p = B/I_p$. Applying $\underline{\mathrm{Hom}}_B(-, B)$ to the exact sequence

$$0 \to I_p \to B \to M_p \to 0$$

shows that $\underline{\mathrm{Ext}}^1_B(M_p, B) \cong I_p^*/B$. By Lemma 2.3

$$I_p^* \cong \bigoplus_{n \geq 0} H^0\big(E, \big(\mathcal{I}_p^{\tau^n}\big)^{-1} \otimes \mathcal{M}_n\big) = \bigoplus_{n \geq 0} H^0\big(E, \mathcal{M}_n\big(\tau^{-n}(p)\big)\big),$$

and so I_p^*/B has Hilbert series $s/(1-s)$. It follows from Lemma 2.2 that $B(I_p^*)_1 = (I_p^*)_{\geq 1}$, and so I_p^*/B is cyclic. In other words, $\underline{\mathrm{Ext}}^1_B(M_p, B)[1]$ is a point module. The fact that this point module is indeed $M^{\ell}_{\tau^{-2}(p)}$ now follows from Remark 3.2 and (3.2). The result for left modules is left to the reader. $\square$

We next want to compute the $\underline{\mathrm{Ext}}$ groups between point modules over $B = B(E, \mathcal{M}, \tau)$. As the next result shows, in qgr-B this follows easily from the equivalence qgr-$B \simeq \mathrm{coh}\, E$. When there is no chance of confusion, given $M \in$ gr-B, the object $\pi(M) \in$ qgr-B will also be written as M.

Lemma 3.4. *Let $B = B(E, \mathcal{M}, \tau)$ as before. Then $\underline{\mathrm{Ext}}^m_{\text{qgr-}B}(M_p, M_q) = 0$ for $m \geq 2$ and*

$$\underline{\mathrm{Hom}}_{\text{qgr-}B}(M_p, M_q) \cong \underline{\mathrm{Ext}}^1_{\text{qgr-}B}(M_p, M_q) \cong \begin{cases} 0 & \textit{for } p, q \textit{ on distinct orbits}, \\ \Bbbk[-j] & \textit{if } p = \tau^j(q), \textit{for } j \in \mathbb{Z}. \end{cases}$$

Remark. This result and its proof also hold when $\deg \mathcal{M} = 2$.

Proof. As noted in Definition 3.1, the point module M_p corresponds to the skyscraper sheaf $\mathcal{O}_p$ at p. Thus, by (3.1),

$$\underline{\mathrm{Ext}}^m_{\text{qgr-}B}(M_p, M_q) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^m_{\text{qgr-}B}\left(M_p, M_q[n]\right) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^m_E\left(\mathcal{O}_p, \mathcal{O}_{\tau^n(q)}\right)$$

for all m. Since E is a smooth curve, $\mathrm{Ext}^m_E(\mathcal{O}_p, \mathcal{O}_r) = 0$ for any closed point $r \in E$ and $m \geq 2$. On the other hand, working locally gives

$$\mathrm{Hom}_E\left(\mathcal{O}_p, \mathcal{O}_r\right) = \mathrm{Ext}^1_E\left(\mathcal{O}_p, \mathcal{O}_r\right) = \begin{cases} 0 & \text{if } p \neq r, \\ \Bbbk & \text{if } p = r. \end{cases}$$

Now apply (2.1). $\square$

We next want to prove the analogue of Lemma 3.4 for homomorphisms in gr-B, for which we need several elementary observations.

Lemma 3.5. *Let $q \in E$. Then the following hold.*

(1) *The only torsionfree extensions of M_q by finite dimensional graded B-modules are the shifted point modules $M_{\tau^{-n}(q)}[n]$ for $n \geq 0$.*

(2) $\underline{\mathrm{Ext}}^1_B(\Bbbk[-n], M_q) = \Bbbk[n+1]$ *for all $n \in \mathbb{Z}$.*

(3) *For all $p, q \in E$, one has $\underline{\mathrm{Ext}}^1_B(M_p, M_q)_{-1} \neq 0$.*

(4) $\mathrm{Ext}^1_{\text{Gr-}B}(M_p, M_p) \neq 0$.

Proof. (1) Note first that, by (3.1), the module $X = M_{\tau^{-n}(q)}[n]$ satisfies $X_{\geq 0} = ((M_{\tau^{-n}(q)})_{\geq n})[n] = M_q$, and so X is an extension of the required form. Conversely, any such extension is necessarily a 1-critical module, and so uniqueness follows, for example, from [18, Corollary 3.7(1)].

(2) Since any non-trivial extension of M_q by a shift of $\Bbbk$ is necessarily torsionfree, it follows from (1) that the only such extension is $0 \to M_q \to M_{\tau^{-1}q}[1] \to \Bbbk[1] \to 0$. Thus $\underline{\mathrm{Ext}}^1_B(\Bbbk[-n], M_q) = \Bbbk[n+1]$.

(3) Consider B/J, where J is the right ideal $\bigoplus_{n\geq 0} H^0(E, \mathcal{M}_n(-p-\tau^{-1}(q)))$. Then the point ideal $I_p = \bigoplus_{n\geq 0} H^0(E, \mathcal{M}_n(-p))$ contains J, with $I_p/J \cong (M_{\tau^{-1}(q)})_{\geq 1} \cong M_q[-1]$. It is clear that this extension of I_p/J by $B/I_p \cong M_p$ is nonsplit, since B/J is cyclic.

(4) By Lemma 2.2(4), the nonsplit extension of $\mathcal{O}_p$ by itself in $\operatorname{coh} E$ gives a nonsplit extension $0 \to \pi(M_p) \to \mathcal{F} \to \pi(M_p) \to 0$ in qgr-B, for some $\mathcal{F} \in$ qgr-B. Note that the section functor ω from Notation 2.2 satisfies $\omega(\pi(N)) = \underline{\operatorname{Hom}}_{\text{qgr-}B}(\pi(B), \pi(N))$. In particular, applying ω to our exact sequence gives an exact sequence

$$0 \to \omega\big(\pi(M_p)\big) \to \omega(\mathcal{F}) \to \omega\big(\pi(M_p)\big) \to \underline{\operatorname{Ext}}^1_{\text{qgr-}B}\big(\pi(B), \pi(M_p)\big) \to \cdots \tag{3.4}$$

Now

$$\begin{aligned}\underline{\operatorname{Ext}}^1_{\text{qgr-}B}\big(\pi(B), \pi(M_p)\big) &= \bigoplus_m \operatorname{Ext}^1_{\text{qgr-}B}\big(\pi(B), \pi\big(M_p[m]\big)\big)\\ &= \bigoplus_m \operatorname{Ext}^1_{\operatorname{coh} E}\big(\mathcal{O}_E, \mathcal{O}_{\tau^m(p)}\big) = 0,\end{aligned}$$

using the equivalence of categories between qgr-B and $\operatorname{coh} E$ and the fact that sheaves with zero-dimensional support have vanishing higher cohomology. However, $\omega(\pi(M_p))_{\geq 0} = M_p$ by construction and so (3.4) becomes the exact sequence

$$0 \to M_p \to \omega(\mathcal{F})_{\geq 0} \to M_p \to 0.$$

This is nonsplit since applying π yields the original nonsplit extension. □

Proposition 3.6. *Let $p, q \in E$. Then*

(1) $$\underline{\operatorname{Hom}}_B\big(M_p, M_q\big) = \begin{cases} \Bbbk[-j] & \text{if } p = \tau^j q \text{ for some } j \geq 0,\\ 0 & \text{otherwise,}\end{cases}$$

while

(2) $$\underline{\operatorname{Ext}}^1_B\big(M_p, M_q\big) = \begin{cases} \Bbbk[1] \oplus \Bbbk[-j] & \text{if } p = \tau^j q \text{ for some } j \geq 0,\\ \Bbbk[1] & \text{otherwise.}\end{cases}$$

Proof. (1) Using (3.1), we have

$$\underline{\operatorname{Hom}}_B\big(M_p, M_q\big)_j \neq 0 \iff M_p[-j] \cong (M_q)_{\geq j} \cong (M_{\tau^j q})[-j].$$

Clearly this happens if and only if $j \geq 0$ and $p = \tau^j q$; in particular it can happen for at most one value of j.

(2) Now consider the Ext groups. Note that

$$\underline{\mathrm{Ext}}^i_{\mathrm{qgr}\text{-}B}\big(M_p, M_q\big) = \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_{\mathrm{gr}\text{-}B}\big((M_p)_{\geq n},\ M_q\big),$$

by [5, Proposition 7.2]. Thus the exact sequences

$$0 \to (M_p)_{\geq n} \to M_p \to M/(M_p)_{\geq n} \to 0$$

induce the exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}_B\big(M_p, M_q\big) &\longrightarrow \underline{\mathrm{Hom}}_{\mathrm{qgr}\text{-}B}\big(M_p, M_q\big) \\ &\xrightarrow{\alpha} \lim_{n\to\infty} \underline{\mathrm{Ext}}^1_B\big(M_p/(M_p)_{\geq n}, M_q\big) \xrightarrow{\beta} \underline{\mathrm{Ext}}^1_B\big(M_p, M_q\big) \\ &\longrightarrow \underline{\mathrm{Ext}}^1_{\mathrm{qgr}\text{-}B}\big(M_p, M_q\big) \longrightarrow \cdots \end{aligned} \tag{3.5}$$

We claim that $X = \lim_{n\to\infty} \underline{\mathrm{Ext}}^1_B(M_p/(M_p)_{\geq n}, M_q)$ is zero in degrees ≥ 0 while $\dim_{\Bbbk} X_{-1} = 1$. To see this, apply $\underline{\mathrm{Hom}}_B(-, M_q)$ to the exact sequence

$$0 \to \Bbbk[-n] \longrightarrow M_p/(M_p)_{\geq n+1} \longrightarrow M_p/(M_p)_{\geq n} \longrightarrow 0.$$

This gives

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Ext}}^1_B\big(M_p/(M_p)_{\geq n}, M_q\big) \longrightarrow \underline{\mathrm{Ext}}^1_B\big(M_p/(M_p)_{\geq n+1}, M_q\big) \\ \xrightarrow{\phi} \underline{\mathrm{Ext}}^1_B\big(\Bbbk[-n], M_q\big) \longrightarrow \cdots \end{aligned}$$

Since $\underline{\mathrm{Ext}}^1_B(\Bbbk[-n], M_q) = \Bbbk[n+1]$ by Lemma 3.5, it follows from this sequence and induction on n that $\underline{\mathrm{Ext}}^1_B(M_p/(M_p)_{\geq n}, M_q)$ lives in negative degrees, and that $\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_B(M_p/(M_p)_{\geq n}, M_q)_{-1} = 1$ for all n. The claim follows.

Next, write $M_p = B/I_p$, where $I_p = \bigoplus_{n\geq 0} H^0(E, \mathcal{I}_p \otimes \mathcal{M}_n)$. Since $\deg(\mathcal{I}_p \otimes \mathcal{M}) \geq 2$, it follows from Lemma 2.2(1) that I_p is generated in degree one. Thus the graded free resolution of M_p begins

$$\cdots \to \bigoplus_{i=1}^{m} B[-1] \longrightarrow B \longrightarrow M_p \longrightarrow 0.$$

Using this to calculate $\underline{\mathrm{Ext}}^1_B(M_p, M_q)$ shows that $\underline{\mathrm{Ext}}^1_B(M_p, M_q)_n = 0$ for $n \leq -2$.

Combining this with the conclusion of the previous paragraph shows that, in (3.5), $\mathrm{Im}\,\beta$ is concentrated entirely in degree -1, and is either 0 or $\Bbbk[1]$. In fact we can be more precise. By comparing Lemma 3.4 with Part (1), it follows that $\alpha = 0$ except when $p = \tau^j(q)$ for some $j < 0$ and in the latter case $\mathrm{Im}(\alpha) = \Bbbk[-j]$. We conclude that

$$\mathrm{Im}\,\beta = \begin{cases} 0 & \text{if } p = \tau^{-1}(q), \\ \Bbbk[1] & \text{otherwise.} \end{cases} \tag{3.6}$$

To complete the proof, we consider several cases. First, if p, q lie on different orbits then Lemma 3.4 implies that $\underline{\mathrm{Ext}}_{\mathrm{qgr}\text{-}B}(M_p, M_q) = 0$. Thus, (3.6) shows that $\underline{\mathrm{Ext}}^1_B(M_p, M_q) = \operatorname{Im}\beta \cong \Bbbk[1]$.

Next, suppose that $p = \tau^{-1}(q)$. Then, $\beta = 0$ by (3.6) and so in this case Lemma 3.4 implies that $\underline{\mathrm{Ext}}^1_B(M_p, M_q) \hookrightarrow \underline{\mathrm{Ext}}^1_{\mathrm{qgr}\text{-}B}(M_p, M_p) \cong \Bbbk[1]$. Since $\Bbbk[1] \hookrightarrow \underline{\mathrm{Ext}}^1_B(M_p, M_q)$ by Lemma 3.5(3), we conclude that $\underline{\mathrm{Ext}}^1_B(M_p, M_q) \cong \Bbbk[1]$, as required.

It remains to consider the case when $p = \tau^j(q)$ for some $-1 \neq j \in \mathbb{Z}$. Comparing (3.6) with Lemma 3.4 gives an exact sequence

$$0 \longrightarrow \Bbbk[1] \longrightarrow \underline{\mathrm{Ext}}^1_B\big(M_p, M_q\big) \longrightarrow \Bbbk[-j] \longrightarrow \cdots$$

If $j \leq -2$, we have shown above that $\underline{\mathrm{Ext}}^1_B(M_p, M_q)_j = 0$. Hence $\underline{\mathrm{Ext}}^1_B(M_p, M_q) \cong \Bbbk[1]$. If $j \geq 0$, then we must show that $\underline{\mathrm{Ext}}^1_B(M_p, M_q) \cong \Bbbk[1] \oplus \Bbbk[-j]$, for which it suffices to show that $0 \neq \underline{\mathrm{Ext}}^1_B(M_p, M_q)_j = \underline{\mathrm{Ext}}^1_B(M_p, M_q[j])_0$. From the exact sequence

$$\begin{aligned} 0 = \operatorname{Hom}_{\mathrm{gr}\text{-}B}\big(M_p, M_q[j]/M_q[j]_{\geq 0}\big) \qquad\qquad& \\ \longrightarrow \operatorname{Ext}^1_{\mathrm{gr}\text{-}B}\big(M_p, M_q[j]_{\geq 0}\big) \longrightarrow \operatorname{Ext}^1_{\mathrm{gr}\text{-}B}\big(M_p, M_q[j]\big)& \end{aligned}$$

it suffices to show that $\underline{\mathrm{Ext}}^1_B(M_p, M_q[j]_{\geq 0})_0 \neq 0$ or, equivalently by (3.1), that $\underline{\mathrm{Ext}}^1_B(M_p, M_p)_0 \neq 0$. In other words, we can reduce to the case $j = 0$, where the result is just Lemma 3.5(4). □

4. Elliptic algebras

In this section, we define elliptic algebras, which are the main objects of interest in this paper, and describe some of their more basic properties.

Definition 4.1. A connected $\mathbb{N}$-graded algebra R is called an *elliptic algebra* if there is a central nonzerodivisor $g \in R_1$ such that $R/gR \cong B(E, \mathcal{M}, \tau)$ for some elliptic curve E, invertible sheaf $\mathcal{M}$, and infinite order automorphism τ. We call $\deg \mathcal{M}$ the *degree* of the elliptic algebra. *In this paper, we will always assume that an elliptic algebra has degree at least* 3 *unless otherwise stated.*

Example 4.2. Some of the most important elliptic algebras arise from the (quadratic) Sklyanin algebra

$$S = \mathrm{Skl}(a,b,c) = \Bbbk\{x_1, x_2, x_3\}/(ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2 : i \in \mathbb{Z}_3),$$

where $(a,b,c) \in \mathbb{P}^2 \setminus \mathcal{S}$ for a (known) finite set $\mathcal{S}$. Here, S contains a canonical central element $g \in S_3$ such that $S/gS \cong B(E, \mathcal{L}, \sigma)$ for an elliptic curve E. In this paper we restrict attention to the case when $|\sigma| = \infty$ since the 3-Veronese ring $T = S^{(3)}$ is then an elliptic algebra, with $T/gT \cong B(E, \mathcal{M}, \tau)$ for $\mathcal{M} = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \mathcal{L}^{\sigma^2}$ and $\tau = \sigma^3$.

Another, related example of an elliptic algebra can be obtained by taking the fourth Veronese ring $T' = (S')^{(4)}$ of the cubic Sklyanin algebra S' from [3]. More generally, there are the (second Veroneses of the) quadrics constructed in [24]. As discussed in [22, Example 8.5], these algebras can all be written as factors of a certain 4-dimensional Sklyanin algebra, although as they will not be needed explicitly in this paper we will omit the definitions.

An elliptic algebra R automatically has a number of good properties, basically because the same properties hold for the factor ring $B = R/Rg$. Before stating the result we need one more definition. Given a noetherian cg $\Bbbk$-algebra A, regard $\Bbbk = A/A_{\geq 1}$ as a right A-module. Then A satisfies *the Artin–Zhang χ-condition* (on the right) provided $\dim_{\Bbbk} \underline{\mathrm{Ext}}^j_A(\Bbbk, M) < \infty$ for all $M \in \text{gr-}R$ and all $j \geq 0$.

Proposition 4.3. *Let R be an elliptic algebra with $B = R/gR$. Then both R and B are noetherian domains generated in degree* 1. *In addition, R and B are Auslander–Gorenstein, CM, AS-Gorenstein, and satisfy the Artin–Zhang χ-condition.*

Proof. By Lemma 2.2, R/gR and hence R are generated in degree one. Now the noetherian, Auslander–Gorenstein and CM properties as well as the χ condition hold for B by [16, Lemma 2.2] and for R by [16, Theorem 6.3]. The proofs of these results also easily imply that $\mathrm{GKdim}(B) = 2$ and $\mathrm{GKdim}(R) = 3$, so that by [12, Theorem 6.3], both B and R are also AS-Gorenstein. $\square$

Notation 4.1. Let R be an elliptic algebra with factor ring $B = R/gR$. For a graded R-module M or, indeed a $\Bbbk[g]$-module M, its *g-torsion submodule* is

$$t(M) = \{m \in M : g^n m = 0 \text{ for } n \gg 1\}.$$

Then M is *g-torsion* if $M = t(M)$ and *g-torsionfree* if $t(M) = 0$. Write $R_{(g)}$ for the homogeneous localisation of R at the completely prime ideal gR; thus $R_{(g)} = R\mathcal{C}^{-1}$, for $\mathcal{C}$ the set of homogeneous elements in $R \setminus gR$. As in [18, Notation 2.5], $R_{(g)}/gR_{(g)} \cong Q_{gr}(B) \cong \Bbbk(E)[t, t^{-1}; \tau]$. We say that a $\Bbbk[g]$-submodule $M \subseteq R_{(g)}$ is *g-divisible* when $M \cap gR_{(g)} = Mg$. It is easy to see that M is g-divisible if and only if $R_{(g)}/M$ is g-torsionfree. The fact that $R/gR \cong B$ is a domain forces R itself to be g-divisible. When M is g-divisible, we can and will identify M/Mg with

$$\overline{M} = (M + gR_{(g)})/gR_{(g)} \subseteq R_{(g)}/gR_{(g)}.$$

The following properties of graded modules over an elliptic algebra will be useful.

Lemma 4.4. *Let R be an elliptic algebra.*

(1) *If $M, N \subseteq R_{(g)}$ are g-divisible graded R-submodules, then $\underline{\mathrm{Hom}}_R(M, N) \subseteq R_{(g)}$ is also g-divisible.*

(2) *If M is a g-torsionfree finitely generated graded right R-module with* $\mathrm{GKdim}\, M \leq 1$, *then $I = \mathrm{Ann}_R(M)$ is a nonzero graded ideal of R with* $\mathrm{GKdim}\, R/I \leq 1$.

Proof. (1) See [18, Lemma 2.12(2)].

(2) As M/Mg is finite dimensional, $M^\circ = M[g^{-1}]_0$ is a finite dimensional module over $R^\circ = R[g^{-1}]_0$. Hence $J = \mathrm{Ann}_{R^\circ}(M^\circ) \neq 0$. Therefore,

$$0 \neq \widehat{J} = \bigcup_n \{a \in R_n : ag^{-n} \in J\}$$

and certainly $\widehat{J} \subseteq \mathrm{Ann}_R M$; whence $I = \mathrm{Ann}_R M \neq 0$. As M is g-torsionfree, $I \not\subseteq gR$. Hence, by [18, Lemma 2.15(3)], $\mathrm{GKdim}\, R/I \leq 1$. □

Lemma 4.5. *Let R be an elliptic algebra with $R/Rg = B$.*

(1) *Let $J \in$ gr-R with $J \subseteq Q = Q_{gr}(R)$. Then J^{**} is the unique largest R-submodule of Q such that* $\mathrm{GKdim}\, J^{**}/J \leq 1$. *In particular, J is reflexive if and only if $Q_{gr}(R)/J$ is 2-pure.*

(2) *Let $J \subseteq R_{(g)}$ be a finitely generated, g-divisible graded right R-submodule. If $\overline{J} = J/Jg$ is saturated as a B-module, then J is reflexive as a right R-module.*

Proof. (1) This follows from the CM property; see, for example, [12, (4.6.6) and Remark 5.8(4)].

(2) We know that J is reflexive if and only if $Q_{gr}(R)/J$ is 2-pure. This is equivalent to $R_{(g)}/J$ being 2-pure, since $Q_{gr}(R)/R_{(g)} = \bigcup_{n\geq 1} g^{-n}R_{(g)}/R_{(g)}$ is 2-pure.

If J is not reflexive, there exists a finitely generated module $J \subsetneq N \subseteq R_{(g)}$ with $\mathrm{GKdim}(N/J) \leq 1$. By Lemma 4.4(2) and the fact that $R_{(g)}/J$ is g-torsionfree, $NI \subseteq J$ for some graded ideal I of R with $\mathrm{GKdim}\, R/I \leq 1$. Let

$$\widehat{N} = \{y \in R_{(g)} : yg^n \in N, \text{ some } n \geq 0\}.$$

Then $\widehat{N}$ is g-divisible, with $\widehat{N}I \subseteq J$. Since $\widehat{N} \subseteq \underline{\mathrm{Hom}}_R(I, J)$, clearly $\widehat{N}$ is left bounded. Since $J \subsetneq \widehat{N}$ and both are left bounded and g-divisible, we must have $\overline{J} \subsetneq \overline{\widehat{N}}$; otherwise $J/Jg = \widehat{N}/\widehat{N}g$ and the graded Nakayama lemma would imply that $J = \widehat{N}$. Moreover, $\overline{\widehat{N}}\,\overline{I} \subseteq \overline{J}$. But $\dim_{\Bbbk} B/\overline{I} < \infty$, since all nonzero ideals of B have finite codimension (see, for example, [2, Lemma 4.4]). Hence $\dim_{\Bbbk} \overline{\widehat{N}}/\overline{J} < \infty$, showing $\overline{J}$ is not saturated, a contradiction. □

The next few lemmas provide useful homological properties for modules over an elliptic algebra. First however we prove an elementary result that will be used several times.

Lemma 4.6. *Let H be a locally finite, left bounded, graded $\Bbbk[g]$-module. If the multiplication map $\bullet g$ has a finite dimensional kernel on H, then the g-torsion submodule of H is also finite dimensional.*

Proof. By hypothesis, $\mathrm{Ann}_H(g)$ is finite-dimensional, say contained in degrees $\leq d$. Now if $0 \neq x \in H$ is g-torsion, then pick $n \geq 1$ minimal with $g^n x = 0$; thus $0 \neq g^{n-1}x \in \mathrm{Ann}_H(g)$. It follows that $\deg x \leq d - n + 1$. In particular, the g-torsion submodule of H is entirely contained in degrees $\leq d$, and so is finite dimensional as H is left bounded and locally finite. □

Lemma 4.7. *Let R be an elliptic algebra, with $B = R/gR$. Suppose that $M \in \text{Gr-}R$ is g-torsionfree and that $N \in \text{Gr-}B$. Then, for all $i \geq 0$, one has $\underline{\mathrm{Ext}}^i_R(M,N) \cong \underline{\mathrm{Ext}}^i_B(M/Mg, N)$ and*

$$\underline{\mathrm{Ext}}^i_{\text{Qgr-}R}\big(\pi(M), \pi(N)\big) \cong \underline{\mathrm{Ext}}^i_{\text{Qgr-}B}\big(\pi(M/Mg), \pi(N)\big).$$

Proof. Both parts are essentially the same easy exercise; cf. [23, Proposition 5.1.2(1)]. □

Lemma 4.8. *Let R be an elliptic algebra with $R/Rg = B = B(E, \mathcal{M}, \tau)$. Let I and J be non-zero g-divisible, reflexive finitely generated graded right R-submodules of $R_{(g)}$. Then*

(1) *The natural inclusion $\overline{\underline{\mathrm{Hom}}_R(I,J)} \subseteq \underline{\mathrm{Hom}}_B(\overline{I}, \overline{J})$ has a finite-dimensional cokernel.*

(2) *The g-torsion subspace of $\underline{\mathrm{Ext}}^1_R(I,J)$ is finite-dimensional over $\Bbbk$.*

Proof. (1) The proof is a variant of [18, Prop. 6.12]. First, replacing I and J by xI and yJ, for some homogeneous elements $x, y \in R \setminus gR$, we can assume without loss that $I, J \subseteq R$. Note that R/I and R/J are g-torsionfree modules and hence, by Lemma 4.5, are either 2-pure or 0.

By Lemma 4.7 we may identify $\underline{\mathrm{Ext}}^i_R(R/I, \overline{J}) = \underline{\mathrm{Ext}}^i_B(B/\overline{I}, \overline{J})$. Thus, applying $\underline{\mathrm{Hom}}_R(R/I, -)$ to the sequence

$$0 \to J \xrightarrow{\bullet g} J \to \overline{J} \to 0$$

gives the exact sequence

$$\begin{aligned} &\underline{\mathrm{Hom}}_B\big(B/\overline{I}, \overline{J}\big) \longrightarrow \underline{\mathrm{Ext}}^1_R\big(R/I, J\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R\big(R/I, J\big) \xrightarrow{\beta} \underline{\mathrm{Ext}}^1_B\big(B/\overline{I}, \overline{J}\big) \\ &\quad \xrightarrow{\alpha} \underline{\mathrm{Ext}}^2_R\big(R/I, J\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^2_R\big(R/I, J\big) \longrightarrow \underline{\mathrm{Ext}}^2_B\big(B/\overline{I}, \overline{J}\big) \longrightarrow \cdots \end{aligned} \tag{4.2}$$

Moreover, $\underline{\mathrm{Hom}}_B(B/\overline{I}, \overline{J}) = 0$ since B is a domain.

We claim that $\dim_{\Bbbk} \underline{\mathrm{Ext}}^2_B(B/\overline{I}, \overline{J}) < \infty$. Indeed, since B is a domain with $\mathrm{GKdim}(B) = 2$, [11, Proposition 5.1(e)] implies that $\mathrm{GKdim}\, B/\overline{I} \leq 1$ and so $B/\overline{I}$ has a finite filtration by point modules and finite-dimensional modules. By Proposition 4.3 B satisfies the Artin–Zhang χ-condition and so $\dim_{\Bbbk} \underline{\mathrm{Ext}}^2_B(\Bbbk, \overline{J}) < \infty$. Thus in order to prove the claim, using the usual long exact sequences in cohomology, it suffices to show that $\dim_{\Bbbk} \underline{\mathrm{Ext}}^2_B(M_p, \overline{J}) < \infty$ for a point module M_p. Consider the exact sequence

$$\cdots \to \underline{\mathrm{Ext}}^1_B\big(M_p, B/\overline{J}\big) \to \underline{\mathrm{Ext}}^2_B\big(M_p, \overline{J}\big) \to \underline{\mathrm{Ext}}^2_B\big(M_p, B\big) \to \cdots$$

Here, $\underline{\mathrm{Ext}}^2_B(M_p, B) = 0$ by Lemma 3.3. Also $B/\overline{J}$ is again filtered by point modules and finite-dimensional modules. Obviously $\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_B(M_p, \Bbbk) < \infty$, while $\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_B(M_p, M_q) < \infty$ for a point module M_q by Proposition 3.6. Thus $\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_B(M_p, B/\overline{J}) < \infty$ and so $\dim_{\Bbbk} \underline{\mathrm{Ext}}^2_B(M_p, \overline{J}) < \infty$, as claimed.

Equation 4.2 now shows that if $N = \underline{\mathrm{Ext}}^2_R(R/I, J)$, then the map $N_{n-1} \xrightarrow{\bullet g} N_n$ is surjective for $n \gg 0$. Since N is left bounded and locally finite (see Notation 2.3) this forces $\dim_{\Bbbk} N_n = d$ for some constant d and any $n \gg 0$. Hence $N_{n-1} \xrightarrow{\bullet g} N_n$ is an isomorphism for $n \gg 0$ and so $\mathrm{Coker}(\beta) \cong \mathrm{Im}(\alpha)$ is finite-dimensional. Identifying $\underline{\mathrm{Ext}}^1_R(R/I, J) \cong \underline{\mathrm{Hom}}_R(I, J)/J$ and $\underline{\mathrm{Ext}}^1_B(B/\overline{I}, \overline{J}) \cong \underline{\mathrm{Hom}}_B(\overline{I}, \overline{J})/\overline{J}$, this means that the natural map $\underline{\mathrm{Hom}}_R(I, J)/J \to \underline{\mathrm{Hom}}_B(\overline{I}, \overline{J})/\overline{J}$ has a finite-dimensional cokernel. As in the proof of [18, Prop 6.12], it easily follows that the natural map $\underline{\mathrm{Hom}}_R(I, J) \to \underline{\mathrm{Hom}}_B(\overline{I}, \overline{J})$ also has a finite-dimensional cokernel.

(2) From the exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}_R\big(I, J\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Hom}}_R\big(I, J\big) \longrightarrow \underline{\mathrm{Hom}}_B\big(\overline{I}, \overline{J}\big) \\ \longrightarrow \underline{\mathrm{Ext}}^1_R\big(I, J\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R\big(I, J\big) \end{aligned}$$

and part (1), the map $\bullet g$ on $H = \underline{\mathrm{Ext}}^1_R(I, J)$ has finite-dimensional kernel. Now apply Lemma 4.6. $\square$

5. Line modules

The main aim of this paper is to create an algebraic analogue of contracting lines of self intersection -1. To this end, in this section we discuss line modules — the appropriate analogues of lines — while in the next section we discuss their intersection theory. Throughout the section, we fix an elliptic algebra R of degree ≥ 3 with $R/Rg = B = B(E, \mathcal{M}, \tau)$.

Definition 5.1. A (*right*) *line module* over the elliptic algebra R is a cyclic graded R-module $L \in \text{gr-}R$ with Hilbert series $\mathrm{hilb}\, L = 1/(1-s)^2$. Because

$\mathrm{Hom}_{\mathrm{gr}\text{-}R}(R, L) = \Bbbk$ for such a module L, there is a unique right ideal J of R with $L \cong R/J$. We refer to J as the *line ideal of* L.

Lemma 5.2. *Let L be a right line module over the elliptic algebra R. Then L is g-torsionfree and 2-critical, and L/Lg is a point module.*

Proof. This follows from [16, Lemma 8.9]. □

Recall from Lemma 2.2 the equivalence of categories $\mathrm{Qcoh}\, E \to \mathrm{Qgr}\text{-}B$. Since the simple objects in $\mathrm{Qcoh}\, E$ are the skyscraper sheaves $\mathcal{O}_p$ for points $p \in E$, following Definition 3.1 the simple objects in $\mathrm{qgr}\text{-}B$ are the images $\pi(M_p)$ of the point modules M_p, and so these are also parametrised by closed points $p \in E$. By a slight abuse of notation we will often write $\pi(M_p) = \mathcal{O}_p$ to emphasise the correspondence.

Definition 5.3. Let M be a right line module or, more generally, a finitely generated g-torsionfree right R-module with $\mathrm{GKdim}\, M = 2$. Then $\mathrm{GKdim}\, M/Mg = 1$, and so $\pi(M/Mg) \in \mathrm{qgr}\text{-}B$ has finite length. Thus $\pi(M/Mg)$ has a filtration with simple factors $\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n}$, for some $p_i \in E$. We define the *divisor* of M to be $\mathrm{Div}\, M = p_1 + \cdots + p_n$. In particular, if L is a line module then $L/Lg \cong M_p$ and $\mathrm{Div}\, L = p$ for some point $p \in E$. The analogous notation will be used for left modules.

Lemma 5.4. *Let $M \in \mathrm{gr}\text{-}R$ be g-torsionfree and assume that M/Mg has a filtration with shifted point module factors $\{M_{p_i}[m_i] : 1 \le i \le d\}$.*

(1) *M is 2-pure and CM, with*

$$\mathrm{Div}\, M = \sum \tau^{m_i}(p_i) \quad \textit{and} \quad \mathsf{hilb}\, M = \sum_{i=1}^{d} s^{-m_i}/(1-s)^2.$$

(2) *Let $N = \underline{\mathrm{Ext}}^1_R(M, R)$. Then $N \in R\text{-}\mathrm{gr}$ and is g-torsionfree, 2-pure, and CM. Moreover, N/gN has a finite filtration with shifted left point module factors $\{M^{\ell}_{\tau^{-2}(p_i)}[-m_i - 1] : 1 \le i \le d\}$. In particular,*

$$\mathrm{Div}\, N = \sum \tau^{m_i - 1}(p_i), \quad \textit{and} \quad \mathsf{hilb}\, N = \sum_{i=1}^{d} s^{m_i+1}/(1-s)^2.$$

Proof. (1) If M is not 2-pure, then it has a submodule H with $\mathrm{GKdim}\, H < 2$. Necessarily $\mathrm{GKdim}\, H = 1$ since finite-dimensional modules are g-torsion. Also, as M is g-torsionfree, $H(n) = \{f \in M : fg^n \in H\}$ satisfies $\mathrm{GKdim}\, H(n) = 1$ for any n. Thus, after replacing H by some such $H(n)$ we can assume that $H \not\subseteq Mg$. In this case, $\dim_{\Bbbk} H/Hg < \infty$ and hence $(H + Mg)/Mg \cong H/H \cap Mg$ is a nonzero, finite-dimensional submodule of M/Mg, contrary to assumption. Thus M is 2-pure.

Recall from Lemma 4.7 that $\underline{\mathrm{Ext}}^j_R(M, B) \cong \underline{\mathrm{Ext}}^j_B(M/Mg, B)$. Thus, applying $\underline{\mathrm{Hom}}_R(M, -)$ to the exact sequence $0 \to R[-1] \xrightarrow{g} R \to B \to 0$, gives the long exact sequence

$$\cdots \longrightarrow \underline{\mathrm{Ext}}^{j-1}_B\big(M/Mg, B\big) \longrightarrow \underline{\mathrm{Ext}}^j_R\big(M, R\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^j_R\big(M, R\big) \longrightarrow \underline{\mathrm{Ext}}^j_B\big(M/Mg, B\big) \longrightarrow \cdots \quad (5.1)$$

Given a point module M_p, then Lemma 3.3 implies that $\underline{\mathrm{Ext}}^j_B(M_p, B) = 0$ for $j \neq 1$. Since M/Mg is filtered by point modules, standard long exact sequences show that $\underline{\mathrm{Ext}}^j_B(M/Mg, B) = 0$ for $j \neq 1$. In particular if $j \neq 1$, then the map $\bullet g$ in (5.1) is surjective. Since $\underline{\mathrm{Ext}}^j_R(M, R)$ is left bounded, this can only happen if $\underline{\mathrm{Ext}}^j_R(M, R) = 0$. Thus M is Cohen–Macaulay.

The computations of $\mathsf{hilb}\, M$ and $\mathrm{Div}\, M$ follow routinely from the hypotheses, using (3.1).

(2) By the proof of part (1), the sequence (5.1) collapses to give the short exact sequence

$$0 \longrightarrow \underline{\mathrm{Ext}}^1_R\big(M, R\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R\big(M, R\big) \longrightarrow \underline{\mathrm{Ext}}^1_B\big(M/Mg, B\big) \longrightarrow 0.$$

Hence $N = \underline{\mathrm{Ext}}^1_R(M, R)$ is a g-torsionfree left R-module with $N/gN \cong \underline{\mathrm{Ext}}^1_B(M/Mg, B)$. Clearly N is finitely generated. Given a point module M_p, then Lemma 3.3 implies that $\underline{\mathrm{Ext}}^1_B(M_p, B) \cong M^\ell_{\tau^{-2}(p)}[-1]$. Since M/Mg is filtered by the $\{M_{p_i}[m_i]\}$, it follows that N/gN is filtered by the left point modules $\{M^\ell_{\tau^{-2}(p_i)}[-m_i - 1]\}$. The values of $\mathsf{hilb}\, N$ and $\mathrm{Div}\, N$ follow, as they did for M, but using (3.2) in place of (3.1). Similarly, N is 2-pure and CM by the arguments from part (1). $\square$

We now consider some properties of line modules.

Lemma 5.5. *Let L, L' be right line modules over an elliptic algebra R, with* $\mathrm{Div}\, L = p$ *and* $\mathrm{Div}\, L' = p'$.

(1) *If $p \neq \tau^j(p')$ for any $j \geq 0$, then $\underline{\mathrm{Hom}}_R(L, L') = 0$.*

(2) *If $p = \tau^j(p')$ for some $j \geq 0$, then either $\underline{\mathrm{Hom}}_R(L, L') = 0$ or else* $\mathsf{hilb}\, \underline{\mathrm{Hom}}_R(L, L') = s^j/(1-s)$.

(3) $\underline{\mathrm{End}}_R(L) \cong \Bbbk[g]$*; in particular* $\mathsf{hilb}\, \underline{\mathrm{End}}_R(L) = 1/(1-s)$.

Proof. By definition, $L/Lg = M_p$ and $L'/L'g = M_{p'}$. By Lemma 4.7, $\underline{\mathrm{Ext}}^i_R(L, M_{p'}) \cong \underline{\mathrm{Ext}}^i_B(M_p, M_{p'})$ for all $i \geq 0$. Applying $\underline{\mathrm{Hom}}_R(L, -)$ to the short exact sequence $0 \to L'g \to L' \to M_{p'} \to 0$ gives

$$0 \longrightarrow \underline{\mathrm{Hom}}_R\big(L, L'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Hom}}_R\big(L, L'\big) \xrightarrow{\gamma} \underline{\mathrm{Hom}}_B\big(M_p, M_{p'}\big) \longrightarrow \cdots . \quad (5.2)$$

(1) In this case, Proposition 3.6 implies that $\underline{\mathrm{Hom}}_B(M_p, M_{p'}) = 0$ and so $\bullet g$ is surjective in (5.2). Since $\underline{\mathrm{Hom}}_R(L, L')$ is left bounded, this forces $\underline{\mathrm{Hom}}_R(L, L') = 0$.

(2) Here, Proposition 3.6 implies that $\underline{\mathrm{Hom}}_B(M_p, M_{p'}) = \Bbbk[-j]$. Thus if $\underline{\mathrm{Hom}}_R(L, L') \neq 0$ then γ is surjective in (5.2). As $\underline{\mathrm{Hom}}_R(L, L')$ is left bounded, this forces $\mathsf{hilb}\,\underline{\mathrm{Hom}}_R(L, L') = s^j(1-s)^{-1}$.

(3) As there is a natural graded inclusion $\Bbbk[g] \hookrightarrow \underline{\mathrm{End}}_R(L)$, part (2) implies that $\mathsf{hilb}\,\underline{\mathrm{Hom}}_R(L, L) = (1-s)^{-1}$ and hence that $\underline{\mathrm{End}}_R(L) = \Bbbk[g]$. $\square$

Lemma 5.6. *Let R be an elliptic algebra, with $R/Rg = B = B(E, \mathcal{M}, \tau)$. Let L and L' be right line modules over R.*

(1) *L is CM, and $L^\vee = \underline{\mathrm{Ext}}^1_R(L, R)[1]$ is a left line module.*

(2) *Under the natural morphism, $L = L^{\vee\vee}$. Further, if J is the line ideal of L, then $J = J_1 R$ is g-divisible, CM and reflexive, while $\overline{J} = J/Jg$ is saturated.*

(3) *Up to isomorphism, there is a unique non-split exact sequence*

$$0 \to R \to M \to L[-1] \to 0.$$

Explicitly, if $L^\vee = R/J^\vee$, then $M = (J^\vee)^*$. This M is g-divisible, CM and reflexive.

(4) *For $j = 0, 1$ one has $\underline{\mathrm{Ext}}^j_R(L', L) \cong \underline{\mathrm{Ext}}^j_R(L^\vee, (L')^\vee)$ as graded vector spaces.*

(5) $\underline{\mathrm{Ext}}^1_R(\Bbbk, L) = 0$.

Proof. (1) Write $L/Lg = M_p$ for some $p \in E$. Then Lemma 5.4 shows that L is CM, and that $N = \underline{\mathrm{Ext}}^1_R(M_p, R)$ has Hilbert series $s/(1-s)^2$, with $N/gN \cong M^{\ell}_{\tau^{-2}(p)}$. In particular, since N/gN is cyclic, N is cyclic by the graded Nakayama lemma, and since $L^\vee = N[1]$ has the Hilbert series of a line module, it is a left line module.

(2) Mimicking the notation from Lemma 3.3, set $E^{ij}(N) = \underline{\mathrm{Ext}}^i_R(\underline{\mathrm{Ext}}^j_R(N, R), R)$ for a graded R-module N. We first note that the natural morphism, $L \to E^{11}(L) = L^{\vee\vee}$ is obtained as follows: applying $\underline{\mathrm{Hom}}(-, R)$ to

$$0 \to R \to J^* \to \underline{\mathrm{Ext}}^i_R(L, R) \to 0$$

gives the exact sequence

$$0 \to J^{**} \to R \to E^{11}(L).$$

Since $J^{**} \supseteq J$ this induces a homomorphism from $L = R/J$ to $R/J^{**} \subseteq E^{11}(L)$.

By Lemma 5.4(2) $L^\vee$ is CM. Thus the Gorenstein spectral sequence (3.3) collapses to show that the natural morphism $L \to L^{\vee\vee}$ is an isomorphism. Since $L = R/J$ is g-torsionfree, and R is g-divisible, J must be g-divisible.

Since $\overline{R}/\overline{J} \cong M_p$, necessarily $\overline{J} = \bigoplus_{n\geq 0} H^0(E, \mathcal{M}_n(-p))$. In particular, $\overline{J}$ is saturated, and so J is reflexive by Lemma 4.5(2). Similarly, $\overline{J}$ is generated in degree one by Lemma 2.2(1) and hence J is generated in degree one by the graded Nakayama lemma. Since $L = R/J$ is CM by part (1), it follows routinely that $\underline{\mathrm{Ext}}^i_R(J, R) = 0$ for $i \geq 1$ and hence that J is CM.

(3) We have an exact sequence

$$0 \to R \to (J^\vee)^* \to L^{\vee\vee}[-1] \to 0.$$

Since $(J^\vee)^*$ is contained in the graded quotient ring $Q_{\mathrm{gr}}(R)$, the inclusion $R \to (J^\vee)^*$ is essential; in particular, the exact sequence is nonsplit. Now $L^{\vee\vee} = L$ by part (2) and $\mathrm{Ext}^1_{\text{gr-}R}(L[-1], R) = \Bbbk$, by part (1). Thus up to isomorphism there is a unique nonsplit degree 1 extension M of R by L, and it is given by $M = (J^\vee)^*$. Since $J^\vee$ is g-divisible by part (2), $M = \underline{\mathrm{Hom}}_R(J^\vee, R)$ is g-divisible by Lemma 4.4. Finally, as $J^\vee$ is CM by the left-sided analogue of part (2), the spectral sequence (3.3) collapses for $J^\vee$ and shows that $\underline{\mathrm{Ext}}^i_R(M, R) = E^{i0}(J^\vee) = 0$ for $i > 0$. In other words, M is CM.

(4) We begin with $\underline{\mathrm{Ext}}^1$. Define a map $(-)^\vee : \underline{\mathrm{Ext}}^1_R(L', L) \to \underline{\mathrm{Ext}}^1_R(L^\vee, (L')^\vee)$ as follows: let

$$\mathsf{E} : 0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} L'[j] \longrightarrow 0$$

be an exact sequence corresponding to an element of $\underline{\mathrm{Ext}}^1_R(L', L)_{-j}$. Applying $\underline{\mathrm{Hom}}_R(-, R)$ to E and using the fact that L and L' are CM gives the dual extension

$$\mathsf{E}^\vee : 0 \longrightarrow (L')^\vee[-j] \xrightarrow{\beta^\vee} \underline{\mathrm{Ext}}^1_R(M, R)[1] \xrightarrow{\alpha^\vee} L^\vee \longrightarrow 0.$$

We leave to the reader the verification using Baer sums [13, Theorem III.2.1] that $(-)^\vee$ is a linear transformation.

The double dual of E is

$$\mathsf{E}^{\vee\vee} : 0 \longrightarrow L \xrightarrow{\alpha^{\vee\vee}} E^{11}(M) \xrightarrow{\beta^{\vee\vee}} L'[j] \longrightarrow 0.$$

Since L and L' are CM, so is M and hence (3.3) implies that $M \cong E^{11}(M)$. The functoriality of (3.3) then ensures that $\alpha^{\vee\vee} = \alpha$ and $\beta^{\vee\vee} = \beta$, whence $\mathsf{E}^{\vee\vee} \cong \mathsf{E}$.

Thus $(-)^\vee$ induces linear maps

$$f : \underline{\mathrm{Ext}}^1_R\big(L', L\big) \to \underline{\mathrm{Ext}}^1_R\big(L^\vee, (L')^\vee\big) \quad \text{and} \quad g : \underline{\mathrm{Ext}}^1_R\big(L^\vee, (L')^\vee\big) \to \underline{\mathrm{Ext}}^1_R\big(L', L\big)$$

such that $g \circ f$ is the identity. The same argument starting with the sequence $\mathsf{E}^\vee$ shows that $f \circ g$ is the identity. Thus $(-)^\vee$ is an isomorphism of graded vector spaces.

In order to prove the result for $\underline{\mathrm{Hom}}$, suppose that $0 \neq f \in \underline{\mathrm{Hom}}_R(L', L)$. Then L and L' are GK-critical by Lemma 5.2 and hence f is an injection. Now applying $\underline{\mathrm{Ext}}^1(-, R)$ produces a monomorphism $f^\vee \in \underline{\mathrm{Hom}}_R(L^\vee, (L')^\vee)$ and so the map $f \mapsto f^\vee$ defines an injection $\underline{\mathrm{Hom}}(L', L) \hookrightarrow \underline{\mathrm{Hom}}(L^\vee, (L')^\vee)$. The fact that this is an isomorphism then follows by applying parts (1, 2).

(5) The exact sequence

$$0 \to R \to (J^\vee)^* \to L[-1] \to 0$$

from part (3) induces the exact sequence

$$\underline{\mathrm{Ext}}^1_R\big(\Bbbk, (J^\vee)^*\big) \to \underline{\mathrm{Ext}}^1_R\big(\Bbbk, L\big)[-1] \to \underline{\mathrm{Ext}}^2_R\big(\Bbbk, R\big).$$

As R is AS-Gorenstein of dimension 3 by Proposition 4.3, the last term is zero. The first term is 0 by reflexivity of $(J^\vee)^*$, and the result follows. □

6. Intersection theory

There is a general notion of intersection product on a noncommutative scheme, due to Mori and Smith [14], that reduces to the usual definition for a commutative scheme but is more convenient when working in a noncommutative setting. In this section we give several alternative formulæ for the intersection product of line modules over elliptic algebras. One drawback of the definition is that it is not always defined for schemes of infinite homological dimension, so we also give a variant that is always defined.

Definition 6.1. Let R be a connected noetherian $\mathbb{N}$-graded algebra. Then the *intersection number* of $\mathcal{M}, \mathcal{N} \in \text{qgr-}R$ is defined to be

$$\big(\mathcal{M} \bullet_{MS} \mathcal{N}\big) = \sum (-1)^{n+1} \dim_{\Bbbk} \mathrm{Ext}^n_{\text{qgr-}R}\big(\mathcal{M}, \mathcal{N}\big),$$

where we declare that the intersection is *undefined* if infinitely many terms are nonzero.

Given $M, N \in \text{gr-}R$, we define $(M \bullet_{MS} N) = (\pi(M) \bullet_{MS} \pi(N))$, although as above since the category will be clear from the context we will usually write M for the image in qgr-R of $M \in \text{gr-}R$.

Notation 6.1. Given an elliptic algebra R, set $R^\circ = R[g^{-1}]_0$. Similarly, for $M \in \text{Gr-}R$, set $M^\circ = M[g^{-1}]_0 \in R^\circ\text{-Mod}$. Since $R[g^{-1}]$ is strongly graded,

$$R[g^{-1}] \cong R^\circ \otimes_{\Bbbk} \Bbbk[g, g^{-1}],$$

and there is an equivalence of categories

$$F : \text{Mod-}R^\circ \to \text{Gr-}R[g^{-1}],$$

given by $F(N) = N \otimes_{\Bbbk} \Bbbk[g, g^{-1}]$.

Finally, write $\mathrm{grk}\, M = \mathrm{grk}_{\Bbbk[g]} M$ for the torsionfree rank of a $\Bbbk[g]$-module M.

Lemma 6.2. *Let R be an elliptic algebra, with $M, N \in \text{gr-}R$. Then, for each $m \geq 0$, $\underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(M, N)$ is a right $\Bbbk[g]$-module with*

$$\underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}\big(M, N\big) \otimes_{\Bbbk[g]} \Bbbk[g, g^{-1}] \cong \underline{\mathrm{Ext}}^m_R\big(M, N\big) \otimes_{\Bbbk[g]} \Bbbk[g, g^{-1}] \\ \cong \mathrm{Ext}^m_{R^\circ}\big(M^\circ, N^\circ\big) \otimes_{\Bbbk} \Bbbk[g, g^{-1}].$$

In particular, $\mathrm{grk}\, \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(M, N) = \mathrm{grk}\, \underline{\mathrm{Ext}}^m_R(M, N) = \dim_{\Bbbk} \mathrm{Ext}^m_{R^\circ}(M^\circ, N^\circ)$.

Proof. The first assertion follows from the fact that g is central.

By Proposition 4.3 and [5, Corollary 7.3(2)], there is a map $\theta : \underline{\mathrm{Ext}}^m_R(M, N) \to \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(M, N)$ with right bounded kernel and cokernel. As g is central, θ is a map of $\Bbbk[g]$-modules, and so the kernel and cokernel of θ are g-torsion. This proves the first isomorphism in the display.

Next, using that Ext commutes with central localisation, we calculate that

$$\underline{\mathrm{Ext}}^m_R\big(M, N\big) \otimes_{\Bbbk[g]} \Bbbk[g, g^{-1}] \cong \underline{\mathrm{Ext}}^m_{R[g^{-1}]}\big(M[g^{-1}], N[g^{-1}]\big) \\ \cong \mathrm{Ext}^m_{R^\circ}\big(M^\circ, N^\circ\big) \otimes_{\Bbbk} \Bbbk[g, g^{-1}],$$

where the final isomorphism uses the equivalence of categories $\text{gr-}R[g^{-1}] \simeq \text{mod-}R^\circ$. This gives the second isomorphism in the display, from which the final equation is an easy consequence. □

We now consider in more detail the homological properties of line modules over the elliptic algebra R.

Lemma 6.3. *Let L and L' be line modules over the elliptic algebra R, with point factors $L/Lg = M_p$ and $L'/L'g = M_{p'}$. Then there is a long exact sequence*

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{qgr}\text{-}R}\big(L, L'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Hom}}_{\mathrm{qgr}\text{-}R}\big(L, L'\big) \\ \longrightarrow \mathbb{F} \longrightarrow \underline{\mathrm{Ext}}^1_{\mathrm{qgr}\text{-}R}\big(L, L'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_{\mathrm{qgr}\text{-}R}\big(L, L'\big) \longrightarrow \mathbb{F} \\ \longrightarrow \underline{\mathrm{Ext}}^2_{\mathrm{qgr}\text{-}R}\big(L, L'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^2_{\mathrm{qgr}\text{-}R}\big(L, L'\big) \longrightarrow 0, \quad (6.2)$$

where $\mathbb{F} \cong \Bbbk[-j]$ if $p = \tau^j(p')$ for some $j \in \mathbb{Z}$, and $\mathbb{F} = 0$ if p and q lie on different orbits.

For $m \geq 3$, multiplication by g induces isomorphisms $\underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(L, L')[-1] \cong \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(L, L')$.

Proof. Applying $\underline{\mathrm{Hom}}_{\mathrm{qgr}\text{-}R}(L, -)$ to $0 \longrightarrow L'[-1] \xrightarrow{\bullet g} L' \longrightarrow M_{p'} \longrightarrow 0$ gives the long exact sequence

$$\cdots \to \underline{\mathrm{Ext}}^{n-1}_{\mathrm{qgr}\text{-}R}\big(L, M_{p'}\big) \to \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'g\big) \\ \to \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'\big) \to \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, M_{p'}\big) \to \cdots \quad (6.3)$$

The lemma now follows by using Lemma 4.7 to identify $\underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}(L, M_{p'}) = \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}B}(M_p, M_{p'})$ and then applying Lemma 3.4. □

Proposition 6.4. *Let R be an elliptic algebra and let $L, L' \in \mathrm{gr}\text{-}R$ be line modules, with $p = \mathrm{Div}\, L$, $p' = \mathrm{Div}\, L'$. Assume that $(L \bullet_{MS} L')$ is defined.*

(1) $\big(L \bullet_{MS} L'[m]\big) = \big(L \bullet_{MS} L'\big)$ *for all $m \in \mathbb{Z}$.*

(2) $$\begin{aligned}\big(L \bullet_{MS} L'\big) &= \sum (-1)^{n+1} \operatorname{grk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'\big)\\ &= \sum (-1)^{n+1} \dim_{\Bbbk} \mathrm{Ext}^n_{R^\circ}\big(L^\circ, (L')^\circ\big).\end{aligned}$$

Proof. (1) Restrict the morphisms in Lemma 6.3 to some degree j and take the alternating sum of the dimensions of the resulting vector spaces in these equations. Since the contributions from $\mathbb{F}$ cancel, this gives

$$\sum_{n\geq 0} (-1)^{n+1} \dim_{\Bbbk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'[-1]\big)_j = \sum_{n\geq 0} (-1)^{n+1} \dim_{\Bbbk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'\big)_j.$$

Thus, by (2.1), $(L \bullet_{MS} L'[j-1]) = (L \bullet_{MS} L'[j])$ and, by induction, $(L \bullet_{MS} L'[m]) = (L \bullet_{MS} L')$ for all $m \in \mathbb{Z}$.

(2) By Lemma 6.2, we only need to prove that

$$\big(L \bullet_{MS} L'\big) = \sum (-1)^{n+1} \operatorname{grk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'\big).$$

Suppose first that $p = \tau^j(p')$ for some $j \in \mathbb{Z}$. Then $(L \bullet_{MS} L'[j]) = (L \bullet_{MS} L')$ by part (1), and

$$\sum (-1)^{n+1} \operatorname{grk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'\big) = \sum (-1)^{n+1} \operatorname{grk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'[j]\big)$$

is obvious since shifting does not affect the rank. Thus it suffices to prove that

$$\big(L \bullet_{MS} L'[j]\big) = \sum (-1)^{n+1} \operatorname{grk} \underline{\mathrm{Ext}}^n_{\mathrm{qgr}\text{-}R}\big(L, L'[j]\big). \tag{6.4}$$

So, consider $N = \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(L, L'[j]) = \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(L, L')[j]$, for some $m \geq 0$. Then after shifting by $[j]$, Lemma 6.3 shows that N fits into an exact sequence

$$0 \to K \to N[-1] \xrightarrow{\bullet g} N \to K' \to 0$$

where K and K' can be zero or $\Bbbk$, depending on the choice of m. Thus the kernel of $\bullet g$ on N is contained in N_{-1} and so $N_{\geq 0}$ is g-torsionfree. Similarly, since K' is concentrated in degree 0, $\bullet g$ gives an isomorphism $N_n \xrightarrow{\sim} N_{n+1}$ for all $n \geq 0$. Since N is locally finite, it follows that $N_{\geq 0} \cong \Bbbk[g]^{\oplus r}$ for some $r \geq 0$. In particular, $\dim_{\Bbbk} N_0 = r = \operatorname{grk} N$ and so (6.4) follows.

If p and p' lie on distinct orbits then the same argument works, since now Lemma 6.3 implies that $N = \underline{\mathrm{Ext}}^m_{\mathrm{qgr}\text{-}R}(L, L') \cong N[-1]$ for each m. □

The projective dimension of an R-module L will be written $\operatorname{pdim}_R(L)$. We make the following easy observation.

Lemma 6.5. *Let R be an elliptic algebra and let L be a line module with $\operatorname{pdim}_{R^\circ}(L^\circ) < \infty$. Then $\operatorname{pdim}_{R^\circ}(L^\circ) = 1$.*

Proof. By Lemma 5.6(1), $\underline{\operatorname{Ext}}^n_R(L, R) = 0$ for $n \neq 1$ and so Lemma 6.2 implies that $\operatorname{Ext}^n_{R^\circ}(L^\circ, R^\circ) = 0$ for $n \neq 1$. If $m = \operatorname{pdim}_{R^\circ}(L^\circ) < \infty$ then it is easy to see that $\operatorname{Ext}^m_{R^\circ}(L^\circ, R^\circ) \neq 0$, and it follows that $m = 1$. □

Corollary 6.6. *Let R be an elliptic algebra with line modules L and L'. Assume that either L° or $(L')^\circ$ has finite projective dimension. Then*

$$\begin{aligned}(L \bullet_{MS} L') &= \operatorname{grk} \underline{\operatorname{Ext}}^1_{\text{qgr-}R}(L, L') - \operatorname{grk} \underline{\operatorname{Hom}}_{\text{qgr-}R}(L, L') \\ &= \operatorname{grk} \underline{\operatorname{Ext}}^1_R(L, L') - \operatorname{grk} \underline{\operatorname{Hom}}_R(L, L') \\ &= \dim_{\Bbbk} \operatorname{Ext}^1_{R^\circ}(L^\circ, (L')^\circ) - \dim_{\Bbbk} \operatorname{Hom}^1_{R^\circ}(L^\circ, (L')^\circ).\end{aligned}$$

Proof. If L° has finite projective dimension, by Lemma 6.5 we have $\operatorname{pdim}_{R^\circ}(L^\circ) = 1$. Thus, by Lemma 6.2, for $i \geq 2$ we have

$$0 = \operatorname{Ext}^i_{R^\circ}(L^\circ, (L')^\circ) = \operatorname{grk} \underline{\operatorname{Ext}}^i_{\text{qgr-}R}(L, L') = \operatorname{grk} \underline{\operatorname{Ext}}^i_R(L, L')$$

and the result follows from Proposition 6.4 and Lemma 6.2.

If instead $(L')^\circ$ has finite projective dimension, then again $(L')^\circ$ has projective dimension 1, which forces its line ideal $(J')^\circ$ to be projective. Lemmas 5.6(1) and 6.2 still imply that $\operatorname{Ext}^i_{R^\circ}(L^\circ, R^\circ) = 0$ for $i \geq 2$. Thus, as $(J')^\circ$ is a direct summand of a free module, $\operatorname{Ext}^i_{R^\circ}(L^\circ, (J')^\circ) = 0$ for $i \geq 2$ as well. Then $\operatorname{Ext}^i_{R^\circ}(L^\circ, (L')^\circ) \cong \operatorname{Ext}^{i+1}_{R^\circ}(L^\circ, (J')^\circ) = 0$ for $i \geq 2$, and again the result follows from Lemma 6.2 and Proposition 6.4. □

Unfortunately, when a (localised) line module L° has infinite projective dimension, the corollary can fail and, indeed, $(L \bullet_{MS} L)$ can even be undefined; see Corollary 10.8 for an example of this phenomenon. However, the higher Ext groups are not really relevant to our applications of self-intersection and so we can define a modified intersection number using just the first two terms in the alternating sum.

Definition 6.7. Let L and L' be line modules over an elliptic algebra R. Then define

$$(L \bullet L') = -\operatorname{grk} \underline{\operatorname{Hom}}_R(L, L') + \operatorname{grk} \underline{\operatorname{Ext}}^1_R(L, L').$$

Corollary 6.6 shows that $(L \bullet L') = (L \bullet_{MS} L')$ provided one of $(L)^\circ$ or $(L')^\circ$ has finite projective dimension. In general, the geometric interpretation of $(L \bullet L')$ is more obscure, but as we see in Section 7, our definition still correlates nicely with several other useful properties of the lines L and L'.

In fact, the main examples we consider in the last part of the paper are elliptic algebras R whose corresponding noncommutative projective schemes are *smooth* in the sense that qgr-R has finite homological dimension. A geometric way of thinking about R° is to note that the noncommutative scheme qgr-R has a closed subscheme qgr-$B \simeq \operatorname{coh} E$ which is a smooth elliptic curve. The category mod-R° then represents the open complement of E. Thus the following result is natural.

Lemma 6.8. *Let R be an elliptic algebra. Then* qgr-R *is smooth if and only if R° has finite global dimension.*

Proof. If qgr-R is smooth then R° has finite global dimension by Lemma 6.2, so suppose that qgr-R has infinite homological dimension. Thus, for any $t \geq 3$ there exist $M, N \in$ gr-R such that $\underline{\operatorname{Ext}}^s_{\text{qgr-}R}(M, N) \neq 0$ for some $s \geq t$. By taking the beginning of a free resolution $0 \to K \to F \to M \to 0$ and replacing M by K we can assume, possibly after increasing s, that M is g-torsionfree. Similarly we may assume that N is g-torsionfree.

Set $B = R/gR$. Then qgr-$B \cong \operatorname{coh} E$ has homological dimension 1. Thus Lemma 4.7 implies that

$$\underline{\operatorname{Ext}}^i_{\text{qgr-}R}(M, N/Ng) \cong \underline{\operatorname{Ext}}^i_{\text{qgr-}B}(M/Mg, N/gN) = 0$$

for $i \geq 2$. Using cohomology arising from the exact sequence

$$0 \longrightarrow N[-1] \xrightarrow{\bullet g} N \longrightarrow N/Ng \longrightarrow 0$$

it follows that $\bullet g$ is injective on $\underline{\operatorname{Ext}}^i_{\text{qgr-}R}(M, N)$ for all $i \geq 3$. In other words, $\underline{\operatorname{Ext}}^s_{\text{qgr-}R}(M, N)$ is g-torsionfree (and non-zero). Finally, by Lemma 6.2 this implies that $\operatorname{Ext}^s_{R^\circ}(M^\circ, N^\circ) \neq 0$. Since t was arbitrary, it follows that R° has infinite global dimension. □

7. Intersections of lines

Fix an elliptic algebra R with $R/gR = B(E, \mathcal{M}, \tau)$, and let $L = R/J$ and $L' = R/J'$ be two right R- line modules, possibly isomorphic, with $\operatorname{Div} L = p$ and $\operatorname{Div} L' = p'$. In this section, we study alternative characterisations of the intersection number $(L{\bullet}L')$, as defined in Definition 6.7. In particular, we show that $(L{\bullet}L) = -1$ if and only if $\underline{\operatorname{Ext}}^1_R(L, L) = 0$, and give similar conditions for when $(L{\bullet}L') = 0$.

We begin with a number of useful observations. First, consider the exact sequences

$$0 \to \underline{\operatorname{Hom}}_R(L, L') \to \underline{\operatorname{Hom}}_R(R, L') \to \underline{\operatorname{Hom}}_R(J, L') \to \underline{\operatorname{Ext}}^1_R(L, L') \to 0,$$

and

$$0 \to \underline{\operatorname{Hom}}_R(J, J') \to \underline{\operatorname{Hom}}_R(J, R) \to \underline{\operatorname{Hom}}_R(J, L') \to \underline{\operatorname{Ext}}^1_R(J, J') \to 0.$$

(The second sequence is exact because J is CM, by Lemma 5.6.) We know that $\operatorname{hilb} \underline{\operatorname{Hom}}_R(J, R) = \operatorname{hilb} R + s/(1-s)^2$ by the left-sided version of Lemma 5.6(3), and of course $\underline{\mathrm{Hom}}_R(R, L') = L'$ has Hilbert series $1/(1-s)^2$. Thus we obtain the useful equation:

$$\begin{aligned}\mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R\big(L, L'\big) - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R\big(L, L'\big) &= \mathsf{hilb}\,\underline{\mathrm{Hom}}_R\big(J, L'\big) - \frac{1}{(1-s)^2} \\ &= \mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R\big(J, J'\big) + h_R - \frac{1}{(1-s)} - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R\big(J, J'\big). \quad (7.1)\end{aligned}$$

The power series on the left of (7.1) will recur often, so we define:

$$X(L, L') = \mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R\big(L, L'\big) - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R\big(L, L'\big). \tag{7.2}$$

Next, by applying $\underline{\mathrm{Hom}}_R(L, -)$ to the short exact sequence

$$0 \to L'[-1] \xrightarrow{\bullet g} L' \to M_{p'} \to 0,$$

and using Lemma 4.7 one obtains the exact sequence:

$$\begin{aligned}0 \longrightarrow \underline{\mathrm{Hom}}_R\big(L, L'\big)[-1] &\xrightarrow{\bullet g} \underline{\mathrm{Hom}}_R\big(L, L'\big) \longrightarrow \underline{\mathrm{Hom}}_B\big(M_p, M_{p'}\big) \\ \longrightarrow \underline{\mathrm{Ext}}^1_R\big(L, L'\big)[-1] &\xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R\big(L, L'\big) \xrightarrow{\delta} \underline{\mathrm{Ext}}^1_B\big(M_p, M_{p'}\big). \quad (7.3)\end{aligned}$$

This gives the Hilbert series equation

$$X\big(L, L'\big) = \frac{C-H}{(1-s)} \quad \text{for } H = \mathsf{hilb}\,\underline{\mathrm{Hom}}_B\big(M_p, M_{p'}\big) \text{ and } C = \mathsf{hilb}\ \mathrm{Im}\,\delta. \tag{7.4}$$

The possibilities for H, C and $E = \mathsf{hilb}\,\underline{\mathrm{Ext}}^1_B(M_p, M_{p'})$ are quite limited: by Proposition 3.6, if $p = \tau^j(p')$ for some $j \geq 0$, then $H = s^j$ and $E = s^{-1} + s^j$; otherwise $H = 0$ and $E = s^{-1}$. In any case $0 \leq C \leq E$.

Note that (7.3) implies that $\underline{\mathrm{Hom}}_R(L, L')$ is g-torsionfree. Moreover, by Proposition 3.6(1) the map $\bullet g$ on $\underline{\mathrm{Ext}}^1_R(L, L')$ has a finite-dimensional kernel. By Lemma 4.6, this implies that the g-torsion submodule of $\underline{\mathrm{Ext}}^1_R(L, L')$ is also finite-dimensional. Since $\dim_{\Bbbk} \mathrm{Im}\,\delta < \infty$ and $\underline{\mathrm{Ext}}^1_R(L, L')$ is left bounded, (7.3) implies that $\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_R(L, L')_n$ is constant for $n \gg 0$. Thus $\mathrm{grk}\,\underline{\mathrm{Ext}}^1_R(L, L') = \dim_{\Bbbk} \underline{\mathrm{Ext}}^1_R(L, L')_n$ for all $n \gg 0$. Obviously the analogous result holds for $\underline{\mathrm{Hom}}_R(L, L')$, and combined with (7.4) this implies that

$$\begin{aligned}\big(L{\bullet}L'\big) &= \mathrm{grk}\,\underline{\mathrm{Ext}}^1_R\big(L, L'\big) - \mathrm{grk}\,\underline{\mathrm{Hom}}_R\big(L, L'\big) \\ &= \dim_{\Bbbk} \underline{\mathrm{Ext}}^1_R\big(L, L'\big)_n - \dim_{\Bbbk} \underline{\mathrm{Hom}}_R\big(L, L'\big)_n, \text{ for } n \gg 0 \\ &= \text{the sum of the coefficients of } C - H.\end{aligned} \tag{7.5}$$

We may think of $X(L, L')$ as a refined version of $(L{\bullet}L')$, since

$$\big(L{\bullet}L'\big) = (1-s)X\big(L, L'\big)|_{s=1}$$

by (7.4) and (7.5).

As a consequence of these calculations, we can already see that the intersection number of two lines on an elliptic algebra lies in a quite limited range. If $p = \tau^j(p')$ for some $j \geq 0$, then $H = s^j$ and $C \leq s^{-1} + s^j$, while otherwise $H = 0$ and $C \leq s^{-1}$. From (7.5) we conclude that

$$\big(L{\bullet}L'\big) \in \{-1, 0, 1\}, \tag{7.6}$$

where the value -1 can only occur if $p = \tau^j(p')$ for some $j \geq 0$. In Lemma 7.4 we will refine this observation to show that in fact $(L{\bullet}L') = -1$ forces L to be isomorphic to L'.

Next, since J and J' are g-divisible by Lemma 5.6(2), we may use Lemma 4.7 to get the exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}_R\big(J, J'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Hom}}_R\big(J, J'\big) \longrightarrow \underline{\mathrm{Hom}}_B\big(\overline{J}, \overline{J'}\big) \\ \xrightarrow{\alpha} \underline{\mathrm{Ext}}^1_R\big(J, J'\big)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R\big(J, J'\big) \longrightarrow \cdots \end{aligned} \tag{7.7}$$

The calculation of the Hilbert series of $\underline{\mathrm{Hom}}_B(\overline{J}, \overline{J'})$ is straightforward. Since

$$\overline{J} = \bigoplus_{n\geq 0} H^0\big(E, \mathcal{M}_n(-p)\big)$$

and similarly for J', by Lemma 2.3 we get

$$\underline{\mathrm{Hom}}_B\big(\overline{J}, \overline{J'}\big) = \bigoplus_{n\geq 0} H^0\big(E, \mathcal{M}_n\big(-p' + \tau^{-n}(p)\big)\big).$$

Thus

$$\mathsf{hilb}\,\underline{\mathrm{Hom}}_B\big(\overline{J}, \overline{J'}\big) = \mathsf{hilb}\,B - 1 + \epsilon_{p,p'}, \quad \text{where } \epsilon_{p,p'} = \begin{cases} 1 & p = p', \\ 0 & p \neq p'. \end{cases} \tag{7.8}$$

From (7.7) and Lemma 4.4(1) we have $\overline{\underline{\mathrm{Hom}}_R(J, J')} \subseteq \underline{\mathrm{Hom}}_B(\overline{J}, \overline{J'})$ with cokernel isomorphic to $\mathrm{Im}\,\alpha$. Thus we conclude that

$$\begin{aligned} \mathsf{hilb}\,\underline{\mathrm{Hom}}_R\big(J, J'\big) &= \frac{\mathsf{hilb}\,B - 1 + \epsilon_{p,p'} - \mathsf{hilb}\,\mathrm{Im}\,\alpha}{(1-s)} \\ &= \mathsf{hilb}\,R + \frac{(\epsilon_{p,p'} - 1)}{(1-s)} - \frac{(\mathsf{hilb}\,\mathrm{Im}\,\alpha)}{(1-s)}. \end{aligned} \tag{7.9}$$

We next want to characterise when the various intersection numbers occur in (7.6). We begin with $(L{\bullet}L)$ for a single line ideal L and are mostly interested in when $(L{\bullet}L) = -1$.

Theorem 7.1. *Let $L = R/J$ be a line module of an elliptic algebra R. Consider the following conditions:*

(1) $\underline{\mathrm{Ext}}^1_R(J, J) = 0$.

(2) $\underline{\mathrm{Ext}}^1_R(L, L) = 0$.

(3) $\mathsf{hilb}\,\underline{\mathrm{End}}_R(J) = \mathsf{hilb}\,R$.

(4) $\mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, L) = s(1-s)^{-2}$.

(5) $(L\bullet L) = -1$.

Then:

(a) (1) $\iff$ (2) $\iff$ (4) $\iff$ (5) $\Rightarrow$ (3).

(b) *If J° is projective then all five are equivalent.*

Proof. (a) By Lemma 5.5(2), $\mathsf{hilb}\,\underline{\mathrm{Hom}}_R(L, L) = (1-s)^{-1}$. Thus adding $(1-s)^{-1}$ to (7.1) we obtain

$$\begin{aligned}\mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R(L, L) &= \mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, L) - \frac{s}{(1-s)^2} \\ &= \mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R(J, J) + \big[h_R - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, J)\big].\end{aligned} \tag{7.10}$$

This shows immediately that (2) $\iff$ (4). The final term $[h_R - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, J)]$ of (7.10) is non-negative by (7.9). Since $\mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R(J, J)$ is obviously non-negative, it follows that (2) implies both (1) and (3).

Now if (1) holds, then $\alpha = 0$ in (7.7) and so (7.9) immediately implies (3). But (7.10) again shows that (1) and (3) together imply (2).

If (2) holds, then since $\underline{\mathrm{Hom}}_R(L, L) = \Bbbk[g]$ by Lemma 5.5, we certainly have

$$(L\bullet L) = \mathrm{grk}\,\underline{\mathrm{Ext}}^1_R(L, L) - \mathrm{grk}\,\underline{\mathrm{Hom}}_R(L, L) = -1$$

and (5) holds. Conversely, if (5) holds then using (7.5) we see that $H = 1$ and $C = 0$ in (7.4). Thus the map $\underline{\mathrm{Ext}}^1_R(L, L)[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R(L, L)$ is surjective, and since $\underline{\mathrm{Ext}}^1_R(L, L')$ is left bounded it must therefore be zero. Thus (2) holds, which completes the proof of (a).

(b) Suppose that J° is projective and that (3) holds. From (7.9) we conclude that $\alpha = 0$ in (7.7) and so $\underline{\mathrm{Ext}}^1_R(J, J)$ is g-torsionfree. But $\underline{\mathrm{Ext}}^1_{R^\circ}(J^\circ, J^\circ) = 0$ and so $\underline{\mathrm{Ext}}^1_R(J, J)$ is g-torsion by Lemma 6.2. Thus $\underline{\mathrm{Ext}}^1_R(J, J) = 0$ and (1) holds. □

Remark 7.2. (1) Suppose that R° has finite global dimension, or, equivalently by Lemma 6.8, that qgr-R is smooth. Then J° is automatically projective by Lemma 6.5 and so part (b) of the theorem applies.

(2) We know of no example of a line module L of an elliptic algebra R, with qgr-R smooth, for which $(L\bullet L) \neq -1$ and we actually conjecture that none exist. For a (non-smooth) example with $(L\bullet L) \neq -1$, see Corollary 10.5.

In the rest of the section, we consider the intersection number of two distinct line modules. Although these results will not be directly relevant for this paper, they will be useful in [19], and as the proofs are similar to that of Theorem 7.1, it is appropriate to include them here.

The following easy lemma will be used a number of times.

Lemma 7.3. *Let R be an elliptic algebra with two line modules $L = R/J$ and $L' = R/J'$. Then $\underline{\mathrm{Hom}}_R(J, J')_0 \subseteq R_0 = \Bbbk$. Moreover*

$$\underline{\mathrm{Hom}}_R\big(J, J'\big)_0 = \Bbbk \iff J = J' \iff L \cong L'.$$

Proof. Certainly $\underline{\mathrm{Hom}}_R(J, J') \subseteq \underline{\mathrm{Hom}}_R(J, R) = J^*$. By the left-handed analogue of Lemma 5.6(3), $(J^*)_0 = R_0 = \Bbbk$. Furthermore, if $J = J'$ then $\Bbbk \subseteq \underline{\mathrm{Hom}}_R(J, J')_0$ and so $\underline{\mathrm{Hom}}_R(J, J')_0 = \Bbbk$. Conversely, if $\Bbbk \subseteq \underline{\mathrm{Hom}}_R(J, J')$ then $J \subseteq J'$ and so $J = J'$ since J and J' have the same Hilbert series. The last equivalence is immediate since a line module is determined up to isomorphism by its line ideal. $\square$

We next refine (7.6) by showing that there are only two possible values of $(L{\bullet}L')$ for non isomorphic lines.

Lemma 7.4. *Let $L \not\cong L'$ be line modules over an elliptic algebra R. Then $X = X(L, L')$ is equal to either 0, $s^{-1}(1-s)^{-1}$, or $s^{-1} + \cdots + s^{j-1}$ for some $j \geq 0$. In particular, $(L{\bullet}L') \in \{0, 1\}$.*

Proof. Adopt the notation of (7.4). If $H = 0$ then (7.4) and Proposition 3.6 imply that $C \leq s^{-1}$ and X is either 0 or $s^{-1}(1-s)^{-1}$.

So suppose that $H \neq 0$. Then by Proposition 3.6(1), $H = s^j$ for some $j > 0$ and this happens precisely when $p = \mathrm{Div}\, L = \tau^j(p')$ for $p' = \mathrm{Div}\, L'$. By Proposition 3.6(2) and (7.3), $C \leq E = s^{-1} + s^j$. Let J, J' be, respectively, the line ideals of L and L' and note that $J \neq J'$ since $L \not\cong L'$. Thus $\underline{\mathrm{Hom}}_R(J, J')_0 = 0$ by Lemma 7.3. It follows from (7.9) that $\mathrm{hilb}\, \underline{\mathrm{Hom}}_R(J, J') \leq h_R - (1-s)^{-1}$. From (7.1), $X \geq \mathrm{hilb}\, \underline{\mathrm{Ext}}^1_R(J, J') \geq 0$. This forces $C \neq 0$, and so C is one of s^j, s^{-1} or $s^{-1} + s^j$. Now use (7.4) again to get the desired possibilities for X.

The possibilities for $(L{\bullet}L')$ are an immediate consequence. $\square$

Lemma 7.5. *Let R be an elliptic algebra. Let J, K be finitely generated g-divisible reflexive right R-submodules of $R_{(g)}$. Assume that either*

(a) *J° is projective; or*

(b) *J is CM and K° is projective.*

Then $\underline{\mathrm{Ext}}^1_R(J, K)$ is finite-dimensional.

Proof. Consider $H = \underline{\mathrm{Ext}}^1_R(J, K)$. By Lemma 4.8(2), the g-torsion subspace of H is finite dimensional. It therefore suffices to prove that H is g-torsion. By Lemma 6.2, it is then enough to show that $\mathrm{Ext}^1_{R^\circ}(J^\circ, K^\circ) = 0$. This is trivial if J° is projective. If K° is projective, then $\mathrm{Ext}^1_{R^\circ}(J^\circ, K^\circ)$ is a direct summand of a sum of copies of $\mathrm{Ext}^1_{R^\circ}(J^\circ, R^\circ)$. As J is CM, Lemma 6.2 again implies that $\mathrm{Ext}^1_{R^\circ}(J^\circ, R^\circ) = 0$ and hence $\mathrm{Ext}^1_{R^\circ}(J^\circ, K^\circ) = 0$. $\square$

We next characterise distinct lines with $(L\bullet L') = 1$.

Theorem 7.6. *Let $L = R/J$, $L' = R/J'$ be line modules over an elliptic algebra R, with $L \ncong L'$. Consider the following conditions:*

(1) $\mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R(J, J') = s^{-1} + 1$.

(2) $X(L, L') = s^{-1}(1-s)^{-1}$.

(3) $\mathsf{hilb}\,R - \mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, J') = (1+s)(1-s)^{-1}$.

(4) $\mathsf{hilb}\,\underline{\mathrm{Hom}}_R(J, L') = s^{-1}(1-s)^{-2}$.

(5) $(L\bullet L') = 1$.

Then:

(a) (1) $\Rightarrow$ (2) $\iff$ (4) $\iff$ (5), *while* (1) $\Rightarrow$ (3).

(b) *If either J° is projective with $(L\bullet L) = -1$, or $(J')^\circ$ is projective with $(L'\bullet L') = -1$, then all five conditions are equivalent.*

Proof. (a) Once again equation (7.1) implies that (2) $\iff$ (4), while Lemma 7.4 gives (2) $\iff$ (5).

(1) $\Rightarrow$ (3). Suppose that (1) holds and consider the part of the exact sequence (7.7) given by

$$\underline{\mathrm{Hom}}_R(J, J') \longrightarrow \underline{\mathrm{Hom}}_B(\overline{J}, \overline{J'}) \xrightarrow{\alpha} \underline{\mathrm{Ext}}^1_R(J, J')[-1] \xrightarrow{\bullet g} \underline{\mathrm{Ext}}^1_R(J, J').$$

By assumption $\underline{\mathrm{Ext}}^1_R(J, J') = \Bbbk + \Bbbk[1]$ is g-torsion, and so the kernel of $\bullet g$ on $\underline{\mathrm{Ext}}^1_R(J, J')$ contains at least the highest degree piece $\underline{\mathrm{Ext}}^1_R(J, J')_0$. Thus $\mathsf{hilb}\ \mathrm{Im}\,\alpha$ equals either s or $1+s$.

Let $p = \mathrm{Div}\,L$ and $p' = \mathrm{Div}\,L'$. If $p \neq p'$, then $\underline{\mathrm{Hom}}_B(\overline{J}, \overline{J'})_0 = 0$ and so $\mathsf{hilb}\ \mathrm{Im}\,\alpha = s$. If $p = p'$ then $\underline{\mathrm{Hom}}_B(\overline{J}, \overline{J'})_0 = \Bbbk$, whereas $\underline{\mathrm{Hom}}_R(J, J')_0 = 0$ by Lemma 7.3. Thus $\mathsf{hilb}\ \mathrm{Im}\,\alpha = 1+s$. In either case, $\mathsf{hilb}\ \mathrm{Im}\,\alpha = \epsilon_{p,p'} + s$, in the notation of (7.8). Thus (3) follows from (7.9).

(1) $\Rightarrow$ (2). If (1) and therefore (3) hold, then (2) follows from (7.1).

(b) (3) $\Rightarrow$ (1). Suppose that (3) holds. Recall that J and J' are CM, g-divisible, and reflexive, by Lemma 5.6(2). Thus the assumption in (b) ensures that the hypothesis of Lemma 7.5 holds. Then $\dim_\Bbbk \underline{\mathrm{Ext}}^1_R(J, J') < \infty$ by that lemma, and so (7.1) implies that

$$\dim_\Bbbk \underline{\mathrm{Ext}}^1_R\big(L, L'\big)_n - \dim_\Bbbk \underline{\mathrm{Hom}}_R\big(L, L'\big)_n = 1$$

for $n \gg 0$. Hence $(L\bullet L') = 1$ by (7.5). Thus (3) implies (5) and hence (2). But (2) and (3) together force $\mathsf{hilb}\,\underline{\mathrm{Ext}}^1_R(J, J') = s^{-1} + 1$ by (7.1), and so (1) holds.

(2) $\Rightarrow$ (3). Finally, assume that (2) holds. Then, as $\dim_{\Bbbk} \underline{\operatorname{Ext}}^1_R(J, J') < \infty$, (7.1) and (7.9) together imply that $\dim_{\Bbbk} \operatorname{Im}\alpha = 1 + \epsilon_{p,p'}$. We claim that this forces $\mathsf{hilb}\ \operatorname{Im}\alpha = \epsilon_{p,p'} + s$. First note that $\dim(\operatorname{Im}\alpha)_0 = \epsilon_{p,p'}$. So if $\mathsf{hilb}\ \operatorname{Im}\alpha \neq \epsilon_{p,p'} + s$, the cokernel of

$$\underline{\operatorname{Hom}}_R(J, J') \to \underline{\operatorname{Hom}}_B(\overline{J}, \overline{J'})$$

is one-dimensional in some degree > 1. Thus

$$\overline{\underline{\operatorname{Hom}}_R(J, J')} \supseteq \underline{\operatorname{Hom}}_B(\overline{J}, \overline{J'})_1 = H^0\big(E, \mathcal{M}_n\big(-p' + \tau^{-1}(p)\big)\big),$$

and it is also an $\big(\overline{\underline{\operatorname{End}}_R(J', J')}, \overline{\underline{\operatorname{End}}_R(J, J)}\big)$-bimodule. By hypothesis, either L or L' satisfies the equivalent conditions in Theorem 7.1; say it is L. Then

$$\overline{\underline{\operatorname{End}}_R(J, J)} = \underline{\operatorname{End}}_B(\overline{J}, \overline{J})$$

is a full TCR, say $B' = B(E, \mathcal{M}(-p), \tau)$. By Lemma 2.3,

$$\underline{\operatorname{Hom}}_B(\overline{J}, \overline{J'})_{\geq 1}$$

is generated in degree 1 as a right B'-module and thus

$$\overline{\underline{\operatorname{Hom}}_R(J, J')} = \underline{\operatorname{Hom}}_B(\overline{J}, \overline{J'})_{\geq 1},$$

a contradiction. Similarly, if $\overline{\underline{\operatorname{End}}_R(J', J')}$ is a full TCR we get a contradiction by viewing $\underline{\operatorname{Hom}}_B(\overline{J}, \overline{J'})_{\geq 1}$ as a left module. This proves the claim that

$$\mathsf{hilb}\ \operatorname{Im}\alpha = \epsilon_{p,p'} + s$$

which, by (7.9), implies (3). $\square$

Finally, we characterise lines L and L' with $(L \bullet L') = 0$, although here we need to assume that $\operatorname{Div} L \neq \operatorname{Div} L'$.

Theorem 7.7. *Let $L = R/J$ and $L' = R/J'$ be line modules over an elliptic algebra R with divisors $\operatorname{Div} L = p \neq p' = \operatorname{Div} L'$. Consider the following conditions.*

(1) $\underline{\operatorname{Ext}}^1_R(J, J') = 0$.

(2) $\mathsf{hilb}\, \underline{\operatorname{Ext}}^1_R(L, L') = \mathsf{hilb}\, \underline{\operatorname{Hom}}_R(L, L')$.

(3) $\mathsf{hilb}\, R - \mathsf{hilb}\, \underline{\operatorname{Hom}}_R(J, J') = (1-s)^{-1}$.

(4) $\mathsf{hilb}\, \underline{\operatorname{Hom}}_R(J, L') = (1-s)^{-2}$.

(5) $(L \bullet L') = 0$.

Then:

(a) (1) $\iff$ (2) $\iff$ (4) $\iff$ (5) $\Rightarrow$ (3).

(b) *If J° or $(J')^\circ$ is projective then all five are equivalent.*

Proof. (a) The proof of this result is very similar to the proof of Theorem 7.1. Recall (7.1):

$$X\big(L, L'\big) = \mathsf{hilb}\, \underline{\mathrm{Hom}}_R\big(J, L'\big) - \frac{1}{(1-s)^2} = \mathsf{hilb}\, \underline{\mathrm{Ext}}^1_R\big(J, J'\big) + \mathscr{H},$$

where $\mathscr{H} = h_R - (1-s)^{-1} - \mathsf{hilb}\, \underline{\mathrm{Hom}}_R(J, J')$. In this case, since $p \neq p'$, (7.9) shows that $\mathscr{H} \geq 0$. Thus just as in the proof of Theorem 7.1, (2) $\iff$ (4) is immediate, and (2) implies (1) and (3). Again, by (7.9), (3) is equivalent to $\alpha = 0$ in (7.7), and so (1) implies (3); thus (1) implies (2) by (7.1).

If (2) holds, so $X(L, L') = 0$, then certainly $(L \bullet L') = 0$, by (7.5), and so (5) holds.

Finally, if (5) holds, then (7.5) implies that

$$\dim_{\Bbbk} \underline{\mathrm{Ext}}^1_R\big(L, L'\big)_n - \dim_{\Bbbk} \underline{\mathrm{Hom}}_R\big(L, L'\big)_n = 0$$

for all $n \gg 0$. This shows that, in (7.1), the non-negative Hilbert series $\mathscr{H}$ and $\mathsf{hilb}\, \underline{\mathrm{Ext}}^1_R(J, J')$ must be polynomials in s. But (7.9) shows that $\mathscr{H}$ is either 0 or a multiple of $1/(1-s)$. Thus $\mathscr{H} = 0$, and so (3) holds. As we already saw, (3) implies that $\alpha = 0$ in (7.7), which in turn implies that $Z = \underline{\mathrm{Ext}}^1_R(J, J')$ is g-torsionfree. But as we have already shown that $\dim_{\Bbbk} Z < \infty$, this forces $Z = 0$. Thus (5) implies (1). This finishes the proof of (a).

(b) In this case, Lemma 5.6(2) and Lemma 7.5 imply that $Z = \underline{\mathrm{Ext}}^1_R(J, J')$ is finite-dimensional. Since we just saw that (3) implies that Z is g-torsionfree, (3) $\Rightarrow$ (1) holds in this case. $\square$

8. Blowing down elliptic algebras and their modules

As mentioned in the introduction (see Proposition 1.1 and Theorem 1.2) two fundamental and inverse constructions in birational geometry are the concepts of blowing up a closed point p on a smooth projective surface and, conversely, blowing down lines of self-intersection -1. In the noncommutative universe one again has a notion of blowing up, coming from [23] and [16] and at least if the set-up is “smooth enough” then one obtains an exceptional line module L with $(L \bullet L) = -1$. In this section we prove one of the main results of the paper by giving a noncommutative analogue of Castelnuovo’s Theorem: if R is an elliptic algebra with a line module L satisfying $(L \bullet L) = -1$, then one can blow L down to get a second elliptic algebra. Moreover, this operation is the inverse of blowing up. In fact we can do a little better and also obtain a result that works without smoothness assumptions, by replacing the requirement that $(L \bullet L) = -1$ by the assumption that the corresponding line ideal J satisfies $\mathsf{hilb}\, \underline{\mathrm{End}}_R(J) = \mathsf{hilb}\, R$. The details are given in Theorems 8.3 and 8.6.

We will need the following technical result about images of direct sums of a single line module.

Proposition 8.1. *Let L be a right line module over an elliptic algebra R. Let $\mathbb{I}$ be an index set, and for each $i \in \mathbb{I}$ choose an integer $a_i \in \mathbb{Z}$. Assume $\{a_i : i \in \mathbb{I}\}$ is bounded below. Let $A \subseteq \mathbb{L} = \bigoplus_{i\in\mathbb{I}} L[-a_i]$ be any submodule such that $N = \mathbb{L}/A$ is GK-2 pure. Then there is $\mathbb{J} \subseteq \mathbb{I}$ such that $\mathbb{L}$ is an internal direct sum $\mathbb{L} = A \oplus \bigoplus_{i\in\mathbb{J}} L[-a_i]$. In particular, $N \cong \bigoplus_{i\in\mathbb{J}} L[-a_i]$.*

Proof. We claim first that there exists a subset $\mathbb{J} \subseteq \mathbb{I}$ which is minimal under inclusion in the family $\mathcal{S} = \{\mathbb{K} \subseteq \mathbb{I} : \mathbb{L} = A + \bigoplus_{i\in\mathbb{K}} L[-a_i]\}$. Indeed, since N is left bounded, the graded Nakayama lemma (which does not require finite generation) applies and shows that $\mathbb{K} \in \mathcal{S}$ if and only if the induced map

$$\psi : \bigoplus_{i\in\mathbb{K}} L[-a_i]/L[-a_i]R_{\geq 1} \to N/NR_{\geq 1}$$

is surjective. Since L is cyclic, $\dim_{\Bbbk} L[-a_i]/L[-a_i]R_{\geq 1} = 1$. Thus any $\mathbb{J}$ for which ψ is an isomorphism of $\Bbbk$-spaces will be minimal in $\mathcal{S}$, proving the claim.

Fix some such $\mathbb{J}$. We will prove that $\mathbb{L} = A \oplus \bigoplus_{i\in\mathbb{J}} L[-a_i]$. Suppose that this fails; thus $0 \neq A \cap \bigoplus_{i\in\mathbb{J}} L[-a_i]$. Write $L[-a_i] = \alpha_i R$, for homogeneous α_i and choose $0 \neq u = (\alpha_i x_i) \in A \cap \bigoplus_{i\in\mathbb{J}} L[-a_i]$, for some homogeneous elements $x_i \in R$. Let $\mathbb{I}^1$ be the (finite) set $\{i \in \mathbb{J} : \alpha_i x_i \neq 0\}$. Without loss of generality, we may choose such u so that $|\mathbb{I}^1|$ is minimal. Let $\mathbb{L}^1 = \bigoplus_{i\in\mathbb{I}^1} L[-a_i]$.

Let $J = \operatorname{Ann}_R(u)$ and, for $i \in \mathbb{I}^1$, set $J_i = \operatorname{Ann}_R(\alpha_i x_i)$. If $r \in J_i$, then ur has strictly more zero entries than u, and so the choice of u forces $ur = 0$. So $J_i \subseteq J \subseteq J_i$. Thus $J_i = J$ is independent of the choice of $i \in \mathbb{I}^1$. Fixing an arbitrary $i \in \mathbb{I}^1$, we have $uR \cong R/J = R/J_i \cong \alpha_i x_i R \subseteq L[-a_i]$. By [16, Lemma 8.9(2)], there is a (shifted) line module $L' \subseteq L[-a_i]$, containing $\alpha_i x_i R$, for which $L'/(\alpha_i x_i R) = F$ is finite-dimensional. Since $uR \cong \alpha_i x_i R$, there is an injective homomorphism $uR \hookrightarrow L'$ with a finite-dimensional cokernel. Since $\underline{\operatorname{Ext}}^1_R(F, \mathbb{L}^1) = 0$ by Lemma 5.6(5), the canonical injection $uR \hookrightarrow \mathbb{L}^1$ lifts to an injection $L' \hookrightarrow \mathbb{L}^1 \subseteq \mathbb{L}$. In other words, there is some $u' = (\alpha_i x'_i)$ so that $uR \subseteq u'R \cong L'$, where $u'R/uR$ is finite-dimensional. As $\mathbb{L}^1$ is torsionfree, $\alpha_i x'_i \neq 0$ if and only if $i \in \mathbb{I}^1$. The argument from the beginning of the paragraph shows that $\operatorname{Ann}_R(\alpha_\ell x'_\ell) = \operatorname{Ann}_R(u')$, and hence that $\alpha_\ell x'_\ell R \cong L'$, for all $\ell \in \mathbb{I}^1$.

Choose $j \in \mathbb{I}^1$ such that $a_j = \max\{a_i : i \in \mathbb{I}^1\}$ and write $L' = \beta R$ for some homogeneous β. For any other $i \in \mathbb{I}^1$, by Lemma 5.5(3) we may identify

$$\Bbbk \cdot g^{a_j - a_i} = \operatorname{Hom}_{\text{gr-}R}\left(L[-a_j], L[-a_i]\right),$$

where $g^{a_j-a_i}$ maps $\alpha_j \mapsto \alpha_i g^{a_j-a_i}$. Now consider the following two maps in $\operatorname{Hom}_{\text{gr-}R}(L', L[-a_i])$:

$$f : \beta x \mapsto \alpha_i x'_i x, \quad \text{and} \quad f' : \beta x \mapsto \alpha_j x'_j x \mapsto \alpha_i g^{a_j-a_i} x'_j x.$$

Since $\operatorname{Hom}_{\text{gr-}R}(L', L[-a_i]) \neq 0$, by Lemma 5.5(2) it is one-dimensional. Thus there is $\lambda_i \in \Bbbk^*$ so that $f = \lambda_i f'$; that is $\alpha_i x_i' = \lambda_i \alpha_i g^{a_j - a_i} x_j'$. Thus $u' = (\lambda_i \alpha_i g^{a_j - a_i}) x_j'$, whence $L' \cong u'R \subseteq K = vR$, where $v = (\lambda_i \alpha_i g^{a_j - a_i})$. Clearly $K \cong L[-a_j]$. Since L is 2-critical, GKdim $K/uR \le 1$. Since $uR \subseteq A$ and $\mathbb{L}/A$ is 2-pure, we must have $K \subseteq A$. Now it is easy to see that

$$\mathbb{L}^1 = \bigoplus_{i \in \mathbb{I}^1} L[-a_i] = K \oplus \bigoplus_{i \in \mathbb{I}^1 \setminus \{j\}} L[-a_i].$$

This implies that

$$A + \bigoplus_{i \in \mathbb{J} \setminus \{j\}} L[-a_i] = \mathbb{L},$$

contradicting the choice of $\mathbb{J}$. Hence

$$\mathbb{L} = A \oplus \bigoplus_{i \in \mathbb{J}} L[-a_i],$$

as required. □

If L is a line module over an elliptic algebra R, we now use it to define the largest extension $\widetilde{K}$ of a reflexive module $K \subset R_{(g)}$ by sums of shifts of L.

Lemma 8.2. *Let R be an elliptic algebra, and let $L = R/J$ be a right R-line module with* $\operatorname{Div} L = p \in E$*. Let $K \subseteq R_{(g)}$ be a graded finitely generated g-divisible reflexive right R-module. Set*

$$\begin{aligned} \widetilde{K} &= \widetilde{K}_L \\ &= \sum_{\alpha} \{ N_\alpha : K \subseteq N_\alpha \subset Q_{\text{gr}}(R) \text{ with } N_\alpha / K \cong L[-i_\alpha] \text{ for some } i_\alpha \in \mathbb{Z} \}. \end{aligned} \tag{8.1}$$

Then the following hold.

(1) $\widetilde{K} = \underline{\operatorname{Hom}}_R(J, K)R$ *and* $\widetilde{K} \subseteq R_{(g)}$.

(2) *As right R-modules,*

$$\widetilde{K}/K \cong \bigoplus_{i \in \mathbb{Z}} L[-i]^{\oplus a_i}$$

for some $a_i \ge 0$. Hence $\operatorname{hilb} \widetilde{K}/K = p(s)(1-s)^{-2}$ *for* $p(s) = \sum_{i \in \mathbb{Z}} a_i s^i$.

(3) *Moreover,* $\operatorname{hilb} \underline{\operatorname{Ext}}^1_R(L, K) = p(s)(1-s)^{-1}$.

(4) *If* $\underline{\operatorname{Ext}}^1_R(L, L) = 0$ *then* $\underline{\operatorname{Ext}}^1_R(L, \widetilde{K}) = 0$.

Proof. Throughout the proof N_α will denote a module satisfying the properties defined by (8.1).

(1) If $N_\alpha \subseteq Q_{gr}(R)$ satisfies $N_\alpha/K \cong L[-i]$, then $N_\alpha = xR + K$ for some $x \in Q_{gr}(R)_i$. Then $xJ \subseteq K$ and so $N_\alpha \subseteq \underline{\mathrm{Hom}}_R(J, K)R$. Thus $\widetilde{K} \subseteq \underline{\mathrm{Hom}}_R(J, K)R$. Conversely, if $x \in \underline{\mathrm{Hom}}_R(J, K)_i \subset Q_{\mathrm{gr}}(R)$ then $(xR + K)/K$ is a homomorphic image of $(R/J)[-i] = L[-i]$. As K is reflexive, $(xR + K)/K$ is either 0 or 2-pure by Lemma 4.5(1). Therefore, because L is 2-critical, either $x \in K$ or else $(xR + K)/K \cong L[-i]$. In either case, $x \in \widetilde{K}$ by the definition of $\widetilde{K}$. Thus $\underline{\mathrm{Hom}}_R(J, K)R \subseteq \widetilde{K}$ since $\widetilde{K}$ is a right R-module.

Since J and K are g-divisible, $\underline{\mathrm{Hom}}_R(J, K) \subseteq R_{(g)}$ by Lemma 4.4(1), and so $\widetilde{K} \subseteq R_{(g)}$.

(2) Clearly $\widetilde{K}/K$ is a homomorphic image of

$$\bigoplus_\alpha N_\alpha/K \cong \bigoplus_\alpha L[-i_\alpha].$$

Once again, Lemma 4.5(1) implies that $\widetilde{K}/K$ is either 2-pure or 0. Since J and K are left bounded and locally finite, so is $\underline{\mathrm{Hom}}_R(J, K)R = \widetilde{K}$. Thus there is a lower bound d such that $d \le i_\alpha$ for all α. Then Proposition 8.1 applies, and shows that

$$\widetilde{K}/K \cong \bigoplus_{i \in \mathbb{Z}} L[-i]^{\oplus a_i},$$

where $a_i = 0$ for $i < d$. Also, the a_i are finite since $\underline{\mathrm{Hom}}_R(J, K)R$ is locally finite. Since $\mathsf{hilb}\, L = (1-s)^{-2}$, it is immediate that $\mathsf{hilb}\, \widetilde{K}/K = p(s)(1-s)^{-2}$.

(3) Consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}_R\big(L, \widetilde{K}/K\big) &\xrightarrow{\alpha} \underline{\mathrm{Ext}}^1_R(L, K) \\ &\xrightarrow{\beta} \underline{\mathrm{Ext}}^1_R\big(L, \widetilde{K}\big) \xrightarrow{\gamma} \underline{\mathrm{Ext}}^1_R\big(L, \widetilde{K}/K\big) \longrightarrow \cdots \end{aligned} \tag{8.2}$$

We will show that $\beta = 0$. So, suppose that $0 \ne \theta \in \underline{\mathrm{Ext}}^1_R(L, K)_{-j}$, corresponding to a nonsplit extension

$$0 \to K \xrightarrow{\iota} N \xrightarrow{\pi} L[j] \to 0.$$

We claim that $\iota : K \to N$ is an essential extension. If not, choose $0 \ne G \subseteq N$ maximal such that $\iota(K) \cap G = 0$. Then there is an exact sequence

$$0 \to K \to N/G \to L[j]/\pi(G) \to 0$$

where $K \to N/G$ is now essential. But $\mathrm{GKdim}\, L[j]/\pi(G) \le 1$ since L is 2-critical, and as K is reflexive this forces $L[j]/\pi(G) = 0$. Thus $N = K \oplus G$, giving the required contradiction. This proves the claim.

Since ι is essential, we may take the extension $K \subseteq N$ inside $Q_{\mathrm{gr}}(R)$. As such, $N \subseteq \widetilde{K}$ by definition and so θ cannot induce a nontrivial extension of $\widetilde{K}$ by L. Thus, $\beta = 0$.

As L_R is finitely generated,

$$\underline{\mathrm{Hom}}_R\Big(L, \bigoplus_{i\in\mathbb{Z}} L[-i]^{\oplus a_i}\Big) = \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{Hom}}_R\big(L, L[-i]\big)^{\oplus a_i}.$$

Therefore, since $\operatorname{hilb} \underline{\mathrm{Hom}}_R(L, L) = 1/(1-s)$ by Lemma 5.5, this implies that

$$\operatorname{hilb} \underline{\mathrm{Hom}}_R\big(L, \widetilde{K}/K\big) = \Big(\sum_{i\in\mathbb{Z}} a_i s^i\Big)/(1-s).$$

Applying α shows that

$$\operatorname{hilb} \underline{\mathrm{Ext}}^1_R\big(L, K\big) = \Big(\sum_{i\in\mathbb{Z}} a_i s^i\Big)/(1-s),$$

as well.

(4) If $\underline{\mathrm{Ext}}^1_R(L, L) = 0$, then $\underline{\mathrm{Ext}}^1_R(L, \widetilde{K}/K) = 0$ in (8.2). Since $\beta = 0$, this forces $\underline{\mathrm{Ext}}^1_R(L, \widetilde{K}) = 0$. □

We now come to one of the main results of the paper by showing that, under mild conditions, in applying the tilde operation to R itself one obtains a ring $\widetilde{R} = \widetilde{R}_L$. As we show later in the section, this operation is a good non-commutative analogue of blowing down a line of self-intersection -1. In fact, the self-intersection condition is not quite the right concept when qgr-R is not smooth, and so the theorem is stated under the weaker condition (8.3).

Theorem 8.3. *Let R be an elliptic algebra with $R/gR \cong B(E, \mathcal{M}, \tau)$. Let $L = R/J$ be a right line module with* Div $L = p$, *satisfying*

$$\operatorname{hilb} \underline{\mathrm{End}}_R(J) = \operatorname{hilb} R. \tag{8.3}$$

Then the module $\widetilde{R} = \widetilde{R}_L$ constructed in Lemma 8.2 is a connected graded subalgebra of $R_{(g)}$. It is also equal to $\underline{\mathrm{Hom}}_R(J, J)R$ and satisfies the following properties.

(1) *As right R-modules,*

$$\widetilde{R}/R \cong \bigoplus_{i\geq 1} L[-i].$$

(2) *As left R-modules,*

$$\widetilde{R}/R \cong \bigoplus_{i\geq 1} L^{\vee}[-i],$$

where $L^{\vee} = \underline{\mathrm{Ext}}^1_R(L, R)[1]$ is the dual line module.

(3) *$\widetilde{R}$ is an elliptic algebra with*

$$\widetilde{R}/g\widetilde{R} \cong B\big(E, \mathcal{M}\big(\tau^{-1}(p)\big), \tau\big).$$

Remark 8.4. (1) In the notation of the theorem, we say that $\widetilde{R}_L$ is obtained by *blowing down* or *contracting* L (or alternatively its dual $L^\vee$).
(2) Note that by Theorem 7.1, if $(L\bullet L) = -1$ then (8.3) holds and so Theorem 8.3 gives a method of contracting a line of self-intersection (-1) on an elliptic algebra.
(3) If qgr-R is smooth then the conditions $(L\bullet L) = -1$ and (8.3) are equivalent (see Remark 7.2). However, when qgr-R is not smooth the later condition can definitely be weaker and there do exist line modules L with $(L\bullet L) \neq -1$ that can still be blown down by the theorem. See Corollary 10.5 for one such example.

Proof. Lemma 5.6(1) implies that

$$\mathsf{hilb}\, \underline{\mathrm{Ext}}^1_R\big(L, R\big) = s/(1-s)^2 = (s + s^2 + \cdots)/(1-s).$$

Thus (1) holds by comparing parts (2) and (3) of Lemma 8.2. Also, by Lemma 8.2(1) we have $\widetilde{R} = \underline{\mathrm{Hom}}_R(J, R)R \subseteq R_{(g)}$. Indeed, since $\underline{\mathrm{Hom}}_R(J, R)$ is automatically a left R-module, $\widetilde{R}$ is actually an R-bimodule. Note that, to this point, we have not used (8.3).

The main part of the proof will be to prove that $\widetilde{R}$ is a subalgebra of $Q_{\mathrm{gr}}(R)$, the first step in which will be to prove that $\widetilde{R} = \underline{\mathrm{End}}_R(J)R$.

Certainly,

$$\underline{\mathrm{End}}_R(J)R \subseteq \underline{\mathrm{Hom}}_R\big(J, R\big)R = \widetilde{R}.$$

Since J is g-divisible, Lemma 4.4(1) implies that $\underline{\mathrm{End}}_R(J)$ is g-divisible. By hypothesis, $\overline{R} = \bigoplus_{n\geq 0} H^0(E, \mathcal{M}_n)t^n$ and $\overline{J} = \bigoplus_{n\geq 0} H^0(E, \mathcal{M}_n(-p))t^n$ and so, by Lemma 2.3,

$$\underline{\mathrm{End}}_{\overline{R}}\big(\overline{J}\big) = \bigoplus_{n\geq 0} H^0\big(E, \mathcal{M}_n\big(-p + \tau^{-n}(p)\big)\big) = B\big(E, \mathcal{M}', \tau\big)$$

for $\mathcal{M}' = \mathcal{M}(-p + \tau^{-1}(p))$. Let $B = B(E, \mathcal{M}, \tau)$. Clearly

$$\overline{\underline{\mathrm{End}}_R(J)} \subseteq \underline{\mathrm{End}}_B\big(\overline{J}\big).$$

Conversely, (8.3) and g-divisibility imply that

$$\mathsf{hilb}\, \overline{\underline{\mathrm{End}}_R(J)} = \mathsf{hilb}\, \overline{R} = \mathsf{hilb}\, B\big(E, \mathcal{M}', \tau\big) = \mathsf{hilb}\, \underline{\mathrm{End}}_B\big(\overline{J}\big).$$

Hence,

$$\overline{\underline{\mathrm{End}}_R(J)} = \underline{\mathrm{End}}_B\big(\overline{J}\big) = B\big(E, \mathcal{M}', \tau\big).$$

This in turn implies that

$$\overline{\underline{\mathrm{End}}_R(J)R} = \overline{\underline{\mathrm{End}}_R(J)} \cdot \overline{R} = B'B,$$

where $B' = B(E, \mathcal{M}', \tau)$.

We claim next that $B'B = B''$, where $B'' = B(E, \mathcal{N}, \tau)$ for $\mathcal{N} = \mathcal{M}(\tau^{-1}(p))$. Indeed, by Lemma 2.2 and the fact that all the relevant invertible sheaves are generated by their global sections,

$$\begin{aligned}
\sum_{i=0}^{n} B'_i B_{n-i} &= \sum_{i=0}^{n} H^0\big(E, \mathcal{M}_i\big(-p+\tau^{-i}(p)\big)\big) \cdot H^0\big(E, \mathcal{M}_{n-i}^{\tau^i}\big) \\
&= \sum_{i=0}^{n} H^0\big(E, \mathcal{M}_n\big(-p+\tau^{-i}(p)\big)\big) \\
&= H^0\Big(E, \sum_{i=0}^{n} \mathcal{M}_n\big(-p+\tau^{-i}(p)\big)\Big) \\
&= H^0\big(E, \mathcal{M}_n\big(\tau^{-1}(p)+\cdots+\tau^{-n}(p)\big)\big) = B''_n,
\end{aligned}$$

proving the claim.

Although we do not know a priori that $\underline{\mathrm{End}}_R(J)R$ is g-divisible, the previous paragraph at least gives the inequality

$$\mathsf{hilb}\,\underline{\mathrm{End}}_R(J)R \geq \big(\mathsf{hilb}\,B''\big)/(1-s).$$

Since Riemann–Roch gives

$$\mathsf{hilb}\,B'' = \mathsf{hilb}\,B + \sum_{i\geq 1} s^i/(1-s),$$

this implies that

$$\big(\mathsf{hilb}\,B''\big)/(1-s) = \mathsf{hilb}\,\widetilde{R}.$$

This forces

$$\underline{\mathrm{End}}_R(J)R = \underline{\mathrm{Hom}}_R\big(J, R\big)R = \widetilde{R}$$

as desired at the beginning of the proof. It follows that $\widetilde{R}/\widetilde{R}g = B''$ and so $\widetilde{R}$ is indeed g-divisible.

Consequently,

$$R \cdot \underline{\mathrm{End}}_R(J) \subseteq R \cdot \underline{\mathrm{Hom}}_R\big(J, R\big) = \underline{\mathrm{Hom}}_R\big(J, R\big) \subseteq \underline{\mathrm{End}}_R(J)R.$$

Thus

$$\big(\underline{\mathrm{End}}_R(J)R\big)^2 = \big(\underline{\mathrm{End}}_R(J)\big)^2 R = \underline{\mathrm{End}}_R(J)R$$

and so $\widetilde{R}$ is indeed a subalgebra of $Q_{\mathrm{gr}}(R)$. Moreover, since $\widetilde{R}/\widetilde{R}g = B''$, it follows that $\widetilde{R}$ is an elliptic algebra.

Finally, all of the above arguments hold for the left line module $L^\vee = R/J^\vee$, and so one obtains a elliptic subalgebra $\widetilde{R}^\ell$ of $Q_{\mathrm{gr}}(R)$ with ${}_R(\widetilde{R}^\ell/R) \cong \bigoplus_{i\geq 1} L^\vee[-i]$.

By Lemma 5.6(3), $M = \underline{\mathrm{Hom}}_R(J^\vee, R)$ is a right R-module such that $M/R \cong L[-1]$. Dually, $M^\vee = \underline{\mathrm{Hom}}_R(J, R)$ is a left R-module with $M^\vee/R \cong L^\vee[-1]$. In particular, $M_1 J \subseteq R$ and so

$$M_1 \subseteq \underline{\mathrm{Hom}}_R(J, R)_1 = M_1^\vee.$$

By symmetry, $M_1 = M_1^\vee$. Now by construction,

$$\widetilde{R}_1 = \left[\underline{\mathrm{Hom}}_R(J, R)R\right]_1 = M_1^\vee$$

and so, dually, $\widetilde{R}_1^\ell = M_1$. Thus $\widetilde{R}_1^\ell = \widetilde{R}_1$. Since $\widetilde{R}$ and $\widetilde{R}^\ell$ are both generated in degree 1 by Proposition 4.3, it follows that they are equal. This proves part (2) and completes the proof. $\square$

Corollary 8.5. *Let R be an elliptic algebra with a line module $L = R/J$ satisfying (8.3), and let $K \subseteq R_{(g)}$ be a reflexive g-divisible finitely generated R-module. Then $\widetilde{K}_L$ is a right $\widetilde{R}_L$-module.*

Proof. By the proof of Theorem 8.3, $\widetilde{R} = \widetilde{R}_L = \underline{\mathrm{End}}_R(J)R$, while $\widetilde{K} = \widetilde{K}_L = \underline{\mathrm{Hom}}_R(J, K)R$ by Lemma 8.2. We also saw that $R \cdot \underline{\mathrm{End}}_R(J) \subseteq \underline{\mathrm{End}}_R(J)R$ in the proof of Theorem 8.3. Thus

$$\begin{aligned}\widetilde{K}\widetilde{R} &= \underline{\mathrm{Hom}}_R(J, K)R \cdot \underline{\mathrm{End}}_R(J)R \\ &\subseteq \underline{\mathrm{Hom}}_R(J, K)\underline{\mathrm{End}}_R(J)R \subseteq \underline{\mathrm{Hom}}_R(J, K)R = \widetilde{K}\end{aligned}$$

as required. $\square$

To conclude this section, we explain how the above construction of ring-theoretic blowing down is formally the inverse of noncommutative blowing up. When R is an elliptic algebra of degree $\mu \geq 3$ with $R/Rg = B(E, \mathcal{M}, \tau)$, then for any $p \in E$ one may define the ring-theoretic blowup of R at p to be the subring $R' = R(p) \subseteq R$ generated by $R'_1 = \{x \in R : \overline{x} \in H^0(E, \mathcal{M}(-p))\}$ (see [16] and [17] for the basic properties of these blowups). Then [17, Theorem 1.1] implies that R' is also elliptic, of degree $\mu - 1$, with $R'/R'g \cong B(E, \mathcal{M}(-p), \tau)$. The ring R' automatically has an *exceptional* line module L satisfying $(R/R') \cong \bigoplus_{i=1}^\infty L[-i]$ as right R'-modules. As we show in the next theorem, one can then recover R by blowing down this line.

As an aside, we note that one can also allow elliptic algebras to have degree 1 or 2 (though in this paper the definition excludes them) and the ring-theoretic blowup at a point of an elliptic algebra of degree $\mu = 2$ can still be defined. However, such a blowup will not now be generated in degree one and so a more complicated definition is necessary (see [17] for the details).

Theorem 8.6. (1) *Let R be an elliptic algebra, of degree at least 4, with $R/Rg = B(E, \mathcal{M}, \tau)$. If $p \in E$, then the exceptional line module $L = R(p)/J$ of the ring-theoretic blowup $R(p)$ satisfies (8.3); in fact, $\underline{\mathrm{End}}_{R(p)}(J) = R(\tau(p))$. Thus the blowdown $\widetilde{R(p)}_L$ of $R(p)$ along L is defined, and it equals R.*

(2) *Conversely, suppose that R' is an elliptic algebra with a line module L satisfying* (8.3) *and set* $\widetilde{R}' = \widetilde{R}'_L$. *If* $\operatorname{Div} L = \tau(p)$*, then the ring-theoretic blowup* $\widetilde{R}'(p)$ *equals* R'.

Proof. We remark that the apparent shift $p \mapsto \tau(p)$ between parts (1) and (2) comes from the fact that, by [16, Lemma 9.1], the line module L in part (1) has $\operatorname{Div} L = \tau(p)$.

(1) We first claim that $R(p)_1 R_1 = R_1 R(\tau(p))_1$. Indeed, since $gR_1 \subseteq R(p)_1 R_1$, it follows that

$$R(p)_1 R_1 = \bigl\{x \in R_2 : \overline{x} \in \overline{R(p)_1 R_1} = H^0\bigl(E, \mathcal{M}_2(-p)\bigr)\bigr\}.$$

A similar calculation shows that $R_1 R(\tau(p))_1$ is equal to the same subspace of R_2, proving the claim.

Since

$$R/R(p) \cong \bigoplus_{i \geq 1} L[-i],$$

clearly

$$R_{\leq 1} R(p)/R(p) \cong L[-1]$$

and so $J = \bigl\{x \in R(p) : R_1 x \subseteq R(p)\bigr\}$. Thus

$$R_1 R\bigl(\tau(p)\bigr)_1 J_1 = R(p)_1 R_1 J_1 \subseteq R(p)_1 R(p)_2 \subseteq R(p),$$

and so $R(\tau(p))_1 J_1 \subseteq J$. By Lemma 5.6(2), the line ideal J is generated in degree 1. Thus $R(\tau(p))_1 J_1 R(p) \subseteq J$ implies $R(\tau(p))_1 J \subseteq J$. Since $R(\tau(p))$ is also generated in degree 1 it follows that $R(\tau(p)) \subseteq \underline{\operatorname{End}}_{R(p)}(J)$, and so

$$\operatorname{hilb} \underline{\operatorname{End}}_{R(p)}(J) \geq \operatorname{hilb} R\bigl(\tau(p)\bigr) = \operatorname{hilb} R(p).$$

Conversely,

$$\operatorname{hilb} \underline{\operatorname{End}}_{R(p)}(J) \leq \operatorname{hilb} R(p)$$

follows, for example, from (7.9). Thus

$$\operatorname{hilb} \underline{\operatorname{End}}_{R(p)}(J) = \operatorname{hilb} R\bigl(\tau(p)\bigr)$$

and hence $\underline{\operatorname{End}}_{R(p)}(J) = R\bigl(\tau(p)\bigr)$.

Now Theorem 8.3 applies to define the blowdown $\widetilde{R(p)}$ of $R(p)$ along L. By that theorem,

$$\widetilde{R(p)} = \underline{\operatorname{End}}_{R(p)}(J) R(p) = R\bigl(\tau(p)\bigr) R(p).$$

Since $R_1 = R(\tau(p))_1 + R(p)_1 = \widetilde{R(p)}_1$ and both R and $\widetilde{R(p)}$ are generated in degree 1 as algebras, necessarily $\widetilde{R(p)} = R$.

(2) In this case, the blowdown $\widetilde{R}'$ satisfies $\widetilde{R}'/\widetilde{R}'g = B(E, \mathcal{M}(p), \tau)$, by Theorem 8.3. The blowup of $\widetilde{R}'$ at the point p is thus the subring of R' generated in degree 1 by $\{x \in \widetilde{R}'_1 : \overline{x} \in H^0(E, \mathcal{M})\}$. This is precisely R'. $\square$

9. Smoothness

A key feature of the commutative geometry described in Proposition 1.1 and Theorem 1.2 is that if $\pi : Y \to X$ is a birational morphism of surfaces where the exceptional locus is a (-1) line, then Y is nonsingular if and only if X is nonsingular. In this section we will prove a natural analogue of this result (see Theorem 9.1). Recall that for a graded ring R we say that qgr-R is *smooth* if this category has finite homological dimension.

Theorem 9.1. *Let T be an elliptic algebra of degree ≥ 4 associated to the elliptic curve E, and let $p \in E$. Let L be the exceptional line module for the blowup $T(p) \subseteq T$. The following are equivalent:*

(1) qgr-$T(p)$ *is smooth.*

(2) qgr-T *is smooth and* $\operatorname{pdim}_{T(p)^\circ} L^\circ < \infty$.

We remark that by Lemma 6.5, we have

$$\operatorname{pdim}_{T(p)^\circ} L^\circ < \infty \iff \operatorname{pdim}_{T(p)^\circ} L^\circ = 1.$$

Note also that we will show later that blowing up the point p in the blowup $T(p)$ leads to a non-smooth noncommutative scheme (see Corollary 10.8 for the details) and so the extra conditions of the theorem are necessary.

As an immediate corollary of Theorem 9.1, we obtain:

Corollary 9.2. *Let R be an elliptic algebra of degree ≥ 3 and suppose that L is a line module with* $\operatorname{pdim} L^\circ < \infty$ *and $(L \bullet L) = -1$. Let $\widetilde{R} = \widetilde{R}_L$ be the blowdown of R constructed by Theorem 8.3. Then* qgr-$\widetilde{R}$ *is smooth if and only if* qgr-R *is smooth.*

Proof of Corollary 9.2. By Theorem 8.6 $R = \widetilde{R}(q)$, where $q = \tau^{-1}(\operatorname{Div} L)$. Thus the result is a direct application of Theorem 9.1. $\square$

The rest of the section is devoted to the proof of Theorem 9.1. We work mostly in the localised category of modules over $U = T^\circ$; note that, by Lemma 6.8, qgr-T is smooth if and only if $\operatorname{gldim} T^\circ < \infty$.

Proposition 9.3. *Let U be a noetherian domain with division ring of fractions $D = Q(U)$ and a projective right ideal J. Set $L = U/J$ and $L^\vee = \operatorname{Ext}^1_U(L, U)$. Let $U \subset V \subset D$ be an overring satisfying:*

(a) $(V/U)_U \cong L^{\oplus \mathbb{J}}$ *for some index set* $\mathbb{J}$*;*

(b) $\operatorname{Hom}_U(L, V) = \operatorname{Ext}^1_U(L, V) = 0$ *and the same for* $L^\vee$.

Then:

(1) $L \otimes_U V = 0 = \operatorname{Tor}_1^U(L, V)$.

(2) *Let $L^\perp$ be the full subcategory of* Mod-U *consisting of modules M satisfying* $\operatorname{Hom}_U(L, M) = \operatorname{Ext}^1_U(L, M) = 0$. *Then* $L^\perp \simeq$ Mod-V.

(3) $\operatorname{gldim} V \leq \operatorname{gldim} U$. *In particular, if* $\operatorname{gldim} U < \infty$ *then* $\operatorname{gldim} V < \infty$.

Proof. (1) We first compute $J \otimes_U V$. Since $V \subset D$, there is an exact sequence

$$\operatorname{Tor}_2^U(L, D/V) \to \operatorname{Tor}_1^U(L, V) \to \operatorname{Tor}_1^U(L, D).$$

Using that $\operatorname{pdim}(L) = 1$ and that D is a flat U-module, the outside terms are zero in this sequence and so $\operatorname{Tor}_1^U(L, V) = 0$. Thus the natural map

$$\phi : J \otimes_U V \to U \otimes_U V = V$$

is injective and we identify $J \otimes_U V$ with $JV = \operatorname{Im} \phi$.

From the exact sequence $0 \to U \to J^* \to L^\vee \to 0$ and (b), the natural map $\operatorname{Hom}_U(J^*, V) \to \operatorname{Hom}_U(U, V) = V$ is an isomorphism. In particular, the inclusion $U \subseteq V$ lifts to an inclusion $J^* \subseteq V$ and so, by the Dual Basis Lemma, $JV \supseteq JJ^* \ni 1$. So $J \otimes_U V = V$. It follows by tensoring the exact sequence

$$0 \to J \to U \to L \to 0$$

with V that $L \otimes_U V = 0$.

(2) Consider the functors

$$F = \operatorname{Hom}_U(V, -) : \text{Mod-}U \to \text{Mod-}V \quad \text{and} \quad G = \operatorname{res}_U : \text{Mod-}V \to \text{Mod-}U.$$

We claim that F and G give inverse equivalences between $L^\perp$ and Mod-V.

Let $M \in L^\perp$, and consider the exact sequence

$$\operatorname{Hom}_U(L^{\oplus \mathbb{J}}, M) \to \operatorname{Hom}_U(V, M) \to \operatorname{Hom}_U(U, M) \to \operatorname{Ext}_U^1(L^{\oplus \mathbb{J}}, M)$$

induced from (a). As the outside terms are zero, this provides a natural isomorphism of U-modules: $GF(M) = \operatorname{Hom}_U(V, M) \xrightarrow{\sim} M$. Hence M carries a natural right V-action. Similarly, if $M' \in L^\perp$ as well then a U-module homomorphism $\theta : M \to M'$ induces a V-module homomorphism

$$\operatorname{Hom}_U(V, M) \to \operatorname{Hom}_U(V, M'),$$

from which it follows that θ is already a V-module map. Thus $GF \cong \operatorname{Id}_{L^\perp}$.

Now let $N \in$ Mod-V. From the spectral sequence [21, Theorem 10.74]

$$\operatorname{Ext}_V^p(\operatorname{Tor}_q^U(L, V), N) \Rightarrow \operatorname{Ext}_U^{p+q}(L, N),$$

we have $\operatorname{Hom}_U(L, N) = 0 = \operatorname{Ext}_U^1(L, N)$, and so $G(N) \in L^\perp$. There is a natural V-module map $N \to \operatorname{Hom}_U(V, N)$ given by $n \mapsto (s \mapsto ns)$, which is the inverse of the natural isomorphism $\operatorname{Hom}_U(V, N) \to N$ discussed above and so it is also an V-module isomorphism. Thus $FG \cong \operatorname{Id}_{\text{Mod-}V}$ on objects. It is routine to check that this respects morphisms and so F, G are indeed inverse equivalences.

(3) Let $M, N \in \text{Mod-}V$, which we identify with $L^{\perp}$, using (2). It is clearly sufficient to prove that

$$\operatorname{Ext}^i_V\left(M, N\right) \cong \operatorname{Ext}^i_U\left(M, N\right), \quad \text{for } i \geq 1. \tag{9.1}$$

To prove this we will use the spectral sequence $(2)_4$ from [7, Section XVI.5, p. 349] for the injection $\phi : U \to V$. We begin with a couple of observations.

By part (2) and the fact that L_U has projective dimension $\operatorname{pdim}_U(L) \leq 1$, we have $\operatorname{Ext}^j_U(L, N) = 0$ for all $j \geq 0$. Now consider the long exact sequence obtained by applying $\operatorname{Hom}_U(-, N)$ to the exact sequence

$$0 \to U \to V \to L^{\oplus \mathbb{J}} \to 0$$

arising from (a). Then certainly $\operatorname{Ext}^q_U(V, N) = 0$ for $q \geq 1$. Moreover, $\operatorname{Hom}_U(V, N) = \operatorname{Hom}_U(U, N) = N$; thus ${}^{(\phi)}N = N$ in the notation of [7]. Therefore, as explained in [7], the cited spectral sequence collapses and the edge homomorphism $(3)_4$ from [7, Section XVI.5, Case 4, p. 349] becomes the desired isomorphism

$$\operatorname{Ext}^i_V\left(M, N\right) = \operatorname{Ext}^i_V\left(M, {}^{(\phi)}N\right) \cong \operatorname{Ext}^i_U\left(M, N\right). \qquad \square$$

We next prove a partial converse to Proposition 9.3(3), for which we need the following result on universal extensions.

Lemma 9.4. *Let U be a noetherian $\Bbbk$-algebra and let L be a finitely generated right U-module satisfying $\operatorname{End}_U(L) = \Bbbk$ and $\operatorname{Ext}^1_U(L, L) = 0$. For any right U-module Q such that $\operatorname{Hom}_U(L, Q) = 0$, there is a short exact sequence*

$$0 \to Q \to N \to \operatorname{Ext}^1_U\left(L, Q\right) \otimes_{\Bbbk} L \to 0,$$

for some $N \in L^{\perp}$.

Proof. Let $E = \operatorname{Ext}^1_U(L, Q)$ and choose a basis $\{e_i\}_{i \in \mathbb{I}}$ for E as a $\Bbbk$-vector space. As in [9, Lemma 4.2], construct a short exact sequence

$$0 \to Q \to N \to E \otimes_{\Bbbk} L \to 0$$

such that the pullback under $e_i \otimes \operatorname{Id}_L : L \to E \otimes_{\Bbbk} L$ is the extension

$$0 \to Q \to N_i \to L \to 0$$

corresponding to e_i. (It can be shown by a diagram chase that N is the element of $\operatorname{Ext}^1_U(E \otimes_{\Bbbk} L, Q)$ corresponding to Id_E via the natural isomorphism $\alpha : \operatorname{Hom}_{\Bbbk}(E, E) \to \operatorname{Ext}^1_U(E \otimes_{\Bbbk} L, Q)$.) By construction, the diagram

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d, equal] & N_i \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "e_i \otimes \operatorname{Id}_L"] & 0 \\
0 \arrow[r] & Q \arrow[r] & N \arrow[r] & E \otimes_{\Bbbk} L \arrow[r] & 0
\end{tikzcd}$$

commutes. Applying $\mathrm{Hom}_U(L,-)$, we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_U(L,Q) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_U(L,N_i) \arrow[r] \arrow[d] & \mathrm{Hom}_U(L,L) \arrow[r, "\delta_i"] \arrow[d, "(e_i\otimes \mathrm{Id}_L)\circ(-)"] & E \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}_U(L,Q) \arrow[r] & \mathrm{Hom}_U(L,N) \arrow[r] & \mathrm{Hom}_U(L,E\otimes_{\Bbbk} L) \arrow[r, "\delta"'] & E.
\end{tikzcd} \tag{9.2}$$

Thus $\delta(e_i \otimes \mathrm{Id}_L) = \delta((e_i \otimes \mathrm{Id}_L)\circ \mathrm{Id}_L) = \delta_i(\mathrm{Id}_L)$, and this equals e_i from the standard way that elements of $\mathrm{Ext}^1_U(L,Q)$ correspond to extensions. Since L is finitely generated and $\mathrm{End}_U(L)\cong \Bbbk$, it follows that δ is an isomorphism.

We now use that $\mathrm{Hom}_U(L,Q) = 0 = \mathrm{Ext}^1_U(L,L)$. Extending the bottom row of (9.2) gives the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_U\big(L,N\big) &\to \mathrm{Hom}_U\big(L,E\otimes_{\Bbbk} L\big) \\ &\xrightarrow{\delta} E \to \mathrm{Ext}^1_U\big(L,N\big) \to \mathrm{Ext}^1_U\big(L,E\otimes_{\Bbbk} L\big). \end{aligned}$$

But $\mathrm{Ext}^1_U(L,E\otimes_{\Bbbk} L)\cong E\otimes_{\Bbbk}\mathrm{Ext}^1_U(L,L) = 0$. Since δ is an isomorphism, it follows that $N\in L^{\perp}$. □

Proposition 9.5. *Suppose that U, V and L satisfy the hypotheses of Proposition 9.3 and, in addition, that $\mathrm{Hom}_U(L,L)=\Bbbk$ and $\operatorname{injdim} U = d<\infty$. If $\operatorname{gldim} V<\infty$ then $\operatorname{gldim} U<\infty$.*

Proof. Let $M,M'\in \text{Mod-}U$. We need to prove that $\mathrm{Ext}^i_U(M,M')=0$ for $i\gg 0$. Take exact sequences

$$0\to Q\to F\to M\to 0 \quad\text{and}\quad 0\to Q'\to F'\to M'\to 0$$

for free modules F, F' and consider the induced long exact sequences of Ext groups. Using that $\operatorname{injdim} F=\operatorname{injdim} U = d$, it follows that

$$\mathrm{Ext}^i_U\big(M,M'\big)\cong \mathrm{Ext}^{i+1}_U\big(M,Q'\big)\cong \mathrm{Ext}^i_U\big(Q,Q'\big)$$

for $i>d$. Thus, it suffices to prove that $\mathrm{Ext}^j_U(Q,Q')=0$ for $j\gg 0$. Since Q and Q' are Goldie torsionfree as defined on page 1471, Lemma 9.4 applies and produces exact sequences

$$0\to Q\to N\to E\otimes_{\Bbbk} L\to 0 \quad\text{and}\quad 0\to Q'\to N'\to E'\otimes_{\Bbbk} L\to 0,$$

where $N,N'\in L^{\perp}$.

Once again, from the induced long exact sequences for Ext groups, it suffices to prove that

$$\mathrm{Ext}^k_U\big(H,H'\big)=0 \quad \text{for } k\gg 0, \quad \text{where } H\in\{L,N\} \text{ and } H'\in\{L,N'\}. \tag{9.3}$$

Since $L = U/J$, with J projective, $\operatorname{injdim}(L) < \infty$ and $\operatorname{pdim}(L) = 1$. So certainly (9.3) holds if either $H = L$ or $H' = L$. The remaining case, where H and $H' \in L^{\perp}$, follows from (9.1). □

We now prove Theorem 9.1.

Proof of Theorem 9.1. Suppose first that qgr-$T(p)$ is smooth. Recall that, by Lemma 6.8, if R is an elliptic algebra then qgr-R is smooth if and only if $\operatorname{gldim} R^{\circ} < \infty$. Thus $\operatorname{gldim} T(p)^{\circ} < \infty$, and in particular $\operatorname{pdim}_{T(p)^{\circ}} L^{\circ} < \infty$ (and thus $\operatorname{pdim}_{T(p)^{\circ}} L^{\circ} = 1$ by Lemma 6.5). By Theorem 8.6, T is the blowdown of $T(p)$ along L, and J satisfies the condition $\operatorname{hilb} \underline{\operatorname{End}}_{T(p)}(J) = \operatorname{hilb} T(p)$ as in (8.3). Since $L^{\circ} = T(p)^{\circ}/J^{\circ}$, the right ideal J° is projective, and by Theorem 7.1 it follows that $\underline{\operatorname{Ext}}^1_{T(p)}(L, L) = 0$. By Lemma 8.2(3), $\underline{\operatorname{Ext}}^1_{T(p)}(L, T) = 0$. Since by Lemma 5.6(4) $\underline{\operatorname{Ext}}^1_{T(p)}(L^{\vee}, L^{\vee}) = 0$, applying Lemma 8.2(3) on the left gives that $\underline{\operatorname{Ext}}^1_{T(p)}(L^{\vee}, T) = 0$. Finally, it is obvious that

$$\underline{\operatorname{Hom}}_{T(p)}(L, T) = 0 = \underline{\operatorname{Hom}}_{T(p)}(L^{\vee}, T).$$

Now using Lemma 6.2, it follows from the above observations that the hypotheses of Proposition 9.3 hold with $V = T^{\circ}, U = T(p)^{\circ}$. Thus $\operatorname{gldim} U < \infty$, Proposition 9.3 implies $\operatorname{gldim} V < \infty$, and so qgr-$T$ is smooth.

Conversely, suppose that qgr-T is smooth and that $\operatorname{pdim}_{T(p)^{\circ}} L^{\circ} < \infty$, so again $\operatorname{pdim}_{T(p)^{\circ}} L^{\circ} = 1$. All of the arguments in the previous paragraph then go through to show that the hypotheses of Proposition 9.3 hold with $V = T^{\circ}$, $U = T(p)^{\circ}$. We also have $\underline{\operatorname{Hom}}_{T(p)}(L, L) = k[g]$ by 5.5(3), and $\operatorname{injdim} T(p) < \infty$ is part of the Auslander–Gorenstein condition which holds by Proposition 4.3. Thus applying Lemma 6.2, the hypotheses of Proposition 9.5 hold. Thus $\operatorname{gldim} V < \infty$ implies $\operatorname{gldim} U < \infty$, and so qgr-$T(p)$ is smooth. □

We do not know how to characterise when $\operatorname{pdim} L^{\circ} = 1$ (equivalently, $\operatorname{pdim} L^{\circ} < \infty$), although we conjecture:

Conjecture 9.6. *Let T be an elliptic algebra with* $\deg T \geq 4$ *and let $T(p)$ be the blowup of T at $p \in E$ with exceptional line L. If there is no T-line module L' with* $\operatorname{Div} L' = \tau(p)$*, then* $\operatorname{pdim}_{T(p)^{\circ}} L^{\circ} = 1$.

10. An example of undefined self-intersection

In this section we describe an elliptic algebra R with an exceptional line module L for which the self-intersection $(L \bullet_{MS} L)$ is undefined. Moreover, $(L \bullet L) \neq -1$ yet the associated line ideal J does satisfy $\operatorname{hilb} \underline{\operatorname{End}}_R(J) = \operatorname{hilb} R$. Thus one can still use Theorem 8.3 to blow down the line L. This justifies the comments made in

Remark 8.4 concerning that theorem and also gives the example promised before Definition 6.7.

In a way that will shortly be made precise, the ring R is obtained by blowing up the same point $p \in E$ twice in the Veronese ring $T = S^{(3)}$ of the Sklyanin algebra S. The key property, here, is that the resulting scheme qgr-R is not smooth. This may be explained by analogy with the commutative situation: iterated ring-theoretic blowups of T are analogs of commutative rings of the form

$$A = \bigoplus_{n \geq 0} H^0\big(\mathbb{P}^2, (\mathcal{I} \otimes \mathcal{O}(9))^{\otimes n}\big),$$

where $\mathcal{I}$ is an ideal sheaf defining a zero-dimensional subscheme Z of $\mathbb{P}^2$. When Z is not reduced, a failure of ampleness means that Proj A need not be isomorphic to the blow-up of $\mathbb{P}^2$ at Z and, moreover, Proj A need not be smooth. Of course, this also shows that the analogy between commutative and noncommutative blowups is less precise in the non-generic situation.

Notation 10.1. We start with the relevant notation, which will be fixed throughout the section. Let $T = S^{(3)}$ be a Sklyanin elliptic algebra, as defined in Example 4.2 for an automorphism σ of infinite order, with quotient ring $T/gT = B = B(E, \mathcal{M}, \tau)$. Fix a point $p \in E$. Following [16] we blow up p once to get a ring $R' = T(p)$ and then blow up R' at p again to give the ring $R = R'(p) = T(2p)$.

Let $L = R/J$ be the exceptional R-line module, with line ideal J, corresponding to the extension $R \subset R'$; this exists by [16, Lemma 9.1]. Similarly, let L' be the exceptional R'-line module corresponding to the extension $R' = T(p) \subset T$. Finally, write

$$X = R'_{\leq 1} R \subseteq Y = T_{\leq 1} R \subseteq Z = T_{\leq 1} R' \subseteq T.$$

The following fact, due to Simon Crawford, will be used several times.

Proposition 10.1 ([8]). *The localised algebra $R^\circ = R[g^{-1}]_0$ is simple.*

We note the following useful facts about the line ideals L and L'.

Lemma 10.2. (1) *The line module L has no proper g-torsionfree factor R-modules; equivalently, L° is a simple R°-module.*

(2) *As R-modules, $Z/R' = (Y + R')/R' \cong L[-1]$ and hence $L \cong L'$.*

Proof. (1) By Proposition 10.1, R° has no finite-dimensional modules. The simplicity of L° is then a consequence of the 1-criticality of L°, which follows from Lemma 5.2.

(2) We first make some calculations in the Sklyanin algebra S. Recall that S_1 may be identified with $H^0(E, \mathcal{L})$ for some invertible sheaf $\mathcal{L}$ on E of degree 3. For $r \in E$, let $W(r) = H^0(E, \mathcal{L}(-r)) \subset S_1$. Then $S_1 W(r) = W(\sigma^{-1}(r)) S_1$, by [16, Lemma 4.1], while $R'_1 = W(p) S_2$ by [16, Lemma 4.2]. We also have $R_1 = W(p) W(\sigma(p)) S_1$ by [16, Lemma 4.6].

Let $V = W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p))$. We show next that $J_1 = V$. We calculate that

$$\begin{aligned} R'_1 V &= W(p)S_2 W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p)) \\ &= W(p)W(\sigma(p))S_1 W(p)W(\sigma(p))S_1 = R_2. \end{aligned}$$

We have $X/R = R'_{\leq 1}R/R \cong L[-1] \cong (R/J)[-1]$, as a consequence of Theorem 8.3, since R' is the blowdown of R along L by Theorem 8.6. Thus $J = \{x \in R : R'_1 x \subseteq R\}$ and so $V \subseteq J_1$. The Hilbert series of J is known, and $\dim_{\Bbbk} J_1 = 6$. On the other hand, using Lemma 2.2 we calculate in $B(E, \mathcal{M}, \tau)$ that $\dim_{\Bbbk} \overline{V} = 6$. Thus $V = J_1$.

We now claim that

$$T_1 J \subseteq X. \tag{10.2}$$

This follows from the calculation

$$\begin{aligned} T_1 J_1 &= S_3 W(\sigma^3(p))W(\sigma(p))W(\sigma^2(p)) \\ &= W(p)S_2 W(p)W(\sigma(p))S_1 = R'_1 R_1 = X_2 \end{aligned} \tag{10.3}$$

and the fact that by Lemma 5.6(2), J is generated in degree one as a right R-ideal.

Now since T is the blowdown of R' along $L' = R'/J'$, similarly to the above we obtain

$$Z/R' = (T_{\leq 1}R')/R' \cong L'[-1] \quad \text{and} \quad J' = \{x \in R' : T_1 x \subseteq R'\}.$$

Since $X \subseteq R'$, (10.2) gives $J \subseteq J'$. By construction, $\dim_{\Bbbk} T_1/R'_1 = 1$, so write $T_1 = R'_1 \oplus \Bbbk a$ for some $a \in T_1$. As $J \subseteq J'$, there is a nonzero homomorphism

$$\pi : L[-1] \cong (R/J)[-1] \to Z/R' \cong L'[-1]$$

sending 1 to a. Since L° is simple, L has no proper g-torsionfree factors, and so π is injective. By comparing Hilbert series, it is an isomorphism. □

We further have:

Lemma 10.3. *As R°-modules, Y° is projective, while J° is not.*

Proof. Let $q = \tau(p)$. By Theorem 8.6, $\underline{\mathrm{End}}_R(J) \cong F = R'(q)$. We now pass to the ring R° and notice that, by standard localisation theory, $F^\circ = \mathrm{End}_{R^\circ}(J^\circ)$. Moreover, $F = T(p + \tau(p))$, the blowup of T at two consecutive points on a τ-orbit, is shown in [16, Proposition 11.2(1)] to have a proper ideal I such that F/I is g-torsionfree. Thus I° is a proper ideal of F° and F° is not simple, whereas by Proposition 10.1 R° is simple. In particular, $R^\circ J^\circ = R^\circ$ and thus J° is an R°-generator; since F° is not Morita equivalent to R°, it follows that J° is not projective as a right R° module.

Now let $\ell = \tau^{-1}(p)$. We claim that $\underline{\mathrm{End}}_R(Y) = T(2\ell)$. This will complete the proof of the lemma since now $\mathrm{End}_{R^\circ}(Y^\circ) \cong T(2\ell)^\circ$, which is again simple by Proposition 10.1. By the Dual Basis Lemma, Y° is therefore projective as a right R°-module.

In order to prove the claim, we note that, from the formulæ from [16, Lemmas 4.1 and 4.6] noted in the proof of Lemma 10.2,

$$\begin{aligned} T(2\ell)_1 T_1 &= W(\sigma^{-3}(p))W(\sigma^{-2}(p))S_1S_3 \\ &= S_3W(p)W(\sigma(p))S_1 = T_1R_1. \end{aligned} \tag{10.4}$$

Moreover, as $T(2\ell) \subseteq T$, certainly $T(2\ell)_1 R \subseteq T_1 R$ and hence

$$T(2\ell)_1 Y = T(2\ell)(R + T_1 R) \subseteq Y.$$

Since $T(2\ell)$ is generated in degree one by definition, it follows that $T(2\ell) \subseteq \underline{\mathrm{End}}_R(Y)$. Equation (10.4) also implies by induction that $T(2\ell)_n T_1 = T_1 R_n$ for all $n \geq 0$, so $T(2\ell)T_1 = T_1 R$. It follows that $T_1 R$ is a finitely generated left $T(2\ell)$-module. In particular, writing a $\Bbbk$-basis $\{x_i\}$ of T_1 as fractions $x_i = y_i z^{-1}$ with a common denominator, where $y_i, z \in T(2\ell)$, we see that $T_1 Rz \subseteq T(2\ell)$, and then $Yz = (k + T_1 R)z \subseteq T(2\ell)$. Thus

$$\underline{\mathrm{End}}_R(Y)Yz \subseteq Yz \subseteq T(2\ell),$$

which means that $\underline{\mathrm{End}}_R(Y)$ and $T(2\ell)$ are equivalent orders. Since $T(2\ell)$ is a maximal order by [16, Theorem 1.1(2)], the inclusion $T(2\ell) \subseteq \underline{\mathrm{End}}_R(Y)$ is actually an equality. $\square$

We next show that $(L{\bullet}L) \neq -1$. In fact, we prove:

Lemma 10.4. *There is a nonsplit exact sequence*

$$0 \to L[-1] \to Y/R \to L[-1] \to 0. \tag{10.5}$$

Proof. By (10.2), $T_1 J \subseteq X$. Thus there is a homomorphism

$$\pi : (R/J)[-1] \to Y/X = T_1 R/R'_1 R$$

which sends 1 to a, where $T_1 = R'_1 \oplus \Bbbk a$. Since $T_1 R = R'_1 R + aR$, π is surjective. Now since $R'/X \cong \bigoplus_{i\geq 2} L[-i]$ as right R-modules by Theorem 8.3, R'/X is g-torsionfree. Since R' is g-divisible, $Q_{gr}(T)/R'$ is g-torsionfree, and so $Q_{gr}(T)/X$ and thus Y/X are also g-torsionfree. As noted in the proof of Lemma 10.2, L has no proper g-torsionfree factor modules, and this forces π to be injective as well. Thus $Y/X \cong L[-1]$ as right R-modules.

We saw that $X/R \cong L[-1]$ in the proof of Lemma 10.2, and so the exact sequence (10.5) exists as claimed. Finally, localising (10.5) gives the exact sequence

$$0 \to L^\circ \to (Y/R)^\circ \to L^\circ \to 0.$$

By Lemma 10.3, $\operatorname{pdim} L^\circ > 1 = \operatorname{pdim}(Y^\circ/R^\circ)$. Thus, neither this sequence nor (10.5) is split. $\square$

Corollary 10.5. *Let $R = T(2p)$ with exceptional line module L, as above. Then $(L{\bullet}L) \neq -1$. On the other hand, $\mathsf{hilb}\, \underline{\mathrm{End}}_R(J) = \mathsf{hilb}\, R$ and so, by Theorem 8.3, one can blow down L.*

Proof. By Lemmas 5.5 and 6.2, $\mathrm{End}_{R^\circ}(L^\circ) = \Bbbk$. On the other hand, $\mathrm{Ext}^1_{R^\circ}(L^\circ, L^\circ) \neq 0$ by Lemma 10.4. Thus, by Lemma 6.2, $(L{\bullet}L) \geq 0 > -1$. Finally, by Theorem 8.6, $\underline{\mathrm{End}}_R(J) = R'(q)$, and so the equality $\mathsf{hilb}\, \underline{\mathrm{End}}_R(J) = \mathsf{hilb}\, R$ follows from [16, Theorem 1.1(1)]. $\square$

The next result shows that there is a particularly interesting self-extension of J°.

Proposition 10.6. *There is a nonsplit exact sequence*

$$0 \to J^\circ \to P \to J^\circ \to 0 \tag{10.6}$$

of R°-modules, where P is projective.

Proof. Recall that

$$X^\circ/R^\circ \cong Y^\circ/X^\circ \cong R^\circ/J^\circ \cong L^\circ,$$

from the proof of Lemma 10.4. Thus the localisation of (10.5) can be written as:

$$0 \to X^\circ/R^\circ \to Y^\circ/R^\circ \to Y^\circ/X^\circ \to 0. \tag{10.7}$$

The natural surjection $\phi' : R^\circ \to R^\circ/J^\circ \cong Y^\circ/X^\circ$ lifts to a homomorphism $\phi : R^\circ \to Y^\circ/R^\circ$, which must be surjective as L° is simple. As was shown in the proof of Lemma 10.4, (10.7) is nonsplit. Let $K = \ker\phi$; thus K is projective since $\operatorname{pdim}(Y^\circ/R^\circ) = 1$.

Clearly $K \subseteq J^\circ = \ker(\phi')$ and $J^\circ/K \cong L^\circ$. This isomorphism lifts to a map $\theta' : R^\circ \to J^\circ$ so that $\theta'(R^\circ) + K = J^\circ$. This induces a surjective homomorphism $\theta : R^\circ \oplus K \to J^\circ$. It is routine to check that

$$\ker\theta = \big\{(r,k) \in R^\circ \oplus K : \theta'(r) = k\big\}$$

and that as an R°-module this is isomorphic to $(\theta')^{-1}(K) = J^\circ$. Thus we have constructed the sequence (10.6) with $P = R^\circ \oplus K$. As $\operatorname{pdim} J^\circ > \operatorname{pdim} P$, it does not split. $\square$

We now examine the higher Ext groups from L° to itself; the ultimate aim being to show that $(L \bullet_{MS} L)$ is undefined.

Lemma 10.7. *Keep the above notation. Then:*

(1) $\mathrm{Ext}^n_{R^\circ}(J^\circ, J^\circ) \cong \mathrm{Ext}^{n+1}_{R^\circ}(J^\circ, J^\circ) \neq 0$ *for $n \geq 1$.*

(2) $\mathrm{Ext}^n_{R^\circ}(J^\circ, J^\circ) \cong \mathrm{Ext}^{n-1}_{R^\circ}(J^\circ, L^\circ) \cong \mathrm{Ext}^n_{R^\circ}(L^\circ, L^\circ)$ *for all $n \geq 2$.*

(3) *In particular,* $\operatorname{Ext}^n_{R^\circ}(L^\circ, L^\circ) \cong \operatorname{Ext}^{n+1}_{R^\circ}(L^\circ, L^\circ) \neq 0$ *for all* $n \geq 2$.

Proof. (1) Applying $\operatorname{Hom}_{R^\circ}(-, J^\circ)$ to (10.6) gives the exact sequence

$$\operatorname{Ext}^m_{R^\circ}\left(P, J^\circ\right) \to \operatorname{Ext}^m_{R^\circ}\left(J^\circ, J^\circ\right) \to \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, J^\circ\right) \to \operatorname{Ext}^{m+1}_{R^\circ}\left(P, J^\circ\right) \quad (10.8)$$

for $m \geq 1$. As P is projective, it follows that

$$\operatorname{Ext}^m_{R^\circ}\left(J^\circ, J^\circ\right) \cong \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, J^\circ\right)$$

for $m \geq 1$. Moreover, (10.6) ensures that $\operatorname{Ext}^1_{R^\circ}(J^\circ, J^\circ) \neq 0$ and hence

$$\operatorname{Ext}^m_{R^\circ}\left(J^\circ, J^\circ\right) \neq 0$$

for $m \geq 1$.

(2,3) Applying $\operatorname{Hom}_{R^\circ}(J^\circ, -)$ to

$$0 \to J^\circ \to R^\circ \to L^\circ \to 0$$

gives

$$\operatorname{Ext}^m_{R^\circ}\left(J^\circ, R^\circ\right) \to \operatorname{Ext}^m_{R^\circ}\left(J^\circ, L^\circ\right) \to \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, J^\circ\right) \to \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, R^\circ\right)$$

for all $m \geq 1$. Now J° is CM by Lemma 5.6(2) and so, as $m \geq 1$, the outside terms are zero in this equation. Hence,

$$\operatorname{Ext}^m_{R^\circ}\left(J^\circ, L^\circ\right) \cong \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, J^\circ\right) \cong \operatorname{Ext}^{m+2}_{R^\circ}\left(J^\circ, J^\circ\right) \cong \operatorname{Ext}^{m+1}_{R^\circ}\left(J^\circ, L^\circ\right) \quad (10.9)$$

for all $m \geq 1$. By part (1) these groups are also non-zero.

From the exact sequence

$$0 \to J^\circ \to R^\circ \to L^\circ \to 0,$$

one also obtains

$$\operatorname{Ext}^m_{R^\circ}\left(J^\circ, L^\circ\right) \cong \operatorname{Ext}^{m+1}_{R^\circ}\left(L^\circ, L^\circ\right)$$

for $m \geq 1$. Combined with (10.9) this implies that

$$\operatorname{Ext}^s_{R^\circ}\left(L^\circ, L^\circ\right) \cong \operatorname{Ext}^{s-1}_{R^\circ}\left(J^\circ, L^\circ\right) \cong \operatorname{Ext}^1_{R^\circ}\left(J^\circ, L^\circ\right) \cong \operatorname{Ext}^2_{R^\circ}\left(L^\circ, L^\circ\right)$$

for all $s \geq 2$. Finally, by (10.9) and part (1),

$$\operatorname{Ext}^s_{R^\circ}\left(L^\circ, L^\circ\right) \cong \operatorname{Ext}^{s-1}_{R^\circ}\left(J^\circ, L^\circ\right) \cong \operatorname{Ext}^s_{R^\circ}\left(J^\circ, J^\circ\right) \neq 0,$$

for all $s \geq 2$. $\square$

Finally, by combining Lemma 10.7 with Proposition 6.4, we get the promised example of an undefined self-intersection.

Corollary 10.8. *Let* $R = T(2p)$ *as above, with exceptional line module* L. *Then the self-intersection* $(L \bullet_{MS} L)$ *is an infinite sum and hence is undefined. Further,* $\operatorname{gldim} R^\circ = \infty$ *and so* qgr-R *is not smooth.* $\square$

Index of notation

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