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Crossed products by compact group actions with the Rokhlin property

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Abstract. We present a systematic study of the structure of crossed products and fixed point algebras by compact group actions with the Rokhlin property on not necessarily unital C^*-algebras. Our main technical result is the existence of an approximate homomorphism from the algebra to its subalgebra of fixed points, which is a left inverse for the canonical inclusion. Upon combining this with results regarding local approximations, we show that a number of classes characterized by inductive limit decompositions with weakly semiprojective building blocks, are closed under formation of crossed products by such actions. Similarly, in the presence of the Rokhlin property, if the algebra has any of the following properties, then so do the crossed product and the fixed point algebra: being a Kirchberg algebra, being simple and having tracial rank zero or one, having real rank zero, having stable rank one, absorbing a strongly self-absorbing C^*-algebra, satisfying the Universal Coefficient Theorem (in the simple, nuclear case), and being weakly semiprojective. The ideal structure of crossed products and fixed point algebras by Rokhlin actions is also studied.

The methods of this paper unify, under a single conceptual approach, the work of a number of authors, who used rather different techniques. Our methods yield new results even in the well-studied case of finite group actions with the Rokhlin property.

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1. Introduction

The Rokhlin property first appeared in the late 1970's and early 1980's, in work of Fack and Maréchal [6], Kishimoto [23], and Herman and Jones [14] on cyclic group actions on UHF-algebras, and in the work of Herman and Ocneanu [15] on integer actions on UHF-algebras.

In [18], Izumi provided a formal definition of the Rokhlin property for an arbitrary finite group action on a unital C^*-algebra. His classification theorems for Rokhlin actions [18,19] are among the major results in the study of finite group actions.

In a different direction, Izumi [18], Hirshberg and Winter [17], Phillips [39], Osaka and Phillips [35], and Pasnicu and Phillips [36], explored the structure of crossed products by finite group actions with the Rokhlin property on unital C^*-algebras, while Santiago [43] addressed similar questions in the non-unital case. The questions and problems addressed in each of these works are different, and consequently the approaches used by the above mentioned authors are substantially distinct in some cases.

In [17], Hirshberg and Winter also introduced the Rokhlin property for a compact group action on a unital C^*-algebra, and their definition coincides with Izumi's in the case of finite groups. They showed that approximate divisibility and $\mathcal{D}$-stability, for a strongly self-absorbing C^*-algebra $\mathcal{D}$, are preserved under formation of crossed product by compact group actions with the Rokhlin property. Extending the results of [35,36,39] to the case of arbitrary compact groups requires new insights, since the main technical tool in all of these works (Theorem 3.2 in [35]) seems not to have a satisfactory analog in the compact group case.

In this paper, we extend the definition of Hirshberg-Winter to actions of compact groups on σ-unital C^*-algebras, and generalize the results on finite group actions with the Rokhlin property of the above mentioned papers to the case of compact group actions. Our contribution is two-fold. First, most of the results we prove here were known only in some special cases (mostly for finite or circle group actions; see [7] and [8] for the circle case), and some of them had not been noticed even in the context of finite groups. Additionally, we do not require our C^*-algebras to be unital, unlike in [17,18,35], or [36]. Finally, our methods represent a uniform treatment of the study of crossed products by actions with the Rokhlin property, where the attention is shifted from the crossed product itself, to the algebra of fixed points.

Our results can be summarized as follows (the list is not exhaustive). We point out that (14) below was first obtained, with different techniques and for unital C^*-algebras, by Hirshberg and Winter as part (1) of Corollary 3.4 in [17]. Also, (10) and (14) were obtained in [9].

Theorem. *The following classes of σ-unital C^*-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of second-countable compact groups with the Rokhlin property:*

(1) *Simple C^*-algebras (Corollary 2.14). More generally, the ideal structure can be completely determined (Theorem 2.13);*

(2) *C^*-algebras that are direct limits of certain weakly semiprojective C^*-algebras (Theorem 3.10). This includes UHF-algebras (or matroid algebras), AF-algebras, AI-algebras, $A\mathbb{T}$-algebras, countable inductive limits of one-dimensional NCCW-complexes, and several other classes (Corollary 3.11);*

(3) *Kirchberg algebras (Corollary 4.11);*

(4) *Simple C^*-algebras with tracial rank at most one (Theorem 4.5);*

(5) *Simple, separable, nuclear C^*-algebras satisfying the Universal Coefficient Theorem (Theorem 3.13);*

(6) *C^*-algebras with nuclear dimension at most n, for $n \in \mathbb{N}$ (Theorem 2.9);*

(7) *C^*-algebras with decomposition rank at most n, for $n \in \mathbb{N}$ (Theorem 2.9);*

(8) *C^*-algebras with real rank zero (Proposition 4.13);*

(9) *C^*-algebras with stable rank one (Proposition 4.13);*

(10) *C^*-algebras with strict comparison of positive elements (Corollary 3.19 in [9]);*

(11) *C^*-algebras whose order on projections is determined by traces (Propsition 4.15);*

(12) *(Not necessarily simple) purely infinite C^*-algebras (Proposition 4.10);*

(13) *Separable $\mathcal{D}$-absorbing C^*-algebras, for a strongly self-absorbing C^*-algebra $\mathcal{D}$ (Theorem 4.3);*

(14) *C^*-algebras whose K-groups are either: trivial, free, torsion-free, torsion, or finitely generated (Corollary 3.4 in [9]);*

(15) *Weakly semiprojective C^*-algebras (Proposition 4.19).*

Our work yields new results even in the case of finite groups. For example, in (14) above, we do not require the algebra A to be simple, unlike in Theorem 3.13 of [18]. In addition, the classes of C^*-algebras considered in Theorem 3.10 may consist of simple C^*-algebras, unlike in Theorem 3.5 in [35] (we also do not impose any conditions regarding corners of our algebras). Additionally, in Proposition 4.19, we show that the inclusion $A^\alpha \to A$ is sequence algebra extendible (Definition 4.16) whenever α has the Rokhlin property, and hence weak semiprojectivity passes from A to A^α. Our conclusion seems not to be obtainable with the methods developed in [35] and related works, since it is not in general true that a corner of a weakly semiprojective C^*-algebra is weakly semiprojective.

Given that our results all follow as easy consequences of our main technical observation, Theorem 2.11, which allows us to deal with the non-unital case as well, we believe that this paper unifies the work of a number of authors, who used rather different methods, under a single systematic and conceptual approach.

In this paper, we take $\mathbb{N} = \{1, 2, \ldots\}$.

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2. An averaging process

We begin by introducing some useful notation and terminology.

2.1. Central sequence algebras and Rokhlin property. Given a C^*-algebra A, let $\ell^\infty(\mathbb{N}, A)$ denote the set of all bounded sequences in A with the supremum norm and pointwise operations. Then $\ell^\infty(\mathbb{N}, A)$ is a C^*-algebra, and it is unital if A is σ-unital, since any countable approximate unit for A determines a unit for $\ell^\infty(\mathbb{N}, A)$. Set

$$c_0(\mathbb{N}, A) = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\}.$$

Then $c_0(\mathbb{N},A)$ is an ideal in $\ell^\infty(\mathbb{N},A)$, and we denote the quotient $\ell^\infty(\mathbb{N},A)/c_0(\mathbb{N},A)$ by A_∞. We write $\eta_A\colon \ell^\infty(\mathbb{N}, A) \to A_\infty$ for the quotient map. We identify A with the subalgebra of $\ell^\infty(\mathbb{N}, A)$ consisting of the constant sequences, and with a subalgebra of A_∞ by taking its image under η_A. If D is any subalgebra of A, then $A_\infty \cap D'$ denotes the relative commutant of D inside of A_∞.

Definition 2.1. For a subalgebra $D \subseteq A$, write $\mathrm{Ann}(D, A_\infty)$ for the annihilator of D in A_∞, which is an ideal in $A_\infty \cap D'$. Following Kirchberg ([20]), we set

$$F(D, A) = A_\infty \cap D'/\mathrm{Ann}(D, A_\infty),$$

and write $\kappa_{D,A}\colon A_\infty \cap D' \to F(D, A)$ for the quotient map. When $D = A$, we abbreviate $F(A, A)$ and $\kappa_{D,A}$ to $F(A)$ and κ_A.

If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A, and D is an α-invariant subalgebra of A, then there are (not necessarily continuous) actions of G on $\ell^\infty(\mathbb{N}, A)$, on A_∞, on $A_\infty \cap D'$ and on $F(D, A)$, respectively denoted, with a slight abuse of notation, by α^∞, α_∞, α_∞ and $F(\alpha)$. Following Kishimoto ([21]), we set

$$\ell^\infty_\alpha(\mathbb{N}, A) = \{a \in \ell^\infty(\mathbb{N}, A) : g \mapsto \alpha^\infty_g(a) \text{ is continuous}\}.$$

We also set $A_{\infty,\alpha} = \eta_A(\ell^\infty_\alpha(\mathbb{N}, A))$ and $F_\alpha(A) = \kappa_{D,A}(A_{\infty,\alpha} \cap D')$.

By construction, $A_{\infty,\alpha}$ and $F_\alpha(D, A)$ are invariant under α_∞ and $F(\alpha)$, so the restrictions of α_∞ and $F(\alpha)$ to $A_{\infty,\alpha}$ and $F_\alpha(D, A)$, which we also denote by α_∞ and $F(\alpha)$, are continuous.

If G is a locally compact group, we denote by $\mathtt{Lt}\colon G \to \mathrm{Aut}(C_0(G))$ the action induced by left translation of G on itself.

The following generalizes Definition 3.2 of [17] to the σ-unital setting. (See Definition 3.2 in [34] for the case of finite groups.) It should also be compared with Definition 1.6 in [45].

Definition 2.2. Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We say that α has the *Rokhlin property* if for every separable α-invariant subalgebra $D \subseteq A$, there is an equivariant unital homomorphism

$$\varphi\colon (C(G), \mathtt{Lt}) \to (F_\alpha(D, A), F(\alpha)).$$

A number of features of the Rokhlin property are studied in [9]. Here, we shall focus on the crossed products and fixed point algebras, with emphasis on their structure and classifiability.

We will repeatedly use the following fact, which is probably standard. Its proof can be found, for example, in [12]. For compact G, we identify $C(G, A)$ and $C(G) \otimes A$ in the usual way.

Proposition 2.3. *Let G be a compact group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Define a homomorphism $\sigma\colon C(G, A) \to C(G, A)$ by $\sigma(a)(g) = \alpha_g(a(g))$ for $a \in C(G, A)$ and $g \in G$. Then*

$$\sigma\colon (C(G, A), \mathtt{Lt} \otimes \mathrm{id}_A) \to (C(G, A), \mathtt{Lt} \otimes \alpha)$$

is an equivariant isomorphism.

We need an easy lifting result. We thank Luis Santiago for pointing it out to us.

Lemma 2.4. *Let G be a locally compact group, let C and A be C^*-algebras, and let $\gamma\colon G \to \mathrm{Aut}(C)$ and $\alpha\colon G \to \mathrm{Aut}(A)$ be actions, and let $D \subseteq A$ be an invariant subalgebra. Give $C \otimes_{\max} A$ the diagonal G-action. Suppose that there exists a unital equivariant homomorphism $\varphi\colon C \to F_\alpha(D, A)$, and choose any function $\theta\colon C \to A_{\infty,\alpha} \cap D'$ satisfying $\kappa_A \circ \theta = \varphi$. Then there exists an equivariant homomorphism*

$$\psi\colon C \otimes_{\max} D \to A_{\infty,\alpha}$$

determined by $\psi(c \otimes d) = \theta(c)d$ for all $c \in C$ and all $a \in A$. Moreover, ψ does not depend on θ.

Proof. We check that ψ is indeed a homomorphism. Let $c_1, c_2 \in C$ and $d_1, d_2 \in A$ be given. Using that $\theta(c_1 c_1)x = \theta(c_1)\theta(c_2)x$ for any $x \in D$ at the second step, and that $\theta(C)$ commutes with D at the third step, we get

$$\begin{aligned}\psi(c_1 c_2 \otimes d_1 d_1) = \theta(c_1 c_2)d_1 d_1 &= \theta(c_1)\theta(c_2)d_1 d_1 = \theta(c_1)d_1\theta(c_2)d_2 \\ &= \psi(c_1 \otimes d_1)\psi(c_2 \otimes d_2),\end{aligned}$$

as desired. Finally, if $\widetilde{\theta}$ is another lift of φ, then clearly $\widetilde{\theta}(c)d = \theta(c)d$ for all $c \in C$ and all $a \in D$, which shows that ψ does not depend on the lift of φ. □

2.2. First results on crossed product and the averaging process. If a compact group G acts on a C^*-algebra A, then A^G is naturally a corner in $A \rtimes G$ (see the Theorem in [42]), even though A is itself not in general a subalgebra of $A \rtimes G$. (When G is discrete, there is a different way of regarding A^G as a subalgebra of the crossed product, since A always sits inside $A \rtimes G$. When G is finite, these two inclusions never agree when G is not trivial, and we will exclusively deal with the corner inclusion considered by Rosenberg.)

Using this corner inclusion, one can many times obtain information about the fixed point algebra through the crossed product. However, since this corner is not in general full, Rosenberg's theorem is not always useful if one is interested in transferring structure from A^G to $A \rtimes G$. Saturation for compact group actions is the basic notion that allows one to do this, up to Morita equivalence. The definition, which is essentially due to Rieffel, is as in Definition 7.1.4 in [38]. What we reproduce below is the equivalent formulation given in Lemma 7.1.9 in [38]. We point out that saturation is equivalent to the corner $A^G \subseteq A \rtimes G$ being full.

Definition 2.5 ([38, Definition 7.1.4]). Let G be a compact group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α is *saturated*, if the set

$$\left\{f_{a,b}\colon G \to A;\, f_{a,b}(g) = a\alpha_g(b) \text{ for all } g \in G, \text{ with } a, b \in A\right\} \subseteq L^1(G, A, \alpha)$$

spans a dense subspace of $A \rtimes_\alpha G$.

It is an easy exercise to check that if a compact group G acts freely on a compact Hausdorff space X, then the induced action on $C(X)$ is saturated. For this, it suffices to prove that the set

$$\left\{\begin{array}{l} f_{a,b} \in C(G \times X) : f_{a,b}(g, x) = a(x)b(g \cdot x) \\ \qquad\qquad\qquad\qquad \text{for all } (g, x) \in G \times X, \text{ with } a, b \in C(X) \end{array}\right\}$$

spans a dense subset of $C(G \times X)$. This linear span is closed under multiplication and contains the constant functions regardless of whether the action of G is free or not, and it is easy to see that it separates the points of $G \times X$ if and only if it is free. The claim then follows from the Stone–Weierstrass theorem. See Theorem 7.2.6 in [38] for a more general result involving $C(X)$-algebras.

Lemma 2.6. *Let $\beta : G \to \mathrm{Aut}(C)$ be a saturated action of a compact group G on a nuclear C^*-algebra C, and let $\mathrm{id}_D\colon G \to \mathrm{Aut}(D)$ denote the trivial action. Then the diagonal action*

$$\gamma = \beta \otimes \mathrm{id}_D\colon G \to \mathrm{Aut}(C \otimes D)$$

is also saturated.

Proof. Then there are canonical identifications

$$(C \otimes D)^\gamma \cong C^\beta \otimes D \quad \text{and} \quad (C \otimes D) \rtimes_\gamma G \cong (C \rtimes_\beta G) \otimes D.$$

Denote by $\iota_C\colon C^\beta \to C \rtimes_\beta G$ the canonical inclusion (see comments above Definition 2.5). Observe that the saturation of β is equivalent to the hereditary subalgebra generated by $\iota_C(C^\beta)$ being all of $C \rtimes_\beta G$ (see [38, Lemma 7.1.9]). Under the above identifications, the inclusion

$$(C \otimes D)^\gamma \hookrightarrow (C \otimes D) \rtimes_\gamma G$$

corresponds to the map

$$\iota_C \otimes \mathrm{id}_D : C^\beta \otimes D \to (C \rtimes_\beta G) \otimes D.$$

Hence the image of $(C \otimes D)^\gamma$ generates all of $(C \otimes D) \rtimes_\gamma G$ as a hereditary subalgebra. We conclude that γ is saturated. □

The following result will be used repeatedly throughout this paper.

Proposition 2.7. *Let G be a second-countable compact group, let A be a σ-unital C^*-algebra, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then α is saturated.*

In particular, the fixed point algebra and the crossed product by a compact group action with the Rokhlin property are Morita equivalent, and thus stably isomorphic whenever the original algebra is separable.

Proof. We begin by proving the statement for finite G and unital, separable A, because we believe the reader will gain better intuition from this particular case. Indeed, finiteness of G allows one to construct the approximations explicitly.

Suppose that G is finite and A is unital and separable. Fix $g \in G$, and denote by u_g the canonical unitary in the crossed product $A \rtimes_\alpha G$ implementing α_g. We claim that it is enough to show that u_g is in the closed linear span of the functions $f_{a,b}$ from Definition 2.5. Indeed, if this is the case, and if $x \in A$, then xu_g also belongs to the closed linear span, and elements of this form span $A \rtimes_\alpha G$.

For $n \in \mathbb{N}$, find projections $e_g^{(n)} \in A$, for $g \in G$, such that

(1) $\|\alpha_g(e_h^{(n)}) - e_{gh}^{(n)}\| < \frac{1}{n}$ for all $g, h \in G$; and

(2) $\sum_{g \in G} e_g^{(n)} = 1$.

For $a, b \in A$, the function $f_{a,b}$ corresponds to the product $a\big(\sum_{h\in G} \alpha_h(b)u_h\big)$. Thus, for $n \in \mathbb{N}$ and $k \in G$, we have

$$f_{e_{gk}^{(n)}, e_k^{(n)}} = e_{gk}^{(n)}\bigg(\sum_{h \in G} \alpha_h\big(e_k^{(n)}\big)u_h\bigg).$$

We use pairwise orthogonality of the projections $e_g^{(n)}$, for $g \in G$, at the third step, to get

$$\begin{aligned} \left\| f_{e_{gk}^{(n)}, e_k^{(n)}} - e_{gk}^{(n)} u_g \right\| &= \left\| e_{gk}^{(n)} \left(\sum_{h \in G} e_{gk}^{(n)} \alpha_h(e_k^{(n)}) u_h \right) - e_{gk}^{(n)} u_g \right\| \\ &\leq \left\| e_{gk}^{(n)} \alpha_g(e_k^{(n)}) u_h - e_{gk}^{(n)} u_h \right\| + \sum_{h \in G, h \neq g} \left\| e_{gk}^{(n)} \alpha_h(e_k^{(n)}) u_h \right\| \\ &< \left\| \alpha_g(e_k^{(n)}) - e_{gk}^{(n)} \right\| + \sum_{h \in G, h \neq g} \left\| \alpha_h(e_k^{(n)}) - e_{hk}^{(n)} \right\| \\ &< \frac{1}{n} + (|G| - 1)\frac{1}{n} = \frac{|G|}{n}. \end{aligned}$$

It follows from condition (2) above that

$$\limsup_{n \to \infty} \left\| \sum_{k \in G} f_{e_{gk}^{(n)}, e_k^{(n)}} - u_g \right\| \leq \limsup_{n \to \infty} \frac{|G|^2}{n} = 0.$$

Hence u_g belongs to the closed linear span of the $f_{a,b}$, and α is saturated.

For G compact and second countable, we are not able to describe the approximating functions $f_{a,b}$ explicitly. (In fact, their existence is a consequence of the Stone–Weierstrass theorem.) Our proof consists in showing that one can build approximating functions in $A \rtimes_\alpha G$ using approximating functions in $C(G) \rtimes_{\mathtt{Lt}} G$.

Suppose that G is compact and A is σ-unital. For an α-invariant subalgebra $D \subseteq A$, denote by $\gamma_D : G \to \mathrm{Aut}(C(G, D))$ the diagonal action $\gamma = \mathtt{Lt} \otimes \alpha|_D$. Then γ is conjugate to $\mathtt{Lt} \otimes \mathrm{id}_D$ by Proposition 2.3. Since $\mathtt{Lt}$ is saturated (see the comments after Definition 2.5), the action $\mathtt{Lt} \otimes \mathrm{id}_D$ is saturated by Lemma 2.6. We deduce that γ_D is also saturated.

Since $\|\cdot\|_{L^1(G,A,\alpha)}$ dominates $\|\cdot\|_{A \rtimes_\alpha G}$, it is enough to show that the span of the functions $f_{a,b}$, with $a, b \in A$, is dense in $L^1(G, A, \alpha)$. Denote by χ_E the characteristic function of a Borel set $E \subseteq G$. It is a standard fact that the linear span of

$$\{x \chi_E : x \in A, E \subseteq G \text{ Borel}\}$$

is dense in $L^1(G, A, \alpha)$. So fix $x \in A$ and a Borel subset $E \subseteq G$. Fix $\varepsilon > 0$. Using that γ_A is saturated, choose $m \in \mathbb{N}$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in C(G, A)$ such that

$$\left\| \sum_{j=1}^m f_{a_j, b_j} - x \chi_E \right\| < \varepsilon,$$

where the norm is taken in $C(G, A) \rtimes_\gamma G$.

Denote by D the separable, α-invariant subalgebra of A generated by the set $\{a_j, b_j : j = 1, \ldots, m\}$. Let $\varphi : C(G) \to F_\alpha(D, A)$ be a unital equivariant homomorphism as in the definition of the Rokhlin property for α. Let $\psi : C(G, D) \to A_{\infty,\alpha}$

be the equivariant homomorphism given by Lemma 2.4. Write

$$\widehat{\psi}\colon C(G,D)\rtimes_{\gamma_D} G \to A_{\infty,\alpha}\rtimes_{\alpha_\infty} G,$$

for the induced map at the level of the crossed products. Under the canonical embedding

$$A_{\infty,\alpha}\rtimes_{\alpha_\infty} G \hookrightarrow (A\rtimes_\alpha G)_\infty$$

provided by [10, Proposition 2.1], we will regard $\widehat{\psi}$ as a homomorphism

$$\widehat{\psi}\colon C(G,D)\rtimes_{\gamma} G \to (A\rtimes_\alpha G)_\infty.$$

It is clear that $\widehat{\psi}(f_{a_j,b_j}) = f_{\psi(a_j),\psi(b_j)}$ for all $j = 1,\dots,m$, and that $\widehat{\psi}(x\chi_E) = x\chi_E$. Hence

$$\begin{aligned}\Big\| \sum_{j=1}^m f_{\psi(a_j),\psi(b_j)} - x\chi_E \Big\|_{(A\rtimes_\alpha G)_\infty} &= \Big\| \widehat{\psi}\Big(\sum_{j=1}^m f_{a_j,b_j} - x\chi_E \Big)\Big\|_{(A\rtimes_\alpha G)_\infty} \\ &\leq \Big\| \sum_{j=1}^m f_{a_j,b_j} - x\chi_E \Big\| < \varepsilon.\end{aligned}$$

To finish the proof, for $j = 1,\dots,m$, choose bounded sequences $(\psi(a_j)_n)_{n\in\mathbb{N}}$ and $(\psi(b_j)_n)_{n\in\mathbb{N}}$ in $C(G,D)$ which represent $\psi(a_j)$ and $\psi(b_j)$, respectively. Then

$$\eta_{A\rtimes_\alpha G}\big(\big(f_{\psi(a_j)_n,\psi(b_j)_n}\big)_{n\in\mathbb{N}}\big) = f_{\psi(a_j),\psi(b_j)}.$$

It follows that for n large enough, we have

$$\Big\| \sum_{j=1}^m f_{\psi(a_j)_n,\psi(b_j)_n} - x\chi_E \Big\|_{A\rtimes_\alpha G} < \varepsilon,$$

showing that α is saturated.

The last part of the statement follows from Rieffel's original definition of saturation ([38, Definition 7.1.4]; see also [38, Proposition 7.1.3]). □

Remark 2.8. In this paper, we will show that a number of properties pass from A to A^α and $A\rtimes_\alpha G$. These properties are all preserved by Morita equivalence. Our strategy will be to show first that the property in question passes to the fixed point algebra. Once this is accomplished, Proposition 2.7 will imply the result for $A\rtimes_\alpha G$. An alternative to this approach is as follows: with $\lambda : G \to \mathcal{U}(L^2(G))$ denoting the left regular representation, the crossed product $A\rtimes_\alpha G$ is isomorphic to the fixed point algebra

$$\big(A\otimes\mathcal{K}\big(L^2(G)\big)\big)^{\alpha\otimes \mathrm{Ad}(\lambda)}.$$

Now, if α has the Rokhlin property, it is immediate to check that so does $\alpha\otimes\mathrm{Ad}(\lambda)$. If the property in question has been shown to pass to fixed point algebras by Rokhlin actions and is invariant under Morita equivalence, then it follows that it also passes to their crossed products.

We recall here that if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a compact group on a σ-unital C^*-algebra A, then we have the following estimates of the nuclear dimension ([48, Definition 2.1]) and decomposition rank ([25, Definition 3.1]) of the crossed product in terms of those of A and the Rokhlin dimension of α ([11, Definition 3.2]):

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (\dim_{\mathrm{nuc}}(A) + 1)(\dim_{\mathrm{Rok}}(\alpha_+1) - 1,$$

and

$$\mathrm{dr}(A \rtimes_\alpha G) \leq (\mathrm{dr}(A) + 1)(\dim_{\mathrm{Rok}}(\alpha_+1) - 1.$$

(For the proofs, see [10, Theorems 3.3 and 3.4] for the case when A is unital, and see [12] for the case of arbitrary σ-unital A.)

Since unital completely positive maps of order zero are necessarily homomorphisms, it is easy to see that the Rokhlin property for a compact group action agrees with Rokhlin dimension zero in the sense of [11, Definition 3.2]. In particular, we deduce the following.

Theorem 2.9. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then*

$$\dim_{\mathrm{nuc}}(A^\alpha) = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \dim_{\mathrm{nuc}}(A),$$

and

$$\mathrm{dr}(A^\alpha) = \mathrm{dr}(A \rtimes_\alpha G) \leq \mathrm{dr}(A).$$

Proof. The equalities $\dim_{\mathrm{nuc}}(A^\alpha) = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G)$ and $\mathrm{dr}(A^\alpha) = \mathrm{dr}(A \rtimes_\alpha G)$ follow from Proposition 2.7, Morita equivalent C^*-algebras have the same nuclear dimension and decomposition rank. The two inequalities follow from the comments before this theorem, since $\dim_{\mathrm{Rok}}(\alpha) = 0$. □

Corollary 2.10. *Let A be an AF-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then A^α and $A \rtimes_\alpha G$ are AF-algebras.*

Proof. Since a separable C^*-algebra has decomposition rank zero if and only if it is an AF-algebra ([25, Example 4.1]), the result follows from Theorem 2.9. □

The following result will be crucial in obtaining further structural properties for crossed products by actions with the Rokhlin property. The proof that we present below was suggested to us by the referee, to whom we are indebted. Our original argument was more technical and involved using certain partitions of unity in $C(G)$ with small enough supports as in [10, Lemma 4.2].

Theorem 2.11. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a compact subset $F_1 \subseteq A$, a compact subset $F_2 \subseteq A^\alpha$ and $\varepsilon > 0$, there exists a completely positive contractive map $\psi\colon A \to A^\alpha$ such that*

(1) *For all $a, b \in F_1$, we have*

$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon;$$

(2) *For all* $a \in F_2$, *we have* $\|\psi(a) - a\| < \varepsilon$.

Moreover, if A *is unital, then we can choose* ψ *so that* $\psi(1) = 1$.

In particular, when A *is separable, there exists an approximate homomorphism* $(\psi_n)_{n\in\mathbb{N}}$ *consisting of completely positive contractive linear maps* $\psi_n\colon A \to A^\alpha$ *for* $n \in \mathbb{N}$, *which can be arranged to be unital if* A *is, such that* $\lim_{n\to\infty} \|\psi_n(a) - a\| = 0$ *for all* $a \in A^\alpha$.

Proof. Denote by D the separable, α-invariant subalgebra generated by $F_1 \cup F_2$. Use the Rokhlin property for α to choose a unital equivariant homomorphism $\varphi\colon C(G) \to F_\alpha(D, A)$. Using Choi–Effros lifting theorem, find a lift $(\varphi_n)_{n\in\mathbb{N}}$ of φ consisting of completely positive, contractive maps $\varphi_n\colon C(G) \to A$, which must then satisfy

(a) $\lim_{n\to\infty} \|\varphi_n(ab)d - \varphi_n(a)\varphi_n(b)d\| = 0$ for all $a, b \in A$ and for all $d \in D$;

(b) $\lim_{n\to\infty} \|\varphi_n(1)d - d\| = 0$ for all $d \in D$ (one can arrange that $\varphi_n(1) = 1$ if A is unital);

(c) $\lim_{n\to\infty} \|\varphi_n(f)d - d\varphi_n(f)\| = 0$ for all $d \in D$ and for all $f \in C(G)$;

(d) $\lim_{n\to\infty} \sup_{g\in G} \|\varphi_n(\mathtt{Lt}_g(f))d - \alpha_g(\varphi_n(f))d\| = 0$ for all $f \in C(G)$ and for all $d \in D$.

(In the last condition, the fact that $\|\varphi_n(\mathtt{Lt}_g(f))d - \alpha_g(\varphi_n(f))d\|$ goes to zero *uniformly* on $g \in G$, and not just pointwise, follows from Dini's theorem using that the image of φ lands in the part of $F(D, A)$ where G acts continuously; see Definition 2.1.)

Denote by μ the normalized Haar measure on G. For $n \in \mathbb{N}$, define $\theta_n\colon C(G) \to A$ by

$$\theta_n(f) = \int_G \alpha_g\big(\varphi_n\big(\mathtt{Lt}_{g^{-1}}(f)\big)\big)\, d\mu(g)$$

for $f \in C(G)$. It is clear that θ_n is completely positive and contractive, and it is easy to check that it is equivariant using translation invariance of μ. Fix $f \in C(G)$ and $d \in D$. We use condition (d) at the last step to obtain

$$\begin{aligned}
\limsup_{n\to\infty} \big\|\theta_n(f)d - \varphi_n(f)d\big\| &= \limsup_{n\to\infty} \left\| \int_G \alpha_g(\varphi_n(\mathtt{Lt}_{g^{-1}}(f)))d - \varphi_n(f)d \, d\mu(g) \right\| \\
&\le \limsup_{n\to\infty} \int_G \big\|\alpha_g(\varphi_n(\mathtt{Lt}_{g^{-1}}(f)))d - \varphi_n(f)d\big\|\, d\mu(g) \\
&\le \limsup_{n\to\infty} \sup_{g\in G} \big\|\alpha_g(\varphi_n(\mathtt{Lt}_{g^{-1}}(f)))d - \varphi_n(f)d\big\| = 0.
\end{aligned}$$

We deduce that $\lim_{n\to\infty} \|\theta_n(f)d - \varphi_n(f)d\|$ exists and equals zero. It follows that the map $\theta\colon C(G) \to F_\alpha(D, A)$ that $(\theta_n)_{n\in\mathbb{N}}$ determines is also a lift for φ. In particular, the maps θ_n satisfy conditions (a), (b) and (c) above, while condition (d) is satisfied exactly for each θ_n.

Now, for $n \in \mathbb{N}$, define $\rho_n\colon C(G) \otimes A \to A$ by

$$\rho_n(f \otimes a) = \theta_n(f^{\frac{1}{2}})a\theta_n(f^{\frac{1}{2}})$$

for $f \in C(G)$ with $f \geq 0$ (and extended linearly), and for all $a \in A$. Then ρ_n is completely positive and contractive. It is also equivariant, since for $f \in C(G)_+$ and $a \in A$, we have

$$\begin{aligned}\alpha_g\big(\rho_n(f \otimes a)\big) &= \alpha_g\big(\theta_n(f^{\frac{1}{2}})a\theta_n(f^{\frac{1}{2}})\big)\\ &= \alpha_g\big(\theta_n(f^{\frac{1}{2}})\big)\alpha_g(a)\alpha_g\big(\theta_n(f^{\frac{1}{2}})\big)\\ &= \theta_n\big(\mathtt{Lt}_g(f^{\frac{1}{2}})\big)\alpha_g(a)\theta_n\big(\mathtt{Lt}_g(f^{\frac{1}{2}})\big)\\ &= \rho_n\big(\mathtt{Lt}_g(f) \otimes \alpha_g(a)\big)\end{aligned}$$

for all $g \in G$. Observe also that

$$\lim_{n\to\infty} \big\|\rho_n(f \otimes d) - \theta_n(f)d\big\| = 0$$

for all $f \in C(G)$ (not just for $f \geq 0$) and for all $d \in D$, by condition (c) above. In particular, for $f_1, f_2 \in C(G)$ and $d_1, d_2 \in D$, we use condition (a) above applied to θ_n to deduce that

$$\begin{aligned}&\limsup_{n\to\infty} \big\|\rho_n(f_1 f_2 \otimes d_1 d_2) - \rho_n(f_1 \otimes d_1)\rho_n(f_2 \otimes d_2)\big\|\\ &\qquad = \limsup_{n\to\infty} \big\|\theta_n(f_1 f_2)d_1 d_2 - \theta_n(f_1)d_1\theta_n(f_2)d_2\big\|\\ &\qquad = \limsup_{n\to\infty} \big\|\left[\theta_n(f_1 f_2) - \theta_n(f_1)\theta_n(f_2)\right] d_1 d_2\big\| = 0.\end{aligned}$$

It follows that the restrictions of the maps ρ_n to $C(G) \otimes D$ determine an asymptotically multiplicative map $C(G) \otimes D \to A$.

By taking fixed point algebras in the conclusion of Proposition 2.3, we deduce that there is an isomorphism $\sigma\colon A \to C(G, A)^{\gamma_A}$ given by $\sigma(a)(g) = \alpha_g(a)$ for $a \in A$ and $g \in G$. In particular, and under the identification of $C(G, A)$ with $C(G) \otimes A$, the isomorphism σ satisfies $\sigma(a) = 1 \otimes a$ for all $a \in A^\alpha$. For $n \in \mathbb{N}$, let $\psi_n\colon A \to A^\alpha$ be given by $\psi_n = \rho_n \circ \sigma$.

Given $a, b \in F_1 \subseteq D$, we have

$$\begin{aligned}\limsup_{n\to\infty} \big\|\psi_n(ab) - \psi_n(a)\psi_n(b)\big\| &= \limsup_{n\to\infty} \big\|\rho_n\big(\sigma(ab)\big) - \rho_n\big(\sigma(a)\big)\rho_n\big(\sigma(b)\big)\big\|\\ &= \limsup_{n\to\infty} \big\|\rho_n\big(\sigma(a)\sigma(b)\big) - \rho_n\big(\sigma(a)\big)\rho_n\big(\sigma(b)\big)\big\|\\ &= 0,\end{aligned}$$

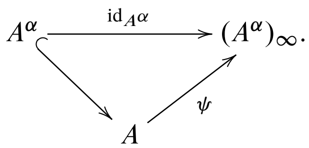
since $\sigma(a), \sigma(b)$, and $\sigma(ab)$ belong to $C(G) \otimes D$ and the maps ρ_n are asymptotically multiplicative on $C(G) \otimes D$.

Finally, given $a \in F_2 \subseteq D$, we use condition (c) above for θ_n at the third step, and condition (b) at the fourth step to get

$$\begin{aligned}\limsup_{n\to\infty} \left\| \psi_n(a) - a \right\| &= \limsup_{n\to\infty} \left\| \rho_n(\sigma(a)) - a \right\| \\ &= \limsup_{n\to\infty} \left\| \rho_n(1 \otimes a) - a \right\| \\ &= \limsup_{n\to\infty} \left\| \theta_n(1) a \theta_n(1) - a \right\| \\ &= \limsup_{n\to\infty} \left\| \theta_n(1) a - a \right\| = 0.\end{aligned}$$

The conclusion then follows by setting $\psi = \psi_n$ for n large enough. It is clear that the ψ_n are unital if the θ_n are, which can always be arranged if A is unital. $\square$

Remark 2.12. Adopt the notation from the theorem above. Then there is a commutative diagram



When A is nuclear, the Choi–Effros lifting theorem shows that the existence of a commutative diagram as above is in fact equivalent to the conclusion in Theorem 2.11. In the general case, however, the existence of such a diagram is a weaker assumption. Barlak and Szabo have independently identified this notion (see, for example, [45]), and have begun a systematic study of this concept in its own right; see [1].

This work consists in showing that a number of properties for A pass to A^α (and $A \rtimes_\alpha G$). We state our results assuming the Rokhlin property, but we really only use the existence of a commutative diagram as in Remark 2.12. As such, our results are valid in a more general context, and the extra flexibility will be needed in [12], where we study crossed products by more general actions.

Our first application of Theorem 2.11 is to the ideal structure of crossed products and fixed point algebras. In the presence of the Rokhlin property, we can describe all ideals: they are naturally induced by invariant ideals in the original algebra.

Theorem 2.13. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.*

(1) *If I is an ideal in A^α, then there exists an α-invariant ideal J in A such that $I = J \cap A^\alpha$.*

(2) *If I is an ideal in $A \rtimes_\alpha G$, then there exists an α-invariant ideal J in A such that $I = J \rtimes_\alpha G$.*

Proof. (1) Let I be an ideal in A^α. Then $J = \overline{AIA}$ is an α-invariant ideal in A. We claim that $J \cap A^\alpha = I$. It is clear that $I \subseteq J \cap A^\alpha$. For the reverse inclusion, let $x \in J \cap A^\alpha$, that is, an α-invariant element in $\overline{AIA}$. For $n \in \mathbb{N}$, choose $m_n \in \mathbb{N}$, elements $a_1^{(n)}, \dots, a_{m_n}^{(n)}, b_1^{(n)}, \dots, b_{m_n}^{(n)}$ in A, and elements $x_1^{(n)}, \dots, x_{m_n}^{(n)}$ in I, such that

$$\Big\| x - \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \Big\| < \frac{1}{n}.$$

Set $M_n = \max_{j=1,\dots,m_n}\{\|a_j^{(n)}\|, \|b_j^{(n)}\|, 1\}$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of completely positive contractive maps $\psi_n \colon A \to A^\alpha$ as in the conclusion of Theorem 2.11 for the choices $\varepsilon_n = 1/n m_n M_n^2$ and

$$F_1^{(n)} = \{a_j^{(n)}, x_j^{(n)}, b_j^{(n)} : j = 1, \dots, m_n\} \cup \{x\}$$

and $F_2^{(n)} = \{x_j^{(n)} : j = 1, \dots, m_n\} \cup \{x\}$. Then

$$\Big\| \psi_n\Big(\sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \Big) - x \Big\| < \frac{1}{n} + \frac{1}{n m_n M_n^2} \leq \frac{2}{n}$$

and

$$\begin{aligned}
&\Big\| \psi_n\Big(\sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \Big) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)}) \Big\| \\
&\quad \leq \frac{1}{n} + \Big\| \psi_n\Big(\sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \Big) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) \psi_n(x_j^{(n)}) \psi_n(b_j^{(n)}) \Big\| \\
&\quad \leq \frac{1}{n} + \frac{1}{n M_n} + \Big\| \psi_n\Big(\sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \Big) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)} x_j^{(n)}) \psi_n(b_j^{(n)}) \Big\| \\
&\quad \leq \frac{1}{n} + \frac{2}{n M_n} \leq \frac{3}{n}.
\end{aligned}$$

We conclude that

$$\Big\| x - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)}) \Big\| < \frac{5}{n}.$$

Since $\sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)})$ belongs to I, it follows that x is a limit of elements in I, and hence it belongs to I itself.

(2) This follows from (1) together with the fact that α is saturated (see Proposition 2.7). Alternatively, use Remark 2.8 together with the fact that the ideals in $A \otimes \mathcal{K}(L^2(G))$ have the form $I \otimes \mathcal{K}(L^2(G))$ for some ideal I in A. □

Corollary 2.14. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If A is simple, then so are A^α and $A \rtimes_\alpha G$.*

In the following corollary, hereditary saturation is as in [38, Definition 7.2.2], while the strong Connes spectrum for an action of a non-abelian compact group (which is a subset of the set $\widehat{G}$ of irreducible representations of the group) is as in [13, Definition 1.2]. (For abelian groups, the notion of strong Connes spectrum was introduced earlier by Kishimoto in [22].) We reproduce both definitions below for the convenience of the reader.

Definition 2.15. Let G be a compact group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α is *hereditarily saturated* if for every nonzero α-invariant hereditary subalgebra $B \subseteq A$, the restriction $\alpha|_B$ of α to B is saturated, in the sense of Definition 2.5.

We need some notation, which we borrow from [13]. For an action $\alpha\colon G \to \mathrm{Aut}(A)$ of a compact group G on a C^*-algebra A, and for a unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$, we set

$$A_2(\pi) = \{x \in A \otimes \mathcal{K}(\mathcal{H}_\pi) : (\alpha_g \otimes \mathrm{id})(x) = x(1_A \otimes \pi_g) \text{ for all } g \in G\}.$$

We denote by $\mathrm{Her}^\alpha(A)$ the family of all nonzero G-invariant hereditary subalgebras of A.

Definition 2.16. Let G be a compact group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We define the following spectra for α:

(1) *Arveson spectrum:*

$$\mathrm{Sp}(\alpha) = \left\{\pi \in \widehat{G} : \overline{A_2(\pi)^* A_2(\pi)} \text{ is an essential ideal in } (A \otimes \mathcal{K}(\mathcal{H}_\pi))^{\alpha \otimes \mathrm{Ad}(\pi)}\right\}.$$

(2) *Strong Arveson spectrum:*

$$\widetilde{\mathrm{Sp}}(\alpha) = \left\{\pi \in \widehat{G} : \overline{A_2(\pi)^* A_2(\pi)} = (A \otimes \mathcal{K}(\mathcal{H}_\pi))^{\alpha \otimes \mathrm{Ad}(\pi)}\right\}.$$

(3) *Connes spectrum:*

$$\Gamma(\alpha) = \bigcap_{B \in \mathrm{Her}^\alpha(A)} \mathrm{Sp}(\alpha|_B).$$

(4) *Strong Connes spectrum:*

$$\widetilde{\Gamma}(\alpha) = \bigcap_{B \in \mathrm{Her}^\alpha(A)} \widetilde{\mathrm{Sp}}(\alpha|_B).$$

Corollary 2.17. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then α has full strong Connes spectrum: $\widetilde{\Gamma}(\alpha) = \widehat{G}$, and it is hereditarily saturated.*

Proof. That $\widetilde{\Gamma}(\alpha) = \widehat{G}$ follows from [13, Theorem 3.3]. Hereditary saturation of actions with full strong Connes spectrum is established in the comments after [13, Lemma 3.1]. □

3. Generalized local approximations

We now turn to the study of preservation of certain structural properties that have proved to be relevant in the context of Elliott's classification program. In order to provide a conceptual approach, it will be necessary to introduce some convenient terminology.

Definition 3.1. Let $\mathcal{C}$ be a class of C^*-algebras and let A be a C^*-algebra.

(1) We say that A is an *(unital) approximate $\mathcal{C}$-algebra*, if A is isomorphic to a direct limit of C^*-algebras in $\mathcal{C}$ (with unital connecting maps).

(2) We say that A is a *(unital) local $\mathcal{C}$-algebra*, if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a C^*-algebra B in $\mathcal{C}$ and a not necessarily injective (unital) homomorphism $\varphi\colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in F$.

(3) We say that A is a *generalized (unital) local $\mathcal{C}$-algebra*, if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a C^*-algebra B in $\mathcal{C}$ and sequence $(\varphi_n)_{n\in\mathbb{N}}$ of asymptotically multiplicative (unital) completely positive contractive maps $\varphi_n\colon B \to A$ that $\mathrm{dist}(a, \varphi_n(B)) < \varepsilon$ for all $a \in F$ and for all n sufficiently large.

Remark 3.2. The term "local $\mathcal{C}$-algebra" is sometimes used to mean that the local approximations are realized by *injective* homomorphisms. For example, in [46] Thiel says that a C^*-algebra A is "$\mathcal{C}$-like," if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a C^*-algebra B in $\mathcal{C}$ and an *injective* homomorphism $\varphi\colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in F$. Finally, we point out that what we call here "approximate $\mathcal{C}$" is called "A$\mathcal{C}$" in [46].

The Rokhlin property is related to the above definition in the following way. Note that the approximating maps for A^α that we obtain in the proof are not necessarily injective, even if we assume that the approximating maps for A are.

Proposition 3.3. *Let $\mathcal{C}$ be a class of C^*-algebras, let A be a C^*-algebra, let G be a second-countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If A is a (unital) local $\mathcal{C}$-algebra, then A^α is a generalized (unital) local $\mathcal{C}$-algebra.*

Proof. Let $F \subseteq A^\alpha$ be a finite subset, and let $\varepsilon > 0$. Find a C^*-algebra B in $\mathcal{C}$ and a (unital) homomorphism $\varphi\colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \frac{\varepsilon}{2}$ for all $a \in F$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of (unital) completely positive contractive maps $\psi_n\colon A \to A^\alpha$ as in the conclusion of Theorem 2.11. Then $(\psi_n \circ \varphi)_{n\in\mathbb{N}}$ is a sequence of (unital) completely positive contractive maps $B \to A^\alpha$ as in the definition of generalized local $\mathcal{C}$-algebra. □

Let $\mathcal{C}$ be a class of C^*-algebras. It is clear that any (unital) approximate $\mathcal{C}$-algebra is a (unital) local $\mathcal{C}$-algebra, and that any (unital) local $\mathcal{C}$-algebra is a generalized (unital) local $\mathcal{C}$-algebra.

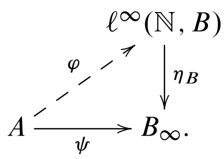
While the converses to these implications are known to fail in general, the notions in Definition 3.1 agree under fairly mild conditions on $\mathcal{C}$; see Proposition 3.9.

Definition 3.4. Let $\mathcal{C}$ be a class of C^*-algebras. We say that $\mathcal{C}$ has *(unital) approximate quotients* if whenever $A \in \mathcal{C}$ (is unital) and I is an ideal in A, the quotient A/I is a (unital) approximate $\mathcal{C}$-algebra, in the sense of Definiton 3.1.

The term "approximate quotients" has been used in [35] with a considerably stronger meaning. Our weaker assumptions still yield an analog of [35, Proposition 1.7]; see Proposition 3.9.

We need to recall a definition due to Loring. The original definition appears in [32], while in [4, Theorem 3.1] it is proved that weak semiprojectivity is equivalent to a condition that is more resemblant of semiprojectivity. For the purposes of this paper, the original definition is better suited.

Definition 3.5. A C^*-algebra A is said to be *weakly semiprojective (in the unital category)* if given a C^*-algebra B and given a (unital) homomorphism $\psi\colon A \to B_\infty$, there exists a (unital) homomorphism $\varphi\colon A \to \ell^\infty(\mathbb{N}, B)$ such that $\eta_B \circ \varphi = \psi$. In other words, the following lifting problem can always be solved:



The proof of the following observation is left to the reader. It states explicitly the formulation of weak semiprojectivity that will be used in our work, specifically in Proposition 3.9.

Remark 3.6. Using the definition of the sequence algebra B_∞, it is easy to show that if A is a weakly semiprojective C^*-algebra, and if $(\psi_n)_{n\in\mathbb{N}}$ is an asymptotically $*$-multiplicative sequence of linear maps $\psi_n\colon A \to B$ from A to another C^*-algebra B, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of homomorphisms $\varphi_n\colon A \to B$ such that

$$\lim_{n\to\infty} \|\varphi_n(a) - \psi_n(a)\| = 0$$

for all $a \in A$. If each ψ_n is unital and A is weakly semiprojective in the unital category, then φ_n can also be chosen to be unital.

We proceed to give some examples of classes of C^*-algebras that will be used in Theorem 3.10. We need a definition first, which appears as [5, Definition 11.2].

Definition 3.7. A C^*-algebra A is said to be a *one-dimensional noncommutative cellular complex*, or one-dimensional NCCW-complex for short, if there exist finite dimensional C^*-algebras E and F, and unital homomorphisms $\varphi, \psi\colon E \to F$, such that A is isomorphic to the pull back C^*-algebra

$$\{(a,b) \in E \oplus C([0,1], F) : b(0) = \varphi(a) \quad \text{and} \quad b(1) = \psi(a)\}.$$

It was shown in [5, Theorem 6.2.2] that one-dimensional NCCW-complexes are semiprojective (in the unital category).

Examples 3.8. The following are examples of classes of weakly semiprojective C^*-algebras (in the unital category) which have approximate quotients.

(1) The class $\mathcal{C}$ of matrix algebras. The (unital) approximate $\mathcal{C}$-algebras are precisely the matroid algebras (UHF-algebras).

(2) The class $\mathcal{C}$ of finite dimensional C^*-algebras. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) AF-algebras.

(3) The class $\mathcal{C}$ of interval algebras, this is, algebras of the form $C([0,1]) \otimes F$, where F is a finite dimensional C^*-algebra. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) AI-algebras.

(4) The class $\mathcal{C}$ of circle algebras, this is, algebras of the form $C(\mathbb{T}) \otimes F$, where F is a finite dimensional C^*-algebra. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) A$\mathbb{T}$-algebras.

(5) The class $\mathcal{C}$ of one-dimensional NCCW-complexes. We point out that certain approximate $\mathcal{C}$-algebras have been classified, in terms of a variant of their Cuntz semigroup, by Robert in [41].

The following result is well known for several particular classes.

Proposition 3.9. *Let $\mathcal{C}$ be a class of C^*-algebras which has (unital) approximate quotients (see Definition 3.4). Assume that the C^*-algebras in $\mathcal{C}$ are weakly semiprojective (in the unital category). For a separable (unital) C^*-algebra A, the following are equivalent:*

(1) *A is an (unital) approximate $\mathcal{C}$-algebra;*

(2) *A is a (unital) local $\mathcal{C}$-algebra;*

(3) *A is a generalized (unital) local $\mathcal{C}$-algebra.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are true in full generality. Weak semiprojectivity of the algebras in $\mathcal{C}$ implies that any generalized local approximation by C^*-algebras in $\mathcal{C}$ can be perturbed to a genuine local approximation by C^*-algebras in $\mathcal{C}$ (see Remark 3.6), showing (3) $\Rightarrow$ (2).

For the implication (2) $\Rightarrow$ (1), note that since $\mathcal{C}$ has approximate quotients, every a local $\mathcal{C}$-algebra is A$\mathcal{C}$-like, in the sense of Definition 3.2 in [46] (see also Paragraph 3.6 there). It then follows from Theorem 3.9 in [46] that A is an approximate $\mathcal{C}$-algebra.

For the unital case, one uses Remark 3.6 to show that (3) $\Rightarrow$ (2) when units are considered. Moreover, for (2) $\Rightarrow$ (1), one checks that in the proof of Theorem 3.9 in [46], if one assumes that the building blocks are weakly semiprojective in the unital category, then the conclusion is that a unital A$\mathcal{C}$-like algebra is a unital A$\mathcal{C}$-algebra. With the notation and terminology of the proof of Theorem 3.9 in [46], suppose that A is a unital $A\mathcal{C}$-like algebra, and suppose that $\varphi\colon C \to A$ is a unital homomorphism, with $C \in \mathcal{C}$. Since C is assumed to be weakly semiprojective in the unital category, the morphism $\alpha\colon C \to B$ can be chosen to be unital. For the same reason, one can arrange that the morphism $\widetilde{\alpha}\colon C \to C_{k_1}$ be unital (possible by changing the choice of k_1). Now, since the connecting maps γ_k are also assumed to be unital, it is easily seen that the one-sided approximate intertwining constructed has unital connecting maps. Finally, when applying Proposition 3.5 in [46], if the algebras A_i, with $i \in I$, are weakly semiprojective in the unital category, then the morphisms $\psi_k\colon A_{i(k)} \to A_{i(k+1)}$ can be chosen to be unital as well. We leave the details to the reader. □

The following is the main application of our approximations results.

Theorem 3.10. *Let $\mathcal{C}$ be a class of separable weakly semiprojective C^*-algebras (in the unital category), and assume that $\mathcal{C}$ has (unital) approximate quotients. Let A be a (unital) local $\mathcal{C}$-algebra, let G be a second-countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then A^α is a (unital) approximate $\mathcal{C}$-algebra.*

Proof. This is an immediate consequence of Proposition 3.3 and Proposition 3.9. □

An alternative proof of part (2) of the corollary below is given in Corollary 2.10.

Corollary 3.11. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.*

(1) *If A is a matroid algebra (UHF), then A^α is a matroid algebra (UHF) and $A \rtimes_\alpha G$ is a matroid algebra. (If G is finite, then $A \rtimes_\alpha G$ is also a UHF-algebra.)*

(2) *If A is an AF-algebra, then so are A^α and $A \rtimes_\alpha G$.*

(3) *If A is an AI-algebra, then so are A^α and $A \rtimes_\alpha G$.*

(4) *If A is an $A\mathbb{T}$-algebra, then so are A^α and $A \rtimes_\alpha G$.*

(5) *If A is a direct limit of one-dimensional NCCW-complexes, then so are A^α and $A \rtimes_\alpha G$.*

Proof. Since the classes in Examples 3.8 have approximate quotients and contain only weakly semiprojective C^*-algebras, the claims follow from Theorem 3.10. □

Theorem 3.10 allows for far more flexibility than [35, Theorem 3.5], since we do not assume our classes of C^*-algebras to be closed under direct sums or by taking corners, nor do we assume that our algebras are semiprojective. In particular, the class $\mathcal{C}$ of weakly semiprojective purely infinite, simple algebras satisfies the assumptions of Theorem 3.10, but appears not to fit into the framework of flexible classes discussed in [35].

Recall that a C^*-algebra is said to be a *Kirchberg algebra* if it is purely infinite, simple, separable and nuclear.

The following lemma is probably standard, but we include its proof here for the sake of completeness.

Lemma 3.12. *Let A be a Kirchberg algebra satisfying the Universal Coefficient Theorem. Then A is isomorphic to a direct limit of weakly semiprojective Kirchberg algebras satisfying the Universal Coefficient Theorem.*

Proof. Since every non-unital Kirchberg algebra is the stabilization of a unital Kirchberg algebra, by Proposition 3.11 in [3] it is enough to prove the statement when A is non-unital. For $j = 0, 1$, set $G_j = K_j(A)$. Write G_j as a direct limit $G_j \cong \varinjlim(G_j^{(n)}, \gamma_j^{(n)})$ of finitely generated abelian groups $G_j^{(n)}$, with connecting maps

$$\gamma_j^{(n)}\colon G_j^{(n)} \to G_j^{(n+1)}.$$

For $j = 0, 1$, use [37, Theorem 4.2.5] to find, for $n \in \mathbb{N}$, Kirchberg algebras A_n satisfying the Universal Coefficient Theorem with $K_j(A_n) \cong G_j^{(n)}$, and homomorphisms

$$\varphi_n\colon A_n \to A_{n+1}$$

such that $K_j(\varphi_n)$ is identified with $\gamma_j^{(n)}$ under the isomorphism $K_j(A_n) \cong G_j^{(n)}$.

The direct limit $\varinjlim(A_n, \varphi_n)$ is isomorphic to A by [37, Theorem 4.2.4]. On the other hand, each of the algebras A_n is weakly semiprojective by [44, Theorem 2.2] (see also [31, Corollary 7.7]), so the proof is complete. □

Theorem 3.13. *Let A be a separable, simple, nuclear C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If A satisfies the Universal Coefficient Theorem, then so do A^α and $A \rtimes_\alpha G$.*

Proof. We claim that it is enough to prove the statement when A is a Kirchberg algebra. Indeed, a C^*-algebra B satisfies the Universal Coefficient Theorem if and only if $B \otimes \mathcal{O}_\infty$ does, since $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$. On the other hand, $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$ has the Rokhlin property, and

$$(A \otimes \mathcal{O}_\infty)^{\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}} = A^\alpha \otimes \mathcal{O}_\infty.$$

Suppose then that A is a Kirchberg algebra. Denote by $\mathcal{C}$ the class of all unital weakly semiprojective Kirchberg algebras satisfying the Universal Coefficient Theorem. Note that $\mathcal{C}$ has approximate quotients. By Lemma 3.12, A is a unital approximate $\mathcal{C}$-algebra. By Theorem 3.10, A^α is also a unital approximate $\mathcal{C}$-algebra. Since the Universal Coefficient Theorem passes to direct limits, we conclude that A^α satisfies it. Since $A \rtimes_\alpha G$ is Morita equivalent to A^α, the same holds for the crossed product. □

4. Further structure results

We now turn to preservation of classes of C^*-algebras that are not necessarily defined in terms of an approximation by weakly semiprojective C^*-algebras. The classes we study can all be dealt with using Theorem 2.11.

Definition 4.1 ([47, Definition 1.3]). A unital, separable C^*-algebra $\mathcal{D}$ is said to be *strongly self-absorbing*, if it is infinite dimensional and the map $\mathcal{D} \to \mathcal{D} \otimes_{\min} \mathcal{D}$, given by $d \mapsto d \otimes 1$ for $d \in \mathcal{D}$, is approximately unitarily equivalent to an isomorphism.

It is a consequence of a result of Effros and Rosenberg that strongly self-absorbing C^*-algebras are nuclear, so that the choice of the tensor product in the definition above is irrelevant. The only known examples of strongly self-absorbing C^*-algebras are the Jiang–Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF-algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ by such UHF-algebras. It has been conjectured that these are the only examples of strongly self-absorbing C^*-algebras. See [47] for the proofs of these and other results concerning strongly self-absorbing C^*-algebras.

The following is a useful criterion to determine when a separable C^*-algebra absorbs a strongly self-absorbing C^*-algebra tensorially. The proof is a straightforward combination of [47, Theorem 2.2] and the Choi–Effros lifting theorem, and we shall omit it. (See also [16, Proposition 4.1].)

Theorem 4.2. *Let A be a separable C^*-algebra, and let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. Then A is $\mathcal{D}$-stable if and only if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$, and every finite subset $E \subseteq \mathcal{D}$, there exists a completely positive map $\varphi\colon \mathcal{D} \to A$ such that*

(1) $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ *for all $a \in F$ and for all $d \in E$;*

(2) $\|\varphi(de)a - \varphi(d)\varphi(e)a\| < \varepsilon$ *for every $d, e \in E$ and every $a \in F$;*

(3) $\|\varphi(1)a - a\| < \varepsilon$ *for all $a \in F$.*

The following result was obtained for unital C^*-algebras as [17, Corollary 3.4, part (1)], using different methods. Our proof of the general case illustrates the generality of our approach.

Theorem 4.3. *Let A be a separable C^*-algebra, let G be a second-countable compact group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra and assume that A is $\mathcal{D}$-stable. Then A^α and $A \rtimes_\alpha G$ are $\mathcal{D}$-stable as well.*

Proof. Since $\mathcal{D}$-stability is preserved under Morita equivalence by [47, Corollary 3.2], it is enough to prove the result for A^α.

Let $\varepsilon > 0$, and let $F \subseteq A^\alpha$ and $E \subseteq \mathcal{D}$ be finite subsets of A and $\mathcal{D}$, respectively. Use Theorem 4.2 to choose a completely positive map $\varphi: \mathcal{D} \to A$ satisfying

(1) $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;

(2) $\|\varphi(de)a - \varphi(d)\varphi(e)a\| < \varepsilon$ for all $d, e \in E$ and all $a \in F$;

(3) $\|\varphi(1)a - a\| < \varepsilon$ and $\|a\varphi(1) - a\| < \varepsilon$ for all $a \in F$.

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of completely positive contractive maps $\psi_n: A \to A^\alpha$ as in the conclusion of Theorem 2.11 for $F_1 = F \cup \{\varphi(1)\}$ and $F_2 = F$. Since $\lim_{n\to\infty} \psi_n(a) = a$ for all $a \in F$, we deduce that

$$\limsup_{n\to\infty} \|a\psi_n(\varphi(d)) - \psi_n(\varphi(d))a\| \le \|a\varphi(d) - \varphi(d)a\| < \varepsilon$$

for all $a \in F$ and all $d \in E$. Likewise,

$$\limsup_{n\to\infty} \|\psi_n(\varphi(de)) - \psi_n(\varphi(d))\psi_n(\varphi(e))\| \le \|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$$

for all $d, e \in E$. Finally, for $a \in F$, we have

$$\limsup_{n\to\infty} \|\psi_n(\varphi(1))a - a\| \le \|\varphi(1)a - a\| < \varepsilon$$

and

$$\limsup_{n\to\infty} \|a\psi_n(\varphi(1)) - a\| \le \|a\varphi(1) - a\| < \varepsilon.$$

We conclude that for n large enough, the completely positive contractive map

$$\psi_n \circ \varphi: \mathcal{D} \to A^\alpha$$

satisfies conditions (1) through (3) of Theorem 4.2, showing that A^α is $\mathcal{D}$-stable. □

Similar methods allow one to prove that the property of being approximately divisible is inherited by the crossed product and the fixed point algebra of a compact group action with the Rokhlin property. (This was first obtained, for unital C^*-algebras, by Hirshberg and Winter as [17, Corollary 3.4, part (2)].) Our proof is completely analogous to that of Theorem 4.3 (using a suitable version of Theorem 4.2), so for the sake of brevity, we shall not present it here.

Our next goal is to show that Rokhlin actions preserve the property of having tracial rank at most one in the simple, unital case.

We will need a definition of tracial rank zero and one. What we reproduce below are not Lin's original definitions ([29, Definition 2.1] and [30, Definition 2.1]). Nevertheless, the notions we define are equivalent in the simple case: for tracial rank zero, this follows from [30, Proposition 3.8], while the argument in the proof of said proposition can be adapted to show the corresponding result for tracial rank one. Recall that an *interval algebra* is a C^*-algebra of the form $C([0,1])\otimes E$, where E is a finite dimensional C^*-algebra. Such algebras have a finite presentation with stable relations; see [32].

Definition 4.4. Let A be a simple, unital C^*-algebra. We say that A has *tracial rank at most one*, and write $\mathrm{TR}(A)\leq 1$, if for every finite subset $F\subseteq A$, for every $\varepsilon>0$, and for every non-zero positive element $x\in A$, there exist a projection $p\in A$, an interval algebra B, and a unital homomorphism $\varphi\colon B\to pAp$, such that

(1) $\|ap-pa\|<\varepsilon$ for all $a\in F$;

(2) $\mathrm{dist}(pap,\varphi(B))<\varepsilon$ for all $a\in F$;

(3) $1-p$ is Murray–von Neumann equivalent to a projection in $\overline{xAx}$.

Additionally, we say that A has *tracial rank zero*, and write $\mathrm{TR}(A)=0$, if the C^*-algebra B as above can be chosen to be finite dimensional.

We will need the following notation. For $t\in\left(0,\frac{1}{2}\right)$, we denote by $f_t\colon[0,1]\to[0,1]$ the continuous function that takes the value 0 on $[0,t]$, the value 1 on $[2t,1]$, and is linear on $[t,2t]$.

Theorem 4.5. *Let A be a unital, separable, simple C^*-algebra with $\mathrm{TR}(A)\leq 1$, let G be a second-countable compact group, and let $\alpha\colon G\to\mathrm{Aut}(A)$ be an action with the Rokhlin property. Then A^α is a unital, separable, simple C^*-algebra with $\mathrm{TR}(A^\alpha)\leq\mathrm{TR}(A)$. If G is finite, then the same holds for the crossed product $A\rtimes_\alpha G$.*

When G is not finite (but compact), then $A\rtimes_\alpha G$ is never unital, and the definition of tracial rank at most one only applies to unital C^*-algebras.

Proof. Let $F\subseteq A^\alpha$ be a finite subset, let $\varepsilon>0$ and let $x\in A^\alpha$ be a non-zero positive element. Without loss of generality, we may assume that $\|a\|\leq 1$ for all $a\in F$, and that $\varepsilon<1$. Find $t\in\left(0,\frac{1}{2}\right)$ such that $(x-2t)_+$ is not zero. Set $y=(x-2t)_+$. Then y belongs to A^α and moreover $f_t(x)y=yf_t(x)=y$.

Using that A has tracial rank zero, find an interval algebra B, a projection $p\in A$, a unital homomorphism $\varphi\colon B\to pAp$, a projection $q\in\overline{yAy}$ and a partial isometry $s\in A$ such that

- $\|ap-pa\|<\frac{\varepsilon}{9}$ for all $a\in F$;
- $\mathrm{dist}(pap,\varphi(B))<\frac{\varepsilon}{9}$ for all $a\in F$;
- $1-p=s^*s$ and $q=ss^*$.

Let $\widetilde{F} \subseteq B$ be a finite subset such that for all $a \in F$, there exists $b \in \widetilde{F}$ with $\|pap - \varphi(b)\| < \frac{\varepsilon}{9}$.

Since $f_t(x)$ is a unit for $\overline{yAy}$, it follows that $q = f_t(x)qf_t(x)$. As A is unital and separable, we can use Theorem 2.11 to find an approximate homomorphism $(\psi_n)_{n\in\mathbb{N}}$ from A to A^α, consisting of unital completely positive maps $\psi_n\colon A \to A^\alpha$ satisfying $\lim_{n\to\infty} \|\psi_n(a) - a\| = 0$ for all $a \in A^\alpha$. (For example, one chooses increasing families $(F_1^{(n)})_{n\in\mathbb{N}}$ and $(F_2^{(n)})_{n\in\mathbb{N}}$ of finite subsets of A and A^α, respectively, whose union is dense in A and A^α, and obtains ψ_n by applying the main part of Theorem 2.11 with tolerance $\varepsilon_n = 1/n$ and sets $F_1^{(n)} \subseteq A$ and $F_2^{(n)} \subseteq A^\alpha$.) We then have

(a) $\limsup_{n\to\infty} \|\psi_n(p)a - a\psi_n(p)\| < \frac{\varepsilon}{9}$ for all $a \in F$;

(b) $\limsup_{n\to\infty} \operatorname{dist}(\psi_n(p)a\psi_n(a), (\psi_n \circ \varphi)(B)) < \frac{\varepsilon}{9}$ for all $a \in F$;

(c) $\lim_{n\to\infty} \|\psi_n(p)a\psi_n(p) - \psi_n(pap)\| = 0$;

(d) $\lim_{n\to\infty} \|\psi_n(p)^*\psi_n(p) - \psi_n(p)\| = 0$;

(e) $\lim_{n\to\infty} \|1 - \psi_n(p) - \psi_n(s)^*\psi_n(s)\| = 0$;

(f) $\lim_{n\to\infty} \|\psi_n(q)\psi_n(s)\psi_n(1-p) - \psi_n(s)\| = 0$;

(g) $\lim_{n\to\infty} \|\psi_n(q)^*\psi_n(q) - \psi_n(q)\| = 0$;

(h) $\lim_{n\to\infty} \|\psi_n(q) - \psi_n(s)\psi_n(s)^*\| = 0$;

(i) $\lim_{n\to\infty} \|\psi_n(q) - f_t(x)\psi_n(q)f_t(x)\| = 0$.

With $r_n = f_t(x)\psi_n(q)f_t(x)$ for $n \in \mathbb{N}$, it follows from conditions (g) and (i) that

(j) $\lim_{n\to\infty} \|r_n^* r_n - r_n\| = 0$.

Find $\delta_1 > 0$ such that whenever e is an element in a C^*-algebra C such that $\|e^*e - e\| < \delta_1$, then there exists a projection f in C such that $\|e - f\| < \frac{\varepsilon}{9}$. Fix a finite set $\mathcal{G} \subseteq B$ of generators for B. Using semiprojectivity of B in the unital category (specifically, the fact that the relations defining it are stable), find $\delta_2 > 0$ such that whenever D is a unital C^*-algebra and $\rho\colon B \to D$ is a unital positive linear map which is δ_2-multiplicative on $\mathcal{G}$, there exists a unital homomorphism $\pi\colon B \to D$ such that $\|\rho(b) - \pi(b)\| < \frac{\varepsilon}{9}$ for all $b \in \widetilde{F}$. (Observe that we are not fixing the target algebra D, which will later be taken to be of the form $fA^\alpha f$ for some projection $f \in A^\alpha$.) Set $\delta = \min\{\delta_1, \delta_2\}$.

Choose $n \in \mathbb{N}$ large enough so that the quantities in conditions (a), (b), (c), (e) and (i) are less than $\frac{\varepsilon}{9}$, the quantities in (d) and (j) are less than δ, the quantities in (e) and (g) are less than $1 - \varepsilon$, and so that $\psi_n \circ \varphi$ is δ-multiplicative on $\mathcal{G}$. Since r_n belongs to $\overline{xA^\alpha x}$ for all $n \in \mathbb{N}$, by the choice of δ there exist a projection e in $\overline{xA^\alpha x}$

such that $\|e - r_n\| < \frac{\varepsilon}{9}$, and a projection $f \in A^\alpha$ such that $\|f - \psi_n(p)\| < \frac{\varepsilon}{9}$. Let $\pi\colon B \to fA^\alpha f$ be a unital homomorphism satisfying

$$\|\pi(b) - (\psi_n \circ \varphi)(b)\| < \frac{\varepsilon}{9}$$

for all $b \in \mathcal{G} \cup \widetilde{F}$.

We claim that the projection f and the homomorphism $\pi\colon B \to fA^\alpha f$ satisfy the conditions in Definition 4.4. Since π is unital, we must have $\pi(1) = f$.

Given $a \in F$, the estimate

$$\|af - fa\| \leq \|a\psi_n(p) - \psi_n(p)a\| + 2\|\psi_n(p) - f\| < \frac{3\varepsilon}{9} < \varepsilon$$

shows that condition (1) is satisfied. In order to check condition (2), given $a \in F$, choose $b \in \widetilde{F}$ such that

$$\|pap - \varphi(b)\| < \frac{\varepsilon}{9}.$$

Then

$$\begin{aligned}\|faf - \pi(b)\| &\leq \|faf - \psi_n(p)a\psi_n(p)\| + \|\psi_n(p)a\psi_n(p) - \psi_n(\varphi(b))\| \\ &\qquad + \|\psi_n(\varphi(b)) - \pi(b)\| \\ &< 2\|f - \psi_n(p)\| + \frac{\varepsilon}{9} + \frac{\varepsilon}{9} < \varepsilon,\end{aligned}$$

so condition (2) is also satisfied. To check condition (3), it is enough to show that $1 - f$ is Murray–von Neumann equivalent (in A^α) to e. We have

$$\begin{aligned}\|(1-f) - \psi_n(s)^*\psi_n(s)\| &\leq \|f - \psi_n(p)\| + \|1 - \psi_n(p) - \psi_n(s)^*\psi_n(s)\| \\ &< \frac{\varepsilon}{9} + 1 - \varepsilon = 1 - \frac{8\varepsilon}{9},\end{aligned}$$

and likewise, $\|e - \psi_n(s)\psi_n(s)^*\| < \frac{\varepsilon}{9} + 1 - \varepsilon$. On the other hand, we use the approximate versions of equation (i) at the second step, and that of equation (f) at the third step, to get

$$\begin{aligned}\|\psi_n(s) - e\psi_n(s)(1-f)\| &< \frac{2\varepsilon}{9} + \|\psi_n(s) - f_t(x)\psi_n(q)f_t(x)\psi_n(s)\psi_n(1-p)\| \\ &< \frac{3\varepsilon}{9} + \|\psi_n(s) - \psi_n(q)\psi_n(s)\psi_n(1-p)\| \\ &< \frac{4\varepsilon}{9}.\end{aligned}$$

Now, it is immediate that

$$\begin{aligned}&\|(1-f) - (e\psi_n(s)(1-f))^*(e\psi_n(s)(1-f))\| \\ &\qquad < 2\|\psi_n(s) - e\psi_n(s)(1-f)\| + \|(1-f) - \psi_n(s)^*\psi_n(s)\| \\ &\qquad < \frac{8\varepsilon}{9} + 1 - \frac{8\varepsilon}{9} = 1.\end{aligned}$$

Likewise,

$$\|e - (e\psi_n(s)(1-f))(e\psi_n(s)(1-f))^*\| < 1.$$

By Lemma 2.5.3 in [26] applied to $e\psi_n(s)(1-f)$, we conclude that $1-f$ and e are Murray–von Neumann equivalent in A^α, and the proof of the first part of the statement is complete.

It is clear that if A has tracial rank zero and we choose B to be finite dimensional, then the above proof shows that A^α has tracial rank zero as well.

Finally, if G is finite, then the last claim of the statement follows from the fact that A^α and $A \rtimes_\alpha G$ are Morita equivalent. □

We believe that a condition weaker than the Rokhlin property ought to suffice for the conclusion of Theorem 4.5 to hold. In view of [39, Theorem 2.8], we presume that fixed point algebras by a suitable version of the tracial Rokhlin property for compact group actions would preserve the class of simple C^*-algebras with tracial rank zero.

We present two consequences of Theorem 4.5. The first one is to simple AH-algebras of slow dimension growth and real rank zero, which do not a priori fit into the general framework of Theorem 3.10, since the building blocks are not necessarily weakly semiprojective.

Corollary 4.6. *Let A be a simple, unital AH-algebra with slow dimension growth and real rank zero. Let G be a second-countable compact group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then A^α is a simple, unital AH-algebra with slow dimension growth and real rank zero.*

Proof. By [30, Proposition 2.6], A has tracial rank zero. Thus A^α is a simple C^*-algebra with tracial rank zero by Theorem 4.5. It is clearly separable, unital, and nuclear. Moreover, it satisfies the Universal Coefficient Theorem by Theorem 3.13. Since AH-algebras of slow dimension growth and real rank zero exhaust the Elliott invariant of C^*-algebras with tracial rank zero, [28, Theorem 5.2] implies that A^α is an AH-algebra with slow dimension growth and real rank zero. □

Denote by $\mathcal{Q}$ the universal UHF-algebra. Recall that a simple, separable, unital C^*-algebra A is said to have *rational tracial rank at most one*, if $\mathrm{TR}(A \otimes \mathcal{Q}) \leq 1$ (see [27, Definition 11.8], and see the comments after it for examples of algebras of rational tracial rank at most one).

Corollary 4.7. *Let A be a simple, separable, unital C^*-algebra, let G be a second-countable compact group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If A has rational tracial rank at most one, then so does A^α (and also $A \rtimes_\alpha G$ if G is finite).*

Proof. The result is an immediate consequence of Theorem 4.5 applied to the action $\alpha \otimes \mathrm{id}_{\mathcal{Q}}: G \to \mathrm{Aut}(A \otimes \mathcal{Q})$. □

We now turn to pure infiniteness in the non-simple case.

Definition 4.8 ([24, Definition 4.1]). A C^*-algebra A is said to be *purely infinite* if the following conditions are satisfied:

(1) There are no non-zero characters (this is, homomorphisms onto the complex numbers) on A, and

(2) For every pair a, b of positive elements in A, with b in the ideal generated by a, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} \|x_n^* b x_n - a\| = 0$.

Theorem 4.9 ([24, Theorem 4.16]; see also [24, Definition 3.2]). *Let A be a C^*-algebra. Then A is purely infinite if and only if for every nonzero positive element $a \in A$, we have $a \oplus a \precsim a$.*

We use the above result to show that, in the presence of the Rokhlin property, pure infiniteness is inherited by the fixed point algebra and the crossed product.

Proposition 4.10. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If A is purely infinite, then so are A^α and $A \rtimes_\alpha G$.*

Proof. By Proposition 2.7 (see also Remark 2.8) and [24, Theorem 4.23], it is enough to prove the result for A^α. Let a be a nonzero positive element in A^α. Since A is purely infinite, by [24, Theorem 4.16] (here reproduced as Theorem 4.9), there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A such that

(a) $\lim\limits_{n\to\infty} \|x_n^* a x_n - a\| = 0$;

(b) $\lim\limits_{n\to\infty} \|x_n^* a y_n\| = 0$;

(c) $\lim\limits_{n\to\infty} \|y_n^* a x_n\| = 0$;

(d) $\lim\limits_{n\to\infty} \|y_n^* a y_n - a\| = 0$.

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of completely positive contractive maps $\psi_n\colon A \to A^\alpha$ as in the conclusion of Theorem 2.11. Easy applications of the triangle inequality yield

(a′) $\lim\limits_{n\to\infty} \|\psi_n(x_n)^* a \psi_n(x_n) - a\| = 0$;

(b′) $\lim\limits_{n\to\infty} \|\psi_n(x_n)^* a \psi_n(y_n)\| = 0$;

(c′) $\lim\limits_{n\to\infty} \|\psi_n(y_n)^* a \psi_n(x_n)\| = 0$;

(d′) $\lim\limits_{n\to\infty} \|\psi_n(y_n)^* a \psi_n(y_n) - a\| = 0$.

Since $\psi_n(x_n)$ and $\psi_n(y_n)$ belong to A^α for all $n \in \mathbb{N}$, we conclude that $a \oplus a \precsim a$ in A^α. It now follows from [24, Theorem 4.16] (here reproduced as Theorem 4.9) that A^α is purely infinite, as desired. ∎

Corollary 4.11. *Let A be a Kirchberg algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then A^α and $A \rtimes_\alpha G$ are Kirchberg algebras.*

Proof. It is well known that A^α and $A \rtimes_\alpha G$ are nuclear and separable. Simplicity follows from Corollary 2.14, and pure infiniteness follows from Proposition 4.10. □

For the sake of comparison, we mention here that stable finiteness passes to fixed point algebras and crossed products by arbitrary compact group actions, since we have $A^\alpha \subseteq A$ and $A \rtimes_\alpha G \subseteq A \otimes \mathcal{K}(L^2(G))$, and stable finiteness passes to subalgebras.

The following definition is standard.

Definition 4.12. Let A be a C^*-algebra.

(1) If A is unital, we say that it has *real rank zero* if the set of invertible self adjoint elements in A is dense in the set of self adjoint elements. If A is not unital, we say that it has real rank zero if so does its unitization $\widetilde{A}$.

(2) If A is unital, we say that it has *stable rank one* if the set of invertible elements is dense in A. If A is not unital, we say that it has stable rank one if so does its unitization $\widetilde{A}$.

In the following proposition, the Rokhlin property is surely stronger than necessary for the conclusion to hold, although some condition on the action must be imposed. We do not know, for instance, whether finite Rokhlin dimension with commuting towers preserves real rank zero and stable rank one.

Proposition 4.13. *Let A be a σ-unital C^*-algebra, let G be a second-countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.*

(1) *If A has real rank zero, then so do A^α and $A \rtimes_\alpha G$.*

(2) *If A has stable rank one, then so do A^α and $A \rtimes_\alpha G$.*

Proof. By Proposition 2.7 (see also Remark 2.8), [40, Theorem 3.3], and [2, Theorem 2.5], it is enough to prove the proposition for A^α. Since the proofs of both parts are similar, we only prove the first one.

Since the commutative diagram in Remark 2.12 can be unitized, it is enough to assume that A is unital. (Equivalently, extend the linear maps $\psi_n\colon A \to A^\alpha$ in the conclusion of Theorem 2.11 to unital maps $\widetilde{\psi_n}\colon \widetilde{A} \to \widetilde{A}^\alpha$.)

Let a be a self-adjoint element in A^α and let $\varepsilon > 0$. Since A has real rank zero, there exists an invertible self-adjoint element b in A such that $\|b - a\| < \frac{\varepsilon}{2}$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $A \to A^\alpha$ as in the conclusion of Theorem 2.11. Then $\psi_n(b)$ is self-adjoint for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} \left\|\psi_n(b)\psi_n(b^{-1}) - 1\right\| = \lim_{n\to\infty} \left\|\psi_n(b^{-1})\psi_n(b) - 1\right\| = 0$$

and

$$\lim_{n\to\infty} \left\|\psi_n(a) - a\right\| = 0.$$

Choose n large enough so that

$$\left\|\psi_n(b)\psi_n(b^{-1})-1\right\|<1 \quad \text{and} \quad \left\|\psi_n(b^{-1})\psi_n(b)-1\right\|<1,$$

and also so that $\|\psi_n(a)-a\|<\frac{\varepsilon}{2}$. Then $\psi_n(b)\psi_n(b^{-1})$ and $\psi_n(b^{-1})\psi_n(b)$ are invertible, and hence so is $\psi_n(b)$. Finally,

$$\left\|a-\psi_n(b)\right\|\leq\left\|a-\psi_n(a)\right\|+\left\|\psi_n(a)-\psi_n(b)\right\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

which shows that A^α has real rank zero. □

We now turn to traces. For a trace τ on a C^*-algebra A, we also denote by τ its amplification to any matrix algebra $M_n(A)$. We denote by $T(A)$ the set of all tracial states on A.

The following is one of Blackadar's fundamental comparability questions:

Definition 4.14. Let A be a simple unital C^*-algebra. We say the the *order on projections (in A) is determined by traces*, if whenever p and q are projections in $M_\infty(A)$ satisfying $\tau(p)\leq\tau(q)$ for all $\tau\in T(A)$, then $p\precsim_{M-vN} q$.

The following extends, with a simpler proof, [35, Proposition 4.8].

Proposition 4.15. *Let A be a simple unital C^*-algebra, and suppose that the order on its projections is determined by traces. Let G be a second-countable compact group, and let $\alpha\colon G\to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the order on projections in A^α is determined by traces.*

Proof. Since $\alpha\otimes \mathrm{id}_{M_n}\colon G\to \mathrm{Aut}(A\otimes M_n)$ has the Rokhlin property and $(A\otimes M_n)^{\alpha\otimes \mathrm{id}_{M_n}}=A^\alpha\otimes M_n$, in Definition 4.14 it is enough to consider projections in the algebra.

Let p and q be projections in A^α, and suppose that it is not the case that $p\precsim_{M-vN} q$ in A^α. We want to show that there exists a tracial state τ on A^α such that $\tau(p)\geq\tau(q)$. By [9, Proposition 3.2, part (1)], it is not the case that $p\precsim_{M-vN} q$ in A, so there exists a tracial state ω on A such that $\omega(p)\geq\omega(q)$. Now take $\tau=\omega|_{A^\alpha}$. □

Finally, we close this section by exploring the extent to which semiprojectivity passes from A to the fixed point algebra and the crossed product by a compact group with the Rokhlin property. Even though we have not been able to answer this question for semiprojectivity, we can provide a satisfying answer for *weak* semiprojectivity (see Definition 3.5).

In order to show this, we introduce the following technical definition, which is inspired in the notion of "corona extendibility" ([33, Definition 1.1]; we are thankful to Hannes Thiel for providing this reference).

Definition 4.16. Let $\theta\colon A \to B$ be a homomorphism between C^*-algebras A and B. We say that θ is *sequence algebra extendible*, if whenever E is a C^*-algebra and $\varphi\colon A \to E_\infty$ is a homomorphism, there exists a homomorphism $\rho\colon B \to E_\infty$ such that $\varphi = \rho \circ \theta$.

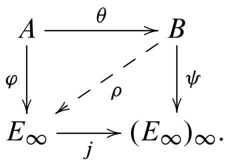
In analogy with [33, Lemma 1.4], we have the following:

Lemma 4.17. *Let $\theta\colon A \to B$ be a sequence algebra extendible homomorphism between C^*-algebras A and B. If B is weakly semiprojective, then so is A.*

Proof. This is straightforward. □

The following lemma will allow us to replace maps from separable C^*-algebras into $(E_\infty)_\infty$ with maps into E_∞. Its proof boils down to a more or less standard reindexation argument.

Lemma 4.18. *Let A and B be separable C^*-algebras, let E be a C^*-algebra. Denote by $j\colon E_\infty \to (E_\infty)_\infty$ the canonical embedding as constant sequences. Suppose that we are given homomorphisms $\theta\colon A \to B$, $\varphi\colon A \to E_\infty$ and $\psi\colon B \to (E_\infty)_\infty$ making the following diagram commute:*



Then there exists a homomorphism $\rho\colon B \to E_\infty$ such that $\rho \circ \theta = \varphi$.

Proof. Let $(\psi^{(n)})_{n\in\mathbb{N}}$ be a sequence of linear maps $\psi^{(n)}\colon B \to E_\infty$ lifting ψ. For $n \in \mathbb{N}$, let $(\psi_m^{(n)})_{m\in\mathbb{N}}$ be a sequence of linear maps $\psi_m^{(n)}\colon B \to E$ lifting $\psi^{(n)}$. Let also $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of linear maps $\varphi_k\colon A \to E$ lifting φ. With the natural representation of elements in $(E_\infty)_\infty$ by doubly indexed sequences in E, the identity $\psi \circ \theta = j \circ \varphi$ can be rephrased as

$$\lim_{n\to\infty} \limsup_{m\to\infty} \left\| \psi_m^{(n)}(\theta(a)) - \varphi_m(a) \right\| = 0$$

for all $a \in A$. Let $(F_r)_{r\in\mathbb{N}}$ and $(G_r)_{r\in\mathbb{N}}$ be sequences of finite subsets of A and B, respectively, such that $\bigcup_{r\in\mathbb{N}} F_r$ is dense in A and $\bigcup_{r\in\mathbb{N}} G_r$ is dense in B. Without loss of generality, we may assume that $F_r^* = F_r$ and $F_k^2 \subseteq F_{r+1}$ for all $r \in \mathbb{N}$, and similarly with the sets G_r for $r \in \mathbb{N}$. Likewise, we may assume that $\theta(F_r) \subseteq G_r$ for all $r \in \mathbb{N}$.

For each $r \in \mathbb{N}$, find an integer n_r such that

(1) $\limsup\limits_{m\to\infty} \left\| \psi_m^{(n_r)}(\theta(a)) - \varphi_m(a) \right\| < \frac{1}{r}$ for all $a \in F_r$;

(2) $\limsup\limits_{m\to\infty} \left\| \psi_m^{(n_r)}(b^*c) - \psi_m^{(n_r)}(b)^*\psi_m^{(n_r)}(c) \right\| < \frac{1}{r}$ for all $b, c \in G_r$;

(3) $\limsup_{m\to\infty} \left\| \psi_m^{(n_r)}(b) \right\| < \|b\| + \frac{1}{r}$ for all $b \in G_r$.

Without loss of generality, we may assume that $n_{r+1} > n_r$ for all $r \in \mathbb{N}$. Similarly, find an increasing sequence $(m_r)_{r\in\mathbb{N}}$ in $\mathbb{N}$ satisfying

(1′) $\left\| \psi_{m_r}^{(n_r)}(\theta(a)) - \varphi_{m_r}(a) \right\| < \frac{1}{r}$ for all $a \in F_r$;

(2′) $\left\| \psi_{m_r}^{(n_r)}(b^*c) - \psi_{m_r}^{(n_r)}(b)^* \psi_{m_r}^{(n_r)}(c) \right\| < \frac{1}{r}$ for all $b, c \in G_r$;

(3′) $\left\| \psi_{m_r}^{(n_r)}(b) \right\| < \|b\| + \frac{1}{r}$ for all $b \in G_r$.

Recall that $\eta_E\colon \ell^\infty(\mathbb{N}, E) \to E_\infty$ denotes the canonical quotient map. Define $\rho\colon B \to \ell^\infty(\mathbb{N}, E)$ by $\rho(b) = \eta_E\big(\psi_{m_r}^{(n_r)}(b)\big)_{r\in\mathbb{N}}$ for $b \in \mathbb{N}$. (One first defines ρ on the union of the G_r, and since it is multiplicative and contractive by construction, it extends to a homomorphism from all of B.) Since the identity $\rho \circ \theta = \varphi$ holds on a dense subspace of A, it holds on all of A. This finishes the proof. □

In the next proposition, we show that weak semiprojectivity passes to fixed point algebras of actions with the Rokhlin property (and to crossed products, whenever the group is finite). Our conclusions seem not to be obtainable with the methods developed in [35], since it is not in general true that a corner of a weakly semiprojective C^*-algebra is weakly semiprojective.

Proposition 4.19. *Let G be a second-countable compact group, let A be a separable C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the canonical inclusion $\iota\colon A^\alpha \to A$ is sequence algebra extendible (Definition 4.16).*

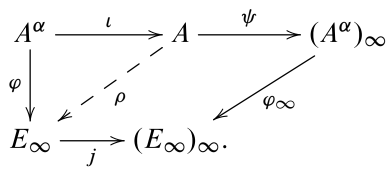
In particular, if A is weakly semiprojective, then so is A^α by Lemma 4.17. If in addition G is finite, then $A \rtimes_\alpha G$ is also weakly semiprojective.

Proof. Use Theorem 2.11 to choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ of asymptotically $*$-multiplicative linear maps $\psi_n\colon A \to A^\alpha$ such that $\lim_{n\to\infty} \|\psi_n(a) - a\| = 0$ for all $a \in A^\alpha$. Regard $(\psi_n)_{n\in\mathbb{N}}$ as a homomorphism $\psi\colon A \to (A^\alpha)_\infty$ such that the restriction $\psi|_{A^\alpha}$ agrees with the canonical inclusion $A^\alpha \hookrightarrow (A^\alpha)_\infty$.

Let E be a C^*-algebra and let $\varphi\colon A^\alpha \to E_\infty$ be a homomorphism. Denote by

$$\varphi_\infty : (A^\alpha)_\infty \to (E_\infty)_\infty$$

the homomorphism induced by φ. There is a commutative diagram



By Lemma 4.18, there exists a homomorphism $\rho\colon A \to E_\infty$ such that $\varphi = \rho \circ \iota$. Thus ι is sequence algebra extendible, as desired.

If G is finite, then $A \rtimes_\alpha G$ can be canonically identified with $M_{|G|} \otimes A^\alpha$, and hence it is also weakly semiprojective. □

Finally, we point out that weak semiprojectivity does not in general pass to crossed products by Rokhlin actions when the group is compact but not finite. Indeed, $C(\mathbb{T}) \rtimes_{\mathtt{Lt}} \mathbb{T} \cong \mathcal{K}(L^2(\mathbb{T}))$ is not weakly semiprojective, while $C(\mathbb{T})$ is even semiprojective.

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