J. Noncommut. Geom. 11 (2017), 1627–1643
DOI 10.4171/JNCG/11-4-12 **Journal of Noncommutative Geometry**

$\mathbb{A}^1$-homotopy invariants of corner skew Laurent polynomial algebras

Gonçalo Tabuada*

Abstract. In this note we prove some structural properties of all the $\mathbb{A}^1$-homotopy invariants of corner skew Laurent polynomial algebras. As an application, we compute the mod-l algebraic K-theory of Leavitt path algebras using solely the kernel/cokernel of the incidence matrix. This leads naturally to some vanishing and divisibility properties of the K-theory of these algebras.

Mathematics Subject Classification (2010). 14A22, 16S36, 19D50, 19D55.

Keywords. Corner skew Laurent polynomial algebra, Leavitt path algebra, algebraic K-theory, noncommutative mixed motives, noncommutative algebraic geometry.

1. Introduction

Given a C^*-algebra C and a "corner" isomorphism ϕ of C, Paschke [13] introduced in the eighties the associated semi-group crossed product C^*-algebra $C \rtimes_\phi \mathbb{N}$. Later, making use of Pimsner-Voiculescu's six-term exact sequence, Rørdam [14, §2] computed the operator K-theory of $C \rtimes_\phi \mathbb{N}$. Paschke's construction and Rørdam's computation still play nowadays a key role in noncommutative geometry.

The main goal of this article is to establish a purely algebraic analogue of Rørdam's computation, where a C^*-algebra is replaced by an algebra A (or, more generally, by a dg category), a semi-group crossed product C^*-algebra is replaced by a corner skew Laurent polynomial algebra $A[t_+, t_-; \phi]$ (see §2), and the operator K-theory is replaced by any $\mathbb{A}^1$-homotopy invariant of dg categories (see §3).

2. Corner skew Laurent polynomial algebras

Let k be a field, A a unital k-algebra, e an idempotent of A, and $\phi\colon A \xrightarrow{\sim} eAe$ a "corner" isomorphism. Following Ara–Barroso–Goodearl–Pardo [3, §2], the

*The author was partially supported by the National Science Foundation CAREER Award #1350472 and the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

associated *corner skew Laurent polynomial algebra* $A[t_+, t_-; \phi]$ is defined as follows: the elements are formal expressions $t_-^m a_{-m} + \cdots + t_- a_{-1} + a_0 + a_1 t_+ + \cdots + a_n t_+^n$ with $a_{-i} \in \phi^i(1)A$ and $a_i \in A\phi^i(1)$ for every $i \geq 0$; the addition is defined componentwise; the multiplication is determined by the distributive law and by the relations $t_- t_+ = 1$, $t_+ t_- = e$, $at_- = t_-\phi(a)$ for every $a \in A$, and $t_+ a = \phi(a) t_+$ for every $a \in A$. Note that $A[t_+, t_-; \phi]$ admits a canonical $\mathbb{Z}$-grading with $\deg(t_\pm) = \pm 1$.

As proved in [3, Lem. 2.4], the corner skew Laurent polynomial algebras can be characterized as those $\mathbb{Z}$-graded algebras $C = \bigoplus_{n \in \mathbb{Z}} C_n$ containing elements $t_+ \in C_1$ and $t_- \in C_{-1}$ such that $t_- t_+ = 1$. Concretely, we have $C = A[t_+, t_-; \phi]$ with $A := C_0$, $e := t_+ t_-$, and $\phi\colon C_0 \to t_+ t_- C_0 t_+ t_-$ given by $c_0 \mapsto t_+ c_0 t_-$.

Example 2.1 (Skew Laurent polynomial algebras). When $e = 1$, $A[t_+, t_-; \phi]$ reduces to the classical skew Laurent polynomial algebra $A \rtimes_\phi \mathbb{Z}$. In the particular case where ϕ is the identity, $A \rtimes_\phi \mathbb{Z}$ reduces furthermore to $A[t, t^{-1}]$.

Example 2.2 (Jacobson algebras). Following [8], the *Jacobson algebra* J_n*,* $n \geq 0$*,* is the k-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$. Note that the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes J_n into a corner skew Laurent polynomial algebra. The algebras J_n are also usually called Cohn algebras (see [1]), and J_0 the (algebraic) Toeplitz algebra.

Example 2.3 (Leavitt algebras). Following [12], the *Leavitt algebra* L_n*,* $n \geq 0$*,* is the k-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$ and $\sum_{i=0}^n x_i y_i = 1$. Note the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes L_n into a corner skew Laurent polynomial algebra. Note also that $L_0 \simeq k[t, t^{-1}]$. In the remaining case $n \geq 1$, L_n is the universal example of a k-algebra of *module type* $(1, n+1)$, i.e. $L_n \simeq L_n^{\oplus(n+1)}$ as right L_n-modules.

Example 2.4 (Leavitt path algebras). Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver; Q_0 and Q_1 stand for the sets of vertex and arrows, respectively, and s and r for the source and target maps, respectively. We assume that Q has no *sources*, i.e. vertices $i \in Q_0$ such that $\{\alpha \mid r(\alpha) = i\} = \emptyset$. Consider the double quiver $\overline{Q} = (Q_0, Q_1 \cup Q_1^*, s, r)$ obtained from Q by adding an arrow α^*, in the converse direction, for each arrow $\alpha \in Q_1$. Following Abrams–Pino [2] and Ara–Moreno–Pardo [5], the *Leavitt path algebra* L_Q *of* Q is the quotient of the quiver algebra $k\overline{Q}$ (which is generated by elements $\alpha \in Q_1 \cup Q_1^*$ and e_i with $i \in Q_0$) by the Cuntz–Krieger's relations: $\alpha^*\beta = \delta_{\alpha\beta} e_{r(\alpha)}$ for every $\alpha, \beta \in Q_1$; $\sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} \alpha\alpha^* = e_i$ for every non-sink $i \in Q_0$. Note that L_Q admits a canonical $\mathbb{Z}$-grading with $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$. For every vertex $i \in Q_0$ choose an arrow α_i such that $r(\alpha_i) = i$ and consider the associated elements $t_+ := \sum_{i \in Q_0} \alpha_i$ and $t_- := t_+^*$. Since $\deg(t_\pm) = \pm 1$ and $t_- t_+ = 1$, L_Q is also an example of a corner skew Laurent polynomial algebra.

In the particular case where Q is the quiver with one vertex and $n+1$ arrows, L_Q is isomorphic to L_n. Similarly, when Q is the quiver with two vertices $\{1, 2\}$ and $2(n+1)$ arrows ($n+1$ from 1 to 1 and $n+1$ from 1 to 2), we have $L_Q \simeq J_n$.

3. $\mathbb{A}^1$-homotopy invariants

A *dg category* $\mathcal{A}$, over a base field k, is a category enriched over cochain complexes of k-vector spaces; see §5.1. Every (dg) k-algebra A gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$. In what follows, we denote by $\mathrm{dgcat}(k)$ the category of (small) dg categories and dg functors.

A functor $E\colon \mathrm{dgcat}(k) \to \mathcal{T}$, with values in a triangulated category, is called a *localizing invariant* if it satisfies the following three conditions:

- it inverts the derived Morita equivalences (see §5.1);
- it sends[1] sequential (homotopy) colimits to sequential homotopy colimits;
- it sends[1] short exact sequences of dg categories, in the sense of Drinfeld [6] and Keller [11] (see [9, §4.6]), to distinguished triangles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \mapsto E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A}) .$$

When E inverts moreover the dg functors $\mathcal{A} \to \mathcal{A}[t]$, where $\mathcal{A}[t]$ stands for the tensor product $\mathcal{A} \otimes k[t]$, we call it an $\mathbb{A}^1$*-homotopy invariant*. Examples of localizing invariants include Hochschild homology HH, topological Hochschild homology THH, cyclic homology HC, the mixed complex C, nonconnective algebraic K-theory $I\!K$, mod-l^ν nonconnective algebraic K-theory $I\!K(-;\mathbb{Z}/l^\nu)$ (with l^ν a prime power) and its variant $I\!K(-)\otimes\mathbb{Z}[1/l]$, homotopy K-theory KH, mod-l^ν homotopy K-theory $KH(-;\mathbb{Z}/l^\nu)$, and étale K-theory $K^{\mathrm{et}}(-;\mathbb{Z}/l^\nu)$; see [16, §8.2]. Among those, $I\!K(-;\mathbb{Z}/l^\nu)$ (with $l \nmid \mathrm{char}(k)$), $I\!K(-)\otimes\mathbb{Z}[1/l]$ (with $l = \mathrm{char}(k)$), KH, $KH(-;\mathbb{Z}/l^\nu)$, and $K^{\mathrm{et}}(-;\mathbb{Z}/l^\nu)$ are $\mathbb{A}^1$-homotopy invariants; see [16, §8.5]. When applied to A, resp. to $\mathrm{perf}_{\mathrm{dg}}(X)$, the preceding invariants reduce to the corresponding invariants of the (dg) k-algebra A, resp. of the k-scheme X.

Example 3.1 (Noncommutative mixed motives). Let $\mathrm{Mot}_{\mathrm{loc}}(k)$, resp. $\mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)$, be the category of noncommutative mixed motives constructed in [16, §8.2], resp. in [16, §8.5.1]. By construction, this triangulated category comes equipped with a localizing invariant $\mathrm{U}_{\mathrm{loc}}\colon \mathrm{dgcat}(k) \to \mathrm{Mot}_{\mathrm{loc}}(k)$, resp. with an $\mathbb{A}^1$-homotopy invariant $\mathrm{U}_{\mathrm{loc}}^{\mathbb{A}^1}\colon \mathrm{dgcat}(k) \to \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)$, which is initial among all localizing invariants, resp. among all $\mathbb{A}^1$-homotopy invariants; consult [16, §8-9] for further details.

[1] In a way which is functorial for strict morphisms of sequential colimits and strict morphisms of short exact sequences of dg categories.

4. Statement of results and applications

Let A be a unital k-algebra, e an idempotent of A, and $\phi\colon A \xrightarrow{\sim} eAe$ a "corner" isomorphism. Out of this data, we can construct the associated corner skew Laurent polynomial algebra $A[t_+, t_-; \phi]$. In the same vein, let us write $A[t_+; \phi]$ for the algebra defined by the formal expressions $a_0 + a_1 t_+ + a_2 t_+^2 + \cdots + a_n t_+^n$ and by the relations $t_+ a = \phi(a) t_+$. Consider the A-A-bimodule ${}_\phi A$ associated to ϕ (see Notation 5.1) and the *square-zero extension* $A[\epsilon] := A \ltimes ({}_\phi A[1])$ of A by the suspension ${}_\phi A[1]$ of ${}_\phi A$. Concretely, $A[\epsilon] = A \oplus (eA)[1]$ with multiplication law $(a, eb) \cdot (a', eb') := (aa', e(ab' + ba'))$. Consider also the dg $A[\epsilon]$-$A[t_+; \phi]$-bimodule B whose restriction to $A[t_+; \phi]$ is the projective resolution $t_+ \cdot -\colon A[t_+; \phi] \to eA[t_+; \phi]$ of the trivial right $A[t_+; \phi]$-module eA. The left dg action is given by the canonical identification of $A[\epsilon]$ with the Ext-algebra $Ext^*_{A[t_+;\phi]}(eA)$. As explained in §5.1, the dg bimodule B corresponds to a morphism from $A[\epsilon]$ to $A[t_+; \phi]$ in the localization of dgcat(k) with respect to derived Morita equivalences. Therefore, given any functor E which inverts derived Morita equivalences, we obtain an induced morphism $E(\mathrm{B})\colon E(A[\epsilon]) \to E(A[t_+; \phi])$.

Theorem 4.1. *For every localizing invariant E, we have the homotopy colimit*

$$\mathrm{hocofib}\, E(\mathrm{B}) \longrightarrow \mathrm{hocofib}\, E(\mathrm{B}) \longrightarrow \cdots \longrightarrow E(A[t_+, t_-; \phi])\,, \tag{4.1}$$

where the transition morphism(s) is induced by the corner isomorphism ϕ.

When E is an $\mathbb{A}^1$-homotopy invariant, $E(\mathrm{B})$ reduces to $\mathrm{id} - E({}_\phi A)\colon E(A) \to E(A)$ and the composition (4.1) is an isomorphism. Consequently, we obtain a triangle

$$E(A) \xrightarrow{\mathrm{id} - E({}_\phi A)} E(A) \longrightarrow E(A[t_+, t_-; \phi]) \xrightarrow{\partial} \Sigma E(A)\,. \tag{4.2}$$

Remark 4.3 (Generalization). Given a dg category $\mathcal{A}$, Theorem 4.1 holds more generally with A and $A[t_+, t_-; \phi]$ replaced by $\mathcal{A} \otimes A$ and $\mathcal{A} \otimes A[t_+, t_-; \phi]$; see §8.

Corollary 4.4. *Given a dg category $\mathcal{A}$, we have a distinguished triangle of spectra:*

$$KH(\mathcal{A} \otimes A) \xrightarrow{\mathrm{id} - KH({}_{\mathrm{id}}\mathcal{A} \otimes_\phi A)} KH(\mathcal{A} \otimes A) \to KH(\mathcal{A} \otimes A[t_+, t_-; \phi]) \xrightarrow{\partial} \Sigma KH(\mathcal{A} \otimes A)\,.$$

Remark 4.5 (Related work). Given a (not necessarily unital) k-algebra C, Ara–Brustenga–Cortiñas proved in [4, Thms. 3.6 and 8.4] the analogue of Corollary 4.4 with $\mathcal{A}$ replaced by C. Their proof, which is inspired from operator K-theory, makes essential use of non-unital algebras. Since these latter objects don't belong to the realm of dg categories, our proof of Corollary 4.4 is necessarily different. Among other ideas used in the proof, we show that every corner skew Laurent polynomial algebra is derived Morita equivalent to a dg orbit category in the sense of Keller; see Proposition 7.6. Finally, note that in contrast with Ara–Brustenga–Cortiñas' result, Corollary 4.4 (and more generally Theorem 4.1) holds for dg algebras and schemes.

Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver, without sources, with v vertices and v' sinks. We assume that the set Q_0 is ordered with the first v' elements corresponding to the sinks. Let I'_Q be the incidence matrix of Q, I_Q the matrix obtained from I'_Q by removing the first v' rows (which are zero), and I^t_Q the transpose of I_Q.

Theorem 4.2. *Let $\mathcal{A}$ be a dg category and Q a finite quiver without sources. For every $\mathbb{A}^1$-homotopy invariant E, we have a distinguished triangle*

$$\oplus_{i=1}^{v-v'} E(\mathcal{A}) \xrightarrow{\binom{0}{\mathrm{id}}-I^t_Q} \oplus_{i=1}^{v} E(\mathcal{A}) \longrightarrow E(\mathcal{A} \otimes L_Q) \xrightarrow{\partial} \oplus_{i=1}^{v-v'} \Sigma E(\mathcal{A})\,.$$

Theorem 4.2 shows that all the information concerning $\mathbb{A}^1$-homotopy invariants of Leavitt path algebras L_Q is encoded in the incidence matrix of the quiver Q.

Example 4.6 (Jacobson algebras). Let Q be the quiver with two vertices $\{1, 2\}$ and $2(n+1)$ arrows ($n+1$ from 1 to 1 and $n+1$ from 1 to 2). In this particular case, the distinguished triangle of Theorem 4.2 reduces to

$$E(\mathcal{A}) \xrightarrow{\binom{n+1}{n}} E(\mathcal{A}) \oplus E(\mathcal{A}) \longrightarrow E(\mathcal{A} \otimes J_n) \xrightarrow{\partial} \Sigma E(\mathcal{A})\,.$$

Since $(n, n+1) = 1$, we conclude that $E(\mathcal{A} \otimes J_n) \simeq E(\mathcal{A})$. This shows that the $\mathbb{A}^1$-homotopy invariants don't distinguish the Jacobson algebras J_n from k. Note that J_n is much bigger than k; for instance, it contains the path algebra kQ.

Example 4.7 (Leavitt algebras). Let Q be the quiver with one vertex and $n+1$ arrows. In this particular case, the distinguished triangle of Theorem 4.2 reduces to

$$E(\mathcal{A}) \xrightarrow{n\cdot\mathrm{id}} E(\mathcal{A}) \longrightarrow E(\mathcal{A} \otimes L_n) \xrightarrow{\partial} \Sigma E(\mathcal{A})\,. \tag{4.8}$$

When $n = 0$, the distinguished triangle (4.8) splits and gives rise to the "fundamental" isomorphism $E(\mathcal{A} \otimes L_0) \simeq E(\mathcal{A}) \oplus \Sigma E(\mathcal{A})$. When $n = 1$, we have $E(\mathcal{A} \otimes L_1) = 0$. In the remaining case $n \geq 2$, $E(\mathcal{A} \otimes L_n)$ identifies with the mod-n Moore object of $E(\mathcal{A})$. Intuitively speaking, this shows that the functor $\mathcal{A} \mapsto \mathcal{A} \otimes L_n$, with $n \geq 2$, is a model of the mod-n Moore construction.

Proposition 4.9. *Let $l_1^{\nu_1} \times \cdots \times l_r^{\nu_r}$ be the prime decomposition of an integer $n \geq 2$. For every dg category $\mathcal{A}$ and $\mathbb{A}^1$-homotopy invariant E, we have a direct sum decomposition $E(\mathcal{A} \otimes L_n) \simeq E(\mathcal{A} \otimes L_{l_1^{\nu_1}}) \oplus \cdots \oplus E(\mathcal{A} \otimes L_{l_r^{\nu_r}})$.*

Roughly speaking, Proposition 4.9 shows that all the $\mathbb{A}^1$-homotopy invariants of Leavitt algebras are "l-local". Note that $L_n \not\simeq L_{l_1^{\nu_1}} \times \cdots \times L_{l_r^{\nu_r}}$.

Remark 4.10 (Homotopy K-theory). By taking $E = KH$ in Theorem 4.2, we obtain a distinguished triangle of spectra

$$\oplus_{i=1}^{v-v'} KH(\mathcal{A}) \xrightarrow{\binom{0}{\mathrm{id}}-I^t_Q} \oplus_{i=1}^{v} KH(\mathcal{A}) \to KH(\mathcal{A} \otimes L_Q) \xrightarrow{\partial} \oplus_{i=1}^{v-v'} \Sigma KH(\mathcal{A})\,.$$

Given a (not necessarily unital) k-algebra C, Ara–Brustenga–Cortiñas constructed in [4, Thm. 8.6] the analogue of the preceding distinguished triangle with $\mathcal{A}$ replaced by C. Our construction is different and applies also to dg categories and schemes.

Remark 4.11. By taking $E = KH$ and $n = l^\nu$ in Example 4.7, we obtain an isomorphism between $KH(\mathcal{A} \otimes L_{l^\nu})$ and the mod-l^ν homotopy K-theory spectrum $KH(\mathcal{A}; \mathbb{Z}/l^\nu)$. When $l \nmid \mathrm{char}(k)$, the latter spectrum is isomorphic to $I\!K(\mathcal{A}; \mathbb{Z}/l^\nu)$.

Mod-l^ν algebraic K-theory of Leavitt path algebras. Let l^ν be a prime power such that $l \neq \mathrm{char}(k)$ and Q a finite quiver without sources. By taking $\mathcal{A} = k$ and $E = I\!K(-; \mathbb{Z}/l^\nu)$ in Theorem 4.2, we obtain a distinguished triangle of spectra

$$\oplus_{i=1}^{v-v'} I\!K(k; \mathbb{Z}/l^\nu) \xrightarrow{\binom{0}{\mathrm{id}} - I_Q^t} \oplus_{i=1}^{v} I\!K(k; \mathbb{Z}/l^\nu) \to I\!K(L_Q; \mathbb{Z}/l^\nu) \xrightarrow{\partial} \oplus_{i=1}^{v-v'} \Sigma I\!K(k; \mathbb{Z}/l^\nu)\,.$$

Remark 4.12. The preceding triangle follows also from the work of Ara–Brustenga–Cortiñas [4]. Indeed, since by hypothesis $l \nmid \mathrm{char}(k)$, the functors $I\!K(-; \mathbb{Z}/l^\nu)$ and $KH(-; \mathbb{Z}/l^\nu)$ are isomorphic. Moreover, as explained in Remark 4.11, the latter functor identifies with $KH(- \otimes L_{l^\nu})$. Therefore, if in Remark 4.10 we take for C the k-algebra L_{l^ν}, we obtain the preceding distinguished triangle of spectra.

Assume that k is algebraically closed. As proved by Suslin[2] in [15, Cor. 3.13], we have $I\!K_n(k; \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l^\nu$ if $n \geq 0$ is even and $I\!K_n(k; \mathbb{Z}/l^\nu) = 0$ otherwise. Consequently, making use of the long exact sequence of algebraic K-theory groups associated to the preceding triangle of spectra, we obtain the following result:

Corollary 4.13. *We have the following computation*

$$I\!K_n(L_Q; \mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel of M} & \text{if } n \geq 0 \text{ even,} \\ \text{kernel of M} & \text{if } n \geq 0 \text{ odd,} \\ 0 & \text{if } n < 0\,, \end{cases}$$

where M *stands for the homomorphism* $\bigoplus_{i=1}^{v-v'} \mathbb{Z}/l^\nu \xrightarrow{\binom{0}{\mathrm{id}} - I_Q^t} \bigoplus_{i=1}^{v} \mathbb{Z}/l^\nu$.

Corollary 4.13 provides a complete and explicit computation of the mod-l^ν (nonconnective) algebraic K-theory of Leavitt path algebras. To the best of the author's knowledge, these computations are new in the literature. In particular, they yield a complete answer to the "mod-l^ν version" of Question 2 raised by Gabe–Ruiz–Tomforde–Whalen in [7, p. 38]. These computations lead also naturally to the following vanishing and divisibility properties of algebraic K-theory:

[2] Given a quiver Q, let $C^*_{\mathbb{C}}(Q)$ be the associated Cuntz–Krieger C^*-algebra. Cortiñas kindly informed the author that the work of Suslin was also used in [4, Thm. 9.4] in order to prove that $I\!K_n(\mathbb{C} \otimes L_Q) \simeq K_n^{\mathrm{top}}(C^*_{\mathbb{C}}(Q)), n \geq 0$, for every quiver Q without sinks such that $\det(\binom{0}{\mathrm{id}} - I_Q^t) \neq 0$.

Proposition 4.14. (i) *If there exists a prime power l^ν and an even (resp. odd) integer $n' \geq 0$ such that $I\!K_{n'}(L_Q;\mathbb{Z}/l^\nu) \neq 0$, then for every even (resp. odd) integer $n \geq 0$ at least one of the groups $I\!K_n(L_Q)$, $I\!K_{n-1}(L_Q)$ is non-zero.*

(ii) *If there exists a prime power l^ν such that $I\!K_n(L_Q;\mathbb{Z}/l^\nu) = 0$ for every $n \geq 0$, then the groups $I\!K_n(L_Q), n \geq 0$, are uniquely l^ν-divisible, i.e. $\mathbb{Z}[1/l^\nu]$-modules.*

Proof. Combine the universal coefficients sequence (see [16, §2.2.2])

$$0 \longrightarrow I\!K_n(L_Q) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \longrightarrow I\!K_n(L_Q;\mathbb{Z}/l^\nu) \longrightarrow {}_{l^\nu} I\!K_{n-1}(L_Q) \longrightarrow 0$$

with the computation of Corollary 4.13. □

Example 4.15 (Quivers without sinks). Let Q be a quiver without sinks. In this case, $\binom{0}{\mathrm{id}} - I_Q^t$ is a square matrix. If l is a prime such that $l \nmid \det(\binom{0}{\mathrm{id}} - I_Q^t)$, then the homomorphism M of Corollary 4.13 is invertible. Consequently, $I\!K_n(L_Q;\mathbb{Z}/l^\nu) = 0$ for every $n \geq 0$. Making use of Proposition 4.14(ii), we then conclude that the algebraic K-theory groups $I\!K_n(L_Q), n \geq 0$, are uniquely l^ν-divisible.

Schemes and stacks. Let X be a quasi-compact quasi-separated k-scheme. By applying the results/examples/remarks of §4 to the dg category $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(X)$, we obtain corresponding results/examples/remarks concerning the scheme X. For instance, Remark 4.11 yields an isomorphism between $KH(\mathrm{perf}_{\mathrm{dg}}(X) \otimes L_{l^\nu})$ and $KH(X;\mathbb{Z}/l^\nu)$. When $l \nmid \mathrm{char}(k)$, the latter spectrum is isomorphic to $I\!K(X;\mathbb{Z}/l^\nu)$. Roughly speaking, the dg category $\mathrm{perf}_{\mathrm{dg}}(X) \otimes L_{l^\nu}$ may be understood as the "noncommutative mod-l^ν Moore object of X". More generally, we can consider the dg category $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(\mathcal{X})$ of perfect complexes of an algebraic stack $\mathcal{X}$. In the particular case of a quotient stack $\mathcal{X} = [X/G]$, with G an algebraic group scheme acting on X, Remark 4.11 yields an isomorphism between $KH(\mathrm{perf}_{\mathrm{dg}}([X/G]) \otimes L_{l^\nu})$ and the mod-l^ν G-equivariant homotopy K-theory spectrum $KH^G(X;\mathbb{Z}/l^\nu)$. When $l \nmid \mathrm{char}(k)$, the latter spectrum is isomorphic to $I\!K^G(X;\mathbb{Z}/l^\nu)$.

5. Preliminaries

5.1. Dg categories. Let $(\mathcal{C}(k), \otimes, k)$ be the category of cochain complexes of k-vector spaces. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [9].

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}(x,y) := \mathcal{A}(y,x)$. A *right dg $\mathcal{A}$-module* is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right dg $\mathcal{A}$-modules. Following [9, §3.2], the *derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$* is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. We write $\mathcal{D}_c(\mathcal{A})$ for the subcategory of compact objects.

A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *derived Morita equivalence* if it induces an equivalence of categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [9, §4.6]. As proved in [16, Thm. 1.37], $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the derived Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of (small) dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y) \otimes \mathcal{B}(w,z)$. As explained in [9, §2.3], this construction gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$, which descends to the homotopy category $\mathrm{Hmo}(k)$.

A *dg* $\mathcal{A}$*-*$\mathcal{B}$*-bimodule* B is a dg functor $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ or equivalently a right dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. Associated to a dg functor $F\colon \mathcal{A} \to \mathcal{B}$, we have the dg $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_F\mathcal{B}\colon \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), (x,z) \mapsto \mathcal{B}(z, F(x))$. Let us write $\mathrm{rep}(\mathcal{A},\mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every object $x \in \mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x,-)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Clearly, the dg $\mathcal{A}$-$\mathcal{B}$-bimodules ${}_F\mathcal{B}$ belongs to $\mathrm{rep}(\mathcal{A},\mathcal{B})$.

As explained in [16, §1.6.3], there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A},\mathcal{B})$ and the set of isomorphism classes of $\mathrm{rep}(\mathcal{A},\mathcal{B})$. Under this bijection, the composition law corresponds to the tensor product of dg bimodules.

Notation 5.1. Given a non-unital homomorphism $\phi\colon A \to B$ between unital k-algebras, let us denote by ${}_\phi B$ the A-B-bimodule $\phi(1)B$ equipped with the A-B-action $a \cdot \phi(1)B \cdot b := \phi(a)Bb$. Note that ${}_\phi B$ belongs to $\mathrm{rep}(A,B)$.

Square-zero extensions. Let $\mathcal{A}$ be a dg category and B a dg $\mathcal{A}$-$\mathcal{A}$-bimodule. The *square-zero extension* $\mathcal{A} \ltimes \mathrm{B}$ *of* $\mathcal{A}$ *by* B is the dg category with the same objects as $\mathcal{A}$ and complexes of k-vector spaces $(\mathcal{A} \ltimes \mathrm{B})(x,y) := \mathcal{A}(x,y) \oplus \mathrm{B}(y,x)$. Given morphisms $(f,f') \in (\mathcal{A} \ltimes \mathrm{B})(x,y)$ and $(g,g') \in (\mathcal{A} \ltimes \mathrm{B})(y,z)$, the composition $(g,g') \circ (f,f')$ is defined as $(g \circ f, g'f + gf')$.

Dg orbit categories. Let $F\colon \mathcal{A} \to \mathcal{A}$ be an equivalence of dg categories. Following Keller [10, §5.1], the associated *dg orbit category* $\mathcal{A}/F^{\mathbb{Z}}$ has the same objects as $\mathcal{A}$ and complexes of k-vector spaces $(\mathcal{A}/F^{\mathbb{Z}})(x,y) := \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y))$. Given objects x, y, z and morphisms

$$\mathrm{f} = \{f_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y)) \quad \text{and} \quad \mathrm{g} = \{g_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(y, F^n(z)),$$

the m^{th}-component of $\mathrm{g} \circ \mathrm{f}$ is defined as $\sum_n F^n(g_{m-n}) \circ f_n$. When $\mathcal{A}$ is a k-algebra A, the dg functor F reduces to an isomorphism $\phi\colon A \xrightarrow{\sim} A$ and the dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$ to the skew Laurent polynomial algebra $A \rtimes_\phi \mathbb{Z}$.

Let us write $\mathcal{A}/F^{\mathbb{N}}$ for the dg category with the same objects as $\mathcal{A}$ and complexes of k-vector spaces $(\mathcal{A}/F^{\mathbb{N}})(x,y) := \bigoplus_{n\geq 0} \mathcal{A}(x, F^n(y))$. The composition law is defined as above. By construction, we have a canonical dg functor $\mathcal{A}/F^{\mathbb{N}} \to \mathcal{A}/F^{\mathbb{Z}}$.

6. Dg categories of idempotents

Let A be a (not necessarily unital) k-algebra.

Definition 6.1. The *dg category of idempotents of* A, denoted by $\underline{A}$, is defined as follows: the objects are the symbols $\mathbf{e}$ with e an idempotent of A; the (complexes of) k-vector spaces $\underline{A}(\mathbf{e}, \mathbf{e}')$ are given by eAe'; the composition law is induced by the multiplication in A; the identity of the object $\mathbf{e}$ is the idempotent e.

Notation 6.2. Let $\mathrm{alg}(k)$ be the category of (not necessarily unital) k-algebras and (not necessarily unital) k-algebra homomorphisms.

Note that the preceding construction gives rise to the following functor:

$$\mathrm{alg}(k) \longrightarrow \mathrm{dgcat}(k) \quad A \mapsto \underline{A} \quad \phi \mapsto \underline{\phi}\,. \tag{6.3}$$

Lemma 6.4. *The functor* (6.3) *preserves filtered colimits.*

Proof. Consider a filtered diagram $\{A_i\}_{i\in I}$ in $\mathrm{alg}(k)$ with colimit A. Given an idempotent element e of A, there exists an index $i' \in I$ and an idempotent $e_{i'} \in A_{i'}$ such that e is the image of $e_{i'}$ under $A_{i'} \to A$. This implies that the induced dg functor $\mathrm{colim}_i \underline{A_i} \to \underline{A}$ is not only (essentially) surjective but also fully-faithful. □

Given a unital k-algebra A with unit 1, let us write $\iota\colon A \to \underline{A}$ for the (unique) dg functor sending the single object of A to the symbol $\mathbf{1}$.

Lemma 6.5. *The dg functor* ι *is a derived Morita equivalence.*

Proof. Note first that the dg functor ι is fully-faithful. Given an idempotent element e of A, the morphisms $\mathbf{1} \stackrel{e}{\to} \mathbf{e}$ and $\mathbf{e} \stackrel{e}{\to} \mathbf{1}$ present the object $\mathbf{e}$ as a direct summand of $\mathbf{1}$. This allows us to conclude that ι is a derived Morita equivalence. □

Remark 6.6. Given a non-unital homomorphism $\phi\colon A \to B$ between unital k-algebras, note that $_\iota\underline{B} \circ {}_\phi B = {}_{\underline{\phi}}\underline{B} \circ {}_\iota\underline{A}$ in the homotopy category $\mathrm{Hmo}(k)$.

Let A be a unital k-algebra and $M_2(A)$ the associated k-algebra of 2×2 matrices. Consider the following non-unital homomorphisms

$$j_1, j_2\colon A \longrightarrow M_2(A) \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Note that if there exist elements t_+, t_- of A such that $t_- t_+ = 1$, then we can also consider the non-unital homomorphism $\phi^{\pm}\colon A \to A, a \mapsto t_+ a t_-$.

Proposition 6.7. (i) *The dg functors $\underline{j_1}$ and $\underline{j_2}$ are derived Morita equivalences. Moreover, their images in the homotopy category* $\mathrm{Hmo}(k)$ *are the same.*

(ii) *The dg functor $\underline{\phi^{\pm}}$ is a derived Morita equivalence. Moreover, its image in the homotopy category* $\mathrm{Hmo}(k)$ *is the identity morphism.*

Proof. (i) Recall first that a dg functor $F: \mathcal{A} \to \mathcal{B}$ is a derived Morita equivalence if and only if its image ${}_F\mathcal{B}$ in the homotopy category $\mathrm{Hmo}(k)$ is invertible. Thanks to Lemma 6.5 and Remark 6.6, it suffices then to show that the A-$M_2(A)$-bimodules ${}_{j_1}M_2(A)$ and ${}_{j_2}M_2(A)$ are invertible in $\mathrm{Hmo}(k)$. Note that their inverses are given by the $M_2(A)$-A-bimodules $M_2(A)j_1(1)$ and $M_2(A)j_2(1)$, respectively. This shows the first claim. The second claim follows from the isomorphism ${}_{j_1}M_2(A) \overset{\sim}{\to} {}_{j_2}M_2(A)$ of A-$M_2(A)$-bimodules given by $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 0 \\ a & b \end{smallmatrix}\right)$.

(ii) Consider the following non-unital homomorphism

$$\varphi^{\pm}: M_2(A) \longrightarrow M_2(A) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t_+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_- & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $j_1 \circ \phi^{\pm} = \varphi^{\pm} \circ j_1$ and $\varphi^{\pm} \circ j_2 = j_2$ in the category $\mathrm{alg}(k)$. By applying the functor (6.3), we hence conclude from item (i) that the dg functor $\underline{\phi^{\pm}}$ is not only a derived Morita equivalence but moreover that its image in the homotopy category $\mathrm{Hmo}(k)$ is the identity morphism. □

7. Proof of Theorem 4.1

Consider the sequential colimit diagram $A \overset{\phi}{\to} A \overset{\phi}{\to} \cdots \to C$ in the category $\mathrm{alg}(k)$. Note that the k-algebra C is non-unital and that the homomorphism ϕ gives rise to an isomorphism $\hat{\phi}: C \overset{\sim}{\to} C$. Let us denote by $C \rtimes_{\hat{\phi}} \mathbb{N}$, resp. $C \rtimes_{\hat{\phi}} \mathbb{Z}$, the associated skew polynomial algebra, resp. skew Laurent polynomial algebra. Note also that ϕ extends to (non-unital) homomorphisms $\phi: A[t_+, ; \phi] \to A[t_+; \phi], a \mapsto t_+ a t_-$, and $\phi^{\pm}: A[t_+, t_-; \phi] \to A[t_+, t_-; \phi], a \mapsto t_+ a t_-$ Under these notations, we have the following sequential colimit diagrams:

$$A[t_+; \phi] \overset{\phi}{\longrightarrow} A[t_+; \phi] \overset{\phi}{\longrightarrow} \cdots \longrightarrow C \rtimes_{\hat{\phi}} \mathbb{N} \tag{7.1}$$

$$A[t_+, t_-; \phi] \overset{\phi^{\pm}}{\longrightarrow} A[t_+, t_-; \phi] \overset{\phi^{\pm}}{\longrightarrow} \cdots \longrightarrow C \rtimes_{\hat{\phi}} \mathbb{Z}\,. \tag{7.2}$$

Lemma 7.3. *The dg category of idempotents of $C \rtimes_{\hat{\phi}} \mathbb{N}$, resp. $C \rtimes_{\hat{\phi}} \mathbb{Z}$, is derived Morita equivalent to the dg category $\underline{C}/\underline{\hat{\phi}}^{\mathbb{N}}$, resp. $\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}$.*

Proof. We focus ourselves in the algebra $C \rtimes_{\hat{\phi}} \mathbb{Z}$ and in the dg orbit category $\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}$; the proof of the other case is similar. Let us denote by 1_n the unit of the n^{th}

copy of $A[t_+, t_-; \phi]$ and by e_n the image of 1_n under the induced homomorphism $A[t_+, t_-; \phi] \to C \rtimes_{\hat{\phi}} \mathbb{Z}$. Given an idempotent element e of $C \rtimes_{\hat{\phi}} \mathbb{Z}$, there exists an integer $n \gg 0$ such that $\mathbf{e}$ is a direct summand of $\mathbf{e_n}$. Since e_n belongs to $C \subset C \rtimes_{\hat{\phi}} \mathbb{Z}$, this allows us to conclude, in particular, that the dg category of idempotents of $C \rtimes_{\hat{\phi}} \mathbb{Z}$ is derived Morita equivalent to its full dg category of symbols $\mathbf{e}$ with e an idempotent element of $C \subset C \rtimes_{\hat{\phi}} \mathbb{Z}$. Given any two such symbols $\mathbf{e}$ and $\mathbf{e}'$, note that we have the following equalities of (complexes of) k-vector spaces:

$$\begin{aligned}\underline{(C \rtimes_{\hat{\phi}} \mathbb{Z})}(\mathbf{e}, \mathbf{e}') &:= e(C \rtimes_{\hat{\phi}} \mathbb{Z})e' \\ &= \bigoplus_{n\in\mathbb{Z}} eC\hat{\phi}^n(e') \\ &= \bigoplus_{n\in\mathbb{Z}} \underline{C}(\mathbf{e}, \underline{\hat{\phi}}^n(\mathbf{e}')) =: (\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}})(\mathbf{e}, \mathbf{e}')\,.\end{aligned}$$

Under these equalities, the composition law of the dg category of idempotents of $C \rtimes_{\hat{\phi}} \mathbb{Z}$ corresponds to the composition law of the dg orbit category $\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}$. □

By combining Lemmas 6.4–6.5 with Remark 6.6, we conclude from Lemma 7.3 that (7.1)–(7.2) give rise to the following sequential (homotopy) colimit diagrams

$$A[t_+;\phi] \xrightarrow{\phi_{A[t_+;\phi]}} A[t_+;\phi] \xrightarrow{\phi_{A[t_+;\phi]}} \cdots \longrightarrow \underline{C}/\underline{\hat{\phi}}^{\mathbb{N}} \tag{7.4}$$

$$A[t_+, t_-;\phi] \xrightarrow{\phi^{\pm}_{A[t_+,t_-;\phi]}} A[t_+, t_-;\phi] \xrightarrow{\phi^{\pm}_{A[t_+,t_-;\phi]}} \cdots \longrightarrow \underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}} \tag{7.5}$$

in the homotopy category $\mathrm{Hmo}(k)$. Proposition 6.7(ii) (with $A[t_+, t_-; \phi]$ instead of A) leads then automatically to the following result:

Proposition 7.6. *The (transfinite) composition (7.5) is an isomorphism. Consequently, the dg categories $A[t_+, t_-; \phi]$ and $\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}$ are derived Morita equivalent.*

Now, consider the square-zero extension $\underline{C}[\epsilon] := \underline{C} \ltimes ({}_{\underline{\hat{\phi}}}\underline{C}[1])$ of $\underline{C}$ by the suspension ${}_{\underline{\hat{\phi}}}\underline{C}[1]$ of the dg $\underline{C}$-$\underline{C}$-bimodule ${}_{\underline{\hat{\phi}}}\underline{C}$ associated to $\underline{\hat{\phi}}$. Consider also the dg $\underline{C}[\epsilon]$-$(\underline{C}/\underline{\hat{\phi}}^{\mathbb{N}})$-bimodule $\hat{\mathrm{B}}$ introduced in [10, §4]; denoted by B' in *loc. cit.*

Lemma 7.7. *We have the following sequential (homotopy) colimit diagram*

$$A[\epsilon] \xrightarrow{\phi_{A[\epsilon]}} A[\epsilon] \xrightarrow{\phi_{A[\epsilon]}} \cdots \longrightarrow \underline{C}[\epsilon]$$

in the homotopy category $\mathrm{Hmo}(k)$.

Proof. Thanks to Lemma 6.4, we have $\underline{A} \xrightarrow{\phi} \underline{A} \xrightarrow{\phi} \cdots \to \underline{C}$ in the category $\mathsf{dgcat}(k)$. Consider the square-zero extension $\underline{A}[\epsilon] := \underline{A} \ltimes ({}_{\phi}\underline{A}[1])$ of $\underline{A}$ by the suspension ${}_{\phi}\underline{A}[1]$ of the dg $\underline{A}$-$\underline{A}$-bimodule ${}_{\phi}\underline{A}$ associated to $\underline{\phi}$. Similarly to the proof of Lemma 6.4, we have the following induced sequential colimit diagram

$$\underline{A}[\epsilon] \xrightarrow{\phi} \underline{A}[\epsilon] \xrightarrow{\phi} \cdots \longrightarrow \underline{C}[\epsilon]\,.$$

Note that the dg functor $A[\epsilon] \to \underline{A}[\epsilon]$ sending the single object $A[\epsilon]$ to the symbol $\mathbf{1}$, where 1 is the unit of A, is a derived Morita equivalence. Under such derived Morita equivalence the dg functor $\underline{\phi}\colon \underline{A}[\epsilon] \to \underline{A}[\epsilon]$ corresponds to the morphism ${}_{\phi}A[\epsilon]\colon A[\epsilon] \to A[\epsilon]$ in the homotopy category $\mathsf{Hmo}(k)$. This concludes the proof. □

By combining Lemma 7.7 with the sequential (homotopy) colimit diagram (7.4), we obtain the following sequential (homotopy) colimit diagram

$$\begin{array}{ccc} \underline{C}[\epsilon] & \xrightarrow{\hat{\mathrm{B}}} & \underline{C}/\underline{\hat{\phi}}^{\mathbb{N}} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ {\scriptstyle {}_{\phi}A[\epsilon]}\uparrow & & \uparrow{\scriptstyle {}_{\phi}A[t_+;\phi]} \\ A[\epsilon] & \xrightarrow{\mathrm{B}} & A[t_+;\phi] \\ {\scriptstyle {}_{\phi}A[\epsilon]}\uparrow & & \uparrow{\scriptstyle {}_{\phi}A[t_+;\phi]} \\ A[\epsilon] & \xrightarrow{\mathrm{B}} & A[t_+;\phi] \end{array} \tag{7.8}$$

in the homotopy category $\mathsf{Hmo}(k)$. As explained in [10, §4], given any localizing invariant E, we have a distinguished triangle

$$E(\underline{C}[\epsilon]) \xrightarrow{E(\hat{\mathrm{B}})} E(\underline{C}/\underline{\hat{\phi}}^{\mathbb{N}}) \longrightarrow E(\underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}) \xrightarrow{\partial} \Sigma E(\underline{C}[\epsilon])\,. \tag{7.9}$$

Therefore, by combining the diagram (7.8) with Proposition 7.6, we obtain the searched homotopy colimit diagram (4.1). This concludes the proof of the first claim.

We now prove the second claim. Let E be an $\mathbb{A}^1$-homotopy invariant. As explained in [10, Prop. 4.6], $E(\mathrm{B})$ reduces to $\mathrm{id} - E({}_{\phi}A)\colon E(A) \to E(A)$. Similarly, the morphism $E(\hat{\mathrm{B}})$ reduces to $\mathrm{id} - E({}_{\underline{\hat{\phi}}}\underline{C})\colon E(\underline{C}) \to E(\underline{C})$. Therefore, making use of Lemma 6.5 and of Remark 6.6, we observe that by applying the functor E to (7.8) we obtain (up to isomorphism) the following sequential colimit diagram:

$$\begin{array}{ccc} E(\underline{C}) & \xrightarrow{\mathrm{id}-E(\hat{\underline{\phi}})} & E(\underline{C}) \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ {\scriptstyle E(\underline{\phi})}\uparrow & & \uparrow{\scriptstyle E(\underline{\phi})} \\ E(\underline{A}) & \xrightarrow{\mathrm{id}-E(\underline{\phi})} & E(\underline{A}) \\ {\scriptstyle E(\underline{\phi})}\uparrow & & \uparrow{\scriptstyle E(\underline{\phi})} \\ E(\underline{A}) & \xrightarrow{\mathrm{id}-E(\underline{\phi})} & E(\underline{A})\,. \end{array} \tag{7.10}$$

We now claim that the induced (transfinite) composition

$$\mathrm{hocofib}(\mathrm{id}-E(\underline{\phi})) \xrightarrow{E(\underline{\phi})} \mathrm{hocofib}(\mathrm{id}-E(\underline{\phi})) \xrightarrow{E(\underline{\phi})} \cdots \longrightarrow \mathrm{hocofib}(\mathrm{id}-E(\hat{\underline{\phi}})) \tag{7.11}$$

is an isomorphism.

As explained in [16, Thm. 8.25], the functor $U_{\mathrm{loc}}^{\mathbb{A}^1}$ is the initial $\mathbb{A}^1$-homotopy invariant. Therefore, it suffices to prove the latter claim in the particular case where $E = U_{\mathrm{loc}}^{\mathbb{A}^1}$. By construction, we have a factorization

$$U_{\mathrm{loc}}^{\mathbb{A}^1}: \mathrm{dgcat}(k) \xrightarrow{U_{\mathrm{add}}} \mathrm{Mot}_{\mathrm{add}}(k) \xrightarrow{\gamma} \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)\,,$$

where $\mathrm{Mot}_{\mathrm{add}}(k)$ is a certain compactly generated triangulated category of noncommutative mixed motives, U_{add} is a certain functor sending sequential (homotopy) colimits to sequential homotopy colimits, and γ is a certain homotopy colimit preserving functor; consult [16, §8.4.2] for details. The triangulated category $\mathrm{Mot}_{\mathrm{add}}(k)$ is moreover enriched over spectra; we write $\mathrm{Hom}_{\mathrm{Spt}}(-,-)$ for this enrichment. Let NM be a compact object of $\mathrm{Mot}_{\mathrm{add}}(k)$. In order to prove our claim, it is then enough to show that the (transfinite) composition obtained by applying the functor $\mathrm{Hom}_{\mathrm{Spt}}(NM,-)$ to (7.11) (with $E = U_{\mathrm{add}}$) is an isomorphism.

Since the spectrum $\mathrm{Hom}_{\mathrm{Spt}}(NM, U_{\mathrm{add}}(\underline{C}))$ is the sequential homotopy colimit of $\mathrm{Hom}_{\mathrm{Spt}}(NM, U_{\mathrm{add}}(\underline{A}))$, with respect to the transition morphism(s) $\mathrm{Hom}_{\mathrm{Spt}}(NM, U_{\mathrm{add}}(\underline{\phi}))$, the proof follows now automatically from the general result [4, Lem. 3.3] concerning spectra. This finishes the proof of Theorem 4.1.

8. Proof of the generalization of Theorem 4.1

The triangulated category $\mathrm{Mot}_{\mathrm{loc}}(k)$ carries a symmetric monoidal structure making the functor U_{loc} symmetric monoidal; see [16, §8.3.1]. Therefore, the distinguished triangle (7.9) (with $E = U_{\mathrm{loc}}$) gives rise to the distinguished triangle:

$$U_{\mathrm{loc}}(\mathcal{A}\otimes \underline{C}[\epsilon]) \xrightarrow{U_{\mathrm{loc}}({}_{\mathrm{id}}\mathcal{A}\otimes \hat{\mathrm{B}})} U_{\mathrm{loc}}(\mathcal{A}\otimes \underline{C}/\underline{\hat{\phi}}^{\mathbb{N}}) \longrightarrow U_{\mathrm{loc}}(\mathcal{A}\otimes \underline{C}/\underline{\hat{\phi}}^{\mathbb{Z}}) \xrightarrow{\partial} \Sigma U_{\mathrm{loc}}(\mathcal{A}\otimes \underline{C}[\epsilon])\,.$$

Since the functor $\mathcal{A}\otimes -$ preserves (sequential) homotopy colimits, the combination of the preceding triangle with the commutative diagram (7.8) and with Proposition 7.6 leads then to the following sequential homotopy colimit diagram

$$\mathrm{hocofib}\, U_{\mathrm{loc}}({}_{\mathrm{id}}\mathcal{A}\otimes \mathrm{B}) \longrightarrow \mathrm{hocofib}\, U_{\mathrm{loc}}({}_{\mathrm{id}}\mathcal{A}\otimes \mathrm{B}) \longrightarrow \cdots \longrightarrow U_{\mathrm{loc}}(\mathcal{A}\otimes A[t_+,t_-;\phi])\,,$$

where the transition morphism(s) is induced by the corner isomorphism ϕ. The proof of the first claim follows now automatically from the fact that U_{loc} is the initial localizing invariant; see [16, Thm. 8.5].

The triangulated category $\mathrm{Mot}^{\mathbb{A}^1}_{\mathrm{loc}}(k)$ carries a symmetric monoidal structure making the functor $U^{\mathbb{A}^1}_{\mathrm{loc}}$ symmetric monoidal; see [16, §8.5.2]. Therefore, the distinguished triangle (4.2) (with $E = U^{\mathbb{A}^1}_{\mathrm{loc}}$) gives rise to the distinguished triangle:

$$U^{\mathbb{A}^1}_{\mathrm{loc}}(\mathcal{A}\otimes A) \xrightarrow{\mathrm{id}-U^{\mathbb{A}^1}_{\mathrm{loc}}({}_{\mathrm{id}}\mathcal{A}\otimes\phi A)} U^{\mathbb{A}^1}_{\mathrm{loc}}(\mathcal{A}\otimes A) \longrightarrow U^{\mathbb{A}^1}_{\mathrm{loc}}(\mathcal{A}\otimes A[t_+,t_-;\phi]) \xrightarrow{\partial} \Sigma U^{\mathbb{A}^1}_{\mathrm{loc}}(\mathcal{A}\otimes A)\,.$$

The proof of the second claim follows now automatically from the fact that $U^{\mathbb{A}^1}_{\mathrm{loc}}$ is the initial $\mathbb{A}^1$-homotopy invariant; see [16, Thm. 8.25].

9. Proof of Theorem 4.2

Similarly to the arguments used in §8, it suffices to prove Theorem 4.2 in the particular case where $E = U^{\mathbb{A}^1}_{\mathrm{loc}}$ and $\mathcal{A} = k$. As mentioned in Example 2.4, the Leavitt path algebra $L := L_Q$ is a corner skew Laurent polynomial algebra. Let L_0 be the homogeneous component of degree 0 and $\phi\colon L_0 \xrightarrow{\sim} eL_0e$ the "corner" isomorphism. Thanks to Theorem 4.1 (with $E = U^{\mathbb{A}^1}_{\mathrm{loc}}$), we have a triangle

$$U^{\mathbb{A}^1}_{\mathrm{loc}}(L_0) \xrightarrow{\mathrm{id}-U^{\mathbb{A}^1}_{\mathrm{loc}}({}_{\phi}L_0)} U^{\mathbb{A}^1}_{\mathrm{loc}}(L_0) \longrightarrow U^{\mathbb{A}^1}_{\mathrm{loc}}(L_Q) \xrightarrow{\partial} \Sigma U^{\mathbb{A}^1}_{\mathrm{loc}}(L_0) \qquad (9.1)$$

in the category $\mathrm{Mot}^{\mathbb{A}^1}_{\mathrm{loc}}(k)$. Following Ara–Brustenga–Cortiñas [4, §5], the k-algebra L_0 admits a "length" filtration $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$. Concretely, $L_{0,n}$ is the k-linear span of the elements of the form $\sigma\varsigma^*$, where σ and ς are paths such that $r(\sigma) = r(\varsigma)$ and $\deg(\sigma) = \deg(\varsigma) = n$. It turns out that the k-algebra $L_{0,n}$ is isomorphic to the product of $(n+1)v'+(v-v')$ matrix algebras with k-coefficients. Making use of the (derived) Morita equivariance between a matrix algebra with k-coefficients and k, we hence conclude that $U_{\mathrm{loc}}^{\mathbb{A}^1}(L_{0,n})$ is isomorphic to the direct sum of $(n+1)v'+(v-v')$ copies of $U_{\mathrm{loc}}^{\mathbb{A}^1}(k)$. Recall from [16, Thm. 8.28] that we we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)}(U_{\mathrm{loc}}^{\mathbb{A}^1}(k), U_{\mathrm{loc}}^{\mathbb{A}^1}(k)) \simeq K_0(k) \simeq \mathbb{Z}.$$

Under this identification, the inclusion $L_{0,n} \subset L_{0,n+1}$ corresponds to the matrix morphism (see [4, §5]):

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & I_Q^t \end{pmatrix} \colon \bigoplus_{i=1}^{(n+1)v'+(v-v')} U_{\mathrm{loc}}^{\mathbb{A}^1}(k) \longrightarrow \bigoplus_{i=1}^{(n+1)v'+v} U_{\mathrm{loc}}^{\mathbb{A}^1}(k)\,. \tag{9.2}$$

In the same vein, the homomorphism $\phi\colon L_{0,n} \to L_{0,n+1}$, which increases the degree of the filtration by 1, corresponds to the matrix morphism

$$\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} \colon \bigoplus_{i=1}^{nv'+v} U_{\mathrm{loc}}^{\mathbb{A}^1}(k) \longrightarrow \bigoplus_{i=1}^{(n+1)v'+v} U_{\mathrm{loc}}^{\mathbb{A}^1}(k)\,. \tag{9.3}$$

Since the functor $U_{\mathrm{loc}}^{\mathbb{A}^1}$ sends sequential (homotopy) colimits to sequential homotopy colimits, we hence obtain the following sequential homotopy colimit diagram

$$\begin{array}{ccccccc} U_{\mathrm{loc}}^{\mathbb{A}^1}(L_{0,0}) & \xrightarrow{(9.2)} & U_{\mathrm{loc}}^{\mathbb{A}^1}(L_{0,1}) & \xrightarrow{(9.2)} & \cdots & \longrightarrow & U_{\mathrm{loc}}^{\mathbb{A}^1}(L_0) \\ \downarrow{\scriptstyle (9.2)-(9.3)} & & \downarrow{\scriptstyle (9.2)-(9.3)} & & & & \downarrow{\scriptstyle \mathrm{id}-U_{\mathrm{loc}}^{\mathbb{A}^1}(\phi L_0)} \\ U_{\mathrm{loc}}^{\mathbb{A}^1}(L_{0,1}) & \xrightarrow[(9.2)]{} & U_{\mathrm{loc}}^{\mathbb{A}^1}(L_{0,2}) & \xrightarrow[(9.2)]{} & \cdots & \longrightarrow & U_{\mathrm{loc}}^{\mathbb{A}^1}(L_0)\,. \end{array} \tag{9.4}$$

Simple matrix manipulations show that the homotopy cofibers of the vertical morphisms of the diagram (9.4) are all equal to the homotopy cofiber of the morphism $\binom{0}{\mathrm{id}} - I_Q^t \colon \bigoplus_{i=1}^{v-v'} U_{\mathrm{loc}}^{\mathbb{A}^1}(k) \to \bigoplus_{i=1}^{v} U_{\mathrm{loc}}^{\mathbb{A}^1}(k)$. This allows us then to conclude that distinguished triangle (9.1) yields the following distinguished triangle

$$\oplus_{i=1}^{v-v'} U_{\mathrm{loc}}^{\mathbb{A}^1}(k) \xrightarrow{\binom{0}{\mathrm{id}}-I_Q^t} \oplus_{i=1}^{v} U_{\mathrm{loc}}^{\mathbb{A}^1}(k) \longrightarrow U_{\mathrm{loc}}^{\mathbb{A}^1}(L_Q) \xrightarrow{\partial} \oplus_{i=1}^{v-v'} \Sigma U_{\mathrm{loc}}^{\mathbb{A}^1}(k)\,.$$

Consequently, the proof is finished. □

10. Proof of Proposition 4.9

By construction, the triangulated category $\mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)$ comes equipped with an action of the homotopy category of spectra (see [16, §A.3]):

$$\mathrm{Spt} \times \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k) \longrightarrow \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k) \qquad (S, NM) \mapsto S \otimes NM\,.$$

Consider the distinguished triangle of spectra $\mathbb{S} \xrightarrow{n\cdot -} \mathbb{S} \to \mathbb{S}/n \to \Sigma\mathbb{S}$, where $\mathbb{S}$ stands for the sphere spectrum. Since $\mathbb{S}/n \otimes U_{\mathrm{loc}}^{\mathbb{A}^1}(\mathcal{A})$ identifies with the mod-n Moore object of $U_{\mathrm{loc}}^{\mathbb{A}^1}(\mathcal{A})$ and we have an isomorphism $\mathbb{S}/n \simeq \mathbb{S}/l_1^{\nu_1} \oplus \cdots \oplus \mathbb{S}/l_r^{\nu_r}$ in Spt, we then conclude from Example 4.7 (with $E = U_{\mathrm{loc}}^{\mathbb{A}^1}$) that

$$U_{\mathrm{loc}}^{\mathbb{A}^1}(\mathcal{A} \otimes L_n) \simeq U_{\mathrm{loc}}^{\mathbb{A}^1}(\mathcal{A} \otimes L_{l_1^{\nu_1}}) \oplus \cdots \oplus U_{\mathrm{loc}}^{\mathbb{A}^1}(\mathcal{A} \otimes L_{l_r^{\nu_r}})\,.$$

The proof follows now automatically from the fact that $U_{\mathrm{loc}}^{\mathbb{A}^1}$ is the initial $\mathbb{A}^1$-homotopy invariant; see [16, Thm. 8.25].

Acknowledgements. The author is very grateful to Guillermo Cortiñas for sharing his views on Leavitt path algebras and their algebraic K-theory, for useful comments on a previous version of this note, and for mentioning the reference [1]. The author would also like to thank the anonymous referee for his/her comments.

References

[1] G. Abrams and M. Kanuni, Cohn path algebras have invariant basis number, *Comm. Algebra*, **44** (2016), no. 1, 371–380. Zbl 1354.16032 MR 3413692

[2] G. Abrams and G. Pino, The Leavitt path algebra of a graph, *J. Algebra*, **293** (2005), no. 2, 319–334. Zbl 1119.16011 MR 2172342

[3] P. Ara, M. Barroso, K. Goodearl, and E. Pardo, Fractional skew monoid rings, *J. Algebra*, **278** (2004), no. 1, 104–126. Zbl 1063.16033 MR 2068068

[4] P. Ara, M. Brustenga, and G. Cortiñas, K-theory of Leavitt path algebras, *Münster J. Math.*, **2** (2009), 5–33. Zbl 1187.19003 MR 2545605

[5] P. Ara, M. Moreno, and E. Pardo, Nonstable K-theory for graph algebras, *Algebr. Represent. Theory*, **10** (2007), no. 2, 157–178. Zbl 1123.16006 MR 2310414

[6] V. Drinfeld, DG quotients of DG categories, *J. Algebra*, **272** (2004), no. 2, 643–691. Zbl 1064.18009 MR 2028075

[7] J. Gabe, E. Ruiz, M. Tomforde, and T. Whalen, K-theory for Leavitt path algebras: Computation and classification, *J. Algebra*, **433** (2015), 35–72. Zbl 1319.16006 MR 3341817

[8] N. Jacobson, Some remarks on one-sided inverses, *Proc. Amer. Math. Soc.*, **1** (1950), 352–355. Zbl 0037.15901 MR 36223

[9] B. Keller, On differential graded categories, in *International Congress of Mathematicians (Madrid). Vol. II*, 151–190, Eur. Math. Soc., Zürich, 2006. Zbl 1140.18008 MR 2275593

[10] B. Keller, On triangulated orbit categories, *Doc. Math.*, **10** (2005), 551–581. Zbl 1086.18006 MR 2184464

[11] B. Keller, Deriving DG categories, *Ann. Sci. École Norm. Sup. (4)*, **27** (1994), no. 1, 63–102. Zbl 0799.18007 MR 1258406

[12] W. Leavitt, The module type of a ring, *Trans. Amer. Math. Soc.*, **103** (1962), 113–130. Zbl 0112.02701 MR 132764

[13] W. L. Paschke, The crossed product of a C^*-algebra by an endomorphism, *Proc. Amer. Math. Soc.*, **80** (1980), 113–118. Zbl 0435.46045 MR 574518

[14] M. Rørdam, Classification of certain infinite simple C^*-algebras, *J. Func. Anal.*, **131** (1995), 415–458. Zbl 0831.46063 MR 1345038

[15] A. Suslin, On the K-theory of local fields, *J. Pure Appl. Algebra*, **34** (1984), no. 2-3, 301–318. Zbl 0548.12009 MR 772065

[16] G. Tabuada, *Noncommutative Motives*, with a preface by Yuri I. Manin, University Lecture Series, 63, American Mathematical Society, Providence, RI, 2015. Zbl 1333.14002 MR 3379910

[17] G. Tabuada, $\mathbb{A}^1$-homotopy invariants of dg orbit categories, *J. Algebra*, **434** (2015), 169–192. Zbl 1351.14009 MR 3342391

Received 02 May, 2016

G. Tabuada, Department of Mathematics, MIT, Cambridge, MA 02139, USA;
Departamento de Matemática, FCT, UNL, Portugal; and
Centro de Matemática e Aplicações (CMA), FCT, UNL, Portugal
E-mail: tabuada@math.mit.edu