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Hopf-dihedral (co)homology and L-theory

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Abstract. We develop an appropriate dihedral extension of the Connes–Moscovici characteristic map for Hopf $*$-algebras. We then observe that one can use this extension together with the dihedral Chern character to detect non-trivial L-theory classes of a $*$-algebra that carry a Hopf symmetry over a Hopf $*$-algebra. Using our machinery we detect a previously unknown L-class of the standard Podleś sphere.

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1. Introduction

In this paper we calculate a new class of finer homological invariants of noncommutative spaces using their Hopf-algebraic symmetries. We borrow our strategy from the study of characteristic classes of topological manifolds where group symmetries of vector or fiber bundles on a manifold are used to obtain topological invariants of the underlying manifold. In noncommutative geometry one can similarly obtain topological invariants of noncommutative spaces by using Hopf symmetries of the underlying space and a canonical characteristic map relating cohomological invariants of the underlying space and its Hopf algebra symmetries. One can see a beautiful and effective execution of this strategy by Connes and Moscovici in their calculation of topological characteristic classes of codimension-1 foliations using the Hopf algebra $\mathscr{H}_1$ and the characteristic map they constructed [11, 12]. Our main contribution in this paper is a new cohomological tool particularly designed to detect finer (more geometric) invariants of noncommutative spaces.

Cohomological invariants have been used to obtain topological invariants of spaces effectively in the past [3, 33, 34]. We observe that geometric invariants require much finer cohomological machineries. From the point of view of noncommutative geometry, this means one must study algebras with additional structures, and use cohomology theories sensitive to these additional structures. Our first candidate is an obvious, but cohomologically underused one: algebras and Hopf-algebras that carry involutive endo-anti-morphisms called $*$-structures, and their Hopf symmetries compatible with these $*$-structures.

The dominant cohomological machinery used in noncommutative geometry appears to be the cyclic cohomology in many manifestations [8,9,15,20,29,44]. This is mainly because of the Chern character that can detect non-trivial K-invariants of the underlying noncommutative space using cyclic cocycles [9, 10]. However, cyclic cohomology is but one cohomology theory among a collection of similarly defined other theories. Each of these theories is indexed by collections of groups including all finite dihedral groups, finite symmetric groups, finite hyperoctahedral groups, and their corresponding artin groups [1, 18]. In the case of cyclic cohomology, we consider finite cyclic groups of all orders. Our choice of dihedral cohomology as the finer replacement of cyclic cohomology is dictated by the fact that combining finite cyclic groups with an involution yield finite dihedral groups.

By introducing a $*$-structure on the underlying algebra A, we jump from the realm of K-theory into the realm of L-theory [41, 45]. Since the natural extension of the Chern character to L-theory uses dihedral cocycles [13, 14, 27], a new strategy emerges for detecting L-classes using a Hopf-dihedral variant of our characteristic map. Using this strategy we managed to detect a previously unknown L-class of the standard Podleś sphere $\mathcal{O}(S_q^2)$ in Section 7. From this point of view, using dihedral cohomology of Hopf $*$-algebras and the associated characteristic map as a refinement of the existing machinery for Hopf-cyclic cohomology is justified since L-theory already yields much finer invariants than K-theory.

Outline of the paper. We start by defining cyclic and dihedral cohomologies as derived functors of diagrams of vector spaces given over the cyclic and dihedral categories in Section 2. Then we recall the definitions of the cyclic and the dihedral cohomologies of $*$-algebras and $*$-coalgebras in Section 3. In Section 4 we develop the Hopf-dihedral cohomology, with a characteristic function built-in, of a Hopf $*$-algebra that has a modular pair in involution (MPI) and an invariant trace. We are going to use this characteristic map, as we described above, to detect L-theory classes of an algebra that carries a specific Hopf symmetry. In the same section we then extend the theory from MPI to arbitrary stable coefficients, and investigate the natural multiplicative structure on the Hopf–Hochschild cohomology, and the *-structure induced on the cohomology. In Section 5 we consider the complexified quantum enveloping algebra $\mathfrak{U}_q(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, and then calculate its Hopf-dihedral cohomology. The next section is devoted to the dual Hopf-dihedral homology together with its own characteristic map. In Section 7, we investigate the interaction of our characteristic map on the Hopf-dihedral cohomology, its dual, and the Chern character from L-theory to dihedral homology on two examples: (i) the group ring $k\pi$ of the fundamental group π of a multiply connected manifold M, and (ii) Podleś sphere $\mathcal{O}(S_{qs}^2)$.

Notation and conventions. We use a base field k of characteristic 0 which carries a complex structure. WLOG one can assume $k = \mathbb{Q}[\sqrt{-1}]$, or any other field containing $\mathbb{Q}[\sqrt{-1}]$. We will use $\mathbb{N}\langle\frac{1}{2}\rangle$ to denote the set of positive half integers $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and $\langle X\rangle$ to denote the k-vector space spanned by a given set X. We use $\mathrm{CH}_\bullet$, $\mathrm{CC}_\bullet$ and $\mathrm{CD}_\bullet$ to denote respectively the Hochschild, the cyclic and dihedral complexes associated with a dihedral module. Similarly, we use $HH_\bullet$, $HC_\bullet$ and $HD_\bullet$ to denote respectively the Hochschild, the cyclic and the dihedral homology of a dihedral module. We use $\mathrm{Tor}^A_\bullet$ and $\mathrm{Ext}^\bullet_A$ to denote respectively the derived functors of the tensor product $\otimes_A$ and the Hom-functor Hom_A for a unital associative algebra A.

2. Cyclic and dihedral (co)modules and their (co)homology

2.1. The cyclic category. Our main references for this subsection are [8, 29].

The cyclic category ΔC is the category with the set of objects $[n]$, $n \in \mathbb{N}$. The morphisms, on the other hand, are generated by the cofaces $\partial_i\colon [n-1] \longrightarrow [n]$, $0 \le i \le n$, the codegeneracies $\sigma_j\colon [n+1] \longrightarrow [n]$, $0 \le j \le n$, and the cyclic operators $\tau_n\colon [n] \longrightarrow [n]$ subject to the relations

$$\partial_j\partial_i = \partial_i\partial_{j-1},\ i < j, \quad \sigma_j\sigma_i = \sigma_i\sigma_{j+1},\ i \le j,$$

$$\sigma_j\partial_i = \begin{cases} \partial_i\sigma_{j-1}, & i < j, \\ \mathrm{Id}_{[n]}, & \text{if } i = j \text{ or } i = j+1, \\ \partial_{i-1}\sigma_j, & i > j+1, \end{cases}$$

$$\tau_n\partial_i = \partial_{i-1}\tau_{n-1},\ 1 \le i \le n, \quad \tau_n\partial_0 = \partial_n,$$
$$\tau_n\sigma_i = \sigma_{i-1}\tau_{n+1},\ 1 \le i \le n, \quad \tau_n\sigma_0 = \sigma_n\tau_{n+1}^2,$$
$$\tau_n^{n+1} = \mathrm{Id}_{[n]}\ .$$

We call a functor of the form $F\colon \Delta C^{\mathrm{op}} \longrightarrow k$-**Mod** a cyclic module, and a functor of the form $G\colon \Delta C \longrightarrow k$-**Mod** a cocyclic module.

2.2. The dihedral category. Our main references for this subsection are [28, 29].

There is a straight extension of ΔC to a larger category ΔD called the dihedral category, with the same set of objects, containing ΔC as a subcategory. The essential difference is that the endomorphisms on each object $[n]$ is the dihedral group $D_{n+1} = \left\langle \tau_n, \omega_n \mid \omega_n^2 = \tau_n^{n+1} = \omega_n\tau_n\omega_n\tau_n = \mathrm{Id}_{[n]}\right\rangle$ of order $2(n+1)$. The rest of the relations, between the morphisms are

$$\partial_i\omega_n = \omega_{n-1}\partial_{n-i}, \quad \sigma_i\omega_n = \omega_{n+1}\sigma_{n-i}, \quad 0 \le i \le n.$$

Similarly as before, a functor $F\colon \Delta D^{\mathrm{op}} \longrightarrow k$-**Mod** is called a dihedral module. In the opposite case, we call a functor $F\colon \Delta D \longrightarrow k$-**Mod** a codihedral module.

2.3. Cohomology. Our main references for this section are [8, 29].

We define a cosimplicial module $\mathrm{CH}^{\Delta}_{\bullet}$

$$\mathrm{CH}^{\Delta}_n = \langle \Delta(n, \cdot) \rangle$$

and the face maps $\partial_i : \mathrm{CH}_n \to \mathrm{CH}_{n-1}$ are defined by pre-composition

$$\partial_i(\psi) = \psi \circ \partial_i \in \Delta(n-1, m)$$

for all $\psi : n \to m$. Then we define a differential $d_n = \sum_{i=0}^{n} (-1)^i \partial_i$. This differential graded cosimplicial module is a resolution of the cosimplicial module $k_\bullet$. If we let $\mathrm{CH}^{\Delta C}_n = \mathrm{CB}^{\Delta C}_n = \langle \Delta C(n, \cdot) \rangle$ and we let

$$d^{\mathrm{CH}}_n = \sum_{i=0}^{n} (-1)^i \partial_i \quad \text{and} \quad d^{\mathrm{CB}}_n = \sum_{i=0}^{n-1} (-1)^i \partial_i$$

with $N_\bullet = \sum_{i=0}^{n} t_n^i$, then the bicomplex $\mathrm{CC}_\bullet$

$$\mathrm{CH}_\bullet \xleftarrow{1-t_\bullet} \mathrm{CB}_\bullet \xleftarrow{N_*} \mathrm{CH}_\bullet \xleftarrow{1-t_\bullet} \cdots$$

is a resolution of the cocyclic module $k_\bullet$ in the category of cocyclic modules. Then we also would see that

$$\mathrm{CC}_\bullet \xleftarrow{1-\omega_\bullet} \mathrm{CC}_\bullet \xleftarrow{1+\omega_\bullet} \mathrm{CC}_\bullet \xleftarrow{1-\omega_\bullet} \cdots$$

is going to be a projective resolution of $k_\bullet$ in the category of codihedral modules, once we replace ΔC with ΔD in the definitions of $\mathrm{CB}_\bullet$ and $\mathrm{CH}_\bullet$.

Definition 2.1. Let $\mathcal{Z}$ be one of Δ, ΔC, or ΔD, $X_\bullet$ a right $\mathcal{Z}$-module, and $Y_\bullet$ a left $\mathcal{Z}$-module. Then the (co)homology of $X_\bullet$ and $Y_\bullet$ are defined as

$$H\mathcal{Z}_\bullet(X_\bullet) = \mathrm{Tor}^{\mathcal{Z}}_\bullet(X_\bullet, k_\bullet) \quad \text{and} \quad H\mathcal{Z}^\bullet(Y_\bullet) = \mathrm{Ext}^\bullet_{\mathcal{Z}}(k_\bullet, Y_\bullet).$$

3. Algebras and coalgebras

3.1. Cyclic (co)homology of algebras. With these notations at hand we can interpret the Hochschild and the cyclic (co)homology as derived functors. Namely, given an algebra $\mathcal{A}$ we define a cyclic module $\mathbf{C}_\bullet(\mathcal{A}) \to k\text{-}\mathbf{Mod}$ by

$$\mathbf{C}_\bullet(\mathcal{A}) = \bigoplus_{n \geq 0} \mathcal{A}^{\otimes n+1},$$

whose structure maps are defined as follows:

$$\partial_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} (\cdots \otimes a_{-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots), & \text{if } 0 \le i \le n-1, \\ (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}), & \text{if } i = n, \end{cases}$$
$$\sigma_j(a_0 \otimes \cdots \otimes a_n) = (\cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots),$$
$$\tau_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

for $0 \le i \le n+1$ and $0 \le j \le n$.

One can realize the Hochschild and the cyclic homologies as derived functors of simplicial and cyclic modules

$$HH_n(\mathcal{A}) = \mathrm{Tor}_n^{\Delta}(\mathbf{C}_\bullet(\mathcal{A}), k_\bullet) \quad \text{and} \quad HC_n(\mathcal{A}) = \mathrm{Tor}_n^{\Delta C}(\mathbf{C}_\bullet(\mathcal{A}), k_\bullet).$$

For the Hochschild and cyclic cohomologies we get

$$HH^n(\mathcal{A}) = \mathrm{Ext}^n_{\Delta}(k_\bullet, \mathbf{C}^\bullet(\mathcal{A})) \quad \text{and} \quad HC^n(\mathcal{A}) = \mathrm{Ext}^n_{\Delta C}(k_\bullet, \mathbf{C}^\bullet(\mathcal{A})),$$

where, this time, our cocyclic module $\mathbf{C}^\bullet(\mathcal{A})$ is given by $\mathbf{C}^n(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n+1}, k)$. More generally, for an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule V, the cocyclic module $\mathbf{C}^n(\mathcal{A}, V) := \mathrm{Hom}(\mathcal{A}^{\otimes n}, V)$ has the coface maps $d_i : \mathbf{C}^{n-1}(\mathcal{A}, V) \longrightarrow \mathbf{C}^n(\mathcal{A}, V)$

$$d_i\varphi(a_1, \ldots, a_n) = \begin{cases} a_1 \cdot \varphi(a_2 \ldots, a_n), & \text{if } i = 0, \\ \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_n), & \text{if } 1 \le i \le n-1, \\ \varphi(a_1, \ldots, a_{n-1}) \cdot a_n, & \text{if } i = n, \end{cases}$$

and the codegeneracy maps $s_j : \mathbf{C}^{n+1}(\mathcal{A}, V) \longrightarrow \mathbf{C}^n(\mathcal{A}, V)$

$$s_j\varphi(a_1, \ldots, a_n) = \varphi(a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n).$$

where $0 \le j \le n$. In particular, for $V = \mathcal{A}^\vee = \mathrm{Hom}(\mathcal{A}, k)$, the above structure is equivalent to the one given by $d_i : \mathbf{C}^{n-1}(\mathcal{A}) \longrightarrow \mathbf{C}^n(\mathcal{A})$

$$d_i\varphi(a_0, \ldots, a_n) = \begin{cases} \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_n), & \text{if } 0 \le i \le n-1, \\ \varphi(a_n a_0, \ldots, a_{n-1}), & \text{if } i = n, \end{cases}$$

and $s_j : \mathbf{C}^{n+1}(\mathcal{A}) \longrightarrow \mathbf{C}^n(\mathcal{A})$

$$s_j\varphi(a_0, \ldots, a_n) = \varphi(a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n),$$

where $0 \le j \le n$. In this case, the cyclic maps $t_n : \mathbf{C}^n(\mathcal{A}) \longrightarrow \mathbf{C}^n(\mathcal{A})$ are given by

$$t_n\varphi(a_0, \ldots, a_n) = \varphi(a_n, a_0, \ldots, a_{n-1}),$$

for every $n \ge 1$.

3.2. Hochschild (co)homology of $*$-algebras. Let $\mathcal{A}$ be an algebra with an involution $*\colon \mathcal{A} \longrightarrow \mathcal{A}$, that is,

$$(ab)^* = b^* a^*, \quad a^{**} = a,$$

for any $a, b \in \mathcal{A}$. We call such an algebra a $*$-algebra. Let also V be an $\mathcal{A}$-bimodule with an involution $*\colon V \longrightarrow V$ satisfying

$$(a \cdot v \cdot b)^* = b^* \cdot v^* \cdot a^*, \quad v^{**} = v,$$

for any $a, b \in \mathcal{A}$, and any $v \in V$. Such a bimodule is called a $*$-bimodule. One then defines on the Hochschild cohomology complex

$$w_n \varphi(a_1, \ldots, a_n) := \varphi(a_n^*, a_{n-1}^*, \ldots, a_1^*)^*,$$

for any $\varphi \in \mathbf{C}^n(\mathcal{A}, V)$.

Lemma 3.1. *On $\mathbf{C}^{n-1}(\mathcal{A}, V)$ we have $w_n d_i = d_{n-i} w_{n-1}$ for $0 \leq i \leq n$.*

Proof. For $1 \leq i \leq n-1$, and for $\varphi \in \mathbf{C}^{n-1}(\mathcal{A}, V)$, we have

$$\begin{aligned} w_n(d_i\varphi)(a_1, \ldots, a_n) &= (d_i\varphi)(a_n^*, \ldots, a_1^*)^* \\ &= \varphi(a_n^*, \ldots, a_{n-i+1}^* a_{n-i}^*, \ldots, a_1^*)^* \\ &= \varphi(a_n^*, \ldots, (a_{n-i} a_{n-i+1})^*, \ldots, a_1^*)^* \\ &= d_{n-i}(w_{n-1}\varphi)(a_1, \ldots, a_n). \end{aligned}$$

Similarly, $w_n d_0 = d_n w_{n-1}$, and $w_n d_n = d_0 w_{n-1}$. □

As a result, we have the following:

Lemma 3.2. *Given a $*$-algebra $\mathcal{A}$ and a $*$-bimodule V, on $\mathbf{C}^{n-1}(\mathcal{A}, V)$ we have*

$$w_n b = (-1)^n b w_{n-1}.$$

Proof. For an arbitrary $\varphi \in \mathbf{C}^{n-1}(\mathcal{A}, V)$ we have

$$\begin{aligned} w_n b\varphi(a_1, \ldots, a_n) &= b\varphi(a_n^*, \ldots, a_1^*)^* \\ &= a_n^* \cdot \varphi(a_{n-1}^*, \ldots, a_1^*)^* \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \varphi(a_n^*, \ldots, a_{n-i+1}^* a_{n-i}^*, \ldots, a_1^*)^* \\ &\quad + (-1)^n \varphi(a_n^*, \ldots, a_2^*) \cdot a_1^* \\ &= (-1)^n b w_{n-1}\varphi(a_1, \ldots, a_n). \end{aligned}$$

□

Now, assuming 2 is invertible in k, based on the decomposition of the Hochschild complex into the $+1$ and -1-eigenspaces of the involution operator it follows that

$$HH^\bullet(\mathcal{A}, V) = HH^\bullet_+(\mathcal{A}, V) \oplus HH^\bullet_-(\mathcal{A}, V),$$

see for instance [29, 5.2.3], and in particular for $V = \mathcal{A}^\vee = \mathrm{Hom}(\mathcal{A}, k)$,

$$HH^\bullet(\mathcal{A}) = HH^\bullet_+(\mathcal{A}) \oplus HH^\bullet_-(\mathcal{A}).$$

Explicitly, $HH^\bullet_+(\mathcal{A}, V)$ (resp., $HH^\bullet_+(\mathcal{A})$) is the cohomology of the w-invariant (real) subcomplex, and $HH^\bullet_-(\mathcal{A}, V)$ (resp., $HH^\bullet_-(\mathcal{A})$) is the cohomology of the w-anti-invariant (imaginary) subcomplex.

3.3. Dihedral cohomology of $*$-algebras. We next record the following on the restriction of the $*$-structure on the cyclic complex.

Lemma 3.3. *On $\mathbf{C}^n(\mathcal{A}) := \mathbf{C}^n(\mathcal{A}, \mathcal{A}^*)$, we have*

$$t_n w_n = w_n t_n^{-1}.$$

Proof. For any $\varphi \in \mathbf{C}^n(\mathcal{A})$, we first recall that $w_n(\varphi)(a_0, \ldots, a_n) = \overline{\varphi(a_0^*, a_n^*, \ldots, a_1^*)}$. Then the claim follows from

$$\begin{aligned} t_n w_n \varphi(a_0, \ldots, a_n) &= w_n \varphi(a_n, a_0, \ldots, a_{n-1}) \\ &= \overline{\varphi(a_n^*, a_{n-1}^*, \ldots, a_0^*)} \\ &= \overline{t_n^{-1} \varphi(a_0^*, a_n^*, \ldots, a_1^*)} \\ &= w_n t_n^{-1} \varphi(a_0, \ldots, a_n). \end{aligned}$$ □

As a result, if $\varphi \in \mathbf{C}^n_\lambda(\mathcal{A})$, i.e. $t_n\varphi = (-1)^n\varphi$, then $t_n w_n \varphi = w_n t_n^{-1}\varphi = (-1)^n w_n\varphi$, that is $w_n\varphi \in \mathbf{C}^n_\lambda(\mathcal{A})$, therefore, we obtain the similar eigen-space decomposition

$$HC^\bullet(\mathcal{A}) = HC^\bullet_+(\mathcal{A}) \oplus HC^\bullet_-(\mathcal{A}).$$

The summands are both called the dihedral cohomology of $\mathcal{A}$, and are denoted by $HD^\bullet_\pm(\mathcal{A})$.

3.4. Dihedral cohomology of $*$-coalgebras. Let us recall from [17, Sect. 2.2] the involutive coalgebras and their (involutive) comodules. A $*$-coalgebra $\mathcal{C}$ is a coalgebra such that

$$\Delta(c^*) = c^*_{(2)} \otimes c^*_{(1)}, \quad \varepsilon(c^*) = \overline{\varepsilon(c)}.$$

An involutive $\mathcal{C}$-bicomodule ($*$-bicomodule) V is a $\mathcal{C}$-bicomodule such that

$$\begin{aligned} (v^*)_{<0>} \otimes (v^*)_{<1>} &= v_{<0>}{}^* \otimes v_{<-1>}{}^*, \\ (v^*)_{<-1>} \otimes (v^*)_{<0>} &= v_{<1>}{}^* \otimes v_{<0>}{}^*. \end{aligned} \tag{3.1}$$

We recall from [16] the coalgebra Hochschild cohomology of a coalgebra $\mathcal{C}$ with coefficients in a $\mathcal{C}$-bicomodule V is given as the homology of the complex

$$\mathbf{C}^\bullet(\mathcal{C}, V) = \bigoplus_{n\geq 0} V \otimes \mathcal{C}^{\otimes n},$$

with the structure maps $d_i\colon \mathbf{C}^{n-1}(\mathcal{C}, V) \longrightarrow \mathbf{C}^n(\mathcal{C}, V)$

$$d_i\varphi(v\otimes c_1\otimes\cdots\otimes c_{n-1}) = \begin{cases} v_{<0>} \otimes v_{<1>} \otimes c_1 \otimes \cdots \otimes c_{n-1}, & \text{if } i = 0,\\ v \otimes c_1 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_{n-1}, & \text{if } 1 \leq i \leq n-1,\\ v_{<0>} \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes v_{<-1>}, & \text{if } i = n,\end{cases}$$

and $s_j\colon \mathbf{C}^{n+1}(\mathcal{C}, V) \longrightarrow \mathbf{C}^n(\mathcal{C}, V)$

$$s_j(v \otimes c_0 \otimes \cdots \otimes c_n) = v \otimes c_0 \otimes \cdots \otimes \varepsilon(c_j) \otimes \cdots \otimes c_n,$$

where $0 \leq j \leq n$.

Lemma 3.4. *On $\mathbf{C}^{n-1}(\mathcal{C}, V)$ we have $w_n d_i = d_{n-i} w_{n-1}$ for $0 \leq i \leq n$.*

Proof. For $1 \leq i \leq n-1$, and for $v \otimes c_1 \otimes \cdots \otimes c_{n-1} \in \mathbf{C}^{n-1}(\mathcal{C}, V)$, we have

$$\begin{aligned} w_n d_i(v \otimes c_1 \otimes \cdots \otimes c_{n-1}) &= w_n(v \otimes c_1 \otimes \cdots \otimes c_{i(1)} \otimes c_{i(2)} \otimes \cdots \otimes c_{n-1})\\ &= v^* \otimes c_{n-1}^* \otimes \cdots \otimes (c_{i(2)})^* \otimes (c_{i(1)})^* \otimes \cdots \otimes c_1^*\\ &= d_{n-i}(v^* \otimes c_{n-1}^* \otimes \cdots \otimes c_i^* \otimes \cdots \otimes c_1^*)\\ &= d_{n-i} w_{n-1}(v \otimes c_1 \otimes \cdots \otimes c_{n-1}).\end{aligned}$$

Similarly, $w_n d_0 = d_n w_{n-1}$, and $w_n d_n = d_0 w_{n-1}$. □

We thus conclude the commutation with the Hochschild coboundary map.

Corollary 3.5. *Given a $*$-coalgebra $\mathcal{C}$ and a $*$-bicomodule V, on $\mathbf{C}^{n-1}(\mathcal{C}, V)$ we have*

$$w_n b = (-1)^n b w_{n-1}.$$

Proof. For an arbitrary $v \otimes c_1 \otimes \cdots \otimes c_{n-1} \in \mathbf{C}^{n-1}(\mathcal{C}, V)$ we have

$$\begin{aligned} &w_n b(v \otimes c_1 \otimes \cdots \otimes c_{n-1})\\ &\qquad = ({v_{<0>}}^* \otimes c_{n-1}^* \otimes \cdots \otimes c_1^* \otimes {v_{<1>}}^*)\\ &\qquad\quad + \sum_{i=1}^{n-1}(-1)^i(v \otimes c_{n-1}^* \otimes \cdots \otimes (c_{i(2)})^* \otimes (c_{i(1)})^* \otimes \cdots \otimes c_1^*)\\ &\qquad\quad + (-1)^n({v_{<0>}}^* \otimes {v_{<-1>}}^* \otimes \cdots \otimes c_{n-1}^* \otimes c_1^*)\\ &\qquad = (-1)^n b w_{n-1}(v \otimes c_1 \otimes \cdots \otimes c_{n-1}).\end{aligned}$$

□

As a result, we have the decomposition

$$HH^\bullet(\mathcal{C}, V) = HH^\bullet_+(\mathcal{C}, V) \oplus HH^\bullet_-(\mathcal{C}, V),$$

of the coalgebra Hochschild cohomology into the dihedral coalgebra Hochschild cohomologies.

4. Hopf-dihedral cohomology

4.1. Hopf-cyclic cohomology. Let us first recall Hopf-cyclic cohomology from [11, 12]. Let $\mathcal{H}$ be a Hopf algebra with a modular pair in involution (MPI) (δ, σ), i.e. δ is a character on $\mathcal{H}$, and $\sigma \in \mathcal{H}$ is a group-like element such that

$$S_\delta^2 = \mathrm{Ad}_\sigma, \quad \delta(\sigma) = 1,$$

where $S_\delta(h) = \delta(h_{(1)})S(h_{(2)})$ is the twisted antipode. Assume also that $\mathcal{A}$ is a (left) $\mathcal{H}$-module algebra,

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \varepsilon(h)1,$$

equipped with a linear form $\tau\colon \mathcal{A} \longrightarrow k$ which is a δ-invariant σ-trace

$$\tau(h \triangleright a) = \delta(h)\tau(a), \quad \tau(ab) = \tau(b(\sigma \triangleright a)).$$

Then the Hopf-cyclic cohomology of $\mathcal{H}$ is defined to satisfy the following, [12].

Ansatz. Let $\mathcal{A}$ be a $\mathcal{H}$-module algebra equipped with a δ-invariant σ-trace $\tau\colon \mathcal{A} \longrightarrow k$. Then the assignment

$$\begin{aligned} &h_1 \otimes \cdots \otimes h_n \mapsto \chi_\tau(h_1 \otimes \cdots \otimes h_n) \in \mathbf{C}^n(\mathcal{A}),\\ &\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a_0, \ldots, a_n) = \tau(a_0(h_1 \triangleright a_1)\ldots(h_n \triangleright a_n)),\end{aligned}$$

defines a cocyclic module $\mathbf{C}^\bullet(\mathcal{H}; \sigma, \delta)$ on $\mathcal{H}$ whose cohomology comes with a canonical map of the form $\chi_\tau\colon HC^n(\mathcal{H}; \delta, \sigma) \longrightarrow HC^n(\mathcal{A})$.

The ansatz then dictates on

$$\mathbf{C}^\bullet(\mathcal{H}; \sigma, \delta) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$$

the cocyclic structure given by the maps

$$d_i(h_1 \otimes \cdots \otimes h_{n-1}) = \begin{cases} 1 \otimes h_1 \otimes \cdots \otimes h_{n-1}, & \text{if } i = 0,\\ h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_{n-1}, & \text{if } 0 < i \leq n,\\ h_1 \otimes \cdots \otimes h_n \otimes \sigma, & \text{if } i = n,\end{cases}$$

and

$$s_j(h_0 \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes \varepsilon(h_j) \otimes \cdots \otimes h_n, \quad 0 \le j \le n.$$

These structure maps encode a simplicial structure $\mathbf{C}^\bullet(\mathcal{H}, \sigma, \delta)\colon \Delta \to k$-**Mod**, and the maps

$$t_n(h_1 \otimes \cdots \otimes h_n) = S_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma)$$

encode the actions of the cyclic groups to the cocyclic module $\mathbf{C}^\bullet(\mathcal{H}; \sigma, \delta)$ which is a functor of the form $\mathbf{C}^\bullet(\mathcal{H}, \sigma, \delta)\colon \Delta C \to k$-**Mod**.

4.2. Hopf $*$-algebras. Let us next recall from [30, Def. 1.7.5] that a Hopf $*$-algebra is a Hopf algebra $\mathcal{H}$, which is a $*$-algebra such that

$$\Delta(h^*) = h_{(1)}{}^* \otimes h_{(2)}{}^*, \quad \varepsilon(h^*) = \overline{\varepsilon(h)}, \quad S(h^*) = S^{-1}(h)^*,$$

for every $h \in \mathcal{H}$.

Let now a $*$-algebra $\mathcal{A}$ be a $\mathcal{H}$-module algebra with the $*$-action, that is, let $\mathcal{A}$ be a $\mathcal{H}$-module algebra with the compatibility

$$(h \triangleright a)^* = S^{-1}(h^*) \triangleright a^* \tag{4.1}$$

between the $*$-structures and the action. Such $\mathcal{A}$ is called a $\mathcal{H}$-module $*$-algebra. See [30, Prop. 6.1.5] for the notion of $*$-action, see also [42, Eqn. (8) & (9)].

4.3. Hopf-dihedral cohomology of a Hopf $*$-algebra. We now investigate the $*$-structure on the Hopf-cyclic complex $\mathbf{C}(\mathcal{H}; \sigma, \delta)$ that makes the characteristic homomorphism of Connes–Moscovici a $*$-homomorphism, that is, for $n \ge 0$

$$w_n(\chi_\tau(h^1 \otimes \cdots \otimes h^n)) = \chi_\tau(w_n(h^1 \otimes \cdots \otimes h^n)).$$

Accordingly,

$$\begin{aligned} w_n(\chi_\tau(h^1 \otimes \cdots \otimes h^n))(a_0, \ldots, a_n) &= \overline{\chi_\tau(h^1 \otimes \cdots \otimes h^n)(a_0^*, a_n^*, \ldots, a_1^*)} \\ &= \overline{\tau(a_0^*(h^1 \triangleright a_n^*) \ldots (h^n \triangleright a_1^*))} \\ &= \overline{\tau(a_0^*(S(h^1)^* \triangleright a_n)^* \ldots (S(h^n)^* \triangleright a_1)^*)} \\ &= \tau((S(h^n)^* \triangleright a_1) \ldots (S(h^1)^* \triangleright a_n)a_0) \\ &= \tau((S^{-1}(h^{n*}) \triangleright a_1) \ldots (S^{-1}(h^{1*}) \triangleright a_n)a_0) \\ &= \tau(a_0(\sigma S^{-1}(h^{n*}) \triangleright a_1) \ldots (\sigma S^{-1}(h^{1*}) \triangleright a_n)) \end{aligned}$$

dictates that

$$w_n(h^1 \otimes \cdots \otimes h^n) = \sigma S^{-1}(h^{n*}) \otimes \cdots \otimes \sigma S^{-1}(h^{1*}). \tag{4.2}$$

Lemma 4.1. *The mapping given by* (4.2) *is an involution on* $\mathbf{C}^\bullet(\mathcal{H};\sigma,\delta)$.

Proof. For $n \geq 1$ we have

$$\begin{aligned} w_n^2(h^1 \otimes \cdots \otimes h^n) &= w_n(\sigma S^{-1}(h^{n*}) \otimes \cdots \otimes \sigma S^{-1}(h^{1*})) \\ &= \sigma S^{-1}(S^{-1}(h^{1*})^*\sigma^*) \otimes \cdots \otimes \sigma S^{-1}(S^{-1}(h^{n*})^*\sigma^*) \\ &= S^{-1}(S^{-1}(h^{1*})^*) \otimes \cdots \otimes S^{-1}(S^{-1}(h^{n*})^*) \\ &= S^{-1}(S(h^{1**})) \otimes \cdots \otimes S^{-1}(S(h^{n**})) \\ &= h^1 \otimes \cdots \otimes h^n. \end{aligned}$$

On the third equation we used the assumption that $\sigma^* = \sigma$, on the second, third and the fourth equations we used (4.1). □

Lemma 4.2. *The mapping given by* (4.2) *is an involution on the coalgebra Hochschild cohomology* $HH^\bullet(\mathcal{H}, {}^\sigma k)$.

Proof. For $1 \leq i \leq n-1$ we observe on $\mathbf{C}^{n-1}(\mathcal{H};\sigma,\delta)$ that

$$\begin{aligned} & w_n d_i(h^1 \otimes \cdots \otimes h^{n-1}) \\ &\quad = w_n(h^1 \otimes \cdots \otimes h^i{}_{(1)} \otimes h^i{}_{(2)} \otimes \cdots \otimes h^{n-1}) \\ &\quad = \sigma S^{-1}(h^{n-1*}) \otimes \cdots \otimes \sigma S^{-1}(h^i{}_{(2)}{}^*) \otimes \sigma S^{-1}(h^i{}_{(1)}{}^*) \otimes \cdots \otimes \sigma S^{-1}(h^{1*}) \\ &\quad = \sigma S^{-1}(h^{n-1*}) \otimes \cdots \otimes \Delta(\sigma S^{-1}(h^{i*})) \otimes \cdots \otimes \sigma S^{-1}(h^{1*}) \\ &\quad = d_{n-i} w_{n-1}(h^1 \otimes \cdots \otimes h^{n-1}). \end{aligned}$$

The equalities $w_n d_0 = d_n w_{n-1}$ and $w_n d_n = d_0 w_{n-1}$ follows similarly from (3.1). □

Lemma 4.3. *The mapping given by* (4.2) *is an involution on the Hopf-cyclic cohomology* $HC^\bullet(\mathcal{H};\sigma,\delta)$.

Proof. Let $\widetilde{h} := h_1 \otimes \cdots \otimes h_n \in \mathbf{C}^n(\mathcal{H};\sigma,\delta)$ be cyclic. Then,

$$\begin{aligned} \chi_\tau(t_n w_n(\widetilde{h})) &= t_n \chi_\tau(w_n(\widetilde{h})) = t_n w_n \chi_\tau(\widetilde{h}) = w_n t_n^{-1} \chi_\tau(\widetilde{h}) = \chi_\tau(w_n t_n^{-1}(\widetilde{h})) \\ &= (-1)^n \chi_\tau(w_n(\widetilde{h})), \end{aligned}$$

that is $\omega_n(\widetilde{h}) \in \mathbf{C}^n(\mathcal{H};\sigma,\delta)$ is also cyclic. □

Consequently,

$$HC^\bullet(\mathcal{H};\sigma,\delta) = HC^\bullet_+(\mathcal{H};\sigma,\delta) \oplus HC^\bullet_-(\mathcal{H};\sigma,\delta).$$

We call the cohomologies on the right hand side, corresponding to the ± 1-eigenspaces of the operator (4.2), the Hopf-dihedral cohomologies of the Hopf algebra $\mathcal{H}$.

4.4. Hopf-dihedral cohomology with general coefficients. Let $\mathcal{H}$ be a Hopf algebra with an invertible antipode, and $\mathcal{A}$ be a left $\mathcal{H}$-module algebra. Let also V be a left $\mathcal{H}$-module and left $\mathcal{H}$-comodule satisfying the stability condition

$$v_{<-1>} \triangleright v_{<0>} = v,$$

for all $v \in V$. Then one can define a para-cyclic module $\mathrm{CC}_{\bullet}^{\mathcal{H}}(\mathcal{A}, V)$ letting

$$\mathrm{CC}_n^{\mathcal{H}}(\mathcal{A}, V) = \mathcal{A}^{\otimes n+1} \otimes V,$$

and defining the structure maps

$$\partial_i(a_0 \otimes \cdots \otimes a_n \otimes v) = \begin{cases} (\cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes v), \\ \hfill \text{if } 0 \leq i \leq n-1, \\ (v_{<-1>} \triangleright a_n) a_0 \otimes \cdots \otimes a_{n-1} \otimes v_{<0>}, \quad \text{if } i = n, \end{cases}$$

$$\sigma_j(a_0 \otimes \cdots \otimes a_n \otimes v) = a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n \otimes v, \quad \text{for } 0 \leq j \leq n-1,$$

$$\tau_n(a_0 \otimes \cdots \otimes a_n \otimes v) = (v_{<-1>} \triangleright a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1} \otimes v_{<0>}.$$

If we let $\mathrm{CC}_{\mathcal{H}}^{\bullet}(\mathcal{A}, V) := \mathrm{Hom}_{\mathcal{H}}(\mathrm{CC}_{\bullet}^{\mathcal{H}}(\mathcal{A}, V), k)$, then we get a cocyclic module. Indeed, for every $\varphi \in \mathrm{CC}_{\mathcal{H}}^n(\mathcal{A}, V)$ we have

$$\begin{aligned} (\tau_n^{n+1}\varphi)(a_0 \otimes \cdots \otimes a_n \otimes v) \\ &= \varphi(\tau_n^{n+1}(a_0 \otimes \cdots \otimes a_n \otimes v)) \\ &= \varphi(v_{<-n-1>} \triangleright a_0 \otimes \cdots \otimes v_{<-1>} \triangleright a_n \otimes v_{<0>}) \\ &= \varphi(v_{<-n-2>} \triangleright a_0 \otimes \cdots \otimes v_{<-2>} \triangleright a_n \otimes v_{<-1>} \triangleright v_{<0>}) \\ &= \varepsilon(v_{<-1>})\varphi(a_0 \otimes \cdots \otimes a_n \otimes v_{<0>}) \\ &= \varphi(a_0 \otimes \cdots \otimes a_n \otimes v). \end{aligned}$$

We call the cyclic cohomology of this cocyclic module $\mathrm{CC}_{\mathcal{H}}^{\bullet}(\mathcal{A}, V)$ as the Hopf-cyclic cohomology of the $\mathcal{H}$-module algebra $\mathcal{A}$ with coefficients in V, and we denote it by $HC_{\mathcal{H}}^{\bullet}(\mathcal{A}, V)$.

Proposition 4.4. *Given any stable $\mathcal{H}$-module/comodule V, and $\tau \in HC_{\mathcal{H}}^0(\mathcal{A}, V)$ we have a pairing of the form $\langle \cdot | \cdot \rangle_\tau : \mathcal{H}^{\otimes n} \otimes V \otimes \mathcal{A}^{\otimes n+1} \to k$ given by*

$$\left\langle h^1 \otimes \cdots \otimes h^n \otimes v \mid a_0 \otimes \cdots \otimes a_n \right\rangle_\tau = \tau(a_0(h^1 \triangleright a_1) \cdots (h^n \triangleright a_n) \otimes v). \tag{4.3}$$

The pairing induces a cocyclic module structure on $\mathrm{CC}_{\mathcal{H}}^{\bullet}(\mathcal{H}, V)$, whose cohomology $HC_{\mathcal{H}}^{\bullet}(\mathcal{H}, V)$ comes with a characteristic map $\chi_\tau : HC_{\mathcal{H}}^{\bullet}(\mathcal{H}, V) \to HC^{\bullet}(\mathcal{A})$. In case V is a left/left SAYD module over $\mathcal{H}$, this cohomology is the Hopf-cyclic cohomology with coefficients in V.

Proof. We need to derive the structure maps on $\mathrm{CC}^\bullet_{\mathscr{H}}(\mathscr{H}, V)$ compatible with the pairing. For that we first observe

$$\begin{aligned}
&\big\langle h^1 \otimes \cdots \otimes h^n \otimes v \mid \partial_i(a_0 \otimes \cdots \otimes a_{n+1})\big\rangle_\tau \\
&= \begin{cases} \tau(a_0 a_1 (h^1 \triangleright a_2) \cdots (h^n \triangleright a_{n+1}) \otimes v), & \text{if } i = 0, \\ \tau(a_0 \cdots (h^{i-1} \triangleright a_{i-1})(h^i \triangleright a_i a_{i+1})(h^{i+1} \triangleright a_{i+2}) \cdots \otimes v), & \text{if } 1 \le i \le n, \\ \tau(a_{n+1} a_0 (h^1 \triangleright a_1) \cdots (h^n \triangleright a_n) \otimes v), & \text{if } i = n+1, \end{cases} \\
&= \begin{cases} \tau(a_0 a_1 (h^1 \triangleright a_2) \cdots (h^n \triangleright a_{n+1}) \otimes v), & \text{if } i = 0, \\ \tau(a_0 \cdots (h^{i-1} \triangleright a_{i-1})(h^i{}_{(1)} \triangleright a_i)(h^i{}_{(2)} \triangleright a_{i+1})(h^{i+1} \triangleright a_{i+2}) \cdots \otimes v), & \\ & \text{if } 1 \le i \le n, \\ \tau(a_0 (h^1 \triangleright a_1) \cdots (h^n \triangleright a_n)(v_{<-1>} \triangleright a_{n+1}) \otimes v_{<0>}), & \text{if } i = n+1, \end{cases}
\end{aligned}$$

which forces the coface maps for the Hopf-cyclic cocyclic module with coefficients to be

$$d_i(h^1 \otimes \cdots \otimes h^n \otimes v) = \begin{cases} (1 \otimes h^1 \otimes \cdots \otimes h^n \otimes v), & \text{if } i = 0, \\ (\cdots \otimes h^{i-1} \otimes \Delta(h^i) \otimes h^{i+1} \otimes \cdots \otimes v), & \text{if } 1 \le i \le n, \\ (h^1 \otimes \cdots \otimes h^n \otimes v_{<-1>} \otimes v_{<0>}), & \text{if } i = n+1. \end{cases}$$

As for the codegeneracies we observe

$$\begin{aligned}
&\big\langle h^1 \otimes \cdots \otimes h^{n+1} \otimes v \mid s_j(a_0 \otimes \cdots \otimes a_{n+1})\big\rangle_\tau \\
&= \begin{cases} \tau(a_0 (h^1 \triangleright 1)(h_2 \triangleright a_1) \cdots (h^{n+1} \triangleright a_n) \otimes v), & \text{if } j = 0, \\ \tau(a_0 (h^1 \triangleright a_1) \cdots (h^j \triangleright a_j)(h^{j+1} \triangleright 1)(h^{j+2} \triangleright a_{j+1}) \cdots (h^{n+1} \triangleright a_n) \otimes v), & \\ & \text{if } 1 \le j \le n-1. \end{cases}
\end{aligned}$$

As a result, the codegeneracies need to be defined as

$$\sigma_j(h^1 \otimes \cdots \otimes h^j) = (\cdots h^j \otimes \varepsilon(h^{j+1}) \otimes h^{j+2} \otimes \cdots \otimes v),$$

for $0 \le j \le n-1$. Finally, for the cyclic maps we get

$$\begin{aligned}
\big\langle h^1 \otimes \cdots \otimes h^{n+1} \otimes v | \tau_n(a_0 \otimes \cdots \otimes a_n)\big\rangle_\tau \\
&= \tau(a_n (h^1 \triangleright a_0) \cdots \otimes (h^n \triangleright a_{n-1}) \otimes v) \\
&= \tau((h^1 \triangleright a_0) \cdots \otimes (h^n \triangleright a_{n-1})(v_{<-1>} \triangleright a_n) \otimes v_{<0>}) \\
&= \tau(a_0 (S(h^1{}_{(n+2)}) h_2 \triangleright a_1) \cdots (S(h^1{}_{(3)}) h_n \triangleright a_{n-1}) \\
&\qquad \cdot (S(h^1{}_{(2)}) v_{<-1>} \triangleright a_n) \otimes S(h^1{}_{(1)}) v_{<0>}),
\end{aligned}$$

which means that the cyclic maps need to be defined as

$$\begin{aligned}
&\tau_n(h^1 \otimes \cdots \otimes h^n \otimes v) \\
&\qquad = S(h^1{}_{(n+2)}) h^2 \otimes \cdots \otimes S(h^1{}_{(3)}) h^n \otimes S(h^1{}_{(2)}) v_{<-1>} \otimes S(h^1{}_{(1)}) v_{<0>}.
\end{aligned}$$

As for the agreement with the original Hopf-cyclic cohomology with coefficients, we observe that following [22] the pairing is defined uniquely in cohomology since the terms

$$HC^p_{\mathcal{H}}(\mathcal{H}, V) \otimes HC^q_{\mathcal{H}}(\mathcal{A}, V) \to HC^{p+q}(\mathcal{A})$$

come from a derived bifunctor. In other words, any cohomological pairing whose 0th term is given in (4.3) for $n = 0$ will be the same with our pairing up to natural equivalence. □

Theorem 4.5. *Let $\mathcal{H}$ be a Hopf $*$-algebra, and $\mathcal{A}$ an $\mathcal{H}$-module $*$-algebra. Let V be a stable $\mathcal{H}$-module/comodule together with a $*$-structure satisfying*

$$(h \triangleright v)^* = S^{-1}(h^*) \triangleright v^*, \quad (v^*)_{<-1>} \otimes (v^*)_{<0>} = v^*{}_{<-1>} \otimes v^*{}_{<0>},$$

for any $h \in \mathcal{H}$ and any $v \in V$. Assume also that there is a cocycle $\tau \in HC^0_{\mathcal{H}}(\mathcal{A}, V)$ which additionally satisfies

$$\overline{\tau(a \otimes v)} = \tau(a^* \otimes v^*).$$

Then the Hopf-cyclic cohomology $HC^\bullet_{\mathcal{H}}(\mathcal{H}, V)$ carries a $*$-structure, and splits into two eigen-spaces

$$HC^\bullet_{\mathcal{H}}(\mathcal{H}, V) = HC^\bullet_{\mathcal{H},+}(\mathcal{H}, V) \oplus HC^\bullet_{\mathcal{H},-}(\mathcal{H}, V).$$

Furthermore, there are characteristic maps

$$\chi_\tau \colon HC^\bullet_{\mathcal{H},\pm}(\mathcal{H}, V) \to HC^\bullet_\pm(\mathcal{A}).$$

Proof. In Proposition 4.4 we proved that our pairing is compatible with the cyclic structure. What remains to be constructed is a *-structure which is compatible with our pairing. For that we observe

$$\begin{aligned}
&\left\langle h^1 \otimes \cdots \otimes h^{n+1} \otimes v \middle| (a_0 \otimes \cdots \otimes a_n)^* \right\rangle_\tau \\
&\qquad = \tau(a_0^*(h^1 \triangleright a_n^*) \cdots (h^n \triangleright a_1^*) \otimes v) \\
&\qquad = \overline{\tau((h^n \triangleright a_1^*)^* \cdots (h^1 \triangleright a_n^*)^* a_0 \otimes v^*)} \\
&\qquad = \overline{\tau((S^{-1}(v^*_{<-1>}) \triangleright a_0)(S(h^n)^* \triangleright a_1) \cdots (S(h^1)^* \triangleright a_n) \otimes v^*_{<0>})} \\
&\qquad = \overline{\tau(a_0(v^*_{<-n>} S^{-1}(h^{n*}) \triangleright a_1) \cdots (v^*_{<-1>} S^{-1}(h^{1*}) \triangleright a_n) \otimes v^*_{<0>})}.
\end{aligned}$$

This means the $*$-structure needs to be defined as

$$(h^1 \otimes \cdots \otimes h^n \otimes v)^* = v^*_{<-n>} S^{-1}(h^{n*}) \otimes \cdots \otimes v^*_{<-1>} S^{-1}(h^{1*}) \otimes v^*_{<0>}.$$

As in the Hopf-cyclic case, the pairing

$$HC^p_{\mathcal{H},\pm}(\mathcal{H}, V) \otimes HC^q_{\mathcal{H},\pm}(\mathcal{A}, V) \to HC^{p+q}_\pm(\mathcal{A})$$

is defined uniquely in cohomology by [22]: any cohomological pairing whose $n = 0$ term is given by (4.3) together with compatibility with the $*$-structures will be the same as ours up to natural equivalence. □

We call the cohomologies $HC^{\bullet}_{\mathcal{H},\pm}(\mathcal{H}, V)$ as the Hopf-dihedral cohomologies of $\mathcal{H}$ with coefficients in a stable H-module/comodule V.

4.5. The differential graded $*$-algebra structure on the Hopf–Hochschild complex. Let $\mathcal{H}$ be a Hopf algebra, let $\mathcal{G}(\mathcal{H})$ denote the set of group-like elements in $\mathcal{H}$.

Given any $\sigma \in \mathcal{G}(\mathcal{H})$, one can think of k both as a right $\mathcal{H}$-comodule k^{σ}, and a left $\mathcal{H}$-comodule ${}^{\sigma}k$ by

$$\rho_{\sigma}: k^{\sigma} \to k^{\sigma} \otimes \mathcal{H}, \quad \rho_{\sigma}(1) = 1 \otimes \sigma, \quad \text{and} \quad \lambda_{\sigma}: {}^{\alpha}k \to {}^{\alpha}k \otimes \mathcal{H}, \quad \lambda_{\sigma}(1) = \sigma \otimes 1. \tag{4.4}$$

It then follows that for any $\sigma \in \mathcal{G}(\mathcal{H})$ the Hochschild complex $\mathrm{CH}^{\bullet}(\mathcal{H}, {}^{\sigma}k)$ is the same as the two sided cobar complex $\mathrm{CB}^{\bullet}(k^1, \mathcal{H}, {}^{\sigma}k)$.

Proposition 4.6. *Let $\mathcal{H}$ be a Hopf algebra, such that the group-like elements $\mathcal{G}(\mathcal{H})$ forms an abelian group. Then the Hochschild complex $\bigoplus_{\sigma \in \mathcal{G}(\mathcal{H})} \mathrm{CH}^{\bullet}(\mathcal{H}, {}^{\sigma}k)$ is a differential graded unital $*$-algebra with the product*

$$\mathrm{CH}^{p}(\mathcal{H}, {}^{\sigma_1}k) \otimes \mathrm{CH}^{q}(\mathcal{H}, {}^{\sigma_2}k) \to \mathrm{CH}^{p+q}(\mathcal{H}, {}^{\sigma_1\sigma_2}k),$$

for any $p, q \in \mathbb{N}$, and any two group-like elements $\sigma_1, \sigma_2 \in \mathcal{G}(\mathcal{H})$. In particular, the $\mathbb{N}$-graded vector space $\mathrm{CH}^{\bullet}(\mathcal{H}, k)$ forms a graded $*$-subalgebra.

Proof. Let us recall the coface maps of the cosimplicial module $\mathbf{C}^{\bullet}(\mathcal{H}, {}^{\sigma_1}k) = \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$ by

$$d_0(1) = 1 - \sigma_1,$$

$$d_i(h^1 \otimes \cdots \otimes h^n) = \begin{cases} 1 \otimes h^1 \otimes \cdots \otimes h^n, & \text{if } i = 0, \\ \cdots \otimes h^{i-1} \otimes h^{i}{}_{(1)} \otimes h^{i}{}_{(2)} \otimes h^{i+1} \otimes \cdots, & \text{if } 1 \le i \le n, \\ h^1 \otimes \cdots \otimes h^n \otimes \alpha, & \text{if } i = n+1. \end{cases}$$

The product on the chain level is defined as

$$\Psi \smile \Phi := \Psi \otimes (\sigma_1 \triangleright \Phi) \in \mathbf{C}^{p+q}(\mathcal{H}, {}^{\sigma_1\sigma_2}k),$$

for any $\Psi \in \mathbf{C}^{p}(\mathcal{H}, {}^{\sigma_1}k)$ and any $\Phi \in \mathbf{C}^{q}(\mathcal{H}, {}^{\sigma_2}k)$, where $\triangleright$ denotes the left diagonal action of $\mathcal{H}$ on $\mathcal{H}^{\otimes n}$. We thus note that

$$\begin{aligned} d_{p+q}(\Psi \smile \Phi) = & \sum_{i=0}^{p}(-1)^i d_i(\Psi) \otimes (\sigma_1 \triangleright \Phi) \\ & + (-1)^{p+1}\Psi \otimes \sigma_1 \otimes (\sigma_1 \triangleright \Phi) + (-1)^{p+2}\Psi \otimes \sigma_1 \otimes (\sigma_1 \triangleright \Phi) \\ & + \sum_{i=1}^{q}(-1)^{p+i}\Psi \otimes (\sigma_1 \triangleright d_i(\Phi)) \\ = & \; d_p(\Psi) \smile \Phi + (-1)^p \Psi \smile d_q(\Phi), \end{aligned}$$

i.e. $\bigoplus_{\sigma \in \mathcal{G}(\mathcal{H})} \mathrm{CH}^\bullet(\mathcal{H}, {}^\sigma k)$ is a differential graded algebra with unit $1 \in \mathbf{C}^0(\mathcal{H}, k)$. In particular, $\mathrm{CH}^\bullet(\mathcal{H}, k)$ is a differential graded subalgebra.

In Lemma 3.4 we showed that $*$-structure defined in (4.2) is compatible with the Hochschild differentials. We now show that it is compatible with the cup product structure above. For that we observe

$$\begin{aligned} & w_{p+q}((h^1 \otimes \cdots \otimes h^p) \smile (h^{p+1} \otimes \cdots \otimes h^{p+q})) \\ & \quad = w_{p+q}(h^1 \otimes \cdots \otimes h^p \otimes \sigma_1 h^{p+1} \otimes \cdots \otimes \sigma_1 h^{p+q}) \\ & \quad = \sigma_2\sigma_1 S^{-1}(h^{p+q*}\alpha) \otimes \cdots \otimes \sigma_2\sigma_1 S^{-1}(h^{p+1*}\sigma_1) \\ & \qquad\qquad \otimes \sigma_2\sigma_1 S^{-1}(h^{p*}) \otimes \cdots \otimes \sigma_2\sigma_1 S^{-1}(h^{1*}) \\ & \quad = \sigma_2 S^{-1}(h^{p+q*}) \otimes \cdots \otimes \beta S^{-1}(h^{p+1*}) \\ & \qquad\qquad \otimes \sigma_2\sigma_1 S^{-1}(h^{p*}) \otimes \cdots \otimes \sigma_2\sigma_1 S^{-1}(h^{1*}) \\ & \quad = w_q(h^{p+1} \otimes \cdots \otimes h^{p+q}) \smile \omega_p(h^1 \otimes \cdots \otimes h^p). \end{aligned}$$ □

We note that the dg-algebra structure on the sum $\bigoplus_{\sigma \in \mathcal{G}(\mathcal{H})} \mathrm{CH}^\bullet(\mathcal{H}, {}^\sigma k)$ works even in the case $\mathcal{G}(\mathcal{H})$ is not abelian. However, we need $\mathcal{G}(\mathcal{H})$ to be an abelian group for the $*$-structure to work.

The definition above is in fact a simplified version of the following. The collection

$$\bigoplus_{\sigma_1, \sigma_2 \in \mathcal{G}(\mathcal{H})} \mathrm{CB}^\bullet(k^{\sigma_1}, \mathcal{H}, {}^{\sigma_2}k)$$

forms a differential graded category where the set of objects is $\mathcal{G}(\mathcal{H})$, and the set of morphisms are defined as $\mathrm{Hom}(\sigma_2, \sigma_1) := \mathrm{CB}^\bullet(k^{\sigma_1}, \mathcal{H}, {}^{\sigma_2}k)$. This category is monoidal where the product comes from the product in $\mathcal{H}$. Then we put an equivalence relation on the set of all morphisms declaring two morphisms

$$\sigma_2 \xrightarrow{\Psi} \sigma_1 \quad \text{and} \quad \sigma_2' \xrightarrow{\Psi'} \sigma_1'$$

to be equivalent if the morphisms

$$1 \xrightarrow{\Psi \triangleleft \sigma_2^{-1}} \sigma_1\sigma_2^{-1} \quad \text{and} \quad 1 \xrightarrow{\Psi' \triangleleft (\sigma_2')^{-1}} \sigma_1'(\sigma_2')^{-1}$$

are identical. Now, the set of equivalence classes of morphisms is the algebra we defined above.

5. Complexified QUE algebras and their Hopf-dihedral cohomology

5.1. Drinfeld–Jimbo QUE algebras. Let $\mathfrak{g}$ be a semi-simple Lie algebra of rank ℓ, $\alpha_1, \ldots, \alpha_n$ an ordered sequence of the simple roots, and $a_{ij} = (\alpha_i, \alpha_j)$ the corresponding Cartan matrix. The Drinfeld–Jimbo quantum enveloping algebra (QUE algebra) $U_q(\mathfrak{g})$ is the Hopf algebra with 4ℓ generators K_i, K_i^{-1}, E_i, F_i, $1 \leq i \leq \ell$, subject to the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad i \neq j,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(n)_q\,!}{(r)_q\,!\,(n-r)_q\,!}, \quad (n)_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The rest of the Hopf algebra structure of $U_q(\mathfrak{g})$ is given by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_j) = F_j \otimes 1 + K_j^{-1} \otimes F_j,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i.$$

5.2. Complexified Drinfeld–Jimbo QUE algebras. Let $\mathfrak{g}$ be a semi-simple Lie algebra of rank ℓ, and $q^4 \neq 0, 1$. The complexified QUE algebra $\mathfrak{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ is the algebra generated by 4ℓ generators K_i, K_i^{-1}, E_i, F_i, $i = 1, \ldots, \ell$, subject to the relation we gave above, except the following:

$$K_i E_j K_i^{-1} = q_i^{a_{ij}/2} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}/2} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}.$$

The rest of the Hopf algebra structure is defined as

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$
$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_j) = F_j \otimes K_j + K_j^{-1} \otimes F_j,$$
$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$
$$S(K_i) = K_i^{-1}, \quad S(E_i) = -q_i E_i, \quad S(F_i) = -q_i^{-1} F_i.$$

The Hopf algebra $\mathfrak{U}_q(\mathfrak{g})$ is a Hopf $*$-algebra by

$$K_i^* = K_i, \quad (K_i^{-1})^* = K_i^{-1}, \quad E_i^* = F_i, \quad F_i^* = E_i.$$

Indeed,

$$\begin{aligned}
\Delta(E_i^*) &= E_{i(1)}^* \otimes E_{i(2)}^* = E_i^* \otimes K_i^* + (K_i^{-1})^* \otimes E_i^* \\
&= F_i \otimes K_i + K_i^{-1} \otimes F_i = \Delta(F_i), \\
\Delta(F_i^*) &= F_{i(1)}^* \otimes F_{i(2)}^* = F_i^* \otimes K_i^* + (K_i^{-1})^* \otimes F_i^* \\
&= E_i \otimes K_i + K_i^{-1} \otimes E_i = \Delta(E_i).
\end{aligned}$$

It is straightforward to check the condition for the group-like elements K_i and K_i^{-1}. As for the counit, we have

$$\varepsilon(E_i) = \varepsilon(E_i^*) = \varepsilon(F_i) = \varepsilon(F_i^*) = 0$$

and

$$\varepsilon(K_i) = \varepsilon(K_i^*) = \varepsilon(K_i^{-1}) = \varepsilon((K_i^{-1})^*) = 1,$$

and finally for the antipodes we see that

$$\begin{aligned}
S(E_i^*) &= S(F_i) = -q_i^{-1} F_i = S^{-1}(E_i)^*, \\
S(F_j^*) &= S(E_j) = -q_j E_j = S^{-1}(F_i)^*.
\end{aligned}$$

We took the definition of the complexified QUE algebra $\mathfrak{U}_q(\mathfrak{g})$ and its $*$-structure from [26], in which the authors use the notation $\check{U}_q(\mathfrak{g})$. We also note from [26] that the Hopf algebras $U_q(\mathfrak{g})$ and $\mathfrak{U}_q(\mathfrak{g})$ are not isomorphic in general. It also follows directly from the new commutator relations that $(K_{2\rho}^2, \varepsilon)$ is a MPI over the Hopf algebra $\mathfrak{U}_q(\mathfrak{g})$, where $K_{2\rho}^2 = K_1^2 \cdots K_\ell^2 \in \mathfrak{U}_q(\mathfrak{g})$.

In view of Theorem 4.5 and Proposition 4.6 we conclude the following.

Corollary 5.1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra of rank ℓ. Then the Hopf-cyclic cohomology $HC^\bullet(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k)$ of the complexified QUE algebra $\mathfrak{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ carries a nontrivial $*$-structure, and thus the Hopf-dihedral cohomologies $HC_\pm^\bullet(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k)$ of $\mathfrak{U}_q(\mathfrak{g})$ are well-defined for any $\mathbf{m} \in \mathbb{N}^{\times \ell}$.*

5.3. Complexified QUE algebra $\mathfrak{U}_q(sl_2)$. The Hopf $*$-algebra $\mathfrak{U}_q(sl_2)$ is the algebra generated by E, F, K and K^{-1} subject to the relations

$$KK^{-1} = 1, \quad KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

The $*$-structure is defined by

$$K^* = K \quad \text{and} \quad E^* = F.$$

The rest of the Hopf $*$-algebra structure is given by

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F$$
$$\varepsilon(K) = \varepsilon(K^{-1}) = 1, \qquad \varepsilon(E) = \varepsilon(F) = 0,$$
$$S(K^{\pm 1}) = K^{\mp 1}, \qquad S(E) = -qE, \qquad S(F) = -q^{-1}F.$$

We finally note that (K^2, ε) is a MPI for the Hopf algebra $\mathfrak{U}_q(sl_2)$.

5.4. Hopf-dihedral cohomology of $\mathfrak{U}_q(sl_2)$. Considering the Hopf subalgebra $T = k[K, K^{-1}]$ of $\mathfrak{U}_q(sl_2)$ generated by K^m, $m \in \mathbb{Z}$, there is a canonical coextension of coalgebras of the form $\pi_T: \mathfrak{U}_q(sl_2) \to T$ given by

$$\pi_T(E^u F^v K^w) = \begin{cases} K^w, & \text{if } u + v = 0, \\ 0, & \text{otherwise,} \end{cases}$$

see also [23, Sect. 4].

Since the projection π_T is a map of coalgebras, we can consider each basis element $E^u F^v K^w \in \mathfrak{U}_q(sl_2)$ as a 1-dimensional T-comodule. Then the spectral sequence associated to the coextension $\pi_T: \mathfrak{U}_q(sl_2) \to T$ (see [23, Sect. 3]) collapses onto the $q = 0$ line as

$$E_1^{0,0} = \begin{cases} \langle 1 \otimes 1 \rangle, & \text{if } m = 0, \\ 0, & \text{otherwise,} \end{cases}$$
$$E_1^{r,0} = \bigoplus_{\mathbf{u},\mathbf{v}} \langle 1 \otimes E^{u_1} F^{v_1} K^{w_1} \otimes \cdots \otimes E^{u_r} F^{v_r} K^{w_r} \otimes K^m \rangle, \quad \text{if } r \geq 1,$$

where $\mathbf{u}$ and $\mathbf{v}$ run over the vectors of dimension p of non-negative integers satisfying the recursive formula

$$w_{i+1} = w_i + (u_{i+1} + v_{i+1}) + (u_i + v_i)$$

with the initial condition $u_0 = v_0 = w_0 = 0$, and the boundary conditions $u_{r+1} = v_{r+1} = 0$ and $w_{r+1} = m$. Hence,

$$w_i = (u_i + v_i) + 2 \sum_{j=1}^{i-1} (u_j + v_j)$$

for $1 \leq i \leq r + 1$, and we note that m must be a positive even number.

Furthermore, $E_2^{r,s} = E_\infty^{r,s} = HH^{r+s}(\mathfrak{U}_q(sl_2), {}^{K^m}k)$, since the E_1-term collapses onto the $s = 0$ row, and we see that the collection $\bigoplus_{r,m\geq 0} HH^r(\mathfrak{U}_q(sl_2), {}^{K^m}k)$ of Hochschild cohomology groups form an algebra generated by

$$HH^1(\mathfrak{U}_q(sl_2), {}^{K^m}k) = \langle (EK)^u(KF)^v |\, u + v = m\rangle\,.$$

Let us *formally* write

$$HH^{\frac{1}{2}}(\mathfrak{U}_q(sl_2), {}^{K^u}k) = \langle (EK)^u, (KF)^u\rangle\,.$$

Then one can easily see that

$$HH^1(\mathfrak{U}_q(sl_2), {}^{K^m}k) = \bigoplus_{u+v=m} HH^{\frac{1}{2}}(\mathfrak{U}_q(sl_2), {}^{K^u}k) \otimes HH^{\frac{1}{2}}(\mathfrak{U}_q(sl_2), {}^{K^v}k).$$

Combining with Proposition 4.6 we have the following result.

Theorem 5.2. *The collection*

$$\bigoplus_{p\in\mathbb{N}\langle\frac{1}{2}\rangle}\bigoplus_{m\in\mathbb{N}} HH^p(\mathfrak{U}_q(sl_2), {}^{K^m}k) \tag{5.1}$$

of Hochschild cohomology groups form a $*$*-algebra generated by* $HH^{\frac{1}{2}}(\mathfrak{U}_q(sl_2), {}^{K^m}k)$ *for* $m \in \mathbb{N}$.

Furthermore, recalling the Hopf-cyclic cohomology with generalized coefficients from Proposition 4.4, we conclude the following.

Corollary 5.3. *The Hopf-cyclic cohomology, with generalized coefficients, of the complexified QUE algebra* $\mathfrak{U}_q(sl_2)$ *is given by*

$$HC^{p+1}_{\mathfrak{U}_q(sl_2)}(\mathfrak{U}_q(sl_2), {}^{K^m}k) = \begin{cases} \left\langle S^{p/2}((EK)^u(KF)^v) |\, u + v = m\right\rangle, & \text{if } p \text{ is even,}\\ 0, & \text{otherwise,}\end{cases}$$

where S *here denotes the iteration with the Bott element in cyclic cohomology. As a result, the Hopf-dihedral cohomology of* $\mathfrak{U}_q(sl_2)$ *is*

$$\begin{aligned} &HC^{p+1}_{\mathfrak{U}_q(sl_2),\pm}(\mathfrak{U}_q(sl_2), {}^{K^m}k)\\ &\quad= \begin{cases} \left\langle S^{p/2}((EK)^u(KF)^v \pm (EK)^v(KF)^u) |\, u + v = m\right\rangle, & \text{if } p \text{ is even,}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

5.5. Hopf-dihedral cohomology of complexified QUE algebras. Considering the Hopf subalgebra $T^\ell = k[K_1^{\pm 1}, \ldots, K_\ell^{\pm 1}]$ of $\mathfrak{U}_q(\mathfrak{g})$, generated by K_i^m where $i = 1, \ldots, \ell$ and $m \in \mathbb{Z}$, there is a canonical coextension of coalgebras of the form $\pi_{T^\ell} : \mathfrak{U}_q(\mathfrak{g}) \to T^\ell$ given by

$$\pi_{T^\ell}(E_1^{u_1} \cdots E_\ell^{u_\ell} F_1^{v_1} \cdots F_\ell^{v_\ell} K_1^{w_1} \cdots K_\ell^{w_\ell}) = \begin{cases} K_1^{w_1} \cdots K_\ell^{w_\ell}, & \text{if } u_1 + \cdots + u_\ell + v_1 + \cdots + v_\ell = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In order to simplify the notation, we use multi-indices $E^{\mathbf{u}} F^{\mathbf{v}} K^{\mathbf{w}}$ for any monomial of the form $E_1^{u_1} \cdots E_\ell^{u_\ell} F_1^{v_1} \cdots F_\ell^{v_\ell} K_1^{w_1} \cdots K_\ell^{w_\ell}$ where **u**, **v** and **w** are elements in $\mathbb{N}^{\times \ell}$.

Theorem 5.4. *For any complexified QUE algebra $\mathfrak{U}_q(\mathfrak{g})$, the Hochschild cohomology groups*

$$\bigoplus_{p \in \mathbb{N}\langle \frac{1}{2} \rangle} \bigoplus_{\mathbf{m} \in \mathbb{N}^{\times \ell}} HH^p(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k)$$

form a $*$-algebra generated by $HH^{\frac{1}{2}}(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k) = \langle (E_j K_j)^u \pm (K_j F_j)^u \rangle$.

As a result, the Hopf-cyclic cohomology with generalized coefficients is obtained as follows.

Corollary 5.5. *For any complexified QUE algebra $\mathfrak{U}_q(\mathfrak{g})$, the Hopf-cyclic cohomology of $\mathfrak{U}_q(\mathfrak{g})$ is given by*

$$HC^{p+1}_{\mathfrak{U}_q(\mathfrak{g})}(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k) = \begin{cases} \langle S^{p/2}((EK)^{\mathbf{u}}(KF)^{\mathbf{v}}) \mid \mathbf{u} + \mathbf{v} = \mathbf{m} \rangle, & \textit{if } p \textit{ is even}, \\ 0, & \textit{otherwise}, \end{cases}$$

where S is the iteration with the Bott element, and hence the Hopf-dihedral cohomology of $\mathfrak{U}_q(\mathfrak{g})$ is

$$HC^{p+1}_{\mathfrak{U}_q(\mathfrak{g}),\pm}(\mathfrak{U}_q(\mathfrak{g}), {}^{K^{\mathbf{m}}}k) = \begin{cases} \langle S^{p/2}((EK)^{\mathbf{u}}(KF)^{\mathbf{v}} \pm (EK)^{\mathbf{v}}(KF)^{\mathbf{u}}) \mid \mathbf{u} + \mathbf{v} = \mathbf{m} \rangle, & \textit{if } p \textit{ is even}, \\ 0, & \textit{otherwise}. \end{cases}$$

6. Hopf-dihedral homology

In this section we develop a $*$-structure on Hopf-cyclic homology, [25] and see also [43], the cyclic dual of Hopf-cyclic cohomology.

6.1. The dual cyclic theory. Let $\mathcal{H}$ be a Hopf algebra with a modular pair in involution (σ, δ). It is shown in [25] that the graded space

$$\mathrm{CC}_\bullet(\mathcal{H};\sigma,\delta) = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n} \tag{6.1}$$

has the structure of a cyclic module via the face operators

$$\partial_i : \mathrm{CC}_n(\mathcal{H},\delta,\sigma) \longrightarrow \mathrm{CC}_{n-1}(\mathcal{H},\delta,\sigma)$$

defined for $i \leq n$ by

$$\partial_i(h^1 \otimes \cdots \otimes h^n) = \begin{cases} \varepsilon(h^1)h^2 \otimes \cdots \otimes h^n, & \text{if } i = 0, \\ h^1 \otimes \cdots \otimes h^i h^{i+1} \otimes \cdots \otimes h^n, & \text{if } 1 \leq i \leq n-1, \\ \delta(h^n)h^1 \otimes \cdots \otimes h^{n-1}, & \text{if } i = n. \end{cases}$$

The degeneracies, on the other hand, are of the form

$$\sigma_j : \mathrm{CC}_n(\mathcal{H},\delta,\sigma) \longrightarrow \mathrm{CC}_{n+1}(\mathcal{H},\delta,\sigma)$$

defined for $0 \leq i \leq n$ as

$$\sigma_j(h^1 \otimes \cdots \otimes h^n) = \begin{cases} 1 \otimes h^1 \otimes \cdots \otimes h^n, & \text{if } j = 0, \\ h^1 \otimes \cdots \otimes h^j \otimes 1 \otimes h^{j+1} \otimes \cdots \otimes h^n, & \text{if } 1 \leq j \leq n-1, \\ h^1 \otimes \cdots \otimes h^n \otimes 1, & \text{if } j = n. \end{cases}$$

Finally, we have the cyclic operators $\tau_n : \mathrm{CC}_n(\mathcal{H},\delta,\sigma) \longrightarrow \mathrm{CC}_n(\mathcal{H},\delta,\sigma)$ given by

$$\tau_n(h^1 \otimes \cdots \otimes h^n) = \delta(h^n_{(2)}) S_\sigma(h^1_{(1)} \ldots h^n_{(1)}) \otimes h^1_{(2)} \otimes \cdots \otimes h^{n-1}_{(2)},$$

where $S_\sigma(h) := \sigma S(h)$ for any $h \in \mathcal{H}$. The cyclic homology of the complex (6.1) is called the Hopf-cyclic homology of the Hopf algebra $\mathcal{H}$, and is denoted by $HC_\bullet(\mathcal{H};\sigma,\delta)$.

6.2. Comodule algebras, invariant traces, and Hopf-cyclic homology. Let $\mathcal{A}$ be a right $\mathcal{H}$-comodule algebra by $\nabla : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{H}$, where we write $\nabla(a) = a_{<0>} \otimes a_{<1>}$, that is,

$$\nabla(ab) = \nabla(a)\nabla(b), \quad \nabla(1) = 1 \otimes 1.$$

Let also $\mathcal{A}$ be equipped with a σ-invariant δ-trace, that is, Tr: $\mathcal{A} \longrightarrow k$ satisfying

$$\mathrm{Tr}(a_{<0>})a_{<1>} = \mathrm{Tr}(a)\sigma, \quad \mathrm{Tr}(ab) = \mathrm{Tr}(ba_{<0>})\delta(a_{<1>}).$$

In the presence of these conditions we have

$$\gamma : \mathrm{CC}_n(\mathcal{A}) \longrightarrow \mathrm{CC}_n(\mathcal{H},\delta,\sigma)$$

a morphism of cyclic modules given by

$$\gamma(a_0 \otimes \cdots \otimes a_n) := \mathrm{Tr}(a_0 a_{1_{<0>}} \dots a_{n_{<0>}}) a_{1_{<1>}} \otimes \cdots \otimes a_{n_{<1>}}. \tag{6.2}$$

Then, just as in the previous section, (6.2) induces the cyclic module structure on $\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n+1}$ as defined in [25].

In particular, a Hopf algebra $\mathcal{H}$ is a $\mathcal{H}$-comodule algebra via its comultiplication. This way one can compare the Hopf-cyclic homology of $\mathcal{H}$ with the algebra cyclic homology of $\mathcal{H}$ regarding it as an algebra. To this end, it is noted in [25, Prop. 3.2] that if $\mathcal{H}$ is a Hopf algebra with a group-like $\sigma \in \mathcal{H}$ such that $S_\sigma^2 = \mathrm{Id}$, then

$$\theta\colon \mathrm{CC}_n(\mathcal{H}, \varepsilon, \sigma) \longrightarrow \mathrm{CC}_n(\mathcal{H}),$$

which is defined as

$$\theta(h_1 \otimes \cdots \otimes h^n) := S_\sigma(h^1{}_{(1)} h^2{}_{(1)} \dots h^n{}_{(1)}) \otimes h^1{}_{(2)} \otimes \cdots \otimes h^n{}_{(2)}, \tag{6.3}$$

is a map of cyclic modules. If furthermore, $\mathcal{H}$ is equipped with a σ-invariant trace $\mathrm{Tr}\colon \mathcal{H} \longrightarrow k$ such that $\mathrm{Tr}(\sigma)$ is invertible in k, then it follows from $\gamma \circ \theta = \mathrm{Tr}(\sigma)\,\mathrm{Id}$ that $HC_\bullet(\mathcal{H}, \varepsilon, \sigma)$ is a direct summand of $HC_\bullet(\mathcal{H})$, [25, Thm. 3.1].

We finally record here that setting $H_\bullet(\mathcal{H}) := \mathrm{Tor}_\bullet^{\mathcal{H}}(k,k)$ for a cocommutative Hopf algebra $\mathcal{H}$, it follows from [25, Thm. 4.1] that

$$HC_n(\mathcal{H}, \varepsilon, 1) = \bigoplus_{i\geq 0} H_{n-2i}(\mathcal{H}),$$

which proven by Karoubi [20] in case $\mathcal{H} = kG$.

6.3. The dihedral structure. Let us now assume that $\mathcal{H}$ is a Hopf $*$-coalgebra, i.e. a $*$-coalgebra

$$\Delta(h^*) = h^*_{(2)} \otimes h^*_{(1)}, \quad \varepsilon(h^*) = \overline{\varepsilon(h)}$$

such that

$$(hg)^* = h^* g^*, \quad 1^* = 1, \quad (S \circ *)^2 = \mathrm{Id},$$

and that $\mathcal{A}$ a $\mathcal{H}$-comodule $*$-algebra, i.e.

$$\nabla(a^*) = a_{<0>}{}^* \otimes S(a_{<1>})^* = a_{<0>}{}^* \otimes S^{-1}(a_{<1>}{}^*), \quad \forall\, a \in \mathcal{A}.$$

We also assume that $\delta(h^*) = \delta(h)$ and that $\sigma^* = \sigma^{-1}$. In order to obtain a $*$-structure on $\mathrm{CC}_\bullet(\mathcal{H}, \delta, \sigma)$ we use (6.2) to transfer the one on $\mathrm{CC}_\bullet(\mathcal{H})$. To this end we introduce cyclic operators

$$\tau_n\colon \mathrm{CC}_n(\mathcal{H}, \delta, \sigma) \longrightarrow \mathrm{CC}_n(\mathcal{H}, \delta, \sigma)$$

by

$$\tau_n(h^1 \otimes \cdots \otimes h^n) = \delta(h^n{}_{(2)}) S_\sigma(h^1{}_{(1)} \dots h^n{}_{(1)}) \otimes h^1{}_{(2)} \otimes \cdots \otimes h^{n-1}{}_{(2)}, \tag{6.4}$$

and involution operators $\omega_n\colon \mathrm{CC}_n(\mathcal{H},\delta,\sigma) \longrightarrow \mathrm{CC}_n(\mathcal{H},\delta,\sigma)$ by

$$\begin{aligned}\omega_n(h^1\otimes\cdots\otimes h^n) &= \delta({h^1}_{(1)}{}^*\dots {h^n}_{(1)}{}^*)S^{-1}({h^n}_{(2)}{}^*)\otimes\cdots\otimes S^{-1}({h^1}_{(2)}{}^*)\\ &= S^{-1}({h^n}^*{}_{(1)})\otimes\cdots\otimes S^{-1}({h^1}^*{}_{(1)})\delta({h^1}^*{}_{(2)}\dots {h^n}^*{}_{(2)}).\end{aligned} \tag{6.5}$$

We then leave it to the reader to verify that we thus get a dihedral module structure on $\bigoplus_{n\geq 0}\mathcal{H}^{\otimes n+1}$.

We next record here that [25, Thm. 3.1] also extends to the dihedral setting. We will make use of the fact that if $\mathcal{H}$ is a Hopf $*$-coalgebra, then $\mathcal{H}$ is a $*$-algebra via $*\circ S$. If, in particular, $\mathcal{H}$ is cocommutative, then we may take $* = \mathrm{Id}$ for the Hopf $*$-coalgebra structure.

We denote the (dihedral) homology of this dihedral module by $HC_*^{\pm}(\mathcal{H};\sigma,\delta)$, and call it the Hopf-dihedral homology.

Proposition 6.1. *The characteristic homomorphism* (6.2) *defines a morphism of dihedral modules of the form* $\gamma\colon \mathrm{CC}_\bullet^{\pm}(\mathcal{A}) \to \mathrm{CC}_\bullet^{\pm}(\mathcal{H};\sigma,\delta)$.

Proof. It follows already from [25, Prop. 3.1] that $\tau_n\gamma = \gamma\tau_n$ on $\mathrm{CC}_n(\mathcal{A})$. It then takes a straightforward calculation to check that (6.2) is compatible with ω_n. □

Theorem 6.2. *Let $\mathcal{H}$ be a Hopf algebra equipped with a modular pair (ε,σ) in involution, and a σ-invariant trace* $\mathrm{Tr}\colon \mathcal{H} \longrightarrow k$ *such that* $\mathrm{Tr}(\sigma)\in k$ *is invertible. Then $HC_n^{\pm}(\mathcal{H};\sigma,\varepsilon)$ is a direct summand of $HC_n^{\pm}(\mathcal{H})$.*

Proof. We have already observed in Proposition 6.1 that (6.2) is a map of dihedral modules. Furthermore, it is checked in [25, Thm. 3.1] that $\gamma\circ\theta = \mathrm{Tr}(\sigma)\,\mathrm{Id}$. On the other hand, it is shown in [25, Prop. 3.2] that (6.3) is a cyclic map. We are then left to observe that

$$\begin{aligned}&\omega_n\theta(h^1\otimes\cdots\otimes h^n)\\ &\quad= \omega_n(S_\sigma({h^1}_{(1)}\dots {h^n}_{(1)})\otimes {h^1}_{(2)}\otimes\cdots\otimes {h^n}_{(2)})\\ &\quad= (S^2({h^1}_{(1)}\dots {h^n}_{(1)})\sigma^{-1})^*\otimes S({h^n}_{(2)})^*\otimes\cdots\otimes S({h^1}_{(2)})^*\\ &\quad= S^{-2}({h^1}_{(1)}{}^*\dots {h^n}_{(1)}{}^*)\sigma\otimes S^{-1}({h^n}_{(2)}{}^*)\otimes\cdots\otimes S^{-1}({h^1}_{(2)}{}^*)\\ &\quad= \sigma {h^1}_{(1)}{}^*\dots {h^n}_{(1)}{}^*\otimes S^{-1}({h^n}_{(2)}{}^*)\otimes\cdots\otimes S^{-1}({h^1}_{(2)}{}^*)\\ &\quad= S_\sigma(S^{-1}({h^n}_{(1)}{}^*)\dots S^{-1}({h^1}_{(1)}{}^*))\otimes S^{-1}({h^n}_{(2)}{}^*)\otimes\cdots\otimes S^{-1}({h^1}_{(2)}{}^*)\\ &\quad= \theta(S^{-1}({h^n}^*)\otimes\cdots\otimes S^{-1}({h^1}^*))\\ &\quad= \theta\omega_n(h^1\otimes\cdots\otimes h^n),\end{aligned}$$

that is (6.3) is a map of dihedral modules. □

6.4. Hopf-dihedral homology of cocommutative Hopf algebras. In this subsection, we extend [25, Sect. 4] to the dihedral setting, and we compute the Hopf-dihedral homology of cocommutative Hopf algebras. To this end, we first recall the path space $E\mathcal{H}_\bullet$ of the Hopf algebra $\mathcal{H}$, which is defined on the graded module level as $E\mathcal{H}_n := \mathcal{H}^{\otimes n+1}$, whose simplicial structure is given by

$$\partial_i(h^0 \otimes \cdots \otimes h^n) = \begin{cases} h^0 \otimes \cdots \otimes h^i h^{i+1} \otimes \cdots \otimes h^n, & \text{if } 0 \le i \le n-1, \\ \delta(h^n) h^0 \otimes \cdots \otimes h^{n-1}, & \text{if } i = n, \end{cases}$$

$$\sigma_j(h^0 \otimes \cdots \otimes h^n) = h^0 \otimes \cdots \otimes h^j \otimes 1 \otimes h^{j+1} \otimes \cdots \otimes h^n, \quad 0 \le j \le n.$$

The simplicial module $E\mathcal{H}_\bullet$ is contractible, and is a resolution for k via $\delta\colon \mathcal{H} \longrightarrow k$.

Lemma 6.3. *If $\mathcal{H}$ is a cocommutative Hopf algebra, then $E\mathcal{H}_\bullet$ is a dihedral module by*

$$\tau_n(h^0 \otimes \cdots \otimes h^n) = h^0 h^1{}_{(1)} \dots h^n{}_{(1)} \otimes S(h^1{}_{(2)} \dots h^n{}_{(2)}) \otimes h^1{}_{(3)} \otimes \cdots \otimes h^{n-1}{}_{(3)} \tag{6.6}$$

and

$$\omega_n(h^0 \otimes \cdots \otimes h^n) = h^0 h^1{}_{(1)} \dots h^n{}_{(1)} \otimes S^{-1}(h^n{}_{(2)}) \otimes \cdots \otimes S^{-1}(h^1{}_{(2)}). \tag{6.7}$$

Proof. It is checked in [25, Lem. 4.2] that $E\mathcal{H}_\bullet$ is a cyclic module via (6.6). As a result, we have

$$\partial_i \tau_n = \tau_{n-1} \partial_{i-1}, \quad \sigma_i \tau_n = \tau_{n+1} \sigma_{n-1}, \qquad 1 \le i \le n$$

already. We next check that

$$\begin{aligned} \omega_n^2(h^0 \otimes \cdots \otimes h^n) &= h^0 h^1_{(1)} \dots h^n_{(1)} S^{-1}(h^n_{(2)}) \dots S^{-1}(h^1{}_{(2)}) \\ &\quad \otimes S^{-1}(S^{-1}(h^1_{(2)})) \otimes \cdots \otimes S^{-1}(S^{-1}(h^n_{(2)})) \\ &= h^0 \otimes S^{-2}(h^1) \otimes \cdots \otimes S^{-2}(h^n) \\ &= h^0 \otimes \cdots \otimes h^n, \end{aligned}$$

that is $\omega_n^2 = \mathrm{Id}$. Moreover we have

$$\begin{aligned} \partial_i \omega_n(h^0 \otimes \cdots \otimes h^n) &= h^0 h^1_{(1)} \dots h^n_{(1)} \otimes S^{-1}(h^n_{(2)}) \otimes \cdots \\ &\quad \cdots \otimes S^{-1}(h^{n-i+1}_{(2)}) S^{-1}(h^{n-i}_{(2)}) \otimes \cdots \otimes S^{-1}(h^1_{(2)}) \\ &= \omega_n \partial_{n-i}(h^0 \otimes \cdots \otimes h^n), \end{aligned}$$

and

$$\begin{aligned} \sigma_j \omega_n(h^0 \otimes \cdots \otimes h^n) &= h^0 h^1_{(1)} \dots h^n_{(1)} \otimes S^{-1}(h^n_{(2)}) \otimes \cdots \otimes S^{-1}(h^{n-j+1}_{(2)}) \\ &\quad \otimes 1 \otimes S^{-1}(h^{n-j}_{(2)}) \otimes \cdots \otimes S^{-1}(h^1_{(2)}) \\ &= \omega_n \sigma_{n-j}(h^0 \otimes \cdots \otimes h^n). \end{aligned}$$

Finally, we see that

$$\begin{aligned}(\tau_n\omega_n)(h^0\otimes\cdots\otimes h^n) &= h^0h^1_{(1)}\dots h^n_{(1)}S^{-1}(h^n_{(2)})\dots S^{-1}(h^1_{(2)})\\ &\quad\otimes S(S^{-1}(h^n_{(3)})\dots S^{-1}(h^1_{(3)}))\otimes S^{-1}(h^n_{(4)})\otimes\cdots\otimes S^{-1}(h^2_{(4)})\\ &= h^0\otimes h^1h^2_{(1)}\dots h^n_{(1)}\otimes S^{-1}(h^n_{(2)})\otimes\cdots\otimes S^{-1}(h^2_{(2)}),\end{aligned}$$

and hence

$$\begin{aligned}(\tau_n\omega_n)^2(h^0\otimes\cdots\otimes h^n) &= h^0\otimes h^1h^2_{(1)}\dots h^n_{(1)}S^{-1}(h^n_{(2)})\dots S^{-1}(h^2_{(2)})\\ &\quad\otimes S^{-2}(h^2_{(3)})\otimes\cdots\otimes S^{-2}(h^n_{(3)})\\ &= h^0\otimes\cdots\otimes h^n,\end{aligned}$$

that is $\tau_n\omega_n\tau_n\omega_n = \mathrm{Id}$, or equivalently $\tau_n\omega_n = \omega_n\tau_n^{-1}$. □

Lemma 6.4. *If $\mathcal{H}$ is a cocommutative Hopf algebra, then $\pi\colon E\mathcal{H}_n \longrightarrow \mathrm{CC}_n(\mathcal{H};1,\varepsilon)$ defined as*

$$\pi(h^0\otimes\cdots\otimes h^n) = \varepsilon(h^0)h^1\otimes\cdots\otimes h^n$$

is a map of dihedral modules.

Proof. As a result of [25, Lem. 4.2] it suffices to observe

$$\begin{aligned}\pi\omega_n(h^0\otimes\cdots\otimes h^n) &= \varepsilon(h^0)\varepsilon(h^1_{(1)}\dots h^n_{(1)})S^{-1}(h^n_{(2)})\otimes\cdots\otimes S^{-1}(h^1_{(2)})\\ &= \varepsilon(h^0)\omega_n(h^1\otimes\cdots\otimes h^n)\\ &= \omega_n\pi(h^0\otimes\cdots\otimes h^n),\end{aligned}$$

as we wanted to show. □

Theorem 6.5. *If $\mathcal{H}$ is a cocommutative Hopf algebra, then for any $n\geq 0$*

$$HC_n^+(\mathcal{H};1,\varepsilon) = \bigoplus_{i\geq 0} H_{n-4i}(\mathcal{H},k) \quad \textit{and} \quad HC_n^-(\mathcal{H};1,\varepsilon) = \bigoplus_{i\geq 0} H_{n-4i-2}(\mathcal{H},k).$$

Proof. Let us first note that $E\mathcal{H}_\bullet$ is a (left) $\mathcal{H}$-module by

$$h\cdot(h^0\otimes\cdots\otimes h^n) = hh^0\otimes\cdots\otimes h^n,$$

and hence $\mathrm{CC}_\bullet(\mathcal{H};1,\varepsilon) = k\otimes_{\mathcal{H}} E\mathcal{H}_\bullet$. Moreover, as a result of Lemma 6.4 we have

$$\mathrm{CC}^{\pm}_\bullet(\mathcal{H};1,\varepsilon)\cong k\otimes_{\mathcal{H}}\mathrm{CC}^{\pm}_\bullet(E\mathcal{H}_\bullet)$$

as bicomplexes where the latter is the dihedral bicomplex of the dihedral module $E\mathcal{H}_\bullet$.

Since $E\mathcal{H}_\bullet$ is contractible the vertical homology vanishes, and since $\varepsilon\colon \mathcal{H} \longrightarrow k$ is the contracting homotopy, the double complex $\mathrm{CC}^+_\bullet(E\mathcal{H}_\bullet)$ is a (Cartan–Eilenberg) resolution for

$$k^+_\bullet : k \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow k \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

and $\mathrm{CC}^-_\bullet(E\mathcal{H}_\bullet)$ is for

$$k^-_\bullet : 0 \longleftarrow 0 \longleftarrow k \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow k \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

As a result,

$$\begin{aligned} HC^+_n(\mathcal{H};1,\varepsilon) &= H_n\big(\mathrm{Tot}(\mathrm{CC}^+(\mathcal{H};1,\varepsilon))\big) = H_n\big(\mathrm{Tot}(k \otimes_{\mathcal{H}} \mathrm{CC}^+(E\mathcal{H}))\big) \\ &= H_n\big(k \otimes_{\mathcal{H}} \mathrm{Tot}(\mathrm{CC}^+(E\mathcal{H}))\big) = \mathbb{H}_n(\mathcal{H}, k^+_\bullet), \end{aligned}$$

and similarly

$$\begin{aligned} HC^-_n(\mathcal{H};1,\varepsilon) &= H_n\big(\mathrm{Tot}(\mathrm{CC}^-(\mathcal{H};1,\varepsilon))\big) = H_n\big(\mathrm{Tot}(k \otimes_{\mathcal{H}} \mathrm{CC}^-(E\mathcal{H}))\big) \\ &= H_n\big(k \otimes_{\mathcal{H}} \mathrm{Tot}(\mathrm{CC}^-(E\mathcal{H}))\big) = \mathbb{H}_n(\mathcal{H}, k^-_\bullet), \end{aligned}$$

where the latter objects are the hyperhomologies of the complexes $k^+_\bullet$ and $k^-_\bullet$, respectively. Since the hyperhomology is independent of the (Cartan–Eilenberg) resolution chosen, we can replace the double complex $\mathrm{CC}^+_\bullet(E\mathcal{H}_\bullet)$ with one having zeros as the horizontal maps, and having resolutions of k on the zeroth (mod 4) column. Similarly, we replace the double complex $\mathrm{CC}^-_\bullet(E\mathcal{H}_\bullet)$ with one having resolutions of k on the second (mod 4) column. Therefore, for the $+1$-eigenspace we get

$$\begin{array}{ccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_1(\mathcal{H},k) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & H_1(\mathcal{H},k) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_0(\mathcal{H},k) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & H_0(\mathcal{H},k) & \longleftarrow & \cdots \end{array}$$

and for the -1-eigenspace we get

$$\begin{array}{ccccccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & 0 & \longleftarrow & H_1(\mathcal{H},k) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & H_1(\mathcal{H},k) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & 0 & \longleftarrow & H_0(\mathcal{H},k) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & H_0(\mathcal{H},k) & \longleftarrow & \cdots \end{array}$$

on the E_2-page (and hence on the E_∞-page, since the horizontal maps are zero) of the bicomplex computing the hyperhomology $\mathbb{H}_n(\mathcal{H}, k_*^\pm)$. The claim then follows. □

We finally relate the Hopf-dihedral homology of a group algebra to the dihedral group homology. We recall the basic facts from [27]. Let G be a (discrete) group, and $\mathcal{A}$ be the group algebra kG. Then the graded space given by

$$E\mathcal{A}_n^\pm := \mathrm{Span}\{\, g_0 \otimes \cdots \otimes g_n \in \mathrm{CC}_n^\pm(\mathcal{A}) \mid g_0 \ldots g_n = 1\} \qquad (6.8)$$

form a dihedral submodule of the standard dihedral module $\mathrm{CC}_\bullet^\pm(\mathcal{A})$. The dihedral homology of this dihedral module is called the dihedral group homology of the group G, and is denoted by $HC_\bullet^\pm(G,k)$.

It follows at once from [27, Cor. 4.4] and Theorem 6.5 that the Hopf-dihedral homology of the Hopf algebra kG is isomorphic to the dihedral group homology of the group G. In the following proposition we record also the explicit isomorphism between the complexes.

Proposition 6.6. *For any (discrete) group G, $HC_\bullet^\pm(kG;1,\varepsilon) \cong HC_\bullet^\pm(G,k)$.*

Proof. It follows from (6.8) that $\theta_n\colon \mathrm{CC}_n^\pm(kG;1,\varepsilon) \longrightarrow EkG_n^\pm$ given by

$$\theta_n(g_1 \otimes \cdots \otimes g_n) := (g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n \qquad (6.9)$$

is an isomorphism. It is also straightforward from the proof of Theorem 6.2 that (6.9) is a map of dihedral modules. □

As a result, we recover [27, Thm. 4.5] from Theorem 6.2. Moreover, from

$$\begin{aligned} HC_n^\alpha(G,k) &\cong HC_n^\alpha(kG;1,\varepsilon) \\ &= H_{n+\alpha-1}(G,k) \oplus H_{n+\alpha-5}(\mathcal{H},k) \oplus H_{n+\alpha-9}(\mathcal{H},k) \oplus \cdots \end{aligned}$$

for $\alpha = \pm 1$ we get the embeddings

$$i_\ell\colon H_n(G,k) \longrightarrow HC_{n+2\ell}^\alpha(G,k), \quad \alpha = (-1)^\ell. \qquad (6.10)$$

7. Hopf-dihedral homology and L-theory

7.1. Hopf-dihedral Chern character. In this subsection we realize the Chern character as a homomorphism with values in the Hopf-dihedral homology. More precisely, given an involutive ring $\mathcal{A}$, we will transfer the KU-groups of $\mathcal{A}$ to the Hopf-dihedral homology of a Hopf algebra $\mathcal{H}$ on which $\mathcal{A}$ is a comodule algebra. Let us first recall the construction of the KU-functor from [4, 27]. For the L-functor, we refer the reader to [13, 14], and the references therein.

Let $\mathcal{A}$ be an algebra with involution, and $U_n^\epsilon(\mathcal{A})$ the set of $2n \times 2n$ matrices with coefficients in $\mathcal{A}$ preserving the quadratic form

$$ {}_\epsilon J = \begin{pmatrix} 0 & I_n \\ \epsilon I_n & 0 \end{pmatrix}. $$

We note from [21] that they are the matrices satisfying $M^\star M = MM^\star = I_{2n}$, where

$$ M^\star = \begin{pmatrix} D^\dagger & \epsilon B^\dagger \\ \epsilon C^\dagger & A^\dagger \end{pmatrix}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. $$

Here $\dagger$ denotes the conjugate transpose of an $n \times n$ matrix. We note also from [21, Ex. 4.16] that $U_n^1(\mathbb{R}) = O(n,n)$, and $U_n^{-1}(\mathbb{R}) = Sp(2n,\mathbb{R})$. Now we set

$$ U^\epsilon(\mathcal{A}) := \operatorname{colim}_n U_n^\epsilon(\mathcal{A}), $$

and following [27] we denote by $BU^\epsilon(\mathcal{A})$ the classifying space of the group $U^\epsilon(\mathcal{A})$. It follows from [4, Prop. 5.1(b) and Thm. 5.2] that the second derived subgroup of $U^\epsilon(\mathcal{A})$ is equal to its first derived subgroup, i.e. it is quasi-perfect. Thus, the commutator subgroup of $U^\epsilon(\mathcal{A})$ is a perfect normal subgroup. As a result, the plus construction [2,24,40] can be applied to the classifying space $BU^\epsilon(\mathcal{A})$ to obtain the space $BU^\epsilon(\mathcal{A})^+$ such that

$$ H_\bullet(BU^\epsilon(\mathcal{A})) \cong H_\bullet(BU^\epsilon(\mathcal{A})^+) $$

and that

$$ \pi_1(BU^\epsilon(\mathcal{A})^+) = \frac{U^\epsilon(\mathcal{A})}{[U^\epsilon(\mathcal{A}), U^\epsilon(\mathcal{A})]}. $$

Then, by definition [21, Def. 4.17], the Hermitian algebraic K-theory of the algebra $\mathcal{A}$ is given by

$$ KU_n^\epsilon(\mathcal{A}) := \pi_n(BU^\epsilon(\mathcal{A})^+), \quad n \geq 1, $$

and for $n = 0$, $KU_0^\epsilon(\mathcal{A})$ is the Grothendieck group of the category of non-degenerate quadratic forms over $\mathcal{A}$. We now consider the composition

$$ \begin{aligned} H_s(U_n^\epsilon(\mathcal{A}), k) \xrightarrow{i_\ell} HC_{s+2\ell}^\alpha(U_n^\epsilon(\mathcal{A}), k) \xrightarrow{j_{s+2\ell}} HC_{s+2\ell}^\alpha(kU_n^\epsilon(\mathcal{A})) \\ \xrightarrow{J_\ell} HC_{s+2\ell}^\alpha(M_{2n}(\mathcal{A})) \xrightarrow{\mathrm{Tr}_\ell} HC_{s+2\ell}^\alpha(\mathcal{A}), \end{aligned} $$

where $\alpha = (-1)^\ell$, and

(i) i_ℓ is the mapping given by (6.10),

(ii) $j_{s+2\ell}$ is the map induced by the inclusion of the dihedral group homology of a group into the dihedral homology of the group algebra of this group,

(iii) J_ℓ is induced by the inclusion of $kU_n^\epsilon(\mathcal{A})$ into $M_{2n}(\mathcal{A})$,

(iv) Tr_ℓ is induced by the trace map inducing the Morita isomorphism.

Taking colimit over n we obtain the mappings

$$L^\ell\colon H_s(U^\epsilon(\mathcal{A}), k) \longrightarrow HC^\alpha_{s+2\ell}(\mathcal{A})$$

for $\ell \geq 0$ and $a = (-1)^\ell$. Finally, the dihedral Chern character

$$\mathrm{Ch}^\ell_s\colon KU^\epsilon_s(\mathcal{A}) \longrightarrow HC^\alpha_s(\mathcal{A}), \quad s \geq 0,\ \alpha = (-1)^\ell, \tag{7.1}$$

is defined to be the composition

$$\begin{aligned} KU^\epsilon_s(\mathcal{A}) = \pi_s(BU^\epsilon(\mathcal{A})^+) &\xrightarrow{h} H_s(BU^\epsilon(\mathcal{A})^+) \\ &\xrightarrow{\sim} H_s(BU^\epsilon(\mathcal{A})) = H_s(U^\epsilon(\mathcal{A}), k) \xrightarrow{L^\ell} HC^\alpha_{s+2\ell}(\mathcal{A}), \end{aligned}$$

where h is the Hurewicz homomorphism. In particular we have

$$\mathrm{Ch}^\ell_0\colon KU^\epsilon_0(\mathcal{A}) \xrightarrow{S_\ell} HC^\alpha_{2\ell}(M_{2n}(\mathcal{A})) \xrightarrow{\mathrm{Tr}_\ell} HC^\alpha_{2\ell}(\mathcal{A}), \tag{7.2}$$

where $S_\ell\colon [p] \mapsto [p \otimes \cdots \otimes p]$.

7.2. Hopf-dihedral homology of a group ring. Following [35], we note from [19, 37] that the problems of modifying even-dimensional multiply-connected manifolds with fundamental group π leads to a need to compute $KU_0(\mathbb{Z}\pi)$ for the group algebra $\mathbb{Z}\pi$. As for the odd-dimensional manifolds, it is shown in [38] that one similarly needs to study $KU_1(\mathbb{Z}\pi)$, where π is the fundamental group of the manifold. See also [45].

Motivated by these discussions, we will study the dihedral Chern character map (7.1) for the group algebra $\mathcal{A} = k\pi$ for a group π. We will show that the dihedral Chern character (7.1) lands in the Hopf-dihedral homology of $k\pi$ which is a direct summand of the algebra dihedral homology of $k\pi$. In view of Theorem 6.5 we gain a computational advantage relating the KU-groups of $k\pi$ to the group homology of the group π.

Theorem 7.1. *Let π be a (discrete) group, and $\mathcal{A} = k\pi$ the group algebra of the group π. Then the dihedral Chern character* (7.1) *lands in the Hopf-dihedral homology of the Hopf algebra $k\pi$.*

Proof. We first note, as a result of Proposition 6.6, that we may replace, up to homology, the inclusion

$$j_\bullet\colon HC^\pm_\bullet(U^\epsilon_n(\mathcal{A}), k) \longrightarrow HC^\pm_\bullet(kU^\epsilon_n(\mathcal{A})),$$

by the section

$$\theta_\bullet\colon HC^\pm_\bullet(kU^\epsilon_n(\mathcal{A}); 1, \varepsilon) \longrightarrow HC^\pm_\bullet(kU^\epsilon_n(\mathcal{A}))$$

of the characteristic homomorphism

$$\gamma_\bullet : HC_\bullet^{\pm}(kU_n^\epsilon(\mathcal{A})) \longrightarrow HC_\bullet^{\pm}(kU_n^\epsilon(\mathcal{A}); 1, \varepsilon).$$

It then follows from Equation (6.9) that the image of $\theta_\bullet$ is the [1]-component of the decomposition of the cyclic homology $HC_\bullet(kU_n^\epsilon(\mathcal{A}))$ along the conjugacy classes of elements $[z]$ in $U_n^\epsilon(\mathcal{A})$

$$\bigoplus_{[z]} HC_\bullet(k[B.(U_n^\epsilon(\mathcal{A})_z, z)])$$

as described in [7] and [29, Sect. 7.4]. Now, the trace map induces a map on homology of the form

$$\mathrm{Tr}_\bullet : HC_\bullet(k[B.(U_n^\epsilon(\mathcal{A})_1, 1)]) \longrightarrow HC_\bullet(k[B.(\pi)_1], 1).$$

Then by [7, Thm. I] we know that the [1]-component has homology of the form

$$HC_\bullet(k[B.(\pi)_1, 1)]) = \bigoplus_{i \geq 0} H_{\bullet - 2i}(\pi).$$

The result then follows. □

7.3. Podleś spheres. We next recall from [36] the (two parameters) quantum Podleś sphere $\mathcal{O}(S_q(c,d))$, along with the coaction of the coordinate (Hopf) algebra $\mathcal{O}(SU_q(2))$ of the quantum group $SU_q(2)$.

The compact quantum group (CQG) algebra $\mathcal{O}(SU_q(2))$ is the algebra generated by x, u, v, y subject to the relations

$$\begin{gathered} ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy, \\ vu = uv, \quad xy - q^{-1}uv = yx - quv = 1, \end{gathered}$$

and its Hopf algebra structure is given by

$$\begin{gathered} \Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y, \\ \Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y, \\ \varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(u) = \varepsilon(v) = 0, \\ S(x) = y, \quad S(y) = x, \quad S(u) = -qu, \quad S(v) = -q^{-1}v, \end{gathered}$$

see also [31]. We next note from [26, Prop. 11.34] that there is a family $\{f_z\}_{z\in\mathbb{C}}$ of characters of $\mathcal{O}(SU_q(2))$, uniquely determined by

(i) the functions $z \mapsto f_z(a)$ is an entire function of exponential growth on the right-half plane for any $a \in \mathcal{O}(SU_q(2))$,

(ii) $f_z f_{z'} = f_{z+z'}$, with $f_0 = \varepsilon$,

(iii) $h(ab) = h(b(f_1 \cdot a \cdot f_1))$, for any $a, b \in \mathcal{O}(SU_q(2))$, and the Haar state h,

satisfying[1]

$$S^2(a) = f_{-1} \cdot a \cdot f_1, \quad f_z(S(a)) = f_{-z}(a)$$

for any $a \in \mathcal{O}(SU_q(2))$. As a result, the pair $(f_1, 1)$ is a MPI for the Hopf algebra $\mathcal{O}(SU_q(2))$. Indeed,

$$\widetilde{S}_1(a) = f_1(a_{(2)})S(a_{(1)}) = S(f_1 \cdot a),$$

and therefore,

$$\begin{aligned}\widetilde{S}_1^2(a) &= f_1(a_{(3)})f_1(S(a_{(1)}))S^2(a_{(2)})\\ &= f_1(a_{(3)})f_{-1}(a_{(1)})S^2(a_{(2)})\\ &= S^2(f_1 \cdot a \cdot f_{-1}) = a.\end{aligned}$$

The Podleś spheres $\mathcal{O}(S_q(c,d))$ on the other hand, is defined to be the algebra generated by z_{-1}, z_0, z_1 subject to the relations

$$\begin{aligned} z_0^2 - qz_1z_{-1} - q^{-1}z_{-1}z_1 &= d1,\\ (1-q^2)z_0^2 + qz_{-1}z_1 - qz_1z_{-1} &= (1-q^2)cz_0,\\ z_{-1}z_0 - q^2z_0z_{-1} &= (1-q^2)cz_{-1},\\ z_0z_1 - q^2z_1z_0 &= (1-q^2)cz_1.\end{aligned} \tag{7.3}$$

In particular, the algebra $\mathcal{O}(S_q(s, 1+s^2))$ is denoted simply by $\mathcal{O}(S^2_{qs})$, and it can be realized as a subalgebra of $\mathcal{O}(SU_q(2))$ via $a_i \mapsto \widetilde{a_i}$ for $i = -1, 0, 1$ where

$$\begin{aligned}\widetilde{z_{-1}} &:= (1+q^2)^{-1/2}x^2 + s(1+q^{-2})^{1/2}xv - q(1+q^2)^{-1/2}v^2,\\ \widetilde{z_0} &:= ux + s(1 + (q+q^{-1})uv) - vy,\\ \widetilde{z_1} &:= (1+q^2)^{-1/2}u^2 + s(1+q^{-2})^{1/2}yu - q(1+q^2)^{-1/2}y^2.\end{aligned}$$

The algebra $\mathcal{O}(S^2_{qs})$ carries the $*$-structure given by

$$z_i{}^* := (-q)^i z_{-i}.$$

[1]It follows from the (non-degenerate) pairing between the CQG algebra $\mathcal{O}(SU_q(2))$ and the QUE algebra $U_q(su_2)$, for the details of which we refer the reader to [42], that the character f_1 corresponds to the evaluation by $K^2 \in U_q(su_2)$.

In case $s = 0$, the algebra $\mathcal{O}(S^2_{q0})$ is called the standard Podleś sphere, and is denoted by $\mathcal{O}(S^2_q)$. Next, we recall the $\mathcal{O}(SU_q(2))$-coaction from [26, Prop. 4.25]. To this end, let

$$W_1 = (w_{i,j})_{i,j\in\{-1,0,1\}} = \begin{pmatrix} w_{-1,-1} & w_{-1,0} & w_{-1,1} \\ w_{0,-1} & w_{0,0} & w_{0,1} \\ w_{1,-1} & w_{1,0} & w_{1,1} \end{pmatrix}$$

$$= \begin{pmatrix} x^2 & (1+q^2)^{1/2}xu & u^2 \\ (1+q^2)^{1/2}xv & 1+(q+q^{-1})uv & (1+q^2)^{1/2}uy \\ v^2 & (1+q^2)^{1/2}vy & y^2 \end{pmatrix}.$$

Then, the comultiplication on $\mathcal{O}(SU_q(2))$ induces (once a_i's are identified with $\widetilde{a}_i$'s) a coaction of the form $\nabla\colon \mathcal{O}(S^2_{qs}) \longrightarrow \mathcal{O}(S^2_{qs}) \otimes \mathcal{O}(SU_q(2))$,

$$\nabla(z_i) = z_j \otimes w_{j,i},$$

where $i, j \in \{-1, 0, 1\}$. A left version of this coaction can be found in [32].

Let us next recall from [42], and from [6, 39], that the Podleś sphere is a (left) coideal subalgebra, as such, taking the quotient by the Hopf ideal

$$I = \mathcal{O}(S^2_{qs})^+ \mathcal{O}(SU_q(2)),$$

we obtain the $\mathcal{O}(U(1))$-comodule algebra structure

$$\begin{array}{ccc} \mathcal{O}(S^2_{qs}) & \xrightarrow{\quad\rho\quad} & \mathcal{O}(S^2_{qs}) \otimes \mathcal{O}(U(1)) \\ & {\scriptstyle\nabla}\searrow \qquad \nearrow{\scriptstyle \mathrm{Id}\otimes\pi_I} & \\ & \mathcal{O}(S^2_{qs}) \otimes \mathcal{O}(SU_q(2)) & \end{array}$$

where $\mathcal{O}(U(1))$ is the Hopf algebra (of regular functions on the circle) generated by two group-likes σ, σ^{-1}. Explicitly, $\rho\colon \mathcal{O}(S^2_{qs}) \longrightarrow \mathcal{O}(S^2_{qs}) \otimes \mathcal{O}(U(1))$ is defined as

$$\begin{aligned} \rho(z_{-1}) &= z_{-1} \otimes \sigma^2, \\ \rho(z_0) &= z_0 \otimes 1, \\ \rho(z_1) &= z_1 \otimes \sigma^{-2}. \end{aligned}$$

We further note from [26, Sect. 4.5] that setting

$$\mathcal{O}(S^2_{qs})[n] := \{a \in \mathcal{O}(S^2_{qs}) \mid \rho(a) = a \otimes \sigma^n\},$$

we have $z^i_{-1} z^j_0 z^k_1 \in \mathcal{O}(S^2_{qs})[2i - 2k]$, where $i, j, k \geq 0$, that

$$\mathcal{O}(S^2_{qs}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(S^2_{qs})[2m], \tag{7.4}$$

and that the Haar state h vanishes on $\mathcal{O}(S^2_{qs})[2m]$ for $m \neq 0$.

Lemma 7.2. *The Haar state h of the CQG algebra $\mathcal{O}(SU_q(2))$ induces a 1-invariant f_1-trace on the algebra $\mathcal{O}(S^2_q)$.*

Proof. For any $a, b \in \mathcal{O}(SU_q(2))$, it follows from the modular property of the Haar functional that

$$\begin{aligned} h(ab) &= h(b(f_1 \cdot a \cdot f_1)) = h(b(a \cdot f_1)) \\ &= h(b(f_1 \cdot a_{(2)}))f_1(a_{(1)}) = h(b(f_1 \cdot a)) \\ &= h(ba_{(1)})f_1(a_{(2)}), \end{aligned}$$

using, on the fourth equality, the fact that $f_1(a_{(1)})a_{(2)} = a$ for any $a \in \mathcal{O}(S^2_{qs})$, see for instance [42, Eq. (22)]. Next, by the invariance property [26, Def. 4.3], see also [36] we readily have

$$h(a_{(1)})a_{(2)} = a_{(1)}h(a_{(2)}) = h(a)1,$$

for any $a \in \mathcal{O}(SU_q(2))$. □

As a result, we have the characteristic homomorphism

$$\gamma_n : \mathrm{CC}_n(\mathcal{O}(S^2_{qs})) \longrightarrow \mathrm{CC}_n(\mathcal{O}(U(1)); 1, \varepsilon)$$

defined as

$$\gamma_n(a_0 \otimes \cdots \otimes a_n) := h(a_0 a_{1<0>} \ldots a_{n<0>}) a_{1<1>} \otimes \cdots \otimes a_{n<1>} \tag{7.5}$$

to transfer the KU-classes of $\mathcal{O}(S^2_{qs})$ to the Hopf-dihedral homology of the (cocommutative) Hopf algebra $\mathcal{O}(U(1))$.

By means of the Hopf-dihedral Chern character

$$\mathrm{Ch}^{\pm}_n : KU_n(\mathcal{O}(S^2_{qs})) \longrightarrow HC^{\pm}_n(\mathcal{O}(U(1)); 1, \varepsilon).$$

we have the Hopf-dihedral homology of $\mathcal{O}(U(1))$ to classify the KU-classes of $\mathcal{O}(S^2_{qs})$.

We consider the composition

$$H_m(U^\epsilon_n(\mathcal{O}(S^2_{qs})), k) \longrightarrow HC^\alpha_{m+2\ell}(U^\epsilon_n(\mathcal{O}(S^2_{qs})), k) \longrightarrow HC^\alpha_{m+2\ell}(\mathcal{O}(U(1)), k),$$

where $\alpha = (-1)^\ell$. Since

$$\begin{aligned} H_0(\mathcal{O}(S^1), k) &\cong H_0(\mathbb{Z}, k) = k, \\ H_1(\mathcal{O}(S^1), k) &\cong H_1(\mathbb{Z}, k) = \langle \sigma \rangle \cong k, \\ H_m(\mathcal{O}(S^1), k) &\cong H_m(\mathbb{Z}, k) = 0, \quad m \geq 2, \end{aligned}$$

for the details of which we refer the reader to [5], we have

$$H_0(U^\epsilon_n(\mathcal{O}(S^2_{qs})), k) = k_{U^\epsilon_n(\mathcal{O}(S^2_{qs}))} = k = k_{\mathbb{Z}} = H_0(\mathbb{Z}, k).$$

As for $H_1(U_n^\epsilon(\mathcal{O}(S_{qs}^2)), k)$, we first recall that since $BU^\epsilon(\mathcal{O}(S_{qs}^2))^+$ is path connected, the Hurewicz map

$$h\colon KU_1^\epsilon(\mathcal{O}(S_{qs}^2)) = \pi_1(BU^\epsilon(\mathcal{O}(S_{qs}^2))^+) = \frac{U^\epsilon(\mathcal{O}(S_{qs}^2))}{[U^\epsilon(\mathcal{O}(S_{qs}^2)),\, U^\epsilon(\mathcal{O}(S_{qs}^2))]}$$
$$\longrightarrow \frac{\pi_1(BU^\epsilon(\mathcal{O}(S_{qs}^2))^+)}{[\pi_1(BU^\epsilon(\mathcal{O}(S_{qs}^2))^+),\, \pi_1(BU^\epsilon(\mathcal{O}(S_{qs}^2))^+)]} \cong H_1(U_n^\epsilon(\mathcal{O}(S_{qs}^2)), k)$$

is the canonical abelianization, and therefore in this case is identity. On the other hand, it follows from the relations (7.3) that the element

$$u = \begin{pmatrix} q^{-1}z_0 & \sqrt{1+q^{-2}}z_1 \\ \sqrt{1+q^2}z_{-1} & qz_0 \end{pmatrix}$$

is a unitary matrix on $\mathcal{O}(S_q^2)$. Hence, for any

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \theta \end{pmatrix} \in U_n^\epsilon(\mathbb{R})$$

we have

$$M = \begin{pmatrix} \alpha\otimes u & \beta\otimes u \\ \gamma\otimes u & \theta\otimes u \end{pmatrix} \in U_{2n}^\epsilon(\mathcal{O}(S_q^2)).$$

For simplicity, we may take

$$M = \begin{pmatrix} 0 & \epsilon I_n\otimes u \\ \epsilon I_n\otimes u & 0 \end{pmatrix} \in U_{2n}^\epsilon(\mathcal{O}(S_q^2)).$$

Then,

$$[M] \mapsto \Big[\sum_{1\le i_0,i_1\le 2n} h({M^{-1}}_{i_0 i_1} M_{i_1 i_{0<0>}}) M_{i_1 i_{0<1>}} \Big]$$
$$= \Big[\sum_{1\le i_0,i_1\le 2n} n h({u^\dagger}_{i_0 i_1} u_{i_1 i_{0<0>}}) u_{i_1 i_{0<1>}} \Big]$$
$$= 2n\big((1+q^{-2})h(z_1^* z_1) - (1+q^2)h(z_{-1}^* z_{-1})\big)[\sigma].$$

This proves that an element $[M] \in KU_0^+(\mathcal{O}(S_q^2))$ is sent to a non-trivial element in $HC_1^+(\mathcal{O}(U(1)); 1, \varepsilon)$ under the dihedral Chern character. Thus we proved

Theorem 7.3. *$KU_1^+(\mathcal{O}(S_q^2))$ is non-trivial.*

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