

## On steady non-commutative crepant resolutions

Osamu Iyama and Yusuke Nakajima

**Abstract.** We introduce special classes of non-commutative crepant resolutions (= NCCR) which we call steady and splitting. We show that a singularity has a steady splitting NCCR if and only if it is a quotient singularity by a finite abelian group. We apply our results to toric singularities and dimer models.

*Mathematics Subject Classification* (2010). 16G50, 14A22; 13C14, 14E15.

*Keywords.* Non-commutative crepant resolutions, class groups, quotient singularities.

### 1. Introduction

Geometric tilting theory gives a framework to study the derived categories of schemes and stacks in terms of non-commutative algebras obtained as the endomorphism algebras of tilting complexes. A well-studied class of geometric tilting is called McKay correspondence, which are given by resolution of singularities [12, 36]. Recently, Van den Bergh introduced the notion of non-commutative crepant resolutions (= NCCR) [43, 44] to explain Bridgeland’s theorem [11] on derived equivalence of crepant resolutions [9], from a viewpoint of geometric tilting theory. For example, in the case of McKay correspondence, skew group algebras are NCCRs of quotient singularities.

For nice classes of singularities, NCCRs provide a method to construct non-commutative rings derived equivalent to their resolutions directly from the given singularities. The notion of NCCRs is quite useful thanks to the fact that it naturally appears in Cohen–Macaulay representation theory initiated by Auslander–Reiten in the 1970s [2]. In fact, the recent notion of cluster tilting subcategories gives a categorical framework to study NCCRs (see [28, 30, 33, 35]).

An interesting family of NCCRs is given by a dimer model, which is a quiver with potential drawn on a torus and gives us an NCCR of a Gorenstein toric singularity in dimension three (see e.g. [5, 7, 13, 24, 27, 39]). For more results and examples of NCCRs, we refer to [15, 16, 19–21, 32, 34, 41, 45]. See also a survey article [37], and the references therein.

In this paper, we introduce nice classes of NCCRs which we call steady and splitting, and discuss existence of such NCCRs. We start with recalling the definition of a non-commutative crepant resolution due to Van den Bergh [44] (see also [43]). For further details on terminologies, see later sections.

**Definition 1.1.** Let  $R$  be a CM normal domain and  $\Lambda$  be a module-finite  $R$ -algebra. We say

- (1)  $\Lambda$  is an  $R$ -order if  $\Lambda$  is a CM  $R$ -module (see Subsection 2.1),
- (2) an  $R$ -order  $\Lambda$  is *non-singular* if  $\text{gl.dim } \Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$  for all  $\mathfrak{p} \in \text{Spec } R$ .

We refer to [33, 2.17] for several conditions which are equivalent to  $\Lambda$  is a non-singular  $R$ -order. Using this notion, Van den Bergh [44] introduced the notion of NCCR. (Note that unlike [44], we do not assume that  $R$  is Gorenstein in this paper.) Also we recall the notion of NCR (see [22]).

**Definition 1.2.** Let  $R$  be a CM normal domain, and  $0 \neq M \in \text{ref } R$ . Let  $E := \text{End}_R(M)$ .

- (1) We say  $E$  is a *non-commutative crepant resolution* (= NCCR) of  $R$  or  $M$  gives an NCCR of  $R$  if  $E$  is a non-singular  $R$ -order.
- (2) We say  $E$  is a *non-commutative resolution* (= NCR) of  $R$  or  $M$  gives an NCR of  $R$  if  $\text{gl.dim } E < \infty$ .

**1.1. Our results.** The existence of an NCCR of  $R$  shows  $R$  has a mild singularity. For example, it was shown that under mild assumptions, any CM normal domain which has an NCCR has at worst rational singularities [40] (see also [22]) and at worst log-terminal singularities [23, 26].

In this paper, we impose extra assumptions on NCCRs, and we will show the existence of such an NCCR characterizes some singularities. Our point is that the size of  $\text{End}_R(M)$  as an  $R$ -module becomes much bigger than that of  $M$ , that is,  $\text{rank}_R \text{End}_R(M) = (\text{rank}_R M)^2$ . Therefore, as an  $R$ -module,  $\text{End}_R(M)$  usually has a direct summand which does not appear in  $M$ . Therefore the following class of NCCRs is of interest.

**Definition 1.3.** We say a reflexive  $R$ -module  $M$  is *steady* if  $M$  is a *generator* (that is,  $R \in \text{add}_R M$ ) and  $\text{End}_R(M) \in \text{add}_R M$  holds. We say an NCCR (resp. NCR)  $\text{End}_R(M)$  is *steady NCCR* (resp. *steady NCR*) if  $M$  is steady.

We refer to Lemma 2.5 for basic properties of steady modules. Also note that the first condition “ $M$  is a generator” is a consequence of the second condition “ $\text{End}_R(M) \in \text{add}_R M$ ” in many cases (see Lemma 2.6).

For example, quotient singularities have steady NCCRs (see Example 2.3). A natural question is the converse, that is, singularities having steady NCCRs are quotient singularities. The aim of this paper is to give a partial answer to this question. To consider such a question, we introduce another class of nice NCCRs called splitting, and show that existence of steady splitting NCCRs implies quotient singularities.

**Definition 1.4.** We say a reflexive  $R$ -module  $M$  is *splitting* if  $M$  is a direct sum of reflexive modules of rank one. We say an NCCR (resp. NCR)  $\text{End}_R(M)$  is *splitting NCCR* (resp. *splitting NCR*) if  $M$  is splitting.

There are several examples of splitting NCCRs (see Example 2.4).

We are now ready to state the main theorem, which will be shown in Section 3.

**Theorem 1.5** (see Theorem 3.1). *Let  $R$  be a  $d$ -dimensional complete local Cohen–Macaulay normal domain containing an algebraically closed field of characteristic zero. Then the following conditions are equivalent.*

- $R$  is a quotient singularity associated with a finite abelian group  $G \subset \text{GL}(d, k)$  (i.e.  $R = S^G$  where  $S = k[[x_1, \dots, x_d]]$ ).
- $R$  has a unique basic module which gives a splitting NCCR.
- $R$  has a steady splitting NCCR.
- There exists a finite subgroup  $G$  of  $\text{Cl}(R)$  such that  $\bigoplus_{X \in G} X$  gives an NCCR of  $R$ .
- $\text{Cl}(R)$  is a finite group and  $\bigoplus_{X \in \text{Cl}(R)} X$  gives an NCCR of  $R$ .

When  $R$  is a toric singularity, we obtain the following simple equivalent condition by applying results in toric geometry.

**Corollary 1.6.** *Let  $R$  be a completion of a toric singularity over an algebraically closed field of characteristic zero. Then all the conditions in Theorem 1.5 and the following condition are equivalent.*

- $\text{Cl}(R)$  is a finite group.

Now we apply this result to dimer models.

Let us recall basic facts on dimer models. For more details, see Example 2.4(b) and references therein. A dimer model is a polygonal cell decomposition of the two-torus whose vertices and edges form a finite bipartite graph, and we can obtain a quiver with the potential  $(Q_\Gamma, W_\Gamma)$  as the dual of a dimer model  $\Gamma$ . From a quiver with the potential  $(Q_\Gamma, W_\Gamma)$ , we define the complete Jacobian algebra  $\mathcal{P}(Q_\Gamma, W_\Gamma)$ . Under a certain condition called “consistency condition”, the center  $R$  of  $\mathcal{P}(Q_\Gamma, W_\Gamma)$  is a complete local Gorenstein toric singularity in dimension three, and  $\mathcal{P}(Q_\Gamma, W_\Gamma)$  is a splitting NCCR of  $R$ .

Thanks to our Theorem 1.5, we have the following result which characterizes when  $\mathcal{P}(Q_\Gamma, W_\Gamma)$  is a steady NCCR of  $R$ .

**Corollary 1.7.** *Let  $\Gamma$  be a consistent dimer model,  $k$  an algebraically closed field of characteristic zero and  $R$  the corresponding complete local Gorenstein toric singularity in dimension three. Then the following conditions are equivalent.*

- $\Gamma$  is homotopy equivalent to a regular hexagonal dimer model (i.e. each face of a dimer model is a regular hexagon).
- $\Gamma$  gives a steady NCCR of  $R$ .

- $R$  is a quotient singularity associated with a finite abelian group  $G \subset \mathrm{SL}(3, k)$  (i.e.  $R = S^G$  where  $S = k[[x_1, x_2, x_3]]$ ).

We show some examples of dimer models.

**Example 1.8.** The following figures are a consistent dimer model which is homotopy equivalent to a regular hexagonal dimer model, and the associated quiver. This quiver is just the McKay quiver of  $G = \langle \mathrm{diag}(\omega, \omega^5, \omega^8) \rangle$  where  $\omega$  is a primitive 14th root of unity. Also, the center of the Jacobian algebra is the quotient singularity associated with  $G$ .



On the other hand, the following consistent dimer model is not homotopy equivalent to a regular hexagonal dimer model, and it gives the toric singularity defined by the cone  $\sigma$ :

$$\sigma = \mathrm{Cone}\{(1, 1, 1), (-1, 1, 1), (-1, -1, 1), (1, -1, 1)\}.$$



In what follows, we will investigate properties of steady NCCRs, and those of splitting NCCRs in Section 2. In Section 3, we will give a proof of Theorem 1.5 in a more detailed form.

**Notations and conventions.** Let  $R$  be a commutative Noetherian ring. We denote  $\mathrm{Mod} R$  to be the category of  $R$ -modules,  $\mathrm{mod} R$  to be the category of finitely generated  $R$ -modules,  $\mathrm{add}_R M$  to be the full subcategory consisting of direct summands of finite direct sums of some copies of  $M \in \mathrm{mod} R$ . When  $R$  is  $G$ -graded for an abelian group  $G$ , we denote by  $\mathrm{Mod}^G R$  the category of  $G$ -graded  $R$ -modules, and  $\mathrm{mod}^G R$  the category of finitely generated  $G$ -graded  $R$ -modules. We say an  $R$ -module  $M = M_1 \oplus \cdots \oplus M_n$  is *basic* if  $M_i$ 's are mutually non-isomorphic. Also, we denote by  $\mathrm{Cl}(R)$  the class group of  $R$ . When we consider a rank one reflexive  $R$ -module  $I$  as an element of  $\mathrm{Cl}(R)$ , we denote it by  $[I]$ .

**Acknowledgements.** The authors thank Ragnar-Olaf Buchweitz for stimulating discussions. The first author thanks Akira Ishii and Kazushi Ueda for kind explanations on dimer models. The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (B) 16H03923, (C) 23540045 and (S) 15H05738. The second author was supported by Grant-in-Aid for JSPS Fellows No. 26–422. Both authors thank the anonymous referee for valuable comments on the paper.

## 2. Steady NCCRs and splitting NCCRs

**2.1. Preliminaries.** We start with preparing basic facts used in this paper. We denote the  $R$ -dual functor by

$$(-)^* := \text{Hom}_R(-, R) : \text{mod } R \rightarrow \text{mod } R.$$

We say that  $M \in \text{mod } R$  is *reflexive* if the natural morphism  $M \rightarrow M^{**}$  is an isomorphism. We denote  $\text{ref } R$  to be the category of reflexive  $R$ -modules.

Let  $(R, \mathfrak{m})$  be a commutative Noetherian local ring. For  $M \in \text{mod } R$ , we define the depth of  $M$  as

$$\text{depth}_R M := \inf\{i \geq 0 \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}.$$

We say  $M$  is a *maximal Cohen–Macaulay* (or *CM* for short)  $R$ -module if  $\text{depth}_R M = \dim R$  or  $M = 0$ . When  $R$  is non-local, we say  $M$  is a CM module if  $M_{\mathfrak{m}}$  is a CM module for any maximal ideal  $\mathfrak{m}$  of  $R$ . Furthermore, we say that  $R$  is a *Cohen–Macaulay ring* (= *CM ring*) if  $R$  is a CM  $R$ -module. We denote  $\text{CM } R$  to be the category of Cohen–Macaulay  $R$ -modules.

The following is well known. For example, see [4, II. 2.1].

**Lemma 2.1.** *Let  $R$  be a commutative Noetherian ring and  $M \in \text{mod } R$ . Suppose  $M$  is a generator. Then the functor  $\text{Hom}_R(M, -) : \text{mod } R \rightarrow \text{mod End}_R(M)$  is fully faithful, restricting to an equivalence  $\text{add}_R M \simeq \text{proj End}_R(M)$ .*

Also, the following property of modules giving NCCRs is important.

**Proposition 2.2** (see [33, 4.5]). *Let  $R$  be a  $d$ -dimensional normal CM local ring. Suppose  $M \in \text{ref } R$  gives an NCCR of  $R$ . Then for any  $X \in \text{ref } R$  with  $\text{End}_R(M \oplus X) \in \text{CM } R$ , we have  $X \in \text{add}_R M$ .*

**2.2. Examples.** We provide examples of steady NCCRs and those of splitting NCCRs.

**Example 2.3.** Let  $S = k[[x_1, \dots, x_d]]$  be a formal power series ring,  $G$  be a finite subgroup of  $\text{GL}(d, k)$  such that  $|G|$  is invertible in  $k$ . Let  $R := S^G$ .

(a) The  $R$ -module  $S$  gives a steady NCCR.

(b) Assume that  $G$  is small (i.e. it contains no pseudo-reflections except the identity). Let  $\{V_i \mid i \in I\}$  be the set of simple  $kG$ -modules. Then we have

$$S \simeq \bigoplus_{i \in I} ((S \otimes_k V_i)^G)^{\oplus \dim_k V_i}$$

as  $R$ -modules, where  $(S \otimes_k V_i)^G$  is an indecomposable CM  $R$ -module with rank  $\dim_k V_i$ .

*Proof.* (i) We prove the statements under the assumption that  $G$  is small.

In this case, we have  $\text{End}_R(S) \simeq S * G$ , where  $S * G$  is the skew group ring [1] (see also [31, 4.2], [38, 5.15]). Since  $S * G$  is a non-singular  $R$ -order (see e.g. [33, 2.12]) and  $\text{End}_R(S) \in \text{add}_R S$  holds, we have the assertion (a).

We have

$$S = (S \otimes_k kG)^G = \left( S \otimes_k \left( \bigoplus_{i \in I} V_i^{\oplus \dim_k V_i} \right) \right)^G = \bigoplus_{i \in I} \left( (S \otimes_k V_i)^G \right)^{\oplus \dim_k V_i}$$

as  $R$ -modules. This is a decomposition into indecomposable  $R$ -modules since  $\text{End}_R(S)/\text{rad End}_R(S) \simeq S * G/\text{rad}(S * G) = kG$  holds. Thus we have the assertion (b).

(ii) We prove the statement (a) for general case. In this case, there exists a formal power series ring  $T$  and a finite small subgroup  $G' \subset \text{GL}(d, k)$  such that  $R \subset T \subset S$ ,  $R = T^{G'}$ , and  $S$  is a free  $T$ -module of finite rank (see [33, Proof of 5.7]). Suppose  $S \simeq T^{\oplus r}$ . Then  $\text{End}_R(S) = M_r(\text{End}_R(T))$  belongs to  $\text{add}_R T = \text{add}_R S$ . Therefore, the assertion follows from (i).  $\square$

Also, we provide Examples of splitting NCCRs.

**Example 2.4.** (a) Let  $R = S^G$  be a quotient singularity by a finite abelian group  $G \subset \text{GL}(d, k)$  such that  $|G|$  is invertible in  $k$ . In this situation,  $S$  gives a steady splitting NCCR of  $R$  by Example 2.3.

(b) Let  $R$  be a Gorenstein toric singularity in dimension three. Then the method of dimer models gives splitting NCCRs of  $R$  (see e.g. [6, 13, 27]). Conversely any splitting NCCR of  $R$  is given by a consistent dimer model [8, Theorem 7.7].

(c) Let  $R = k[[x, y, u, v]]/(f(x, y) - uv)$  be a  $cA_n$  singularity. If  $f$  is a product  $f_1 \cdots f_m$  with  $f_i \notin (x, y)^2$ , then  $R$  has a splitting NCCR [35].

**2.3. Basic properties.** We show some basic properties of steady NCCRs and splitting NCCRs.

**Lemma 2.5.** *Let  $M$  be a steady  $R$ -module.*

(a)  $\text{add}_R M = \text{add}_R \text{End}_R(M)$ .

(b)  $\text{add}_R M = \text{add}_R M^*$ .

(c) If  $M$  gives an NCCR, then  $M \in \text{CM } R$ .

*Proof.* (a) Since  $M$  is a generator, we have  $M \simeq \text{Hom}_R(R, M) \in \text{add}_R \text{End}_R(M)$ .

(b) We have  $M^* \simeq \text{Hom}_R(M, R) \in \text{add}_R \text{End}_R(M) = \text{add}_R M$  by (a). Similarly we have  $M \simeq \text{Hom}_R(M^*, R) \in \text{add}_R \text{Hom}_R(M, R) = \text{add}_R M^*$ .

(c) This is clear since  $M \simeq \text{Hom}_R(R, M) \in \text{add}_R \text{End}_R(M) \subset \text{CM } R$ . □

**Lemma 2.6.** *Let  $R$  be a CM normal domain. Suppose  $M$  satisfies  $\text{End}_R(M) \in \text{add}_R M$ . Then  $M$  is a generator if one of the following conditions is satisfied.*

- $R$  contains a field of characteristic zero,
- $M$  has a rank one reflexive module as a direct summand.

*In particular, if  $M$  is splitting, then  $M$  is a generator.*

*Proof.* Firstly, if  $R$  contains a field of characteristic zero, we have

$$R \in \text{add}_R \text{End}_R(M) \subset \text{add}_R M$$

by [3, 5.6]. Next, we suppose that  $I$  is a rank one reflexive  $R$ -module such that  $I \in \text{add}_R M$ . Then we have  $R \simeq \text{Hom}_R(I, I) \in \text{add}_R M$ . □

The following characterization of steady splitting modules is important.

**Proposition 2.7.** *Let  $R$  be a complete local normal domain, and let  $M$  be a finite subset of  $\text{Cl}(R)$  and  $M := \bigoplus_{X \in M} X$ . Then  $M$  is steady if and only if  $M$  is a subgroup of  $\text{Cl}(R)$ .*

*Proof.* Let  $M := \{[M_1], \dots, [M_n]\}$ . Then  $\text{End}_R(M) \in \text{add}_R M$  means that  $[M_i] - [M_j] \in M$  holds for any  $i$  and  $j$ . This is clearly equivalent to that  $M$  forms a subgroup of  $\text{Cl}(R)$ . □

We denote by  $K_0(R)$  the Grothendieck group of  $R$ . The following propositions play the crucial role to prove the main theorem.

**Proposition 2.8.** *Let  $R$  be a complete local CM normal domain. Suppose  $M = \bigoplus_{i=1}^n M_i$  gives a splitting NCR of  $R$ , and  $M$  is a generator.*

- (a)  $K_0(R)$  and  $\text{Cl}(R)$  are generated by  $[M_1], \dots, [M_n]$ .
- (b) If  $M$  is steady, then  $\text{Cl}(R) = \{[M_1], \dots, [M_n]\}$ .

*Proof.* (a) We have a surjection  $K_0(R) \twoheadrightarrow \text{Cl}(R)$  by [10, VII.4.7]. Thus, it is enough to show that  $K_0(R)$  is generated by  $[M_1], \dots, [M_n]$ . Let  $E := \text{End}_R(M)$ . For any  $X \in \text{mod } R$ , let  $Y := \text{Hom}_R(M, X) \in \text{mod } E$ . Since  $\text{gl.dim } E < \infty$ , there exists a projective resolution of the  $E$ -module  $Y$  of the form

$$0 \rightarrow P_r \rightarrow \dots \rightarrow P_0 \rightarrow Y \rightarrow 0.$$

Since  $M$  is a generator,  $P_i = \text{Hom}_R(M, N_i)$  for some  $N_i \in \text{add}_R M$  and we have an exact sequence below (see Lemma 2.1),

$$0 \rightarrow N_r \rightarrow \cdots \rightarrow N_0 \rightarrow X \rightarrow 0.$$

Thus  $X$  belongs to the subgroup  $\langle [M_1], \dots, [M_n] \rangle$  of  $K_0(R)$ .

(b) The assertion follows from (a) and Proposition 2.7.  $\square$

### 3. Proof of the main theorem

In this section, we give a proof of Theorem 1.5 in a detailed form (see Theorem 3.1). To state it, we prepare general observations on graded rings. Let  $G$  be a finite abelian group and  $S = \bigoplus_{i \in G} S_i$  a  $G$ -graded ring. We say that  $S$  is *strongly  $G$ -graded* if the map  $S_j \rightarrow \text{Hom}_{S_0}(S_i, S_{i+j})$  sending  $x \in S_j$  to  $(y \mapsto yx) \in \text{Hom}_R(S_i, S_{i+j})$  is an isomorphism for any  $i, j \in G$ .

Now we restate Theorem 1.5 in a detailed form.

**Theorem 3.1.** *Let  $R$  be a  $d$ -dimensional complete local Cohen–Macaulay normal domain. Consider the following conditions.*

- (1)  *$R$  is a quotient singularity associated with a finite (small) abelian group  $G \subset \text{GL}(d, k)$  (i.e.  $R = S^G$  where  $S = k[[x_1, \dots, x_d]]$  for a field  $k$ ).*
- (2) *There exist a finite abelian group  $G$  and a complete regular local ring  $S$  which is strongly  $G$ -graded such that  $R = S_0$ .*
- (3)  *$R$  has a unique basic module which gives a splitting NCCR.*
- (4)  *$R$  has a steady splitting NCCR.*
- (5)  *$R$  has a steady splitting NCR.*
- (6) *There exists a finite subgroup  $G$  of  $\text{Cl}(R)$  such that  $\bigoplus_{X \in G} X$  gives an NCCR of  $R$ .*
- (7) *There exists a finite subgroup  $G$  of  $\text{Cl}(R)$  such that  $\bigoplus_{X \in G} X$  gives an NCR of  $R$ .*
- (8)  *$\text{Cl}(R)$  is a finite group and  $\bigoplus_{X \in \text{Cl}(R)} X$  gives an NCCR of  $R$ .*
- (9)  *$\text{Cl}(R)$  is a finite group and  $\bigoplus_{X \in \text{Cl}(R)} X$  gives an NCR of  $R$ .*

*Then the conditions (2)–(9) are equivalent. If  $R$  contains an algebraically closed field of characteristic zero, then all the conditions (1)–(9) are equivalent.*



In order to prove this, we need some preparations. Let  $G$  be a finite abelian group and  $S = \bigoplus_{i \in G} S_i$  a  $G$ -graded ring. Then the *smash product* [17]

$$S\#G := (S_{j-i})_{i,j \in G}$$

is a ring whose multiplication is given by the multiplication in  $S$  and the matrix multiplication rule. We have a natural morphism

$$\phi : S\#G \rightarrow \text{End}_{S_0}(S) \tag{3.1}$$

of rings sending  $(x_{ij})_{i,j \in G} \in S\#G$  to  $(S \ni (y_i)_{i \in G} \mapsto (\sum_{i \in G} y_i x_{ij})_{j \in G} \in S) \in \text{End}_R(S)$ . Clearly  $S$  is strongly  $G$ -graded if and only if  $\phi$  is an isomorphism.

We need the following observations to prove Theorem 3.1.

**Proposition 3.2.** (a) *Let  $G$  be a finite abelian group and  $S$  a  $G$ -graded ring. Then we have an equivalence  $\text{Mod}^G S \simeq \text{Mod}(S\#G)$ , which preserves finitely generated modules.*

(b) *Let  $R$  be a normal domain and  $G$  a finite subgroup of  $\text{Cl}(R)$ . Then there exists a strongly  $G$ -graded ring  $S$  such that  $S_0 = R$  and  $[S_i] = i$  in  $\text{Cl}(R)$  for any  $i \in G$ . Moreover, if  $R$  is complete local, then so is  $S$ .*

(c) *Let  $G$  be a finite abelian group,  $S$  a local normal domain which is  $G$ -graded and  $R = S_0$ . If  $\text{Cl}(S) = 0$  holds and the  $R$ -module  $S_i$  has rank one for any  $i \in G$ , then  $\text{Cl}(R) = \{[S_i] \mid i \in G\}$ .*

*Proof.* For (a), we refer to [29, Theorem 3.1]. For (b), we refer to [23].

(c) Let  $I$  be a divisorial ideal of  $R$ . Then  $X = (S \otimes_R I)^{**}$  is a divisorial ideal of  $S$  which is  $G$ -graded. Since  $\text{Cl}(S) = 0$ , there exists an isomorphism  $f : S \rightarrow X$  of  $S$ -modules. Let  $f(1) = (x_i)_{i \in G}$ . Since  $X$  has a unique maximal submodule and  $f(1)$  generates  $X$ , some  $x_i$  also generates  $X$ . Thus  $x_i : S \simeq X(i)$  as  $G$ -graded  $S$ -modules. Taking the degree  $i$  part, we have an isomorphism  $S_i \simeq X_0 = I$  of  $R$ -modules. □

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* We prove the following implications, where the implications written by the arrows  $\implies$  are clear.

$$\begin{array}{ccccccc} (4) & \implies & (8) & \implies & (6) & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ (5) & \implies & (9) & \implies & (7) & \implies & (2) \implies (3) \implies (4) \end{array}$$

(4) $\implies$ (8) (resp. (5) $\implies$ (9)). This follows from Proposition 2.8(b).

(7)⇒(2). By Proposition 3.2(b), there exists a strongly  $G$ -graded ring  $S = \bigoplus_{X \in G} X$  such that  $S$  is a complete local and  $S_0 = R$  and  $[S_i] = i$  hold in  $\text{Cl}(R)$  for any  $i \in G$ . We only have to show that  $S$  is a regular, or equivalently, the residue field  $k$  has a finite projective dimension as an  $S$ -module. Note that  $k$  is a  $G$ -graded  $S$ -module concentrated in degree 0. By Proposition 3.2(a) and (3.1), we have equivalences

$$\text{mod}^G S \simeq \text{mod}(S\#G) \simeq \text{mod End}_R(S).$$

Since  $S$  gives an NCR of  $R$ , the abelian category  $\text{mod}^G S$  has a finite global dimension. Thus  $k$  has a projective resolution in  $\text{mod}^G S$  which has finite length. Forgetting the grading, the  $S$ -module  $k$  has a finite projective dimension.

(2)⇒(3). Since  $S$  is strongly  $G$ -graded, we have an isomorphism  $S\#G \simeq \text{End}_R(S)$  of rings in (3.1). In particular  $\text{End}_R(S)$  belongs to  $\text{CM } R$ . By Proposition 3.2(a), we have an equivalence  $\text{mod}^G S \simeq \text{mod End}_R(S)$ . Since  $S$  is regular, the categories  $\text{mod } S$  and  $\text{mod}^G S$  have finite global dimension at most  $\dim S$ . Therefore  $S$  gives an NCCR of  $R$ .

It remains to prove that there are no other splitting  $R$ -modules giving NCCRs up to additive equivalences. By Proposition 3.2(c), any rank one reflexive  $R$ -module is a direct summand of  $S$ . Thus the assertion follows from Proposition 2.2.

(3)⇒(4). We suppose  $N = N_1 \oplus \cdots \oplus N_n$  is basic, and it gives the unique splitting NCCR. Since  $\text{End}_R(\text{Hom}_R(N_i, N)) \simeq \text{End}_R(N)$ ,  $\text{Hom}_R(N_i, N)$  also gives a splitting NCCR. By the uniqueness,  $\text{Hom}_R(N_i, N) \simeq N$  for any  $i$ . Thus,  $\text{End}_R(N) = \bigoplus_{i=1}^n \text{Hom}_R(N_i, N) \simeq N^{\oplus n}$  is a steady NCCR.

Thus we have shown that all the conditions (2)–(9) are equivalent. In the rest, we assume  $R$  contains an algebraically closed field  $k$  of characteristic zero.

(1)⇒(4). This is shown in Example 2.4(a).

(2)⇒(1). The dual abelian group  $G^\vee = \text{Hom}(G, k^\times)$  acts on  $S$  by  $f(x) = f(i)x$  for any  $f \in G^\vee, i \in G$  and  $x \in S_i$ . Moreover, the action of  $G^\vee$  is linearizable (see e.g. [38, 5.3]), and we can assume that  $G^\vee$  is a small subgroup of  $\text{GL}(d, k)$  (see the proof of Example 2.3). Since  $R = S_0 = S^{G^\vee}$ , we have the assertion.  $\square$

Next we prove Corollary 1.6.

*Proof of Corollary 1.6.* Suppose that  $R \simeq \widehat{A}$  is the  $\mathfrak{m}$ -adic completion of a toric singularity  $A$  where  $\mathfrak{m}$  is the irrelevant maximal ideal. That is, let  $S$  be a positive affine normal semigroup, and let  $A := k[S]$ . Here, we note the relationship of class groups. We define a natural morphism

$$\text{Cl}(A) \rightarrow \text{Cl}(R) \quad ([I] \mapsto [\widehat{I} \simeq I \otimes_A R]).$$

This is injective (see e.g. [46, 15.2]).

It is clear that the equivalent conditions in Theorem 3.1 implies  $\text{Cl}(R)$  is a finite group. Conversely, we assume  $\text{Cl}(R)$  is a finite group. Since we have  $\text{Cl}(A) \hookrightarrow \text{Cl}(R)$ ,  $\text{Cl}(A)$  is also a finite group. Therefore, a semigroup  $S$  is simplicial, and it is equivalent to  $A$  is a quotient singularity associated with a finite abelian group (see e.g. [14, 4.59], [18, 1.3.20]). Consequently,  $R$  is also a quotient singularity associated with a finite abelian group.  $\square$

Finally we prove Corollary 1.7.

*Proof of Corollary 1.7.* The second and the third conditions are equivalent to each other by Theorem 3.1 (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). It remain to show that they are equivalent to the first condition.

Assume that  $\Gamma$  is a regular hexagonal dimer model. Then by [25, Lemma 5.3], there exists a finite abelian subgroup  $G$  of  $\text{SL}(3, k)$  such that  $\mathcal{P}(Q_\Gamma, W_\Gamma) = k[[x_1, x_2, x_3]] * G$ . Thus  $R = k[[x_1, x_2, x_3]]^G$  holds.

Assume that  $R = S^G$  with  $S = k[[x_1, x_2, x_3]]$  and a finite small abelian subgroup  $G$  of  $\text{SL}(3, k)$ . By Theorem 3.1(1) $\Rightarrow$ (3),  $S$  is a unique  $R$ -module which gives an NCCR up to additive equivalence. Thus we have  $\mathcal{P}(Q_\Gamma, W_\Gamma) \simeq \text{End}_R(S) \simeq S * G$  as  $R$ -algebras. This implies that  $Q_\Gamma$  is the McKay quiver of  $G$ , and therefore each face of  $\Gamma$  is hexagon (see [42, Section 5]) and it is homotopy equivalent to a regular hexagonal dimer model.  $\square$

#### 4. Remarks and questions

We end this paper with a few remarks. We do not know an answer to the following question.

**Question 4.1.** *Assume that  $R$  contains an algebraically closed field of characteristic zero. Does existence of steady NCCRs of  $R$  imply that  $R$  is a quotient singularity?*

This is true in dimension two. More strongly, we have the following observation.

**Proposition 4.2.** *For a two dimensional complete local normal domain  $R$  containing an algebraically closed field of characteristic zero, the following conditions are equivalent.*

- (a)  *$R$  is a quotient singularity associated with a finite group  $G \subset \text{GL}(2, k)$  (i.e.  $R = S^G$  where  $S = k[[x_1, x_2]]$  for a field  $k$ ).*
- (b)  *$R$  has a steady NCCR.*
- (c)  *$R$  has an NCCR.*
- (d)  *$R$  has only finitely many isomorphism classes of indecomposable MCM  $R$ -modules.*

*Moreover, in this case, NCCRs of  $R$  are precisely additive generators of  $\text{CM } R$ , and hence any NCCR of  $R$  is steady.*

*Proof.* (a) $\Rightarrow$ (b). This is shown in Example 2.3 (a).

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (d). It is well known that, in dimension two, modules giving NCCRs are precisely additive generators of  $\text{CM } R$  (see Proposition 2.2).

(d) $\Rightarrow$ (a). This is well known (see [3, 2.1 and 4.9], [46, Chapter 11]).  $\square$

We pose the following question, which we do not know an answer even for quotient singularities.

**Question 4.3.** *Are all steady NCCRs Morita equivalent? More strongly, are all modules giving steady NCCRs additive equivalent?*

This is true in dimension two by Proposition 4.2.

## References

- [1] M. Auslander, On the purity of the branch locus, *Amer. J. Math.*, **84** (1962), 116–125. [Zbl 0112.13101](#) [MR 137733](#)
- [2] M. Auslander, Functors and morphisms determined by objects, in *Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976)*, 1–244, Lecture Notes in Pure Appl. Math., 37, Dekker, New York, 1978. [Zbl 0383.16015](#) [MR 480688](#)
- [3] M. Auslander, Rational singularities and almost split sequences, *Trans. Amer. Math. Soc.*, **293** (1986), no. 2, 511–531. [Zbl 0594.20030](#) [MR 816307](#)
- [4] M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36, Cambridge University Press, Cambridge, 1995. [Zbl 0834.16001](#) [MR 1314422](#)
- [5] C. Beil, On the noncommutative geometry of square superpotential algebras, *J. Algebra*, **371** (2012), 207–249. [Zbl 1271.14005](#) [MR 2975394](#)
- [6] R. Bocklandt, Consistency conditions for dimer models, *Glasgow Math. J.*, **54** (2012), 429–447. [Zbl 1244.14042](#) [MR 2911380](#)
- [7] R. Bocklandt, Generating toric noncommutative crepant resolutions, *J. Algebra*, **364** (2012), 119–147. [Zbl 1263.14006](#) [MR 2927051](#)
- [8] R. Bocklandt, Calabi–Yau algebras and weighted quiver polyhedra, *Math. Z.*, **273** (2013), 311–329. [Zbl 1278.16017](#) [MR 3010162](#)
- [9] A. Bondal and D. Orlov, Derived categories of coherent sheaves, in *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, Vol. II, 47–56, Higher Ed. Press, Beijing, 2002. [Zbl 0996.18007](#) [MR 1957019](#)
- [10] N. Bourbaki, *Commutative algebra. Chapters 1–7*, translated from the French, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998. [Zbl 0902.13001](#) [MR 1727221](#)
- [11] T. Bridgeland, Flops and derived categories, *Invent. Math.*, **147** (2002), no. 3, 613–632. [Zbl 1085.14017](#) [MR 1893007](#)

- [12] T. Bridgeland, A. King, and M. Reid, The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.*, **14** (2001), no. 3, 535–554. [Zbl 0966.14028](#) [MR 1824990](#)
- [13] N. Broomhead, Dimer model and Calabi–Yau algebras, *Mem. Amer. Math. Soc.*, **215** (2012), no. 1011, viii+86pp. [Zbl 1237.14002](#) [MR 2908565](#)
- [14] W. Bruns and J. Gubeladze, *Polytopes, rings and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009. [Zbl 1168.13001](#) [MR 2508056](#)
- [15] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh, Non-commutative desingularization of determinantal varieties. I, *Invent. Math.*, **182** (2010), no. 1, 47–115. [Zbl 1204.14003](#) [MR 2672281](#)
- [16] I. Burban, O. Iyama, B. Keller, and I. Reiten, Cluster tilting for one-dimensional hypersurface singularities, *Adv. Math.*, **217** (2008), no. 6, 2443–2484. [Zbl 1143.13014](#) [MR 2397457](#)
- [17] C. Cibils and E. Marcos, Skew category, Galois covering and smash product of a  $k$ -category, *Proc. Amer. Math. Soc.*, **134** (2006), no. 1, 39–50. [Zbl 1098.18002](#) [MR 2170541](#)
- [18] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124, American Mathematical Society, 2011. [Zbl 1223.14001](#) [MR 2810322](#)
- [19] H. Dao, Remarks on non-commutative crepant resolutions of complete intersections, *Adv. Math.*, **224** (2010), no. 3, 1021–1030. [Zbl 1192.13011](#) [MR 2628801](#)
- [20] H. Dao and C. Huneke, Vanishing of Ext, cluster tilting modules and finite global dimension of endomorphism rings, *Amer. J. Math.*, **135** (2013), no. 2, 561–578. [Zbl 1274.13031](#) [MR 3038721](#)
- [21] H. Dao, E. Faber, and C. Ingalls, Noncommutative (Crepant) Desingularizations and the Global Spectrum of Commutative Rings, *Algebr. Represent. Theory*, **18** (2015), no. 3, 633–664. [Zbl 1327.14020](#) [MR 3357942](#)
- [22] H. Dao, O. Iyama, R. Takahashi, and C. Vial, Non-commutative resolutions and Grothendieck groups, *J. Noncommut. Geom.*, **9** (2015) no. 1, 21–34. [Zbl 1327.13053](#) [MR 3337953](#)
- [23] H. Dao, O. Iyama, R. Takahashi, and M. Wemyss, Gorenstein modifications and  $\mathbb{Q}$ -Gorenstein rings, 2016. [arXiv:1611.04137](#)
- [24] B. Davison, Consistency conditions for brane tilings, *J. Algebra*, **338** (2011), 1–23. [Zbl 1250.14028](#) [MR 2805177](#)
- [25] M. Herschend, O. Iyama, and S. Oppermann,  $n$ -representation infinite algebras, *Adv. Math.*, **252** (2014), 292–342. [Zbl 1339.16020](#) [MR 3144232](#)
- [26] C. Ingalls and T. Yasuda, Log centres of noncommutative crepant resolutions are Kawamata log terminal: remarks on a paper of Stafford and Van den Bergh, *preprint*, 2014.
- [27] A. Ishii and K. Ueda, Dimer models and the special McKay correspondence, *Geom. Topol.*, **19** (2015) 3405–3466. [Zbl 1338.14019](#) [MR 3447107](#)
- [28] O. Iyama, Auslander correspondence, *Adv. Math.*, **210** (2007), no. 1, 51–82. [Zbl 1115.16006](#) [MR 2298820](#)
- [29] O. Iyama and B. Lerner, Tilting bundles on orders on  $\mathbb{P}^d$ , *Israel J. Math.*, **211** (2016), no. 1, 147–169. [Zbl 1365.14004](#) [MR 3474959](#)

- [30] O. Iyama and I. Reiten, Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras, *Amer. J. Math.*, **130** (2008), no. 4, 1087–1149. [Zbl 1162.16007](#) [MR 2427009](#)
- [31] O. Iyama and R. Takahashi, Tilting and cluster tilting for quotient singularities, *Math. Ann.*, **356** (2013), 1065–1105. [Zbl 06181402](#) [MR 3063907](#)
- [32] O. Iyama and M. Wemyss, On the Noncommutative Bondal–Orlov Conjecture, *J. Reine Angew. Math.*, **683** (2013), 119–128. [Zbl 1325.14007](#) [MR 3181550](#)
- [33] O. Iyama and M. Wemyss, Maximal modifications and Auslander–Reiten duality for non-isolated singularities, *Invent. Math.*, **197** (2014), no. 3, 521–586. [Zbl 1308.14007](#) [MR 3251829](#)
- [34] O. Iyama and M. Wemyss, Singular derived categories of  $\mathbb{Q}$ -factorial terminalizations and maximal modification algebras, *Adv. Math.*, **261** (2014), 85–121. [Zbl 1326.14033](#) [MR 3213296](#)
- [35] O. Iyama and M. Wemyss, Reduction of triangulated categories and maximal modification algebras for  $cA_n$  singularities, to appear in *J. Reine Angew. Math.*
- [36] M. Kapranov and E. Vasserot, Kleinian singularities, derived categories and Hall algebras, *Math. Ann.*, **316** (2000), no. 3, 565–576. [Zbl 0997.14001](#) [MR 1752785](#)
- [37] G. J. Leuschke, Non-commutative crepant resolutions: scenes from categorical geometry, in *Progress in commutative algebra. 1*, 293–361, de Gruyter, Berlin, 2012. [Zbl 1254.13001](#) [MR 2932589](#)
- [38] G. J. Leuschke and R. Wiegand, *Cohen–Macaulay Representations*, Mathematical Surveys and Monographs, 181, American Mathematical Society, 2012. [Zbl 1252.13001](#) [MR 2919145](#)
- [39] S. Mozgovoy and M. Reineke, On the noncommutative Donaldson–Thomas invariants arising from brane tilings, *Adv. Math.*, **223** (2010), no. 5, 1521–1544. [Zbl 1191.14008](#) [MR 2592501](#)
- [40] J. T. Stafford and M. Van den Bergh, Noncommutative resolutions and rational singularities, *Michigan Math. J.*, **57** (2008), special volume in honor of Melvin Hochster, 659–674. [Zbl 1177.14026](#) [MR 2492474](#)
- [41] Y. Toda and H. Uehara, Tilting generators via ample line bundles, *Adv. Math.*, **223** (2010), no. 1, 1–29. [Zbl 1187.18009](#) [MR 2563209](#)
- [42] K. Ueda and M. Yamazaki, A note on dimer models and McKay quivers, *Comm. Math. Phys.*, **301** (2011), no. 3, 723–747. [Zbl 1211.81090](#) [MR 2784278](#)
- [43] M. Van den Bergh, Three-dimensional flops and noncommutative rings, *Duke Math. J.*, **122** (2004), no. 3, 423–455. [Zbl 1074.14013](#) [MR 2057015](#)
- [44] M. Van den Bergh, Non-Commutative Crepant Resolutions, in *The Legacy of Niels Henrik Abel*, 749–770, Springer-Verlag, Berlin, 2004. [Zbl 1082.14005](#) [MR 2077594](#)
- [45] M. Wemyss, Flops and clusters in the homological minimal model programme, to appear in *Invent. Math.* [arXiv:1411.7189](#)
- [46] Y. Yoshino, *Cohen–Macaulay modules over Cohen–Macaulay rings*, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990. [Zbl 0745.13003](#) [MR 1079937](#)

Received 25 March, 2016; revised 02 March, 2017

O. Iyama, Graduate School of Mathematics, Nagoya University,  
Furocho, Chikusaku, Nagoya 464-8602, Japan

E-mail: [iyama@math.nagoya-u.ac.jp](mailto:iyama@math.nagoya-u.ac.jp)

Y. Nakajima, Graduate School of Mathematics, Nagoya University,  
Furocho, Chikusaku, Nagoya 464-8602, Japan;

Current address: Kavli Institute for the Physics and Mathematics of the Universe (WPI),  
UTIAS, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

E-mail: [m06022z@math.nagoya-u.ac.jp](mailto:m06022z@math.nagoya-u.ac.jp); [yusuke.nakajima@ipmu.jp](mailto:yusuke.nakajima@ipmu.jp)