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Modular curvature for toric noncommutative manifolds

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Abstract. In this paper, we extend recent results on the modular geometry on noncommutative two tori to a larger class of noncommutative manifolds: toric noncommutative manifolds. We first develop a pseudo differential calculus which is suitable for spectral geometry on toric noncommutative manifolds. As the main application, we derive a general expression for the modular curvature with respect to a conformal change of metric on toric noncommutative manifolds. By specializing our results to the noncommutative two and four tori, we recovered the modular curvature functions found in the previous works. An important technical aspect of the computation is that it is free of computer assistance.

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1. Introduction

In the noncommutative differential geometry program (cf. for instance, Connes's book [10]), the geometric data is given in the form of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a $*$-algebra which serves as the algebra of coordinate functions of the underlying space, and D is an unbounded self-adjoint operator such that the commutators $[D, a]$ are bounded operators on $\mathcal{H}$ for all $a \in \mathcal{A}$. More than the topological structure, the spectral data also reflects the metric and differential structure of the geometric space. The prototypical example comes from spin geometry: $(C^\infty(M), L^2(\$), \not{D})$, where M is a closed spin manifold with spinor bundle $\$$ and $\not{D}$ is the Dirac operator. In Riemannian geometry, local geometric invariants, such as the scalar curvature function can be recovered from the asymptotic expansion of Schwartz kernel function of the heat operator $e^{-t\Delta}$:

$$K(x, x, t) \backsim \sum_{j \geq 0} V_j(x) t^{(j-d)/2}, \quad \text{where } d \text{ is the dimension of the manifold.}$$

Equivalently, one turns to the asymptotic expansion of the heat kernel trace

$$\operatorname{Tr}(f e^{-tD^2}) \backsim_{t\searrow 0} \sum_{j\geq 0} V_j(f, D^2) t^{(j-d)/2}, \quad f \in \mathcal{A}, \tag{1.1}$$

which makes perfect sense in the operator theoretic setting. In the spirit of Connes and Moscovici's work [14], the coefficients $V_j(\cdot, D^2)$ above (in (1.1)), viewed as functionals on the algebra of coordinate functions, encode the local geometry, such as intrinsic curvatures, with respect to the metric implemented by the operator D. This approach was carried out in great depth on noncommutative two tori. The technical tool for the computation is the pseudo differential calculus associated to a C^*-dynamical system which was developed in Connes' seminal paper [8], meanwhile the computation was initiated, see [7]. The first application of such calculus is the Gauss–Bonnet theorem for noncommutative two torus [15]. The major progress occurred in Connes and Moscovici's recent work [14]. The high lights of the paper contains not only the full local expression for the functional of the second heat coefficient, but also several geometric applications of the local formulas which demonstrate the great significance of the approach. The appearance of the modular curvature functionals in those closed formulas gives vivid reflections of the noncommutativity. An independent calculation for the Gauss–Bonnet theorem and the full expressions of the modular curvature functions, was carried out in [20] and [21], with a different CAS (Computer Algebra System). Modular scalar curvature on noncommutative four tori was studied in [22] and [19]. Recently, in [32], the computation was extended to Heisenberg modules over noncommutative two torus and the whole calculation was greatly simplified so that CAS was no longer need. See also [3] and [36] for other related work on noncommutative two tori.

It is natural to investigate how to implement the program for other noncommutative manifolds. An interesting class of examples comes from deformation of classical Riemannian manifolds, such is the Connes–Landi deformations (cf. [12, 13]), also called toric noncommutative manifolds in [5]. The underlying deformation theory, called θ-deformation in the literature, was first developed in Rieffel's work [35].

Following the spirit of the previous work on noncommutative tori, we use a pseudo differential calculus to tackle the heat kernel coefficients in this paper. The construction is the first main outcome of this paper. Our pseudo differential calculus is designed to handle two families of noncommutative manifolds simultaneously: tori and spheres obtained by the Connes–Landi deformation. In contrast to noncommutative two tori, the noncommutative four-spheres are different in two essential ways:

(1) The dimension of the action torus (which is two) is less than the dimension of the underlying manifold (which is four);

(2) The underlying manifold is not parallelizable.

The first one implies that the torus action is not transitive, hence the correspondent C^*-dynamical system will not be able to reveal the entire geometry. The second fact indicates that one should expect a more sophisticated asymptotic formula for the product of two symbols than the one appears in Connes' construction. The method taken in this paper is to apply the deformation theory not only to the algebra of smooth functions on the underlying Riemannian manifold, but also to the whole pseudo differential calculus. The resulting symbol calculus blends the commutative and the noncommutative coordinates in a simplest fashion.

In order for the deformation theory to apply, both the symbol map and the quantization map in the calculus have to be equivariant with respect to the torus action. This leads us to work with global pseudo differential calculus on closed manifolds in which all the ingredients are given in a coordinate-free way. Such calculus, which appeared first in Widom's work [41] and [42], turned out to be the perfect tool to develop the deformation process.

In the rest of the paper, we devote the attention to applications. In contrast to the work [14, 22, 32], we skip the construction of the spectral triple since only pseudo differential operators acting on functions are consider in this paper. As a consequence, we use scalar Laplacian operator (instead of the Dirac operator) to define the metric and the noncommutative conformal change of metric is implemented by a perturbation of the scalar Laplacian operator via a Weyl factor k. The first consequence of the pseudo differential calculus is the existence of the asymptotic expansion (1.1). The associated modular curvature is defined to be the functional density with respect to the V_2 term in the heat kernel asymptotic (1.1). It is worth to point out that the modular curvature defined here is only part of the full intrinsic scalar curvature in [9, Definition 1.147].

In this paper, we only test our pseudo differential calculus on the simplest but totally nontrivial perturbed Laplacian: $k\Delta$, which is obtained from the degree zero Laplacian $k\Delta k$ in [14] by a conjugation. Here k is a Weyl factor as before, and Δ is the scalar Laplacian associated to the Riemannian metric. As an instance of [14, Theorem 2.2], we prove that the zeta function at zero is independent of the conformal perturbation, namely:

$$\zeta_{k\Delta}(0) = \zeta_{\Delta}(0). \tag{1.2}$$

The main result is the local formula for the modular curvature $\mathcal{R} \in C^\infty(M_\Theta)$ with respect to a perturbed Laplacian $\pi^\Theta(P_k)$ (i.e. a noncommutative conformal change of metric):

$$\begin{aligned}\mathcal{R}(k) = \big(k^{-m/2}\mathcal{K}(\triangle)(\nabla^2 k) + k^{-(m+2)/2}\mathcal{G}(\triangle_{(1)}, \triangle_{(2)})\,(\nabla k \nabla k)\big)g^{-1} \\ + ck^{-(\frac{m}{2}-1)}\mathcal{S}_\Delta.\end{aligned} \tag{1.3}$$

Let us explain the notations. First, $k \in C^\infty(M_\Theta)$ is a Weyl factor, $m = \dim M$ is an even integer, g^{-1} is the metric tensor on the cotangent bundle and ∇ is the Levi-Civita connection so that the contraction $(\nabla^2 k)g^{-1}$ is equal to $-\Delta k$ and $(\nabla k \nabla k)g^{-1}$, which equals the squared length of the covector ∇k in the commutative situation, generalizes the Dirichlet quadratic form appeared in [14, Eq. (0.1)]. The scalar curvature function $\mathcal{S}_\Delta$ associated to the metric g appears naturally if the metric is nonflat, the coefficient c is a constant depends only on the dimension of the manifold M. The triangle $\triangle$ (compare to Δ, the Laplacian operator) is the modular operator (see (7.10)), while for $j = 1, 2$, $\triangle_{(j)}$ indicates that the operator $\triangle$ applied only to the jth factor. The modular curvature functions $\mathcal{K}$ and $\mathcal{G}$ are computed explicitly in the last section. A crucial property of the modular curvature functions is that they can be written as linear combinations of simple divided differences[1] of the modified logarithm $\mathcal{L}_0 = \log s/(s-1)$, which is the generating function of Bernoulli numbers after the substitution $s \mapsto e^s$. The significance of this feature was explained in [31].

The second main outcome of this paper is obtained by specializing the result above onto dimension two. We show that the expressions of $\mathcal{K}$ and $\mathcal{G}$ agree with the result in [32, Theorem 3.2] which gives further validation for our pseudo differential calculus and the computation performed in the last section as in [32] and as a significant improvement of the previous work, the computation does not require aid from CAS.

In dimension four, we show that the modular curvature functions are both zero with respect to the operator $k\Delta$. Since $k\Delta$ is the leading part of the Laplacian adapted in [22], the non-zero contributions to the modular functions come from the symbols of degree one and zero. This fact can be observed in [19] in which the computation was simplified.

The other significant feature of our approach is that the computation is no longer require computer assistance. The efficiency of our computation relies on a tensor calculus over the toric noncommutative manifolds which is obtained from a deformation of tensor calculus over the toric manifolds. On a smooth manifold M, a tensor calculus consists of three parts: the pointwise tensor product and contraction between tensor fields, and a connection ∇ which is characterized by the Leibniz property. For instance, a differential operator on $C^\infty(M)$ can be represented by a finite sum $f \mapsto \sum_\alpha \rho_{\alpha_j} \cdot \nabla^j f$, where ρ_{α_j} is a contravariant tensor field ρ of rank j so that the contraction $\rho_{\alpha_j} \cdot \nabla^j f$ produces a smooth function. One of the merits is this observation is that it has a straightforward generalization to our noncommutative setting: the tensor product and contractions between tensor fields are pointwise, like functions on a manifold, therefore the deformation procedure for functions extends naturally to tensor fields. To obtain a calculus, we show that the Leibniz property of the Levi-Civita connection still holds in the deformed setting. As an example, we see that the Dirichlet quadratic form appeared in [14, Eq. (0.1)] with respect to the complex structure associated to the modular parameter $\sqrt{-1}$ has the following

[1] "simple" means that at most the third divided difference occurs. For the notion of divided differences, we refer to [31, Appendix A] and a classical reference [33].

counterpart in terms of the deformed tensor calculus:

$$\Box_{\mathfrak{R}}(h) = (\nabla h \otimes_\Theta \nabla h) \cdot_\Theta g^{-1}, \tag{1.4}$$

where g^{-1} is the metric tensor on the cotangent bundle and $\otimes_\Theta$ and $\cdot_\Theta$ are deformed tensor product and contraction respectively. Being technical tools, such deformed tensor calculus and pseudo differential calculus have many other potential applications, for instance:

(i) the gauge theory on toric noncommutative manifolds studied in [5];

(ii) exploring generalizations of Riemannian metrics on noncommutative manifolds (cf. [36]).

We end this introduction with a brief outline of the paper. Section 2 consists of functional analytic backgrounds of the deformation theory. We split the discussion into two parts: deformation of algebras and deformation of operators according to their roles as "symbols" and "operators" in the general framework of pseudo differential calculi.

In Section 3, we explain that how apply the deformation process to the whole tensor calculus, which serves as preparation for Sections 4 and 5, which consist of the construction of pseudo differential calculus for toric noncommutative manifolds.

The remaining two sections are devoted to applications. We first sketch the proof of the existence of the heat kernel asymptotic following [26] and [6] in Section 6. Finally, Section 7 consists of explicit computation of the local formula of the associated modular curvature. Some technical parts of the computation are moved to the appendixes.

2. Deformation along $\mathbb{T}^n$

2.1. Deformation of Fréchet algebras. In this section, we will provide the functional analytic framework which is necessary for our later discussion on toric noncommutative manifold. We refer to Rieffel's monograph [35] for further details, also [5, 24] and [43]. All the topological vectors spaces appeared in this paper are over the field of complex numbers.

Definition 2.1. Let V be Fréchet space whose topology is defined by an increasing family of semi-norms $\|\cdot\|_k$. We say V is a smooth $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ module if V admits a n-torus action $\alpha_t \colon V \to V$ such that the function $t \mapsto \alpha_t(v)$ belongs to $C^\infty(\mathbb{T}^n, V)$ for all $v \in V$, moreover, we require the action is strongly continuous in the following sense: $\forall v \in V$, given a multi-index μ, we can find another integer j' such that

$$\left\| \partial_t^\mu \alpha_t(v) \right\|_j \le C_{\mu,j,j'} \left\| v \right\|_{j'}, \quad \forall t \in \mathbb{T}^n, \tag{2.1}$$

where the constant $C_{j'}$ depends on j' and the vector v.

Due to the duality between $\mathbb{Z}^n$ and $\mathbb{T}^n$, Fourier theory tells us that all smooth $\mathbb{T}^n$-module in definition 2.1 are $\mathbb{Z}^n$-graded:

$$V = \overline{\bigoplus_{r \in \mathbb{Z}^n} V_r},$$

where V_r is the image of the projection $p_r : V \to V$:

$$p_r(v) = \int_{\mathbb{T}^n} \alpha_t(v) e^{-2\pi i r \cdot t}\, dt, \quad v \in V. \tag{2.2}$$

Namely, any vector in V admits a isotypical decomposition:

$$v = \sum_{r \in \mathbb{Z}^n} v_r, \quad \text{with } v_r = p_r(v) \text{ as above.} \tag{2.3}$$

The sequence $\{v_r\}_{r \in \mathbb{Z}^n}$ is of rapidly decay in r due to an integration by parts argument on (2.2). The precise estimate is given below:

Proposition 2.2. *Let V be a smooth $\mathbb{T}^n$-module as in definition 2.1, whose topology is given by an countable increasing family of semi-norms $\|\cdot\|_j$ with $j \in \mathbb{N}$. Then for any element $v = \sum_{r \in \mathbb{Z}^n} v_r \in V$ with its isotypical decomposition, then the sequence of the jth semi-norms: $\|p_r(v)\|_j$ is of rapidly decay in $r \in \mathbb{Z}^n$. More precisely, for any integer $k, j > 0$, there exist a degree k polynomial $Q_k(x_1, \ldots, x_n)$ and another large integer j' such that $\forall v \in V$,*

$$\|p_r(v)\|_j \leq \frac{C_{k,j'}}{|Q_k(r)|} \|\alpha_t(v)\|_{j'}, \tag{2.4}$$

In particular, the isotypical decomposition $\sum_{r \in \mathbb{Z}^n} v_r$ converges absolutely to v.

The proof can be found in, for instance, [35, Lemma 1.1].

Conversely, suppose V admits a smooth $\mathbb{Z}^n$ grading:

$$V = \overline{\bigoplus_{r \in \mathbb{Z}^n} V_r},$$

then the $\mathbb{T}^n$ action is given by on each homogeneous component V_r

$$t \cdot v_r = e^{2\pi i t \cdot r} v_r, \quad t \in \mathbb{T}^n. \tag{2.5}$$

A vector $v = \sum_{r \in \mathbb{Z}^n} v_r \in V$ is smooth respect to the torus action if and only if for each semi-norm $\|\cdot\|_j$, the sequence $\|v_r\|_j$ decays faster than any polynomial in r, that is for each semi-norm $\|\cdot\|_j$ and integer k, there is an integer l and a constant $C_{j,k}$, such that

$$\|v_r\|_j \leq C_{j,k} \frac{\|v\|_l}{r^k}. \tag{2.6}$$

Definition 2.3. A $\mathbb{T}^n$ smooth algebra $\mathcal{A}$ is a smooth $\mathbb{T}^n$ module as in Definition 2.1 such that the multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is $\mathbb{T}^n$-equivariant and jointly continuous, that is for every j, there is a k and a constant C_j such that

$$\|ab\|_j \leq C_k \|a\|_k \|b\|_k , \quad \forall a, b \in A. \tag{2.7}$$

Additionally, if $\mathcal{A}$ is a $*$-algebra, we required the $*$-operator is continuous and $\mathbb{T}^n$-equivariant. Similarly, a $\mathbb{T}^n$-smooth left (right) $\mathcal{A}$-module V is a $\mathbb{T}^n$-smooth module while the left (right) module structure is $\mathbb{T}^n$-equivariant and the jointly continuous as in (2.7).

Definition 2.4. Let $\mathcal{A}$ be a $\mathbb{T}^n$ smooth algebra as above. For a skew symmetric $n \times n$ matrix Θ, we denote the corresponding bi-character:

$$\chi_\Theta(r, l) = e^{\pi i \langle r, \Theta l\rangle}, \quad r, l \in \mathbb{Z}^n, \tag{2.8}$$

where the pairing $\langle \cdot, \cdot \rangle$ is the usual dot product in $\mathbb{R}^n$. The deformation of $\mathcal{A}$ is a family of algebras $\mathcal{A}_\Theta$ parametrized by Θ, whose underlying topological vector space is equal to $\mathcal{A}$ while the multiplication $\times_\Theta$ is deformed as follow:

$$a \times_\Theta b = \sum_{r,s\in\mathbb{Z}^n} \chi_\Theta(r, l)\, a_r b_s, \quad \forall a, b \in \mathcal{A}, \tag{2.9}$$

and $a = \sum_r a_r$, $b = \sum_s b_s$ are the isotypical decomposition as in (2.2).

Each $\mathcal{A}_\Theta$ inherits the smooth $\mathbb{T}^n$ module structure from $\mathcal{A}$ and since $\chi_\Theta(r, l)$ in (2.9) are complex numbers of length 1, the new multiplication is jointly continuous as well. Hence, for any $n \times n$ skew symmetric matrix Θ, the deformation $\mathcal{A}_\Theta$ are all smooth $\mathbb{T}^n$ algebra as in definition 2.3.

Proposition 2.5. *The deformed product $\times_\Theta$ on $\mathcal{A}_\Theta$ is associative. That is, for any $a, b, c \in \mathcal{A}$,*

$$(a \times_\Theta b) \times_\Theta c = a \times_\Theta (b \times_\Theta c)\,. \tag{2.10}$$

If the algebra $\mathcal{A}$ is a $*$-algebra, the deformation $\mathcal{A}_\Theta$ are $*$-algebras as well with respect to the original $*$-operator, that is $\forall a, b \in \mathcal{A}$,

$$(a \times_\Theta b)^* = b^* \times_\Theta a^*. \tag{2.11}$$

Proof. Let $a, b, c \in \mathcal{A}_\Theta$ with their isotypical decomposition: $a = \sum_r a_r$, $b = \sum_s b_s$ and $c = \sum_l c_l$, where r, s, l are summed over $\mathbb{Z}^n$. We compute the left hand side of (2.10),

$$\begin{aligned}(a \times_\Theta b) \times_\Theta c &= \sum_{k,l} \chi_\Theta(k,l)\Big(\sum_{r+s=k} \chi_\Theta(r,s) a_r b_s\Big) c_l \\ &= \sum_{r,s,l} \chi_\Theta(r+s,l)\chi_\Theta(r,s) a_r b_s c_l \\ &= \sum_{r,s,l} \chi_\Theta(r,l)\chi_\Theta(s,l)\chi_\Theta(r,s) a_r b_s c_l,\end{aligned}$$

here we have used the estimate (2.6) to exchange the order of summation. Similar computation gives us the right hand side:

$$a \times_\Theta (b \times_\Theta c) = \sum_{r,s,l} \chi_\Theta(r,s)\chi_\Theta(r,l)\chi_\Theta(s,l) a_r b_s c_l.$$

Thus we have proved the associativity. Notice that we have not yet used the skew-symmetric property of Θ. In fact, the skew-symmetric property is only necessary for the $*$-operator to survive after deformation. In particular, it implies that for the bi-character χ_Θ defined in (2.8),

$$\chi_\Theta(r,l) = \chi_\Theta(l,r)^*, \quad \forall r, l \in \mathbb{Z}^n,$$

here the $*$ operator is the conjugation on complex numbers. Since the $*$ operator is $\mathbb{T}^n$-equivariant, it flips the $\mathbb{Z}^n$-grading of $\mathcal{A}$, that is, it sends the r component to the $-r$ component: $(a_r)* = a^*_{-r}$, where $a = \sum_{r \in \mathbb{Z}^n} a_r \in \mathcal{A}$. Indeed,

$$(a_r)* = \left(\int_{\mathbb{T}^n} e^{-2\pi i r\cdot t}\alpha_t(a)\, dt\right)^* = \int_{\mathbb{T}^n} e^{2\pi i r\cdot t}\alpha_t(a^*)\, dt = a^*_{-r}.$$

Therefore:

$$\begin{aligned}(a \times_\Theta b)^* &= \Big(\sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l) a_r b_l\Big)^* = \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l)^*(b_l)^*(a_r)^* \\ &= \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(l,r) b^*_{-l} a^*_{-r} = \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(-l,-r) b^*_{-l} a^*_{-r} \\ &= \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(l,r) b^*_l a^*_r = b^* \times_\Theta a^*.\end{aligned}$$

The proof is complete. □

Proposition 2.6. *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a $\mathbb{T}^n$-equivariant continuous algebra homomorphism, where A, B are two $\mathbb{T}^n$ smooth algebras which admit deformation as above. If we identify $\mathcal{A}$ and $\mathcal{A}_\Theta$, $\mathcal{B}$ and $\mathcal{B}_\Theta$ by the identity maps respectively, then*

$$\phi\colon \mathcal{A}_\Theta \to \mathcal{B}_\Theta \tag{2.12}$$

is still an $\mathbb{T}^n$-equivariant algebra homomorphism with respect to the new product $\times_\Theta$.

Proof. For any $a, a' \in \mathcal{A}$ with the isotypical decomposition $a = \sum_r a_r$, $b = \sum_l b_l$, thanks to the equivariant property of ϕ, we have $\phi(a_r) = \phi(a)_r$ and $\phi(b_l) = \phi(b)_l$ for any $r, l \in \mathbb{Z}^n$. Use the continuity of ϕ, we compute:

$$\begin{aligned}
\phi(a \times_\Theta a') &= \phi\Big(\sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l) a_r a'_l\Big) = \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l)\phi(a_r)\phi(a'_l) \\
&= \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l)\phi(a)_r\phi(a')_l \\
&= \phi(a) \times_\Theta \phi(a'). \qquad \square
\end{aligned}$$

The next proposition shows that any $\mathbb{T}^n$-equivariant trace on a smooth $\mathbb{T}^n$-algebra $\mathcal{A}$ extends naturally to a trace on all the deformations $\mathcal{A}_\Theta$.

Proposition 2.7. *Let $\tau\colon \mathcal{A} \to \mathbb{C}$ be a $\mathbb{T}^n$-equivariant trace and Θ is a $n \times n$ skey symmetric matrix as before. Then $\tau\colon \mathcal{A}_\Theta \to \mathbb{C}$ is a continuous linear functional for the deformation $\mathcal{A}_\Theta$ and $\mathcal{A}$ are identical as topological vector spaces. However, τ is indeed a trace on $\mathcal{A}_\Theta$, that is $\forall a, b \in \mathcal{A}_\Theta$*

$$\tau(a \times_\Theta b) = \tau(ab) = \tau(ba) = \tau(b \times_\Theta a). \tag{2.13}$$

Proof. From the $\mathbb{T}^n$-equivariant property of τ, we know that for any isotypical component a_r, $(r \in \mathbb{Z}^n)$, of $a \in \mathcal{A}$,

$$\tau(a_r) = \tau(\alpha_t(a_r)) = \tau(e^{2\pi i t\cdot r} a_r) = e^{2\pi i t\cdot r}\tau(a_r), \quad \forall t \in \mathbb{T}^n.$$

Therefore $\tau(a_r) = 0$ for all $r \neq 0$. Follows from the continuity, for any $a \in \mathcal{A}$,

$$\tau(a) = \tau\Big(\sum_{r\in\mathbb{Z}^n} a_r\Big) = \sum_{r\in\mathbb{Z}^n} \tau\,(a_r) = \tau(a_0),$$

that is, the trace of a depends only on its $\mathbb{T}^n$-invariant component. Since Θ is skew symmetric, we get

$$\tau(a \times_\Theta b) = \tau\,((a \times_\Theta b)_0) = \sum_{r\in\mathbb{Z}^n} \chi_\Theta(r,-r)\tau(a_r b_{-r}) = \sum_{r\in\mathbb{Z}^n} \tau(a_r b_{-r}) = \tau(ab).$$

Similar computation gives that $\tau(ba) = \tau(b \times_\Theta a)$. Therefore if τ is a trace on $\mathcal{A}$, then it is a trace on $\mathcal{A}_\Theta$ as well. $\square$

In this paper, we have to deal with certain topological algebras which are not Fréchet. For instance, the algebra of pseudo differential operators of integer orders, and the associated algebra of symbols, whose topology is certain inductive limit of Fréchet topologies. More precisely, we consider a filtered algebra $\mathcal{A}$ with a filtration:

$$\cdots \subset \mathcal{A}_{-j} \subset \cdots \subset \mathcal{A}_0 \subset \cdots \mathcal{A}_j \cdots \subset \mathcal{A}, \tag{2.14}$$

where each A_j ($j \in \mathbb{Z}$) is a smooth $\mathbb{T}^n$-module as defined before, in particular, a Fréchet space. As a topological vector space, the total space $\mathcal{A}$ is a countable strict inductive limit of $\{\mathcal{A}_j\}_{j\in\mathbb{Z}}$, the topology is just called strict inductive limit topology (cf. for instance [40, Sec. 13] for more details). This topology is never metrizable unless the filtration is stabilized starting from some $\mathcal{A}_j$, therefore it is not Fréchet. Nevertheless, we shall not looking at the topology of the whole algebra $\mathcal{A}$ even when considering the continuity of the multiplication map. Instead, we focus on each $\mathcal{A}_j$, assume that the Fréchet topology is defined by a countable family of increasing semi-norms $\{\|\cdot\|_{l,j}\}_{l\in\mathbb{N}}$. The multiplication preserves the filtration:

$$m\colon \mathcal{A}_{j_1} \times \mathcal{A}_{j_2} \to \mathcal{A}_{j_1+j_2} \tag{2.15}$$

such that the continuity condition holds: for fixed j_1, j_2 and a positive integer l, one can find a integer k and constant C_{k,j_1,j_2} such that

$$\|m(a_1 a_2)\|_{l,j_1+j_2} \le C_{k,j_1,j_2} \|a_1\|_{k,j_1} \|a_2\|_{k,j_2}, \quad \forall a_1 \in \mathcal{A}_{j_1},\ a_2 \in \mathcal{A}_{j_2}. \tag{2.16}$$

The multiplication m is deformed in a similar fashion as in (2.9):

$$m_\Theta\colon A_{j_1} \times A_{j_2} \to A_{j_1+j_2} \tag{2.17}$$

$$(a_1, a_2) \mapsto \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r,l) m\left((a_1)_r, (a_2)_l\right). \tag{2.18}$$

Examples are provided in the next section.

2.2. Deformation of operators. The associativity of the $\times_\Theta$ multiplication proved in proposition 2.5 is a special instance of certain "functoriality" in the categorical framework explained in [5]. Let us start with deformation of operators.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces which are both strongly continuous unitary representation of $\mathbb{T}^n$, denoted by $t \mapsto U_t \in B(\mathcal{H}_1)$ and $t \mapsto \widetilde{U}_t \in B(\mathcal{H}_2)$ respectively, where $t \in \mathbb{T}^n$. If no confusions arise, both representations will be denoted by U_t. Then $B(\mathcal{H}_1, \mathcal{H}_2)$, the space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, becomes a $\mathbb{T}^n$-module via the adjoint action:

$$P \in B(\mathcal{H}_1, \mathcal{H}_2) \mapsto \mathrm{Ad}_t(P) := \widetilde{U}_t P U_{-t}, \quad t \in \mathbb{T}^n. \tag{2.19}$$

Denote by $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$, the space of all the $\mathbb{T}^n$ smooth vectors in $B(\mathcal{H}_1, \mathcal{H}_2)$. It is a Fréchet space on which the topology is defined by the semi-norms $\{\|\cdot\|_j\}_{j\in\mathbb{N}}$:

$$q_j(P) := \sum_{|\beta|\le j} \frac{1}{\beta!} \left\| \partial_t^\beta \operatorname{Ad}_t(P) \right\|_{B(\mathcal{H}_1,\mathcal{H}_2)}. \tag{2.20}$$

The semi-norms above are constructed in such a way that the continuity estimate (2.1) for the torus action holds automatically. Following from Proposition 2.2, we see that any $\mathbb{T}^n$-smooth operator P admits an isotypical decomposition $P = \sum_{r\in\mathbb{Z}^n} P_r$, where the operator norms $\{\|P_r\|\}_{r\in\mathbb{Z}^n}$ decays faster than polynomial in r, in particular the converges of the infinite sum is absolute with respect to the operator norm in $B(\mathcal{H}_1, \mathcal{H}_2)$.

Now we are ready to define the deformation map $\pi^\Theta: B(\mathcal{H}_1, \mathcal{H}_2)_\infty \to B(\mathcal{H}_1, \mathcal{H}_2)_\infty$.

Definition 2.8. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert space with strongly continuous unitary $\mathbb{T}^n$ actions as above, that is $\forall v \in \mathcal{H}_j$, $(j = 1, 2)$, $t \mapsto t \cdot v$ is continuous in $t \in \mathbb{T}^n$. We denote the actions by $t \to U_t$ and $t \to \widetilde{U}_t$ respectively. For a fixed $n\times n$ skew symmetric matrix Θ, we recall the associated bi-character $\chi_\Theta(r, l) = e^{\pi i \langle r, \Theta l\rangle}$. Then the deformation map

$$\pi^\Theta: B(\mathcal{H}_1, \mathcal{H}_2)_\infty \to B(\mathcal{H}_1, \mathcal{H}_2)_\infty : P \mapsto \pi^\Theta(P)$$

is defined as follows,

$$\pi^\Theta(P)(f) = \sum_{r,l\in\mathbb{Z}^n} \chi_\Theta(r, l) P_r(f_l), \quad P \in B(\mathcal{H}_1, \mathcal{H}_2)_\infty, \tag{2.21}$$

where $P = \sum_{r\in\mathbb{Z}^n} P_r$ and $f = \sum_{l\in\mathbb{Z}^n} f_l \in \mathcal{H}_1$ with their isotypical decomposition. We can assume that f is a $\mathbb{T}^n$-smooth vector for the subspace of all $\mathbb{T}^n$-smooth vectors in dense in $\mathcal{H}_1$. Alternatively, $\pi^\Theta(P)$ is given by

$$\pi^\Theta(P) = \sum_{r\in\mathbb{Z}^n} P_r U_{r\cdot\Theta/2}, \tag{2.22}$$

here $r \cdot \Theta/2$ stands for the matrix multiplication between a row vector r and Θ whose result is a point in $\mathbb{T}^n$.

Remark. The deformed operator $\pi^\Theta(P)$ belongs to $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$. Indeed, in (2.22), the each isotypical component of P is perturbed by a unitary operator $U_{r\cdot\Theta/2}$, therefore the right hand side of (2.22) is a sum of rapidly decay sequence, which implies not only the boundedness of $\pi^\Theta(P)$, but also the $\mathbb{T}^n$-smoothness.

Let us give a precise estimate of the operator norm of $\pi^\Theta(P)$.

Lemma 2.9. *Let $B(\mathcal{H}_1, \mathcal{H}_2)_\infty$ denote the Fréchet algebra of $\mathbb{T}^n$-smooth vectors in $B(\mathcal{H}_1, \mathcal{H}_2)$ whose topology is given by the seminorms in* (2.20)*. Then the deformation*

$$\pi^\Theta: B(\mathcal{H}_1, \mathcal{H}_2)_\infty \to B(\mathcal{H}_1, \mathcal{H}_2)_\infty$$

is a continuous linear map with respect to the Fréchet topology with the estimate: for any multi-index μ, one can find any integer l large enough such that

$$\left\| \partial_t^{\mu} \big(\mathrm{Ad}_t(\pi^{\Theta}(P)) \big) \right\| \leq C_{\mu} q_l(P). \tag{2.23}$$

Proof. Given $P \in B(\mathcal{H}_1, \mathcal{H}_2)_{\infty}$ a $\mathbb{T}^n$-smooth operator with the isotypical decomposition $P = \sum_{r \in \mathbb{Z}^n} P_r$. From the definition, $\pi^{\Theta}(P)$ has the isotypical decomposition $\pi^{\Theta}(P) = \sum_{r \in \mathbb{Z}^n} P_r U_{r \cdot \Theta/2}$, thus

$$\mathrm{Ad}_t(\pi^{\Theta}(P)) = \sum_{r \in \mathbb{Z}^n} \mathrm{Ad}_t(P_r U_{r \cdot \Theta/2}) = \sum_{r \in \mathbb{Z}^n} e^{2\pi i r \cdot t} P_r U_{r \cdot \Theta/2}.$$

If we let $h(r)$ be the polynomial in r such that $\partial_t^{\mu}(e^{2\pi i r \cdot t}) = h(r) e^{2\pi i r \cdot t}$, the degree of $h(r)$ is equal to $|\mu|$, compute

$$\partial_t^{\mu}(\mathrm{Ad}_t(\pi^{\Theta}(P))) = \sum_{r \in \mathbb{Z}^n} \partial_t^{\mu}(e^{2\pi i r \cdot t}) P_r U_{r \cdot \Theta/2} = \sum_{r \in \mathbb{Z}^n} h(r) e^{2\pi i r \cdot t} P_r U_{r \cdot \Theta/2}. \tag{2.24}$$

Since that $\|P_r\|$ is of rapidly decay in r, we can find a large integer l such that

$$\|h(r) P_r\| \leq \frac{C q_l(P)}{|r|^{n+1}}.$$

Therefore

$$\left\| \partial_t^{\mu} \big(\mathrm{Ad}_t(\pi^{\Theta}(P)) \big) \right\| \leq \left(\sum_{r \in \mathbb{Z}^n} \frac{C}{|r|^{n+1}} \right) q_l(P).$$

□

Similar to (2.11), we have the compatibility between the deformation map and the $*$-operation (taking the adjoint) on operators.

Lemma 2.10. *Let $P \in B(\mathcal{H}_1, \mathcal{H}_2)_{\infty}$, then its adjoint $P^* \in B(\mathcal{H}_1, \mathcal{H}_2)_{\infty}$ as well, we have*

$$\pi^{\Theta}(P^*) = \pi^{\Theta}(P)^* \tag{2.25}$$

Proof. Since the torus action is unitary, the adjoint operation is equivariant:

$$(\mathrm{Ad}_t(P))^* = (U_t P U_{-t})^* = U_t P^* U_{-t} = \mathrm{Ad}_t(P^*),$$

therefore for the isotypical components, $P_r^* = (P_{-r})^*$ for all $r \in \mathbb{Z}^n$,

$$\begin{aligned}
\pi^{\Theta}(P^*) &= \sum_{r \in \mathbb{Z}^n} P_r^* U_{r \cdot \Theta/2} = \sum_{r \in \mathbb{Z}^n} (P_{-r})^* (U_{-r \cdot \Theta/2})^* \\
&= \sum_{r \in \mathbb{Z}^n} (U_{-r \cdot \Theta/2} P_{-r})^* = \sum_{r \in \mathbb{Z}^n} (P_{-r} U_{-r \cdot \Theta/2})^* \\
&= \sum_{r \in \mathbb{Z}^n} (P_r U_{r \cdot \Theta/2})^* \\
&= (\pi^{\Theta}(P))^*,
\end{aligned}$$

here we have used the facts that $U_{-r \cdot \Theta/2}$ and T_r commute. □

The next lemma says that the deformation is somehow invertible.

Lemma 2.11. *Let Θ and Θ' be two $n \times n$ skew symmetric matrices and for any $P \in B(\mathcal{H}_1, \mathcal{H}_2)_\infty$, we have*

$$\pi^\Theta \circ \pi^{\Theta'}(P) = \pi^{\Theta+\Theta'}(P).$$

In particular, we see that the deformation process is invertible, namely π^Θ and $\pi^{-\Theta}$ are inverse to each other.

Proof. Given $P = \sum_{r\in\mathbb{Z}^n} P_r$, $\pi^\Theta(T) = \sum_{r\in\mathbb{Z}^n} P_r U_{r\cdot\Theta/2}$ is the isotypical decomposition of $\pi^\Theta(T)$, therefore

$$\pi^{\Theta'}\big(\pi^\Theta(P)\big) = \sum_{r\in\mathbb{Z}^n} P_r U_{r\cdot\Theta/2} U_{r\cdot\Theta'/2} = \sum_{r\in\mathbb{Z}^n} P_r U_{r\cdot(\Theta+\Theta')/2} = \pi^{\Theta+\Theta'}(P). \qquad \square$$

If we take $\mathcal{H}_1$ and $\mathcal{H}_2$ above to be the same Hilbert space, $B(\mathcal{H}_\infty)$ becomes an $\mathbb{T}^n$ smooth algebra as in definition 2.3. Following from definition 2.4, we obtain a family of deformed algebras $(B(\mathcal{H}_\infty), \times_\Theta)$ parametrized by skew symmetric matrices Θ. The multiplication map is obviously $\mathbb{T}^n$-equivariant, that is $\mathrm{Ad}_t(P_1)\mathrm{Ad}_t(P_2) = \mathrm{Ad}_t(P_1 P_2)$, for all $t \in \mathbb{T}^n$ and for any $P_1, P_2 \in B(\mathcal{H}_\infty)$. The associativity of the $\times_\Theta$ multiplication has the following analogy.

Proposition 2.12. *Keep the notations as above. The deformation map*

$$\pi^\Theta \colon (B(\mathcal{H})_\infty, \times_\Theta) \to \pi^\Theta\,(B(\mathcal{H}_\infty)) \subset B(\mathcal{H})$$

is an algebra isomorphism, namely, for any $P_1, P_2 \in B(\mathcal{H})_\infty$,

$$\pi^\Theta(P_1)\pi^\Theta(P_2) = \pi^\Theta(P_1 \times_\Theta P_2), \tag{2.26}$$

recall that the deformed product $\times_\Theta$ is defined in (2.9).

Proof. The invertiblity of π^Θ is proved in Lemma 2.11. It remains to show that it is an algebra morphism, that is for any $\mathbb{T}^n$-smooth vector $v \in \mathcal{H}$, we have

$$\pi^\Theta(P_1)\big(\pi^\Theta(P_2)(v)\big) = \pi^\Theta\,(P_1 \times_\Theta P_2)\,(v). \tag{2.27}$$

Observe that $\pi^\Theta(P)(v)$ can be formally written as $P \times_\Theta v$ according to (2.21). Therefore the left hand side and the right hand side of (2.27) becomes $P_1 \times_\Theta (P_2 \times_\Theta v)$ and $(P_1 \times_\Theta P_2) \times_\Theta v$ respectively, thus equation (2.27) is exactly the same as the associativity of the $\times_\Theta$-multiplication proved in (2.10). $\square$

We have seen that the isotypical decomposition of an operator $P = \sum_r P_r$ converges with respect to operator norms. The normality of the trace somehow allows itself to pass the summation, namely $\mathrm{Tr}(P) = \sum_r \mathrm{Tr}(P_r)$ whenever P is a trace-class operator.

Lemma 2.13. *Let $\mathcal{H}$ be a saperable Hilbert space with a strongly continuous unitary $\mathbb{T}^n$-action and $P = \sum_r P_r \in B(\mathcal{H})_\infty$ is $\mathbb{T}^n$-smooth operator with its isotypical decomposition. Suppose P is a trace-class operator, then so is $\pi^\Theta(P)$. Moreover,* $\operatorname{Tr} P = \operatorname{Tr} P_0 = \operatorname{Tr} \pi^\Theta(P)$, *where P_0 is the $\mathbb{T}^n$-invariant part of both P and $\pi^\Theta(P)$.*

Proof. Since $\mathcal{H}$ is a strongly continuous unitary representation of $\mathbb{T}^n$, it admits a orthonormal decomposition

$$\mathcal{H} = \overline{\bigoplus_{l\in\mathbb{Z}^n} H_l},$$

in which each H_l consists of eigenvector of the torus action:

$$H_l = \left\{v \in \mathcal{H} \mid t \cdot v = e^{2\pi i t\cdot l} v\right\}.$$

For each H_l, one can pick a orthonormal basis $\{\varepsilon_{k,l}\}_{k\in\mathbb{N}}$, then $\{\varepsilon_{k,l}\}_{l\in\mathbb{Z}^n, k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. Since $\sum_{r\in\mathbb{Z}^n} P_r$ convergence absolutely in the operator norm,

$$\sum_{\substack{l\in\mathbb{Z}^n,\\ k\in\mathbb{N}}} \left\langle \left(\sum_{r\in\mathbb{Z}^n} P_r\right)(\varepsilon_{k,l}), \varepsilon_{k,l}\right\rangle = \sum_{\substack{l\in\mathbb{Z}^n,\\ k\in\mathbb{N}}} \sum_{r\in\mathbb{Z}^n} \langle P_r(\varepsilon_{k,l}), \varepsilon_{k,l}\rangle,$$

observe that for all $r \in \mathbb{Z}^n$, $P_r(H_l) \subset H_{r+l}$, therefore $\langle P_r(\varepsilon_{k,l}), \varepsilon_{k,l}\rangle = 0$ except the case when $r = 0$. We continue the computation above:

$$\sum_{\substack{l\in\mathbb{Z}^n,\\ k\in\mathbb{N}}} \sum_{r\in\mathbb{Z}^n} \langle P_r(\varepsilon_{k,l}), \varepsilon_{k,l}\rangle = \sum_{\substack{l\in\mathbb{Z}^n,\\ k\in\mathbb{N}}} \langle P_0(\varepsilon_{k,l}), \varepsilon_{k,l}\rangle.$$

Since P is traceable, the left hand side above converges absolutely, therefore P_0 is traceable as well and has the same trace as P. Recall equation (2.22): $\pi^\Theta(P) = \sum_{r\in\mathbb{Z}^n} P_r U_{r\cdot\Theta/2}$. Notice that the computation above still works if P_r is replaced by $P_r U_{r\cdot\Theta/2}$, therefore $\pi^\Theta(P)$ is of trace class and $\operatorname{Tr}\pi^\Theta(P) = \operatorname{Tr}(\pi^\Theta(P))_0$, where $(\pi^\Theta(P))_0$ is the $\mathbb{T}^n$-invariant part of $\pi^\Theta(P)$. Since P and $\pi^\Theta(P)$ have the same invariant part, we have finished the proof. □

The following corollary is important for our later discussion.

Corollary 2.14. *Let $\mathcal{H}$ be a separable Hilbert space. Let $P_1, P_2 \in B(\mathcal{H})$ be two $\mathbb{T}^n$-smooth vectors such that at least one of them is traceable, then $P_1 P_2$ and $P_1 \times_\Theta P_2$ are both traceable with*

$$\operatorname{Tr}(P_1 P_2) = \operatorname{Tr}(P_1 \times_\Theta P_2).$$

Combine the equation above with (2.26)*, we obtain:*

$$\operatorname{Tr}\left(\pi^\Theta(P_1)\pi^\Theta(P_2)\right) = \operatorname{Tr}(P_1 P_2).$$

2.3. Deformation of functions. Let M be a smooth manifold without boundary. For any diffeomorphism $\varphi \in \mathrm{Diff}(M)$, we defined the pull back action on $C^\infty(M)$:

$$U_\varphi(f)(x) = f(\varphi^{-1}(x)), \quad f \in C^\infty(M), \tag{2.28}$$

which is $*$-automorphism of $C^\infty(M)$. Assume that M admits a n-torus action: $\mathbb{T}^n \subset \mathrm{Diff}(M)$, then one can quickly verify that $\mathbb{T}^n$ acts smoothly (cf. Eq. (2.1)) on $C^\infty(M)$ with respect to the smooth Fréchet topology, also the pointwise multiplication is jointly continuous (cf. Eq. (2.7)) , therefore we can deform the multiplication to $\times_\Theta$ following definition 2.3 with respect to a skew symmetric matrix Θ, and the new algebra

$$C^\infty(M_\Theta) := (C^\infty(M), \times_\Theta)$$

plays the role of smooth coordinate functions on a noncommutative manifold M_Θ.

Later, we will assume the manifold M is compact. The non-compact examples we are interested in is the cotangent bundle T^*M. One can easily lift the torus action to T^*M by the natural extension of diffeomorphims: $\varphi \mapsto \varphi^*$, where φ^* is the differential of φ. Thus the cotangent bundle of the noncommutative manifold M_Θ is given by the deformed algebra:

$$C^\infty(T^*M_\Theta) := (C^\infty(T^*M), \times_\Theta).$$

Another crucial example is $S\Sigma(M) \subset C^\infty(T^*M)$, the spaces of symbols of pseudo differential operators on M. It is a filtered algebra:

$$S\Sigma = \bigcup_{j=-\infty}^{\infty} S\Sigma^j(M),$$

where each $S\Sigma^j(M)$ consists of smooth functions with the estimate in local coordinates (x, ξ),

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{j-|\beta|}, \tag{2.29}$$

the optimized constants $C_{\alpha,\beta}$ define a family of semi-norms that makes $S\Sigma(M)$ into a Fréchet space. The smoothing symbols $S\Sigma^{-\infty}$ is the intersection:

$$S\Sigma^{-\infty} = \bigcap_{j=-\infty}^{\infty} S\Sigma^j(M),$$

and the quotient $\mathrm{CL} = S\Sigma / S\Sigma^{-\infty}$ is called the space of complete symbols.

For any $t \in \mathbb{T}^n$ viewed as a diffeomorphism on M, let $p \in S\Sigma^j(M)$ be a symbol of order j, $t \mapsto U_t(p)$ is a function valued in $C^\infty(T^*M)$. Observe that the partial derivatives in t can be written as a finite sum in local coordinates:

$$\partial_t^\gamma U_t(p) = U_t\left(\sum_j \partial_x^{\alpha_j} \partial_\xi^{\beta_j} p\right),$$

where γ, α_j, β_j are multi-indices. This shows not only that $U_t(p)$ still belongs to $S\Sigma^j(M)$ but also the torus action is smooth (cf. Definition 2.1). Therefore we can twist the pointwise multiplication on the filtered algebra $S\Sigma(M)$ as explained in Subsection 2.1, the deformed version is denoted by

$$S\Sigma(M_\Theta) = (S\Sigma(M), \times_\Theta),$$

where $\times_\Theta$ is given in (2.17).

3. Deformation of tensor calculus

Let M be a compact toric manifold M as before, that is $\mathrm{Diff}(M)$ contains a n-torus. So far, we have deformed the "smooth structure" of M, namely we have found the counterpart of the algebra of smooth coordinate functions in the noncommutative setting in the previous section. We will extend the deformation process further by restricting the torus action. For example, if we assume that the torus action is affine: $\mathbb{T}^n \subset \mathrm{Affine}(M) \subset \mathrm{Diff}(M)$, then the whole tensor calculus can be deformed. Since our ultimate goal is to study curvatures, one can assume that M is Riemannian and the torus acts as isometries. We make a formal definition here.

Definition 3.1. A toric Riemannian manifolds M is a closed (compact without boundary) Riemannian manifolds whose isometry group contains a n-torus. In other words, M admits an $\mathbb{T}^n$-action as isometries.

Example 3.2. Let $M = \mathbb{T}^n$ be the n-torus with the usual flat metric, while the n-torus acts on itself by translations. The deformation gives the well-known family called noncommutative n-torus.

Example 3.3. Consider the two torus $\mathbb{T}^2$ acts on the four-sphere S^4 by embedding $\mathbb{T}^2$ into SO(5) as rotations in the first four coordinates:

$$(t_1, t_2) \in \mathbb{R}^2 \mapsto \begin{pmatrix} e^{-\pi i t_1} & & \\ & e^{-\pi i t_2} & \\ & & 1 \end{pmatrix},$$

where we identify $\mathbb{R}^5$ with $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{R}$. Thus any Riemannian metric which is invariant under the rotations above provides an instance of a toric manifold, for example, the Robertson–Walker metrics with a cosmic scale factor $a(t)$:

$$ds^2 = dt^2 + a(t)^2 d\omega^2,$$

where $d\omega^2$ is the round metric on S^3, when $a(t) = \sin t$, we recover the round metric on S^4.

3.1. Isospectral deformations of toric Riemannian manifolds. To finished the story of deforming smooth functions on M, we shall represent the deformed algebra $C^\infty(M_\Theta)$ as bounded operators, whose underlying Hilbert space is metric-related. Therefore we assume that M is a toric Riemannian manifold. It is straightforward to see that the integration against the Riemannian metric g

$$\int_M : C^\infty(M) \to \mathbb{C} \tag{3.1}$$

is $\mathbb{T}^n$-invariant, where $\mathbb{C}$ is viewed a trivial $\mathbb{T}^n$ module. The pairing

$$\langle f, h\rangle := \int_M f\bar{h}\, dg, \quad \forall f, h \in C^\infty(M),$$

makes $C^\infty(M)$ into a pre-Hilbert space, and the completion is $\mathcal{H} = L^2(M)$. For each $t \in \mathbb{T}^n$, the pull-back action $U_t: C^\infty(M) \to C^\infty(M)$ defined in (2.28) can be extended to a unitary operator on $\mathcal{H}$. It is easy to check the equivariant property of the representation $f \in C^\infty(M) \mapsto L_f \in B(\mathcal{H})$, where L_f is the left multiplication operator by f:

$$U_t L_f U_{-t} = L_{U_t(f)}, \quad \forall t \in \mathbb{T}^n.$$

Recall from Section 2.3 that for a fixed skew symmetric matrix Θ, the deformed algebra is given by replacing the multiplication in $C^\infty(M)$: $(C^\infty(M), \times_\Theta)$. The deformation map $\pi^\Theta: B(\mathcal{H})_\infty \to B(\mathcal{H})_\infty$ in Section 2.2 gives a new description of the algebra $C^\infty(M_\Theta)$ as a subalgebra of $B(\mathcal{H})$.

Proposition 3.4. *Let M be a toric Riemannian manifold and Θ is a skew symmetric matrix. We denote by $C^\infty(M_\Theta) = \pi^\Theta(C^\infty(M)) \subset B(\mathcal{H})$, the image of $C^\infty(M)$ under the deformation map π^Θ. Then*

$$\pi^\Theta: (C^\infty(M), \times_\Theta) \to C^\infty(M_\Theta)$$

is an $*$-algebra isomorphism:

As an example, we first compare two sets of notations on noncommutative two tori.

Example 3.5 (Noncommutative two torus). Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the induce flat metric and (x_1, x_2) be the coordinates on $\mathbb{T}^2$, put $e_1(x_1, x_2) = e^{2\pi i x_1}$, $e_2(x_1, x_2) = e^{2\pi i x_2}$. By elementary Fourier theory on $\mathbb{T}^2$, $\{e_1^k e_2^l\}_{k,l\in\mathbb{Z}}$ serves as basis for the $C^\infty(\mathbb{T}^2)$, that is

$$f = \sum_{(k,l)\in\mathbb{Z}} f_{(k,l)} e_1^k e_2^l, \quad f \in C^\infty(\mathbb{T}^2), \tag{3.2}$$

moreover the Fourier coefficients $\{f_{(k,l)}\}$ are of rapidly decay in (k,l). For $t = (t_1, t_2) \in \mathbb{R}^2$, the torus action is given by

$$\alpha_t(e_1^k e_2^l) = e^{2\pi i t_1 k + t_2 l} e_1^k e_2^l. \tag{3.3}$$

Hence, the right hand side of (3.2) is the isotypical decomposition of the function f.

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, denote $\Theta = \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$. The deformed algebra $C^\infty(\mathbb{T}^2_\theta)$ is identical to $C^\infty(\mathbb{T}^2)$ as a topological vector space with the deformed product

$$\begin{aligned} f \times_\theta g &= \sum_{r,s \in \mathbb{Z}^2} e^{\pi i \langle r, \Theta s\rangle} f_r g_s e_1^{r_1+s_1} e_2^{r_2+s_2} \\ &= \sum_{r,s \in \mathbb{Z}^2} e^{-\pi i \theta (r_1 s_2 - r_2 s_1)} f_r g_s e_1^{r_1+s_1} e_2^{r_2+s_2}, \end{aligned} \tag{3.4}$$

where $r = (r_1, r_2)$, $s = (s_1, s_2)$, f_r and g_s are the Fourier coefficients of f and g. Take $f = e_1$, $g = e_2$ we see that

$$e_1 \times_\theta e_2 = e^{-2\pi i \theta} e_2 \times_\theta e_1.$$

Therefore $C^\infty(\mathbb{T}_\theta)$ is isomorphic to the smooth noncommutative two torus[2] A_θ^∞ as a topological algebra.

Example 3.6 (Noncommutative four-sphere). Let $M = S^4 \subset \mathbb{R}^5$ be the unit four-sphere with the two torus rotation action defined in Example 3.3. Let $\mathbb{R}^5 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$ with coordinates (z_1, z_2, x), then the polynomial functions on S^4 is generated by the coordinate functions z_1, z_2 and x with the relation

$$z_1 z_1^* + z_2 z_2^* + x^2 = 1.$$

While the pull-back action on functions on generators is given by

$$z_1 \mapsto e^{2\pi i t_1} z_1, \quad z_2 \mapsto e^{2\pi i t_2} z_2, \quad x \mapsto x,$$

where $(e^{2\pi i t_1}, e^{2\pi i t_2}) \in \mathbb{T}^2$. Choose Θ to be

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \quad \theta \in \mathbb{R},$$

then the resulting multiplication $\times_\Theta$ is presented by the relations on generators: $z_1 \times_\Theta z_2 = e^{2\pi i \theta} z_2 \times_\Theta z_1$ while x_0 is central.

Example 3.7 (Higher dimensional noncommutative tori and spheres). Let Θ be a $n \times n$ skew symmetric matrix. Consider $\mathbb{T}^n$ acts on itself via translations, the resulting deformed algebra $C^\infty(\mathbb{T}^n_\Theta)$ is called a noncommutative n-torus.

To construct noncommutative spheres, we consider the rotation action of $\mathbb{T}^n$ on $\mathbb{R}^{2n}$ (resp., $\mathbb{R}^{2n+1}$):

$$t = (t_1, \ldots, t_n) \mapsto \begin{pmatrix} e^{2\pi i t_1} & & \\ & \ddots & \\ & & e^{2\pi i t_n} \end{pmatrix},$$

[2] See [14] for the definition.

respectively,

$$t = (t_1, \dots, t_n) \mapsto \begin{pmatrix} e^{2\pi i t_1} & & & \\ & \ddots & & \\ & & e^{2\pi i t_n} & \\ & & & 1 \end{pmatrix}.$$

The induced action on S^{2n-1} (resp., S^{2n}) gives rise to the noncommutative sphere $C^\infty(S^{2n-1}_\Theta)$ (resp., $C^\infty(S^{2n}_\Theta)$).

The complete geometric space of M_Θ is given by a spectral triple $(C^\infty(M_\Theta), \mathcal{H}, D)$ (or a twisted version, cf. [14, Sec. 1]). For example, if we assume that the toric Riemannian manifold M is moreover a toric spin manifold with the spinor bundle $\not{S}$ and the Dirac operator $\not{D}$, then $(C^\infty(M_\Theta), L^2(\not{S}), \not{D})$ is indeed a spectral triple. Moreover it satisfies all axioms of a noncommutative spin geometry proposed by Connes (for instance, in [11]). We shall not touch the detail construction of the spectral triple and examination of those axioms in this paper, and refer to [12, 13, 43].

3.2. Deformation of tensor calculus. For a diffeomorphism $\varphi: M \to M$, one can lift it to the tesnor bundle over M by its differential $d\varphi$ and the dual $(d\varphi)^*$:

$$d\varphi_x : T_x M \to T_{\varphi(x)} M, \quad (d\varphi_x)^* : T^*_{\varphi(x)} M \to T^*_x M, \quad \forall x \in M. \tag{3.5}$$

For vector fields X and one forms ω we have the similar pull-back action:

$$U_\varphi(X)|_p := d\varphi_{\varphi^{-1}(x)}(X|_{\varphi^{-1}(p)}), \quad U_\varphi(\omega)|_p := (d\varphi_x^{-1})^* \omega|_{\varphi^{-1}(p)}, \tag{3.6}$$

where $p \in M$. We verify that the contraction is equivariant, indeed,

$$\begin{aligned} U_\varphi(X) U_\varphi(\omega)|_p &= \omega|_{\varphi^{-1}(p)} \cdot \big(d\varphi^{-1} \circ d\varphi(X|_{\varphi^{-1}(p)})\big) \\ &= \omega|_{\varphi^{-1}(p)} \cdot X|_{\varphi^{-1}(p)} \\ &= U_\varphi(\omega \cdot X)|_p. \end{aligned}$$

One extends U_φ to all tensor fields $\Gamma(\mathcal{T}M)$ in a natural way such that the tensor product and the contraction are equivariant. In particular, if M is a toric Riemannian manifold and $\mathcal{T}M$ be the tensor bundle of all ranks over M, thus any tensor field s admits a isotypical decomposition $s = \sum_{r \in \mathbb{Z}^n} s_r$. If we treat the tensor product $\otimes$ and the contraction $\cdot$ as generalized versions (to all tensor fields) of the multiplication between functions, then (2.9) gives rise to a deformed version: $\otimes_\Theta$ and $\cdot_\Theta$ respectively.

The essential component of the tensor calculus is a connection ∇ with the Leibniz property (see Prop. 3.9). For this purpose, we required that the torus acts as affine transformations on M. Such transformations φ is define by the property of preserving linear connections. For example, let ∇ be a linear connection and φ be a affine transformation on M, for any vector field X and Y, we have $\nabla_X Y = \nabla_{U_\varphi(X)} U_\varphi(Y)$.

Rewrite this according to the notations above: $U_\varphi(\nabla Y) = \nabla U_\varphi(Y)$. In general, we have the following lemma.

Lemma 3.8. *Given an affine transformation $\varphi: M \to M$ with the pull-back action U_φ, for any tensor field $s \in \Gamma(\mathcal{T}M)$, we have:*

$$U_\varphi(\nabla s) = \nabla U_\varphi(s). \tag{3.7}$$

The proof is skipped here. See Section 4, Chapter 1 of [28] for a short exploration, the classical textbook on this subject is [30].

A straightforward consequence of Lemma 3.8 is that the covariant derivative ∇ preserves the isotypic decomposition, namely, for any tensor field with its isotypic decomposition $s = \sum_{r\in\mathbb{Z}^n} s_r$, we have:

$$(\nabla^j s)_r = \nabla^j (s)_r. \tag{3.8}$$

Now we are ready to prove the Leibniz property.

Proposition 3.9 (The Leibniz property). *Given two tensor fields on M, $s_1 \in \Gamma(\mathcal{T}_s^r M)$ and $s_2 \in \Gamma(\mathcal{T}_{s'}^{r'} M)$, we have*

$$\nabla(s_1 \otimes_\Theta s_2) = \nabla s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta \nabla s_2. \tag{3.9}$$

For the contraction map $\times_\Theta$, given a vector field X and a one-form ω, we get

$$d(X \cdot_\Theta \omega) = (\nabla X) \cdot_\Theta \omega + X \cdot_\Theta \nabla\omega. \tag{3.10}$$

Proof. The proof of (3.9) and (3.10) are almost identical, thus we only show the first one.

$$\begin{aligned}
\nabla(s_1 \otimes_\Theta s_2) &= \nabla\Bigg(\sum_{\mu,\nu\in Z^n} \chi_\Theta(\mu,\nu)(s_1)_\mu \otimes (s_2)_\nu\Bigg) \\
&= \sum_{\mu,\nu\in Z^n} \chi_\Theta(\mu,\nu)\left(\nabla(s_1)_\mu \otimes (s_2)_\nu + (s_1)_\mu \otimes \nabla(s_1)_\mu\right), \\
&= \sum_{\mu,\nu\in Z^n} \chi_\Theta(\mu,\nu)(\nabla s_1)_\mu \otimes (s_2)_\nu + \sum_{\mu,\nu\in Z^n} \chi_\Theta(\mu,\nu)(s_1)_\mu \otimes (\nabla s_2)_\nu \\
&= \nabla s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta \nabla s_2.
\end{aligned}$$

The crucial step is from the second equal sign to the third one, which requires the $\mathbb{T}^n$-equivariant property (3.8) for the connection. Also we can switch ∇ and the summation $\sum_{\mu,\nu\in Z^n}$ for ∇ is a continuous map with respect to the smooth Fréchet topologies. $\square$

3.3. Lifting the tensor calculus to the cotangent bundle T^*M. The symbol calculus of pseudo differential operator involves not only the smooth functions but also the whole pull-back tensor fields (smooth sections of pull-back tensor bundles) on the cotangent bundle T^*M:

$$\mathcal{B}^r_s M := \pi^* \mathcal{T}^r_s M \subset \mathcal{T}^r_z(T^*M), \tag{3.11}$$

where $\pi: T^*M \to M$ is the natural projection, and r, s are the contravariant and the covariant rank respectively as before, $\mathcal{B}M$ denotes the collection of tensor fields of all ranks. From analytical point of view, the only difference between sections of $\mathcal{B}^r_s M$ and ordinary tensor fields on M (sections of $\mathcal{T}^r_s M$)) is that the base point coordinates of the former tensor fields depend on (x, ξ).

As in the previous section, we assume that $\varphi: M \to M$ is an affine transformation. Keep in mind that $d\varphi_x: T_x M \to T_{\varphi(x)} M$ for all $x \in M$. The “inverse dual” gives rise a lift of φ to T^*M, denoted by φ^*

$$\varphi^*: \xi_x \in T^*_x M \mapsto (d\varphi^*_x)^{-1}(\xi_x) = (d\varphi^{-1}_x)^*(\xi_x) \in T^*_{\varphi(x)} M, \tag{3.12}$$

for all $x \in M$. For a function $f \in C^\infty(T^*M)$ on T^*M, we define $U_\varphi(f)(\xi_x) := f((\varphi^*)^{-1}(\xi_x))$ and similar to (3.6), we extend U_φ to pull-back tensor fields:

$$U_\varphi(X)(\xi_x) := d\varphi_{\varphi^{-1}x}\big(X|_{(\varphi^{-1})^*(\xi_x)}\big), \quad U_\varphi(\omega)(\xi_x) := (d\varphi^{-1}_x)^*\big(\omega|_{(\varphi^{-1})^*(\xi_x)}\big), \tag{3.13}$$

where X is a pull back vector field, hence the evaluation $X|_{(\varphi^{-1})^*(\xi_x)}$ belongs to $T_{\varphi^{-1}x} M$, thus

$$d\varphi_{\varphi^{-1}x}\big(X|_{(\varphi^{-1})^*(\xi_x)}\big) \in T^*_x M$$

as expected. Similar explanation for the one form ω. As a consequence, we can quickly verify that $U_\varphi(X) U_\varphi(\omega) = U_\varphi(X \cdot \omega)$, which means the natural pairing is equivariant. We extend U_φ to pull-back tensor fields of all ranks in such a way as before that the pointwise tensor product and contraction are both equivariant. As a result, the deformed tensor product and contraction $\otimes_\Theta$ and $\cdot_\Theta$ can be extended to all pull-back tensor fields.

Since we are on the cotangent bundle, the horizontal differential (along the fibers) and the vertical differential which is similar to the connection on the underlying manifold, appear naturally. Before that, we have to introduce another key ingredient for the calculus.

Consider a pseudo differential operator P acting on $C^\infty_c(\mathbb{R}^n)$ with symbol $p(x, \xi) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$:

$$(Pf)(x) = \int_{\mathbb{R}^n} e^{-i\xi \cdot (x-y)} p(x, \xi) f(y)\, dy\, d\eta, \quad \forall f \in C^\infty_c(\mathbb{R}^n). \tag{3.14}$$

The function $l(x, \xi, y) = \xi \cdot (y - x)$ plays a significant role in the quantization map above. Its generalization to manifolds is a smooth function $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$.

The linearity in ξ becomes the linearity of ℓ on each fiber of T^*M, but, the linearity in x has no straightforward analogy. However, when the manifold M is equipped with a connection ∇ (on the cotangent bundle of M), the linearity in x can be described as the vanishing of higher order (≥ 2) symmetrized covariant derivatives (along the y variable) $\partial^k \ell(\xi_x, y)$ at $x = y$ for any $k \geq 2$.

Another motivation is related to Hömander's perspective on pseudo differential operators in [29], which is well-explained in [17].

Definition 3.10. Let M be a smooth manifold with a connection ∇ (on the cotangent bundle). Let $\partial^j := \mathrm{Sym} \circ \nabla^j$ be the jth symmetrized covariant derivative. A phase function with respect to a given connection is a real-valued smooth function $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$ such that for fixed base point x, ℓ is linear in $\xi_x \in T_x^*M$ and such that for all ξ_x, the symmetrized covariant derivatives (along y) satisfies:

$$\partial^j \ell(\xi_x, y)|_{y=x} = \begin{cases} \xi_x, & j = 1, \\ 0, & j \neq 1. \end{cases} \tag{3.15}$$

The existence of such functions was proved in [42, Proposition 2.1]. Phase functions defined by (3.15) are, by no means unique. Geometrical, ℓ can be constructed locally using the exponential map associated to the given connection: $\ell(\xi_x, y) = \langle \xi_x, \exp_x^{-1} y \rangle$, where exp is the exponential map associated to the connection ∇ (cf. [17] and [23]). Observe that the property (3.15) is invariant under affine transformations. Namely, for an affine transformation φ on M, we define the action on $C^\infty(T^*M \times M)$ in the following way to make things equivariant:

$$U_\varphi(\ell)(\xi_x, y) = \ell((\varphi^{-1})^*\xi_x, \varphi^{-1}y), \quad \ell \in C^\infty(T^*M \times M). \tag{3.16}$$

Follows from Lemma 3.8, $U_\varphi(\nabla^j \ell) = \nabla^j U_\varphi(\ell)$. In particular, if $\ell(\xi_x, y)$ satisfies (3.15), so does $U_\varphi(\ell)$. Therefore when the torus acting as affine transformations $\mathbb{T}^n \subset \mathrm{Affine}(M)$, we start with any phase function $\tilde{\ell}$, the average over $\mathbb{T}^n$

$$\ell := \int_{\mathbb{T}^n} U_t(\tilde{\ell})\, dt \tag{3.17}$$

is $\mathbb{T}^n$-invariant. Hence from now on, we simply assume that ℓ is invariant under the torus action.

Two important consequences follows from the invariant property.

(1) The mixed derivatives are a family of invariant tensor fields on T^*M

$$D^j \nabla^i \ell(\xi_x, x) := D^j \nabla_y^i \ell(\xi_x, y)|_{y=x}.$$

Therefore after deformation, they become center elements (commute with everything else) in the deformed tensor calculus.

(2) In particular, $d\ell = \nabla\ell$ is invariant. Follows from (3.13) and (3.12), the pointwise version can be written as

$$d\ell((\varphi^{-1})^*\xi_x, \varphi^{-1}y) = (\varphi^{-1})^* d\ell(\xi_x, y). \tag{3.18}$$

The left hand side is a covector in $T^*_{\varphi^{-1}y}M$, meanwhile, in the right hand side, we have $d\ell(\xi_x, y) \in T^*_y M$ and $(\varphi^{-1})^*: T^*_y M \to T^*_{\varphi^{-1}y}M$.

We are ready to define the horizontal and the vertical differential on pull-back tensor fields on T^*M that play the role of ∂_x^α and ∂_ξ^β respectively in equation (2.29).

Definition 3.11. The vertical derivative D is the differential along the fibers of on T^*M. For any $x \in M$, $p \in \mathcal{B}M$ a pull-back tensor field, the jth derivative $D^j p$ evaluate at point $\xi_x \in T^*_x M$ gives rise to a j-linear function on $T^*_x M$, thus $D^j p$ is a contravariant j-tensor. The precise definition is given as follows: for an integer $j \geq 1$, $D^j: C^\infty(T^*M) \to \Gamma(B_0^j M)$

$$\begin{aligned}(D^j p)|_{\xi_x} \cdot (\omega_1 \otimes \cdots \otimes \omega_j) \\ = \frac{d}{ds_1}\Big|_{s_1=0} \cdots \frac{d}{ds_j}\Big|_{s_j=0} p(\xi_x + s_1\omega_1 + \cdots + s_j\omega_j),\end{aligned} \tag{3.19}$$

where $p \in \mathcal{B}M, \omega_1, \ldots, \omega_j \in T^*_x M$.

The vertical differential D is $\mathbb{T}^n$-equivariant:

Proposition 3.12. *The vertical differential D is equivariant with respect to diffeomorphisms of M. Namely, let $\varphi: M \to M$ be a diffeomorphism, U_φ is the induced pull-back action, then*

$$D^j U_\varphi(p) = U_\varphi\big((D^j p)\big). \tag{3.20}$$

Proof. We only prove the case in which $p \in C^\infty(T^*M)$ is a smooth function and $j = 1$. The general stituation can be handled in a similar way.

Let $\omega(\xi_x) = \omega(x)$ be a pull-back one form, where $\omega(x)$ is a one-form on M. We would like to show that $(DU_\varphi(p)) \cdot \omega = (U_\varphi(Dp)) \cdot \omega$. For the left hand side, according to (3.19) and (3.12):

$$\begin{aligned}\big((DU_\varphi(p)) \cdot \omega\big)\Big|_{\xi_x} &= \frac{d}{ds}\Big|_{s=0} U_\varphi(p)\big(\xi_x + s\omega|_x\big) \\ &= \frac{d}{ds}\Big|_{s=0} p\big((d\varphi^{-1})^*\big(\xi_x + s\omega|_x\big)\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\big((U_\varphi(Dp)) \cdot \omega\big)\big|_{\xi_x} &= U_\varphi\big(Dp \cdot U_{\varphi^{-1}}(\omega)\big)\big|_{\xi_x} = \big(Dp \cdot U_{\varphi^{-1}}(\omega)\big)\big|_{(d\varphi^{-1})^*\xi_x} \\ &= \frac{d}{ds}\Big|_{s=0} p\big((d\varphi^{-1})^*\xi_x + U_{\varphi^{-1}}(s\omega)|_{(d\varphi^{-1})^*\xi_x}\big).\end{aligned}$$

From (3.12), we see that

$$U_{\varphi^{-1}}(s\omega)|_{(d\varphi^{-1})^*\xi_x} = (d\varphi^{-1})^*\big(s\omega|_{d\varphi^*\circ(d\varphi^{-1})^*\xi_x}\big) = (d\varphi^{-1})^*\big(s\omega|_x\big).$$

Therefore the proof is complete. □

Finding candidates for horizontal derivative is not as straightforward as for the vertical one. As a price to pay for a coordinate free construction, the horizontal derivative involves both x and ξ in local coordinate. The two directions are linked by the phase function $\ell(\xi_x, y)$.

For a fixed $\xi_x \in T^*M$, the exterior derivative in $y \in M$: $d_y\ell(\xi_x, y)$ gives rise a one form supported near by x, then for any smooth function $p \in C^\infty(T^*M)$, the evaluation $p(d\ell(\xi_x, y))$ produces a smooth function in y, which extends the value $p(\xi_x)$ to a small neighborhood of x. Indeed, at $x = y$, $d\ell(\xi_x, y)|_{y=x} = \xi_x$. Hence the covariant derivatives $\nabla_y^j p(d\ell(\xi_x, y))$, $j = 0, 1, 2, \ldots$, make sense near by x.

Definition 3.13. Keep the notations as above. The jth horizontal covariant derivative of a symbol p is given by:

$$(\nabla^j p)(\xi_x) = \nabla_y^j p(d\ell(\xi_x, y))|_{y=x}, \quad p(\xi_x) \in C^\infty(T^*M). \tag{3.21}$$

Remarks. (1) When evaluating at $y = x$, the value in the right hand side of (3.21) does not depend on the choice of the phase function ℓ as long as the property (3.15) is fulfilled.

(2) The vertical and horizontal derivatives D and ∇ can be extended to all pull-back tensor fields (cf. (3.11)): $p(\xi_x) \in C^\infty(\mathcal{B}^r_s M)$, where (r, s) is the rank.

(3) The vertical D and horizontal ∇ derivatives commute with each other, thus the *mixed derivatives* $D^j\nabla^l p(\xi_x)$ is well-defined.

Let us prove the equivariant property for the horizontal derivative ∇.

Let $\varphi: M \to M$ be an affine transformation on M. According to (3.17), we can assume that the phase function ℓ is U_φ-invariant:

$$\ell(\widetilde{\xi}_{\tilde{x}}, \varphi^{-1}(y)) = \ell(\xi_x, y), \quad y \text{ is near by } x.$$

It follows that $d\ell$ is U_φ-invariant as well: $U_\varphi(d\ell)(\xi_x, y) = d\ell(\xi_x, y)$. Pointwisely,

$$(d\varphi^{-1})^*\big(d\ell\big((d\varphi^{-1})^*\xi_x, \varphi^{-1}y\big)\big) = d\ell(\xi_x, y). \tag{3.22}$$

Proof. The one form $d\ell(\xi_x, y)$ can be treated as a pull-back one form via the projection $\mathrm{pr}_2: T^*M \times M \to M$, similar to (3.13), the U_φ action looks like:

$$U_\varphi(d\ell)(\xi_x, y) := (d\varphi^{-1})^*\big(d\ell\big((d\varphi^{-1})^*\xi_x, \varphi^{-1}y\big)\big).$$

Let Y be a vector field on M supported near by a point x. Denote $\widetilde{x} = \varphi^{-1}(x)$, $\widetilde{y} = \varphi^{-1}(y)$ and $\widetilde{\xi}_{\widetilde{x}} = (d\varphi^{-1})^*(\xi_x)$.

$$\begin{aligned}\big(U_\varphi(d\ell)\cdot Y\big)\big|_{(\xi_x,y)} &= (d\ell)\big(\widetilde{\xi}_{\widetilde{x}}, \varphi^{-1}y\big)\cdot d\varphi^{-1}(Y|_y)\\ &= \frac{d}{ds}\Big|_{s=0}\ell\big(\widetilde{\xi}_{\widetilde{x}}, \exp_{\widetilde{y}}\left(sd\varphi^{-1}(Y|_y)\right)\big).\end{aligned}$$

Since φ is affine transformation, $\exp_{\widetilde{y}}(sd\varphi^{-1}(Y|_y)) = \varphi^{-1}(\exp_y(sY|_y))$. We continue the calculation and use the invaraint property of ℓ:

$$\begin{aligned}\big(U_\varphi(d\ell)\cdot Y\big)\big|_{(\xi_x,y)} &= \frac{d}{ds}\Big|_{s=0}\ell\big(\widetilde{\xi}_{\widetilde{x}}, \exp_{\widetilde{y}}\left(sd\varphi^{-1}(Y|_y)\right)\big)\\ &= \frac{d}{ds}\Big|_{s=0}\ell\big((d\varphi^{-1})^*(\xi_x), \varphi^{-1}\left(\exp_y(sY|_y)\right)\big)\\ &= \frac{d}{ds}\Big|_{s=0}\ell\big(\xi_x, \exp_y(sY|_y)\big) = (d\ell\cdot Y)|_{(\xi_x,y)}.\end{aligned}$$ □

Proposition 3.14. *For any integer $j \geq 1$, the jth vertical covariant derivative ∇^j is equivariant with respect to the group of affine transformations on M. Namely, for any $\varphi \in \mathrm{Affine}(M)$:*

$$U_\varphi\left(\nabla^j p\right) = \nabla^j U_\varphi(p), \tag{3.23}$$

where $p \in \mathcal{B}^r_s M$ is a pull-back tensor field over T^*M of rank (r,s).

Proof. We will verify the case when $p \in C^\infty(T^*M)$ is only a function and $j = 1$. The gerneral cases can be work out in a similar way with the Leibniz property of the connection. We also assume that ℓ is φ invariant so that (3.18) holds.

Let Y be a pull-back vector field and we will identify $Y|_{\xi_x}$ with $Y|_x$ in the rest of the computation. We need to show that

$$U_\varphi(\nabla p)\cdot Y = \nabla U_\varphi(p)\cdot Y.$$

Start with the left hand side,

$$\begin{aligned}\big(\nabla U_\varphi(p)\cdot Y\big)\big|_{\xi_x} &= \frac{d}{ds}\Big|_{s=0}U_\varphi(p)\big(d\ell(\xi_x, \exp_x sY|_x)\big)\\ &= \frac{d}{ds}\Big|_{s=0}p\big((\varphi^{-1})^*\big(d\ell(\xi_x, \exp_x sY|_x)\big)\big)\\ &= \frac{d}{ds}\Big|_{s=0}p\big(d\ell\big((\varphi^{-1})^*\xi_x, \varphi^{-1}\exp_x sY|_x\big)\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\big(U_\varphi(\nabla p)\cdot Y\big)\big|_{\xi_x} &= U_\varphi\big(\nabla p\cdot U_{\varphi^{-1}}(Y)\big)\big|_{\xi_x} = \big(\nabla p\cdot U_{(\varphi^{-1}}(Y)\big)\big|_{\varphi^{-1})^*\xi_x}\\ &= \frac{d}{ds}\Big|_{s=0}p\big(d\ell\big((\varphi^{-1})^*\xi_x, \exp_{\varphi^{-1}x} sU_{\varphi^{-1}}(Y)|_{\varphi^{-1}x}\big)\big).\end{aligned}$$

To finish the proof, we just have to observe that $U_{\varphi^{-1}}(Y)|_{\varphi^{-1}x} = d\varphi^{-1}(Y|_x)$ and

$$\exp_{\varphi^{-1}x} s d\varphi^{-1}(Y|_x) = \varphi^{-1} \exp_x sY|_x, \quad s \geq 0,$$

provided that φ is an affine transformation. □

Similar to tensor fields on M, we define the deformed contraction $\cdot_\Theta$ and tensor product $\otimes_\Theta$ between pull-back tensor fields and the Leibniz rule below follows from the equivariant property proved above.

Proposition 3.15. . *Given two pull-back tensor fields $s_1, s_2 \in \mathcal{B}M$, we have*

$$\begin{aligned} \nabla(s_1 \otimes_\Theta s_2) &= \nabla s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta \nabla s_2, \\ D(s_1 \otimes_\Theta s_2) &= D s_1 \otimes_\Theta s_2 + s_1 \otimes_\Theta D s_2. \end{aligned} \tag{3.24}$$

Same results holds for the deformed contraction $\cdot_\Theta$.

4. Pseudo differential operators on toric noncommutative manifolds

Let M be a closed manifold with a torus action $\mathbb{T}^n \subset \mathrm{Diff}(M)$ as before. We would like to apply the construction in Section 2.2 to $\Psi(M)$, the algebra of pseudo differential operators acting on $C^\infty(M)$. Namely, the algebra of pseudo differential operators on the noncommutative manifold M_Θ is just the image of $\Psi(M)$ under the deformation map π^Θ (see Definition 2.8). We shall describe a Fréchet topology for pseudo differential operators so that the convergences of the series in (2.9) make sense.

Denote by $\Psi^j(M)$, $j \in \mathbb{Z}$, consists of pseudo differential operators whose symbol $p(x, \xi)$, when localized on some open chart, belongs to $S\Sigma^j(M)$ defined in (2.29). As usual, we put

$$\Psi(M) = \cup_{d\in\mathbb{Z}} \Psi^d(M), \quad \Psi^{-\infty}(M) = \cap_{d\in\mathbb{Z}} \Psi^d(M). \tag{4.1}$$

In this paper, we only need pseudo differential operators of integer orders. The following characterization (called Beals–Cordes type in [39]) of zero-order pseudo differential operators was proved in [1] with a correction [2], also in [16, 18].

Proposition 4.1. *Let M be a closed manifold and $\Psi^0(M)$ be the space of all zero order pseudo differential operators. Given a operator $P: C^\infty(M) \to C^\infty(M)$, then $P \in \Psi^0(M)$ if and only if for any finite collection of first order differential operators $\mathfrak{F} = \{F_1, \dots, F_l\}$, we have*

$$\mathrm{ad} F_l \cdots \mathrm{ad} F_1(P) \in B(L^2(M)), \tag{4.2}$$

where $\mathrm{ad} F_j(P) = [F_j, P]$, $1 \leq j \leq l$.

The existence of elliptic operators allows us to extend the characterization above to pseudo differential operators of all integer orders.

Corollary 4.2. *Let M be a closed manifold with associated Sobolev spaces $\{\mathcal{H}_s\}_{s\in\mathbb{R}}$ and $\Psi^d(M)$ be the space of pseudo differential operators of order $d \in \mathbb{Z}$. Given a continuous linear operator $P\colon C^\infty(M) \to C^\infty(M)$ that admits a bounded extension from $\mathcal{H}_d$ to $\mathcal{H}_0$, P belongs to $\Psi^d(M)$ if and only if for any finite collection of first order differential operators $\mathfrak{F} = \{F_1, \ldots, F_l\}$, we have*

$$\operatorname{ad} F_l \cdots \operatorname{ad} F_1(P) \in B(\mathcal{H}_d, \mathcal{H}_0). \tag{4.3}$$

Corollary 4.2 leads us to consider the following family of semi-norms on $\Psi^d(M)$ indexed by $(j, \mathfrak{F})$, where $j \in \mathbb{Z}$ and $\mathfrak{F} = (F_1, \ldots, F_k)$ is a finite collection of first order differential operators. A crucial property for pseudo differential operators acting on functions is that the commutator of two operators $[P, Q]$ is of order $\operatorname{ord}(P) + \operatorname{ord}(Q) - 1$. In particular, if $P \in \Psi^d(M)$, the iterated commutator: $[F_k, \ldots, [F_1, P]]$ still belong to $\Psi^d(M)$, thus defines a bounded operator on $\mathcal{H}_{j+d} \to \mathcal{H}_j$. We define a semi-norm $\|\cdot\|_{(j,\mathfrak{F})}$ on $\Psi^d(M)$ as follows:

$$\|P\|_{(j,\mathfrak{F})} = \|[F_k, \ldots, [F_1, P]]\|_{j+d,j}\,, \tag{4.4}$$

where on the right hand side, $\|\cdot\|_{j+d,j}$ is the operator norm from $\mathcal{H}_{j+d}(M) \to \mathcal{H}_j(M)$.

Corollary 4.2 implies that any Cauchy sequence in $\Psi^d(M)$ with respect to the family of semi-norms above converges to a pseudo differential operator. Due to the compactness of M, one can find a a countable increasing subfamily in the seminorms that define the same Fréchet topology. We summarize the facts as below:

Proposition 4.3. *Keep the notations as above.*

(1) *For all $d \in \mathbb{Z}$, the semi-norms $\|\cdot\|_{(j,\mathfrak{F})}$ defined in (4.4) make $\Psi^d(M)$ into a Fréchet space.*

(2) *For $d_1 < d_2 \in \mathbb{Z}$, then the inclusion $(\Psi^{d_1}(M), \|\cdot\|_{(d_1,\mathfrak{F})}) \to (\Psi^{d_2}(M), \|\cdot\|_{(d_2,\mathfrak{F})})$ is continuous, which makes $\Psi^{d_1}(M)$ into a closed sub-Fréchet space of $\Psi^{d_2}(M)$.*

(3) *The subspace of smooth operators $\Psi^{-\infty}(M) = \cap_{s\in\mathbb{R}} \Psi^s$ is a two-sided closed ideal in $\Psi(M) = \cup_{s\in\mathbb{R}} \Psi^s$.*

It is well known that pseudo differential operators on a closed manifold M is stable under the action of the diffeomorphism group of M, more precisely, given any diffeomorphism $\varphi\colon M \to M$, let $U_\varphi\colon C^\infty(M) \to C^\infty(M)$ be the pull-back operator defined in (2.28), for any $P \in \Psi^d(M)$, then the conjugation $U_\varphi P U_\varphi^{-1}$ still belongs to $\Psi^d(M)$. Therefore the Fréchet spaces $\Psi^d(M)$ $(d \in \mathbb{Z})$ become $\mathbb{T}^n$-modules via the adjoint action:

$$t \cdot P := \operatorname{Ad}_t(P) = U_t P U_{-t}, \quad \forall t \in \mathbb{T}^n,\ P \in \Psi^d(M). \tag{4.5}$$

In order to apply the deformation machinary, we need to show that the function: $t \mapsto \mathrm{Ad}_t(P)$ is smooth in $t \in \mathbb{T}^n$ with respect to the Féchet topology on $\Psi^d(M)$ (cf. estimate (2.1)), for all $d \in \mathbb{Z}$.

Let $t = (t_1, \ldots, t_n)$ be a coordinate system on $\mathbb{T}^n$, and then $\{\partial_{t_1}, \ldots, \partial_{t_n}\}$ constitute a basis of the Lie algebra, the push-forward vector fields via the action on M are denoted by $\{X_1, \ldots, X_n\}$.

Proposition 4.4. *Keep the notations as above. Given $P \in \Psi^d(M)$ be a pseudo differential operator of order d, the operator-valued function $t \to \mathrm{Ad}_t(P)$ is smooth in t. Moreover, for any finite collection of first order differential operators $\mathfrak{F} = \{F_1, \ldots, F_k\}$ and any multi-index $\mu = (\mu_1, \ldots, \mu_j)$, one can find another finite collection of first order operators $\mathfrak{F}'$ such that:*

$$\left\|\partial_t^\mu \mathrm{Ad}_t(P)\right\|_{(s,\mathfrak{F})} \leq C \left\|P\right\|_{(s,\mathfrak{F}')}, \quad t \in \mathbb{T}^n,\ s \in \mathbb{R}, \tag{4.6}$$

where the constant C depends on $\mathfrak{F}$ and μ. The subscript $s \in \mathbb{R}$ means that the operator norm is the one from the $s + d$th to the sth Sobolev space.

Proof. Apply the product rule onto (4.5), we see that

$$\partial_{t_i}\left(\mathrm{Ad}_t(P)\right) = \mathrm{Ad}_t(\mathrm{ad}(X_i)(P)),$$

where $\mathrm{ad}(X_i)$ is the commutator: $\mathrm{ad}(X_i)(P) = [X_i, P]$. Similarly, the higher order partial derivatives are given by:

$$\partial_{t_{\mu_1}} \cdots \partial_{t_{\mu_j}}\left(\mathrm{Ad}_t(P)\right) = \mathrm{Ad}_t\big(\mathrm{ad}(X_{\mu_1}) \cdots \mathrm{ad}(X_{\mu_j})(P)\big). \tag{4.7}$$

Since the right hand side above is a pseudo differential operator, we have proved that the function $t \to \mathrm{Ad}_t(P)$ is smooth.

To show the estimate (4.6), we observe that for any vector field F,

$$\mathrm{ad}(F) \circ \mathrm{Ad}_t = \mathrm{Ad}_t \circ \mathrm{ad}(\mathrm{Ad}_{-t}(F)).$$

Thus for $\mathfrak{F} = \{F_1, \ldots, F_l\}$, a finite collection of vector fields on M,

$$\begin{aligned}
&\mathrm{ad}(F_1) \cdots \mathrm{ad}(F_l)(\partial_t^\mu \mathrm{Ad}_t(P)) \\
&\qquad = \mathrm{ad}(F_1) \cdots \mathrm{ad}(F_l)\big(\mathrm{Ad}_t(\mathrm{ad}(X_{i_1}) \cdots \mathrm{ad}(X_{i_j})(P))\big) \\
&\qquad = \mathrm{Ad}_t\big(\mathrm{ad}(\mathrm{Ad}_{-t}(F_1)) \cdots \mathrm{ad}(\mathrm{Ad}_{-t}(F_1))\mathrm{ad}(X_{\mu_1}) \cdots \mathrm{ad}(X_{\mu_j})(P)\big).
\end{aligned}$$

Since the torus action is unitary, we can drop Ad_t when computing operator norms:

$$\|\partial_t^\mu \mathrm{Ad}_t(P)\|_{(s,\mathfrak{F})} \leq C \|P\|_{(s,\mathfrak{F}')}$$

with

$$\mathfrak{F}' = \{F_1, \ldots, F_l, X_{\mu_1}, \ldots, X_{\mu_k}\}. \qquad \square$$

Follows from the smoothness of the torus action, the right hand side of the isotypical decomposition

$$P = \sum_{r \in \mathbb{Z}^n} P_r, \quad \forall P \in \Psi^d(M),$$

with $P_r = \int_{\mathbb{T}^n} \mathrm{Ad}_t(P) e^{-2\pi i r \cdot t} dt$, converges to P with respect to the Fréchet topology defined above. Moreover, for each semi-norm $\|\cdot\|_{d,\mathfrak{F}}$, the sequence $\|P_r\|_{d,\mathfrak{F}}$ is of rapidly decay in r. Fixed a skew symmetric $n \times n$ matrix Θ, the definition 2.8 of the deformation map π^Θ is extended to pseudo differential operators of all orders, namely, for all $P \in \Psi(M)$, $\pi^\Theta(P)$ is defined by

$$\pi^\Theta(P)(f) := \sum_{r,l \in \mathbb{Z}^n} P_r(f_l), \quad \forall f \in C^\infty(M). \tag{4.8}$$

Alternatively,

$$\pi^\Theta(P) := \sum_{r \in \mathbb{Z}^n} P_r U_{r.\Theta/2}, \tag{4.9}$$

where $r \cdot \Theta/2$ denotes the matrix multiplication in which r is a row vector.

For each $d \in \mathbb{Z}$, we denote by $\Psi^d(M_\Theta)$ the image of $\Psi^d(M)$ under π^Θ, $\Psi(M_\Theta)$ and $\Psi^{-\infty}(M_\Theta)$ are the union and intersection as in (4.1) respectively. Due to the continuity of the map π^Θ (cf. (2.23)) we see that the order of P is stable under the deformation:

Lemma 4.5. *Let $P \in \Psi^d(M)$ is a pseudo differential operator on M of order $d \in \mathbb{Z}$. Then $\pi^\Theta(P)\colon C^\infty(M) \to C^\infty(M)$ define in (4.9) extends to a bounded operator from $\mathcal{H}_s \to \mathcal{H}_{s-d}$ for all $s \in \mathbb{R}$.*

We summarize some crucial properties of $\Psi(M_\Theta)$ in the following proposition.

Proposition 4.6. *Let Θ be a $n \times n$ skew symmetric matrix. The filtrated algebra $\Psi(M)$ of all pseudo differential operators admits the following deformation. For any pseudo differential operators P and Q, order d_1 and d_2 respectively, the $\times_\Theta$ multiplication*

$$\times_\Theta\colon \Psi^{d_1} \times \Psi^{d_2} \to \Psi^{s_1+s_2} \tag{4.10}$$

$$(P, Q) \mapsto P \times_\Theta Q = \sum_{r,l \in \mathbb{Z}^n} e^{\pi i \langle r, \Theta l\rangle} P_r Q_l \tag{4.11}$$

is well-defined. Due to the skew symmetric property of Θ, the $\times_\Theta$ multiplication is compatible with the original $*$-operation in $\Psi(M)$, namely:

$$(P \times_\Theta Q)^* = Q^* \times_\Theta P^*. \tag{4.12}$$

Therefore $(\Psi(M), \times_\Theta)$ is a filtrated $*$-algebra. Follows from Lemma 2.10 and Proposition 2.12, we obtain that the deformation map π^Θ makes $(\Psi(M_\Theta), \cdot)$ into *a filtrated $*$-algebra, where $\cdot$ denotes the composition between operators. More explicitly, we have for any $P, Q \in \Psi(M)$,*

$$\pi^{\Theta}(P \times_{\Theta} Q) = \pi^{\Theta}(P)\pi^{\Theta}(P), \quad \pi^{\Theta}(P^*) = \pi^{\Theta}(P)^*. \tag{4.13}$$

Proof. Notice that composition of operators

$$\Psi^{s_1} \times \Psi^{s_2} \to \Psi^{s_1+s_2} : (P, Q) \mapsto PQ, \quad s_1, s_2 \in \mathbb{R},$$

satisfies the jointly continuity (cf. (2.7)) with respect to the operator norms, plus the rapidly decay property in the components of the isotypical decomposition, we conclude that the infinite sum in the right hand side of (4.11) converges with respect to the Fréchet topology.

After justifying the convergence, (4.12) is an instance of Proposition 2.5, while (4.13) is a straightforward generalization of Lemma 2.10 and Proposition 2.12. At last, the deformation π^{Θ} has an inverse $\pi^{-\Theta}$ (cf. 2.11), therefore, it is an filtrated $*$-algebra isomorphism between $(\Psi(M), \times_{\Theta})$ and $(\Psi(M_{\Theta}), \cdot)$. □

Similar to the commutative case, we define the deformed algebra of classical pseudo differential operators to be the quotient $\pi^{\Theta}(\Psi(M))/\pi^{\Theta}(\Psi^{-\infty}(M))$, where $\Psi^{-\infty}(M)$ is the space of smoothing operators. Given a pseudo differential operator P, the deformation $\pi^{\Theta}(P)$ is not a pseudo differential operator anymore. Indeed, $\pi^{\Theta}(P)$ is not pseudo-local (cf. [26, Lemma 1.2.7]) in general due to the fact that the support of P is distorted by the torus action. However, for smoothing operators, we do have $\pi^{\Theta}(\Psi^{-\infty}(M)) = \Psi^{-\infty}(M)$.

Lemma 4.7. *Let M be a compact smooth manifold without boundary. Given $P : C^{\infty}(M) \to C^{\infty}(M)$ a continuous (with respect to the Fréchet topology given by the partial derivatives in local coordinate) linear operator with the smoothing property that for any $s, t \in \mathbb{R}$, P can be extended to a bounded operator between associated Sobolev spaces: from $\mathcal{H}^s$ to $\mathcal{H}^t$. In other words, there exits a constant $C_{s,t}$ such that for all $f \in C^{\infty}(M)$*

$$\|Pf\|_s \leq C_{s,t} \|f\|_t . \tag{4.14}$$

Then the the distributional kernel of P, $K(x, y)$ over $M \times M$, belongs to $C^{\infty}(M \times M)$.

Proof. The proof can be achieved by applying the Sobolev lemma to the estimate in [26, Lemma 1.2.9] □

Let $P \in \Psi^{-\infty}(M)$ be a smoothing operator, then the estimate (4.14) holds for $\pi^{\Theta}(P)$ for all real number s. Hence the operator $\pi^{\Theta}(P)$ has a smooth Schwartz kernel by lemma 4.7 above. We summarize the fact in the proposition below:

Proposition 4.8. *Let $\mathcal{H}$ be the Hilbert space of L^2-functions on a toric Riemannian manifold M. As a $\mathbb{T}^n$-smooth subspace of $B(\mathcal{H})$, $\Psi^{-\infty}(M)$ is stable under the deformation map π^Θ, that is*

$$\pi^\Theta(\Psi^{-\infty}(M)) \subset \Psi^{-\infty}(M).$$

We define the deformation of classical pseudo differential operators $\mathrm{CL}(M_\Theta)$ to be the quotient:

$$\mathrm{CL}(M_\Theta) = \Psi^\infty(M_\Theta)/\Psi^{-\infty}(M) = \pi^\Theta(\Psi^\infty(M))/\Psi^{-\infty}(M) \tag{4.15}$$

$$= \pi^\Theta(\Psi^\infty(M)/\Psi^{-\infty}(M)) = \pi^\Theta(\mathrm{CL}(M)). \tag{4.16}$$

5. Widom's pseudo differential calculus

In the literature, symbol calculus for pseudo differential operators on manifolds was developed by pasting Fourier integral operators on open sets of $\mathbb{R}^n$. In such construction, only the leading symbol of an operator is well-defined as a function on the cotangent bundle, the rest of them depend heavily on the chosen local coordinates and the transformation rules between different coordinate systems are quite cumbersome. It is a natural question to ask for an invariant (independent of the choice of coordinates) construction of such calculus. The construction was developed long time ago, in Widom's work [42] and [41], also [4], later various modifications were suggested [17, 23, 25, 37, 38]. As a price to pay, the resulting symbol calculus is quite sophisticated geometrically and combinatorially: tensor calculus is heavily involved several multi-indices are required in the expression of the product formula of two symbols. In this section, we will follows Widom's work, focus on explaining the notations appeared in the main results and refer most of the technical proofs to [42] and [41].

In this section, M is a always a compact smooth manifold without boundary endowed with a torsion free connection ∇.

5.1. The symbol map and the quantization map.

We start with an observation on differential operators. Using the connection ∇, any differential operator P on M can be defined in a coordinate-free way as of the form of a finite sum:

$$P(f) = \sum_\alpha \rho_\alpha \cdot (\nabla^\alpha f), \tag{5.1}$$

where each ρ_α is a contravariant tensor fields such that the contraction $\rho_\alpha \cdot (\nabla^\alpha f)$ gives rise a smooth function on M. If the tensor field ρ_α is symmetric, then the polarization gives rise a polynomial on the cotangent bundle, which leads to the classical notion of symbols of differential operators. The bottom line here is that if we replace the contraction in (5.1) by the deformed version $\cdot_\Theta$, then we recovered the deformed operator $\pi^\Theta(P)$.

Let $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$ be a phase function as in Definition 3.10 with respect to the given connection ∇. Then $\ell(\cdot, y)$ can be thought as vector field on M (supported near by the point y). With the interpretation, we denote:

$$\ell(\cdot, y)^k = \ell(\cdot, y) \otimes \cdots \otimes \ell(\cdot, y), \tag{5.2}$$

which is a symmetric kth order contravariant tensor field and so the pairing

$$\nabla^k f(x_0) \cdot \ell(x_0, x)^k \tag{5.3}$$

is well-defined and, by the symmetry of $l(x_0, x)^k$,

$$\nabla^k f(x_0) \cdot \ell(x_0, x)^k = \partial^k f(x_0) \ell(x_0, x)^k, \tag{5.4}$$

where ∂^k is the symmetrization of ∇^k as before. An interesting consequence of such construction is the analogy of the Taylor's expansion formula on manifolds (cf. [42, Prop. 2.2]):

$$f(x) \backsim \sum_{j=0}^{\infty} \frac{1}{j!} \nabla^j f(x_0) \cdot \ell(x_0, x)^j, \quad f \in C^\infty(M). \tag{5.5}$$

Definition 5.1. Let M be a smooth manifold with a linear connection ∇, and $\ell(\xi_x, y) \in C^\infty(T^*M \times M)$ be a phase function in Definition 3.10 with respect to ∇. Denote by $\psi_\Delta \in C^\infty(M \times M)$ a cut-off function such that $\psi_\Delta = 1$ is equal to 1 on a small neighborhood of the diagonal and also supp ψ_Δ is still closed to the diagonal so that $\psi_\Delta(x, y) \neq 0$ implies $d_y \ell(\xi_x, y) \neq 0$ for all $\xi_x \neq 0$. For any pseudo differential operator $P : C^\infty(M) \to C^\infty(M)$, the symbol $\sigma(P)$ of P is a smooth function on T^*M:

$$\sigma(P)(\xi_x) = P \psi_\Delta(x, y) e^{i\ell(\xi_x, y)}\big|_{y=x}, \tag{5.6}$$

where the operator P acts on the y variable.

Remark. Up to smoothing operators, the symbol map σ is independent of the choice of the cut-off function ψ_Δ, and the choice of the connection ∇ and the phase function $\ell(\xi_x, y)$.

On the other hand, pseudo differential operators are obtained by quantizing symbols.

Definition 5.2. Let ψ_Δ be the cut-off function used in Definition 5.1. For any $f \in C^\infty(M)$, put $y = \exp_x Y$, the quantization map $\mathrm{Op}: S\Sigma^d(M) \to \Psi^d(M)$, $d \in \mathbb{Z}$ is defined as follows:

$$(\mathrm{Op}(p) f)(x) = \frac{1}{(2\pi)^m} \int_{T_x^*M} \int_{T_x M} e^{-i\langle \xi_x, Y\rangle} p(\xi_x) \psi_\Delta(x, y) f(\exp_x Y)\, dY\, d\xi, \tag{5.7}$$

where m is the dimension of the manifold M, $\langle \xi_x, Y\rangle$ is the canonical pairing: $T_x^*M \times T_xM \to \mathbb{R}$, and dY, $d\xi$ denote the densities that are dual to each other. Different choices of the cut-off function ψ_Δ give rise the same quantization map modular smoothing operators.

It is well known that σ and Op are inverse to each other up to smoothing operators.

As an exmaple, we compute the symbol of the scalar Laplacian Δ.

Lemma 5.3. *The symbol of the scalar Laplacian $\Delta\colon C^\infty(M) \to C^\infty(M)$ is equal to $\sigma(\Delta) = |\xi|^2$, where $|\xi|^2$ is the squared length function on T^*M.*

Proof. In local coordinates,

$$\Delta f = -(\nabla^2 f)_{ij} g^{ij}. \tag{5.8}$$

To compute the symbol, we can ignore the cut-off function ψ_Δ because Δ is a differential operator:

$$\nabla^2 e^{i\ell(\xi_x,y)} = e^{i\ell(\xi_x,y)}\big(i\nabla^2\ell(\xi_x,y) - \nabla\ell(\xi_x,y)\nabla\ell(\xi_x,y)\big).$$

Note that ∇ is torsion-free, thus ∇^2 is equal to its symmetrization ∂^2. Therefore by the definition of ℓ, at $y = x$, $e^{i\ell(\xi_x,y)} = 1$, $\nabla^2\ell = 0$ and $\nabla_j\ell(\xi_x,y) = \xi_j$. Pair $\nabla^2 e^{i\ell(\xi_x,y)}$ with the metric on T^*M: $g^{-1} = g^{ij}$, we obtain:

$$\sigma(\Delta)(\xi_x) = \xi_i\xi_j g^{ij} = |\xi|^2. \tag{5.9}$$

where (g^{ij}) is the metric tensor on T^*M. □

Given two pseudo differential operators P and Q with symbols p and q respectively, the symbol of the composition PQ is given by an asymptotic product $\sum_j a_j(p,q)$, where the $a_j(\cdot,\cdot)$ are bi-differential operators. Similar to the case of differential operators in $C^\infty(M)$ described (5.1), the bi-differential operators $a_j(p,q)$ on $C^\infty(T^*M)\times C^\infty(T^*M)$ are obtained as the contraction between mixed derivatives of p and q (cf. (3.13)) and tensor fields of suitable ranks so that the contraction produces a smooth function.

Proposition 5.4. *Keep the notations as above, for any two pseudo differential operators P ans Q with $p = \sigma(P)$ and $q = \sigma(Q)$, then $\sigma(PQ) = p \star q$ has the following asymptotic form:*

$$p \star q \backsim \sum_{j=0}^{\infty} a_j(p,q), \tag{5.10}$$

where $a_j(\cdot,\cdot)$ are bi-differential operators reducing the total degree by j, namely, for any $d, d' \in \mathbb{Z}$,

$$a_j(\cdot,\cdot)\colon S\Sigma^d(M) \times S\Sigma^{d'}(M) \to S\Sigma^{d+d'-j}(M).$$

Here $\backsim$ means that, if we truncate the sum to the first K terms, then the remainder is of order $d + d' - K$, where d and d' are the order of P and Q, respectively.

The bi-differential operators a_j are given by:

$$a_j(p,q) = \sum \frac{i^{-j}}{k!\alpha_0!\alpha_1!\cdots\alpha_k!\beta_1!\cdots\beta_k!} \big(D^{\alpha_0+\sum_1^k \alpha_k} p\big)\big(D^{\sum_1^k \beta_s}\nabla^{\alpha_0} q\big)\big(\nabla^{\alpha_1+\beta_1}\ell\big)\cdots\big(\nabla^{\alpha_k+\beta_k}\ell\big), \quad (5.11)$$

where the summation is taken over:

$$j = -\left(k - \alpha_0 - \sum_1^k(\alpha_s + \beta_s)\right) \geq 0, \quad \alpha_0 \geq 0,\ \alpha_1,\ldots,\alpha_k \geq 1,\ \beta_1,\ldots,\beta_k \geq 2.$$

The operation between all factors in (5.11) is mixed type of tensor product and contraction bewteen tensor fields. The contraction occurs between the contravariant and covariant tensors with the same index, thus eventually yields a smooth function.

Remarks. (1) The vertical and the horizontal differentials D and ∇ are defined in Section 3.3.

(2) In (5.11), the order of the multiplication of p, q and ℓ is arranged in such a way that it is works for pseudo differential operators on vector bundles.

(3) The constrains $\alpha_1,\ldots\alpha_k \geq 1$ and $\beta_1,\ldots,\beta_k \geq 2$ gives great simplification for the first a few a_j's.

(4) The first a few a_j's are listed below:

$$a_0(p,q) = pq, \quad (5.12)$$
$$a_1(p,q) = -iDp\nabla q, \quad (5.13)$$
$$a_2(p,q) = -\frac{1}{2}D^2 p\nabla^2 q - \frac{1}{2}(Dp)(D^2 q)(\nabla^3\ell). \quad (5.14)$$

5.2. Schwartz Kernels and the trace formula. The last piece we need in the symbol calculus is the trace formula of an operator from its symbol, provided the operator is of trace-class. Since the symbol only defines an operator upto smoothing operators. We measure the error by a dilation parameter $t \in [1,\infty)$.

Definition 5.5. A dilation of a pseudo differential operator P is a family of pseudo differential operators P_t with $t \in [1,\infty)$ given by dilation of the symbols functions, in local coordinate:

$$p_t(x,\xi) = p(x,\xi/t). \quad (5.15)$$

Let (M,g) be a m dimensional closed Riemannian manifold with the metric tensor g. The canonical $2m$-form on the cotangent bundle T^*M is given in local coordinates (x,ξ) by

$$\Omega = dx_1 \wedge \cdots \wedge dx_m \wedge d\xi_1 \wedge \cdots \wedge d\xi_m, \quad (5.16)$$

while $\Omega = dg d\xi_{x,g^{-1}}$ with the volume form $dg = (\det g) dx_1 \wedge \cdots \wedge dx_m$ and

$$d\xi_{x,g^{-1}} = (\det g)^{-1} d\xi_1 \wedge \cdots \wedge d\xi_m \tag{5.17}$$

defines a measure on the fiber $T^*_x M$.

Proposition 5.6. *Keep the notations as above, let P be pseudo differential operator whose order is less than m, the dimension of the manifolds such that the Schwartz kernel function $k_P(x,y)$ exists, then the Schwartz kernel of the dilation P_t (see Definition 5.5) on the diagonal is given by*

$$k_{P_t}(x,x) = \frac{t^m}{(2\pi)^m} \int_{T^*_x M} \sigma(P)\, d\xi_{g^{-1}} + O\big(t^{-N}\big), \quad \forall N \in \mathbb{N}. \tag{5.18}$$

In particular, we obtain the trace formula for P_t:

$$\operatorname{Tr}(P_t) = \frac{t^m}{(2\pi)^m} \int_{T^* M} \sigma(P)\Omega + O\big(t^{-N}\big), \quad \forall N \in \mathbb{N}, \tag{5.19}$$

where $\Omega = dg d\xi_{x,g^{-1}}$ is the canonical $2m$-form defined in (5.16).

See [41, Theorem 5.7] for the proof.

5.3. Deformation of the symbol calculus. Now let us take the torus action into account. The symbol calculus depends on the choice of a linear connection, therefore if we assume that the torus acts as affine transformations $\mathbb{T}^n \subset \operatorname{Affine}(M)$ so that the the connection is preserved as explain in Section 3, the all the constructions, such as the symbol map and the quantization map, are $\mathbb{T}^n$-equivariant. The Fréchet topologies on symbols and pseudo differential operators are constructed in such a way that both the symbol map and the quantization map are continuous. As a consequence, for any pseudo differential operator $P = \sum_{r \in \mathbb{Z}^n} P_r$ and its symbol $p = \sum_{r \in \mathbb{Z}^n} p_r$ with their isotypical decomposition, we have

$$\begin{gathered}
\sigma(P_r) \backsim \sigma(P)_r \backsim p_r, \quad \operatorname{Op}(p_r) \backsim \operatorname{Op}(p)_r \backsim P_r, \\
\sigma(P) \backsim \sum_{r \in \mathbb{Z}^n} p_r, \quad \operatorname{Op}(q) \backsim \sum_{r \in \mathbb{Z}^n} P_r.
\end{gathered}$$

Given two deformed pseudo differential operators $\pi^\Theta(P)$ and $\pi^\Theta(Q)$ with symbols p and q respectively, our goal is to find an deformed the asymptotic symbol product $\star_\Theta$ such that the error

$$\pi^\Theta(P)\pi^\Theta(Q) - \pi^\Theta(\operatorname{Op}(p \star_\Theta q))$$

belongs to $\Psi^{-\infty}(M)$. Since $\pi^{\Theta}(P)\pi^{\Theta}(Q) = \pi^{\Theta}(P \times_{\Theta} Q)$, $p \star_{\Theta} q$ should be $\sigma(P \times_{\Theta} Q)$ upto a smoothing operator. We formally compute:

$$\begin{aligned}\sigma(P \times_{\Theta} Q) &= \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu)\sigma(P_\mu Q_\nu) \backsim \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu) p_\mu \star q_\mu \\ &\backsim \sum_{\mu,\nu\in\mathbb{Z}^n} \sum_{j=0}^{\infty} \chi_{\Theta}(\mu,\nu) a_j(p_\mu, q_\nu) \backsim \sum_{j=0}^{\infty} \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu\nu) a_j(p_\mu, q_\nu) \\ &= \sum_{j=0}^{\infty} a_j(p,q)_{\Theta}.\end{aligned}$$

The summation $\sum_{j=0}^{\infty}$ simply means if we truncate the sum to first N terms, the remainder belongs to symbol of order $-J_N$, where $J_N \to \infty$ as $N \to \infty$. Therefore $\sum_{j=0}^{\infty}$ behaves like a finite sum.

It remains to compute $a_j(p,q)_{\Theta}$. Recall from (5.11), each $a_j(p,q)_{\Theta}$ is obtained by contraction and tensor product between tensor fields. Both operations are equivariant with respect to the torus action. Take a_1, for instance,

$$\begin{aligned}a_1(p,q)_{\Theta} = \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu) a_1(p_\mu, q_\nu) &= \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu)(-i)(Dp_r)(\nabla q_l) \\ \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu)(-i)(Dp)_r(\nabla q)_l &= (-i)(Dp)\cdot_{\Theta}(\nabla q).\end{aligned}$$

In general, for all $j \geq 0$, one can quickly verify that $a_j(p,q)_{\Theta}$ is of the same form as $a_j(p,q)$ in (5.11), in which the tensor product and the contraction are replaced by the deformed version $\otimes_{\Theta}$ and $\cdot_{\Theta}$ respectively.

We summarize the discussion above in the proposition:

Proposition 5.7. *Keep the notations in Proposition 5.4. Let M be a closed manifold with a n-torus action so that the previous deformation machinery applies. Let $\pi^{\Theta}(P)$ and $\pi^{\Theta}(Q)$ be the deformation of pseudo differential operators. Denote $p = \sigma(P)$ and $q = \sigma(Q)$, then*

$$\pi^{\Theta}(P)\pi^{\Theta}(Q) \backsim \pi^{\Theta}(\mathrm{Op}(p \star_{\Theta} q)),$$

with

$$p \star_{\Theta} q \backsim \sum_{j=0}^{\infty} a_j(p,q)_{\Theta},$$

where the bi-differential operators $a_j(\cdot,\cdot)$ are the deformation of $a_j(\cdot,\cdot)$ in Proposition 5.4:

$$a_j(\cdot,\cdot)_{\Theta} = \sum_{\mu,\nu\in\mathbb{Z}^n} \chi_{\Theta}(\mu,\nu) a_j(p_\mu, q_\nu), \quad \chi_{\Theta}(\mu,\nu) = e^{i\langle\mu,\Theta\nu\rangle}.$$

Precisely, $a_j(p,q)_\Theta$ can be obtained from $a_j(p,q)$ (in Equation (5.11)) by replacing the pointwise tensor product and contraction by the deformed version $\otimes$ and $\times_\Theta$ (cf. Sec. 2.3):

$$a_j(p,q)_\Theta = \sum \frac{i^{-j}}{k!\alpha_0!\alpha_1!\cdots\alpha_k!\beta_1!\cdots\beta_k!} \big(D^{\alpha_0+\sum_1^k \alpha_k} p\big) \cdot_\Theta \big(D^{\sum_1^k \beta_s}\nabla^{\alpha_0} q\big)\big(\nabla^{\alpha_1+\beta_1}\ell\big)\cdots\big(\nabla^{\alpha_k+\beta_k}\ell\big), \quad (5.20)$$

where the summation is over:

$$j = -\Big(k-\alpha_0-\sum_1^k(\alpha_s+\beta_s)\Big) \geq 0, \quad \alpha_0 \geq 0,\ \alpha_1,\dots\alpha_k \geq 1,\ \beta_1,\dots,\beta_k \geq 2.$$

Remark. Notice that the phase function ℓ is $\mathbb{T}^n$-invariant, so are the covariant derivatives $\nabla^k\ell$ ($k=0,1,2,\dots$), as a result, when $\nabla^k\ell$ are involved, the deformed and undeformed tensor product and contraction make no difference.

6. Heat kernel asymptotic

6.1. Resolvent approximation. The deformation symbol calculus can be quickly upgraded to the parametric version. Let $\pi^\Theta(P)$ be the deformation of a second order differential operator P, whose symbol equals $\sum_0^2 p_j$ with p_j of degree j ($j=0,1,2$). Let λ be the resolvent parameter that lies in some cone region in the complex plane, denote $p_2(\lambda)=p_2-\lambda$, and $p_j(\lambda)=p_j$ for $j=0,1$. We would like to solve the resolvent equation

$$(p_2(\lambda)+p_1+p_0) \star_\Theta (b_0(\lambda)+b_1(\lambda)+\cdots) \backsim 1,$$

where $b_\kappa(\lambda)$ are parametric symbols of order $-2-\kappa$, $\kappa=0,1,2,\dots$. With the symbol calculus in hand, if the inverse of $p_2(\lambda)$ exists, which will be $b_0(\lambda)$, that is

$$a_0(b_0(\lambda),p_2(\lambda))_\Theta \backsim a_0(p_2(\lambda),b_0(\lambda))_\Theta \backsim 1,$$

then b_κ can be constructed inductively as follows:

$$b_\kappa(\lambda) = -b_0(\lambda) \times_\Theta \Bigg(\sum_{\substack{-\kappa=\mu-2-\nu-j\\ \nu<\kappa}} a_j(p_\mu(\lambda),b_\nu(\lambda))_\Theta\Bigg) \quad (6.1)$$

or

$$b_\kappa(\lambda) = -\Bigg(\sum_{\substack{-\kappa=\mu-2-\nu-j\\ \nu<\kappa}} a_j(b_\nu(\lambda),p_\mu(\lambda))_\Theta\Bigg) \times_\Theta b_0(\lambda), \quad (6.2)$$

where $\mu=0,1,2$ and $\nu=0,1,2,\dots$.

We will need b_1 and b_2 later:

$$b_1 = \big((-i)Db_0 \times_\Theta \nabla p_2 + b_0 \times_\Theta p_1\big) \times_\Theta (-b_0) \tag{6.3}$$

$$\begin{aligned} b_2 = \big(&b_0 \times_\Theta p_0 + p_1 \times_\Theta b_1(\lambda) + (-i)Db_0 \times_\Theta \nabla p_1 + (-i)Db_1 \times_\Theta \nabla p_2 \\ &\tfrac{1}{2}D^2 b_0 \times_\Theta \nabla^2 p_2 - \tfrac{1}{2}(\nabla^3 \ell)Db_0 \times_\Theta D^2 p_2\big) \times_\Theta (-b_0). \end{aligned} \tag{6.4}$$

6.2. Perturbation of the scalar Laplacian Δ via a Weyl factor. From now on, we shall focus a specific family of deformed elliptic geometric differential operators which represents a family of noncommutative metrics. Ellipticity simply means we know how to inverse its leading symbol in the deformed symbol algebra.

Parallel to the work on noncommutative tori (cf. [14, 15, 19, 21]), the new family of noncommutative metrics is obtained by a noncommutative conformal change of the Riemannian metric that we started with before the deformation. In terms of spectral point of view, the new metrics we are looking at in this paper are given by a perturbation of the scalar Laplacian Weyl factor in the noncommutative coordinate algebra.

Definition 6.1. Let M be a toric Riemannian manifold as in definition (3.1) and $C^\infty(M_\Theta) = \pi^\Theta(C^\infty(M)) \subset B(\mathcal{H})$ is the algebra of smooth functions on its deformation M_Θ with respect to a skew-symmetric $n \times n$ matrix Θ. Here, $\mathcal{H}$ is the Hilbert space of L^2-functions on M as before. Denote by $C(M_\Theta)$ the C^*-algebra of the operator norm completion of $C^\infty(M_\Theta)$. A *Weyl factor* $\pi^\Theta(k)$ is an element in $C^\infty(M_\Theta) \subset C(M_\Theta)$ (that is $k \in C^\infty(M)$) which is invertible and positive.

The Weyl factor k is always of the form $k = e^h$ for some self-adjoint operator h. To be precise, let $h \in C^\infty(M)$ real-valued smooth function so that is it self-adjoint as an operator. Follows from Lemma 2.10, $\pi^\Theta(h)$ is also self-adjoint. Let $k = \exp_\Theta(h)$ so that $\pi^\Theta(k) = e^{\pi^\Theta(h)}$. In the rest of the computation, we shall drop $\pi^\Theta(\cdot)$ and apply the deformed calculus on to smooth function k and h directly.

Definition 6.2. Fixed a Weyl factor $\pi^\Theta(k)$ with $k \in C^\infty(M)$, a deformed differential operator $\pi^\Theta(P_k)$ is called of perturbed Laplacian type if its spectrum is contained in $[0, \infty)$, and $P_k \in \Psi^2(M)$ is a differential operator whose symbol is of the form $\sigma(P_k) = p_2 + p_1 + p_0$, when $j = 0, 1$, $p_j(\xi_x) \in S\Sigma^j(M)$ is polynomial in ξ of degree j. For $j = 2$, we require the leading symbol is of the form:

$$p_2(\xi_x) = k\,|\xi|^2\,,$$

where $|\xi|^2 = \langle \xi, \xi \rangle_{g^{-1}}$ is the squared length function on T^*M with respect to the given Riemannian metric g^{-1}.

Notice that the function $|\xi|^2$ is $\mathbb{T}^n$-invariant, we have $k \times_\Theta |\xi|^2 = k\,|\xi|^2$. Hence $p_2(\xi_x)$ is the symbol of the operator $\pi^\Theta(k)\Delta$. When the complex parameter λ is off the positive real line, $k\,|\xi|^2 - \lambda$ is invertible in $S\Sigma(M_\Theta, \Lambda)$, the inverse is denoted by $b_0(\xi, \lambda) = (k\,|\xi|^2 - \lambda)^{-1}$. For the smoothness of the exposition, we move the detail construction of $b_0(\xi, \lambda)$ to Appendix A.

6.3. Some estimates. Parametric pseudo differential operators is more that just a map $\lambda \mapsto P(\lambda)$. For instance, we require that our families of operators is holomorphic in λ, moreover, when parametric symbols are considered, differentiating its symbol in the parameter reduces the order of the operator:

$$\left|D_x^\alpha D_\xi^\beta D_\lambda^j p(x, \xi, \lambda)\right| \leq C_{\alpha,\beta,j} \big(1 + |\xi|^2 + |\lambda|\big)^{(d-|\beta|-2j)/2}. \tag{6.5}$$

As a result, the operator norm has some control in the resolvent parameter:

Lemma 6.3. *Given any positive integer k, one can find another integer $K > 0$ such that for any $\pi^\Theta(Q(\lambda)) \in \Psi^{-K}(M_\Theta, \Lambda)$, the operator norm of $\pi^\Theta(Q(\lambda)) \in B(\mathcal{H}^{-k}, \mathcal{H}^k)$ has the estimate:*

$$\left\|\pi^\Theta(Q(\lambda))\right\|_{-j,j} \leq C\big(1 + |\lambda|\big)^{-j}. \tag{6.6}$$

Proof. This fact is well known, see [26, Lemma 1.7.1(b)] for instance, for parametric pseudo differential operators on closed smooth manifolds. Namely, we have $Q(\lambda) \in \Psi^{-K}$ implies

$$\left\|Q(\lambda)\right\|_{-j,j} \backsim O\big(|\lambda|^{-j}\big) \quad \text{as } |\lambda| \to \infty.$$

Now we take the torus action into account, notice that for any partial derivative in $t \in \mathbb{T}^n \mapsto \mathrm{Ad}_t(Q(\lambda))$, the resulting operator $\partial_t^\mu \mathrm{Ad}_t(Q(\lambda))$ with a multi-index μ, still lies in Ψ^{-K} if $Q(\lambda)$ does. In particular,

$$\left\|\partial_t^\mu \mathrm{Ad}_t(Q(\lambda))\right\|_{-j,j} \backsim O\big(|\lambda|^{-j}\big) \quad \text{as } |\lambda| \to \infty,\ \forall t \in \mathbb{T}^n.$$

From the standard Fourier theory on torus,

$$\left\|\pi^\Theta(Q)\right\|_{-j,j} \leq C \sup_{t \in \mathbb{T}^n} \left\|\partial_t^\mu \mathrm{Ad}_t(Q(\lambda))\right\|_{-j,j},$$

provided that the total order of the partial derivative μ is greater that the dimension of the torus. The proof is complete. □

Let us consider a smooth family $k_s = \exp_\Theta(sh)$ of Weyl factors, where $s \in [0, 1]$ and h is a real valued function on M so that it is a self-adjoint operator. Recall that we have dropped the deformation map $\pi^\Theta(\cdot)$ when dealing with k and its logarithm h, the calculation is taken in the deformed version. So we will simply write $k_s = e^{sh}$ in the rest of the computation.

We fix a h and consider the family of Laplacian operators perturbed by $k_s = e^{sh}$:

$$P_s = k_s \Delta + \text{lower order terms}, \tag{6.7}$$

where Δ is the scalar Laplacian operator. In our application, the lower order terms involve the Weyl factor k upto its second covariant derivative. In particular, the symbol of P_s depends smoothly in s and differentiating in s do not increase the order of the symbol (or the operator).

Denote the resolvent by $R(\lambda, s) = (P_s - \lambda)^{-1}$. Base on the fact that the spectrum of P_s is contained in $[0, \infty)$ for all $s \in [0, 1]$, one can repeat the argument in [26, Lemma 1.6.6] to show that for any integer $j \geq 0$,

$$\left\| R(\lambda, s) \right\|_{-j,j} \backsim O\big(|\lambda|^{l(j)}\big), \quad \text{as } |\lambda| \to \infty.$$

The power $l(j)$ is positive and uniform with respect to s.

The resolvent approximation in Section 6.1 gives us a sequence of deformed pseudo differential operators $\{R_j(\lambda, s)\}_{j=0}^{\infty}$ such that the difference $R - R_j$ is of order $-j-1$ for each j. With Lemma 6.3, arguing as in [26, Lemma 1.7.2], we conclude that for any integer $j \geq 0$,one can choose l larger enough so that

$$\left\| R(\lambda, s) - R_l(\lambda, s) \right\|_{-j,j} \backsim O\big(|\lambda|^{-j}\big), \quad \text{as } |\lambda| \to \infty. \tag{6.8}$$

We also need the variation in s:

$$\begin{aligned} \frac{d}{ds}\big(R(\lambda, s) - R_l(\lambda, s)\big) &= R(\lambda, s)\big(I - (P_s - \lambda) R_l(\lambda, s)\big) \\ &\quad - R(\lambda, s) \frac{dP_s}{ds} R(\lambda, s)\big(I - (P_s - \lambda) R_l(\lambda, s)\big). \end{aligned}$$

When apply the operator norm $\|\cdot\|_{-j,j}$ ($j \geq 0$) on both sides, the error term has the estimate:

$$\left\| I - (P_s - \lambda) R_l(\lambda, s) \right\|_{-j,j} \backsim O\big(|\lambda|^{-j}\big),$$

provide l large enough. As a result,

$$\left\| \frac{d}{ds}\big(R(\lambda, s) - R_l(\lambda, s)\big) \right\|_{-j,j} \backsim O\big(|\lambda|^{-j}\big), \quad \text{as } |\lambda| \to \infty. \tag{6.9}$$

For the complete argument, see [6, Eqs. (3.12)–(3.14)].

6.4. Heat kernels and the variation. Notice that the spectrum of P_s is contained in $[0, \infty)$, therefore the heat operator e^{-tP_s} can be defined by a contour integral:

$$e^{-tP_s} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (P_s - \lambda)^{-1}\, d\lambda, \tag{6.10}$$

where C is a curve in the complex plane that circle around $[0,\infty)$ in such a way that

$$e^{-ts} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(s-\lambda)^{-1}\,ds, \quad \forall s \geq 0.$$

If one replaces the resolvent $(P_s-\lambda)^{-1}$ by its lth approximation $R_l(\lambda,s)$ in the integral above, denote $E_l(t,s) = \int_C e^{-t\lambda}R_l(\lambda,s)\,d\lambda$, then the estimate (6.8) gives rise to

$$\left\|e^{-tP_s} - E_l(t,s)\right\|_{-j,j} \backsim o\big(t^j\big) \quad \text{as } t \to 0, \tag{6.11}$$

provided l large enough. In order to study the trace, it is easier to consider the associated pseudo differential operators before the deformation, for which the Schwartz kernel functions make sense. Write e^{-tP_s} and $E_l(t,s)$ as

$$e^{-tP_s} = \pi^\Theta\big(\widetilde{E}(t,s)\big), \quad E_l(t,s) = \pi^\Theta\big(\widetilde{E}_l(t,s)\big),$$

where $\widetilde{E}$ and $\widetilde{E}_l$ are smoothing operators (means of order $-\infty$) on M. Thanks to the continuity of π^Θ, (6.11) implies that: $\|\widetilde{E}_s - \widetilde{E}_l(\lambda,s)\|_{-j,j} \backsim o(t^j)$, in terms of their Schwartz kernel functions (see [26, Lemma 1.7.3]): we have

$$\left|H_{\widetilde{E}_s}(x,y,t) - H_{\widetilde{E}_l(t,s)}(x,y,t)\right|_{L^2_j} \backsim o\big(t^j\big) \quad \text{as } t \to 0.$$

In particular, for the L^2-trace,

$$\operatorname{Tr}\big(\widetilde{E}_s(t) - \widetilde{E}_l(\lambda,s)\big) \backsim o\big(t^j\big), \quad \text{as } t \to 0.$$

Since the trace is preserved under the deformation process (Lemma 2.13), we have proved that $\operatorname{Tr} e^{-tP_s}$ has an asymptotic expansion as $t \to 0$ given by $\operatorname{Tr} E_l(t,s)$, $l = 0,1,2,\ldots$. To be precise: for given integer $j \geq 0$, one can find l large enough such that

$$\operatorname{Tr}\big(e^{-tP_s} - E_l(t,s)\big) \backsim o\big(t^j\big), \quad \text{as } t \to 0. \tag{6.12}$$

The variation in s can be performed in a similar fashion. Start with (6.9), we have the operator norm estimate:

$$\left\|\frac{d}{ds}\big(e^{-tP_s} - E_l(t,s)\big)\right\|_{-j,j} \backsim o\big(t^j\big) \quad \text{as } t \to 0,$$

which leads to the Schwartz kernel estimate:

$$\left|\frac{d}{ds}\big(H_{\widetilde{E}_s}(x,y,t) - H_{\widetilde{E}_l(t,s)}(x,y,t)\big)\right|_{L^2_j} \backsim o\big(t^j\big) \quad \text{as } t \to 0.$$

Finally we can conclude that the heat kernel asymptotic can be differentiated term by term in the parameter s:

$$\operatorname{Tr}\left(\frac{d}{ds}\big(e^{-tP_s} - E_l(t,s)\big)\right) \backsim o\big(t^j\big), \quad \text{as } t \to 0, \tag{6.13}$$

provide l large enough. More details can be found in [6]. Again, (6.13) means that the heat kernel asymptotic defined by $E_l(t,s)$ can be differentiate term by term in s.

To complete the formula of the heat kernel asymptotic, it remains to compute the trace: $\mathrm{Tr}(E_l) = \mathrm{Tr}(\widetilde{E}_l)$ for each $l = 0, 1, 2, \ldots$. Let $\{b_j(\lambda, s)\}_{j=0}^{\infty}$ be the sequence of symbols defined inductively in (6.1) with respect to the perturbed Laplacian P_s. Although the symbols b_j are constructed out of deformed calculus, the following homogeneity (of degree $-2 - j$) in (ξ, λ) is preserved as in the classical situation:

$$b_j(c\xi, c^2\lambda) = c^{-2-j} b_j(\xi, \lambda), \quad \forall c > 0, \ j = 0, 1, 2 \ldots. \tag{6.14}$$

As a consequence (see [26, Section 1.7]), the following symbols

$$\widetilde{e}_j(t, s) = \frac{1}{2\pi i} \int_C e^{-t\lambda} b_j(\lambda, s)\, d\lambda, \quad j = 0, 1, 2 \ldots, \tag{6.15}$$

belong to $S\Sigma^{-\infty}$. We define $\widetilde{E}_l = \sum_{j=1}^{l} \mathrm{Op}(e_j)$ and $E_l = \pi^{\Theta}(\widetilde{E}_l)$. When taking the homogeneity property (6.14) into account, the trace formula in Proposition 5.6 becomes:

$$H_{b_j(\lambda)}(x, x, \lambda) = \frac{1}{(2\pi)^m} \int_{T_x^* M} b_j(\xi_x, \lambda)\, d\xi_{x, g^{-1}} + O\big(|\lambda|^{-N}\big)$$

for all positive integer N. Here $H_{b_j(\lambda)}$ is the Schwartz kernel of the quantization operator of b_j. After the contour integral, we get, for any positive integer N:

$$H_{e_j(\lambda)}(x, x, t) = \frac{1}{2\pi i} \frac{1}{(2\pi)^m} \int_C e^{-t\lambda} \int_{T_x^* M} b_j(\xi_x, \lambda)\, d\xi_{x, g^{-1}}\, d\lambda + o\big(t^{-N}\big),$$
$$\text{as } t \to 0.$$

Use the homogeneity property (6.14) again, one can perform a substitution $t\lambda \to \lambda$ to show that

$$\int_C e^{-t\lambda} \int_{T_x^* M} b_j(\xi_x, \lambda)\, d\xi_{x, g^{-1}}\, d\lambda = t^{(j-n)/2} \int_C e^{-\lambda} \int_{T_x^* M} b_j(\xi_x, \lambda)\, d\xi_{x, g^{-1}}\, d\lambda.$$

Finally, we have proved that

$$H_{\widetilde{E}_l}(x, x, t) = \sum_{j=0}^{l} t^{(j-n)/2} V_j(x) + o\big(t^{-N}\big), \quad \forall N \in \mathbb{N},$$

as $t \to 0$, with

$$V_j(x) = \frac{1}{2\pi i} \frac{1}{(2\pi)^m} \int_C e^{-\lambda} \int_{T_x^* M} b_j(\xi_x, \lambda)\, d\xi_{x, g^{-1}}\, d\lambda.$$

We summarize the long discussion above into the theorem below.

Theorem 6.4. *Let P_s with $s \in [0, \varepsilon)$ for small $\varepsilon > 0$ be a family of perturbed Laplacians defined in (6.7). For any $f = \pi^{\Theta}(\tilde{f})$ with $\tilde{f} \in C^\infty(M)$, viewed as a deformed zero-order pseudo differential operator by left-multiplication, then*

$$\operatorname{Tr}\left(fe^{-tP_s}\right) \backsim \sum_{j=0}^{\infty} t^{(j-m)/2} V_j(f, P_s) \tag{6.16}$$

where upto a factor $(2\pi)^{-m}$, $m = \dim M$,

$$\begin{aligned} V_j\left(f, \pi^{\Theta}(P_k)\right) &= \int_M f(x) V_j(x)\, dg = \int_{T^*M} \left(\frac{1}{2\pi i} \int_C e^{-\lambda} b_j(\xi_x, \lambda)\, d\lambda\right) \Omega, \\ &= \int_M \int_{T_x^*M} \left(\frac{1}{2\pi i} \int_C e^{-\lambda} b_j(\xi_x, \lambda)\, d\lambda\right) d\xi_{x,g^{-1}}\, dg, \end{aligned} \tag{6.17}$$

here Ω is the canonical volume form on T^*M defined in (5.16) and C is the contour that defines the heat operator.

Moreover, the heat asymptotic (6.16) can be differentiated term by term in s:

$$\frac{d}{ds} \operatorname{Tr}\left(fe^{-tP_s}\right) \backsim \sum_{j=0}^{\infty} t^{(j-m)/2} \frac{d}{ds} V_j(f, P_s). \tag{6.18}$$

We end this section with a quick application of the technical fact (6.18).

6.5. Zeta functions and conformal indices. The original notion of conformal index for manifolds was introduced in [6], which admits a generalization in the setting of spectral triples [14]. As an instance, let us state the result for toric noncommutative manifolds. Denote $k_s = e^{sh}$ with $s \in [0, 1]$ and $h = h^* \in C^\infty(M_\Theta)$.

Theorem 6.5. *Consider the perturbed Laplacian in (6.7) without lower order terms $P_s = k_s \Delta$. We assume that $m = \dim M$ is even and denote $V_j(P_s) := V_j(1, P_s)$ defined in (6.17), then*

$$\frac{d}{ds} V_m(P_s) = 0, \quad \forall s \in [0, 1].$$

In particular, at $j = m$, the coefficient

$$V_m(\Delta) = V_m(P_0) = V_m(P_1) = V_m(k\Delta), \tag{6.19}$$

that is the value $V_m(k\Delta)$ does not depend on the Weyl factor k.

After verifying the technical assumption that the heat kernel asymptotic can be differentiated term by term in s (Eq. (6.18)), the theorem is a special case of Section 2.1 in [14].

The result can be rephrased using zeta functions. Again, let $P_k = k\Delta \in \Psi^2(M_\Theta)$. To define the complex power P_k^z where z is a complex number, we need to remove zero from the spectrum of P_k. To do so, we consider

$$\widetilde{\mathcal{P}}_k := P_k(I - \mathcal{P}_{\ker P_k}),$$

where $\mathcal{P}_{\ker P_k}$ is the projection on to the kernel of P_k. For $z \in \mathbb{C}$, the complex power is defined by the contour integral

$$P_k^z := \frac{1}{2\pi i} \int_C \lambda^z (\widetilde{\mathcal{P}}_k - \lambda)^{-1} \, d\lambda.$$

For $\Re z$ large enough, P_k^{-z} is of trace-class, so the corespondent zeta function is well-defined:

$$\zeta(P_k, z) = \operatorname{Tr} P_k^{-z}.$$

It is well known that, for instance, see [26, Lemma 1.10.1], the heat kernel asymptotic (6.16) gives rise to a meromorphic extension of $\zeta(P_k, z)$ to $\mathbb{C}$ with at most simples poles. Moreover, the coefficients $V_j(P_k)$ correspond to the value or the residue of the zeta function at $z_j = (m - j)/2$, $j = 0, 1, 2, \ldots$, where $m = \dim M$. In particular, at $z = 0$, the zeta function is regular and

$$\zeta(k\Delta, 0) = V_j(k\Delta) - \dim \ker k\Delta.$$

Since k is invertible, $\dim \ker k\Delta = \dim \ker \Delta$, combine this with (6.19), we conclude:

Theorem 6.6. *Let k be a Weyl factor, then the zeta function of $k\Delta$ at zero is independent of k, that is*

$$\zeta_{k\Delta}(0) = \zeta_\Delta(0). \tag{6.20}$$

7. Modular curvature

Definition 7.1. Let P_k be a perturbed Laplacian via a Weyl factor k as before, which stands for a noncommutative metric for the noncommutative manifold M_Θ. Via analogy, we define the associated modular curvature to be the functional density of the second heat coefficient. Precisely, the modular curvature $\mathcal{R} = \pi^\Theta(\widetilde{\mathcal{R}})$ with with $\widetilde{\mathcal{R}} \in C^\infty(M)$, is defined by the property: for any $f = \pi^\Theta(\tilde{f})$,

$$V_2(f, P_k) = \int_M \tilde{f} \times_\Theta \widetilde{\mathcal{R}}, \quad \forall f \in C^\infty(M).$$

We have shown in Theorem 6.4 that

$$\mathcal{R}(x) = (2\pi)^{-\dim M} \int_{T_x^* M} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\lambda} b_2(\xi_x, \lambda) \, d\lambda \, d\xi_{x, g^{-1}}, \quad x \in M,$$

where b_2 is the second term in the resolvent approximation of $(P_k - \lambda)^{-1}$. We will process the integration as follows:

$$\widetilde{\mathcal{R}}(x) = \int_0^\infty \frac{1}{2\pi i} \int_{\mathcal{C}} \left(\int_{S_x^* M} e^{-\lambda} b_2(\xi_x, \lambda) \, d\omega_{x,g^{-1}} \right) d\lambda (r^{m-1} \, dr), \tag{7.1}$$

where $d\omega_{x,g^{-1}}$ is the volume form on the unit cosphere inside $T_x^* M$ associated to the metric g^{-1} and m is the dimension of the underlying manifold.

At the end, we showed that the results agree with the previous work [14, 32] and [22]. In the rest of the computation, the tensor calculus is always the deformed version, we will suppress all $\otimes_\Theta$, $\times_\Theta$ to simplify the notations.

7.1. Set up and the complete expression of the b_2 term. We perform the calculation with respect to a perturbed Laplacian $\pi^\Theta(P_k)$ (cf. Definition 6.2) whose symbol is of the form:

$$\sigma(\pi^\Theta(P_k)) = k\,|\xi|^2 + p_1(\xi_x) + p_0(\xi_x). \tag{7.2}$$

Here $p_1(\xi_x)$ and $p_0(\xi_x)$ are the degree one and zero parts respectively whose explicit expressions will be determined in specific examples. Let us consider, at first, the simplest perturbation $\pi^\Theta(k)\Delta$, whose symbol is the leading term of all the perturbed Laplacians appeared in the previous work [14, 21] and [22]:

Lemma 7.2. *The symbol of the perturbed Laplacian $k\Delta$ is equal to*

$$k \times_\Theta |\xi|^2 = k\,|\xi|^2 \,,$$

where $|\xi|$ is the length function on T^*M.

Proof. We have seen in Lemma 5.3 that $\sigma(\Delta) = |\xi|^2$. Since k is independent of the fiber direction variable ξ, $a_j(k, \cdot) = 0$ for all $j > 0$, thus,

$$k \star_\Theta |\xi|^2 = a_0(k, |\xi|^2)_\Theta = k \times_\Theta |\xi|^2 = k\,|\xi|^2 \,,$$

the last equal sign holds because $|\xi|^2$ is $\mathbb{T}^n$-invariant. □

Denote by

$$p_2(\xi_x, \lambda) = k\,|\xi|^2 - \lambda \tag{7.3}$$

the parametric leading symbol, and its inverse in the deformed algebra $C^\infty(T^*M_\Theta)$.

$$b_0(\xi_x, \lambda) = (k\,|\xi|^2 - \lambda)^{-1}.$$

The construction of b_0 is explained in Appendix A in detail. Recall (6.3) and (6.4), the explicit expressions of b_1 and b_2 can be calculated by repeatedly applying the Leibniz property of D and ∇. We shall leave the lengthy calculation in Appendix B and C, instead, we start with Proposition C.7: in (7.1), the integral over the unit sphere $\int_{S^{m-1}} b_2 d\sigma_{S^{m-1}}$ is equal to, up to an overall factor $\mathrm{Vol}(S^{m-1})$:

$$\begin{aligned}
&\tfrac{4}{m}2b_0^3k^2(\nabla k)b_0(\nabla k)b_0\,|\xi|^6\,g^{-1} - (2+\tfrac{4}{m})b_0^2k(\nabla k)b_0(\nabla k)b_0\,|\xi|^4\,g^{-1}\\
&+\tfrac{4}{m}b_0^2k(\nabla k)b_0^2k(\nabla k)b_0\,|\xi|^6\,g^{-1} - b_0^2k(\nabla^2 k)b_0\,|\xi|^2\,g^{-1}\\
&+\tfrac{4}{m}b_0^3k^2(\nabla^2 k)b_0\,|\xi|^4\,g^{-1} + \tfrac{1}{m}\tfrac{2}{3}b_0^2k^2\mathcal{S}_\Delta b_0\,|\xi|^2\,.
\end{aligned} \tag{7.4}$$

7.2. Integration in λ. As explained in [14, Section 6], the resolvent parameter can be taken to be -1 due to the homogeneity of the symbol. The argument works in higher dimensions in the following way.

Let $m = \dim M$ be even. We fix a point $x \in M$ and identify $T_x^*M \cong \mathbb{R}^m$ so that the Riemannian metric is the usual Euclidean metric. Put

$$b_2(r,\lambda) := \int_{S^{m-1}} b_2(\xi,\lambda)\,d\sigma_{S^{m-1}}, \tag{7.5}$$

where $r \in [0,\infty)$ and $\lambda \in \Lambda$, a cone region in $\mathbb{C}$ in which $\sqrt{\lambda}$ is well-defined. We would like to switch the order of $d\lambda(r^{m-1}\,dr)$ in (7.1). Integration by parts gives us: for any integer $j > 0$,

$$\begin{aligned}
&\int_0^\infty \frac{1}{2\pi i}\int_C e^{-\lambda}b_2(r,\lambda)\,d\lambda(r^{m-1}\,dr)\\
&\qquad\qquad = \int_0^\infty \frac{1}{2\pi i}\int_C e^{-\lambda}\frac{d^j}{d\lambda^j}b_2(r,\lambda)\,d\lambda(r^{m-1}\,dr).
\end{aligned}$$

Recall that by saying "$b(r,\lambda)$ is homogeneous in $(r,\sqrt{\lambda})$ of degree $d \in \mathbb{Z}$" we mean that for any $c > 0$, $b(cr, c^2\lambda) = c^d b(r,\lambda)$. Observe that differentiating in λ lower the homogeneity by 2, thus if we take j to be $j_0 = m/2-1$, the smallest integer so that the homogeneity of $\frac{d^j}{d\lambda^j}b_2(r,\lambda)$ is strictly less than $-m$, then integral

$$B_{j_0}(\lambda) = \int_0^\infty \frac{d_0^j}{d\lambda_0^j}b_2(r,\lambda)(r^{m-1}\,dr)$$

exists and $B_{j_0}(\lambda)$ is homogeneous in λ of degree -1, that is, $B_{j_0}(c\lambda) = c^{-1}B_{j_0}(\lambda)$ for any $c > 0$. Indeed,

$$\begin{aligned}
B_{j_0}(c\lambda) &= \int_0^\infty \Big(\frac{d^{j_0}}{d\lambda^{j_0}}b_2\Big)\Big(\sqrt{c}\frac{r}{\sqrt{c}}, c\lambda\Big)r^{m-1}\,dr\\
&= \int_0^\infty c^{\frac{-4-2j_0}{2}}\Big(\frac{d^{j_0}}{d\lambda^{j_0}}b_2\Big)\Big(\frac{r}{\sqrt{c}},\lambda\Big)r^{m-1}\,dr\\
&= c^{-2-j_0+m/2}\int_0^\infty \Big(\frac{d^{j_0}}{d\lambda^{j_0}}b_2\Big)(r,\lambda)r^{m-1}\,dr,
\end{aligned}$$

and $-2-j_0+m/2 = -1$ since $j_0 = m/2-1$.

Lemma 7.3. *Keep the notations as above. Assume that* $m = \dim M$ *is even and set* $j_0 = m/2 - 1$. *We have*

$$\frac{1}{2\pi i}\int_C e^{-\lambda} B_{j_0}(\lambda)\, d\lambda = B_{j_0}(-1). \tag{7.6}$$

Therefore:

$$\frac{1}{2\pi i}\int_C e^{-\lambda}\Big(\int_0^\infty b_2(r,\lambda) r^{m-1}\, dr\Big)\, d\lambda = \int_0^\infty \frac{d_0^j}{d\lambda_0^j}\Big|_{\lambda=-1} b_2(r,\lambda) r^{m-1}\, dr. \tag{7.7}$$

Proof. Let $C \in \mathbb{C}$ be a contour around $[0,\infty)$ used before to define the heat operator via holomorphic functional calculus, then

$$\frac{1}{2\pi i}\int_C e^{-t\lambda}\frac{1}{s-\lambda}\, d\lambda = e^{-ts}, \quad \forall s,t \geq 0. \tag{7.8}$$

Upto homotopy equivalence in the region $\mathbb{C}\setminus[0,\infty)$, we can force the contour C to be contained in a cone region $U_\delta = \{z \in \mathbb{C} \mid \pi - \delta < \arg z < \pi + \delta\}$ for any $\delta > 0$. Write $\lambda = re^{i\theta}$ in its polar form, due to the homogeneity:

$$B_{j_0}(\lambda) = \frac{1}{re^{i\theta}} e^{i\theta} B_{j_0}\big(e^{i\theta}\big) = \frac{1}{\lambda} e^{i\theta} B_{j_0}\big(e^{i\theta}\big),$$

where θ can be chosen to be contained in $(\pi-\delta, \pi+\delta)$ for any $\delta > 0$, hence:

$$\frac{1}{2\pi i}\int_C e^{-t\lambda} B_{j_0}(\lambda)\, d\lambda = \Big(\frac{1}{2\pi i}\int_C e^{-t\lambda}\Big(-\frac{1}{\lambda}\Big)\, d\lambda\Big) B_{j_0}(-1) = B_{j_0}(-1).$$

Here we have used (7.8) to conclude that $\frac{1}{2\pi i}\int_C e^{-t\lambda}(-\frac{1}{\lambda})\, d\lambda = e^0 = 1$. □

7.3. The rearrangement lemma. Integration in r is handle by the rearrangement lemma which will be explained below. We will feel free to use the notations in [32] and [31]. From now on, the parameter λ is taken to be -1. Put $r = |\xi|$. After a substitution $r \mapsto r^2$, the summands in (7.4) contain two types:

$$kf_0(rk)\rho f_1(rk) \quad \text{or} \quad kf_0(rk)\rho_1 f_1(rk)\rho_2 f_2(rk), \tag{7.9}$$

here k is the Weyl factor and f_j's are some smooth functions on $\mathbb{R}_+$, while ρ_j's are tensor fields over M on which $C^\infty(M_\Theta)$ acts from both sides. Introduce the modular operator $\triangle$:

$$\triangle\,(\rho) := k^{-1}\rho k, \tag{7.10}$$

then the rearrangement lemma (cf. [14, Lemma 6.2]; [31, Corollary 3.9]) yields:

$$\int_0^\infty kf_0(rk)\rho f_1(rk)\, dr = \mathcal{K}(\triangle)(\rho), \tag{7.11}$$

with
$$\mathcal{K}(s) = \int_0^\infty f_0(r) f_1(rs)\, dr, \quad s \in (0,\infty). \tag{7.12}$$

For the second type,

$$\int_0^\infty k f_0(rk)\rho_1 f_1(rk)\rho_2 f_2(rk)\,dr = \mathscr{G}(\triangle_{(1)}, \triangle_{(2)})(\rho_1 \cdot \rho_2), \tag{7.13}$$

with $$\mathscr{G}(s,t) = \int_0^\infty f_0(r) f_1(rs) f_2(rst)\,dr, \quad s,t \in (0,\infty), \tag{7.14}$$

where $\triangle_{(j)}$ indicates that $\triangle$ acts on the jth factor with $j = 1, 2$. We introduce the following families of modular curvature functions:

$$K_{(p,q)}(s,r) = r^{p+q-2}(r+1)^{-p}(sr+1)^{-q}, \tag{7.15}$$

$$H_{(p,q,l)}(s,t,r) = r^{p+q+l-2}(r+1)^{-p}(sr+1)^{-q}(str+1)^{-l}, \tag{7.16}$$

where the parameter $r \in [0,\infty)$ and the arguments $s,t \in (0,\infty)$. For instance, when applying the lemma to $b_0^2 k(\nabla^2 k) b_0 |\xi|^2 g^{-1}$ in (7.4), the associated function is $K_{(2,1)}(s,r)$. For the term $b_0^3 k^2 (\nabla k) b_0 (\nabla k) b_0 |\xi|^6 g^{-1}$, the function is $H_{(3,1,1)}(s,t,r)$.

7.4. Modular curvature on noncommutative two tori. Let $M = \mathbb{T}^2$ be the two torus with the Euclidean metric, thus $m = \dim M = 2$ and the scalar curvature function $\mathcal{S}_\Delta = 0$. Recall that the modular curvature $\mathcal{R}$ is obtained by applying $\int_0^\infty (\cdot) r\,dr$ to all terms in (7.4) while the resolvent parameter λ is replaced by -1. After a substitution $r \mapsto r^2$, $r\,dr$ becomes $dr/2$. View $r \mapsto k|\xi|^2$, then for example, the b_0 yields the function $f_1(r) = (r+1)^{-1}$. We would like to apply (7.9) to

$$2b_0^3 k^2 (\nabla^2 k) b_0 |\xi|^4 g^{-1} = k^{-1} 2k b_0^3 k^2 (\nabla^2 k) b_0 |\xi|^4 g^{-1},$$

therefore

$$k^{-1}\int_0^\infty \left(2k b_0^3 k^2 (\nabla^2 k) b_0 |\xi|^4 g^{-1}\right)(dr/2) = \mathcal{K}_1(\triangle)(\nabla^2 k) g^{-1}$$

with $\mathcal{K}_1(s) = \int_0^\infty K_{(3,1)}(s,r)\,dr$. Other terms in (7.4) can be handled in a similar way, for instance, $b_0^2 k(\nabla k) b_0^2 k (\nabla k) b_0 |\xi|^6 g^{-1}$ a term yields a modular function with two variables: we first bring the k in the middle of ∇k in front via the modular operator, that is: $(\nabla k)k = k \triangle (\nabla k)$, so we rewrite

$$b_0^2 k(\nabla k) b_0^2 k(\nabla k) b_0 |\xi|^6 g^{-1} = k^{-2} k b_0^2 k^3 \triangle (\nabla k) b_0^2 (\nabla k) b_0 |\xi|^6 g^{-1}$$

and apply (7.13):

$$\begin{aligned} k^{-2}\int_0^\infty & k b_0^2 k^3 \triangle (\nabla k) b_0^2 (\nabla k) b_0 |\xi|^6 g^{-1}\,(dr/2) \\ &= k^{-2}\widetilde{\mathscr{G}}_1(\triangle_{(1)}, \triangle_{(2)})\,(\triangle(\nabla k)k)\,g^{-1} \\ &= k^{-2}\mathscr{G}_1(\triangle_{(1)}, \triangle_{(2)})\,(\nabla k \nabla k)\,g^{-1}, \end{aligned}$$

with $\mathscr{G}_1(s,t) = s\widetilde{\mathscr{G}}_1(s,t)$ and $\widetilde{\mathscr{G}}_1(s,t) = \int_0^\infty H_{(2,2,1)}(s,t,r)\,(dr/2)$.

We collect the terms that involve $\nabla^2 k$ and their associated functions as below: which are $\frac{4}{m} b_0^3 k^2 (\nabla^2 k) b_0 |\xi|^4 g^{-1}$ and $-b_0^2 k (\nabla^2 k) b_0 |\xi|^2 g^{-1}$,

$$\begin{aligned} &\frac{4}{m} b_0^3 k^2 (\nabla^2 k) b_0 |\xi|^4 g^{-1}, && \frac{4}{m} K_{(3,1)}(s,r) \\ &- b_0^2 k (\nabla^2 k) b_0 |\xi|^2 g^{-1}, && - K_{(2,1)}(s,r), \end{aligned} \tag{7.17}$$

they yield the following term in (7.22):

$$k^{-1} \mathcal{K}(\triangle)(\nabla^2 k) g^{-1},$$

where

$$\mathcal{K}(s) = \frac{1}{2} \int_0^\infty \frac{4}{m} K_{(3,1)}(s,r) - K_{(2,1)}(s,r)\, dr.$$

The constant $1/2$ comes from the substitution $r \mapsto r^2$. Plug in (7.15) and $m = 2$, we obtained

$$\mathcal{K}(s) = \frac{-2s + (s+1)\log(s) + 2}{2(s-1)^3}. \tag{7.18}$$

Similarly, according to (7.13) and (7.14), we collect terms in (7.4) that contribute to the function $\mathcal{G}$ in (7.22) in the following table:

$$\begin{aligned} &\frac{4}{m} 2 b_0^3 k^2 (\nabla k) b_0 (\nabla k) b_0 |\xi|^6 g^{-1}, && \frac{4}{m} H_{(3,1,1)}(s,t,r), \\ &-\Big(2 + \frac{4}{m}\Big) b_0^2 k (\nabla k) b_0 (\nabla k) b_0 |\xi|^4 g^{-1}, && -\Big(2 + \frac{4}{m}\Big) H_{(2,1,1)}(s,t,r), \\ &\frac{4}{m} b_0^2 k (\nabla k) b_0^2 k (\nabla k) b_0 |\xi|^6 g^{-1}, && \frac{4}{m} s H_{(2,2,1)}(s,t,r). \end{aligned} \tag{7.19}$$

When $m = 2$, the two variable function $\mathcal{G}$ is given by:

$$\mathcal{G}(s,t) = \frac{1}{2} \int_0^\infty \big(4H_{(3,1,1)} - 4H_{(2,1,1)} + 4sH_{(2,2,1)}\big)(s,t,r)\, dr$$

The explicit expression is given by:

$$\mathcal{G}(s,t) = \frac{\left(\begin{aligned} &(st-1)^3 \log(s) - (s-1)((t-1)(s(t-2)+1)(st-1) \\ &\qquad + (s-1)(st(2t-1)-1)\log(st)) \end{aligned} \right)}{(s-1)^2 s (t-1)^2 (st-1)^3} \tag{7.20}$$

Finally, taking two overall factors into account: $\mathrm{Vol}(S^{m-1})$ and $(2\pi)^{-m}$ (cf. (7.4) and Theorem 6.4), we have proved the following theorem.

Theorem 7.4. *Let $M_\Theta = \mathbb{T}^2_\Theta$, the noncommutative two torus and $P_k = k\Delta$. Then the functional $V_2(\cdot, P_k)$ of the second heat coefficient can be express in in the following way: $\forall f \in C^\infty(M)$, put $f = \pi^\Theta(\tilde{f})$,*

$$V_2(f, P_k) = \int_M \tilde{f} \times_\Theta \tilde{\mathcal{R}}\, dg. \tag{7.21}$$

Keep the notations as in (7.10)–(7.14), then upto a constant factor $(2\pi)^{-1}$,

$$\widetilde{\mathcal{R}} = k^{-1}\mathcal{K}(\triangle)(\nabla^2 k)g^{-1} + k^{-2}\mathcal{G}(\triangle_{(1)}, \triangle_{(2)})\,(\nabla k \nabla k)\, g^{-1}, \tag{7.22}$$

where the modular curvature functions $\mathcal{K}$ *and* $\mathcal{G}$ *are given in* (7.18) *and* (7.20).

Remarks. (1) $\mathcal{R} = \pi^{\Theta}(\widetilde{\mathcal{R}}) \in C^{\infty}(M_\Theta)$ is the associated modular curvature.

(2) Originally, the integrand of the right hand side of (7.21) should be $\bar{\mathcal{R}} \times_\Theta \tilde{f}$, but since $\mathcal{R}$ is self-adjoint and $\int_M$ is a trace with respect to the deformed product, we see that

$$\int_M \bar{\mathcal{R}} \times_\Theta \tilde{f}\, dg = \int_M \tilde{f} \times_\Theta \mathcal{R}\, dg.$$

(3) One can verify that the functions $\mathcal{K}$ and $\mathcal{G}$ agree with the modular curvature functions $-F_{0,0}(s)$ and $G^{\mathfrak{R}}_{0,0}(s,t)$ in [32, Thm. 3.2] in the following way:

$$\mathcal{K}(s) = -F_{0,0}(s), \quad \mathcal{G}(s,t) = \frac{1}{s} G^{\mathfrak{R}}_{0,0}(s,t).$$

The appearance of the negative sign in front of $F_{0,0}$ is due to the fact that $(\nabla^2 k)g^{-1} = -\Delta k$. In [32, Thm 3.2], the quadratic form is defined as $(k^{-1}\nabla k)(k^{-1}\nabla k)g^{-1}$, compare to our quadratic form $k^{-2}(\nabla k)(\nabla k)g^{-1}$. The factor $1/s$ in front of $G^{\mathfrak{R}}_{0,0}$ that stands for the inverse modular operator $\triangle^{-1} = k(\cdot)k^{-1}$ is exactly the price to pay to move k^{-1} in front of ∇k.

(4) The Laplacian $k\Delta$ considered in the theorem is related to the degree zero Laplacian with complex structure $\sqrt{-1}$ appeared in [14] via the conjugation by k: $k^2\Delta \mapsto k\Delta k$. After converting the Weyl factor k and the modular operator $\triangle$ to their own logarithms: $h = \log k$ (defined by $k = e^h$) and $\bigtriangledown = \log \triangle = -[h, \cdot]$, our modular curvature functions agree with those in [14]. This issue is explained in [32, Sec. 4.7] in detail. Therefore our calculation gives a new confirmation of the results for noncommutative two tori which is independent of the aid of CAS.

7.5. Modular curvature for even dimensional toric noncommutative manifolds. Now let us assume that M is even dimensional and $m = \dim M \geq 4$. From Lemma 7.3, is modular curvature is obtained by integrating b_2 defined in (7.4) in the following way: set $j_0 = m/2 - 1$,

$$\widetilde{\mathcal{R}} = \int_0^\infty \Big(\frac{d^{j_0}}{d\lambda^{j_0}} b_2\Big)\Big|_{\lambda=-1} (r^{m-1}\, dr).$$

As before, we perform replace r by r^2 so that the volume form $r^{m-1}\,dr$ becomes $r^{m/2-1}dr/2$. Recall the functions $K_{(p,q)}(s,r)$ and $H_{(p,q,l)}(s,t,r)$ defined in (7.15) and (7.16), take again the term

$$\frac{4}{m} b_0^3 k^2 (\nabla^2 k) b_0\, |\xi|^4\, g^{-1}$$

as an example, it leads to the function in s:

$$\frac{4}{m}\frac{1}{2}\int_0^\infty \frac{d^{j_0}}{d\lambda^{j_0}}\Big|_{\lambda=-1}\frac{r^2}{(r-\lambda)^3}\frac{1}{sr-\lambda}r^{m/2-1}\,dr, \quad j_0 = m/2-1.$$

Let us consider in general

$$\int_0^\infty \frac{d^{j_0}}{d\lambda^{j_0}}\Big|_{\lambda=-1}\frac{r^{p+q-2}}{(r-\lambda)^p}\frac{1}{(sr-\lambda)^q}r^{m/2-1}\,dr, \quad j_0 = m/2-1. \tag{7.23}$$

Via a substitution $u = 1/r$, the integral becomes:

$$\begin{aligned}
&\int_0^\infty u^{-j_0}\frac{d^{j_0}}{d\lambda^{j_0}}\Big|_{\lambda=-1}\frac{1}{(1-u\lambda)^p}\frac{1}{(s-u\lambda)^q}(-du)\\
&= \int_0^\infty \frac{d^{j_0}}{dx^{j_0}}\Big|_{x=-u}\frac{1}{(1-x)^p}\frac{1}{(s-x)^q}(-du)\\
&= \int_0^\infty \frac{d^{j_0}}{du^{j_0}}\frac{1}{(1-u)^p}\frac{1}{(s-u)^q}(-du) = \Big(\frac{d}{du}\Big)^{j_0-1}\Big|_{u=0}\frac{1}{(1-u)^p}\frac{1}{(s-u)^q},
\end{aligned}$$

here we need the fact that p, q are both positive integers so that the limit at infinity equals zero. Due to the homogeneity of b_2 in r, all the terms in (7.4) can be handle in a similar way, therefore, we upgrade functions in (7.15) and (7.16) as follows:

$$\widetilde{K}_{(p,q)}(s,m) = \Big(\frac{d}{du}\Big)^{m/2-2}\Big|_{u=0}\frac{1}{(1-u)^p}\frac{1}{(s-u)^q}, \tag{7.24}$$

$$\widetilde{H}_{(p,q,l)}(s,t,m) = \Big(\frac{d}{du}\Big)^{m/2-2}\Big|_{u=0}\frac{1}{(1-u)^p}\frac{1}{(s-u)^q}\frac{1}{(st-u)^l}. \tag{7.25}$$

Base on (7.17) and (7.19), we can write down the modular curvature function for dimension $m \geq 4$:

$$\mathcal{K}(s,m) = \frac{1}{2}\Big(\frac{4}{m}\widetilde{K}_{(3,1)}(s,m) - \widetilde{K}_{(2,1)}(s,m)\Big), \tag{7.26}$$

$$\mathcal{G}(s,t,m) = \frac{1}{2}\Big(\frac{8}{m}\widetilde{H}_{(3,1,1)} - \Big(2+\frac{4}{m}\Big)\widetilde{H}_{(2,1,1)} + \frac{4}{m}s\widetilde{H}_{(2,2,1)}\Big)(s,t,m). \tag{7.27}$$

The only term left in (7.4) is

$$\frac{1}{m}\frac{2}{3}b_0^2 k^2 \mathcal{S}_\Delta b_0\,|\xi|^2\,.$$

Apply the rearrangement lemma, we see that it becomes

$$\frac{1}{m}\frac{2}{3}k^{-m/2+1}F(\triangle)(\mathcal{S}_\Delta)$$

after integration, where the function $F(s) = \frac{1}{2}\widetilde{K}_{(2,1)}(s,m)$. Since $\mathcal{S}_\Delta$ is $\mathbb{T}^n$-invariant, in particular, it commutes with k, in other words, $\triangle(\mathcal{S}_\Delta) = \mathcal{S}_\Delta$, therefore $F(\triangle)(\mathcal{S}_\Delta) = F(1)\mathcal{S}_\Delta$ and we denote

$$
\begin{aligned}
F(1) &= \frac{1}{2}\widetilde{K}_{(2,1)}(1,m) = \frac{1}{2}\left((1-u)^{-3}\right)^{(m/2-2)}\Big|_{u=0} \\
&= \begin{cases} \frac{1}{4}(-1)^{m/2-2}(m/2)!, & m = 6, 8, 10, \ldots, \\ \frac{1}{2}, & m = 4. \end{cases}
\end{aligned}
\tag{7.28}
$$

Theorem 7.5. *Let M_Θ be a noncommutative toric manifold whose dimension is an even integer m and $\pi^\Theta(P_k) = \pi^\Theta(k)\Delta$. Then the associated modular curvature $\widetilde{\mathcal{R}}$, up to an overall constant $\operatorname{Vol}(S^{m-1})(2\pi)^{-m}$, is of the form:*

$$
\begin{aligned}
\widetilde{\mathcal{R}} = \big(k^{-m/2}\mathcal{K}(\triangle, m)(\nabla^2 k) + k^{-m/2-1}\mathcal{G}(\triangle_{(1)}, \triangle_{(2)}, m)\,(\nabla k \nabla k)\big)g^{-1} \\
+ c_m k^{-m/2+1}\mathcal{S}_\Delta.
\end{aligned}
\tag{7.29}
$$

When $m \geq 4$, the modular curvature functions $\mathcal{K}$ and $\mathcal{G}$ are given in (7.26), (7.27) respectively. While the constant c_m is equal to $\frac{1}{m}\frac{2}{3}F(1)$, where $F(1)$ is calculated in (7.28).

In dimension four, $m/2-2 = 0$, thus no differentiation involves when computing $\widetilde{K}_{(p,q)}$ and $\widetilde{H}_{(p,q,l)}$. So one can quickly compute $\widetilde{K}_{(3,1)}(s,4) = \widetilde{K}_{(1,1)}(s,4) = 1/s$, therefore $\mathcal{K}(s,4) = 0$. Meanwhile,

$$
\widetilde{H}_{(3,1,1)}(s,t,4) = \widetilde{H}_{(2,1,1)}(s,t,4) = \frac{1}{s^2 t}, \quad \widetilde{H}_{(2,2,1)}(s,t,4) = \frac{1}{s^3 t},
$$

as a result, $\mathcal{G}(s,t,4) = 0$.

Corollary 7.6. *Let $m = \dim M = 4$. For the perturbed Laplacian $\pi^\Theta(P_k) = \pi^\Theta(k)\Delta$, the modular curvature is simply:*

$$
\mathcal{R}(k) = ck^{-1}\mathcal{S}_\Delta, \tag{7.30}
$$

where

$$
c = (4\pi)^{-2}\frac{1}{6}.
$$

Remark. The value of c agrees with the classical result: upto a factor $(4\pi)^{-\dim M/2}$, the density of the second heat coefficient for the scalar Laplacian operator equals $1/6$ times the scalar curvature.

Proof. It remains to determine the coefficient for the scalar curvature term. We recall

$$
F(1) = \frac{1}{2}, \quad \operatorname{Vol}(S^3) = \frac{2\pi^{4/2}}{\Gamma(2)} = 2\pi^2,
$$

thus

$$
c = \operatorname{Vol}(S^3)(2\pi)^{-4}\frac{1}{m}\frac{2}{3}F(1) = (4\pi)^{-2}\frac{1}{6}. \qquad \square
$$

7.6. Comparison with [22] and [19]. In this section, we would like to reproduce the results on noncommutative four tori in [22] and [19]. The perturbed Laplacian $\pi^{\Theta}(P_k)$ considered in [22, Lemma 3.3] has the following symbol:

$$\sigma(P_k) = p_2 + p_1 + p_0$$

with

$$p_2 = k\,|\xi|^2, \quad p_1 = \frac{-i}{2}(\nabla k)D\,|\xi|^2, \quad p_0 = -\Delta k + (\nabla k)k^{-1}(\nabla k)g^{-1}, \tag{7.31}$$

where the Laplacian Δ is associated to the flat metric on $\mathbb{T}^4$. Based on the previous computation, the contribution to the modular curvature functions $\mathcal{K}$ and $\mathcal{G}$ from the leading term p_2 is equal to zero. It remains to count the contribution from p_1 and p_0 which is in the last line of the b_2 term in (B.8):

$$-iD(b_0 p_1 b_0)(\nabla p_2)b_0 - b_0 p_0 b_0 - b_1 p_1 b_0 + iDb_0(\nabla p_1)b_0. \tag{7.32}$$

Similar to the computation in Appendix B, we list the result of each summand as below: For $-iD(b_0 p_1 b_0)(\nabla p_2)b_0$:

$$\frac{1}{2}\big(b_0^2 k(\nabla k)b_0(\nabla k)b_0 + b_0(\nabla k)b_0^2 k(\nabla k)b_0\big)(D\,|\xi|^2)^2\,|\xi|^2$$
$$-\frac{1}{2}\big(b_0(\nabla k)b_0(\nabla k)b_0(D^2\,|\xi|^2)\,|\xi|^2\big).$$

For $-b_1 p_1 b_0$:

$$\frac{1}{2}b_0^2 k(\nabla k)b_0(\nabla k)b_0(D\,|\xi|^2)^2\,|\xi|^2 + \frac{1}{4}b_0(\nabla k)b_0(\nabla k)b_0(D\,|\xi|^2)^2.$$

For $-b_0 p_0 b_0$:

$$b_0(\Delta k)b_0 - b_0(\nabla k)k^{-1}(\nabla k)b_0 g^{-1}.$$

The last one:

$$iDb_0(\nabla p_1)b_0 = \frac{1}{2}b_0^2 k(\nabla^2 k)b_0(D\,|\xi|^2)^2.$$

Sum up the terms above and perform integrating over the cosphere bundle, namely the following substitution:

$$(D\,|\xi|^2)^2 \mapsto \frac{4\,|\xi|^2}{m}\mathrm{Vol}(S^{m-1})g^{-1}, \quad D^2\,|\xi|^2 \mapsto 2\mathrm{Vol}(S^{m-1})g^{-1},$$

We group the sum into two parts associated to the modular curvature function $\mathcal{K}(s)$ and $\mathcal{G}(s,t)$ respectively. More precisely, (7.32) can be written as $I_1 + I_2$ upto an overall factor $\mathrm{Vol}(S^3)$, with

$$I_1 = b_0(\Delta k)b_0 + \frac{1}{2}\frac{4}{m}b_0^2 k(\nabla^2 k)b_0\,|\xi|^2\,g^{-1}, \tag{7.33}$$

and:

$$I_2 = \frac{4}{m} b_0^2 k(\nabla k) b_0 (\nabla k) b_0 \left|\xi\right|^4 g^{-1} + \frac{1}{2}\frac{4}{m} b_0 (\nabla k) b_0^2 k (\nabla k) b_0 \left|\xi\right|^4 g^{-1} + \Big(\frac{1}{4}\frac{4}{m} - 1\Big) b_0 (\nabla k) b_0 (\nabla k) b_0 \left|\xi\right|^2 g^{-1} - b_0 (\nabla k) k^{-1} (\nabla k) b_0 g^{-1}. \quad (7.34)$$

Notice that $\Delta k = -(\nabla^2 k) g^{-1}$, therefore I_1 is of the form

$$I_1 \mapsto k^{-2} \mathcal{K}(\triangle)(\nabla^2 k) g^{-1}$$

after integration, with

$$\mathcal{K}(s) = \frac{1}{2}\Big(- K_{(1,1)}(s,4) + \frac{1}{2}\frac{4}{m} K_{(2,1)}(s,4)\Big) = -\frac{1}{4}\frac{1}{s}. \quad (7.35)$$

Meanwhile,

$$I_2 \mapsto k^{-3} \mathcal{G}(\triangle_{(1)}, \triangle_{(2)})(\nabla k \nabla k)$$

where $\mathcal{G}(s,t)$ equals $(8s^2t)^{-1}$, which comes from the sum:

$$\frac{1}{2}\Big(\Big(\frac{4}{m} H_{(2,1,1)} + \frac{1}{2}\frac{4}{m} s H_{(1,2,1)} + \Big(\frac{1}{4}\frac{4}{m} - 1\Big) H_{(1,1,1)}\Big)(s,t,4) - \frac{1}{s} K_{(1,1)}(st,4)\Big),$$

here we have used the fact that

$$H_{(2,1,1)}(s,t,4) = s H_{(1,2,1)}(s,t,4) = H_{(1,1,1)}(s,t,4) = \frac{1}{s} K_{(1,1)}(st,4) = \frac{1}{s^2 t}.$$

We summarize the computation as below.

Theorem 7.7. *Let $M_\Theta = \mathbb{T}^4_\Theta$ be a noncommutative four torus. With respect to the perturbed Laplacian $\pi^\Theta(P_k)$ whose symbol is defined in Eq. (7.31), the modular curvature is of the form:*

$$\mathcal{R} = k^{-2} \mathcal{K}(\triangle)(\nabla^2 k) g^{-1} + k^{-3} \mathcal{G}(\triangle_{(1)}, \triangle_{(2)})(\nabla k \nabla k)$$

upto an overall factor $\mathrm{Vol}(S^3)(2\pi)^{-4}$*, where*

$$\mathcal{K}(s) = \frac{1}{4s}, \quad \mathcal{G}(s,t) = -\frac{1}{8s^2 t}. \quad (7.36)$$

The result should be compared with [19, Eq. (1)].

A. Inverse of the leading symbol $(k\left|\xi\right|^2 - \lambda)$

For $\lambda \in \mathbb{C} \setminus [0,\infty)$, we would like to show that the leading symbol of a perturbed Laplacian $k\left|\xi\right|^2 - \lambda$ (see Definitions 6.1 and 6.2) is invertible in the symbol algebra $S\Sigma(M_\Theta, \Lambda)$, upto smoothing symbols.

First, we recall a technical result. Given a toric Spin manifold M with spinor bundle $\not{S}$ and the Dirac operator $\not{D}$, the deformed spectral triple $(C^\infty(M_\Theta), L^2(\not{S}), D)$ is a regular spectral triple in the sense of, for instance, [27, 34]. In particular, the subalgebra $C^\infty(M_\Theta)$ inside $B(L^2(\not{S}))$ is closed under holomorphic functional calculus. We refer to [43, Prop. 5] for the proof.

Given a Weyl factor $\pi^\Theta(k)$ (with $k \in C^\infty(M)$), whose spectrum is contained in $(0,\infty)$, thus for any $s > 0$, and $\lambda \in \mathbb{C} \setminus (0,\infty)$, the resolvent $(s\pi^\Theta(k) - \lambda)^{-1}$ still lies in $C(M_\Theta) = \pi^\Theta(C^\infty(M))$. In particular, there exists a unique smooth function in $C^\infty(M)$, denoted by $(sk-\lambda)^{-1}$ (here the -1 power stands for the deformed inverse, other than the reciprocal of a function), such that $\pi^\Theta((sk-\lambda)^{-1}) = (s\pi^\Theta(k)-\lambda)^{-1}$.

As a $C^\infty(M)$-valued function, $(s,\lambda) \mapsto (sk-\lambda)^{-1}$ is smooth in $s \in (0,\infty)$ and holomorphic in $\lambda \in \mathbb{C} \setminus (0,\infty)$. To avoid the singularity caused by s being zero, we choose a cut-off function

$$\rho(s) = \begin{cases} 0, & \text{for } t \le 1/2, \\ 1, & \text{for } t \ge 1, \end{cases} \tag{A.1}$$

and denote

$$r(s,\lambda) = \rho(s)(sk-\lambda)^{-1}.$$

Finally, we claim that the inverse (upto smoothing symbols) of $k\,|\xi|^2 - \lambda$ in $S\Sigma(M_\Theta)/S\Sigma^{-\infty}$ is given by composing $r(s,\lambda)$ with the squared length function on T^*M:

$$b_0(\xi_x,\lambda) = r\big(|\xi_x|^2,\lambda\big), \quad \xi_x \in T^*_x M, \tag{A.2}$$

which is a well-defined smooth functions on T^*M.

Proposition A.1. *The function $b_0(\xi_x,\lambda)$ defined in (A.2) belongs to $S\Sigma^{-2}(M_\Theta,\lambda)$ and serves as the inverse of $p_2(\xi_x,\lambda) = k\,|\xi|^2 - \lambda$ in the deformed symbol algebra $S\Sigma(M_\Theta,\Lambda)$, that is $p_2 \times_\Theta b_0 \backsim 1$ and $b_0 \times_\Theta p_2 \backsim 1$. Here $\backsim$ means "upto smoothing symbols".*

Proof. Let $\times_\Theta$ and $*_\Theta$ denote the multiplication in $S\Sigma(M_\Theta,\Lambda)$ and $C^\infty(M_\Theta)$ respectively. Since the length function $|\xi|$ is $\mathbb{T}^n$-invariant, let $s \in (0,\infty)$ as before and fix a λ, we see that

$$(p_2 \times_\Theta b_0)|_{\xi_x} = \big((sk-\lambda) *_\Theta r(s,\lambda)\big)\big|_{s=|\xi_x|^2}.$$

In the right hand side above, $(sk-\lambda)$ and $r(s,t)$ are functions in s valued in $C^\infty(M)$, thus the $*_\Theta$-multiplication makes sense. By the construction of $r(s,\lambda)$, $(sk-\lambda) *_\Theta r(s,\lambda) = 1$ when $s \ge 1$, hence $(p_2 \times_\Theta b_0)|_{\xi_x} = 1$ for all $|\xi_x| \ge 1$, thus $p_2 \times_\Theta b_0 \backsim 1$. Same argument shows that $b_0 \times_\Theta p_2 \backsim 1$.

One can quickly verify that for any $c > 0$,

$$b_0(c\xi_x, c^2\lambda) = c^{-2} b_0(\xi_x,\lambda),$$

provided $|\xi_x|$ large enough. Hence b_0 belongs to $S\Sigma^{-2}(M_\Theta,\lambda)$. □

B. Computation of the b_2 term

The goal of this section is to compute the full expression of the b_2 term in the resolvent aproximation. Recall from (6.3) and (6.4), we need to compute:

$$b_1 = \big(b_0 p_1 + (-i) D b_0 \nabla p_2\big)(-b_0), \tag{B.1}$$

$$\begin{aligned} b_2 = \big(&b_0 p_0 + b_1 p_1 + (-i) D b_0 \nabla p_1 + (-i) D b_1 \nabla p_2 \\ &- \tfrac{1}{2} D^2 b_0 \nabla^2 p_2 - \tfrac{1}{2} D b_0 D^2 p_2 (\nabla^3 \ell)\big)(-b_0), \end{aligned} \tag{B.2}$$

where $b_0 = p_2^{-1} = (k\,|\xi|^2 - \lambda)^{-1}$.

Since $\nabla\,|\xi|^2 = 0$ (proved in Lemma C.4), the Leibniz rule gives us

$$\nabla k\,|\xi|^2 = (\nabla k)\,|\xi|^2\,, \tag{B.3}$$

$$\nabla^2 k\,|\xi|^2 = (\nabla^2 k)\,|\xi|^2 + k(\nabla^2\,|\xi|^2). \tag{B.4}$$

The derivatives of $b_0 = (k\,|\xi|^2 - \lambda)^{-1}$ are obtained by applying the standard resolvent identity:

$$(D b_0) = -b_0 (D p_2) b_0,$$

which holds as well when D is replaced by ∇. We notice that $D p_2$ commutes with b_0, but ∇p_0 does not, because ∇k and k do not commute. We continue the computation:

$$D b_0 = -(b_0 b_0) D p_2 = -(b_0^2 k) D\,|\xi|^2\,, \tag{B.5}$$

and

$$\begin{aligned} D^2 b_0 &= D(-b_0 b_0 D p_2) \\ &= -b_0 b_0 D^2 p_2 + 2 b_0 b_0 b_0 D p_2 D p_2 \\ &= -b_2^2 k D^2\,|\xi|^2 + 2 b_0^3 k^2 (D\,|\xi|^2)^2. \end{aligned} \tag{B.6}$$

Combine (B.1), (B.5), and (B.3), we get:

$$b_1 = -i b_0^2 k (D\,|\xi|^2 \cdot \nabla k) b_0\,|\xi|^2 - b_0 p_1 b_0. \tag{B.7}$$

Proposition B.1. *The b_2 term is given by:*

$$\begin{aligned} b_2 = 2 b_0^3 k^2 (\nabla k) &b_0 (\nabla k) b_0\,|\xi|^4\,(D\,|\xi|^2)^2 \\ &- b_0^2 k (\nabla k) b_0 (\nabla k) b_0 \big(|\xi|^2\,(D\,|\xi|^2)^2 + |\xi|^4\,D^2\,|\xi|^2\big) \\ &+ b_0^2 k (\nabla k) b_0^2 k (\nabla k) b_0\,|\xi|^4\,(D\,|\xi|^2)^2 \\ &- \frac{1}{2}(b_0^3 k^2)(\nabla^3 \ell)(D\,|\xi|^2)(D^2\,|\xi|^2) - \frac{1}{2} b_0^3 k^2 D^2\,|\xi|^2\,\nabla^2\,|\xi|^2 \\ &- \frac{1}{2} b_0^2 k (\nabla^2 k) b_0\,|\xi|^2\,D^2\,|\xi|^2 + b_0^3 k^2 (\nabla^2 k) b_0\,|\xi|^2\,(D\,|\xi|^2)^2 \\ &- i D(b_0 p_1 b_0)(\nabla p_2) b_0 - b_0 p_0 b_0 - b_1 p_1 b_0 + i D b_0 (\nabla p_1) b_0. \end{aligned} \tag{B.8}$$

Proof. We first compute the last two terms (terms with $1/2$ in front) in Equation (B.2). From (B.6) and (B.4):

$$\begin{aligned} D^2 b_0 \times_\Theta \nabla^2 p_2 &= \big(-b_2^2 k D^2 |\xi|^2 + 2b_0^3 k^2 (D|\xi|^2)^2\big)\big((\nabla^2 k)|\xi|^2 + k(\nabla^2 |\xi|^2)\big) \\ &= -b_0^2 k (D^2|\xi|^2 \cdot \nabla^2 k)|\xi|^2 - b_0^2 k^2 D^2 |\xi|^2 \nabla^2 |\xi|^2 \\ &\quad + 2b_0^3 k^2 \big((D|\xi|^2)^2 \nabla^2 k\big)|\xi|^2 + 2b_0^3 k^3 (D|\xi|^2)^2 \nabla^2 |\xi|^2 . \end{aligned} \tag{B.9}$$

Combine (B.5) and (B.4):

$$\begin{aligned} (\nabla^3 \ell) D b_0 D^2 p_2 &= (\nabla^3 \ell)(-b_0^2 k)(D|\xi|^2) k (D^2 |\xi|^2) \\ &= (-b_0^2 k^2)(\nabla^3 \ell)(D|\xi|^2)(D^2|\xi|^2). \end{aligned} \tag{B.10}$$

Apply D onto (B.7),

$$\begin{aligned} Db_1 = &-i\Big[D(b_0^2 k)(D\xi^2 \cdot \nabla k) b_0 \xi^2 + b_0^2 k D(D|\xi|^2 \cdot \nabla k) b_0 |\xi|^2 \\ &+ b_0^2 k (D|\xi|^2 \cdot \nabla k)(Db_0)|\xi|^2 + b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0 D|\xi|^2\Big] \\ &- D(b_0 p_1 b_0) \\ = &-i\Big[-2b_0^3 k^2 D|\xi|^2 (D|\xi|^2 \cdot \nabla k) b_0 |\xi|^2 + b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0 D|\xi| \\ &+ b_0^2 k (D^2|\xi|^2 \cdot \nabla k) b_0 |\xi|^2 - b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0^2 k (D|\xi|^2)|\xi|^2\Big] \\ &- D(b_0 p_1 b_0), \end{aligned}$$

thus

$$\begin{aligned} Db_1 \nabla p_2 = -i\Big[&-2b_0^3 k^2 (D|\xi|^2 \cdot \nabla k) b_0 (D|\xi|^2 \cdot \nabla k)|\xi|^4 \\ &+ b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0 (D|\xi|^2 \cdot \nabla k)|\xi|^2 \\ &+ b_0^2 k (D^2|\xi|^2 \cdot \nabla k) b_0 (\nabla k)|\xi|^4 \\ &- b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0^2 k (D|\xi|^2 \cdot \nabla k)|\xi|^4\Big] \\ &- D(b_0 p_1 b_0)\nabla p_2. \end{aligned} \tag{B.11}$$

We substitute (B.9), (B.10) and (B.11) into (B.2):

$$\begin{aligned} b_2 = 2b_0^3 k^2 &(D|\xi|^2 \cdot \nabla k) b_0 (D|\xi|^2 \cdot \nabla k) b_0 |\xi|^4 \\ &- b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0 (D|\xi|^2 \cdot \nabla k) b_0 |\xi|^2 \\ &- b_0^2 k (D^2|\xi|^2 \cdot \nabla k) b_0 (\nabla k) b_0 |\xi|^4 \\ &+ b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0^2 k (D|\xi|^2 \cdot \nabla k) b_0 |\xi|^4 \\ &+ \frac{1}{2}(-b_0^2 k^2)(\nabla^3 \ell)(D|\xi|^2)(D^2|\xi|^2) b_0 \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}b_0^2 k(D^2|\xi|^2\cdot\nabla^2 k)b_0|\xi|^2\\
&-\frac{1}{2}b_0^2k^2D^2|\xi|^2\nabla^2|\xi|^2 b_0\\
&+b_0^3k^2((D|\xi|^2)^2\nabla^2 k)b_0|\xi|^2\\
&-iD(b_0p_1b_0)(\nabla p_2)b_0-b_0p_0b_0-b_1p_1b_0+iDb_0(\nabla p_1)b_0.
\end{aligned}
$$

To obtain (B.8), we just need to move the vertical and the horizontal derivatives of $|\xi|^2$ to the right end for each summand above. This operation is valid because of the $\mathbb{T}^n$-invariant property of the function $|\xi|^2$. □

C. Integration over the cosphere bundle S^*M

Using symmetries, one can quickly compute the surface integral of the function x^2 over the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$:

$$
\int_{x^2+y^2+z^2=1} x^2\,dS=\frac{1}{3}\int_{x^2+y^2+z^2=1}(x^2+y^2+z^2)\,dS=\frac{1}{3}\mathrm{Vol}(S^2).
$$

In general, we have

Lemma C.1. *Let $\xi=(\xi_1,\dots,\xi_m)$ be a coordinate system of $\mathbb{R}^m$ and S^{m-1} is the unit sphere with induced measure $d\sigma_{S^{m-1}}$. Then*

$$
\int_{S^{m-1}}\xi_s\xi_a\,d\sigma_{S^{m-1}}=\begin{cases}0, & \textit{when } s\neq a,\\ \frac{|\xi|^2}{m}\mathrm{Vol}(S^{m-1}), & \textit{when } s=a.\end{cases} \tag{C.1}
$$

In an orthonormal local coordinate system: $g^{ij}=\xi_{ij}$ and $|\xi|^2=\sum_j\xi_j^2$, hence

$$
((D|\xi|^2)^2)_{ij}=\big((D|\xi|^2)\otimes(D|\xi|^2)\big)_{ij}=4\xi_i\xi_j,\quad (D^2|\xi|^2)_{ij}=2\delta_{ij}.
$$

Apply the lemma above, one can quickly conclude:

Lemma C.2. *Let $S^{m-1}\subset T_x^*M\cong\mathbb{R}^m$ be the unit sphere with respect to the Riemannian metric g^{-1} and $d\sigma_{S^{m-1}}$ be the induced measure as before. Then*

$$
\int_{S_x^*M}(D|\xi|^2)^2\,d\sigma_{S^{m-1}}=\frac{4|\xi|^2}{m}\mathrm{Vol}(S^{m-1})(g^{-1}|_x), \tag{C.2}
$$

$$
\int_{S_x^*M}D^2|\xi|^2\,d\sigma_{S^{m-1}}=2\mathrm{Vol}(S^{m-1})(g^{-1}|_x). \tag{C.3}
$$

Now we have to deal with the horizontal derivatives $\nabla^j|\xi|^2$ with $j=1,2$ which leads to the appearance of the scalar curvature function $\mathcal{S}_\Delta$. Start with $\nabla^3\ell$, the following lemma was proved in [41], which is recalled here for completeness:

Lemma C.3. *In local coordinates at a given point $x \in M$,*

$$(\nabla^3 \ell)_{ijk} = -\frac{1}{3} \sum_p \xi_p \big(R^p{}_{ikj} + R^p{}_{jki} \big).$$

Proof. We adopt the following convention for the curvature tensor on the tangent bundle TM:

$$\big(R(X,Y)Z \big)^l = R^l{}_{ijk} X^j Y^k Z^l,$$

where X, Y and Z are vector fields. In local coordinates:

$$\begin{aligned}(\nabla^3 \ell)_{ikj} - (\nabla^3 \ell)_{ijk} &= \big((\nabla_j \nabla_k - \nabla_k \nabla_j)(\nabla \ell) \big)_i \\ &= -(\nabla \ell)_p R^p{}_{ijk} = \xi_p R^p{}_{ikj}.\end{aligned}$$

The negative sign appears because the covariant derivatives are taken on the cotangent bundle. Since the connection is torsion free, $(\nabla^3 \ell)$ is symmetric in the first two indices, thus

$$(\nabla^3 \ell)_{jki} = (\nabla^3 \ell)_{jik} + \xi_p R^p{}_{jki} = (\nabla^3 \ell)_{ijk} + \xi_p R^p{}_{jki}.$$

Again, due to the symmetry in the first two indices, all the components of the tensor $\nabla^3 \ell$ fall into $\{ (\nabla^3 \ell)_{ijk}, (\nabla^3 \ell)_{ikj}, (\nabla^3 \ell)_{jki} \}$. Since $\partial^3 \ell = 0$, summing up the three terms gives us

$$\begin{aligned} 0 &= 3(\nabla^3 \ell)_{ijk} + \xi_p R^p{}_{ikj} + \xi_p R^p{}_{jki}, \\ (\nabla^3 \ell)_{ijk} &= -\frac{1}{3} \big(\xi_p R^p{}_{ikj} + \xi_p R^p{}_{jki} \big). \end{aligned}$$

□

Lemma C.4. *Evaluating at at given point $x \in M$, the horizontal and the vertical derivatives of $|\xi|^2$ are given by:*

$$\nabla |\xi|^2 = 0, \tag{C.4}$$

$$(\nabla^2 |\xi|^2) = (\nabla^3 \ell)(\nabla \ell) g^{-1} + (\nabla \ell)(\nabla^3 \ell) g^{-1}, \tag{C.5}$$

where the contraction is implemented in the following way:

$$(\nabla^2 |\xi|^2)_{jk} = (\nabla^3 \ell)_{kij} (\nabla \ell)_a g^{ka} + (\nabla \ell)_a (\nabla^3 \ell)_{kij} g^{ak}.$$

In particular, when $g^{ij} = \delta_{ij}$,

$$(\nabla^2 |\xi|^2)_{jk} = \frac{2}{3} \sum_{p,i} \xi_p \xi_i R_{pjik}. \tag{C.6}$$

Proof. According to the definition of the horizontal differential ∇, we have to compute:

$$\nabla^k \big| d_y \ell(\xi_x, y) \big|^2 \big|_{y=x} = \nabla^k \langle d_y \ell(\xi_x, y), d_y \ell(\xi_x, y) \rangle_{g^{-1}}, \quad k = 1, 2.$$

Set $d\ell = d_y\ell(\xi_x, y)$, we rewrite the metric pairing $\langle d\ell, d\ell\rangle_{g^{-1}}$ as $d\ell \otimes d\ell \cdot g^{-1}$. Using the Leibniz's rule and the fact that the connection is Levi-Civita: $\nabla g^{-1} = 0$:

$$\begin{aligned}\nabla\,|\xi|^2 &= \nabla(d\ell \otimes d\ell \cdot g^{-1}) = (2\nabla d\ell) \otimes d\ell \cdot g^{-1} \\ &= \nabla^2\ell \otimes \ell \cdot g^{-1}.\end{aligned}$$

Notice that ∇ is torsion-free, when evaluating at $y = x$

$$(\nabla^2\ell)|_{y=x} = (\partial^2\ell)|_{y=x} = 0,$$

here ∂^2 is the symmetrization of ∇^2. Thus we have proved (C.4). However, the second covariant derivative is nonzero. Again, since $\nabla g^{-1} = 0$:

$$\begin{aligned}\nabla^2\big((d\ell \otimes d\ell)\cdot g^{-1}\big) = (\nabla^2 d\ell) \otimes d\ell \cdot g^{-1} + d\ell \otimes (\nabla^2 d\ell)\cdot g^{-1} \\ + (\nabla d\ell) \otimes (\nabla d\ell)\cdot g^{-1}.\end{aligned}$$

At $y = x$: $(\nabla d\ell) = \nabla^2\ell = 0$ as before, thus we have reached (C.5).

At last, we need to compute the contraction (C.6). When $g^{ij} = \delta_{ij}$, apply Lemma C.3:

$$(\nabla^3\ell)_{ijk}(d\ell)_a g^{aj} = -\frac{1}{3}\sum_{i,p}\xi_p\xi_i\big(R_{pikj} + R_{pjki}\big).$$

Since the curvature tensor R_{pikj} is anti-symmetric in (p, i) and in (k, j) respectively, the first term summed to zero and for the second term, we use the minus sign to switch k and j, thus the result becomes:

$$(\nabla^3\ell)_{ijk}(d\ell)_a g^{aj} = \frac{1}{3}\sum_{i,p}\xi_p\xi_i R_{pjik}.$$

The second term in (C.5) provides the same answer, therefore we have proved (C.6). □

Corollary C.5. *In an orthonormal local coordinate system, we compute the following contractions:*

$$(D\,|\xi|^2)(D^2\,|\xi|^2)(\nabla^3\ell) = -\frac{8}{3}\sum_{k,p,i}\xi_p\xi_k R_{piki}, \tag{C.7}$$

$$(D^2\,|\xi|^2)(\nabla^2\,|\xi|^2) = \frac{4}{3}\sum_{k,p,i}\xi_p\xi_k R_{piki}, \tag{C.8}$$

$$(D\,|\xi|^2)^2(\nabla^2\,|\xi|^2) = 0. \tag{C.9}$$

Proof. Since $(D^2\,|\xi|^2)_{ij} = 2g^{ij} = 2\delta_{ij}$, take (C.6) into account:

$$(D^2\,|\xi|^2)(\nabla^2\,|\xi|^2) = \frac{4}{3}\sum_{p,i,k}\xi_p\xi_i R_{pkik},$$

which is (C.8). For (C.7), we need $(D\,|\xi|^2)_j = 2\xi_j$ and lemma C.3:

$$\begin{aligned}(D\,|\xi|^2)_k(D^2\,|\xi|^2)_{ij}(\nabla^3\ell)_{ijk} &= -\frac{1}{3}(2\xi_k)(2\delta_{ij})\sum_p\xi_p\big(R_{pikj}+R_{pjki}\big)\\ &= -\frac{8}{3}\sum_p\xi_p\xi_k R_{piki}.\end{aligned}$$

At last,

$$\big((D\,|\xi|^2)^2\big)_{ij}(\nabla^2\,|\xi|^2)_{ij} = \sum_{p,l}(4\xi_i\xi_j)\frac{2}{3}(\xi_p\xi_l R_{pilj}).$$

Due to the anti-symmetries of the curvature tensor, the right hand side vanishes when summing over all indices i, l, p, j. □

Corollary C.6. *Keep the notations as above,*

$$\int_{S^*M}(D\,|\xi|^2)(D^2\,|\xi|^2)(\nabla^3\ell)\,d\sigma_{S^{m-1}} = -\frac{8}{3m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta, \tag{C.10}$$

$$\int_{S^*M}(D^2\,|\xi|^2)(\nabla^2\,|\xi|^2)\,d\sigma_{S^{m-1}} = \frac{4}{3m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta. \tag{C.11}$$

Proof. According to (C.7) and (C.8), it suffices to show that

$$\int_{S_x^*M}\Big(\sum_{p,i,k}\xi_p\xi_i R_{pkik}\Big)\,d\sigma_{S^{m-1}} = \frac{1}{m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta,$$

here $\mathcal{S}_\Delta = \sum_{p,k}R_{pkpk}$ is the trace of the Ricci tensor. Indeed, we apply Lemma C.1 again,

$$\begin{aligned}\int_{S_x^*M}\Big(\sum_{p,i,k}\xi_p\xi_i R_{pkik}\Big)\,d\sigma_{S^{m-1}} &= \int_{S_x^*M}\Big(\sum_{p,k}\xi_p^2 R_{pkpk}\Big)\,d\sigma_{S^{m-1}}\\ &= \frac{1}{m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\sum_{p,k}R_{pkpk}\\ &= \frac{1}{m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta.\end{aligned}$$

□

Apply the following substitution rules (Eqs. (C.10), (C.11), (C.2), and (C.3))

$$(D\,|\xi|^2)(D^2\,|\xi|^2)(\nabla^3\ell) \mapsto -\frac{8}{3m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta,$$
$$(D^2\,|\xi|^2)(\nabla^2\,|\xi|^2) \mapsto \frac{4}{3m}\,|\xi|^2\,\mathrm{Vol}(S^{m-1})\mathcal{S}_\Delta$$
$$(D\,|\xi|^2)^2 \mapsto \frac{4\,|\xi|^2}{m}\mathrm{Vol}(S^{m-1})(g^{-1}|_x), \quad D^2\,|\xi|^2 \mapsto 2\mathrm{Vol}(S^{m-1})(g^{-1}|_x),$$

to the b_2 term in Proposition B.1. The result is summarized below.

Proposition C.7. *Keep the notations as above. Assume that the lower order symbols of the Laplacian is zero, that is $p_1 = p_0 = 0$. Along the fiber T_x^*M for some $x \in M$, the integral over the unit sphere $\int_{S^{m-1}} b_2 d\sigma_{S^{m-1}}$ is equal to, up to an overall factor* $\mathrm{Vol}(S^{m-1})$*:*

$$\begin{aligned}
&\frac{4}{m}2b_0^3k^2(\nabla k)b_0(\nabla k)b_0\,|\xi|^6\,g^{-1} - \Big(2+\frac{4}{m}\Big)b_0^2k(\nabla k)b_0(\nabla k)b_0\,|\xi|^4\,g^{-1}\\
&+\frac{4}{m}b_0^2k(\nabla k)b_0^2k(\nabla k)b_0\,|\xi|^6\,g^{-1} - b_0^2k(\nabla^2 k)b_0\,|\xi|^2\,g^{-1}\\
&+\frac{4}{m}b_0^3k^2(\nabla^2 k)b_0\,|\xi|^4\,g^{-1} + \frac{1}{m}\frac{2}{3}b_0^2k^2\mathcal{S}_\Delta b_0\,|\xi|^2\,,
\end{aligned} \tag{C.12}$$

where m is the dimension of the manifold.

Remark. The contraction $(\nabla^2 k)g^{-1} = -\Delta k$ is nothing but the Laplacian of the conformal factor k.

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