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Toeplitz operators and the Roe–Higson type index theorem

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Abstract. Let M be a complete Riemannian manifold and assume that M is partitioned by a hypersurface N. In this paper we introduce a novel class of functions $C_{\mathrm{w}}(M)$ on noncompact manifolds, which is slightly larger than the algebra of Higson functions. Out of ϕ that belongs to $C_{\mathrm{w}}(M)$ we construct an index class $\mathrm{Ind}(\phi, D)$ in K_1-group of the Roe algebra of M by using the Kasparov product. It is supposed to be a counterpart of Roe's odd index class. We finally prove that Connes' pairing of $\mathrm{Ind}(\phi, D)$ and Roe's cyclic 1-cocycle is equal to the Fredholm index of a Toeplitz operator on N. This is an extension of the Roe–Higson index theorem to even-dimensional partitioned manifolds.

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1. Introduction

Let M be a complete Riemannian manifold and assume that M is a partitioned manifold. That is, there exists a closed submanifold N of codimension one such that M is decomposed by N into two components M^+ and M^-, and one has $N = M^+ \cap M^- = \partial M^+ = \partial M^-$; see Definition 2.1. Let $S \to M$ be a Clifford bundle in the sense of J. Roe [16, Definition 3.4] and D the Dirac operator of S. Denote by S_N the restriction of S to N and ν a unit normal vector field on N pointing from M^- into M^+. Then we can equip S_N with a $\mathbb{Z}_2(= \mathbb{Z}/2\mathbb{Z})$-graded Clifford bundle structure, where the $\mathbb{Z}_2$-grading on S_N is defined by using the Clifford action of ν. Let D_N be the graded Dirac operator on S_N.

Let $C^*(M)$ be the Roe algebra of M, which is a non-unital C^*-algebra introduced by Roe in [16]. In [16], Roe also defined the odd index class odd-ind$(D) = [u_D] \in K_1(C^*(M))$, where u_D is the Cayley transform of D.

Roe also defined the cyclic 1-cocycle ζ on a dense subalgebra $\mathcal{X}$ of $C^*(M)$, which is called the Roe cocycle. The Roe cocycle is constructed to be the Connes–Chern character of the Fredholm module defined by the partition of M which corresponds to the Wiener–Hopf extension. Thus Connes' pairing $\langle x, \zeta\rangle$ of ζ

with $x \in K_1(C^*(M))$ is equal to the index map defined by the Wiener–Hopf extension; see [13, Remark 4.14] and Section 4.3.

The Roe cocycle ζ is related to the Poincaré dual of N, $pd(N) \in H_c^1(M)$; see [15, Section 6.1]. In fact, there uniquely exists a coarse cohomology class $\alpha \in HX^1(M)$ such that the character map $HX^1(M) \to H_c^1(M)$ sends α to $pd(N)$. Moreover, the character map $HX^1(M) \to HC^1(\mathcal{X})$ sends α to $[\zeta]$. By this relationship of ζ and N, it is expected that we could pick up some informations of N by using ζ. Indeed, Roe proved Connes' pairing $\langle \text{odd-ind}(D), \zeta \rangle$ is equal to the Fredholm index of D_N^+ up to a certain constant multiple [16]. In [10], N. Higson gave a simplified proof of a variation of Roe's theorem, thus we call it the Roe–Higson index theorem in this paper.

On the other hand, $\text{index}(D_N^+)$ is 0 for N is of odd dimension; see, for instance [16, Proposition 11.14]. This implies that the Roe–Higson index $\langle \text{odd-ind}(D), \zeta \rangle$ is trivial when M is of even dimension. However, Connes' pairing of ζ with $x \in K_1(C^*(M))$ is non trivial in general.

In this paper, we shall develop an index theorem on even dimensional partitioned manifolds, which is analogous to the Roe–Higson index theorem. For this purpose, we need to replace two ingredients, odd-ind(D) and the Dirac operator D_N^+ by an index class $\text{Ind}(\phi, D) = [\phi]\widehat{\otimes}[D]$ and a Toeplitz operator on N, respectively. In order to define this index class $\text{Ind}(\phi, D)$ on M, we need to introduce a new class of C^*-algebra $C_{\mathrm{w}}(M)$, which is larger than the Higson functions on M and smaller than the bounded continuous functions on M; see Definition 3.1. In fact, we use $\phi \in GL_l(C_{\mathrm{w}}(M))$ and $[D] \in KK^0(C_{\mathrm{w}}(M), C^*(M))$. By using the algebra $C_{\mathrm{w}}(M)$, this index class $\text{Ind}(\phi, D)$ can be regarded as a counterpart of Roe's odd index; see Subsection 4.1. It turns out Connes' pairing $\langle \text{Ind}(\phi, D), \zeta \rangle$ is equal to the Fredholm index of a Toeplitz operator on N up to a certain constant multiple. The precise statement as follows:

Main Theorem (see Theorem 3.6). *Let M be a complete Riemannian manifold which is partitioned by N as previously. Let $S \to M$ be a graded Clifford bundle with the grading ϵ and denote by D the graded Dirac operator of S. Take $\phi \in GL_l(C_{\mathrm{w}}(M))$; see Definition 3.1. Then the following formula holds:*

$$\langle \text{Ind}(\phi, D), \zeta \rangle = -\frac{1}{8\pi i} \text{index}(T_{\phi|_N}).$$

Applying a topological formula of the Fredholm index for Toeplitz operators proved by P. Baum and R. G. Douglas [2], we obtain the following:

Corollary (see Corollary 3.7). *Let M be a partitioned manifold partitioned by (M^+, M^-, N). Denote by Π the characteristic function of M^+. Let $S \to M$ be a graded Clifford bundle with the grading ϵ, and denote by D the graded Dirac operator of S. We assume $\phi \in C^\infty(M; GL_l(\mathbb{C}))$ is bounded with bounded gradient* *and ϕ^{-1} is also a bounded function. Then one has*

$$\operatorname{index}\Big(\Pi(D+\epsilon)^{-1}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}(D+\epsilon)\Pi\colon \Pi(L^2(S))^l \to \Pi(L^2(S))^l\Big) = \int_{S^*N} \pi^*\mathrm{Td}(TN\otimes\mathbb{C})ch(\mathcal{S}^+)\pi^*ch(\phi).$$

The idea of the proof is as follows. Firstly, we calculate the Kasparov product $[\phi]\widehat{\otimes}[D]$ by using the Cuntz picture of $[D]$. Secondly, we calculate $\langle \mathrm{Ind}(\phi, D), \zeta\rangle$ explicitly by using the Hilbert transformation and a homotopy of Fredholm operators the case for $M = \mathbb{R}\times N$ and $\phi = 1\otimes\psi$ for $\psi \in C^\infty(N; GL_l(\mathbb{C}))$. Finally, we reduce the general case to $\mathbb{R}\times N$ by applying a similar argument in Higson [10].

Set $M = \mathbb{R}\times N$ and assume that N is of odd dimension. Let $i\colon N \ni x \mapsto (0, x) \in \mathbb{R}\times N$ be the inclusion map. Connes [5, 7] defined an element $i! \in KK^1(N, M)$. In this case, the main theorem is derived from the Roe–Higson index theorem by applying $i!$ *formally* as follows. The Dirac operator D on M defines an element $[D] \in KK^0(M, \mathrm{pt})$. Let us take a function $\phi\colon M \to GL_l(\mathbb{C})$ and suppose that ϕ determines an element $[\![\phi]\!] \in KK^1(M, M)$. By the Kasparov product, we have an element

$$[\![\phi]\!]\widehat{\otimes}[D] \in KK^1(M, \mathrm{pt})$$

and also

$$i!\widehat{\otimes}\big([\![\phi]\!]\widehat{\otimes}[D]\big) = [\![\phi|_N]\!]\widehat{\otimes}[D_N] \in KK^0(N, \mathbb{C}).$$

On the other hand, the Roe–Higson index theorem implies $\langle A(x), \zeta\rangle = q_*(i!\widehat{\otimes}x)$ with $x \in KK^1(M, \mathrm{pt})$ the fundamental class of the Dirac operator, where $A\colon KK^1(M, \mathrm{pt}) \to K_1(C^*(M))$ is the assembly map and $q_*\colon K^0(N) \to \mathbb{Z}$ is the homomorphism induced by the mapping q from N to a point. Thus we have

$$\langle A\big([\![\phi]\!]\widehat{\otimes}[D]\big), \zeta\rangle = q_*\big(i!\widehat{\otimes}\big([\![\phi]\!]\widehat{\otimes}[D]\big)\big) = \operatorname{index}(T_{\phi|_N}),$$

which is a statement of the main theorem for $M = \mathbb{R}\times N$.

This formal argument is correct only if ϕ is an element in $GL_l(C_0(M))$ since the above KK groups are defined as $KK^1(M, \mathrm{pt}) = KK^1(C_0(M), \mathbb{C})$, for instance. However, if ϕ were chosen as an element in $GL_l(C_0(M))$, ϕ should take a constant value outside a compact set of M. This implies that $\phi|_N$ is homotopic to a constant function in $GL_l(C(N))$ and thus $\operatorname{index}(T_{\phi|_N})$ should vanish. Therefore, in order to obtain non-trivial index, we have to employ a larger algebra than $C_0(M)$.

Higson [9] introduced such a C^*-algebra $C_h(M)$ that contains $C_0(M)$, which is now called the Higson algebra. It plays an important role in a K-homological proof of the Roe–Higson index theorem. The Higson algebra is defined as follows: $C_h(M)$ is the C^*-algebra generated by all smooth and bounded functions defined on M of which gradient is vanishing at infinity [9, p. 26]. $C_h(M)$ contains $C_0(M)$ as an ideal and is contained in $C_{\mathrm{w}}(M)$ by definition. Given $\psi \in C^\infty(N)$, we note that $\phi = 1 \otimes \psi$ does not belong to $C_h(M)$ in general. Thus the Higson algebra is not large enough to prove our main theorem. On the other hand, we have $\phi \in C_{\mathrm{w}}(M)$. Moreover, $C_{\mathrm{w}}(M)$ is the largest C^*-algebra A for which we can define $[D]$ as an element in $KK^0(A, C^*(M))$. They are reasons why we introduced the C^*-algebra $C_{\mathrm{w}}(M)$ in our main theorem.

This paper contains more general method than that of author's previous paper [17], which proves the case when two dimension by elementary method and contains a non-trivial example.

2. Preliminaries

2.1. Partitioned manifolds. We firstly describe a partitioned manifold, which is a main object in our main theorem.

Definition 2.1. Let M be an oriented complete Riemannian manifold. Assume that the triple (M^+, M^-, N) satisfies the following conditions:

- M^+ and M^- are submanifolds of M of the same dimension as M, $\partial M^+ \neq \emptyset$ and $\partial M^- \neq \emptyset$;
- $M = M^+ \cup M^-$;
- N is a closed submanifold of M of codimension one;
- $N = M^+ \cap M^- = -\partial M^+ = \partial M^-$.

Then we call (M^+, M^-, N) a partition of M. M is also called a partitioned manifold.

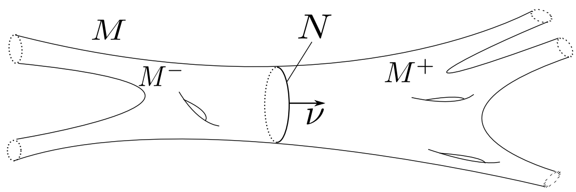

Figure 1. Partitioned manifold.

For example, we can consider $\mathbb{R} \times N$ is partitioned by $(\mathbb{R}_+ \times N, \mathbb{R}_- \times N, \{0\} \times N)$, where we set $\mathbb{R}_+ = \{t \in \mathbb{R} \,;\, t \geq 0\}$ and $\mathbb{R}_- = \{t \in \mathbb{R} \,;\, t \leq 0\}$.

We fix the notation of two functions which are defined by a partition.

Definition 2.2. Assume that M is partitioned by (M^+, M^-, N). Denote by Π the characteristic function of M^+ and set $\Lambda = 2\Pi - 1$.

2.2. The Roe algebra. In this subsection we recall the definition of the Roe algebra $C^*(M)$ and describe some properties of it.

Definition 2.3 ([14, p. 191]). Let M be a complete Riemannian manifold and $S \to M$ a Hermitian vector bundle. Denote by $L^2(S)$ the L^2-sections of S and $\mathcal{L}(L^2(S))$ the bounded operators on $L^2(S)$. Denote by $\mathcal{X}$ the algebra of bounded integral operators on $L^2(S)$ which have smooth kernels and finite propagation. Denote by $C^*(M)$ the closure of $\mathcal{X}$ and call it the Roe algebra.

We collect some properties of the Roe algebra which we shall need. Let $C_0(M)$ be the C^*-algebra of all continuous functions on M vanishing at infinity.

Proposition 2.4 ([11,14,15]). *Let M be a complete Riemannian manifold and $S \to M$ a Hermitian vector bundle. Denote by $\mathcal{D}^*(M)$ the $*$-subalgebra of $\mathcal{L}(L^2(S))$ which contains pseudolocal operators with finite propagation, where $T \in \mathcal{L}(L^2(S))$ is pseudolocal if $[f, T] \sim 0$ for all $f \in C_0(M)$, that is, $[f, T]$ is a compact operator. Denote by $D^*(M)$ the closure of $\mathcal{D}^*(M)$. Then the following holds:*

(i) *$D^*(M)$ is a unital C^*-algebra;*

(ii) *For all $u \in C^*(M)$ and $f \in C_0(M)$, one has $uf \sim 0$ and $fu \sim 0$;*

(iii) *$C^*(M)$ is equal to the closure of $\{u \in \mathcal{L}(L^2(S))\,;$ finite propagation and $uf \sim 0$ and $fu \sim 0$ for all $f \in C_0(M)\}$;*

(iv) *$C^*(M)$ is a closed $*$-bisided ideal in $D^*(M)$;*

(v) *Let D be a self-adjoint first order elliptic differential operator with finite propagation. Then one has $f(D) \in C^*(M)$ for all $f \in C_0(\mathbb{R})$ and $\chi(D) \in D^*(M)$ for any chopping function $\chi \in C(\mathbb{R}; [-1, 1])$. Here $\chi \in C(\mathbb{R}; [-1, 1])$ is a chopping function if χ is an odd function and $\lim_{x\to\infty} \chi(x) = 1$.*

Moreover, we assume M is a partitioned manifold. Then we can get the following properties.

Proposition 2.5. *If M is a partitioned manifold, then the following holds:*

(i) *For all $u \in C^*(M)$, one has $[\Pi, u] \sim 0$ and $[\Lambda, u] \sim 0$.*

(ii) *For all $u \in C^*(M)$ and $\varphi \in C(M)$ satisfies $\varphi = \Pi$ on the complement of a compact set in M, one has $[\varphi, u] \sim 0$.*

Proof. Due to [14, Lemma 1.5], $[\Pi, u]$ is of trace class for all $u \in \mathcal{X}$. So (i) is proved by the definition of $C^*(M)$. Since the support of $\Pi - \varphi$ is compact, there exists $f \in C_0(M)$ such that $f(\Pi - \varphi) = (\Pi - \varphi)f = \Pi - \varphi$. Therefore, we get (ii). □

2.3. The Roe cocycle. We define a certain cyclic 1-cocycle on $\mathcal{X}$, which is called the Roe cocycle.

Definition 2.6. For any $A, B \in \mathcal{X}$, set

$$\zeta(A, B) = \frac{1}{4}\mathrm{Tr}\big(\Lambda[\Lambda, A][\Lambda, B]\big).$$

We call $\zeta\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ the Roe cocycle.

Proposition 2.7 ([14, Proposition 1.6]). *ζ is a cyclic 1-cocycle on $\mathcal{X}$.*

In our main theorem, we would like to take the pairing of ζ with the index class in $K_1(C^*(M))$. For this purpose, we have to extend a domain of ζ.

Definition 2.8. Let M be a partitioned manifold and $S \to M$ a Hermitian vector bundle. Then we define a subalgebra $\mathcal{A}$ in $C^*(M)$ such that one has $u \in \mathcal{A}$ if $[\Lambda, u]$ is of trace class.

We note that $\mathcal{A}$ is a Banach algebra with norm $\|u\|_{\mathcal{A}} = \|u\| + \|[\Lambda, u]\|_1$, where $\|\cdot\|$ is the operator norm and $\|\cdot\|_1$ is the trace norm.

Proposition 2.9. *Let M be a partitioned manifold and $S \to M$ a Hermitian vector bundle. Then $\mathcal{A}$ is dense and closed under holomorphic functional calculus in $C^*(M)$.*

Proof. Since $\mathcal{X} \subset \mathcal{A} \subset C^*(M)$, $\mathcal{A}$ is dense in $C^*(M)$. By [6, p. 92 Proposition 3] and Proposition 2.5(ii), $\mathcal{A}$ is closed under holomorphic functional calculus in $C^*(M)$. □

Remark 2.10. Due to Definition 2.8, we can extend a domain of ζ to $\mathcal{A}^+$, the unitization of $\mathcal{A}$.

2.4. Pairing of the Roe cocycle with an element in K_1-group.

For any Banach algebra A, we denote by A^+ the unitization of A. Denote by $GL_l(A)$ the set of invertible elements u in $M_l(A^+)$ such that we have $u - 1 \in M_l(A)$. Set $K_1(A) = \pi_0(GL_\infty(A))$, the K_1-group of A. In this subsection, we describe Connes' pairing of the Roe cocycle with an element in $K_1(C^*(M))$.

Proposition 2.11. *Let M be a partitioned manifold and $S \to M$ be a Hermitian vector bundle. Then the inclusion $i\colon \mathcal{A} \to C^*(M)$ induces an isomorphism*

$$i_*\colon K_1(\mathcal{A}) \cong K_1(C^*(M)).$$

Proof. Use Proposition 2.9 and [6, p. 92 Proposition 3]. □

Due to Proposition 2.11, we can take the pairing of the Roe cocycle with an element in $K_1(C^*(M))$ through the isomorphism $i_*: K_1(\mathcal{A}) \cong K_1(C^*(M))$ as follows:

Definition 2.12 ([6, p. 109]). Define the map

$$\langle \cdot, \zeta \rangle : K_1(C^*(M)) \to \mathbb{C}$$

by $\langle [u], \zeta \rangle = \frac{1}{8\pi i} \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$, where we assume $[u]$ is represented by an element $u \in GL_l(\mathcal{A})$ and u_{ij} is the (i, j)-component of u. We note that this is Connes' pairing of cyclic cohomology with K-theory, and $1/8\pi i$ is a constant multiple in Connes' pairing.

We can write its pairing by a Fredholm index.

Proposition 2.13. *For any $u \in GL_l(C^*(M))$, one has*

$$\langle [u], \zeta \rangle = -\frac{1}{8\pi i} \operatorname{index}\big(\Pi u \Pi : \Pi(L^2(S))^l \to \Pi(L^2(S))^l\big).$$

Proof. Since both sides of this equation do not change by homotopy of $u \in GL_l(C^*(M))$, it suffices to show the case when $u \in GL_l(\mathcal{A})$. Then we obtain

$$8\pi i \langle [u], \zeta \rangle = \frac{1}{4} \sum_{i,j} \operatorname{Tr}\big(\Lambda[\Lambda, (u^{-1})_{ij}][\Lambda, u_{ji}]\big) = \frac{1}{4} \operatorname{Tr}\big(\Lambda[\Lambda, u^{-1}][\Lambda, u]\big).$$

Due to an equality $\Pi - \Pi u^{-1} \Pi u \Pi = -\Pi[\Pi, u^{-1}][\Pi, u]\Pi$, these two operators $\Pi - \Pi u^{-1} \Pi u \Pi$ and $\Pi - \Pi u \Pi u^{-1} \Pi$ are of trace class on $\Pi(L^2(S))^l$. Thus we get

$$\begin{aligned} &\operatorname{index}\big(\Pi u \Pi : \Pi(L^2(S))^l \to \Pi(L^2(S))^l\big) \\ &\qquad = \operatorname{Tr}\big(\Pi - \Pi u^{-1} \Pi u \Pi\big) - \operatorname{Tr}\big(\Pi - \Pi u \Pi u^{-1} \Pi\big) \end{aligned}$$

by [6, p. 88]. The above calculations implies desired equality. □

2.5. Toeplitz operators.

We recall the definition of Toeplitz operators for Dirac operators and its index theorem. The Fredholm index of the Toeplitz operator appears in our main theorem.

Definition 2.14. Let N be a closed Riemannian manifold. Let $S_N \to N$ be a Clifford bundle in the sense of [16, Definition 3.4] and D_N the Dirac operator of S_N. Denote by H_+ the subspace of $L^2(S_N)$ generated by non-negative eigenvectors of D_N and let $P: L^2(S_N) \to H_+$ be the projection.

Let $\phi \in C(N; M_l(\mathbb{C}))$ be a continuous map from N to $M_l(\mathbb{C})$. Then for any $s \in H_+^l$, we define the Toeplitz operator $T_\phi: H_+^l \to H_+^l$ by $T_\phi s = P(\phi s)$.

Toeplitz operators are Fredholm when the range of ϕ is contained in the set of invertible matrices.

Proposition 2.15 ([3, Lemma 2.10]). *Assume that ϕ is a smooth map, then $[\phi, P]$ is a pseudodifferential operator of order -1. Therefore $[\phi, P]$ is a compact operator on $L^2(S_N)^l$ for all $\phi \in C(N\,;\, M_l(\mathbb{C}))$. This implies a Toeplitz operator T_ϕ is a Fredholm operator for $\phi \in C(N\,;\, GL_l(\mathbb{C}))$.*

There exists an index theorem for Toeplitz operators. We can consider that this index theorem is a corollary of the Atiyah–Singer index theorem. Let $\pi\colon S^*N \to N$ be the unit sphere bundle of T^*N. Denote by $\sigma(x,\xi) \in \mathrm{End}((\pi^*S_N)_{(x,\xi)})$ the principal symbol of D_N for all $(x,\xi) \in S^*N$ and $\mathcal{S}^+_{(x,\xi)}$ the 1-eigenspace of $\sigma(x,\xi) = ic(\xi)$. Set $\mathcal{S}^+ = \bigcup_{(x,\xi)} \mathcal{S}^+_{(x,\xi)}$, then $\mathcal{S}^+$ turns out to be a subbundle of π^*S_N.

Proposition 2.16 ([2, Cororally 24.8]; [3, Theorem 4]). *The Fredholm index of Toeplitz operators satisfies the following:*

$$\mathrm{index}(T_\phi) = \langle \pi^*\mathrm{Td}(TN \otimes \mathbb{C})ch(\mathcal{S}^+)\pi^*ch(\phi), [S^*N]\rangle.$$

3. Main theorem

3.1. The index class. In this subsection, we define the odd index class in $K_1(C^*(M))$. After that, we take the pairing of the Roe cocycle with this class.

Let (M, g) be a complete Riemannian manifold and $S \to M$ a graded Clifford bundle with the Clifford action c and the grading ϵ. Denote by D the graded Dirac operator of S. Set $\|f\| = \sup_{x\in M} |f(x)|$ for $f \in C(M)$ and $\|X\| = \sup_{x\in M} \sqrt{g_x(X,X)}$ for $X \in C^\infty(TM)$. Denote by $C_b(M)$ the C^*-algebra of continuous bounded functions on M.

Definition 3.1. Define $\mathcal{W}(M)$ by the subset of $C^\infty(M)$ such that one has $f \in \mathcal{W}(M)$ if $\|f\| < +\infty$, $\|\mathrm{grad}(f)\| < +\infty$. Define $C_{\mathrm{w}}(M)$ by the closure of $\mathcal{W}(M)$ by the uniform norm on M.

Of course, $\mathcal{W}(M)$ is a unital $*$-subalgebra of $C_b(M)$. Therefore, $C_{\mathrm{w}}(M)$ is a unital C^*-algebra.

Remark 3.2. Let $C_h(M)$ be the Higson algebra of M, that is, $C_h(M)$ is the C^*-algebra generated by all smooth and bounded functions defined on M with which gradient is vanishing at infinity [9, p.26]. By definition, one has $C_h(M) \subset C_{\mathrm{w}}(M)$.

We assume $M = \mathbb{R} \times N$ and $\phi \in C^\infty(N)$. In this case, we have $1 \otimes \phi \in C_{\mathrm{w}}(M)$ but $1 \otimes \phi \notin C_h(M)$ in general. This is a merit of using $C_{\mathrm{w}}(M)$ (see Section 6).

We define a Kasparov $(C_{\mathrm{w}}(M), C^*(M))$-module which is made of the Dirac operator D. We assume that the Roe algebra $C^*(M)$ is an evenly graded C^*-algebra and a graded Hilbert $C^*(M)$-module simultaneously, where the grading is induced by ϵ. Since $\chi_0(x) = x(1 + x^2)^{-1/2}$ is a chopping function, the left composition of $F_D = D(1 + D^2)^{-1/2} \in D^*(M)$ on an element of $C^*(M)$ is an odd operator on $C^*(M)$.

Proposition 3.3. *Let $\mu: C_{\mathrm{w}}(M) \to \mathbb{B}(C^*(M))$ be the left composition of the multiplication operator: $\mu(f)u = fu \in C^*(M)$ for $f \in C_{\mathrm{w}}(M)$ and $u \in C^*(M)$. Then one has $[C^*(M), \mu, F_D] \in KK(C_{\mathrm{w}}(M), C^*(M))$.*

Proof. Our proof is similar to the Baaj-Julg picture of Kasparov modules [1, Proposition 2.2]. Firstly, we obtain $F_D \in \mathbb{B}(C^*(M))$, since F_D is a self-adjoint bounded operator on $L^2(S)$ and one has $F_D u \in C^*(M)$ for any $u \in C^*(M)$. Because of $1 - F_D^2 = 1 - D^2(1+D^2)^{-1} = (1+D^2)^{-1} \in C^*(M) = \mathbb{K}(C^*(M))$ and $F_D^* = F_D$, it suffices to show $[\mu(f), F_D] \in C^*(M)$.

Now, the following integral formula

$$[\mu(f), F_D] = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda)(1+D^2+\lambda)^{-1}[f, D](1+D^2+\lambda)^{-1} d\lambda + \frac{1}{\pi}\int_0^\infty \lambda^{-1/2} D(1+D^2+\lambda)^{-1}[D, f]D(1+D^2+\lambda)^{-1} d\lambda$$

is uniformly integrable for any $f \in \mathcal{W}(M)$ by

$$\|(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1} \quad \text{and} \quad \|D(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1/2}$$

for any $\lambda \geq 0$, and

$$[\mu(f), D] = -c(\mathrm{grad}(f)) \in D^*(M)$$

for any $f \in \mathcal{W}(M)$. So we obtain

$$[\mu(f), F_D] \in C^*(M)$$

for any $f \in \mathcal{W}(M)$ by $D(1+D^2+\lambda)^{-1} \in C^*(M)$ for any $\lambda \geq 0$. Thus we obtain

$$[\mu(f), F_D] \in C^*(M)$$

for any $f \in C_{\mathrm{w}}(M)$, since we have $\|[\mu(f), F_D]\| \leq 2\|f\|$ for any $f \in \mathcal{W}(M)$ and $\mathcal{W}(M)$ is dense in $C_{\mathrm{w}}(M)$. This implies $(C^*(M), \mu, F_D)$ is a Kasparov $(C_{\mathrm{w}}(M), C^*(M))$-module. □

Remark 3.4. Set $[D] = [C^*(M), \mu, F_D] \in KK(C_{\mathrm{w}}(M), C^*(M))$. Let χ be a chopping function. Then one has $\chi(D) - F_D \in C^*(M)$ by $\chi - \chi_0 \in C_0(M)$. Therefore, we obtain $[D] = [C^*(M), \mu, \chi(D)]$, that is, $[D]$ is independent of the choice of a chopping function χ.

This class $[D] \in KK(C_{\mathrm{w}}(M), C^*(M))$ goes to the E-theoretic class introduced by C. Wulff [19] by using the canonical map

$$KK\big(C_{\mathrm{w}}(M), C^*(M)\big) \to E\big(C_{\mathrm{w}}(M), C^*(M)\big).$$

Any $\phi \in GL_l(C_{\mathrm{w}}(M))$ determines $[\phi] \in K_1(C_{\mathrm{w}}(M))$. Due to the Kasparov product

$$\hat{\otimes}_{C_{\mathrm{w}}(M)}: K_1\big(C_{\mathrm{w}}(M)\big) \times KK\big(C_{\mathrm{w}}(M), C^*(M)\big) \to K_1\big(C^*(M)\big),$$

we get the index class in $K_1(C^*(M))$ as follows.

Definition 3.5. For any $\phi \in GL_l(C_{\mathrm{w}}(M))$, set

$$\mathrm{Ind}(\phi, D) = [\phi]\hat{\otimes}_{C_{\mathrm{w}}(M)}[D] \in K_1\big(C^*(M)\big).$$

3.2. The operator on N. Roughly speaking, our main theorem is Connes' pairing of the Roe cocycle with $\mathrm{Ind}(\phi, D) \in K_1(C^*(M))$ is calculated by the Fredholm index of a Toeplitz operator on a hypersurface N. In this subsection, we define its operator.

Let M be a partitioned manifold partitioned by (M^+, M^-, N). Let $S = S^+ \oplus S^-$, c and D are same in Subsection 3.1. Let $\nu \in C^\infty(TN)$ be the outward pointing normal unit vector field on $N = \partial M^-$.

Set $S_N = S^+|_N$ and define $c_N: C^\infty(TN) \to C^\infty(\mathrm{End}(S_N))$ by $c_N(X) = c(\nu)c(X)$. Then S_N can be equipped with a Clifford bundle structure with the Clifford action c_N. Denote by D_N the Dirac operator of S_N. We denote the restriction of $\phi \in GL_l(C_{\mathrm{w}}(M))$ to N by the same letter ϕ. Let T_ϕ be the Toeplitz operator with symbol ϕ. This Toeplitz operator T_ϕ is the operator on N in our main theorem.

3.3. The index theorem. We recall that we can take Connes' paring of the Roe cocycle with $\mathrm{Ind}(\phi, D) \in K_1(C^*(M))$. Our main theorem gives the result of its paring.

Theorem 3.6. *Let M be a partitioned manifold partitioned by (M^+, M^-, N). Let $S \to M$ be a graded Clifford bundle with the grading ϵ and denote by D the graded Dirac operator of S. We denote the restriction of $\phi \in GL_l(C_{\mathrm{w}}(M))$ to N by the same letter ϕ. Then the following formula holds:*

$$\langle \mathrm{Ind}(\phi, D), \zeta \rangle = -\frac{1}{8\pi i}\mathrm{index}(T_\phi).$$

If a function $\phi \in C^\infty(M; GL_l(\mathbb{C}))$ satisfies $\|\phi\| < \infty$, $\|\mathrm{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$, one has $\phi \in GL_l(C_{\mathrm{w}}(M))$ since the gradient of ϕ^{-1} is also bounded. The index theorem for Toeplitz operators (see Proposition 2.16) implies the following:

Corollary 3.7. *Let M be a partitioned manifold partitioned by (M^+, M^-, N), and Π the characteristic function of M^+. Let $S \to M$ be a graded Clifford bundle with the grading ϵ and denote by D the graded Dirac operator of S. Assume that $\phi \in C^\infty(M; GL_l(\mathbb{C}))$ satisfies $\|\phi\| < \infty$, $\|\mathrm{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$.*

Then one has

$$\operatorname{index}\Big(\Pi(D+\epsilon)^{-1}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}(D+\epsilon)\Pi\colon \Pi(L^2(S))\to\Pi(L^2(S))\Big) = \int_{S^*N}\pi^*\mathrm{Td}(TN\otimes\mathbb{C})ch(\mathcal{S}^+)\pi^*ch(\phi).$$

The proof of Theorem 3.6 and Corollary 3.7 is provided in Sections 6 and 7.

4. Suspensions and extensions

4.1. A relationship with Roe's odd index. In this subsection, we give a formal discussion about a relationship with Roe's odd index. Firstly, we recall the definition of Roe's odd index class odd-ind(D) [14, Definition 2.7]. Let M be a complete Riemannian manifold, $S\to M$ a Clifford bundle, D the Dirac operator of S and χ a chopping function. Then we have $\chi(D)\in D^*(M)$ and $q(\chi(D))$ is independent of a choice of χ, where $q\colon D^*(M)\to D^*(M)/C^*(M)$ is a quotient map. Moreover, we have

$$\big[q((\chi(D)+1)/2)\big]\in K_0\big(D^*(M)/C^*(M)\big)$$

by $\chi^2-1\in C_0(\mathbb{R};\mathbb{R})$. Let $\delta\colon K_0(D^*(M)/C^*(M))\to K_1(C^*(M))$ be a connecting homomorphism of the six-term exact sequence in operator K-theory. Set

$$\text{odd-ind}(D)=\delta\big(\big[q((\chi(D)+1)/2)\big]\big)\in K_1\big(C^*(M)\big).$$

Remark that we have

$$\text{odd-ind}(D)=\big[(D-i)(D+i)^{-1}\big]$$

if we choose

$$\chi(x)=\frac{1}{\pi}\mathrm{Arg}\Big(-\frac{x-i}{x+i}\Big),$$

where we choose the principal value of the argument is $(-\pi,\pi]$. Note that the map defining the odd index class is called the assembly map

$$A\colon K^1\big(C_0(M)\big)\to K_1\big(C^*(M)\big).$$

Secondly, we reconstruct this odd index in terms of KK-theory. Define $c.\colon\mathbb{C}\to C_{\mathrm{w}}(M)$ by $c_z(x)=z$ for $z\in\mathbb{C}$ and $x\in M$. Then we have

$$c.\in KK\big(\mathbb{C},C_{\mathrm{w}}(M)\big)$$

since this map $c.$ is a $*$-homomorphism. On the other hand, we have

$$\big[C^*(M)\oplus C^*(M),\mu\oplus\mu,\chi(D)\oplus(-\chi(D))\big]\in KK^1\big(C_{\mathrm{w}}(M),C^*(M)\big)$$

since $\chi_0(x)=x(x^2+1)^{-1/2}$ is a chopping function and we have $\chi-\chi_0\in C_0(\mathbb{R})$.

We denote by $[D]$ this KK-element. Then we obtain

$$c.\widehat{\otimes}_{C_{\mathrm{w}}(M)}[D] = \text{odd-ind}(D).$$

Finally, we may suppose our Kasparov product is a counterpart of Roe's odd index as follows. We composite the suspension isomorphism

$$KK\big(\mathbb{C}, C_{\mathrm{w}}(M)\big) \to KK^1\big(\mathbb{C}, C_{\mathrm{w}}(M) \otimes C_0(\mathbb{R})\big)$$

and the induced homomorphism by an inclusion

$$C_{\mathrm{w}}(M) \otimes C_0(\mathbb{R}) \to C_{\mathrm{w}}(M) \otimes C(S^1) \to C_{\mathrm{w}}(M \times S^1).$$

Thus we get a homomorphism

$$\sigma\colon KK\big(\mathbb{C}, C_{\mathrm{w}}(M)\big) \to KK^1\big(\mathbb{C}, C_{\mathrm{w}}(M \times S^1)\big).$$

On the other hand, there is a homomorphism

$$KK^1\big(C_{\mathrm{w}}(M), C^*(M)\big) \to KK^1\big(C_{\mathrm{w}}(M), C^*(M \times S^1)\big)$$

since KK is stable. Let D_{S^1} be a Dirac operator on S^1. D_{S^1} determines $[D_{S^1}] \in KK^1(C(S^1), \mathbb{C})$. By the composition of the Kasparov product $[D_{S^1}] \otimes_{\mathbb{C}} -$ and the induced map of this $*$-homomorphism

$$C_{\mathrm{w}}(M \times S^1) \ni f \mapsto f|_{M \times \{1\}} \otimes 1 \in C_{\mathrm{w}}(M) \otimes C(S^1),$$

we get a homomorphism

$$\tau\colon KK^1\big(C_{\mathrm{w}}(M), C^*(M)\big) \to KK\big(C_{\mathrm{w}}(M \times S^1), C^*(M \times S^1)\big).$$

Consequently, by using homomorphisms σ and τ, we may suppose our Kasparov product is a counterpart of Roe's odd index.

4.2. Wrong way functoriality. In this subsection, we see a correspondence between an index theorem for partitioned manifolds with Connes' wrong way functoriality. For the simplicity, we assume $M = \mathbb{R} \times N$ with N closed. Let $i\colon \{\mathrm{pt}\} \to \mathbb{R}$ be an inclusion map defined by $i(\mathrm{pt}) = 0$, and $p\colon \mathbb{R} \to \{\mathrm{pt}\}$ a constant map. Due to Connes (see, for instance, [5,7]), they define wrong way functoriality $i! \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$, $(i \times \mathrm{id}_N)! \in KK^1(C(N), C_0(M))$ and $p! \in KK^1(C_0(\mathbb{R}), \mathbb{C})$, respectively. We note the following:

$$i! \otimes_{C_0(\mathbb{R})} p! = (p \circ i)! = 1_{\mathbb{C}} \in KK^0(\mathbb{C}, \mathbb{C}).$$

Let D_N be the Dirac operator on N and $D_{\mathbb{R}}$ the Dirac operator on $\mathbb{R}$ defined by a spin structure of $\mathbb{R}$. These Dirac operators define elements in K-homology, that is, they define

$$[D_N] \in KK^*\big(C(N), \mathbb{C}\big) \quad \text{and} \quad [D_{\mathbb{R}}] = p! \in KK^1\big(C_0(\mathbb{R}), \mathbb{C}\big),$$

respectively. Moreover, D_N and $D_{\mathbb{R}}$ determine the Dirac operator D_M on $M = \mathbb{R} \times N$ satisfies

$$[D_M] = [D_{\mathbb{R}}] \otimes_{\mathbb{C}} [D_N] \in KK^{*+1}\big(C_0(M), \mathbb{C}\big).$$

Firstly, we assume $* = 0$. Let $[\![E]\!] \in KK^0(C_0(M), C_0(M))$ be a KK-element defined by a vector bundle $E \to M$ by using the inclusion map

$$KK^0\big(\mathbb{C}, C_0(M)\big) \to KK^0\big(C_0(M), C_0(M)\big).$$

Then we have

$$\begin{aligned}(i \times \mathrm{id}_N)! \otimes_{C_0(M)} &\big([\![E]\!] \otimes_{C_0(M)} [D_M]\big)\\ &= [\![E|_N]\!] \otimes_{C(N)} (i \times \mathrm{id}_N)! \otimes_{C_0(M)} \big([D_{\mathbb{R}}] \otimes_{\mathbb{C}} [D_N]\big)\\ &= [\![E|_N]\!] \otimes_{C(N)} [D_N].\end{aligned}$$

Therefore, by using the map

$$q_*\colon KK^0\big(C(N), \mathbb{C}\big) \to KK^0\big(\mathbb{C}, \mathbb{C}\big) \cong \mathbb{Z},$$

which is the homomorphism induced by the mapping q from N to a point, we have

$$q_*\big(i! \otimes_{C_0(M)} \big([\![E]\!] \otimes_{C_0(M)} [D_M]\big)\big)$$

is equal to the Fredholm index of the Dirac operator on N twisted by $E|_N$. This is a similar formula to the Roe–Higson index theorem. Combine the Roe–Higson index theorem, this implies the composition of the assembly map A with Connes' pairing of ζ is equal to $q_*(i! \otimes_{C_0(M)} -)$.

On the other hand, we assume $* = 1$. Take $\phi \in GL_l(C_0(M))$, then it defines an element $[\![\phi]\!] \in KK^1(C_0(M), C_0(M))$ by using the inclusion map

$$KK^1\big(\mathbb{C}, C_0(M)\big) \to KK^1\big(C_0(M), C_0(M)\big).$$

The similar argument in $* = 0$ implies

$$q_*\big(i! \otimes_{C_0(M)} \big([\![\phi]\!] \otimes_{C_0(M)} [D_M]\big)\big) = \mathrm{index}(T_{\phi|_N}).$$

However, since $\phi - 1$ vanishes at infinity, $\phi|_N$ is homotopic to a constant function in $GL_l(C(N))$. Therefore the right hand side is always 0.

One might suspect that this computation can be (partially) extended to a larger class of ϕ. This suspicious is natural, however, the equality $\langle A(x), \zeta\rangle = q_*(i! \otimes_{C_0(M)} x)$ is only proved when $x \in KK^1(C_0(M), \mathbb{C})$ is the fundamental class of the Dirac operator by using the Roe–Higson index theorem. In particular, the left hand side vanishes when $\dim M$ is even, that is, the above x is an element in the kernel of $\langle A(-), \zeta\rangle$. Therefore, we have to directly prove at least the equality

$$\langle \mathrm{Ind}(D, 1 \otimes \psi), \zeta\rangle = q_*\big(i! \otimes_{C_0(M)} \big([D_{\mathbb{R}}] \otimes_{\mathbb{C}} \big([\![\psi]\!] \otimes_{C(N)} [D_N]\big)\big)\big)$$

for $\psi \in GL_l(C^\infty(N))$ in order to prove our main theorem by using wrong way functoriality. Of course, our main theorem implies this formula.

4.3. The Roe cocycle and an extension. In this subsection, we see a relationship between the Roe cocycle ζ and an extension. Let M be a partitioned manifold and Π the characteristic function of M^+. Set $H = \Pi(L^2(S))$. Let $q: \mathcal{L}(H) \to \mathcal{Q}(H)$ be the quotient map to the Calkin algebra. Define $\sigma: C^*(M) \to \mathcal{L}(H)$ by $\sigma(A) = \Pi A \Pi$ and $\tau: C^*(M) \to \mathcal{Q}(H)$ by $\tau = q \circ \sigma$. Set

$$E = \big\{(A, T) \in C^*(M) \oplus \mathcal{L}(H)\,;\, \tau(A) = q(T)\big\}.$$

Then we get an extension τ of $C^*(M)$:

$$0 \to \mathcal{K}(H) \hookrightarrow E \to C^*(M) \to 0.$$

This extension τ corresponds to the Fredholm module $(L^2(S), \Lambda)$ on $\mathcal{X}$ and the Connes–Chern character of $(L^2(S), \Lambda)$ equals the Roe cocycle.

By the definition of a pairing $\langle \cdot, \cdot \rangle_{\mathrm{ind}}: K_1(C^*(M)) \times \mathrm{Ext}(C^*(M)) \to \mathbb{Z}$ and Proposition 2.13, we obtain

$$\langle [u], \zeta \rangle = \langle [u], [\tau] \rangle_{\mathrm{ind}} = \mathrm{index}(\Pi u \Pi)$$

up to a certain constant multiple for any $[u] \in K_1(C^*(M))$.

Moreover, these are equal to the connecting homomorphism of this extension:

$$\partial: K_1\big(C^*(M)\big) \to K_0\big(\mathcal{K}(H)\big) \cong \mathbb{Z}.$$

In fact, for any unitary

$$u \in U\big(C^*(M)\big) = \big\{u \in U(C^*(M)^+)\,;\, u - 1 \in C^*(M)\big\},$$

denote by $v(u)$ the partial isometry part of the polar decomposition of $\sigma(u)$. Then we have $\tau(u) = q(v(u))$ since $\sigma(u)$ is an essential unitary operator on H. Therefore, $(u, v(u)) \in E$ is a partial isometry lift of u. So we obtain

$$\partial\big([u]\big) = \big[\Pi - v(u)^* v(u)\big] - \big[\Pi - v(u) v(u)^*\big] \in K_0\big(\mathcal{K}(H)\big).$$

By the identification $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$, we have

$$\partial\big([u]\big) = \mathrm{index}\big(v(u)\big) = \mathrm{index}\big(\sigma(u)\big).$$

Therefore, we obtain

$$\langle [u], \zeta \rangle = \langle [u], [\tau] \rangle_{\mathrm{ind}} = \partial\big([u]\big) = \mathrm{index}(\Pi u \Pi)$$

up to a certain constant multiple.

5. Calculation of the Kasparov product in the index class

5.1. Explicit formula of the index class. In this subsection, we represent the index class by an element in $GL_l(C^*(M))$. For this purpose, we present $[D]$ by the Cuntz picture of $KK(C_{\mathrm{w}}(M), C^*(M))$ and then we calculate Kasparov product $[\phi]\widehat{\otimes}_{C_{\mathrm{w}}(M)}[D]$. Set

$$C_b^*(M) = \left\{u + \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix};\, u \in C^*(M),\ f, g \in C_b(M)\right\}.$$

Then $C_b^*(M)$ is a C^*-subalgebra of $D^*(M)$ and contains $C^*(M)$ as an essential ideal. Let $\chi \in C(\mathbb{R}; [-1, 1])$ be a chopping function. Set $\eta(x) = (1 - \chi(x)^2)^{1/2} \in C_0(\mathbb{R})$. Then η is a positive even function and we have $\eta(D) \in C^*(M)$.

Proposition 5.1. *Let $\iota\colon C_b^*(M) \to M_\infty(C_b^*(M))$ be the standard inclusion and $\mathcal{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Set $\mathcal{D}_\chi = \chi(D) + \epsilon\eta(D) \in D^*(M)$,*

$$\psi_{\chi,+}(f) = \iota\left(\mathcal{D}_\chi \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}_\chi\right) \quad \text{and} \quad \psi_-(f) = \iota\left(\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}\right).$$

Then

$$(\psi_+, \psi_-)\colon C_{\mathrm{w}}(M) \to \mathbb{B}\big(\mathbb{H}_{C^*(M)}\big) \triangleright C^*(M) \otimes \mathcal{K}$$

is a prequasihomomorphism from $C_{\mathrm{w}}(M)$ to $C^*(M) \otimes \mathcal{K}$ in the sense of [8, Definition 2.1] and one has $[D] = [\psi_+, \psi_-]$ in $KK(C_{\mathrm{w}}(M), C^*(M))$. We note that (ψ_+, ψ_-) is a quasi-homomorphism in the sense of [12, Definition 3.3.1]. Here, we omit the subscript χ for the simplicity.

Proof. We assume $C^*(M)^{\mathrm{op}}$ is equipped with the interchanged grading of $C^*(M)$. Then $(C^*(M)^{\mathrm{op}}, 0, \chi(D))$ is a degenerate Kasparov $(C_{\mathrm{w}}(M), C^*(M))$-module. So we obtain

$$[D] = \left[C^*(M) \oplus C^*(M)^{\mathrm{op}}, \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \chi(D) & 0 \\ 0 & \chi(D) \end{bmatrix}\right].$$

Since the following difference

$$\begin{bmatrix} \chi(D) & \epsilon\eta(D) \\ \epsilon\eta(D) & \chi(D) \end{bmatrix} - \begin{bmatrix} \chi(D) & 0 \\ 0 & \chi(D) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon\eta(D) \\ \epsilon\eta(D) & 0 \end{bmatrix} \in M_2\big(C^*(M)\big)$$

is an $C^*(M)$-compact operator, we obtain $[D] = [C^*(M) \oplus C^*(M)^{\mathrm{op}}, \mu \oplus 0, G]$, where G is the operator of the first term of the above difference.

The even grading of $C^*(M)$ is defined by the decomposition of $S^+ \oplus S^-$, so we have

$$[D] = \left[E = C^*(M)^{\mathrm{tri}} \oplus C^*(M)^{\mathrm{tri}}, \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{bmatrix}\right]$$

under the canonical isomorphism (see [4, p.119]):

$$KK^0\big(C_{\mathrm{w}}(M), C^*(M)\big) \cong KK^0\big(C_{\mathrm{w}}(M), C^*(M)^{\mathrm{tri}}\big),$$

where $^{\mathrm{tri}}$ means the trivially grading. Now, we conjugate by $\mathcal{D} \oplus 1 \in \mathbb{B}(E)$. Then we obtain

$$[D] = \left[E, \begin{bmatrix} \mathcal{D}(\mu \oplus 0)\mathcal{D} & 0 \\ 0 & 0 \oplus \mu \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right].$$

By adding a degenerate $(C_{\mathrm{w}}(M), C^*(M)^{\mathrm{tri}})$-module $(\mathbb{H}_{C^*(M)} \oplus \mathbb{H}_{C^*(M)}, 0, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$, we obtain

$$[D] = \left[(C^*(M)^{\mathrm{tri}} \oplus \mathbb{H}_{C^*(M)})^2, \begin{bmatrix} (\mathcal{D}(\mu \oplus 0)\mathcal{D}) \oplus 0 & 0 \\ 0 & (0 \oplus \mu) \oplus 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right],$$

where $\mathbb{H}_{C^*(M)}$ is a countably generated Hilbert space over $C^*(M)^{\mathrm{tri}}$. We define a unitary operator

$$W : C^*(M)^{\mathrm{tri}} \oplus \mathbb{H}_{C^*(M)} \to \mathbb{H}_{C^*(M)}$$

by $W(a_0, (a_i)_{i=1}^{\infty}) = (a_i)_{i=0}^{\infty}$ and conjugate by $W \oplus W$. So we obtain

$$[D] = \left[\mathbb{H}_{C^*(M)} \oplus \mathbb{H}_{C^*(M)}, \begin{bmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right].$$

We can show $\psi_+(f) \in M_\infty(C_b^*(M))$ by using

$$\left[\begin{bmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \in \mathbb{K}\big(\widehat{\mathbb{H}}_{C^*(M)^{\mathrm{tri}}}\big).$$

Therefore,

$$(\psi_+, \psi_-) : C_{\mathrm{w}}(M) \to \mathbb{B}\big(\mathbb{H}_{C^*(M)}\big) \triangleright C^*(M) \otimes \mathcal{K}$$

is a prequasihomomorphism from $C_{\mathrm{w}}(M)$ to $C^*(M) \otimes \mathcal{K}$ and we obtain

$$[D] = [\psi_+, \psi_-]. \qquad \square$$

Remark 5.2. By definition, one has

$$\mathcal{D}\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}\mathcal{D} - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = \mathcal{D}\begin{bmatrix} f\eta(D)^+ & [f, \chi(D)^-] \\ 0 & \eta(D)^- f \end{bmatrix} \in C^*(M)$$

for any $f \in C_{\mathrm{w}}(M)$. We get another proof of $\psi_+(f) \in M_\infty(C_b^*(M))$.

The Cuntz picture of Kasparov modules suits the Kasparov product with an element in K_1-group [8, Remark 1, Theorem 3.3]. See also [12, p. 60], which contains an explicit formula.

Proposition 5.3. *For any $\phi \in GL_l(C_{\mathrm{w}}(M))$, one has*

$$\mathrm{Ind}(\phi, D) = \left[\mathcal{D}\begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix}\mathcal{D}\begin{bmatrix} 1 & 0 \\ 0 & \phi^{-1} \end{bmatrix}\right] \in K_1\big(C^*(M)\big).$$

Proof. Firstly, we obtain

$$\psi_+(\phi-1)+1 = j\left(\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\right) \quad\text{and}\quad \psi_-(\phi-1)+1 = j\left(\begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix}\right),$$

where $j\colon GL_l(C_b^*(M)) \to GL_\infty(C_b^*(M))$ is the standard inclusion. Thus we get

$$\mathrm{Ind}(\phi, D) = \left[\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\right] \in K_1(C^*(M)). \qquad \square$$

The last of this subsection, we back to Connes' pairing in our main theorem.

Remark 5.4. By Proposition 2.13, one has

$$\begin{aligned}&\langle \mathrm{Ind}(\phi, D), \zeta\rangle\\ &= -\frac{1}{8\pi i}\mathrm{index}\left(\Pi\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\Pi\colon \Pi\big(L^2(S)\big)^l \to \Pi\big(L^2(S)\big)^l\right).\end{aligned}$$

On the other hand, $\Pi u \Pi$ is Fredholm for any $u \in GL_l(C_b^*(M))$ because of $[f, \Pi] = 0$ for any $f \in C_b(M)$. This implies

$$\langle \mathrm{Ind}(\phi, D), \zeta\rangle = -\frac{1}{8\pi i}\mathrm{index}\left(\Pi\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\Pi\right).$$

In order to use bellow sections, we fix notation here. Set

$$u_{\chi,\phi} = \mathcal{D}_\chi\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}_\chi \quad\text{and}\quad v_{\chi,\phi} = u_{\chi,\phi} - \begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix}.$$

Then we obtain

$$v_{\chi,\phi} = \mathcal{D}_\chi\begin{bmatrix}(\phi-1)\eta(D)^+ & [\phi, \chi(D)^-]\\ 0 & \eta(D)^-(\phi-1)\end{bmatrix}.$$

5.2. Another formula in the special case. By Remark 5.4, our main theorem is the coincidence of two Fredholm indices:

$$\mathrm{index}\big(\Pi u_{\chi,\phi}\Pi\big) = \mathrm{index}(T_\phi).$$

Both sides of this equation do not change a homotopy of ϕ. Therefore, it suffices to show the case when $\phi \in GL_l(\mathcal{W}(M))$. In this case, $\phi\colon M \to GL_l(\mathbb{C})$ is a smooth function such that $\|\phi\| < \infty$, $\|\mathrm{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$. Moreover, we also assume that ϕ satisfies $[|D|, \phi] \in \mathcal{L}(L^2(S))$. This condition is a technical assumption in this subsection. For example, if $S \to M$ has bounded geometry and all derivatives of ϕ are bounded, then ϕ satisfies this technical assumption. Set

$$\mathcal{W}_1(M) = \big\{f \in \mathcal{W}(M); \big[|D|, f\big] \in \mathcal{L}\big(L^2(S)\big)\big\}.$$

In this subsection, we use $\chi_0(x) = x(1+x^2)^{-1/2}$ as a chopping function, that is, we use $\mathcal{D} = \mathcal{D}_{\chi_0}$.

In order to prove our main theorem, we perturb the operator $\mathcal{D}\left[\begin{smallmatrix}\phi & 0\\ 0 & 1\end{smallmatrix}\right]\mathcal{D}$ by a homotopy. Firstly, for any $t \in [0,1]$, set

$$F_t = t + (1-t)(1+D^2)^{-1/2} \in D^*(M).$$

For any $t \in (0,1]$ and $x \in \mathbb{R}$, set

$$f_t(x) = \frac{1}{t + (1-t)(1+x^2)^{-1/2}}.$$

Then we obtain $f_t(D) \in D^*(M)$ since we have $f_t - 1/t \in C_0(M)$. Thus F_t has a bounded inverse $F_t^{-1} = f_t(D)$ for any $t \in (0,1]$.

Secondly, because of

$$(D+\epsilon)^{-1}\begin{bmatrix} f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon)\sigma - \begin{bmatrix} 0 & 0\\ 0 & f\end{bmatrix}\sigma = (D+\epsilon)^{-1}\begin{bmatrix} f & -c(\mathrm{grad}(f))^- \\ 0 & f\end{bmatrix}\sigma$$

for any $f \in M_l(\mathcal{W}(M))$ and $\sigma \in C_c^\infty(S)$, we obtain

$$\left\| (D+\epsilon)^{-1}\begin{bmatrix} f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon)\sigma\right\|_{L^2} \leq \bigl(2\|f\| + \|\mathrm{grad}(f)\|\bigr)\|\sigma\|_{L^2}.$$

This implies

$$\rho(f) = (D+\epsilon)^{-1}\begin{bmatrix} f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon) \in \mathcal{L}\bigl(L^2(S)\bigr)$$

since $C_c^\infty(S)$ is dense in $L^2(S)$. Moreover, we obtain $\rho(f) \in C_b^*(M)$ by $(D+\epsilon)^{-1} \in C^*(M)$ and

$$\begin{bmatrix} f & -c(\mathrm{grad}(f))^- \\ 0 & f\end{bmatrix} \in D^*(M).$$

Finally, set $\rho_0(f) = \mathcal{D}\left[\begin{smallmatrix} f & 0\\ 0 & 0\end{smallmatrix}\right]\mathcal{D}$ and $\rho_t(f) = F_t^{-1}\rho(f)F_t$ for any $t \in (0,1]$ and $f \in \mathcal{W}(M)$. Formally, we set $F_0^{-1} = (1+D^2)^{1/2}$. Then we obtain

$$\rho_t(f) = F_t^{-1}\rho(f)F_t \in \mathcal{L}\bigl(L^2(S)\bigr)$$

for any $t \in [0,1]$ and $f \in \mathcal{W}(M)$. We note that we have

$$\rho_0(f) = \mathcal{D}\begin{bmatrix} f & 0\\ 0 & 0\end{bmatrix}\mathcal{D} \quad \text{and} \quad \rho_1(f) = \rho(f).$$

This family of bounded operators $t \mapsto \rho_t(f)$ is continuous in $C_b^*(M)$ for $f \in \mathcal{W}_1(M)$.

Proposition 5.5. *For any* $t \in [0,1]$ *and* $f \in M_l(\mathcal{W}_1(M))$*, one has* $\rho_t(f) \in M_l(C_b^*(M))$*. Moreover,*

$$[0,1] \ni t \mapsto \rho_t(f) \in M_l\big(C_b^*(M)\big) \subset M_l\big(\mathcal{L}\big(L^2(S)\big)\big)$$

is continuous.

Proof. It suffices to show the case when $l = 1$.

Firstly we show $\rho_t(f) \in C_b^*(M)$. When $t = 0, 1$, we already proved. We assume $t \in (0,1)$. We have

$$\rho_t(f) - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = F_t^{-1}(D+\epsilon)^{-1}$$
$$\cdot \begin{bmatrix} tf + (1-t)f(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- + (1-t)[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf + (1-t)(1+D^2)^{-1/2} f \end{bmatrix}.$$

Because of $F_t^{-1} \in D^*(M)$, $(D+\epsilon)^{-1} \in C^*(M)$ and

$$\begin{bmatrix} tf + (1-t)f(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- + (1-t)[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf + (1-t)(1+D^2)^{-1/2} f \end{bmatrix} \in D^*(M),$$

we obtain $\rho_t(f) \in C_b^*(M)$.

Next we show continuity of $t \mapsto \rho_t(f)$. F_t^{-1}, $\rho(f)$, and F_t are bounded operators for any $t \in (0,1]$, and $[0,1] \ni t \mapsto F_t \in \mathcal{L}(L^2(M))$ is continuous. Thus $t \mapsto \rho_t(f)$ is continuous on $(0,1]$. The rest of proof is continuity at $t = 0$. First, we show $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$ for any $t \in [0,1]$. Set

$$g_t(x) = \frac{x}{(1+x^2)(t + (1-t)(1+x^2)^{-1/2})}$$

and

$$h_t(x) = \frac{1}{(1+x^2)(t + (1-t)(1+x^2)^{-1/2})}.$$

Then we have

$$|g_t(x)| = \frac{1}{t(|x| + 1/|x|) + (1-t)\sqrt{1 + 1/x^2}} \leq \frac{1}{2t + 1 - t} \leq 1$$

and $|h_t(x)| \leq 1$. Thus we obtain $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$ by

$$(D+\epsilon)^{-1}F_t^{-1} = D(1+D^2)^{-1}F_t^{-1} + \epsilon(1+D^2)^{-1}F_t^{-1} = g_t(D) + \epsilon h_t(D).$$

By using $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$, we can prove continuity at $t = 0$. For any $t > 0$, a difference $\rho_t(f) - \rho_0(f)$ equals

$$(D+\epsilon)^{-1}F_t^{-1}\begin{bmatrix} tf - tf(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- - t[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf - t(1+D^2)^{-1/2}f \end{bmatrix}$$
$$+ \{(D+\epsilon)^{-1}F_t^{-1} - \mathcal{D}\}\begin{bmatrix} f(1+D^2)^{-1/2} & [f, D^-(1+D^2)^{-1/2}] \\ 0 & (1+D^2)^{-1/2}f \end{bmatrix}.$$

The first term converges to 0 with the operator norm as $t \to 0$.

We show the second term converges to 0 with the operator norm as $t \to 0$. Due to

$$\mathcal{D} - (D+\epsilon)F_t = t(D+\epsilon)\big\{1 - (1+D^2)^{-1/2}\big\},$$

the second term is equal to

$$t(D+\epsilon)^{-1}F_t^{-1}\big\{(1+D^2)^{-1/2} - 1\big\}(1+D^2)^{1/2}$$
$$\cdot \begin{bmatrix} f(1+D^2)^{-1/2} & [f, D^-(1+D^2)^{-1/2}] \\ 0 & (1+D^2)^{-1/2}f \end{bmatrix}.$$

Therefore, if $(1+D^2)^{1/2}f(1+D^2)^{-1/2}$ and $(1+D^2)^{1/2}[f, D(1+D^2)^{-1/2}]$ are bounded, the second term converges to 0 with the operator norm as $t \to 0$. We show that $(1+D^2)^{1/2}f(1+D^2)^{-1/2}$ and $(1+D^2)^{1/2}[f, D(1+D^2)^{-1/2}]$ are bounded. By using the following equalities

$$(D^2+1)^{1/2}f(D^2+1)^{-1/2} = \big[(D^2+1)^{1/2}, f\big](D^2+1)^{-1/2} + f$$

and

$$(D^2+1)^{1/2}\big[f, D(D^2+1)^{-1/2}\big] = \big[(D^2+1)^{1/2}, f\big]D(D^2+1)^{-1/2} + [f, D],$$

it suffices to show that $[(D^2+1)^{1/2}, f]$ is a bounded operator. Because of $\alpha(x) = \sqrt{x^2+1} - |x| \in C_0(\mathbb{R})$, we have $\alpha(D) \in \mathcal{L}(L^2(S))$. This implies $[(D^2+1)^{1/2}, f]$ is bounded if and only if $[|D|, f]$ is bounded. We note that boundness of $[|D|, f]$ is required the definition of the algebra $\mathcal{W}_1(M)$. Hence,

$$(D^2+1)^{1/2}f(D^2+1)^{-1/2} \quad \text{and} \quad (D^2+1)^{1/2}\big[f, D(D^2+1)^{-1/2}\big]$$

are bounded. Thus the second term converges to 0 as $t \to 0$. Therefore $t \mapsto \rho_t(f)$ is continuous. □

Due to Proposition 5.5, the following maps

$$\Pi\big\{\rho_t(\phi - 1) + 1\big\}\Pi\colon \Pi\big(L^2(S)\big)^l \to \Pi\big(L^2(S)\big)^l$$

determines a continuous family of Fredholm operators for any $\phi \in GL_l(\mathcal{W}_1(M))$. Therefore, we obtain

$$\langle \mathrm{Ind}(\phi, D), \zeta \rangle = -\frac{1}{8\pi i} \mathrm{index}\left(\Pi (D+\epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D+\epsilon)\Pi \right)$$

for any $\phi \in GL_l(\mathcal{W}_1(M))$.

Remark 5.6. In the definition of ρ_t, we do not use the assumption $[|D|, f] \in \mathcal{L}(L^2(S))$. In particular, one has $\rho(f) \in C_b^*(M)$ for $f \in \mathcal{W}(M)$. Set $\varrho(\phi) = \rho(\phi - 1) + 1$ for any $\phi \in GL_l(\mathcal{W}(M))$. Then the operator $\Pi \varrho(\phi) \Pi$ is Fredholm for all $\phi \in GL_l(\mathcal{W}(M))$.

6. The case for $\mathbb{R} \times N$

Let N be a closed manifold. In this section, we prove Theorem 3.6 in the case that $M = \mathbb{R} \times N$. Recall that $\mathbb{R} \times N$ is partitioned by $(\mathbb{R}_+ \times N, \mathbb{R}_- \times N, \{0\} \times N)$. Let $S_N \to N$ be a Clifford bundle, c_N the Clifford action on S_N and D_N the Dirac operator on S_N. Given $\phi \in C^\infty(N; GL_l(\mathbb{C}))$, we define the map $\widetilde{\phi}: \mathbb{R} \times N \to GL_l(\mathbb{C})$ by $\widetilde{\phi}(t, x) = \phi(x)$. We often denote $\widetilde{\phi}$ by ϕ in the sequel. Note that we have $\phi \in GL_l(\mathcal{W}_1(\mathbb{R} \times N))$.

Let $p: \mathbb{R} \times N \to N$ be the projection to N. Set $S = p^* S_N \oplus p^* S_N$ and $\epsilon = 1 \oplus (-1)$, where ϵ is the grading operator on S. Then we define a Clifford action

$$c: C^\infty(TM) \to C^\infty\big(\mathrm{End}(S)\big)$$

by

$$c(\mathrm{d}/\mathrm{d}t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad c(X) = \begin{bmatrix} 0 & c_N(X) \\ c_N(X) & 0 \end{bmatrix} \quad \text{for all } X \in C^\infty(TN).$$

Here $\mathrm{d}/\mathrm{d}t$ is a coordinate unit vector field on $\mathbb{R}$. Then $S \to M$ is a Clifford bundle and the Dirac operator D of S is given by

$$D = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + D_N \\ -\mathrm{d}/\mathrm{d}t + D_N & 0 \end{bmatrix}.$$

Denote by H_+ the subspace of $L^2(S_N)$ which is generated by non-negative eigenvectors of D_N. Also denote by H_- the orthogonal complement of H_+ in $L^2(S)$. Set $F = 2P - 1$, where P is the projection to H_+.

Due to Subsection 5.2, it suffices to show

$$\mathrm{index}\left(\Pi (D+\epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D+\epsilon)\Pi \right) = \mathrm{index}(T_\phi).$$

For this purpose, we perturb the operator $\Pi \varrho(\phi) \Pi$ by a homotopy. We firstly estimate the supremum of some functions to prove a continuity of the homotopy.

Lemma 6.1.

(i) *Set*

$$f_s(x) = \frac{x}{x^2 + (1-s)^2} \quad \text{and} \quad g_s(x) = \frac{1}{x^2 + (1-s)^2}$$

for all $s \in [0,1]$ *and* $x \in \mathbb{R} \setminus (-s, s)$. *Then one has* $\sup_x |f_s(x)| \leq 2$ *and* $\sup_x |g_s(x)| \leq 2$ *for all* $s \in [0,1]$.

(ii) *Set*

$$\mu_{\lambda,s}(x) = \frac{1}{x^2 + \{(1-s)\lambda + s\mathrm{sgn}(\lambda)\}^2 + (1-s)^2}$$

and $\nu_{\lambda,s}(x) = x\mu_{\lambda,s}(x)$ *for all* $\lambda \in \mathbb{R}$, $s \in [0,1)$ *and* $x \in \mathbb{R}$, *where* $\mathrm{sgn}(\lambda)$ *is* 1 *if* $\lambda \geq 0$ *or* -1 *if* $\lambda < 0$. *Then one has*

$$\sup_x |\mu_{\lambda,s}(x)| \leq \frac{1}{(1-s)^2(\lambda^2+1)} \quad \text{and} \quad \sup_x |\nu_{\lambda,s}(x)| \leq \frac{1}{2(1-s)\sqrt{\lambda^2+1}}$$

for all $\lambda \in \mathbb{R}$, $s \in [0,1)$.

Proof. (i) For $0 \leq s \leq 1/2$, we have

$$|f_s(x)| \leq f_s(1-s) \leq 1.$$

For $1/2 \leq s \leq 1$, we have

$$|f_s(x)| \leq f_s(s) \leq 2.$$

This implies $\sup_x |f_s(x)| \leq 2$. On the other hand, we have $|g_s(x)| \leq g_s(s) \leq 2$.

(ii) For $\lambda \geq 0$, we have

$$(1-s)\lambda + s\mathrm{sgn}(\lambda) \geq (1-s)\lambda \geq 0.$$

On the other hand, for $\lambda < 0$, we have

$$(1-s)\lambda + s\mathrm{sgn}(\lambda) \leq (1-s)\lambda < 0.$$

So we obtain $|\mu_{\lambda,s}(x)| \leq h_{\lambda,s}(0) \leq 1/(1-s)^2(\lambda^2+1)$.

On the other hand, we obtain

$$\begin{aligned} |\nu_{\lambda,s}(x)| &\leq \nu_{\lambda,s}\left(\sqrt{\{(1-s)\lambda + s\mathrm{sgn}(\lambda)\}^2 + (1-s)^2}\right) \\ &\leq \frac{1}{2(1-s)\sqrt{\lambda^2+1}}. \end{aligned}$$

□

Proposition 6.2. *Set*

$$D_s = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + (1-s)D_N + sF \\ -\mathrm{d}/\mathrm{d}t + (1-s)D_N + sF & 0 \end{bmatrix}$$

for all $s \in [0,1]$ *and*

$$u_{\phi,s} = \big(D_s + (1-s)\epsilon\big)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} \big(D_s + (1-s)\epsilon\big).$$

Then the map $[0,1] \ni s \mapsto u_{\phi,s} \in \mathcal{L}(L^2(S)^l)$ *is continuous.*

Proof. It suffices to show the case when $l = 1$. Since we have $(\mathrm{d}/\mathrm{d}t)^* = -\mathrm{d}/\mathrm{d}t$ and D_N is a Dirac operator on N, D_s is a self-adjoint closed operator densely defined on $\mathrm{domain}(D_s) = \mathrm{domain}(D)$.

Next we show $\sigma(D_s) \cap (-s,s) = \emptyset$ for all $s \in (0,1]$. Set

$$T_s = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + (1-s)D_N \\ -\mathrm{d}/\mathrm{d}t + (1-s)D_N & 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & F \\ F & 0 \end{bmatrix}.$$

These operators T_s and J are self-adjoint and we have

$$D_s = T_s + sJ \quad \text{and} \quad T_sJ + JT_s = 2(1-s)D_NF \geq 0$$

on $\mathrm{domain}(D)$. So for any $\sigma \in \mathrm{domain}(D)$, we obtain

$$\begin{aligned} \|D_s\sigma\|_{L^2}^2 &= \|T_s\sigma\|_{L^2}^2 + s^2\|J\sigma\|_{L^2}^2 + s\langle (T_sJ + JT_s)\sigma, \sigma\rangle_{L^2} \\ &\geq s^2\|J\sigma\|_{L^2}^2 = s^2\|\sigma\|_{L^2}^2. \end{aligned}$$

This implies $\sigma(D_s) \cap (-s,s) \neq \emptyset$. In particular, D_1 has a bounded inverse.

On the other hand, when $s \in [0,1)$, we have $(D_s + (1-s)\epsilon)^{-1} \in \mathcal{L}(L^2(S))$ since $(D_s + (1-s)\epsilon)^2 = D_s^2 + (1-s)^2$ is invertible. Therefore $u_{\phi,s}$ is well defined as a closed operator on $L^2(S)$ with $\mathrm{domain}(u_{\phi,s}) = \mathrm{domain}(D)$ for all $s \in [0,1]$. Thus we obtain $u_{\phi,s} \in \mathcal{L}(L^2(S))$ by

$$u_{\phi,s} = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + (D_s + (1-s)\epsilon)^{-1} \cdot \begin{bmatrix} (1-s)(\phi - 1) & -(1-s)c_N(\mathrm{grad}(\phi)) + s[\phi, F] \\ 0 & (1-s)(\phi - 1) \end{bmatrix}.$$

Next we show continuity of $[0,1] \ni s \mapsto u_{\phi,s} \in \mathcal{L}(L^2(S))$. First, because

$$\big(D_s + (1-s)\epsilon\big)^{-1} = f_s(D_s) + (1-s)\epsilon g_s(D_s),$$

we have

$$\|\big(D_s + (1-s)\epsilon\big)^{-1}\| \leq \sup_x |f_s(x)| + (1-s)\sup_x |g_s(x)| \leq 4 \qquad (*)$$

by Lemma 6.1. Therefore $\{\|(D_s + (1-s)\epsilon)^{-1}\|\}_{s\in[0,1]}$ is a bounded set.

Next, for any $s, s' \in [0,1]$, a difference $u_{\phi,s} - u_{\phi,s'}$ equals

$$(D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} (s'-s)(\phi-1) & (s-s')c_N(\mathrm{grad}(\phi)) + (s-s')[\phi, F] \\ 0 & (s'-s)(\phi-1) \end{bmatrix}$$
$$+ \left\{(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\right\}$$
$$\cdot \begin{bmatrix} (1-s')(\phi-1) & -(1-s')c_N(\mathrm{grad}(\phi)) + s'[\phi, F] \\ 0 & (1-s')(\phi-1) \end{bmatrix} =: \alpha_{s,s'} + \beta_{s,s'}.$$

The first term $\alpha_{s,s'}$ converges to 0 with the operator norm as $s \to s'$.

The rest of proof is the proof of the convergence to 0 of the second term $\beta_{s,s'}$. Firstly, we assume $s' = 1$. Then we obtain

$$\beta_{s,1} = \left\{(D_s + (1-s)\epsilon)^{-1} - D_1^{-1}\right\} \begin{bmatrix} 0 & [\phi, F] \\ 0 & 0 \end{bmatrix}$$

and

$$(D_s + (1-s)\epsilon)^{-1} - D_1^{-1} = (s-1)(D_s + (1-s)\epsilon)^{-1} D_1^{-1} \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix}$$
$$+ (1-s)(D_s + (1-s)\epsilon)^{-1}(J - \epsilon)D_1^{-1}$$

since D_N commutes F and $\mathrm{d}/\mathrm{d}t$ on domain(D), respectively. Therefore, we have

$$\beta_{s,1} = (s-1)(D_s + (1-s)\epsilon)^{-1} D_1^{-1} \begin{bmatrix} 0 & 0 \\ 0 & D_N[\phi, F] \end{bmatrix}$$
$$+ (1-s)(D_s + (1-s)\epsilon)^{-1}(J - \epsilon)D_1^{-1} \begin{bmatrix} 0 & [\phi, F] \\ 0 & 0 \end{bmatrix}$$

and thus $\beta_{s,1}$ converges to 0 with the operator norm as $s \to 1$ since $D_N[\phi, F]$ is a pseudo-differential operator of order 0 on N and $\|(D_s + (1-s)\epsilon)^{-1}\|$, $\|J\|$, $\|\epsilon\|$, and $\|D_1^{-1}\|$ are uniformly bounded.

We assume $0 \leq s' < 1$. Since an operator

$$\begin{bmatrix} (1-s')(\phi-1) & -(1-s')c_N(\mathrm{grad}(\phi)) + s'[\phi, F] \\ 0 & (1-s')(\phi-1) \end{bmatrix}$$

is bounded, it suffices to show

$$\left\|\big(D_s + (1-s)\epsilon\big)^{-1} - \big(D_{s'} + (1-s')\epsilon\big)^{-1}\right\| \to 0$$

as $s \to s'$. We have

$$\big(D_s + (1-s)\epsilon\big)^{-1} - \big(D_{s'} + (1-s')\epsilon\big)^{-1}$$
$$= (s-s')\big(D_s + (1-s)\epsilon\big)^{-1} \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix} \big(D_{s'} + (1-s')\epsilon\big)^{-1}$$
$$+ (s'-s)\big(D_s + (1-s)\epsilon\big)^{-1}(J - \epsilon)\big(D_{s'} + (1-s')\epsilon\big)^{-1}$$

and the second term converges to 0 with the operator norm as $s \to s'$ by (∗). So it suffices to show that an operator U defined by

$$\begin{aligned} U &= \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix} (D_{s'} + (1-s')\epsilon)^{-1} \\ &= \begin{bmatrix} D_N A_{s'}^{-1}(\mathrm{d}/\mathrm{d}t + (1-s')D_N + s'F) & -(1-s')D_N A_{s'}^{-1} \\ (1-s')D_N A_{s'}^{-1} & D_N A_{s'}^{-1}(-\mathrm{d}/\mathrm{d}t + (1-s')D_N + s'F) \end{bmatrix} \end{aligned}$$

is a bounded operator on $L^2(S) = L^2(\mathbb{R})^2 \otimes L^2(S_N)$, where set

$$A_{s'} = -\mathrm{d}^2/\mathrm{d}t^2 + \left\{(1-s')D_N + s'F\right\}^2 + (1-s')^2.$$

Now, if $D_N A_{s'}^{-1}$, $iD_N A_{s'}^{-1}\mathrm{d}/\mathrm{d}t$, and $D_N A_{s'}^{-1} D_N$ are bounded, then U is also bounded. We show $D_N A_{s'}^{-1} D_N$ is bounded. Denote by E_λ the λ-eigenspace of D_N. Then $D_N A_{s'}^{-1} D_N$ acts as

$$\lambda^2\left\{-\mathrm{d}^2/\mathrm{d}t^2 + \left((1-s')\lambda + s'\mathrm{sgn}(\lambda)\right)^2 + (1-s')^2\right\}^{-1}$$

on $L^2(\mathbb{R}) \otimes E_\lambda$. This operator is equal to $\lambda^2\mu_{\lambda,s'}(i\,\mathrm{d}/\mathrm{d}t)$ and we have

$$\|\lambda^2\mu_{\lambda,s'}(i\,\mathrm{d}/\mathrm{d}t)\| \le 1/(1-s')^2$$

by Lemma 6.1. Therefore we obtain

$$\|D_N A_{s'}^{-1} D_N\| \le 1/(1-s')^2.$$

Similarly, we can show

$$\|D_N A_{s'}^{-1}\| \le 1/(1-s')^2 \quad (\text{use } \mu_{\lambda,s'})$$

and

$$\|iD_N A_{s'}^{-1}\mathrm{d}/\mathrm{d}t\| \le 1/2(1-s') \quad (\text{use } \nu_{\lambda,s'}).$$

Thus U is bounded. Therefore we obtain

$$\left\|\left(D_s + (1-s)\epsilon\right)^{-1} - \left(D_{s'} + (1-s')\epsilon\right)^{-1}\right\| \to 0$$

as $s \to s'$ as required. □

By Proposition 6.2, $\Pi u_{\phi,s}\Pi$ is a continuous path in $\mathcal{L}(\Pi(L^2(S))^l)$. In fact, this continuous path is a desired homotopy of Fredholm operators.

Proposition 6.3. *Set*

$$v_{\phi,s} = u_{\phi,s} - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix}$$

for all $s \in [0,1]$. *One has* $[\Pi, v_{\phi,s}] \sim 0$. *Therefore,*

$$\Pi u_{\phi,s}\Pi \colon \Pi\left(L^2(S)\right) \to \Pi\left(L^2(S)\right)$$

is a Fredholm operator.

Proof. It suffices to show the case when $l = 1$. Due to Proposition 6.2 and closedness of $\mathcal{K}(L^2(S))$, we may assume $s \in [0, 1)$.

First, we show $g(D_s + (1-s)\epsilon)^{-1} \sim 0$ for any $g \in C_0(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, it suffices to show the case when $g \in C_c^\infty(\mathbb{R})$. Because T_s (see in the proof of Proposition 6.2) is a first order elliptic differential operator and g commutes with a operator on N, we have

$$\begin{aligned}
&\big\|g\big(D_s + (1-s)\epsilon\big)^{-1}u\big\|_{H^1} \\
&\qquad \le C\Big(\big\|g\big(D_s + (1-s)\epsilon\big)^{-1}u\big\|_{L^2} + \big\|T_s g\big(D_s + (1-s)\epsilon\big)^{-1}u\big\|\Big) \\
&\qquad \le C'\|u\|_{L^2}
\end{aligned}$$

for any $u \in L^2(S)$. Here, $\|\cdot\|_{H^1}$ is the Sobolev first norm on a compact set $\mathrm{Supp}(g) \times N$. By the Rellich lemma, we have $g(D_s + (1-s)\epsilon)^{-1} \sim 0$. Thus we also have

$$\big(D_s + (1-s)\epsilon\big)^{-1} g = \big(\bar{g}\big(D_s + (1-s)\epsilon\big)^{-1}\big)^* \sim 0.$$

Second, we show $[\varphi, (D_s + (1-s)\epsilon)^{-1}] \sim 0$ for any $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi = \Pi$ on the complement of a compact set in M. Since φ commutes with a operator on N, we have

$$\big[\varphi, \big(D_s + (1-s)\epsilon\big)^{-1}\big] = \big(D_s + (1-s)\epsilon\big)^{-1} \begin{bmatrix} 0 & \varphi' \\ -\varphi' & 0 \end{bmatrix} \big(D_s + (1-s)\epsilon\big)^{-1} \sim 0.$$

By a similar proof in the proof of Proposition 2.5(ii), we have

$$[\Pi, (D_s + (1-s)\epsilon)^{-1}] \sim 0.$$

This proves $\Pi v_{\phi,s} \sim v_{\phi,s}\Pi$, and thus

$$\Pi u_{\phi,s}\Pi \colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)$$

is a Fredholm operator. □

Due to Propositions 6.2 and 6.3,

$$\mathrm{index}\big(\Pi\varrho(\phi)\Pi \colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big)$$

is equal to $\mathrm{index}(\Pi u_{\phi,1}\Pi)$. Let $H \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert transformation:

$$Hf(t) = -\frac{i}{\pi}\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(y)}{t-y}\,dy.$$

Then the eigenvalues of H are only 1 and -1 by $H^2 = 1$ and $H \neq \pm 1$. Let $\mathcal{H}_-$ be the (-1)-eigenspace of H and $\hat{P} \colon L^2(\mathbb{R}) \to \mathcal{H}_-$ the projection to $\mathcal{H}_-$.

Proposition 6.4. *Set* $\mathcal{T}_\phi = (-it + F)^{-1}\phi(-it + F)$*. Then* $\hat{P}\mathcal{T}_\phi\hat{P}^*$ *is a Fredholm operator and one has*

$$\operatorname{index}\big(\Pi\varrho(\phi)\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big) = \operatorname{index}\big(\hat{P}\mathcal{T}_\phi\hat{P}^*\colon X \to X\big),$$

where $X = \mathcal{H}_- \otimes L^2(S_N)$*.*

Proof. Due to Propositions 6.2 and 6.3, we have

$$\begin{aligned}\operatorname{index}&\big(\Pi\varrho(\phi)\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big)\\ &= \operatorname{index}\big(\Pi u_{\phi,1}\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big).\end{aligned}$$

Because of

$$u_{\phi,1} = \begin{bmatrix} 1 & 0 \\ 0 & (\mathrm{d}/\mathrm{d}t + F)^{-1}\phi(\mathrm{d}/\mathrm{d}t + F)\end{bmatrix},$$

the quantity

$$\operatorname{index}\big(\Pi\varrho(\phi)\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big)$$

equals

$$\operatorname{index}\big(\Pi(\mathrm{d}/\mathrm{d}t+F)^{-1}\phi(\mathrm{d}/\mathrm{d}t+F)\Pi\colon \Pi\big(L^2(\mathbb{R})\big)\otimes L^2(S_N) \to \Pi\big(L^2(\mathbb{R})\big)\otimes L^2(S_N)\big).$$

Let $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transformation:

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

Then, we have $\mathcal{F}^{-1}\Pi\mathcal{F} = (1 - H)/2 = \hat{P}$ and $\mathcal{F}^{-1}\mathrm{d}/\mathrm{d}t\mathcal{F} = -it$. This implies

$$\operatorname{index}\big(\Pi\varrho(\phi)\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big) = \operatorname{index}\big(\hat{P}\mathcal{T}_\phi\hat{P}^*\colon X \to X\big). \qquad \square$$

Thus it suffices to calculate $\operatorname{index}(\hat{P}\mathcal{T}_\phi\hat{P}^*)$ in order to prove the main theorem. For this purpose, we use eigenfunctions of the Hilbert transformation.

Lemma 6.5 ([18, Theorem 1]). *Define* $a_n \in L^2(\mathbb{R})$ *by*

$$a_n(t) = \frac{(t - i)^n}{(t + i)^{n+1}}$$

for all $n \in \mathbb{Z}$*. Then* $\{a_n/\sqrt{\pi}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$ *and*

$$Ha_n = \begin{cases} a_n, & \text{if } n < 0,\\ -a_n, & \text{if } n \geq 0.\end{cases}$$

This implies $\mathcal{H}_- = \operatorname{Span}_{\mathbb{C}}\{a_n\}_{n\geq 0}$*.*

Proposition 6.6. *One has* $\mathrm{index}(\widehat{P}\mathcal{T}_\phi\widehat{P}^*) = \mathrm{index}(T_\phi)$. *Therefore Theorem 3.6 in the case when* $M = \mathbb{R} \times N$ *holds.*

Proof. Set $X_0 = \mathbb{C}\{a_0\} \otimes H_+$ and $X_1 = (\mathrm{Span}_{\mathbb{C}}\{a_n\}_{n\geq 1} \otimes H_+) \oplus (\mathcal{H}_- \otimes H_-)$. We note that we have

$$X_0 \oplus X_1 = \mathcal{H}_- \otimes L^2(S_N) = X.$$

Let $p\colon \mathcal{H}_- \to \mathbb{C}\{a_0\}$ be the projection to $\mathbb{C}\{a_0\}$. Then $p_0 = p \otimes P\colon X \to X_0$ is the projection to X_0 and $p_1 = \mathrm{id}_X - p_0\colon X \to X_1$ is the projection to X_1.

By the decomposition of $L^2(S_N) = H_+ \oplus H_-$, we have

$$\mathcal{T}_\phi = \begin{bmatrix} \mathrm{id}_{L^2(\mathbb{R})} \otimes P\phi P^* & \frac{t-i}{t+i} \otimes P\phi(1-P)^* \\ \frac{t+i}{t-i} \otimes (1-P)\phi P^* & \mathrm{id}_{L^2(\mathbb{R})} \otimes (1-P)\phi(1-P)^* \end{bmatrix}.$$

So we obtain

$$\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_0^* = p^* \otimes P\phi P^* = \mathrm{id}_{\mathbb{C}\{a_0\}} \otimes T_\phi$$

and

$$\mathcal{T}_\phi\widehat{P}^* p_1^* = \begin{bmatrix} (\widehat{P}-p)^* \otimes P\phi P^* & \frac{t-i}{t+i}\widehat{P}^* \otimes P\phi(1-P)^* \\ \frac{t+i}{t-i}(\widehat{P}-p)^* \otimes (1-P)\phi P^* & \widehat{P}^* \otimes (1-P)\phi(1-P)^* \end{bmatrix}.$$

This implies $\mathrm{Image}(\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_0^*) \subset X_0$, $\mathrm{Image}(\mathcal{T}_\phi\widehat{P}^* p_1^*) \subset X_1$ and

$$\big(\widehat{P}\mathcal{T}_{\phi^{-1}}\widehat{P}^* p_1^*\big)\big(\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_1^*\big) = \widehat{P}\mathcal{T}_{\phi^{-1}}\mathcal{T}_\phi\widehat{P}^* p_1^* = \mathrm{id}_{X_1}.$$

So $\widehat{P}\mathcal{T}_\phi\widehat{P}^*$ forms a direct sum of an invertible part $\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_1^*$ and another part $\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_0^*$:

$$\widehat{P}\mathcal{T}_\phi\widehat{P}^* = \begin{bmatrix} \widehat{P}\mathcal{T}_\phi\widehat{P}^* p_0^* & 0 \\ 0 & \widehat{P}\mathcal{T}_\phi\widehat{P}^* p_1^* \end{bmatrix} \text{ on } X_0 \oplus X_1.$$

This proves $\mathrm{index}(\widehat{P}\mathcal{T}_\phi\widehat{P}^*) = \mathrm{index}(\widehat{P}\mathcal{T}_\phi\widehat{P}^* p_0^*) = \mathrm{index}(T_\phi)$. □

We note that we also get

$$\mathrm{index}\big(\Pi u_{\chi,\phi}\Pi\big) = \mathrm{index}(T_\phi).$$

7. The general case

In this section we reduce the proof for the general partitioned manifold to that of $\mathbb{R} \times N$. Our argument is similar to Higson's argument in [10]. By above sections, it suffices to show the case when $\phi \in GL_l(\mathcal{W}(M))$. Firstly, we shall show a cobordism invariance. See also [10, Lemma 1.4].

Lemma 7.1. *Let (M^+, M^-, N) and $(M^{+\prime}, M^{-\prime}, N')$ be two partitions of M. Assume that these two partitions are cobordant, that is, symmetric differences $M^\pm \triangle M^{\pm\prime}$ are compact. Let Π and Π' be the characteristic function of M^+ and $M^{+\prime}$, respectively. Take $\phi \in GL_l(\mathcal{W}(M))$. Then one has*

$$\operatorname{index}\big(\Pi u_{\chi,\phi}\Pi\big) = \operatorname{index}\big(\Pi' u_{\chi,\phi}\Pi'\big) \quad \text{and} \quad \operatorname{index}\big(\Pi\varrho(\phi)\Pi\big) = \operatorname{index}\big(\Pi'\varrho(\phi)\Pi'\big).$$

Proof. It suffices to show the case when $l = 1$. Since we have $[\phi, \Pi] = 0$ and $[u_{\chi,\phi}, \Pi] \sim 0$, we obtain

$$\begin{aligned}\operatorname{index}\big(\Pi u_{\chi,\phi}\Pi\colon \Pi\big(L^2(S)\big) \to \Pi\big(L^2(S)\big)\big) \\ &= \operatorname{index}\left((1-\Pi)\begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix} + \Pi u_{\chi,\phi}\colon L^2(S) \to L^2(S)\right) \\ &= \operatorname{index}\left(\begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix} + \Pi v_{\chi,\phi}\colon L^2(S) \to L^2(S)\right).\end{aligned}$$

Therefore, it suffices to show $\Pi v_{\chi,\phi} \sim \Pi' v_{\chi,\phi}$. Now, since $M^\pm \triangle M^{\pm\prime}$ are compact, there exists $f \in C_0(M)$ such that $\Pi - \Pi' = (\Pi - \Pi')f$. So we obtain

$$\Pi v_{\chi,\phi} - \Pi' v_{\chi,\phi} = (\Pi - \Pi') f v_{\chi,\phi} \sim 0.$$

By the similar argument, we can prove

$$\operatorname{index}\big(\Pi\varrho(\phi)\Pi\big) = \operatorname{index}\big(\Pi'\varrho(\phi)\Pi'\big). \qquad \square$$

Secondly, we shall prove an analogue of Higson's Lemma [10, Lemma 3.1].

Lemma 7.2. *Let M_1 and M_2 be two partitioned manifolds and $S_j \to M_j$ a Hermitian vector bundle. Let Π_j be the characteristic function of M_j^+. We assume that there exists an isometry $\gamma\colon M_2^+ \to M_1^+$ which lifts an isomorphism $\gamma^*\colon S_1|_{M_1^+} \to S_2|_{M_2^+}$. We denote the Hilbert space isometry defined by γ^* by the same letter*

$$\gamma^*\colon \Pi_1\big(L^2(S_1)\big) \to \Pi_2\big(L^2(S_2)\big).$$

Take $u_j \in GL_l(C_b^*(M_j))$ such that $\gamma^* u_1 \Pi_1 \sim \Pi_2 u_2 \gamma^*$. Then one has

$$\operatorname{index}\big(\Pi_1 u_1 \Pi_1\big) = \operatorname{index}\big(\Pi_2 u_2 \Pi_2\big).$$

Similarly, if there exists an isometry $\gamma\colon M_2^- \to M_1^-$, which lifts an isomorphism

$$\gamma^*\colon S_1|_{M_1^-} \to S_2|_{M_2^-} \quad \text{and} \quad \gamma^* u_1 \Pi_1 \sim \Pi_2 u_2 \gamma^*,$$

then one has

$$\operatorname{index}\big(\Pi_1 u_1 \Pi_1\big) = \operatorname{index}\big(\Pi_2 u_2 \Pi_2\big).$$

Proof. It suffices to show the case when $l = 1$. Let

$$v\colon (1-\Pi_1)\big(L^2(S_1)\big) \to (1-\Pi_2)\big(L^2(S_2)\big)$$

be any invertible operator. Then

$$V = \gamma^*\Pi_1 + v(1-\Pi_1)\colon L^2(S_1) \to L^2(S_2)$$

is also invertible operator. Hence we obtain

$$\begin{aligned} &V\big((1-\Pi_1)+\Pi_1 u_1 \Pi_1\big) - \big((1-\Pi_2)+\Pi_2 u_2 \Pi_2\big)V \\ &\qquad = \gamma^*\Pi_1 u_1 \Pi_1 - \Pi_2 u_2 \Pi_2 \gamma^* \sim \gamma^* u_1 \Pi_1 - \Pi_2 u_2 \gamma^* \sim 0. \end{aligned}$$

Therefore, we obtain

$$\operatorname{index}\big(\Pi_1 u_1 \Pi_1\big) = \operatorname{index}\big(\Pi_2 u_2 \Pi_2\big)$$

since V is an invertible operator and one has

$$\operatorname{index}\big(\Pi_j u_j \Pi_j\big) = \operatorname{index}\big((1-\Pi_j) + \Pi_j u_j \Pi_j\big)$$

for $j = 1, 2$. □

Applying Lemma 7.2, we prove the following:

Corollary 7.3. *Let M_1 and M_2 be two partitioned manifolds. Let $S_j \to M_j$ be a graded Clifford bundle with the grading ϵ_j, and denote by D_j the graded Dirac operator of S_j. We assume that there exists an isometry $\gamma\colon M_2^+ \to M_1^+$ which lifts isomorphism $\gamma^*\colon S_1|_{M_1^+} \to S_2|_{M_2^+}$ of graded Clifford structures. Moreover, we assume that $\phi_j \in GL_l(\mathcal{W}(M))$ satisfies $\phi_1(\gamma(x)) = \phi_2(x)$ for all $x \in M_2^+$. Then one has*

$$\operatorname{index}\big(\Pi_1 u_{\chi,\phi_1} \Pi_1\big) = \operatorname{index}\big(\Pi_2 u_{\chi,\phi_2} \Pi_2\big).$$

Proof. Fix small $R > 0$. It suffices to show $\gamma^* u_{\chi,\phi_1}\Pi_1 \sim \Pi_2 u_{\chi,\phi_2}\gamma^*$ the case when a chopping function $\chi \in C(\mathbb{R};[-1,1])$ satisfies $\operatorname{Supp}(\widehat{\chi}\,) \subset (-R, R)$. Set $N_{2R} = \{x \in M_1^+ ;\, d(x, N_1) \leq 2R\}$. Let φ_1 be a smooth function on M_1 such that $\operatorname{Supp}(\varphi_1) \subset M_1^+ \setminus N_{2R}$ and assume that there exists a compact set $K \subset M_1$ such that $\varphi_1 = \Pi_1$ on $M_1 \setminus K$. Set $\varphi_2(x) = \varphi_1(\gamma(x))$ for all $x \in M_2^+$ and $\varphi_2 = 0$ on M_2^-. Then we have

$$\gamma^* v_{\chi,\phi_1}\Pi_1 \sim \gamma^* v_{\chi,\phi_1}\varphi_1 \quad \text{and} \quad \Pi_2 v_{\chi,\phi_2}\gamma^* \sim \varphi_2 v_{\chi,\phi_2}\gamma^*.$$

Thus, if we have $\gamma^* v_{\chi,\phi_1}\varphi_1 \sim \varphi_2 v_{\chi,\phi_2}\gamma^*$, then we obtain

$$\begin{aligned} \gamma^* u_{\chi,\phi_1}\Pi_1 &\sim \gamma^* v_{\chi,\phi_1}\varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \Pi_1 \\ &\sim \varphi_2 v_{\chi,\phi_2}\gamma^* + \Pi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \Pi_2 u_{\chi,\phi_2}\gamma^*. \end{aligned}$$

We shall show $\gamma^* v_{\chi,\phi_1}\varphi_1 \sim \varphi_2 v_{\chi,\phi_2}\gamma^*$. Now, we have $\gamma^* v_{\chi,\phi_1}\varphi_1 = v_{\chi,\phi_2}\gamma^*\varphi_1$ since the propagation of $\chi(D)$ and $\eta(D)$ is less than R, respectively, and $\gamma^* D = D\gamma^*$ on M^+. Moreover, we have $[v_{\chi,\phi_2}, \varphi_2] \sim 0$ since $v_{\chi,\phi_2} \in M_l(C^*(M))$. Therefore, we obtain

$$\gamma^* v_{\chi,\phi_1}\varphi_1 = v_{\chi,\phi_2}\gamma^*\varphi_1 = v_{\chi,\phi_2}\varphi_2\gamma^* \sim \varphi_2 v_{\chi,\phi_2}\gamma^*. \qquad \square$$

In order to prove Corollary 3.7, we apply Lemma 7.2 as follows.

Corollary 7.4. *We also assume as in Corollary 7.3. Then one has*

$$\operatorname{index}\bigl(\Pi_1\varrho(\phi_1)\Pi_1\bigr) = \operatorname{index}\bigl(\Pi_2\varrho(\phi_2)\Pi_2\bigr).$$

Proof. It suffices to show $\gamma^*\varrho(\phi_1)\Pi_1 \sim \Pi_2\varrho(\phi_2)\gamma^*$. Let φ_1 be a smooth function on M_1 such that $\mathrm{Supp}(\varphi_1) \subset M_1^+$ and assume that there exists a compact set $K \subset M_1$ such that $\varphi_1 = \Pi_1$ on $M_1 \setminus K$. Set $\varphi_2(x) = \varphi_1(\gamma(x))$ for all $x \in M_2^+$ and $\varphi_2 = 0$ on M_2^-. Set $v_{\phi_j} = \varrho(\phi_j) - \begin{bmatrix} 1 & 0 \\ 0 & \phi_j \end{bmatrix}$. Then we have

$$\gamma^* v_{\phi_1}\Pi_1 \sim \gamma^* v_{\phi_1}\varphi_1 \quad \text{and} \quad \Pi_2 v_{\phi_2}\gamma^* \sim \varphi_2 v_{\phi_2}\gamma^*.$$

Thus, if one has $\gamma^* v_{\phi_1}\varphi_1 \sim \varphi_2 v_{\phi_2}\gamma^*$, then we obtain

$$\begin{aligned}
\gamma^*\varrho(\phi_1)\Pi_1 &\sim \gamma^* v_{\phi_1}\varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \Pi_1 \\
&\sim \varphi_2 v_{\phi_2}\gamma^* + \Pi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \Pi_2\varrho(\phi_2)\gamma^*.
\end{aligned}$$

We shall show $\gamma^* v_{\phi_1}\varphi_1 \sim \varphi_2 v_{\phi_2}\gamma^*$. In fact, we obtain

$$\begin{aligned}
&\gamma^* v_{\phi_1}\varphi_1 - \varphi_2 v_{\phi_2}\gamma^* \\
&\quad \sim \bigl\{\gamma^*\varphi_1(D_1+\epsilon_1)^{-1} - (D_2+\epsilon_2)^{-1}\gamma^*\varphi_1\bigr\} \begin{bmatrix} \phi_1 - 1 & -c(\mathrm{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\
&\quad \sim (D_2+\epsilon_2)^{-1}\gamma^*[D_1,\varphi_1](D_1+\epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\mathrm{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\
&\quad \sim 0
\end{aligned}$$

since $\mathrm{grad}(\varphi_1)$ has a compact support and $[D_1, \varphi_1] = c(\mathrm{grad}(\varphi_1))$. Thus, we get $\gamma^* u_{\phi_1}\Pi_1 \sim \Pi_2 u_{\phi_2}\gamma^*$. Therefore, we obtain

$$\operatorname{index}\bigl(\Pi_1 u_{\phi_1}\Pi_1\bigr) = \operatorname{index}\bigl(\Pi_2 u_{\phi_2}\Pi_2\bigr)$$

by Lemma 7.2. $\square$

Proof of Theorem 3.6, the general case. We assume $\phi \in GL_l(\mathcal{W}(M))$. Firstly, let $a \in C^\infty([-1,1];[-1,1])$ satisfy

$$a(t) = \begin{cases} -1, & \text{if } -1 \le t \le -3/4, \\ 0, & \text{if } -2/4 \le t \le 2/4, \\ 1, & \text{if } 3/4 \le t \le 1. \end{cases}$$

Let $(-4\delta, 4\delta) \times N$ be diffeomorphic to a tubular neighborhood of N in M satisfies

$$\sup_{(t,x),(s,y) \in [-3\delta, 3\delta] \times N} |\phi(t,x) - \phi(s,y)| < \|\phi^{-1}\|^{-1}.$$

Set $\psi(t,x) = \phi(4\delta a(t), x)$ on $(-4\delta, 4\delta) \times N$ and $\psi = \phi$ on $M \setminus (-4\delta, 4\delta) \times N$. Then we obtain $\psi \in GL_l(\mathcal{W}(M))$ and $\|\psi - \phi\| < \|\phi^{-1}\|^{-1}$. Thus a map

$$[0,1] \ni t \mapsto \psi_t = t\psi + (1-t)\phi \in GL_l\big(\mathcal{W}(M)\big)$$

is continuous with the uniform norm. Therefore it suffices to show the case when $\phi \in GL_l(\mathcal{W}(M))$ satisies $\phi(t,x) = \phi(0,x)$ on $(-2\delta, 2\delta) \times N$. Due to Lemma 7.1, we may change a partition of M to

$$\big(M^+ \cup \big([-\delta, 0] \times N\big), M^- \setminus \big((-\delta, 0] \times N\big), \{-\delta\} \times N\big)$$

without changing index$(\Pi u_{\chi,\phi} \Pi)$. Then, due to Corollary 7.3, we may change $M^+ \cup ([-\delta, 0] \times N)$ to $[-\delta, \infty) \times N$ without changing index$(\Pi u_{\chi,\phi} \Pi)$. Here, ϕ is equal to $\phi(0,x)$ on $[-\delta, \infty) \times N$ and the metric on $[0,\infty) \times N$ is product. We denote this manifold by

$$M' = \big([-\delta, \infty) \times N\big) \cup \big(M^- \setminus \big((-\delta, 0] \times N\big)\big).$$

M' is partitioned by

$$\big([-\delta, \infty) \times N, M^- \setminus \big((-\delta, 0] \times N\big), \{-\delta\} \times N\big).$$

We apply a similar argument to M', we may change M' to a product $\mathbb{R} \times N$ without changing index$(\Pi u_{\chi,\phi} \Pi)$. Now we have changed M to $\mathbb{R} \times N$. □

Proof of Corollary 3.7, the general case. Similar. □

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