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# Brauer–Severi motives and Donaldson–Thomas invariants of quantized threefolds

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**Abstract.** Motives of Brauer–Severi schemes of Cayley-smooth algebras associated to homogeneous superpotentials are used to compute inductively the motivic Donaldson–Thomas invariants of the corresponding Jacobian algebras. This approach can be used to test the conjectural exponential expressions for these invariants, proposed in [3]. As an example we confirm the second term of the conjectured expression for the motivic series of the homogenized Weyl algebra.

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## 1. Introduction

We fix a homogeneous degree $d$ superpotential $W$ in $m$ non-commuting variables $X_1, \dots, X_m$. For every dimension $n \geq 1$, $W$ defines a regular functions, sometimes called the Chern–Simons functional

$$\mathrm{Tr}(W)\colon \mathbb{M}_{m,n} = \underbrace{M_n(\mathbb{C}) \oplus \cdots \oplus M_n(\mathbb{C})}_{m} \longrightarrow \mathbb{C}$$

obtained by replacing in $W$ each occurrence of $X_i$ by the $n \times n$ matrix n the $i$th component, and taking traces.

We are interested in the (naive, equivariant) motives of the fibers of this functional which we denote by

$$\mathbb{M}^W_{m,n}(\lambda) = \mathrm{Tr}(W)^{-1}(\lambda).$$

Recall that to each isomorphism class of a complex variety $X$ (equipped with a good action of a finite group of roots of unity) we associate its naive equivariant motive $[X]$, which is an element in the ring $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1/2}]$ (see [4] or [3]) and is subject to the scissor- and product-relations

$$[X] - [Z] = [X - Z] \quad \text{and} \quad [X].[Y] = [X \times Y]$$

whenever $Z$ is a Zariski closed subvariety of $X$. A special element is the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}, \mathrm{id}]$ and we recall from [12, Lemma 4.1] that

$$[\mathrm{GL}_n] = \prod_{k=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^k)$$

and from [3, 2.2] that

$$[\mathbb{A}^n, \mu_k] = \mathbb{L}^n$$

for a linear action of $\mu_k$ on $\mathbb{A}^n$. This ring is equipped with a plethystic exponential Exp, see for example [2] and [4].

The representation theoretic interest of the degeneracy locus $Z = \{d\,\mathrm{Tr}(W) = 0\}$ of the Chern–Simons functional is that it coincides with the scheme of $n$-dimensional representations

$$Z = \mathtt{rep}_n(R_W) \quad \text{where } R_W = \frac{\mathbb{C}\langle X_1, \dots, X_m \rangle}{(\partial_{X_i}(W) : 1 \leq i \leq m)}$$

of the corresponding Jacobi algebra $R_W$ where $\partial_{X_i}$ is the cyclic derivative with respect to $X_i$. As $W$ is homogeneous it follows from [4, Thm. 1.3] (or [1] if the superpotential allows "a cut") that its virtual motive is equal to

$$[\mathtt{rep}_n(R_W)]_{\mathrm{virt}} = \mathbb{L}^{-\frac{mn^2}{2}} \left( [\mathbb{M}^W_{m,n}(0)] - [\mathbb{M}^W_{m,n}(1)] \right)$$

where $\hat{\mu}$ acts via $\mu_d$ on $\mathbb{M}^W_{m,n}(1)$ and trivially on $\mathbb{M}^W_{m,n}(0)$. These virtual motives can be packaged together into the motivic Donaldson–Thomas series

$$U_W(t) = \sum_{n=0}^{\infty} \mathbb{L}^{-((m-1)n^2)/2} \frac{[\mathbb{M}^W_{m,n}(0)] - [\mathbb{M}^W_{m,n}(1)]}{[\mathrm{GL}_n]} t^n$$

In [3], A. Cazzaniga, A. Morrison, B. Pym, and B. Szendröi conjecture that this generating series has an exponential expression involving simple rational functions of virtual motives determined by representation theoretic information of the Jacobi algebra $R_W$

$$U_W(t) \overset{?}{=} \mathtt{Exp}\left( - \sum_{i=1}^{k} \frac{M_i}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \frac{t^{m_i}}{1 - t^{m_i}} \right),$$

where $m_1 = 1, \dots, m_k$ are the dimensions of simple representations of $R_W$ and $M_i \in \mathcal{M}_{\mathbb{C}}$ are motivic expressions without denominators, with $M_1$ the virtual motive of the scheme parametrizing (simple) 1-dimensional representations. Evidence for this conjecture comes from cases where the superpotential admits a cut and hence one can use dimensional reduction, introduced by A. Morrison in [12], as in the case of quantum affine three-space [3].

The purpose of this paper is to introduce an inductive procedure to test the conjectural exponential expressions given in [3] in other interesting cases such as the homogenized Weyl algebra and elliptic Sklyanin algebras. To this end we introduce the following quotient of the free necklace algebra on $m$ variables

$$\mathbb{T}_m^W(\lambda) = \frac{\mathbb{C}\langle X_1, \dots, X_m\rangle \otimes \mathtt{Sym}(V_m)}{(W - \lambda)},$$

where

$$V_m = \frac{\mathbb{C}\langle X_1, \dots, X_m\rangle}{[\mathbb{C}\langle X_1, \dots, X_m\rangle, \mathbb{C}\langle X_1, \dots, X_m\rangle]_{\text{vect}}}$$

is the vectorspace space having as a basis all cyclic words in $X_1, \dots, X_m$. Note that any superpotential is an element of $\mathtt{Sym}(V_m)$. Substituting each $X_k$ by a generic $n \times n$ matrix and each cyclic word by the corresponding trace we obtain a quotient of the trace ring of $m$ generic $n \times n$ matrices

$$\mathbb{T}_{m,n}^W(\lambda) = \frac{\mathbb{T}_{m,n}}{(\text{Tr}(W) - \lambda)} \quad \text{with } \mathbb{M}_{m,n}^W(\lambda) = \mathtt{trep}_n(\mathbb{T}_{m,n}^W)$$

such that its scheme of trace preserving $n$-dimensional representations is isomorphic to the fiber $\mathbb{M}_{m,n}^W(\lambda)$. We will see that if $\lambda \neq 0$ the algebra $\mathbb{T}_{m,n}^W(\lambda)$ shares many ringtheoretic properties of trace rings of generic matrices, in particular it is a Cayley-smooth algebra, see [10]. As such one might hope to describe $\mathbb{M}_{m,n}^W(\lambda)$ using the Luna stratification of the quotient and its fibers in terms of marked quiver settings given in [10]. However, all this is with respect to the étale topology and hence useless in computing motives.

For this reason we consider the Brauer–Severi scheme of $\mathbb{T}_{m,n}^W(\lambda)$, as introduced by M. Van den Bergh in [17] and further investigated by M. Reineke in [16], which are quotients of a principal $\text{GL}_n$-bundles and hence behave well with respect to motives. More precisely, the Brauer–Severi scheme of $\mathbb{T}_{m,n}^W(\lambda)$ is defined as

$$\mathtt{BS}_{m,n}^W(\lambda) = \left\{(v, \phi) \in \mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}_{m,n}^W(\lambda) \mid \phi(\mathbb{T}_{m,n}^W(\lambda))v = \mathbb{C}^n\right\} / \, \text{GL}_n$$

and their motives determine inductively the motives in the Donaldson–Thomas series. In Proposition 5 we will show that

$$(\mathbb{L}^n - 1)\frac{[\mathbb{M}_{m,n}^W(0)] - [\mathbb{M}_{m,n}^W(1)]}{[\text{GL}_n]}$$

is equal to

$$\begin{aligned}&[\mathtt{BS}_{m,n}^W(0)] - [\mathtt{BS}_{m,n}^W(1)] \\ &\quad + \sum_{k=1}^{n-1} \frac{\mathbb{L}^{(m-1)k(n-k)}}{[\text{GL}_{n-k}]}\big([\mathtt{BS}_{m,k}^W(0)] - [\mathtt{BS}_{m,k}^W(1)]\big)\big([\mathbb{M}_{m,k}^W(0)] - [\mathbb{M}_{m,k}^W(1)]\big).\end{aligned}$$

In Section 4 we will compute the first two terms of $U_W(t)$ in the case of the quantized 3-space in a variety of ways. In the final section we repeat the computation for the homogenized Weyl algebra and show that it coincides with the conjectured expression of [3]. In a forthcoming paper [11] we will compute the first two terms of the series for elliptic Sklyanin in the case of 2-torsion points.

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## 2. Brauer–Severi motives

With $\mathbb{T}_{m,n}$ we will denote the *trace ring of m generic n × n matrices*. That is, $\mathbb{T}_{m,n}$ is the $\mathbb{C}$-subalgebra of the full matrix-algebra

$$M_n\big(\mathbb{C}[x_{ij}(k) \mid 1 \leq i,j \leq n,\ 1 \leq k \leq m]\big)$$

generated by the $m$ generic matrices

$$X_k = \begin{bmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{bmatrix}$$

together with all elements of the form $\mathrm{Tr}(M)1_n$ where $M$ runs over all monomials in the $X_i$. These algebras have been studied extensively by ringtheorists in the ’80s and some of the results are summarized in the following result

**Proposition 1.** *Let $\mathbb{T}_{m,n}$ be the trace ring of m generic n × n matrices, then*

(1) *$\mathbb{T}_{m,n}$ is an affine Noetherian domain with center $Z(\mathbb{T}_{m,n})$ of dimension $(m-1)n^2+1$ and generated as $\mathbb{C}$-algebra by the $\mathrm{Tr}(M)$ where $M$ runs over all monomials in the generic matrices $X_k$.*

(2) *$\mathbb{T}_{m,n}$ is a maximal order and a noncommutative UFD, that is all twosided prime ideals of height one are generated by a central element and $Z(\mathbb{T}_{m,n})$ is a commutative UFD which is a complete intersection if and only if $n = 1$ or $(m,n) = (2,2), (2,3)$ or $(3,2)$.*

(3) *$\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra unless $(m,n) = (2,2)$, that is, every localization at a central height one prime ideal is an Azumaya algebra.*

*Proof.* For (1) see, for example, [13] or [15]. For (2) see, for example, [8], and for (3), for example, [7]. □

A Cayley–Hamilton algebra of degree $n$ is a $\mathbb{C}$-algebra $A$ , equipped with a linear trace map tr: $A \longrightarrow A$ satisfying the following properties:

(1) $\mathrm{tr}(a).b = b.\mathrm{tr}(a)$;

(2) $\mathrm{tr}(a.b) = \mathrm{tr}(b.a)$;

(3) $\mathrm{tr}(\mathrm{tr}(a).b) = \mathrm{tr}(a).\mathrm{tr}(b)$;

(4) $\mathrm{tr}(a) = n$;

(5) $\chi_a^{(n)}(a) = 0$ where $\chi_a^{(n)}(t)$ is the formal Cayley–Hamilton polynomial of degree $n$, see [14].

For a Cayley–Hamilton algebra $A$ of degree $n$ it is natural to look at the scheme $\mathtt{trep}_n(A)$ of all *trace preserving* $n$-dimensional representations of $A$, that is, all trace preserving algebra maps $A \longrightarrow M_n(\mathbb{C})$. A Cayley–Hamilton algebra $A$ of degree $n$ is said to be a *smooth Cayley–Hamilton algebra* if $\mathtt{trep}_n(A)$ is a smooth variety. Procesi has shown that these are precisely the algebras having the smoothness property of allowing lifts modulo nilpotent ideals in the category of all Cayley–Hamilton algebras of degree $n$, see [14]. The étale local structure of smooth Cayley–Hamilton algebras and their centers have been extensively studied in [10].

**Proposition 2.** *Let $W$ be a homogeneous superpotential in m variables and define the algebra*

$$\mathbb{T}^W_{m,n}(\lambda) = \frac{\mathbb{T}_{m,n}}{(\mathrm{Tr}(W) - \lambda)} \quad \textit{then} \quad \mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda)).$$

*If* $\mathrm{Tr}(W) - \lambda$ *is irreducible in the UFD* $Z(\mathbb{T}_{m,n})$*, then for* $\lambda \neq 0$

(1) $\mathbb{T}^W_{m,n}(\lambda)$ *is a reflexive Azumaya algebra;*

(2) $\mathbb{T}^W_{m,n}(\lambda)$ *is a smooth Cayley–Hamilton algebra of degree n and of Krull dimension* $(m-1)n^2$*;*

(3) $\mathbb{T}^W_{m,n}(\lambda)$ *is a domain;*

(4) *The central singular locus is the the non-Azumaya locus of* $\mathbb{T}^W_{m,n}(\lambda)$ *unless* $(m, n) = (2, 2)$.

*Proof.* (1) As $\mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ is a smooth affine variety for $\lambda \neq 0$ (due to homogeneity of $W$) on which $\mathrm{GL}_n$ acts by automorphisms, we know that the ring of invariants,

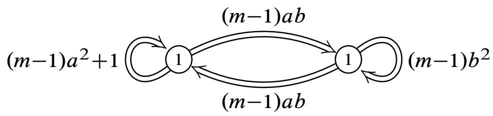
$$\mathbb{C}\big[\mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))\big]^{\mathrm{GL}_n} = Z(\mathbb{T}^W_{m,n}(\lambda))$$

which coincides with the center of $\mathbb{T}^W_{m,n}(\lambda)$ by e.g. [10, Prop. 2.12], is a normal domain. Because the non-Azumaya locus of $\mathbb{T}_{m,n}$ has codimension at least 3 (if $(m, n) \neq (2, 2)$) by [7], it follows that all localizations of $\mathbb{T}^W_{m,n}(\lambda)$ at height one prime ideals are Azumaya algebras. Alternatively, using (2) one can use the theory of local quivers as in [10].

(2) That the Cayley–Hamilton degree of the quotient $\mathbb{T}^W_{m,n}(\lambda)$ remains $n$ follows from the fact that $\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra and irreducibility of $\mathrm{Tr}(W) - \lambda$. Because $\mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ is a smooth affine variety, $\mathbb{T}^W_{m,n}(\lambda)$ is a smooth Cayley–Hamilton algebra. The statement on Krull dimension follows from the fact that the Krull dimension of $\mathbb{T}_{m,n}$ is known to be $(m-1)n^2 + 1$.

(3) After taking determinants, this follows from factoriality of $Z(\mathbb{T}_{m,n})$ and irreducibility of $\mathrm{Tr}(W) - \lambda$.

(4) This follows from the theory of local quivers as in [10]. The most general non-simple representations are of representation type $(1, a; 1, b)$ with the dimensions of the two simple representations $a, b$ adding up to $n$. The corresponding local quiver is




and as $(m-1)ab \geq 2$ under the assumptions, it follows that the corresponding singular point is singular. □

Let us define for all $k \leq n$ and all $\lambda \in \mathbb{C}$ the locally closed subscheme of $\mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$

$$\mathtt{X}_{k,n,\lambda} = \left\{(v, \phi) \in \mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda)) \mid \dim_{\mathbb{C}}(\phi(\mathbb{T}^W_{m,n}(\lambda)).v) = k\right\}.$$

Sending a point $(v, \phi)$ to the point in the Grassmannian $\mathtt{Gr}(k, n)$ determined by the $k$-dimensional subspace $V = \phi(\mathbb{T}^W_{m,n}(\lambda)).v \subset \mathbb{C}^n$ we get a Zariskian fibration as in [12]

$$\mathtt{X}_{k,n,\lambda} \twoheadrightarrow \mathtt{Gr}(k, n).$$

To compute the fiber over $V$ we choose a basis of $\mathbb{C}^n$ such that the first $k$ base vectors span $V = \phi(\mathbb{T}^W_{m,n}(\lambda)).v$. With respect to this basis, the images of the generic matrices $X_i$ all are of the following block-form

$$\phi(X_i) = \begin{bmatrix} \phi_k(X_i) & \sigma(X_i) \\ 0 & \phi_{n-k}(X_i) \end{bmatrix} \quad \text{with} \begin{cases} \phi_k(X_i) \in M_k(\mathbb{C}), \\ \phi_{n-k}(X_i) \in M_{n-k}(\mathbb{C}), \\ \sigma(X_i) \in M_{n-k \times k}(\mathbb{C}). \end{cases}$$

Using these matrix-form it is easy to see that

$$\mathrm{Tr}\left(\phi\big(W(X_1, \ldots, X_m)\big)\right) = \mathrm{Tr}\left(\phi_k\big(W(X_1, \ldots, X_m)\big)\right) + \mathrm{Tr}\left(\phi_{n-k}\big(W(X_1, \ldots, X_m)\big)\right).$$

That is, if $\phi_k \in \mathtt{trep}_k(\mathbb{T}^W_{m,k}(\mu))$ then $\phi_{n-k} \in \mathtt{trep}(\mathbb{T}^W_{m,n-k}(\lambda - \mu))$ and moreover we have that $(v, \phi_k) \in \mathtt{X}_{k,k,\mu}$. Further, the $m$ matrices $\sigma(X_i) \in M_{n-k \times k}(\mathbb{C})$ can be taken arbitrary. Rephrasing this in motives we get

$$[\mathtt{X}_{k,n,\lambda}] = \mathbb{L}^{mk(n-k)}[\mathtt{Gr}(k,n)] \sum_{\mu\in\mathbb{C}} [\mathtt{X}_{k,k,\mu}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(\lambda-\mu))\big].$$

Here the summation $\sum_{\mu\in\mathbb{C}}$ is shorthand for distinguishing between zero and non-zero values of $\mu$ and $\lambda-\mu$. For example, with $\sum_{\mu\in\mathbb{C}}[\mathtt{X}_{k,k,\mu}][\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(\lambda-\mu))]$ we mean for $\lambda \neq 0$

$$(\mathbb{L}-2)[\mathtt{X}_{k,k,1}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(1))\big] + [\mathtt{X}_{k,k,0}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(\lambda))\big] + [\mathtt{X}_{k,k,\lambda}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(0))\big]$$

and when $\lambda = 0$

$$(\mathbb{L}-1)[\mathtt{X}_{k,k,1}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(1))\big] + [\mathtt{X}_{k,k,0}]\big[\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(0))\big].$$

Further, we have

$$[\mathtt{Gr}(k,n)] = \frac{[\mathrm{GL}_n]}{[\mathrm{GL}_k][\mathrm{GL}_{n-k}]\mathbb{L}^{k(n-k)}} \quad \text{and} \quad [\mathtt{X}_{k,k,\mu}] = [\mathrm{GL}_k]\big[\mathtt{BS}^W_{m,k}(\mu)\big]$$

and substituting this in the above, and recalling that $\mathbb{M}^W_{m,l}(\alpha) = \mathtt{trep}_l(\mathbb{T}^W_{m,l}(\alpha))$, we get:

**Proposition 3.** *With notations as before we have for all* $0 < k < n$ *and all* $\lambda \in \mathbb{C}$ *that*

$$[\mathtt{X}_{k,n,\lambda}] = [\mathrm{GL}_n]\mathbb{L}^{(m-1)k(n-k)} \sum_{\mu\in\mathbb{C}} \big[\mathtt{BS}^W_{m,k}(\mu)\big]\frac{[\mathbb{M}^W_{m,n-k}(\lambda-\mu)]}{[\mathrm{GL}_{n-k}]}.$$

*Further, we have*

$$[\mathtt{X}_{0,n,\lambda}] = \big[\mathbb{M}^W_{m,n}(\lambda)\big] \quad \textit{and} \quad [\mathtt{X}_{n,n,\lambda}] = [\mathrm{GL}_n]\big[\mathtt{BS}^W_{m,n}(\lambda)\big].$$

We can also express this in terms of generating series. Equip the commutative ring $\mathcal{M}_{\mathbb{C}}[\![t]\!]$ with the modified product

$$t^a * t^b = \mathbb{L}^{(m-1)ab}t^{a+b}$$

and consider the following two generating series for all $\frac{1}{2} \neq \lambda \in \mathbb{C}$

$$\mathtt{B}_\lambda(t) = \sum_{n=1}^{\infty}\big[\mathtt{BS}^W_{m,n}(\lambda)\big]t^n \quad \text{and} \quad \mathtt{R}_\lambda(t) = \sum_{n=1}^{\infty}\frac{\big[\mathbb{M}^W_{m,n}(\lambda)\big]}{[\mathrm{GL}_n]}t^n,$$

$$\mathtt{B}_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty}\big[\mathtt{BS}^W_{m,n}(\tfrac{1}{2})\big]t^n \quad \text{and} \quad \mathtt{R}_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty}\frac{\big[\mathbb{M}^W_{m,n}(\frac{1}{2})\big]}{[\mathrm{GL}_n]}t^n.$$

**Proposition 4.** *With notations as before we have the functional equation*

$$1 + \mathtt{R}_1(\mathbb{L}t) = \sum_\mu \mathtt{B}_\mu(t) * \mathtt{R}_{1-\mu}(t).$$

*Proof.* The disjoint union of the strata of the dimension function on $\mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ gives

$$\mathbb{C}^n \times \mathbb{M}^W_{m,n}(\lambda) = \mathtt{X}_{0,n,\lambda} \sqcup \mathtt{X}_{1,n,\lambda} \sqcup \cdots \sqcup \mathtt{X}_{n,n,\lambda}.$$

Rephrasing this in terms of motives gives

$$\mathbb{L}^n\big[\mathbb{M}^W_{m,n}(\lambda)\big] = \big[\mathbb{M}^W_{m,n}(\lambda)\big] + \sum_{k=1}^{n-1}[\mathtt{X}_{k,n,\lambda}] + [\mathrm{GL}_n]\big[\mathtt{BS}^W_{m,n}(\lambda)\big]$$

and substituting the formula of Proposition 3 into this we get

$$\begin{aligned}\frac{\big[\mathbb{M}^W_{m,n}(\lambda)\big]}{[\mathrm{GL}_n]}\mathbb{L}^n t^n &= \frac{\big[\mathbb{M}^W_{m,n}(\lambda)\big]}{[\mathrm{GL}_n]}t^n \\ &+ \sum_{k=1}^{n-1}\sum_{\mu\in\mathbb{C}}\big(\big[\mathtt{BS}^W_{m,k}(\mu)\big]t^k\big) * \left(\frac{\big[\mathbb{M}^W_{m,n-k}(\lambda-\mu)\big]}{[\mathrm{GL}_{n-k}]}t^{n-k}\right) + \big[\mathtt{BS}^W_{m,n}(\lambda)\big]t^n.\end{aligned}$$

Now, take $\lambda = 1$ then on the left hand side we have the $n$th term of the series $1+\mathtt{R}_1(\mathbb{L}t)$ and on the right hand side we have the $n$th factor of the series $\sum_\mu \mathtt{B}_\mu(t)*\mathtt{R}_{1-\mu}(t)$. The outer two terms arise from the product $\mathtt{B}_{\frac{1}{2}}(t) * \mathtt{R}_{\frac{1}{2}}(t)$, using that $W$ is homogeneous whence for all $\lambda \neq 0$

$$\mathtt{BS}^W_{m,n}(\lambda) \simeq \mathtt{BS}^W_{m,n}(1) \quad \text{and} \quad \mathbb{M}^W_{m,n}(\lambda) \simeq \mathbb{M}^W_{m,n}(1).$$

This finishes the proof. □

These formulas allow us to determine the motive $[\mathbb{M}^W_{m,n}(\lambda)]$ inductively from lower dimensional contributions and from the knowledge of the motive of the Brauer–Severi scheme $[\mathtt{BS}^W_{m,n}(\lambda)]$.

**Proposition 5.** *For all n we have the following inductive description of the motives in the Donaldson–Thomas series*

$$(\mathbb{L}^n - 1)\frac{\big[\mathbb{M}^W_{m,n}(0)\big] - \big[\mathbb{M}^W_{m,n}(1)\big]}{[\mathrm{GL}_n]}$$

*is equal to*

$$\begin{aligned}&\big[\mathtt{BS}^W_{m,n}(0)\big] - \big[\mathtt{BS}^W_{m,n}(1)\big] \\ &+ \sum_{k=1}^{n-1}\frac{\mathbb{L}^{(m-1)k(n-k)}}{[\mathrm{GL}_{n-k}]}\big(\big[\mathtt{BS}^W_{m,k}(0)\big] - \big[\mathtt{BS}^W_{m,k}(1)\big]\big)\big(\big[\mathbb{M}^W_{m,k}(0)\big] - \big[\mathbb{M}^W_{m,k}(1)\big]\big).\end{aligned}$$

*Proof.* Follows from Proposition 3 and the fact that for all $\mu \neq 0$ we have that

$$\left[\mathbb{M}^W_{m,k}(\mu)\right] = \left[\mathbb{M}^W_{m,k}(1)\right] \quad \text{and} \quad \left[\mathtt{BS}^W_{m,k}(\mu)\right] = \left[\mathtt{BS}^W_{m,k}(1)\right]. \qquad \square$$

## 3. Deformations of affine 3-space

The commutative polynomial ring $\mathbb{C}[x, y, z]$ is the Jacobi algebra associated with the superpotential $W = XYZ - XZY$. For this reason we restrict in the rest of this paper to cases where the superpotential $W$ is a cubic necklace in three non-commuting variables $X, Y$ and $Z$, that is $m = 3$ from now on. As even in this case the calculations become quickly unmanageable we restrict to $n \leq 2$, that is we only will compute the coefficients of $t$ and $t^2$ in $U_W(t)$. We will have to compute the motives of fibers of the Chern–Simons functional

$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \longrightarrow^{\mathrm{Tr}(W)} \mathbb{C}$$

so we want to express $\mathrm{Tr}(W)$ as a function in the variables of the three generic $2 \times 2$ matrices

$$X = \begin{bmatrix} n & p \\ q & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

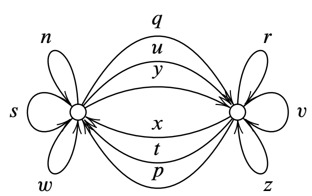
We will call $\{n, r, s, v, w, x\}$ (resp. $\{p, t, x\}$ and $\{q, u, y\}$) the diagonal- (resp. upper- and lower-) variables. We claim that

$$\mathrm{Tr}(W) = C + Q_q.q + Q_u.u + Q_y.y,$$

where $C$ is a cubic in the diagonal variables and $Q_q, Q_u$ and $Q_y$ are bilinear in the diagonal and upper variables, that is, there are linear terms $L_{ab}$ in the diagonal variables such that

$$\begin{cases} Q_q = L_{qp}.p + L_{qt}.t + L_{qx}.x, \\ Q_u = L_{up}.p + L_{ut}.t + L_{ux}.x, \\ Q_y = L_{yp}.p + L_{yt}.t + L_{yx}.x. \end{cases}$$

This follows from considering the two diagonal entries of a $2 \times 2$ matrix as the vertices of a quiver and the variables as arrows connecting these vertices as follows




and observing that only an oriented path of length 3 starting and ending in the same vertex can contribute something non-zero to $\mathrm{Tr}(W)$. Clearly these linear and cubic terms are fully determined by $W$. If we take

$$W = \alpha X^3 + \beta Y^3 + \gamma Z^3 + \delta XYZ + \epsilon XZY$$

then we have $C = W(n,s,w) + W(r,v,z)$ and

$$\begin{cases} L_{qp} = 3\alpha(n+r), \\ L_{qt} = \epsilon w + \delta z, \\ L_{qx} = \delta s + \epsilon v, \end{cases} \quad \begin{cases} L_{up} = \delta w + \epsilon z, \\ L_{ut} = 3\beta(s+v), \\ L_{ux} = \epsilon n + \delta r, \end{cases} \quad \begin{cases} L_{yp} = \epsilon s + \delta v, \\ L_{yt} = \delta n + \epsilon r, \\ L_{yx} = 3\gamma(w+z). \end{cases}$$

By using the cellular decomposition of the Brauer–Severi scheme of $\mathbb{T}_{3,2}$ one can simplify the computations further by specializing certain variables. From [16] we deduce that $\mathtt{BS}_2(\mathbb{T}_{3,2})$ has a cellular decomposition as $\mathbb{A}^{10} \sqcup \mathbb{A}^8 \sqcup \mathbb{A}^8$ where the three cells have representatives

$$\begin{cases} \mathtt{cell}_1 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & p \\ 1 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}, \\ \mathtt{cell}_2 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & t \\ 1 & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}, \\ \mathtt{cell}_3 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ 0 & v \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & x \\ 1 & z \end{bmatrix}. \end{cases}$$

It follows that $\mathtt{BS}^W_{3,2}(1)$ decomposes as $\mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$ where the subschemes $\mathbf{S_i}$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{cases} \mathbf{S_1} : (C + Q_u.u + Q_y.y + Q_q)|_{n=0} = 1, \\ \mathbf{S_2} : (C + Q_y.y + Q_u)|_{s=0} = 1, \\ \mathbf{S_3} : (C + Q_y)|_{w=0} = 1. \end{cases}$$

Note that in using the cellular decomposition, we set a variable equal to 1. So, in order to retain a homogeneous form we let $\mathbb{G}_m$ act on $n, s, w, r, v, z$ with weight one, on $q, u, y$ with weight two and on $x, t, p$ with weight zero. Thus, we need a slight extension of [4, Thm. 1.3] as to allow $\mathbb{G}_m$ to act with weight two on certain variables.

From now on we will assume that $W$ is as above with $\delta = 1$ and $\epsilon \neq 0$. In this generality we can prove:

**Proposition 6.** *With assumptions as above*

$$[\mathbf{S_3}] = \begin{cases} \mathbb{L}^7 - \mathbb{L}^4 + \mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, 0) = 1]_{\mathbb{A}^2}, & \text{if } \gamma \neq 0, \\ \mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, z) = 1]_{\mathbb{A}^3}, & \text{if } \gamma = 0. \end{cases}$$

*Proof.* $\mathbf{S_3}$: The defining equation in $\mathbb{A}^8$ is equal to

$$W(n,s,0) + W(r,v,z) + (\epsilon s + v)p + (n + \epsilon r)t + 3\gamma(z)x = 1.$$

If $\epsilon s + v \neq 0$ we can eliminate $p$ and get a contribution $\mathbb{L}^5(\mathbb{L}^2 - \mathbb{L})$. If $v = -\epsilon s$ but $n + \epsilon r \neq 0$ we can eliminate $t$ and get a term $\mathbb{L}^4(\mathbb{L}^2 - \mathbb{L})$. From now on we may assume that $v = -\epsilon s$ and $r = -\epsilon^{-1} n$.

$\gamma \neq 0$: Assume first that $z \neq 0$ then we can eliminate $x$ and get a contribution $\mathbb{L}^4(\mathbb{L} - 1)$. If $z = 0$ then we get a term

$$\mathbb{L}^3\big[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, 0) = 1\big]_{\mathbb{A}^2}$$

$\gamma = 0$: Then we have a remaining contribution

$$\mathbb{L}^3\big[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, z) = 1\big]_{\mathbb{A}^3}.$$

Summing up all contributions gives the result. □

Calculating the motives of $\mathbf{S_2}$ and $\mathbf{S_1}$ in this generality quickly leads to a myriad of subcases to consider. For this reason we will defer the calculations in the cases of interest to the next sections. Specializing Proposition 5 to the case of $n = 2$ we get

**Proposition 7.** *For $n = 2$ we have that*

$$(\mathbb{L}^2 - 1)\frac{\big[\mathbb{M}^W_{3,2}(0)\big] - \big[\mathbb{M}^W_{3,2}(1)\big]}{[\mathrm{GL}_2]}$$

*is equal to*

$$\big[\mathtt{BS}^W_{3,2}(0)\big] - \big[\mathtt{BS}^W_{3,2}(1)\big] + \frac{\mathbb{L}^2}{(\mathbb{L}-1)}\big(\big[\mathbb{M}^W_{3,1}(0)\big] - \big[\mathbb{M}^W_{3,1}(1)\big]\big)^2.$$

*Proof.* The result follows from Proposition 5 and from the fact that $\mathbf{BS}^W_{3,1}(1) = \mathbb{M}^W_{3,1}(1)$ and $\mathbf{BS}^W_{3,1}(0) = \mathbb{M}^W_{3,1}(0)$. □

## 4. Quantum affine three-space

For $q \in \mathbb{C}^*$ consider the superpotential $W_q = XYZ - qXZY$, then the associated algebra $R_{W_q}$ is the quantum affine 3-space

$$R_{W_q} = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(XY - qYX, ZX - qXZ, YZ - qZY)}.$$

It is well known that $R_{W_q}$ has finite dimensional simple representations of dimension $n$ if and only if $q$ is a primitive $n$th root of unity. For other values of $q$ the only finite dimensional simples are 1-dimensional and parametrized by $XYZ = 0$ in $\mathbb{A}^3$. In this case we have

$$\begin{cases} \big[\mathbb{M}^{W_q}_{3,1}(1)\big] = \big[(q-1)XYZ = 1\big]_{\mathbb{A}^3} = (\mathbb{L}-1)^2, \\ \big[\mathbb{M}^{W_q}_{3,1}(0)\big] = \big[(1-q)XYZ = 0\big]_{\mathbb{A}^3} = 3\mathbb{L}^2 - 3\mathbb{L} + 1. \end{cases}$$

That is, the coefficient of $t$ in $U_{W_q}(t)$ is equal to

$$\mathbb{L}^{-1}\frac{\big[\mathbb{M}^{W_q}_{3,1}(0)\big] - \big[\mathbb{M}^{W_q}_{3,1}(1)\big]}{[\mathrm{GL}_1]} = \mathbb{L}^{-1}\frac{2\mathbb{L}^2 - \mathbb{L}}{\mathbb{L}-1} = \frac{2\mathbb{L}-1}{\mathbb{L}-1}.$$

In [3, Thm. 3.1] it is shown that in case $q$ is not a root of unity, then

$$U_{W_q}(t) = \mathtt{Exp}\Big(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t}\Big)$$

and if $q$ is a primitive $n$th root of unity then

$$U_{W_q}(t) = \mathtt{Exp}\Big(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t} + (\mathbb{L}-1)\frac{t^n}{1-t^n}\Big).$$

In [3, 3.4.1] a rather complicated attempt is made to explain the term $\mathbb{L}-1$ in case $q$ is an $n$th root of unity in terms of certain simple $n$-dimensional representations of $R_{W_q}$. Note that the geometry of finite dimensional representations of the algebra $R_{W_q}$ is studied extensively in [5] and note that there are additional simple $n$-dimensional representations not taken into account in [3, 3.4.1].

Perhaps a more conceptual explanation of the two terms in the exponential expression of $U_{W_q}(t)$ in case $q$ is an $n$th root of unity is as follows. As $W_q$ admits a cut $W_q = X(YZ - qZY)$ it follows from [12] that for all dimensions $m$ we have

$$\big[\mathbb{M}^{W_q}_{3,m}(0)\big] - \big[\mathbb{M}^{W_q}_{3,m}(1)\big] = \mathbb{L}^{m^2}\big[\mathtt{rep}_m\big(\mathbb{C}_q[Y,Z]\big)\big],$$

where $\mathbb{C}_q[Y,Z] = \mathbb{C}\langle Y,Z\rangle/(YZ - qZY)$ is the quantum plane. If $q$ is an $n$th root of unity the only finite dimensional simple representations of $\mathbb{C}_q[Y,Z]$ are of dimension 1 or $n$. The 1-dimensional simples are parametrized by $YZ = 0$ in $\mathbb{A}^2$ having as motive $2\mathbb{L}-1$ and as all have $\mathrm{GL}_1$ as stabilizer group, this explains the term $(2\mathbb{L}-1)/(\mathbb{L}-1)$. The center of $\mathbb{C}_q[Y,Z]$ is equal to $\mathbb{C}[Y^n,Z^n]$ and the corresponding variety $\mathbb{A}^2 = \mathtt{Max}(\mathbb{C}[Y^n,Z^n])$ parametrizes $n$-dimensional semi-simple representations.The $n$-dimensional simples correspond to the Zariski open set $\mathbb{A}^2 - (Y^nZ^n = 0)$ which has as motive $(\mathbb{L}-1)^2$. Again, as all these have as $\mathrm{GL}_2$-stabilizer subgroup $\mathrm{GL}_1$, this explains the term

$$\mathbb{L}-1 = \frac{(\mathbb{L}-1)^2}{[\mathrm{GL}_1]}.$$

As the superpotential allows a cut in this case we can use the full strength of [1] and can obtain $[\mathbb{M}^W_{3,2}(0)]$ from $[\mathbb{M}^W_{3,2}(1)]$ from the equality

$$\mathbb{L}^{12} = \big[\mathbb{M}^W_{3,2}(0)\big] + (\mathbb{L}-1)\big[\mathbb{M}^W_{3,2}(1)\big].$$

To illustrate the inductive procedure using Brauer–Severi motives we will consider the case $n = 2$, that is $q = -1$ with superpotential $W = XYZ + XZY$. In this case we have from [3, Thm. 3.1] that

$$U_W(t) = \mathtt{Exp}\Big(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t} + (\mathbb{L}-1)\frac{t^2}{1-t^2}\Big).$$

The basic rules of the plethystic exponential on $\mathcal{M}_{\mathbb{C}}[\![t]\!]$ are

$$\mathtt{Exp}\bigg(\sum_{n\geq 1}[A_n]t^n\bigg) = \prod_{n\geq 1}(1-t^n)^{-[A_n]} \quad \text{where } (1-t)^{-\mathbb{L}^m} = (1-\mathbb{L}^m t)^{-1}$$

and one has to extend all infinite products in $t$ and $\mathbb{L}^{-1}$. One starts by rewriting $U_W(t)$ as a product

$$U_W(t) = \mathtt{Exp}\Big(\frac{t}{1-t}\Big)\mathtt{Exp}\Big(\frac{\mathbb{L}}{\mathbb{L}-1}\frac{t}{1-t}\Big)\mathtt{Exp}\Big(\frac{\mathbb{L}t^2}{1-t^2}\Big)\mathtt{Exp}\Big(\frac{t^2}{1-t^2}\Big)^{-1},$$

where each of the four terms is an infinite product

$$\mathtt{Exp}\Big(\frac{t}{1-t}\Big) = \prod_{m\geq 1}(1-t^m)^{-1}, \quad \mathtt{Exp}\Big(\frac{\mathbb{L}}{\mathbb{L}-1}\frac{t}{1-t}\Big) = \prod_{m\geq 1}\prod_{j\geq 0}(1-\mathbb{L}^{-j}t^m)^{-1},$$
$$\mathtt{Exp}\Big(\frac{\mathbb{L}t^2}{1-t^2}\Big) = \prod_{m\geq 1}(1-\mathbb{L}t^{2m})^{-1}, \quad \mathtt{Exp}\Big(\frac{t^2}{1-t^2}\Big)^{-1} = \prod_{m\geq 1}(1-t^{2m}).$$

That is, we have to work out the infinite product

$$\prod_{m\geq 1}\big((1-t^{2m-1})^{-1}(1-\mathbb{L}t^{2m})^{-1}\big)\prod_{m\geq 1}\prod_{j\geq 0}(1-\mathbb{L}^{-j}t^m)^{-1}$$

as a power series in $t$, at least up to quadratic terms. One obtains

$$U_W(t) = 1 + \frac{2\mathbb{L}-1}{\mathbb{L}-1}t + \frac{\mathbb{L}^4+3\mathbb{L}^3-2\mathbb{L}^2-2\mathbb{L}+1}{(\mathbb{L}^2-1)(\mathbb{L}-1)}t^2 + \cdots$$

That is, if $W = XYZ + XZY$ one must have the relation:

$$\big[\mathbb{M}^W_{3,2}(0)\big] - \big[\mathbb{M}^W_{3,2}(1)\big] = \mathbb{L}^5\big(\mathbb{L}^4+3\mathbb{L}^3-2\mathbb{L}^2-2\mathbb{L}+1\big).$$

**4.1. Dimensional reduction.** It follows from the dimensional reduction argument of [12] that

$$\big[\mathbb{M}^{W}_{3,2}(0)\big] - \big[\mathbb{M}^{W}_{3,2}(1)\big] = \mathbb{L}^4\big[\mathtt{rep}_2\,\mathbb{C}_{-1}[X,Y]\big],$$

where $\mathbb{C}_{-1}[X,Y]$ is the quantum plane at $q = -1$, that is, $\mathbb{C}\langle X,Y\rangle/(XY+YX)$. The matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

gives us the following system of equations

$$\begin{cases} 2ae + bg + fc = 0, \\ 2hd + bg + fc = 0, \\ f(a+d) + b(e+h) = 0, \\ c(h+e) + g(a+d) = 0, \end{cases}$$

where the two first are equivalent to $ae = hd$ and $2ae + bg + fc = 0$. Changing variables

$$x = \frac{1}{2}(a+d), \quad y = \frac{1}{2}(a-d), \quad u = \frac{1}{2}(e+h), \quad v = \frac{1}{2}(e-h),$$

the equivalent system then becomes (in the variables $b, c, f, g, u, v, x, y$)

$$\begin{cases} xv + yu = 0, \\ xu + yv + bg + fc = 0, \\ fx + bu = 0, \\ cu + gx = 0. \end{cases}$$

**Proposition 8.** *The motive of* $R_2 = \mathtt{rep}_2\,\mathbb{C}_{-1}[x,y]$ *is equal to*

$$[R_2] = \mathbb{L}^5 + 3\mathbb{L}^4 - 2\mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}.$$

*Proof.* If $x \neq 0$ we obtain

$$v = -\frac{yu}{x}, \quad f = -\frac{bu}{x}, \quad g = -\frac{cu}{x}$$

and substituting these in the remaining second equation we get the equation(s)

$$u(y^2 - x^2 + 2bc) = 0 \quad \text{and} \quad x \neq 0.$$

If $u \neq 0$ then $y^2 - x^2 + 2bc = 0$. If in addition $b \neq 0$ then $c = \frac{x^2-y^2}{2b}$ and $y$ is free. As $x, u$ and $b$ are non-zero this gives a contribution $(\mathbb{L}-1)^3\mathbb{L}$. If $b = 0$ then $c$ is free and $x^2 - y^2 = 0$, so $y = \pm x$. This together with $x \neq 0 \neq u$ leads to a contribution of $2\mathbb{L}(\mathbb{L}-1)^2$. If $u = 0$ then $y, b$ and $c$ are free variables, and together with $x \neq 0$ this gives $(\mathbb{L}-1)\mathbb{L}^3$.

Remains the case that $x = 0$. Then the system reduces to

$$\begin{cases} yu = 0, \\ yv + bg + fc = 0, \\ bu = 0, \\ cu = 0. \end{cases}$$

If $u \neq 0$ then $y = 0, b = 0$ and $c = 0$ leaving $c, g, v$ free. This gives $(\mathbb{L}-1)\mathbb{L}^3$. If $u = 0$ then the only remaining equation is $yv + bg + fc = 0$. That is, we get the cone in $\mathbb{A}^6$ of the Grassmannian $\mathrm{Gr}(2,4)$ in $\mathbb{P}^5$. As the motive of $\mathrm{Gr}(2,4)$ is

$$\big[\mathrm{Gr}(2,4)\big] = (\mathbb{L}^2+1)(\mathbb{L}^2+\mathbb{L}+1)$$

we get a contribution of

$$(\mathbb{L}-1)(\mathbb{L}^2+1)(\mathbb{L}^2+\mathbb{L}+1)+1.$$

Summing up all contributions gives the desired result. □

**4.2. Brauer–Severi motives.** In the three cells of the Brauer–Severi scheme of $\mathbb{T}_{3,2}$ of dimensions resp. 10, 9, and 8 the superpotential $\mathrm{Tr}(XYZ + XZY)$ induces the equations:

$$\begin{cases} \mathbf{S_1} : 2rvz + puz + pvy + rty + psy + rux + puw + tz + vx + sx + tw = 1, \\ \mathbf{S_2} : 2rvz + pvy + rty + nty + pz + rx + nx + pw = 1, \\ \mathbf{S_3} : 2rvz + pv + rt + nt + ps = 1. \end{cases}$$

**Proposition 9.** *With notations as above, the Brauer–Severi scheme of* $\mathbb{T}^W_{3,2}(1)$ *has a decomposition*

$$\mathbf{BS}^W_{3,2}(1) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3},$$

*where the schemes* $\mathbf{S_i}$ *have motives*

$$\begin{cases} [\mathbf{S_1}] = \mathbb{L}^9 - \mathbb{L}^6 - 2\mathbb{L}^5 + 3\mathbb{L}^4 - \mathbb{L}^3, \\ [\mathbf{S_2}] = \mathbb{L}^8 - 2\mathbb{L}^5 + \mathbb{L}^4, \\ [\mathbf{S_3}] = \mathbb{L}^7 - 2\mathbb{L}^4 + \mathbb{L}^3. \end{cases}$$

*Therefore, the Brauer–Severi scheme has motive*

$$\big[\mathbf{BS}^W_{3,2}(1)\big] = \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 - \mathbb{L}^6 - 4\mathbb{L}^5 + 2\mathbb{L}^4.$$

*Proof.* $\mathbf{S_1}$: From Proposition 6 we obtain

$$[\mathbf{S_3}] = \mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3 \big[W(n,s,0) + W(-n,-s,z) = 1\big]_{\mathbb{A}^3}$$

and as $W(n,s,0) + W(-n,-s,z) = 2nsz$ we get $\mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3(\mathbb{L}-1)^2$.

$\mathbf{S_2}$: The defining equation is

$$2rvz + y\big(pv + (r+n)t\big) + p(z+w) + x(r+n) = 1.$$

If $r + n \neq 0$ we can eliminate $x$ and have a contribution $\mathbb{L}^6(\mathbb{L}^2 - \mathbb{L})$. If $r + n = 0$ we get the equation

$$2rvz + p(yv + z + w) = 1.$$

If $yv + z + w \neq 0$ we can eliminate $p$ and get a term $\mathbb{L}^3(\mathbb{L}^4 - \mathbb{L}^3)$. If $r + n = 0$ and $yv + z + w = 0$ we have $2rvz = 1$ so a term $\mathbb{L}^4(\mathbb{L}-1)^2$. Summing up gives us

$$[\mathbf{S_2}] = \mathbb{L}^4(\mathbb{L}-1)(\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} - 1) = \mathbb{L}^8 - 2\mathbb{L}^5 + \mathbb{L}^4.$$

$\mathbf{S_1}$: The defining equation is

$$2rvz + p\big(u(z+w) + y(v+s)\big) + t(z + w + ry) + x(v + s + ru) = 1.$$

If $v + s + ru \neq 0$ we can eliminate $x$ and get $\mathbb{L}^5(\mathbb{L}^4 - \mathbb{L}^3)$. If $v + s + ru = 0$ and $z + w + ry \neq 0$ we can eliminate $t$ and have a term $\mathbb{L}^4(\mathbb{L}^4 - \mathbb{L}^3)$. If $v + s + ru = 0$ and $z + w + ry = 0$, the equation becomes (in $\mathbb{A}^8$, with $t, x$ free variables)

$$2r(vz - puy) = 1$$

giving a term $\mathbb{L}^2(\mathbb{L}^5 - [vz = puy])$. To compute $[vz = puy]_{\mathbb{A}^5}$ assume first that $v \neq 0$, then this gives $\mathbb{L}^3(\mathbb{L}-1)$ and if $v = 0$ we get $\mathbb{L}(3\mathbb{L}^2 - 3\mathbb{L} + 1)$. That is,

$$[vz = puy]_{\mathbb{A}^5} = \mathbb{L}^4 + 2\mathbb{L}^3 - 3\mathbb{L}^2 + \mathbb{L}.$$

In total this gives us

$$[\mathbf{S_1}] = \mathbb{L}^3(\mathbb{L}-1)(\mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 - 2\mathbb{L} + 1) = \mathbb{L}^9 - \mathbb{L}^6 - 2\mathbb{L}^5 + 3\mathbb{L}^4 - \mathbb{L}^3$$

finishing the proof. □

**Proposition 10.** *From the Brauer–Severi motive we obtain*

$$\begin{cases} \big[\mathbb{M}^W_{3,2}(1)\big] = \mathbb{L}^{11} - \mathbb{L}^8 - 3\mathbb{L}^7 + 2\mathbb{L}^6 + 2\mathbb{L}^5 - \mathbb{L}^4, \\ \big[\mathbb{M}^W_{3,2}(0)\big] = \mathbb{L}^{11} + \mathbb{L}^9 + 2\mathbb{L}^8 - 5\mathbb{L}^7 + 3\mathbb{L}^5 - \mathbb{L}^4. \end{cases}$$

*As a consequence we have,*

$$\big[\mathbb{M}^W_{3,2}(0)\big] - \big[\mathbb{M}^W_{3,2}(1)\big] = \mathbb{L}^4(\mathbb{L}^5 + 3\mathbb{L}^4 - 2\mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}).$$

*Proof.* We have already seen that $\mathbb{M}^W_{3,1}(1) = \{(x,y,z) \mid 2xyz = 1\}$ and $\mathbb{M}^W_{3,1}(0) = \{(x,y,z) \mid xyz = 0\}$ whence

$$\big[\mathbb{M}^W_{3,1}(1)\big] = (\mathbb{L}-1)^2 \quad \text{and} \quad \big[\mathbb{M}^W_{3,1}(0)\big] = 3\mathbb{L}^2 - 3\mathbb{L} + 1.$$

Plugging this and the obtained Brauer–Severi motive into Proposition 5 gives $[\mathbb{M}^W_{3,2}(1)]$. From this $[\mathbb{M}^W_{3,2}(0)]$ follows from the equation

$$\mathbb{L}^{12} = (\mathbb{L}-1)\big[\mathbb{M}^W_{3,2}(1)\big] + \big[\mathbb{M}^W_{3,2}(0)\big]. \qquad \square$$

## 5. The homogenized Weyl algebra

If we consider the superpotential $W = XYZ - XZY - \frac{1}{3}X^3$ then the associated algebra $R_W$ is the homogenized Weyl algebra

$$R_W = \frac{\mathbb{C}\langle X,Y,Z\rangle}{(XZ-ZX, XY-YX, YZ-ZY-X^2)}.$$

In this case we have $\mathbb{M}^W_{3,1}(1) = \{x^3 = -3\}$ and $\mathbb{M}^W_{3,1}(0) = \{x^3 = 0\}$, whence

$$\big[\mathbb{M}^W_{3,1}(1)\big] = \mathbb{L}^2[\mu_3], \quad \text{and} \quad \big[\mathbb{M}^W_{3,1}(0)\big] = \mathbb{L}^2$$

where, as in [3, 3.1.3] we denote by $[\mu_3]$ the equivariant motivic class of $\{x^3 = 1\} \subset \mathbb{A}^1$ carrying the canonical action of $\mu_3$. Therefore, the coefficient of $t$ in $U_W(t)$ is equal to

$$\mathbb{L}^{-1}\frac{\big[\mathbb{M}^W_{3,1}(0)\big] - \big[\mathbb{M}^W_{3,1}(0)\big]}{[\mathrm{GL}_1]} = \frac{\mathbb{L}\big(1-[\mu_3]\big)}{\mathbb{L}-1}.$$

As all finite dimensional simple representations of $R_W$ are of dimension one, this leads to the conjectural expression [3, Conjecture 3.3]

$$U_W(t) \overset{?}{=} \mathrm{Exp}\Big(\frac{\mathbb{L}\big(1-[\mu_3]\big)}{\mathbb{L}-1}\frac{t}{1-t}\Big).$$

Balazs Szendrői kindly provided the calculation of the first two terms of this series. Denote with $\widetilde{\mathbf{M}} = 1 - [\mu_3]$, then

$$U_W(t) \overset{?}{=} 1 + \frac{\mathbb{L}\widetilde{\mathbf{M}}}{\mathbb{L}-1}t + \frac{\mathbb{L}^2\widetilde{\mathbf{M}}^2 + \mathbb{L}(\mathbb{L}^2-1)\widetilde{\mathbf{M}} + \mathbb{L}^2(\mathbb{L}-1)\sigma_2(\widetilde{\mathbf{M}})}{(\mathbb{L}^2-1)(\mathbb{L}-1)}t^2 + \cdots$$

As was pointed out by B. Pym and B. Davison it follows from [4, Defn. 4.4 and Prop. 4.5(4)] that $\sigma_2(\widetilde{\mathbf{M}}) = \mathbb{L}$, so the second term is equal to

$$\frac{\mathbb{L}^3(\mathbb{L}-1) + \widetilde{\mathbf{M}}\mathbb{L}(\mathbb{L}^2-1) + \widetilde{\mathbf{M}}^2\mathbb{L}^2}{(\mathbb{L}^2-1)(\mathbb{L}-1)}.$$

We will now compute the this second term using Brauer–Severi motives.

Recall that $\mathtt{BS}^{W}_{3,2}(i)$, for $i = 0,1$, decomposes as $\mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$ where the subschemes $\mathbf{S_i}$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{cases} \mathbf{S_1} : \frac{1}{3}r^3 + \big((w-z)p + rx\big)u + \big((v-s)p - rt\big)y - rp + (z-w)t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +(s-v)x = \delta_{i1}, \\ \mathbf{S_2} : -\frac{1}{3}n^3 - \frac{1}{3}r^3 + (vp + (n-r)t)y + (w-z)p + (r-n)x = \delta_{i1}, \\ \mathbf{S_3} : -\frac{1}{3}n^3 - \frac{1}{3}r^3 + (v-s)p + (n-r)t = \delta_{i1}. \end{cases}$$

If we let the generator of $\mu_3$ act with weight one on the variables $n, s, w, r, v, z$, with weight two on $x, t, p$ and with weight zero on $q, u, y$ we see that the schemes $S_j$ for $i = 1$ are indeed $\mu_3$-varieties. We will now compute their equivariant motives:

**Proposition 11.** *With notations as above, the Brauer–Severi scheme of* $\mathbb{T}^{W}_{3,2}(1)$ *has a decomposition*

$$\mathbf{BS}^{W}_{3,2}(1) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3},$$

*where the schemes* $\mathbf{S_i}$ *have equivariant motives*

$$\begin{cases} [\mathbf{S_1}] = \mathbb{L}^9 - \mathbb{L}^6, \\ [\mathbf{S_2}] = \mathbb{L}^8 + ([\mu_3] - 1)\mathbb{L}^6 = \mathbb{L}^8 - \tilde{\mathbf{M}}\mathbb{L}^6, \\ [\mathbf{S_3}] = \mathbb{L}^7 + ([\mu_3] - 1)\mathbb{L}^5 = \mathbb{L}^7 - \tilde{\mathbf{M}}\mathbb{L}^5. \end{cases}$$

*Therefore, the Brauer–Severi scheme* $\mathbf{BS}^{W}_{3,2}(1)$ *has equivariant motive*

$$\big[\mathbf{BS}^{W}_{3,2}(1)\big] = \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 + \big([\mu_3] - 2\big)\mathbb{L}^6 + \big([\mu_3] - 1\big)\mathbb{L}^5.$$

*Proof.* $\mathbf{S_3}$: If $v - s \neq 0$ we can eliminate $p$ and obtain a contribution $\mathbb{L}^5(\mathbb{L}^2 - \mathbb{L})$. If $v = s$ and $n - r \neq 0$ we can eliminate $t$ and obtain a term $\mathbb{L}^4(\mathbb{L}^2 - \mathbb{L})$. Finally, if $v = s$ and $n = r$ we have the identity $-\frac{2}{3}n^3 = 1$ and a contribution $\mathbb{L}^5[\mu_3]$.

$\mathbf{S_2}$: If $r - n \neq 0$ we can eliminate $x$ and get a term $\mathbb{L}^6(\mathbb{L}^2 - \mathbb{L})$. If $r - n = 0$ we get the equation in $\mathbb{A}^8$

$$-\frac{2}{3}n^3 + p(vy + w - z) = 1.$$

If $vy + w - z \neq 0$ we can eliminate $p$ and get a contribution $\mathbb{L}^3(\mathbb{L}^4 - \mathbb{L}^3)$. Finally, if $vy + w - z = 0$ we get the equation $-\frac{2}{3}n^3 = 1$ and hence a term $\mathbb{L}^3.\mathbb{L}^3[\mu_3]$.

$\mathbf{S_1}$: If $(w - z)p + rx \neq 0$ then we can eliminate $u$ and get a contribution

$$\mathbb{L}^4\big(\mathbb{L}^5 - \big[(w-z)p + rx = 0\big]_{\mathbb{A}^5}\big) = \mathbb{L}^6(\mathbb{L} - 1)(\mathbb{L}^2 - 1).$$

If $(w - z)p + rx = 0$ but $(v - s)p - rt \neq 0$ we can eliminate $y$ and get a term

$$\mathbb{L}.\big[(w-z)p + rx = 0, (v-s)p - rt \neq 0\big]_{\mathbb{A}^8}.$$

To compute the equivariant motive in $\mathbb{A}^8$ assume first that $r \neq 0$ then we can eliminate $x$ from the equation and obtain

$$\begin{aligned}\mathbb{L}^2\big[r \neq 0, (v-s)p - rt \neq 0\big]_{\mathbb{A}^5} \\ &= \mathbb{L}^2\big(\mathbb{L}^4(\mathbb{L}-1) - \big[r \neq 0, (v-s)p - rt = 0\big]_{\mathbb{A}^5}\big) \\ &= \mathbb{L}^5(\mathbb{L}-1)^2.\end{aligned}$$

If $r = 0$ we have to compute

$$\big[(w-z)p = 0, (v-s)p \neq 0\big]_{\mathbb{A}^7} = \mathbb{L}^2(\mathbb{L}-1)(\mathbb{L}^2-\mathbb{L})\mathbb{L} = \mathbb{L}^4(\mathbb{L}-1)^2.$$

So, in total this case gives a contribution

$$\mathbb{L}.\big[(w-z)p + rx = 0, (v-s)p - rt \neq 0\big]_{\mathbb{A}^8} = \mathbb{L}^5(\mathbb{L}-1)(\mathbb{L}^2-1).$$

If $(w-z)p + rx = 0$, $(v-s)p - rt = 0$, and $r \neq 0$ we can eliminate $x = \frac{z-w}{r}p$ and $t = \frac{v-s}{r}p$ and substituting in the defining equation of $\mathbf{S_1}$ we get

$$-\frac{1}{3}r^3 - rp = 1,$$

so we can eliminate $p$ and obtain a contribution $\mathbb{L}^6(\mathbb{L}-1)$. Finally, if $(w-z)p+rx=0$, $(v-s)p - rt = 0$, and $r = 0$ we get the system of equations

$$\begin{cases}(w-z)p = 0, \\ (v-s)p = 0, \\ (z-w)t + (s-v)x = 1.\end{cases}$$

If $p \neq 0$ we must have $w-z = 0$ and $v-s = 0$ which is impossible, so we must have $p = 0$ and the remaining equation is $(z-w)t + (s-v)x = 1$ giving a contribution $\mathbb{L}^5(\mathbb{L}^2-1)$. Summing up these contributions gives the claimed motive. □

**Proposition 12.** *With notations as above, the Brauer–Severi scheme of* $\mathbb{T}^W_{3,2}(0)$ *has a decomposition*

$$\mathbf{BS}^W_{3,2}(0) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3},$$

*where the schemes* $\mathbf{S_i}$ *have (equivariant) motives*

$$\begin{cases}[\mathbf{S_1}] = \mathbb{L}^9 + \mathbb{L}^7 - \mathbb{L}^6, \\ [\mathbf{S_2}] = \mathbb{L}^8, \\ [\mathbf{S_3}] = \mathbb{L}^7.\end{cases}$$

*Therefore, the Brauer–Severi scheme* $\mathbf{BS}^W_{3,2}(0)$ *has (equivariant) motive*

$$\big[\mathbf{BS}^W_{3,2}(0)\big] = \mathbb{L}^9 + \mathbb{L}^8 + 2\mathbb{L}^7 - \mathbb{L}^6.$$

*Proof.* $\mathbf{S_3}$: If $v-s \neq 0$ we can eliminate $p$ and obtain a contribution $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$. If $v=s$ and $n-r \neq 0$ we can eliminate $t$ and obtain a term $\mathbb{L}^4(\mathbb{L}^2-\mathbb{L})$. Finally, if $v=s$ and $n=r$ we have the identity $n^3=0$ and a contribution $\mathbb{L}^5$.

$\mathbf{S_2}$: If $r-n \neq 0$ we can eliminate $x$ and get a term $\mathbb{L}^6(\mathbb{L}^2-\mathbb{L})$. If $r-n=0$ we get the equation in $\mathbb{A}^8$

$$-\frac{2}{3}n^3 + p(vy+w-z) = 1.$$

If $vy+w-z \neq 0$ we can eliminate $p$ and get a contribution $\mathbb{L}^3(\mathbb{L}^4-\mathbb{L}^3)$. Finally, if $vy+w-z=0$ we get the equation $n^3=0$ and hence a term $\mathbb{L}^6$.

$\mathbf{S_1}$: If $(w-z)p+rx \neq 0$ we can eliminate $u$ and obtain a term

$$\mathbb{L}^4\big(\mathbb{L}^5 - \big[(w-z)p+rx=0\big]_{\mathbb{A}^5}\big) = \mathbb{L}^6(\mathbb{L}-1)(\mathbb{L}^2-1).$$

If $(w-z)p+rx=0$ but $(v-s)p-rt \neq 0$ then we can eliminate $y$ and obtain a contribution

$$\mathbb{L}\big[(w-z)p+rx=0, (v-s)p-rt \neq 0\big]_{\mathbb{A}^8} = \mathbb{L}^5(\mathbb{L}-1)(\mathbb{L}^2-1).$$

Now, assume that $(w-z)p+rx=0$ and $(v-s)p-rt=0$. If $r \neq 0$ then we can eliminate $p, t$ as before and substituting them in the defining equation of $\mathbf{S_1}$ we get

$$-\frac{1}{3}r^3 - rp = 0$$

and we can eliminate $p$ giving a contribution $\mathbb{L}^6(\mathbb{L}-1)$. Finally, if $(w-z)p+rx=0$ and $(v-s)p-rt=0$ and $r=0$ we have the system of equations

$$\begin{cases}(w-z)p=0,\\ (v-s)p=0,\\ (z-w)t+(s-v)x=0.\end{cases}$$

If $p \neq 0$ we get $w-z=0$ and $v-s=0$ giving a contribution $\mathbb{L}^6(\mathbb{L}-1)$. If $p=0$ the only remaining equation is $(z-w)t+(s-v)x=0$ which gives a contribution $\mathbb{L}^5(\mathbb{L}^2+\mathbb{L}-1)$. Summing up all terms gives the claimed motive. □

Now, we have all the information to compute the second term of the motivic Donaldson–Thomas series. We have

$$\begin{cases}\big[\mathbf{BS}^W_{3,2}(0)\big] - \big[\mathbf{BS}^W_{3,2}(1)\big] = \mathbb{L}^7 + \tilde{\mathbf{M}}\mathbb{L}^6 + \tilde{\mathbf{M}}\mathbb{L}^5,\\ \big[\mathbb{M}^W_{3,1}(0)\big] - \big[\mathbb{M}^W_{3,1}(1)\big] = \tilde{\mathbf{M}}\mathbb{L}^2.\end{cases}$$

By Proposition 7 this implies that

$$(\mathbb{L}^2-1)\frac{\big[\mathbb{M}^W_{3,2}(0)\big] - \big[\mathbb{M}^W_{3,2}(1)\big]}{[\mathrm{GL}_2]} = \mathbb{L}^7 + \tilde{\mathbf{M}}\mathbb{L}^6 + \tilde{\mathbf{M}}\mathbb{L}^5 + \tilde{\mathbf{M}}^2\frac{\mathbb{L}^6}{(\mathbb{L}-1)}.$$

Therefore the virtual motive is equal to

$$\mathbb{L}^{-4}\frac{\big[\mathbb{M}^{W}_{3,2}(0)\big]-\big[\mathbb{M}^{W}_{3,2}(1)\big]}{[\mathrm{GL}_2]}=\frac{\mathbb{L}^3(\mathbb{L}-1)+\tilde{\mathbf{M}}\mathbb{L}(\mathbb{L}^2-1)+\tilde{\mathbf{M}}^2\mathbb{L}^2}{(\mathbb{L}^2-1)(\mathbb{L}-1)},$$

which coincides with the conjectured term in [3, Conjecture 3.3].

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