Symmetric noncommutative birational transformations

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Abstract. In [11] certain birational transformations were constructed between the noncommutative schemes associated to quadratic and cubic three dimensional Sklyanin algebras. In the current paper we consider the inverse birational transformations and show that they are of the same type. Moreover we extend everything to the $\mathbb{Z}$-algebras context, which allows us to incorporate the noncommutative quadrics introduced by Van den Bergh in [18].

1. Introduction

Throughout this paper k will be an algebraically closed field and all rings will be algebras over k. Following tradition [5] we associate a noncommutative projective scheme $X = \mathrm{Proj}(A)$ to a connected graded k-algebra $A = k \oplus A_1 \oplus \cdots$, which is generated by A_1. $\mathrm{Proj}(A)$ is defined via its category of "quasicoherent sheaves":

$$\mathrm{Qcoh}(X) := \mathrm{QGr}(A) = \mathrm{Gr}(A)/\mathrm{Tors}(A),$$

where $\mathrm{Gr}(A)$ is the category of graded right A-modules and $\mathrm{Tors}(A)$ is the full subcategory of torsion A-modules, i.e. those graded right A-modules that have locally right bounded grading [5]. A non-trivial class of noncommutative surfaces is given by three dimensional Artin–Schelter regular algebras as defined in [1]. Recall that a connected graded algebra A is called AS-regular of dimension d if it satisfies the following conditions:

(i) The Hilbert series $h_n(A) := \dim_k(A_n)$ is bounded by a polynomial in n;

(ii) A has finite global dimension d;

(iii) A has the Gorenstein property with respect to d.

Such algebras with $d = 3$ and which are generated in degree 1 are classified in [1,2]. There are two possibilities for the number of generators and relations:

(i) A is generated by three elements satisfying three quadratic relations (the "quadratic case"). In this case A has Hilbert series $1/(1 - t)^3$, i.e. the same Hilbert series as a polynomial ring in three variables. Therefore we think of $\mathrm{Proj}(A)$ as being a noncommutative $\mathbb{P}^2$;

(ii) A is generated by two elements satisfying two cubic relations (the "cubic case"). In this case A has Hilbert series $1/(1 - t)^2(1 - t^2)$ and we can think of $\mathrm{Proj}(A)$ as being a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$. (The rationale for this is explained in [18].)

We write (r, s) for the number of generators of A and the degrees of the relations. Thus $(r, s) = (3, 2)$ or $(2, 3)$ depending on whether A is quadratic or cubic.

The classification of three-dimensional AS-regular algebras A is in terms of suitable geometric data $(Y, \mathcal{L}, \sigma)$ where Y is a k-scheme, σ is an automorphism of Y and $\mathcal{L}$ is a line bundle on Y. More precisely: starting from $(Y, \mathcal{L}, \sigma)$ one can construct a twisted homogeneous coordinate ring $B(Y, \mathcal{L}, \sigma)$ (see, for example, [4]). This is a connected graded k-algebra with $r := \dim_k(B_1)$ equal to 2 or 3. Setting $s = 5 - r$ as above, we can then find the AS-regular algebra $A(Y, \mathcal{L}, \sigma)$ by dropping all relations in degree $s + 1$ and higher. In particular there is a surjective morphism $A(Y, \mathcal{L}, \sigma) \to B(Y, \mathcal{L}, \sigma)$ giving rise to a fully faithful functor

$$\mathrm{QGr}(B) \hookrightarrow \mathrm{QGr}(A). \tag{1.1}$$

Moreover, there is an equivalence of categories $\mathrm{QGr}(B) \cong \mathrm{Qcoh}(Y)$ (see, for example, [4]). Therefore one often says there is a commutative curve Y contained inside the noncommutative surface X.

Below we say that A is a (quadratic or cubic) Sklyanin algebra if Y is smooth and σ is a translation.

Inspired by the commutative case it makes sense to expect that in a suitable sense noncommutative $\mathbb{P}^2$s are birationally equivalent to noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$s. Such birational equivalences were constructed in [12] and [11].

In [11] we provide a noncommutative version of the standard birational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ by showing that for each cubic Sklyanin algebra A there exists a quadratic Sklyanin algebra A' and an inclusion

$$\check{A}' \hookrightarrow \check{A}^{(2)}, \tag{1.2}$$

where $\check{A}$ and $\check{A}'$ are the associated $\mathbb{Z}$-algebras. (See §2 for more on $\mathbb{Z}$-algebras.) We then check that this inclusion gives rise to an isomorphism of the function fields $\mathrm{Frac}_0(A') \cong \mathrm{Frac}_0(A)$, a result which was already announced in [15] and [14]. In [11], we also provide a noncommutative version of the Cremona transform $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by showing that for each quadratic Sklyanin algebra A there exists a quadratic Sklyanin algebra A' and an inclusion

$$\check{A}' \hookrightarrow \check{A}^{(2)}. \tag{1.3}$$

The construction was based on the choice of 3 non-collinear points p, q, r on Y and if $A = A(Y, \mathcal{L}, \psi)$ then

$$A' = A\big(Y, \mathcal{L} \otimes \psi^* \mathcal{L} \otimes \mathcal{O}_Y(-p - q - r), \psi\big).$$

In §3 we recapitulate the construction of (1.2) and (1.3).

In §4 we extend the above to the level of $\mathbb{Z}$-algebras (see §2 and Remark 4.2 for the appropriate definitions) and prove the following:

Theorem (Theorem 4.1). *Let A be a cubic Sklyanin $\mathbb{Z}$-algebra. Then there is a quadratic Sklyanin $\mathbb{Z}$-algebra A' and an inclusion $A' \hookrightarrow A^{(2)}$ inducing an isomorphism between their function fields. This inclusion is constructed with respect to a point p on Y such that if $A = A(Y, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ then*

$$A' = A\big(Y, \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{O}_Y(-p), \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \mathcal{O}_Y(-\tau^{-1}p)\big),$$

where $\tau = \alpha^2 \in \mathrm{Aut}(Y)$ is some nontrivial translation (see §4 for the appropriate definitions. In the special case where A comes from a $\mathbb{Z}$-graded cubic Sklyanin algebra $A(Y, \mathcal{L}, \sigma)$, one has $\tau = \sigma^4$).

Remark 1.1. The existence of the function field a cubic Sklyanin $\mathbb{Z}$-algebra (or more generally a "quadric") is proven in Appendix A.

In §4 we also provide noncommutative versions of the inverse birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as follows:

Theorem (Theorem 4.4). *Let A be a quadratic Sklyanin $\mathbb{Z}$-algebra, then there is a cubic Sklyanin $\mathbb{Z}$-algebra A' and an inclusion $A' \hookrightarrow A$ inducing an isomorphism between their function fields. This inclusion is constructed with respect to points p, q on Y such that if $A = A(Y, \mathcal{L}, \psi)$ then*

$$A' = A\big(Y, \mathcal{L} \otimes \mathcal{O}_Y(-p), \psi^* \mathcal{L} \otimes \mathcal{O}_Y(-q), \psi^{*2} \mathcal{L} \otimes \mathcal{O}_Y(-\psi^{-3}p)\big).$$

In §5 we show (Theorem 5.2) that, modulo some technical hypothesis, the non-commutative Cremona as in (1.3) factors through the noncommutative $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. As such all of these are examples of "quadratic transforms", a more general type of noncommutative birational transformations introduced in §5.

In the last sections we show that quadratic transforms are invertible in the following sense (for simplicity we omit some technical hypotheses):

Theorem (Theorem 6.4). *Let $\gamma: A' \hookrightarrow A^{(w)}$ be a quadratic transform. Then there exists a quadratic transform $\delta: A \hookrightarrow A'^{(v)}$ such that the compositions $\gamma \circ \delta: A \hookrightarrow A^{(vw)}$ and $\delta \circ \gamma: A' \hookrightarrow A'^{(vw)}$ induce the identity map on the function fields.*

Remark 1.2. In the above theorem the possible values for w are determined by the definition of a quadratic transform. Moreover v can be chosen as a function of w and $\frac{v}{w} \in \{\frac{1}{2}, 1, 2\}$.

In §6 we show that it suffices to prove this theorem in case γ is as in Theorem 4.1 or Theorem 4.4. Both of these cases are covered in §7. The proof is quite technical and uses a $\mathbb{Z}^2$-algebra which in a certain sense glues A and A'. These $\mathbb{Z}^2$-algebras have been further investigated in [10], where it was shown that they are related to noncommutative blowups. On the other hand the geometric picture is rather simple. For example if $A = A(Y, \mathcal{L}, \psi)$ is a quadratic Sklyanin algebra and $\gamma\colon A' \hookrightarrow A$ is constructed with respect to points p, q on Y as in Theorem 4.4, then the inverse $\delta\colon A \hookrightarrow A'^{(2)}$ is constructed with respect to p' where $p + q + p' \sim [\psi_* \mathcal{L}]$ in the Picard group of Y. Conversely if $A = A(Y, (\mathcal{L}_i)_i)$ is a cubic Sklyanin $\mathbb{Z}$-algebra and $\gamma\colon A' \hookrightarrow A^{(2)}$ is constructed with respect to a point p on Y as in Theorem 4.1, then the inverse $\delta\colon A \hookrightarrow A'$ is constructed with respect to p', q' where $p + \tau q' \sim [\mathcal{L}_0]$ and $p + p' \sim [\mathcal{L}_1]$ in the Picard group of Y.

2. $\mathbb{Z}$-algebras

In this section we recall some definitions and facts on $\mathbb{Z}$-algebras. We refer the reader to [13] or Sections 3 and 4 of [18] for a more thorough introduction. Recall that a $\mathbb{Z}$-algebra is defined as an algebra R (without unit) with a decomposition

$$R = \bigoplus_{(m,n) \in \mathbb{Z}^2} R_{m,n}$$

such that addition is degree-wise and multiplication satisfies $R_{m,n} R_{n,j} \subset R_{m,j}$ and $R_{m,n} R_{i,j} = 0$ if $n \neq i$. Moreover, there are local units $e_n \in R_{n,n}$ such that for each $x \in R_{m,n} : e_m x = x = x e_n$.

Notation. If A is a graded algebra, then it gives rise to a $\mathbb{Z}$-algebra $\check{A}$ via

$$\check{A}_{m,n} = A_{n-m}.$$

In particular the notion of a $\mathbb{Z}$-algebra is a generalization of a ($\mathbb{Z}$-)graded algebra. Based on this, most graded notions have a natural $\mathbb{Z}$-algebra counterpart. For example we say that a $\mathbb{Z}$-algebra R is positively graded if $R_{m,n} = 0$ for $m > n$.

A graded R-module is an R-module M together with a decomposition $M = \oplus_n M_n$ such that the R-action on M satisfies $M_m R_{m,n} \subset M_n$ and $M_m R_{i,n} = 0$ if $i \neq m$. The category of graded R-modules is denoted $\mathrm{Gr}(R)$ and similar to the graded case we use the notation $\mathrm{QGr}(R) := \mathrm{Gr}(R)/\mathrm{Tors}(R)$.

Remark 2.1. If A is a graded algebra, then obviously $\mathrm{Gr}(A) = \mathrm{Gr}(\check{A})$ by identifying $A(n) \in \mathrm{Gr}(A)$ with $e_{-n}\check{A} \in \mathrm{Gr}(\check{A})$. Similarly, $\mathrm{QGr}(A) = \mathrm{QGr}(\check{A})$.

Definition 2.2. Let R be a $\mathbb{Z}$-algebra. Then for each $n \in \mathbb{Z}$ we define $R(n)$ by setting $(R(n))_{i,j} = R_{i+n,j+n}$ with obvious multiplication. We say R is n-periodic if there is a $\mathbb{Z}$-algebra-isomorphism $R \cong R(n)$.

Lemma 2.3. *Let R be a $\mathbb{Z}$-algebra. Then there exists a graded algebra A such that $R = \check{A}$ if and only if R is 1-periodic.*

Proof. The "only if" part is obvious from the definition of $\check{A}$. The "if" part is proven in [18, Lemma 3.4]. $\qquad\square$

Remark 2.4. From a categorical point of view a $\mathbb{Z}$-algebra R is nothing but a k-linear category $\mathcal{R}$ whose objects are given by the integers. The homogeneous elements of the algebra then correspond to morphisms between two such integers via $\mathrm{Hom}_{\mathcal{R}}(-j, -i) = R_{i,j}$ and multiplication in R corresponds to composition of morphisms in $\mathcal{R}$.

We would like to point out that this categorical notation is slightly different from the notation used in [18, §3] where a $\mathbb{Z}$-algebra is said to be a pre-additive category $\mathcal{R}'$ by identifying $\mathrm{Hom}_{\mathcal{R}'}(i, j)$ with $R_{i,j}$ and using concatenation of morphisms in $\mathcal{R}'$ to define a multiplication on R. Obviously for a $\mathbb{Z}$-algebra R with associated categories $\mathcal{R}$ and $\mathcal{R}'$ (using these 2 different interpretations), we have an isomorphism of categories $\mathcal{R}' \cong \mathcal{R}^{\mathrm{op}}$ by sending object i in $\mathcal{R}'$ to $-i$ in $\mathcal{R}$.

In the light of (4.3) however, our notation makes more sense for this paper than the one used in [18, §3].

2.1. AS-regular $\mathbb{Z}$-algebras.

Definition 2.5. Let R be a $\mathbb{Z}$-algebra, then R is said to be connected, if it is positively graded, $\dim_k(R_{m,n}) < \infty$ for each m, n and $R_{m,m} \cong k$ for all m. We say R is generated in degree 1 if $R_{m,m+1}R_{m+1,n} = R_{m,n}$ holds for all $m < n$. If R is a connected $\mathbb{Z}$-algebra, generated in degree 1, then we denote $S_{n,R} = e_n R/(e_n R)_{\geq n+1}$. i.e. $S_{n,R}$ is the unique R-module concentrated in degree n where it is equal to the base field k.

We can now give the definition of an AS-regular $\mathbb{Z}$-algebra as in [18]

Definition 2.6. A $\mathbb{Z}$-algebra R over k is said to be AS-regular if the following conditions are satisfied:

(1) R is connected and generated in degree 1;

(2) $\dim_k(R_{m,n})$ is bounded by a polynomial in $n - m$;

(3) The projective dimension of $S_{n,R}$ is finite and bounded by a number independent of n;

(4) $\forall n \in \mathbb{N} : \sum_{i,j} \dim_k \left(\mathrm{Ext}^i_{\mathrm{Gr}(R)}(S_{j,R}, e_n R) \right) = 1$ (the "Gorenstein condition").

It is immediate that if a graded algebra A is AS-regular, then $\check{A}$ is AS-regular in the above sense.

$\mathbb{Z}$-algebra analogues of three dimensional quadratic and cubic AS-regular algebras were classified in [18], it is shown in loc.cit. that every quadratic AS-regular $\mathbb{Z}$-algebra is of the form $\check{A}$ for some quadratic AS-regular algebra A. However, most cubic AS-regular $\mathbb{Z}$-algebras are not 1-periodic. To distinguish cubic AS-regular algebras from the more general cubic AS-regular $\mathbb{Z}$-algebras, one often refers to the latter as *quadrics*.

Similar to the graded case, the classification of three-dimensional quadratic and cubic AS-regular $\mathbb{Z}$-algebras in terms of geometric data $(Y, (\mathcal{L}_i)_{i \in \mathbb{Z}})$ where Y is a k-scheme and $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ is an elliptic helix of line bundles on Y (see [6] for more information on helices). Starting from the geometric data one first constructs a $\mathbb{Z}$-algebra analogue $B = B(Y, (\mathcal{L}_i)_i)$ of the twisted homogeneous coordinate ring:

$$B_{i,j} := \begin{cases} \Gamma(Y, \mathcal{L}_i \otimes \mathcal{L}_{i+1} \otimes \ldots \mathcal{L}_{j-1}), & \text{if } i \le j, \\ 0, & \text{if } i > j. \end{cases}$$

Again there is an equivalence of categories:

$$\operatorname{Qcoh}(Y) \cong \operatorname{QGr}(B) \colon \mathcal{F} \mapsto \bigoplus_{i \ge 0} \Gamma\big(Y, \mathcal{F} \otimes \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{i-1}\big) \qquad (2.1)$$

(see [18, Corollary 5.5.9]). $r = \dim_k(B_{i,i+1})$ does not depend on i and equals 3 (in the quadratic case) or 2 (in the cubic case). $A(Y, (\mathcal{L}_i)_i)$ is then obtained from B by only preserving the relations in degree $(i, i + s)$ for $s = 5 - r$. It is shown that $\dim_k(A_{i,i+n})$ does not depend on i; it hence makes sense to write $h(n) := \dim_k(A_{i,i+n})$ and one checks that $h(n)$ coincides with the Hilbert series in the graded case (even though AS-regular $\mathbb{Z}$-algebras need not be 1-periodic).

2.2. $\mathbb{Z}$-domains and $\mathbb{Z}$-fields of fractions.
In this section we give the natural generalizations of "domain" and "field of fractions" for $\mathbb{Z}$-algebras. Among other generalizations, these notions can also be found in [7, §2].

Definition 2.7. Let R be a $\mathbb{Z}$-algebra. Then we say that R is a $\mathbb{Z}$-domain if the following condition is satisfied:

$$\forall i, j, k \in \mathbb{Z}, \ \forall x \in R_{i,j}, \ \forall y \in R_{j,k} : xy = 0 \Rightarrow x = 0 \ \vee \ y = 0$$

It is known that three dimensional AS-regular algebras are domains ([3, Theorem 3.9] and [3, Theorem 8.1]). We extend this result to the level of quadrics and show:

Theorem 2.8. *Let $A = A(Y, (\mathcal{L}_i)_i)$ be a quadric. Then A is a $\mathbb{Z}$-domain.*

Proof. The proof of this theorem is postponed to Appendix A. $\qquad \square$

Let R be a $\mathbb{Z}$-algebra and $\mathcal{R}$ the associated category as in Remark 2.4. Let W be a collection of homogeneous elements and let $\mathcal{W}$ be the corresponding collection of morphisms. We then say that R is localizable at W if $(\mathcal{R}, \mathcal{W})$ admits a calculus of fractions (see for example [8]). In this case we define $R[W^{-1}]$ as the $\mathbb{Z}$-algebra associated to $\mathcal{R}[\mathcal{W}^{-1}]$.

Using the theory of (right) fractions for a category one easily checks that the following definition makes sense.

Definition 2.9. Let R be a $\mathbb{Z}$-domain, then R admits a $\mathbb{Z}$-field of (right) fractions if the following condition is satisfied:

$$\forall r \in R_{l,i}, \ s \in R_{l,j} \setminus \{0\} : \exists n \in \mathbb{Z} : \exists r' \in R_{j,n}, s' \in R_{i,n} \setminus \{0\} : rs' = sr'.$$

The elements of $\text{Frac}(R)_{i,j}$ are equivalence classes of couples (r, s) where $r \in R_{i,l}$, $s \in R_{j,l} \setminus \{0\}$ for some $l \in \mathbb{Z}$. The equivalence relation is given by

$$\forall l_1, l_2 \in \mathbb{Z}, \ \forall r_1 \in R_{i,l_1}, \ r_2 \in R_{i,l_2}, \ s_1 \in R_{j,l_1} \setminus \{0\}, \ s_2 \in R_{j,l_2} \setminus \{0\} :$$
$$(r_1, s_1) \sim (r_2, s_2)$$
$$\Updownarrow$$
$$\exists l_3 \in \mathbb{Z}, \ \exists x \in R_{l_1,l_3} \setminus \{0\}, \ \exists y \in R_{l_2,l_3} \setminus \{0\} : r_1 x = r_2 y \text{ and } s_1 x = s_2 y.$$

The following is obvious from the definition:

Proposition 2.10. *Let A be a graded domain. Then:*

A admits a (graded) field of (right) fractions

$$\Leftrightarrow \check{A} \text{ admits a } \mathbb{Z}\text{-field of (right) fractions.}$$

Moreover in this case $\widetilde{\text{Frac}(A)} = \text{Frac}(\check{A})$

Recall that graded domains admit fields of (right-)fractions when they are graded (right-)noetherian or when they have subexponential growth. A similar result can be found in [7]:

Proposition 2.11. *Let R be a $\mathbb{Z}$-domain such that each $e_i R$ is a uniform module (i.e. for all nonzero $M, N \subset e_i R : M \cap N \neq 0$). Then R admits a $\mathbb{Z}$-field of (right) fractions.*

Proof. We need to show that for all $r \in R_{l,i}, s \in R_{l,j} \setminus \{0\}$ there exists an $n \in \mathbb{Z}$ and elements $r' \in R_{j,n}, s' \in R_{i,n} \setminus \{0\}$ such that $rs' = sr'$. If $r = 0$ then it suffices to take $r' = 0$. If $r \neq 0$ the existence of r' and s' follows from the fact that rR and sR are nonzero submodules of the uniform module $e_l R$. $\square$

Similarly the following will be shown in Appendix A:

Theorem 2.12. *Let A be a quadric. Then A admits a $\mathbb{Z}$-field of fractions.*

Remark 2.13. As three dimensional, quadratic AS-regular $\mathbb{Z}$-algebras are 1-periodic, the existence of their $\mathbb{Z}$-field of fractions is automatic from Proposition 2.10.

Remark 2.14. When $Q = \mathrm{Frac}(A)$ it is customary to refer to $Q_{0,0}$ as the function field of $\mathrm{QGr}(A)$. We will do so throughout this paper.

3. Summary of the results in [11]

In [11] we construct noncommutative versions of the birational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and the Cremona transform $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ using the following recipe:

Step 1. Let A be a 3-dimensional Sklyanin algebra. Let $(Y, \mathcal{L}, \sigma)$ be the associated geometric data, B be the associated twisted homogeneous coordinate ring and $\mathrm{Qcoh}(X) = \mathrm{QGr}(A)$. Denote $\mathrm{Bimod}(Y - Y) \hookrightarrow \mathrm{Bimod}(X - X)$ for the categories of bimodules (recall: $\mathrm{Bimod}(X_1 - X_2)$ is the category of right exact functors $\mathrm{Qcoh}(X_1) \to \mathrm{Qcoh}(X_2)$ commuting with direct limits) with o_Y, o_X corresponding to the identity functors. Let $o_X(-Y) = \ker(o_X \twoheadrightarrow o_Y)$[1].

Step 2. Choose a divisor d on Y in the following way:

- $d = p + q + r$ for p, q, r distinct points, non-collinear with respect to the embedding in $\mathbb{P}^2$ given by $\mathcal{L}$, in case A is quadratic;
- $d = p$ in case A is cubic.

Let $\mathcal{O}_d$ be it's structure sheaf and denote $o_d \in \mathrm{Bimod}(Y - Y) \subset \mathrm{Bimod}(X - X)$ for the bimodule

$$o_d := (- \otimes_Y \mathcal{O}_d)$$

and

$$\begin{aligned} m_d &:= \ker(o_X \twoheadrightarrow o_d), \\ m_{d,Y} &:= \ker(o_Y \twoheadrightarrow o_d). \end{aligned} \tag{3.1}$$

Step 3. Define $\mathbb{Z}$-algebras D_Y and D as follows:

$$(D_Y)_{m,n} = \begin{cases} \mathrm{Hom}\left(\mathcal{O}_Y(-2n), \mathcal{O}_Y(-2m) \otimes_{o_Y} m_{\tau^{-m}d,Y} \ldots m_{\tau^{-n+1}d,Y}\right), \\ \hspace{6cm} \text{if } n \geq m, \\ 0, \hspace{5cm} \text{if } n < m, \end{cases}$$

$$D_{m,n} = \begin{cases} \mathrm{Hom}\left(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m) \otimes_{o_X} m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d}\right), & \text{if } n \geq m, \\ 0, & \text{if } n < m, \end{cases} \tag{3.2}$$

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[1]Unfortunately $\mathrm{Bimod}(X_1 - X_2)$ appears not to be an abelian category. This technical difficulty is solved in [17] by embedding $\mathrm{Bimod}(X - Y)$ into a larger abelian category $\mathrm{BIMOD}(X_1 - X_2)$ consisting of "weak bimodules". In particular we will always construct kernels and cokernels as elements in $\mathrm{BIMOD}(X_1 - X_2)$. This being said, these technical complication will be invisible in this paper as all bimodules we construct will be elements of the (smaller) category $\mathrm{Bimod}(X_1 - X_2)$.

where $\tau = \sigma^{s+1}$,

$$\mathcal{O}_X(i) = \pi(A(i)), \quad \mathcal{O}_Y(i) = \pi(B(i)), \tag{3.3}$$

π is the quotient functor $\mathrm{Gr}(A) \to \mathrm{QGr}(A)$ and $m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d} \subset o_X$ is the image of the morphism of bimodules

$$m_{\tau^{-m}d} \otimes m_{\tau^{-m-1}d} \otimes \cdots \otimes m_{\tau^{-n+1}d} \to o_X \otimes o_X \otimes \cdots \otimes o_X \to o_X.$$

By [17, Lemma 8.2.1] the inclusions $m_{\tau^{-i}d} \hookrightarrow o_X$ give rise to an inclusion $D \hookrightarrow \check{A}^{(2)}$.

Step 4. One shows that D_Y is a twisted homogeneous coordinate ring, i.e. there is a sequence of line bundles $(\mathcal{G}_i)_{i \in \mathbb{Z}}$ such that

$$(D_Y)_{m,n} = \Gamma(Y, \mathcal{G}_m \otimes \cdots \otimes \mathcal{G}_{n-1}). \tag{3.4}$$

Moreover, the $\mathcal{G}_i$ must form an "elliptic helix" (for a quadratic AS regular $\mathbb{Z}$-algebra), i.e.

- $\deg \mathcal{G}_i = 3$ for all i;
- $\mathcal{G}_0 \not\cong \mathcal{G}_1$;
- $\mathcal{G}_i \otimes \mathcal{G}_{i+1}^{\otimes -2} \otimes \mathcal{G}_{i+2} \equiv \mathcal{O}_Y$.

Step 5. One shows that D is generated in degree 1. For this we need some natural vanishing results on $\mathcal{O}_X(a) \otimes m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d}$ (We will recapitulate and slightly extend these vanishing results in Lemmas 4.8 and 4.10.)

Step 6. One shows that the canonical map $D \to D_Y$ is surjective and that D has the correct Hilbert series. It then follows that D is the AS-regular $\mathbb{Z}$-algebra corresponding to the elliptic helix $(\mathcal{G}_i)_{i \in \mathbb{Z}}$ on Y. As D is quadratic, there is an AS-regular quadratic algebra A' such that $D = \check{A}'$ (see [18, Theorem 4.2.2]).

Step 7. Check that the induced morphism

$$\mathrm{Frac}_0(A') = \mathrm{Frac}_{0,0}(D) \to \mathrm{Frac}_{0,0}(\check{A}) = \mathrm{Frac}_0(A)$$

is an isomorphism. This check is immediate as injectivity comes for free and surjectivity follows by a Hilbert series argument.

4. Addendum to [11]

As explained in the previous section, in [11] we construct noncommutative versions of the birational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. In this section we slightly generalize this (Theorem 4.1) and in addition we discuss a noncommutative version of the inverse birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ (Theorem 4.4).

Theorem 4.1. *Let $A = A(Y, (\mathcal{L}_i)_{i \in \mathbb{Z}})$ be a quadric such that:*

- *Y is a smooth elliptic curve;*

- *$\mathcal{L}_2 \cong \alpha^* \mathcal{L}_0$ for some $\alpha \in \mathrm{Aut}(Y)$ which is given by translation by a point of order at least 3 (i.e. $\alpha^2 \neq \mathrm{Id}$),*

then there exists a quadratic Sklanin algebra A' and an inclusion $\breve{A}' \hookrightarrow A^{(2)}$ inducing an isomorphism of function fields.

Remark 4.2. We refer to a quadric as in Theorem 4.1 as a cubic Sklyanin $\mathbb{Z}$-algebra.

Remark 4.3. Theorem 4.1 extends [11] in the sense that it does not require A to be 1-periodic.

Theorem 4.4. *Let $A = A(Y, \psi, \mathcal{L})$ be a quadratic Sklyanin algebra. Then there exists a cubic Sklyanin $\mathbb{Z}$-algebra D and an inclusion $D \hookrightarrow \breve{A}$ inducing an isomorphism of function fields. The construction of D depends on two points, $p, q \in Y$ and D is 1-periodic, i.e. of the form $D = \breve{A}'$, if $q = \sigma p$ for σ a translation such that $\sigma^2 = \psi^{-3}$.*

As both Theorem 4.1 and 4.4 contain statements on AS-regular $\mathbb{Z}$-algebras which need not be 1-periodic we introduce some notation for an AS-regular $\mathbb{Z}$-algebra A: Inspired by (3.3) and Remark 2.1 we define

$$\mathcal{O}_X(i) = \pi(e_{-i}A), \quad \mathcal{O}_Y(i) = \pi(e_{-i}B). \tag{4.1}$$

Under the equivalence of categories $\mathrm{QGr}(B) \cong \mathrm{Qcoh}(Y)$ (see (2.1)) $\mathcal{O}_Y(i)$ can be identified with

$$\begin{cases} \mathcal{L}_0 \mathcal{L}_1 \ldots \mathcal{L}_{i-1}, & \text{if } i > 0, \\ \mathcal{O}_Y, & \text{if } i = 0, \\ (\mathcal{L}_0 \mathcal{L}_1 \ldots \mathcal{L}_{-i-1})^{-1}, & \text{if } i < 0. \end{cases} \tag{4.2}$$

Similar to the graded case [11, §4] we have

$$\begin{aligned} \mathrm{Hom}\left(\mathcal{O}_X(-j), \mathcal{O}_X(-i)\right) &= \mathrm{Hom}_{\mathrm{QGr}(A)}\left(\pi(e_j A), \pi(e_i A)\right) \\ &= \lim_{n \to \infty} \mathrm{Hom}_{\mathrm{Gr}(A)}\left(e_j A_{\geq n}, e_i A\right) \\ &= \mathrm{Hom}_{\mathrm{Gr}(A)}\left(e_j A, e_i A\right) \\ &= A_{i,j}, \end{aligned} \tag{4.3}$$

where the second equality follows from the fact that $e_j A$ and $e_i A$ are finitely generated A-modules (see [5, Proposition 7.2] for the graded case). The third equality follows from the Gorenstein condition as this implies

$$\mathrm{Hom}_{\mathrm{Gr}(A)}\left(e_j A / e_j A_{\geq n}, e_i A\right) = \mathrm{Ext}^1_{\mathrm{Gr}(A)}\left(e_j A / e_j A_{\geq n}, e_i A\right) = 0.$$

We also have the following standard vanishing result:

Lemma 4.5. *Let A be an AS-regular $\mathbb{Z}$-algebra of dimension 3 with Hilbert function h and let $X = \mathrm{QGr}(A)$ and $\mathcal{O}_X(i)$ be as above, then we have*

$$\mathrm{Ext}^n\left(\mathcal{O}_X(i), \mathcal{O}_X(j)\right) = 0 \quad \text{for } n \neq 0, 2,$$

$$\dim_k\left(\mathrm{Ext}^2(\mathcal{O}_X(i), \mathcal{O}_X(j))\right) = \begin{cases} 0, & \text{if } j \geq i - s, \\ h(i - s - 1 - j), & \text{if } j \leq i - s - 1. \end{cases}$$

Proof. Similar to [5, Theorem 8.1]. □

Remark 4.6. Throughout the rest of the paper X and $\mathcal{O}_X(i)$ will be defined as in Lemma 4.5.

We now give an overview of both proofs separately. The recipe is as in §3 and we only highlight the steps that need to be adapted.

4.1. Noncommutative $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. (Proof of Theorem 4.1).

Step 1. Let $A = A(Y, (\mathcal{L}_i)_{i \in \mathbb{Z}})$ be a cubic Sklyanin $\mathbb{Z}$-algebra as in the statement of Theorem 4.1.

Steps 2 and 3. Let p be a point on Y and set $\tau = \alpha^2 \neq \mathrm{Id}$ and $d = p$ in the definition of D and D_Y as in (3.2).

Step 4. Computations similar to the ones in [11, §5] show that D_Y is a twisted homogeneous coordinate ring, i.e.

$$D_{Y,m,n} = \Gamma\left(Y, \mathcal{G}_m \otimes \cdots \otimes \mathcal{G}_{n-1}\right)$$

for some new collection of line bundles on Y:

$$\mathcal{G}_i = \mathcal{O}_Y(-\tau^{-i}p) \otimes \mathcal{L}_{2i} \otimes \mathcal{L}_{2i+1}. \tag{4.4}$$

It follows easily that these line bundles satisfy:

(a) $\deg \mathcal{G}_i = 3$;

(b) $\mathcal{G}_0 \not\cong \mathcal{G}_1$;

(c) $\mathcal{G}_i \otimes \mathcal{G}_{i+1}^{\otimes -2} \otimes \mathcal{G}_{i+2} \cong \mathcal{O}_Y$;

(d) $\psi^*(\mathcal{G}_i) \cong \mathcal{G}_{i+1}$, where ψ is an arbitrary translation satisfying $\psi^3 = \tau$.

Steps 5, 6, and 7. We need the following generalizations of [11, Lemmas 6.2, 6.4, and 6.5]:

Lemma 4.7. *Let A be as above and let p be a point in Y. Let $i \in \mathbb{Z}$ and let P_i be the point module generated in degree i corresponding to the point p (i.e. the point module truncated at degree i).*

Then there is a complex of the following form:

$$0 \to e_{i+5}A \xrightarrow{(\zeta,0)} e_{i+4}A^{\oplus 2} \oplus e_{i+3}A \to e_{i+2}A^{\oplus 3} \to e_i A \to P_i \to 0, \qquad (4.5)$$

where ζ is part of the minimal resolution of S_{i+1} as given in [17, Definition 4.1.1.]

$$0 \to e_{i+5}A \xrightarrow{\zeta} e_{i+4}A^{\oplus 2} \xrightarrow{\varepsilon} e_{i+2}A^{\oplus 2} \xrightarrow{\delta_0} e_{i+1}A \xrightarrow{\gamma} S_{i+1} \to 0.$$

Moreover, the complex (4.5) is exact everywhere except at $e_i A$ where it has one-dimensional cohomology, concentrated in degree $i + 1$.

Proof. Similar to the proof of [11, Lemma 6.2] this is based on the fact that the minimal projective resolution for P_i is given by

$$0 \to e_{i+3}A \to e_{i+1}A \oplus e_{i+2}A \to e_i A \to P_i \to 0. \qquad (4.6)$$

The latter is a $\mathbb{Z}$-algebra analogue of the projective resolution of a point module as in [3, Proposition 6.7.i]. The proof in loc. cit. can be adapted to a $\mathbb{Z}$-algebra version as follows:

– First we note that by [17, §5] there is a 1-1-correspondence between point modules (truncated in degree i) and points on Y, so the correspondence between $p \in Y$ and P_i is well defined.

– Next we generalize line modules to modules of the form $e_j A / a A$, where $a \in A_{j,j+1}$. As A is a $\mathbb{Z}$-domain (Theorem 2.8) we know that line modules have the desired Hilbert series and have projective dimension 1. The characterization of line modules as in [3, Corollary 2.43] generalizes to this new definition of line modules if we replace the modules $A(-i)$ in the graded case by $e_i A$.

– Finally we find (4.6) by generalizing the proof of [3, Proposition 6.7.i]. This proof uses the fact that point modules in the graded case have projective dimension 2. A careful observation however shows that projective dimension ≤ 2 suffices for the proof. This in turn can be shown by a variation of [3, Proposition 2.46.i]. $\qquad \square$

In particular [11, Lemma 6.4, Lemma 6.5] generalize to the following

Lemma 4.8. *With the notations as above we have for $i - j \leq 3$:*

$$\mathrm{Ext}^2 \left(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d} \right) = 0,$$

and for $i - j \leq 2m - 2n + 2$:

$$\mathrm{Ext}^1 \left(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{\tau^{-m}d} \ldots m_{\tau^{-n+1}d} \right) = 0.$$

Proof. Similar to [11, Lemma 6.4, Lemma 6.5] by using Lemmas 4.7 and 4.5. $\qquad \square$

One can now copy the proofs in [11, §6] to find that D and D_Y are generated in degree 1, that the canonical map $D \to D_Y$ is surjective and an isomorphism in degree 0, 1, and 2, that D has the correct Hilbert series and hence $D \cong A(Y, (\mathcal{G}_i)_{i \in \mathbb{Z}}) = \breve{A}'$ for the graded algebra $A' = A'(Y, \mathcal{G}_0, \psi)$. Similarly, the results in [11, §7] extend to an isomorphism of function fields $\mathrm{Frac}_{0,0}(A) \cong \mathrm{Frac}_0(A')$.

4.2. Noncommutative $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ (Proof of Theorem 4.4).

Step 1. Throughout this section $A = A(Y, \mathcal{L}, \psi)$ is a quadratic Sklyanin algebra, hence Y is a smooth elliptic curve, $\mathcal{L}$ is a degree 3 line bundle on Y and $\psi \in \mathrm{Aut}(Y)$ is given by translation by a point with $\tau = \psi^3 \neq \mathrm{Id}$.

Steps 2 and 3. The construction of D is slightly different from the one in (3.2): Let p, q be points on Y and denote

$$d_i = \begin{cases} \tau^{-j} p, & \text{if } i = 2j, \\ \tau^{-j} q, & \text{if } i = 2j + 1, \end{cases} \tag{4.7}$$

where $\tau = \psi^3$. We define $m_{d_i, Y}$ and m_{d_i} as in (3.1) and introduce $\mathbb{Z}$-algebra D_Y and D via

$$(D_Y)_{m,n} = \begin{cases} \mathrm{Hom}(\mathcal{O}_Y(-n), \mathcal{O}_Y(-m) \otimes_{o_Y} m_{d_m, Y} \ldots m_{d_{n-1}, Y}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } n \geq m, \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } n < m, \end{cases} \tag{4.8}$$

$$D_{m,n} = \begin{cases} \mathrm{Hom}(\mathcal{O}_X(-n), \mathcal{O}_X(-m) \otimes_{o_X} m_{d_m} \ldots m_{d_{n-1}}), & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases} \tag{4.9}$$

Again using [17, Lemma 8.2.1], there is an inclusion $D \hookrightarrow \check{A}$.

Step 4. Computations similar to the ones in [11, §5] show

$$D_{Y,m,n} = \Gamma\big(Y, \mathcal{G}_m \otimes \cdots \otimes \mathcal{G}_{n-1}\big)$$

for some collection of line bundles on Y given by:

$$\mathcal{G}_i = \mathcal{O}_Y(-d_i) \otimes \psi^{i*} \mathcal{L} \tag{4.10}$$

and these line bundles satisfy:

(a) $\deg \mathcal{G}_i = 2$;

(b) $\mathcal{G}_0 \not\cong \mathcal{G}_2$;

(c) $\mathcal{G}_i \otimes \mathcal{G}_{i+1}^{-1} \otimes \mathcal{G}_{i+2}^{-1} \otimes \mathcal{G}_{i+3} \cong \mathcal{O}_Y$;

(d) $\alpha^*(\mathcal{G}_i) \cong \mathcal{G}_{i+2}$, where α is an arbitrary translation satisfying $\alpha^2 = \tau$.

Moreover, if $q = \alpha^{-1} p$ then $\mathcal{G}_{i+1} \cong \sigma^* \mathcal{G}_i$ for an arbitrary translation $\sigma: Y \to Y$ such that $\sigma^2 = \alpha$.

Steps 5 and 6. This is completely analogous to [11, §6]. The only difference lies in the resolution of $\mathcal{O}(-i) \otimes m_{d_n}$: this time it is of the form:

$$0 \to \mathcal{O}(-i - 3) \to \mathcal{O}(-i - 2)^{\oplus 2} \to \mathcal{O}(-i) \otimes m_{d_n} \to 0.$$

This is based on the following:

Lemma 4.9. *Let $A = A(Y, \mathcal{L}, \psi)$ be a quadratic AS-regular algebra of dimension 3, let $p \in Y$ be a point and let P be the corresponding point module. Then the minimal resolution of P has the following form:*

$$0 \rightarrow A(-2) \rightarrow A(-1)^{\oplus 2} \rightarrow A \rightarrow P \rightarrow 0.$$

Proof. This follows immediately from [3, Proposition 6.7]. $\qquad\square$

From this we find the following standard vanishing results:

Lemma 4.10. *With the notations as above we have for $i - j \leq 2$:*

$$\mathrm{Ext}^2\left(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{m_d} \ldots m_{d_{n-1}}\right) = 0,$$

and for $i - j \leq m - n + 1$:

$$\mathrm{Ext}^1\left(\mathcal{O}_X(i), \mathcal{O}_X(j) \otimes m_{d_m} \ldots m_{d_{n-1}}\right) = 0.$$

Proof. Similar to [11, Lemma 6.4, Lemma 6.5] by using Lemmas 4.9 and 4.5. $\qquad\square$

For checking that D has the correct Hilbert series we see that (using computation similar to [17, Corollary 5.2.4]) the colength of $m_{d_m} \ldots m_{d_{m+i-1}}$ inside o_X is

$$\begin{cases} 2 \cdot \dfrac{a(a + 1)}{2} = a(a + 1), & \text{if } i = 2a, \\ \dfrac{(a + 2)(a + 1)}{2} + \dfrac{a(a + 1)}{2} = (a + 1)^2, & \text{if } i = 2a + 1. \end{cases} \tag{4.11}$$

Using Lemma 4.10 this implies

$$\dim D_{m,m+i} = \begin{cases} \dfrac{(2a + 1)(2a + 2)}{2} - a(a + 1) = (a + 1)^2, & \text{if } i = 2a, \\ \dfrac{(2a + 2)(2a + 3)}{2} - (a + 1)^2 = (a + 1)(a + 2), & \text{if } i = 2a + 1. \end{cases}$$

i.e. D has the Hilbert series of a cubic AS-regular $\mathbb{Z}$-algebra and hence we can conclude that D is AS-regular.

Step 7. Injectivity of $\mathrm{Frac}_{0,0}(D) \rightarrow \mathrm{Frac}_0(A)$ is again immediate. For surjectivity we need to prove that for any fixed $n \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that

$$\mathrm{Hom}\left(\mathcal{O}_X(-i - n - N), \mathcal{O}_X(-i - n) \otimes m_{d_i} \ldots m_{d_{i+n+N-1}}\right) \neq 0$$

holds for all $i \in \mathbb{Z}$. Now note that the colength of $m_{d_i} \ldots m_{d_{i+n+N-1}}$ inside o_X is given by a^2 or $a(a + 1)$ if $n + N = 2a$ or $2a + 1$, respectively. In particular, this colength grows like $\frac{N^2}{4}$ for N large. Hence the codimension of

$$\mathrm{Hom}\left(\mathcal{O}_X(-i - n - N), \mathcal{O}_X(-i - n) \otimes m_{d_i} \ldots m_{d_{i+n+N-1}}\right)$$

inside $A_{i+n,i+n+N}$ grows at most like $\frac{N^2}{4}$. As $\dim_k A_{i+n,i+n+N}$ grows like $\frac{N^2}{2}$ it hence follows that

$$\Gamma\big(\mathcal{O}_X(-i-n-N), \mathcal{O}_X(-i-n) \otimes m_{d_i} \ldots m_{d_{i+n+N-1}}\big) \neq 0$$

for large N.

5. Cremona transformations and quadratic transforms

Convention 5.1. *In the following sections we will always identify graded algebras with their associated $\mathbb{Z}$-algebras. The $\mathbb{Z}$-algebra D which was introduced in Sections 3 and 4 will now be denoted by A'. Moreover we will use the terminology "(three dimensional) Sklyanin $\mathbb{Z}$-algebras" for both quadratic and cubic Sklyanin $\mathbb{Z}$-algebras.*

The goal of this section is to link the noncommutative version of the Cremona transform $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as constructed in [11] (see also §3 for a reminder of this construction) to the noncommutative versions of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in Theorems 4.1, 4.4. This is inspired by the following commutative picture:
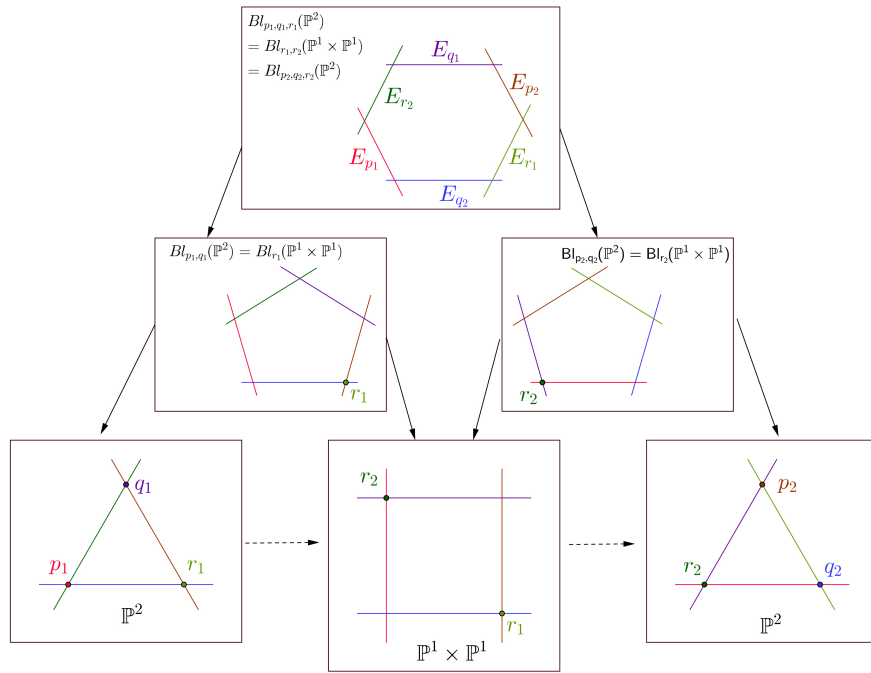

Figure 1. The commutative Cremona transform.

i.e. the (commutative) Cremona transform $\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ (given by blowing up three non-collinear points p_1, q_1, r_1, see for example [9, Example V.4.2.3.]) factors as $\gamma = \gamma_1 \circ \gamma_2$ where $\gamma_1\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and $\gamma_2\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ are the standard birational transformations obtained by blowing up one point in $\mathbb{P}^1 \times \mathbb{P}^1$ and two points in $\mathbb{P}^2$.

We prove that (modulo some technical assumptions) the same holds in the noncommutative setting:

Theorem 5.2. *Let $A = A(Y, \mathcal{L}, \psi)$ and A' be quadratic Sklyanin algebras and let $\gamma\colon A' \hookrightarrow A^{(2)}$ be a noncommutative Cremona transformation as in [11]. Denote $d = p + q + r$ as in loc. cit. and assume p, q and r are non-collinear points lying in different τ-orbits where $\tau = \psi^3$. Then there exists a cubic Sklyanin $\mathbb{Z}$-algebra A'' and inclusions $\gamma_1\colon A' \hookrightarrow A''^{(2)}$, $\gamma_2\colon A'' \hookrightarrow A$ such that γ_1 is as in Theorem 4.1, γ_2 is as in Theorem 4.4 and $\gamma = \gamma_2 \circ \gamma_1$.*

Before we can prove this we need the following technical result:

Lemma 5.3. *Let p and q be points such that $p \neq \tau^i q$ for $i \in \{-1, 0, 1\}$. Then $m_{p+q} = m_p m_q$.*

Proof. Consider the following diagram:

The middle row and column are the exact sequences as in (3.1). The last row is obtained by applying $- \otimes m_q$ to the middle row, it hence is automatically right exact. Similarly the first column is right exact.

We can identify $m_p m_q$ both with the image of f_1 and a_3 (and hence the kernel of g_1 and b_3). To see this recall that $m_p m_q$ is defined as the image of

$$f\colon m_p \otimes m_q \to o_X \otimes o_X \to o_X$$

The identification then follows as $f = f_2 \circ a_3 = a_2 \circ f_1$ and as a_2 and f_2 are monomorphisms.

Next we claim that a_1 and f_3 are in fact isomorphisms. The proof follows from this claim as it implies $m_p m_q$ serves both as a kernel for $b_2 \circ f_2$ and $g_2 \circ a_2$. As such it is the pullback of

$$
\begin{array}{ccc}
m_p & \xrightarrow{\ a_2\ } & o_X \\
 & & \Big\uparrow{\scriptstyle f_2} \\
 & & m_q
\end{array}
$$

On the other hand

$$o_{p+q} = o_p \oplus o_q$$

such that m_{p+q}, being the kernel of $o_X \xrightarrow{\ b_2 \oplus g_2\ } o_{p+q}$, is also a pullback of the above diagram. In particular, $m_{p+q} \cong m_p m_q$.

It hence remains to prove the above claim. As the argument is the same for both morphisms we only explain this for a_1. Note that this map is obtained by tensoring $m_p \hookrightarrow o_X$ with o_q. As such the cokernel of a_1 is given by $o_p \otimes o_q$ whereas the kernel is a subobject of $\mathcal{T}or_1(o_p, o_q)$ (see [17, §3] for the definition of $\mathcal{T}or_i$). In particular it suffices to show $o_p \otimes o_q = 0 = \mathcal{T}or_1(o_p, o_q)$. By [17, Lemma 5.5.1] we need to show $\mathrm{Hom}_X(\mathcal{O}_p, \mathcal{O}_q) = 0 = \mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q)$. For Hom_X this follows from the fact that $\mathcal{O}_p$ and $\mathcal{O}_q$ are simple. For Ext^1, we can use [17, Proposition 5.1.2 and (5.3)] to reduce the computations to showing $\mathrm{Ext}^1_Y(\mathcal{O}_p, \mathcal{O}_q) = 0 = \mathrm{Hom}_Y(\mathcal{O}_p, \mathcal{O}_{\tau q})$. The latter follows p, q and τq are different points by assumption. $\qquad\square$

We can now finish the main result of this section:

Proof of Theorem 5.2. Recall from §3 that $\gamma \colon A' \hookrightarrow A^{(2)}$ was constructed as follows:

$$
\begin{array}{c}
(A')_{m,n} = \mathrm{Hom}_X\big(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m) \otimes_{o_X} m_{\tau^{-m}d} \dots m_{\tau^{-n+1}d}\big) \\
\Big\downarrow \\
A_{2m,2n} = \mathrm{Hom}_X\big(\mathcal{O}_X(-2n), \mathcal{O}_X(-2m)\big)
\end{array}
\tag{5.1}
$$

We construct $\gamma_2 \colon A'' \hookrightarrow A$ with respect to the points p, q as in Theorem 4.4. i.e.

$$(A'')_{m,n} = \mathrm{Hom}_X\big(\mathcal{O}_X(-n), \mathcal{O}_X(-m) \otimes_{o_X} m_{d_m} \dots m_{d_{n-1}}\big),$$

where

$$
d_i = \begin{cases} \tau^{-j} p, & \text{if } i = 2j, \\ \tau^{-j} q, & \text{if } i = 2j + 1. \end{cases}
$$

By Lemma 5.3 for each i we can write $m_{d_{2i}} m_{d_{2i+1}} = m_{\tau^{-i}(p+q)}$. In particular, the inclusions $m_{\tau^{-i}(p+q+r)} \hookrightarrow m_{\tau^{-i}(p+q)}$ give rise to an inclusion $\gamma_1 \colon A' \hookrightarrow A''^{(2)}$ such that $\gamma = \gamma_2 \circ \gamma_1$. It hence remains to show that γ_1 is in fact an inclusion as in Theorem 4.1. For this we need to prove the existence of a point p' such that

$$\gamma_1\big((A')_{m,n}\big) = \mathrm{Hom}_{X'}\big(\mathcal{O}_{X'}(-2n), \mathcal{O}_{X'}(-2m) \otimes_{o_{X'}} m'_{\tau^{-m}p'} \ldots m'_{\tau^{-n+1}p'}\big) \quad (5.2)$$

where $m'_{p'} = \ker(o_{X'} \to o_{p'})$. As A' is generated in degree 1, it suffices to check (5.2) for $n = m + 1$ in which case the left hand side of (5.2) equals

$$\Gamma\big(Y, \psi^{2m*}\mathcal{L} \otimes \psi^{2m+1*}\mathcal{L} \otimes \mathcal{O}_Y(-\tau^{-m}p - \tau^{-m}q - \tau^{-m}r)\big)$$

and the right hand side equals

$$\Gamma\big(Y, \psi^{2m*}\mathcal{L} \otimes \mathcal{O}_Y(\tau^{-m}p) \otimes \psi^{2m+1*}\mathcal{L} \otimes \mathcal{O}_Y(-\tau^{-m}q) \otimes \mathcal{O}_Y(\tau^{-m}p')\big).$$

Hence the theorem is proven by choosing $p' = r$. $\square$

Inspired by Theorem 5.2 we make the following definition:

Definition 5.4. Let A, A' be three dimensional Sklyanin algebras. An inclusion $A' \hookrightarrow A^{(v)}$ is called a quadratic transform if it can be written as a composition of inclusions as in Theorems 4.1 and 4.4.

Remark 5.5. By construction a quadratic transform always induces an isomorphism of function fields.

It immediately follows from the definition that if $A' \hookrightarrow A^{(v)}$ is a quadratic transform, then $v = 2^n$ for some nonnegative integer n. By construction our noncommutative versions of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in Theorems 4.1 and 4.4 are quadratic transforms. Theorem 5.2 implies the noncommutative Cremona transform as in §3 is a quadratic transform as well. Similar to the construction in Figure 1 one can introduce a (commutative) cubic Cremona transform $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by blowing up two points in each copy of $\mathbb{P}^1 \times \mathbb{P}^1$, provided that these points do not lie on one ruling; see, for example, Figure 2. As this cubic Cremona factors through $\mathbb{P}^2$ using the classical birational transformation, there is an obvious noncommutative generalization $A' \hookrightarrow A^{(2)}$ where A and A' are cubic Sklyanin $\mathbb{Z}$-algebras. By definition this noncommutative cubic Cremona transformation is a quadratic transform as well.

6. Inner morphisms

The main goal of the following sections is to prove that in a suitable sense quadratic transforms are invertible:
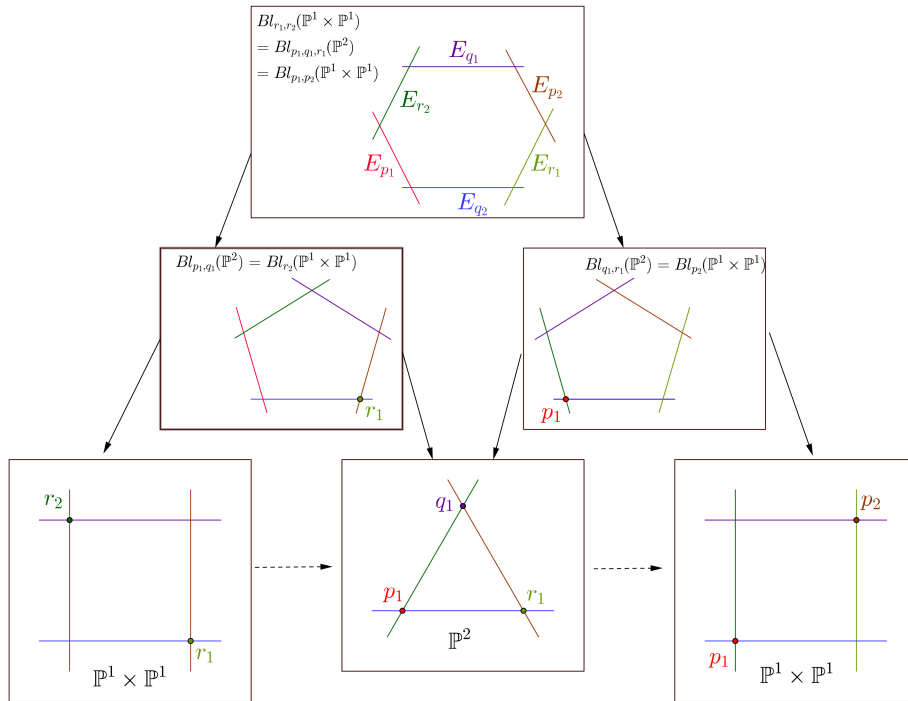
Figure 2. The commutative cubic Cremona transform.

Definition 6.1. Let A be a $\mathbb{Z}$-algebra such that $Q := \operatorname{Frac}(A)$ exists. We say that an injective morphism of $\mathbb{Z}$-algebras $\phi: A \hookrightarrow A^{(v)}$ is *inner* if there exist $z_m \in Q_{vm,m} - \{0\}$ such that for $a \in A_{m,n}$ we have $\phi(a) = z_m a z_n^{-1}$. Moreover, we require:

$$
\begin{aligned}
z_m &\in A_{vm,m}, &&\text{if } m < 0, \\
z_0 &= 1, &&(6.1) \\
z_m^{-1} &\in A_{m,vm}, &&\text{if } m > 0.
\end{aligned}
$$

The following is clear:

Proposition 6.2. *If $A \to A^{(v)}$ is inner then the induced map*

$$ Q_{0,0} \to (Q^{(v)})_{0,0} = Q_{0,0} $$

is the identity.

Definition 6.3. Let A and A' be $\mathbb{Z}$-algebras such that $\operatorname{Frac}(A)$ and $\operatorname{Frac}(A')$ exist. An inclusion $\gamma: A' \hookrightarrow A^{(w)}$ is said to be *invertible* if there exists an inclusion $\delta: A \hookrightarrow A'^{(v)}$ such that $\delta \circ \gamma: A' \hookrightarrow A'^{(vw)}$ and $\gamma \circ \delta: A \hookrightarrow A^{(vw)}$ are inner.

We can now state the main result of this paper:

Theorem 6.4. *Assume that $\gamma \colon A' \hookrightarrow A^{(2^m)}$ is a quadratic transform between (three dimensional) Sklyanin $\mathbb{Z}$-algebras. Write $\gamma = \gamma_1 \circ \cdots \circ \gamma_t$ with the γ_j as in Theorems 4.1 and 4.4. Moreover, we assume that for each factor $\gamma_i \colon A_i \hookrightarrow A_{i+1}^{(2)}$ (with A_{i+1} necessarily cubic) the points p_i, q_i used in the construction of γ_i lie in different τ-orbits. Then γ is invertible and the "inverse" δ can be chosen as a quadratic transform $\delta \colon A \to A'^{(2^n)}$ with $|n - m| \leq 1$.*

We will call δ as in the previous theorem an inverse quadratic transform to γ.

The following reduces the amount of work for proving Theorem 6.4 dramatically.

Lemma 6.5. *Let $\gamma_1 \colon A' \hookrightarrow A^{(w_1)}$ and $\gamma_2 \colon A'' \hookrightarrow A'^{(w_2)}$ be invertible as in Definition 6.3. Then $\gamma_1 \circ \gamma_2$ is invertible as well.*

Proof. By assumption there exist inclusion $\delta_1 \colon A \hookrightarrow A'^{(v_1)}$ and $\delta_2 \colon A' \hookrightarrow A''^{(v_2)}$ such that $\delta_1 \circ \gamma_1$, $\delta_2 \circ \gamma_2$, $\gamma_1 \circ \delta_1$ and $\gamma_2 \circ \delta_2$ are inner. We now claim $(\delta_2 \circ \delta_1) \circ (\gamma_1 \circ \gamma_2)$ and $(\gamma_1 \circ \gamma_2) \circ (\delta_2 \circ \delta_1)$ are inner as well. As both proofs are analogous, we only prove the latter. Let $z_{1,m} \in \operatorname{Frac} A_{w_1 v_1 m, m} - \{0\}$ and $z_{2,m} \in \operatorname{Frac} A'_{w_2 v_2 m, m} - \{0\}$ be as in Definition 6.1. Then for each $a \in A_{m,n}$:

$$
\begin{aligned}
\big((\gamma_1 \circ \gamma_2) \circ (\delta_2 \circ \delta_1)\big)(a) &= \gamma_1\big((\gamma_2 \circ \delta_2)(\delta_1(a))\big) \\
&= \gamma_1\big(z_{2,v_1 m} \delta_1(a) z_{2,v_1 n}^{-1}\big) \\
&= \gamma_1(z_{2,v_1 m})(\gamma_1 \circ \delta_1)(a)\gamma_1(z_{2,v_1 n})^{-1} \\
&= \gamma_1(z_{2,v_1 m})(z_{1,m} a z_{1,n}^{-1})\gamma_1(z_{2,v_1 n})^{-1} \\
&= \big(\gamma_1(z_{2,v_1 m})z_{1,m}\big)a\big(\gamma_1(z_{2,v_1 n})z_{1,n}\big)^{-1}.
\end{aligned}
$$

Moreover, obviously

$$
\begin{aligned}
\gamma_1(z_{2,v_1 m})z_{1,m} &\in \operatorname{Frac}(A)_{w_1 v_1 w_2 v_2 m, m} \setminus \{0\}, \\
\gamma_1(z_{2,v_1 m})z_{1,m} &\in A, && \text{if } m < 0, \\
\gamma_1(z_{2,0})z_{1,0} &= 1, \\
\big(\gamma_1(z_{2,v_1 m})z_{1,m}\big)^{-1} &\in A, && \text{if } m > 0. \qquad \square
\end{aligned}
$$

The proof of Theorem 6.4 then follows from the following 2 theorems which are proven in the next section:

Theorem 6.6. *Let $\gamma \colon A' \hookrightarrow A$ be a quadratic transform as in Theorem 4.4. Then γ is invertible and the inverse can be chosen as $\delta \colon A \hookrightarrow A'^{(2)}$ as in Theorem 4.1.*

Theorem 6.7. *Let $\gamma \colon A' \hookrightarrow A^{(2)}$ be a quadratic transform as in Theorem 4.1 and assume that the points p, q used in the construction of γ lie in different τ-orbits. Then γ is invertible and the inverse can be chosen as $\delta \colon A \hookrightarrow A'$ as in Theorem 4.4.*

Remark 6.8. Our approach to proving Theorems 6.6 and 6.7 is as follows: we first construct an inclusion of $\mathbb{Z}$-algebras $\delta: A \hookrightarrow A'^{(v)}$ such that the composition $\gamma \circ \delta: A \to A^{(wv)}$ is inner. Afterwards we will show that δ is in fact a quadratic transform and that $\delta \circ \gamma$ is inner as well.

7. Inverting quadratic transforms between quadratic Sklyanin algebras and cubic Sklyanin $\mathbb{Z}$-algebras

In this section we finish the proof of Theorem 6.4 by proving Theorem 6.6 and Theorem 6.7. The proofs of these theorems are intertwined:

In §7.1 we prove that if $\gamma: A' \hookrightarrow A$ is as in Theorem 4.4, then there is a quadratic transform $\delta: A \hookrightarrow A'^{(2)}$ such that $\gamma \circ \delta$ is inner. In §7.2 we prove that if $\gamma': A' \hookrightarrow A^{(2)}$ is as in Theorem 4.1, then there is a quadratic transform $\delta': A \hookrightarrow A'$ such that $\gamma' \circ \delta'$ is inner. Finally in §7.3 we prove that these constructions are each others inverses. i.e. if we were to construct δ out of γ as in §7.1, set $\gamma' = \delta$ and compute δ' as in §7.2, then $\delta' = \gamma$. This allows us to conclude that not only $\gamma \circ \delta$, but also $\delta \circ \gamma$ is inner (as it is equal to $\gamma' \circ \delta'$). The analogous results are true if we were to start from γ'.

7.1. The $\mathbb{Z}^2$-algebra associated to a noncommutative $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Throughout this subsection $\gamma: A' \hookrightarrow A$ will be a quadratic transform between a quadratic Sklyanin algebra $A = A(Y, \mathcal{L}, \psi)$ and a cubic Sklyanin $\mathbb{Z}$-algebra A'. Recall from §4.2 that the construction of γ is based on the choice of two points $p, q \in Y$. We will use the notation from this section, in particular d_i is defined as in (4.7). Moreover we define $\tau = \psi^3$, $\mathcal{L}_i = \psi^{*i}\mathcal{L}$ and assume p and q lie different τ-orbits.

We "glue" the algebras A and A' into a single $\mathbb{Z}^2$-algebra:

$$\widetilde{A}_{(i,j),(m,n)} := \begin{cases} \operatorname{Hom}_X\left(\mathcal{O}_X(-m-n), \mathcal{O}_X(-i-j) \otimes_{o_X} m_{d_j} \ldots m_{d_{n-1}}\right), \\ \hspace{6cm} \text{if } n > j, \\ \operatorname{Hom}_X\left(\mathcal{O}_X(-m-n), \mathcal{O}_X(-i-j)\right), \hspace{0.8cm} \text{if } n \leq j, \end{cases}$$

with X and $\mathcal{O}_X(i)$ as in Lemma 4.5.

Note that

$$\widetilde{A}_{(i,0),(m,0)} = A_{i,m}$$

and

$$\widetilde{A}_{(0,j),(0,n)} = \begin{cases} \operatorname{Hom}_X\left(\mathcal{O}_X(-n), \mathcal{O}_X(-j)m_{d_j} \ldots m_{d_{n-1}}\right), & \text{if } n \geq j, \\ 0, & \text{if } n < j, \end{cases}$$

such that

$$\widetilde{A}_{(0,j),(0,n)} = A'_{j,n}$$

In other words $\widetilde{A}$ "contains" both A and A'.

Remark 7.1. By construction $\widetilde{A}_{(i,j),(m,n)} \subset A_{i+j,m+n}$ and as such $\widetilde{A}$ contains no nontrivial zero divisors.

We now give some results on the dimensions of certain $\widetilde{A}_{(i,j),(m,n)}$. For this let $h(n)$ be the Hilbert function of A and $h'(n)$ be the Hilbert function of A'. i.e.

$$\forall i \in \mathbb{Z} : \dim_k(A_{i,i+n}) = h(n) \text{ and } \dim_k(A'_{i,i+n}) = h'(n). \tag{7.1}$$

The following easy properties of $\widetilde{A}$ are immediate from the definition:

Proposition 7.2. *Let $\widetilde{A}$ be as above then*

(1) $\dim_k \widetilde{A}_{(i,j),(i,j+b)} = h'(b)$ *holds for all $b, i, j \in \mathbb{Z}$ with $b \geq 0$;*

(2) $\dim_k \widetilde{A}_{(i,j),(i+a,j+b)} = h(a + b)$ *holds for all $a, b, i, j \in \mathbb{Z}$ with $a \geq 0, b \leq 0$.*

More interesting is the following:

Lemma 7.3. *Let $a \leq 0$, then:*

$$\dim_k \widetilde{A}_{(i,j),(i+a,j+b)} = \dim_k \widetilde{A}_{(i,j),(i,j+b+2\cdot a)} = h'(b + 2 \cdot a). \tag{7.2}$$
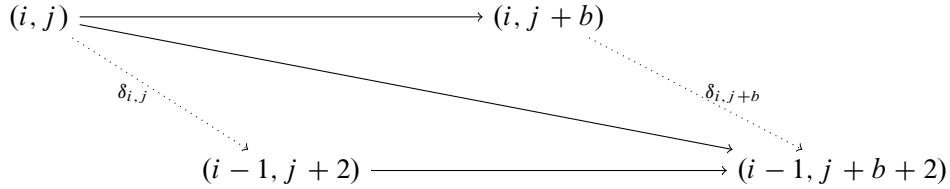
Proof. The case $a = -1$ can be done analogously to §4.2 Steps (5) and (6). For $a \leq -2$ we can no longer use Lemma 4.10 and the proof is based on the existence of an "I-basis" (see [16] for the definition and construction of an I-basis). As we will only use the case $a = -1$, we refer the interested reader to Appendix B for the details of the proof for $a \leq -2$. $\square$

As a result of Lemma 7.3 both $\widetilde{A}_{(i,j),(i-1,j+2)}$ and $\widetilde{A}_{(i,j),(i+1,j-1)}$ are one dimensional. Let $\delta_{i,j}$ and $\gamma_{i,j}$ be nonzero elements in these spaces. We can then visualise $\widetilde{A}$ on a 2-dimensional square grid.
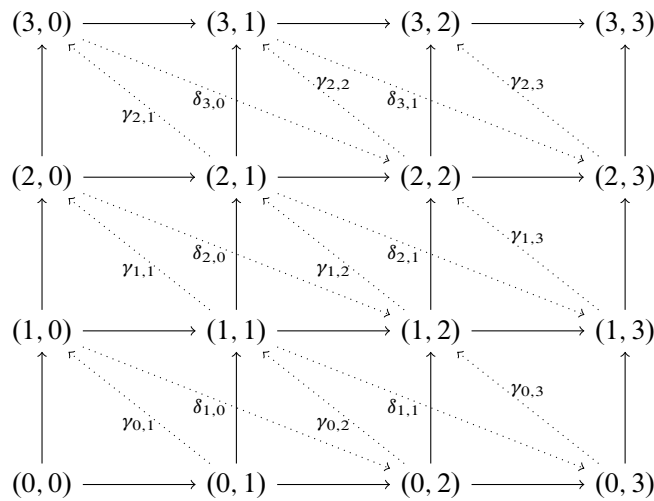
The vertical arrows represent three dimensional vector spaces, whereas the horizontal arrows represent two dimensional vector spaces and the dotted arrows represent one dimensional vector spaces (labeled by $\gamma_{i,j}$ and $\delta_{i,j}$).

Now consider the following diagram

$$
\begin{array}{ccc}
(i,j) & \longrightarrow & (i,j+b) \\[2em]
(i-1,j+2) & \longrightarrow & (i-1,j+b+2)
\end{array}
$$

with $\delta_{i,j}$ on the left dotted arrow and $\delta_{i,j+b}$ on the right dotted arrow.

From Lemma 7.3 we conclude that the vector spaces on the solid arrows all have the same dimension. Hence since A is a domain we have an isomorphism of vector spaces:

$$\delta_{i,j}^{-1} \cdot \delta_{i,j+b} \colon \widetilde{A}_{(i,j),(i,j+b)} \to \widetilde{A}_{(i-1,j+2),(i-1,j+b+2)}. \tag{7.3}$$

Whenever $2i + j = 2m + n$ and $i \geq m$ we write:

$$\delta_{(i,j),(m,n)} = \delta_{i,j}\delta_{i-1,j+2}\ldots\delta_{m+1,n-2} \in \widetilde{A}_{(i,j),(m,n)}. \tag{7.4}$$

When $i < m$ we define:

$$\delta_{(i,j),(m,n)} := \delta_{(m,n),(i,j)}^{-1} \in \operatorname{Frac}(A)_{i+2j,m+2n} \tag{7.5}$$

In particular, we always have

$$\delta_{(i,j),(k,l)}\delta_{(k,l),(m,n)} = \delta_{(i,j),(m,n)}.$$

From (7.3) we obtain an isomorphism

$$\delta_{(0,2i+j),(i,j)} \cdot \delta_{(i,j+b),(0,2i+j+b)} \colon \widetilde{A}_{(i,j),(i,j+b)} \to \widetilde{A}_{(0,2i+j),(0,2i+j+b)}.$$

Now note that there is always an inclusion

$$\cdot\, \delta_{(m,n),(i,n+2m-2i)} \colon \widetilde{A}_{(i,j),(m,n)} \to \widetilde{A}_{(i,j),(i,n+2m-2i)}. \tag{7.6}$$

If $m \geq i$ this follows from the fact that A is a domain. Lemma 7.3 tells us the map is also well defined if $m < i$, in which case case (7.6) even is an isomorphism.

Summarizing we obtain an inclusion

$$\delta_{(0,2i+j),(i,j)} \cdot \delta_{(m,n),(0,2m+n)} \colon \widetilde{A}_{(i,j),(m,n)} \to \widetilde{A}_{(0,2i+j),(0,2m+n)}.$$

And hence an inclusion

$$\delta_{(0,2i),(i,0)} \cdot \delta_{(m,0),(0,2m)} \colon A_{i,m} = \widetilde{A}_{(i,0),(m,0)} \to \widetilde{A}_{(0,2i),(0,2m)} = A'_{2i,2m}.$$

One easily checks that these inclusions are compatible with multiplication on A and A' such that we get an inclusion of algebras

$$\delta\colon A \hookrightarrow A'^{(2)}.$$

Our goal is to show that δ is in fact a quadratic transform as in Definition 5.4. Moreover, we want $\gamma \circ \delta$ to be inner (as stated in the introduction of this section, the proof to show that $\delta \circ \gamma$ is inner is postponed to §7.3). We first prove the latter:

Let $\gamma_{(i,j),(m,n)}$ be defined like $\delta_{(i,j),(m,n)}$ but using $\gamma_{i,j}$ instead of $\delta_{i,j}$ (and $i + j = m + n$ in stead of $2i + j = 2m + n$; recall $\gamma_{i,j}$ lies in the 1-dimensional space

$$\widetilde{A}_{(i,j),(i+1,j-1)} = \operatorname{Hom}\big(\mathscr{O}_X(-i-j), \mathscr{O}_X(-i-j)\big).$$

One then sees that the transform $\gamma\colon A' \to A$ we started with is given by

$$\gamma_{(j,0),(0,j)} \cdot \gamma_{(0,n),(n,0)}\colon A'_{j,n} = \widetilde{A}_{(0,j),(0,n)} \to \widetilde{A}_{(j,0),(n,0)} = A_{j,n}.$$

One way to see this is to note that the above map factors as

$$\gamma_{(j,0),(0,j)} \cdot \gamma_{(0,n),(n,0)} = \big(\gamma_{(j,0),(0,j)} \cdot \gamma_{(n-j,j),(n,0)}\big) \circ \big(\cdot \gamma_{(0,n),(n-j,j)}\big),$$

where

$$\cdot \gamma_{(0,n),(n-j,j)}\colon \operatorname{Hom}\big(\mathscr{O}_X(-n), \mathscr{O}_X(-j) \otimes_{o_X} m_{-j} \ldots m_{-n-1}\big)$$
$$\hookrightarrow \operatorname{Hom}\big(\mathscr{O}_X(-n), \mathscr{O}_X(-j)\big)$$

is the actual inclusion $A'_{j,n} \hookrightarrow A_{j,n}$ and

$$\gamma_{(j,0),(0,j)} \cdot \gamma_{(n-j,j),(n,0)}\colon \operatorname{Hom}\big(\mathscr{O}_X(-n), \mathscr{O}_X(-j)\big) \to \operatorname{Hom}\big(\mathscr{O}_X(-n), \mathscr{O}_X(-j)\big)$$

are identity morphisms (potentially up to scalars, which can simultaneously be set to 1).

In particular, $\gamma \circ \delta$ is given by

$$\gamma_{(2i,0),(0,2i)}\delta_{(0,2i),(i,0)} \cdot \delta_{(m,0),(0,2m)}\gamma_{(0,2m),(2m,0)}\colon A_{i,m} \to A_{2i,2m}, \qquad (7.7)$$

which is inner with

$$z_i = \gamma_{(2i,0),(0,2i)}\delta_{(0,2i),(i,0)} \in \operatorname{Frac}(A)_{2i,i}.$$

One easily checks that the elements z_i indeed satisfy the conditions in (6.1).

We now prove that δ is a quadratic transform. For this we need to show the existence of a point $p' \in Y$ such that

$$\delta(A_{i,i+1}) = \Gamma\big(Y, \mathscr{G}_{2i}\mathscr{G}_{2i+1}(-\tau^{-i}p')\big)$$
$$= \Gamma\big(Y, \mathscr{L}_{2i}\mathscr{L}_{2i+1}(-d_{2i} - d_{2i+1} - \tau^{-i}p')\big). \qquad (7.8)$$

We first define $\widetilde{B}$ like $\widetilde{A}$ but starting from B instead of from A. We find (using (4.1) and (4.2))

$$\widetilde{B}_{(i,j),(m,n)} := \begin{cases} \Gamma\big(Y, \mathcal{L}_{i+j}\mathcal{L}_{i+j+1}\ldots\mathcal{L}_{m+n-1}(-d_j - \cdots - d_{n-1})\big), & \text{if } n > j, \\ \Gamma\big(Y, \mathcal{L}_{i+j}\mathcal{L}_{i+j+1}\ldots\mathcal{L}_{m+n-1}\big), & \text{if } n \leq j. \end{cases}$$

Similar to [11, Lemma 6.7] one can show:

Lemma 7.4. *The canonical map*

$$\widetilde{A}_{(i,j),(m,n)} \to \widetilde{B}_{(i,j),(m,n)}$$

is an epimorphism in the first quadrant (i.e. $m \geq i$, $n \geq j$).

Recall that by the definition of δ we have for each $x \in A_{i,i+1}$

$$\delta_{i,0}\delta_{i-1,2}\ldots\delta_{1,2i-2}\delta(x) = x\delta_{i+1,0}\delta_{i,2}\ldots\delta_{1,2i} \tag{7.9}$$

when $i \geq 0$, and

$$\delta(x)\delta_{0,2i+2}\delta_{-1,2i+4}\ldots\delta_{i+2,-2} = \delta_{0,2i}\delta_{-1,2i+2}\ldots\delta_{i+1,-2}x \tag{7.10}$$

when $i < 0$. Hence in order to prove the existence of p' in (7.8) we have to understand (the product of) the image(s) $\overline{\delta_{i,j}}$ of $\delta_{i,j}$ in

$$\widetilde{B}_{(i,j),(i-1,j+2)} = \Gamma\big(Y, \mathcal{L}_{i+j}(-d_j - d_{j+1})\big). \tag{7.11}$$

As $\mathcal{L}_{i+j}(-d_j - d_{j+1})$ has degree 1 on Y we can choose a point $p'_{i,j}$ defined by

$$d_j + d_{j+1} + p'_{i,j} \sim [\mathcal{L}_{i+j}], \tag{7.12}$$

such that

$$\begin{aligned} \widetilde{B}_{(i,j),(i-1,j+2)} &= \Gamma\big(Y, \mathcal{L}_{i+j}(-d_j - d_{j+1})\big) \\ &= \Gamma\big(Y, \mathcal{L}_{i+j}(-d_j - d_{j+1} - p'_{i,j})\big). \end{aligned} \tag{7.13}$$

Lemma 7.5. *Define p' by the following identitiy*

$$p + q + \tau p' \sim [\mathcal{L}_0].$$

Then $p'_{i,j} = \tau^{-i+1}p'$.

Proof. As ψ is a translation such that $\mathcal{L}_{i+1} \cong \psi^*\mathcal{L}_i$, there is an invertible sheaf $\mathcal{N}$ of degree zero (see for example [18, Theorem 4.2.3]) such that

$$[\mathcal{L}_n] = [\mathcal{L}_0] + 3n[\mathcal{N}]$$

and

$$\tau^{-i}p \sim p + 3i[\mathcal{N}].$$

As $p'_{i,j}$ is uniquely defined by (7.12) this proves the lemma. $\qquad\square$

In particular, if p' is as in the above lemma, then (7.13) gives rise to

$$
\begin{aligned}
\widetilde{B}_{(i,j),(i-1,j+2)} &= \Gamma\big(Y, \mathcal{L}_{i+j}(-d_j - d_{j+1})\big) \\
&= \Gamma\big(Y, \mathcal{L}_{i+j}(-d_j - d_{j+1} - \tau^{-i+1}p')\big).
\end{aligned}
\tag{7.14}
$$

In particular, $\overline{\delta_{i,j}}$ is a non-zero section of $\mathcal{L}_{i+j}(-d_j - d_{j+1} - \tau^{-i+1}p')$. As the latter has degree zero on Y $\overline{\delta_{i,j}}$ is everywhere non-zero on Y.

In particular, going back to (7.9) (and hence assuming $i \geq 0$, the case $i < 0$ being completely similar) we see that

$$
\overline{\delta}_{i,0}\overline{\delta}_{i-1,2}\ldots\overline{\delta}_{1,2i-2}
$$

is an everywhere non-zero section of

$$
\mathcal{L}_i \ldots \mathcal{L}_{2i-1}\big(-d_0 - d_1 - \tau^{-i+1}p' - d_2 - d_3 - \tau^{-i+2}p' - \cdots - d_{2i-2} - d_{2i-1} - p'\big).
$$

Likewise

$$
\overline{x}\,\overline{\delta}_{i+1,0}\overline{\delta}_{i,2}\ldots\overline{\delta}_{1,2i}
$$

is a section of

$$
\mathcal{L}_i \ldots \mathcal{L}_{2i+1}\big(-d_0 - d_1 - \tau^{-i}p' - d_2 - d_3 - \tau^{-i+1}p' - \ldots - d_{2i} - d_{2i+1} - p'\big)
$$

so that $\overline{\delta(x)}$ is a section of $\mathcal{L}_{2i}\mathcal{L}_{2i+1}(-d_{2i} - d_{2i+1} - \tau^{-i}p')$. This is precisely what we had to show according to (7.8).

7.2. The $\mathbb{Z}^2$ algebra associated to a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Throughout this subsection $\gamma\colon A' \hookrightarrow A^{(2)}$ will be a quadratic transform between a cubic Sklyanin $\mathbb{Z}$-algebra $A = A(Y, (\mathcal{L}_i)_{i\in\mathbb{Z}})$ and a quadratic Sklyanin algebra A'. Recall from §4.1 that the construction of γ is based on the choice of a points $p \in Y$. We will use the notation from this section, in particular $\tau = \alpha^2$.

We define the $\mathbb{Z}^2$-algebra $\widetilde{A}$ as follows:

$$
\widetilde{A}_{(i,j),(m,n)} := \begin{cases} \mathrm{Hom}_X\big(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)m_{\tau^{-j}p}\ldots m_{\tau^{-n+1}p}\big), \\ \hspace{8cm} \text{if } n > j, \\ \mathrm{Hom}_X\big(\mathcal{O}_X(-m-2n), \mathcal{O}_X(-i-2j)\big), \hspace{1.5cm} \text{if } n \leq j. \end{cases}
$$

As in the previous section the following easy properties of $\widetilde{A}$ are immediate from the definition:

Proposition 7.6. *Let $\widetilde{A}$ be as above then:*

(1) $\widetilde{A}_{(i,0),(m,0)} = A_{i,m}$;

(2) $\widetilde{A}_{(0,j),(0,n)} = A'_{j,n}$;

(3) $\widetilde{A}$ *contains no nontrivial zero divisors;*

(4) $\dim_k \widetilde{A}_{(i,j),(i,j+b)} = h'(b)$ *holds for all* $b, i, j \in \mathbb{Z}$ *with* $b \geq 0$ *and* h' *the Hilbert series of* A';

(5) $\dim_k \widetilde{A}_{(i,j),(i+a,j+b)} = h(a+2b)$ *holds for all* $a, b, i, j \in \mathbb{Z}$ *with* $a \geq 0, b \leq 0$ *and* h *the Hilbert series of* A.

Proof. (1), (2), and (5) are immediate from the construction.

For (3) we used the fact that A is a $\mathbb{Z}$-domain as in Theorem 2.8.

For (4) we use can use computations as in §4.1 because

$$
\begin{aligned}
\widetilde{A}&_{(i,j),(i,j+b)} \\
&= \mathrm{Hom}\left(\mathcal{O}_X(-i-j-b), \mathcal{O}_X(-i-j) \otimes m_{\tau^{-j}p} \ldots m_{\tau^{-j-b+1}p}\right) \\
&= \mathrm{Hom}\left(\mathcal{O}_X(-i-j-b), \mathcal{O}_X(-i-j) \otimes m_{\tau^{-i-j}(\tau^i p)} \ldots m_{\tau^{-i-j-b+1}(\tau^i p)}\right).
\end{aligned}
$$

Hence after replacing p by $\tau^{-i}p$ this equals $D_{i+j,i+j+b}$ as in (3.2), which is known to have the correct Hilbert series. $\square$

We also have the following partial analogue of Lemma 7.3:

Lemma 7.7. $\dim_k \widetilde{A}_{(i,j),(i-1,j+b)} = \dim_k \widetilde{A}_{(i,j),(i,j+b-1)} = h'(b-1)$.

Proof. The computation is completely similar to the $a = -1$ case of Lemma 7.3 using Lemma 4.10 and [11, Lemma 6.5]. $\square$

Remark 7.8. Although one cannot use an I-basis in the classical sense we expect $\dim_k \widetilde{A}_{(i,j),(i+a,j+b)} = h'(b+a)$ to hold for all $a \leq 0$.

As a corollary of Lemma 7.7 and Proposition 7.6(3) we know $\widetilde{A}_{(i,j),(i-1,j+1)}$ and $\widetilde{A}_{(i,j),(i+2,j-1)}$ are one dimensional. Let $\delta_{i,j}$ and $\gamma_{i,j}$ be nonzero elements in these spaces. We can then visualise $\widetilde{A}$ on a 2-dimensional square grid.
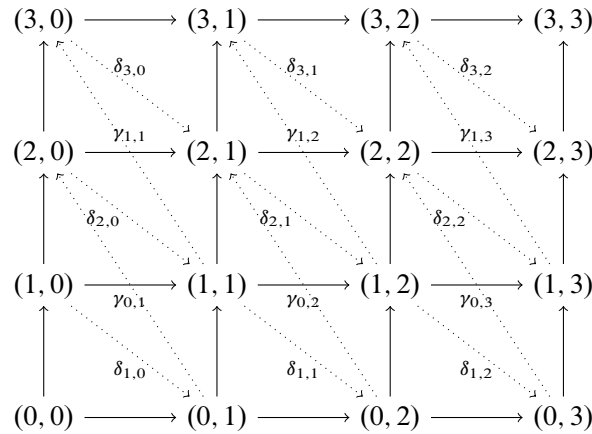
All horizontal arrows represent three dimensional vector spaces whereas the vertical arrows represent two dimensional vector spaces and dotted arrows represent one dimensional vector spaces (labeled by $\gamma_{i,j}$ and $\delta_{i,j}$).

Completely identical to previous sections there is an inclusion

$$\delta_{(0,i),(i,0)} \cdot \delta_{(m,0),(0,m)} \colon A_{i,m} = \widetilde{A}_{(i,0),(m,0)} \to \widetilde{A}_{(0,i),(0,m)} = A'_{i,m}$$

(where the elements $\delta_{(i,j),(m,n)}$ are defined as in (7.4) and (7.5). The only thing which essentially changed is that $\delta_{(i,j),(m,n)}$ is now only defined when $i + j = m + n$ in stead of $i + 2j = m + 2n$.)

The induced inclusion of algebras

$$\delta \colon A \hookrightarrow A'$$

is such that the composition $\gamma \circ \delta$ is inner with

$$z_i = \gamma_{(i,0),(0,2i)} \delta_{(0,i),(i,0)}.$$

Our next aim is to show that δ is a quadratic transform. For this we need to show the existence of two points $p', q' \in Y$ such that if we define d'_i as

$$d'_i = \begin{cases} \tau^{-l} p', & \text{if } i = 2l, \\ \tau^{-l} q', & \text{if } i = 2l + 1, \end{cases} \tag{7.15}$$

then we have

$$\delta(A_{i,i+1}) = \Gamma\big(Y, \mathcal{G}_i(-d'_i)\big) = \Gamma\big(Y, \mathcal{L}_{2i} \mathcal{L}_{2i+1}(-\tau^{-i} p - d'_i)\big). \tag{7.16}$$

We again start by defining a $\mathbb{Z}^2$-algebra $\widetilde{B}$. This time it takes the following form:

$$\widetilde{B}_{(i,j),(m,n)} := \begin{cases} \Gamma\big(Y, \mathcal{L}_{i+2j} \mathcal{L}_{i+2j+1} \ldots \mathcal{L}_{m+2n-1}(-\tau^{-j} p - \cdots - \tau^{-n+1} p)\big), \\ \hspace{8cm} \text{if } n > j. \\ \Gamma\big(Y, \mathcal{L}_{i+2j} \mathcal{L}_{i+2j+1} \ldots \mathcal{L}_{m+2n-1}\big), \hspace{1.2cm} \text{if } n \le j. \end{cases}$$

Similar to [11, Lemma 6.7] one can show:

Lemma 7.9. *The canonical map*

$$\widetilde{A}_{(i,j),(m,n)} \to \widetilde{B}_{(i,j),(m,n)}$$

is an epimorphism in the first quadrant (i.e. $m \ge i$, $n \ge j$).

Recall that for each $x \in A_{i,i+1}$, $\delta(x)$ is related to x and elements $\delta_{(i,j),(m,n)}$ via

$$\delta_{i,0}\delta_{i-1,1}\ldots\delta_{1,i-1}\delta(x) = x\delta_{i+1,0}\delta_{i,1}\ldots\delta_{1,i} \tag{7.17}$$

when $i \geq 0$, and

$$\delta(x)\delta_{0,i+1}\delta_{-1,i+2}\ldots\delta_{i+2,-1} = \delta_{0,i}\delta_{-1,i+1}\ldots\delta_{i+1,-1}x \tag{7.18}$$

when $i < 0$. Hence in order to understand $\delta(x)$ in $A'_{2i,2i+2} = B'_{2i,2i+2}$ we need to understand (the product of) the image(s) $\overline{\delta_{i,j}}$ of $\delta_{i,j}$ in $\tilde{B}$. First remark that if we choose $p', q' \in Y$ such that

$$\begin{aligned} p + \tau q' &\sim [\mathcal{L}_0], \\ p + p' &\sim [\mathcal{L}_1], \end{aligned} \tag{7.19}$$

then similar to Lemma 7.5 we then have for all i, j:

$$\tau^{-j}p + d'_{i-1} \sim [\mathcal{L}_{i+2j}]$$

with d'_i as in (7.15).

$\overline{\delta_{i,j}}$ is a nonzero element of

$$\Gamma\big(Y, \mathcal{L}_{i+2j}(-\tau^{-j}p - d'_{i-1})\big).$$

As $\mathcal{L}_{i+2j}(-\tau^{-j}p - d'_{i-1})$ has degree zero on Y, $\overline{\delta_{i,j}}$ is everywhere non-zero on Y.

In particular, going back to (7.9) (and hence assuming $i \geq 0$, the case $i < 0$ being completely similar) we see that

$$\overline{\delta_{i,0}}\,\overline{\delta_{i-1,1}}\ldots\overline{\delta_{1,i-1}}$$

is an everywhere non-zero section of

$$\mathcal{L}_i \ldots \mathcal{L}_{2i-1}\big(-p - d'_{i-1} - \tau^{-1}p - d'_{i-2} - \cdots - \tau^{-i+1}p - d'_0\big).$$

Likewise

$$\overline{x}\,\overline{\delta_{i+1,0}}\,\overline{\delta_{i,1}}\ldots\overline{\delta_{1,i}}$$

is a section of

$$\mathcal{L}_i \ldots \mathcal{L}_{2i+1}\big(-p - d'_i - \tau^{-1}p - d'_{i-1} - \cdots - \tau^{-i}p - d'_0\big),$$

so that $\overline{\delta(x)}$ is a section of $\mathcal{L}_{2i}\mathcal{L}_{2i+1}(-\tau^{-i}p - d'_i)$. This is precisely what we had to show according to (7.8).

7.3. Invertability of the quadratic transforms. We now show that if γ and δ are as in §7.1 or §7.2, then $\delta \circ \gamma$ is inner. This boils down to computations on the geometric data associated to γ and δ. First assume $A = A(Y, \mathcal{L}, \psi)$ is quadratic and $\gamma: A' \hookrightarrow A$ is constructed with respect to $p, q \in Y$. Then according to Lemma 7.5 the quadratic transform $\delta: A \hookrightarrow A'^{(2)}$ is constructed with respect to a point $p' \in Y$ satisfying

$$p + q + \tau p' \sim [\mathcal{L}_0].$$

Using the techniques in §7.2 we find a quadratic transform $\widetilde{\gamma}: A' \to A$ such that $\widetilde{\gamma} \circ \delta$ is inner. By (7.19) we know $\widetilde{\gamma}$ is constructed with respect to points $p'', q'' \in Y$ satisfying

$$p' + \tau q'' \sim [\mathcal{G}_0],$$
$$p' + p'' \sim [\mathcal{G}_1],$$

with $\mathcal{G}_i$ as in (4.10). Moreover §7.1 constructs A out of an elliptic helix $(\mathcal{L}_i'')_{i \in \mathbb{Z}}$ given by

$$\mathcal{L}_i'' = \mathcal{G}_{2i} \otimes \mathcal{G}_{2i+1} \otimes \mathcal{O}_Y(-\tau^{-i} p).$$

We need to check that $p'' = p$, $q'' = q$, and $\mathcal{L}_i'' \cong \mathcal{L}_i$. For this recall from [18, Theorem 4.2.3] that there exists a linebundle $\mathcal{N}$ of degree zero on Y such that for each linebundle $\mathcal{M}$ we have $[\psi^* \mathcal{M}] = [\mathcal{M}] + \deg(\mathcal{M}) \cdot [\mathcal{N}]$. Using this we find:

$$p' + \tau q'' \sim [\mathcal{G}_0] = [\mathcal{L}_0 \otimes \mathcal{O}_Y(-p)]$$
$$\Downarrow$$
$$p' + q'' - 3[\mathcal{N}] + p \sim [\mathcal{L}_0]$$
$$\Downarrow$$
$$p + q'' + \tau p' \sim p + q + \tau p'$$
$$\Downarrow$$
$$q'' = q.$$

Similarly, $p'' = p$. Next we show that the elliptic helix $(\mathcal{L}_i'')_{i \in \mathbb{Z}}$ coincides with $(\mathcal{L}_i)_{i \in \mathbb{Z}}$:

$$
\begin{aligned}
[\mathcal{L}_i''] &= [\mathcal{G}_{2i}] + [\mathcal{G}_{2i+1}] + \left[\mathcal{O}_Y(-\tau^{-i} p')\right] \\
&= [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + \left[\mathcal{O}_Y(-d_{2i} - d_{2i+1})\right] + \left[\mathcal{O}_Y(-\tau^{-i} p')\right] \\
&= [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + \left[\mathcal{O}_Y(-d_i - d_{i+1})\right] - 3i \cdot [\mathcal{N}] + \left[\mathcal{O}_Y(-\tau^{-i} p')\right] \\
&= \left([\mathcal{L}_{2i}] - 3i \cdot [\mathcal{N}]\right) + \left([\mathcal{L}_{2i+1}] + \left[\mathcal{O}_Y(-d_i - d_{i+1} - \tau^{-i} p')\right]\right) \\
&= [\psi_*^i \mathcal{L}_{2i}] = [\mathcal{L}_i].
\end{aligned}
$$

Next we do similar computations in case $A = A(Y, (\mathcal{L}_i)_{i \in \mathbb{Z}})$ is a quadric and $\gamma \colon A' \hookrightarrow A^{(2)}$ is constructed with respect to a point $p \in Y$ as in §7.1. Similar to the above it suffices to prove $p'' = p$ and $\mathcal{L}_i'' = \mathcal{L}_i$ where

$$p' + q' + \tau p'' \sim [\mathcal{G}_0],$$
$$\mathcal{L}_i'' = \mathcal{G}_i \otimes \mathcal{O}_Y(-d_i'),$$

where $\mathcal{G}_i$ is as in (4.4) and p', q', d_i' are as in (7.19) and (7.15). First we prove $p'' = p$, for this we take $\mathcal{N}$ a degree zero line bundle on Y such that $[\alpha^* \mathcal{M}] = [\mathcal{M}] + \deg(\mathcal{M}) \cdot [\mathcal{N}]$ with α as in Theorem 4.1

$$p' + q' + \tau p'' \sim [\mathcal{G}_0] = [\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{O}_Y(-p)]$$
$$\Downarrow$$
$$p' + q' + p'' - 2[\mathcal{N}] + p \sim [\mathcal{L}_0] + [\mathcal{L}_1]$$
$$\Downarrow$$
$$p' + \tau q' + p'' + p \sim p + \tau q' + p + p'$$
$$\Downarrow$$
$$p'' = p.$$

We now show that the elliptic helix $(\mathcal{L}_i'')_{i \in \mathbb{Z}}$ coincides with $(\mathcal{L}_i)_{i \in \mathbb{Z}}$:

$$[\mathcal{L}_i''] = [\mathcal{G}_i] + \left[\mathcal{O}_Y(-d_i)\right]$$
$$= [\mathcal{L}_{2i}] + [\mathcal{L}_{2i+1}] + \left[\mathcal{O}_Y(-d_i)\right] + \left[\mathcal{O}_Y(-\tau^{-i} p')\right]$$
$$= [\mathcal{L}_i] + [\mathcal{L}_{3i+1}] + \left[\mathcal{O}_Y(-d_i)\right] + \left[\mathcal{O}_Y(-\tau^{-i} p')\right]$$
$$= [\mathcal{L}_i].$$

Finishing the proof of Theorem 6.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A. Quadrics admit $\mathbb{Z}$-fields of fractions

In this appendix we prove the following:

Theorem (Theorem 2.8 and 2.12). *Let A be a quadric, then A is a $\mathbb{Z}$-domain and A admits a $\mathbb{Z}$-field of fractions.*

The proof of this theorem is based on several preliminary results.

Notation. Throughout this appendix A will always be a quadric. Moreover, for any A-module M we let $\mathrm{pd}(M)$ and $\mathrm{GKdim}(M)$ denote the projective and Gelfand–Kirillov dimension respectively.

A.1. Preliminary results.

A.1.1. Some lemmas.

Lemma A.1. *Let M be a finitely generated left- or right-A-module and assume* $\mathrm{pd}(M) \leq 1$, *then* $\mathrm{GKdim}(M) \geq 2$.

Proof. Upon replacing the projective modules $A(-i)$ by $e_i A$ or Ae_i one can copy the proof of [3, Proposition 2.41]. □

Lemma A.2. *Let M be a finitely generated right-A-module and let S_i denote the simple module $e_i A/e_i A_{>i}$, then*

$$pd(M) \leq 2 \implies \forall i \in \mathbb{Z} : \mathrm{Hom}_A(S_i, M) = 0.$$

Proof. Upon replacing the projective modules $A(-i)$ by $e_i A$ one can copy the proof of [3, Proposition 2.46 (i)] □

Lemma A.3. *Let $i \in \mathbb{Z}$ be any integer and M be some graded submodule of $e_i A$ then* $\mathrm{GKdim}(M) = 3 \Leftrightarrow \mathrm{GKdim}(e_i A/M) < 3$.

Proof. $\mathrm{GKdim}(e_i A/M) < 3 \Rightarrow \mathrm{GKdim}(M) = 3$ is trivial. Let us prove the other direction.

Assume by way of contradiction that $\mathrm{GKdim}(M) = \mathrm{GKdim}(e_i A/M) = 3$. As both M and $e_i A/M$ are nonzero we have $e(M) > 0$ and $e(e_i A/M) > 0$. However as they have equal GKdim, we have $e(e_i A) = e(M) + e(e_i A/M)$. A direct computation shows that $e(e_j A) = \frac{1}{2}$ holds for all j. Similar to the proof of [3, Proposition 2.21 (iii)] we then know that $e(M)$ and $e(e_i A/M)$ must be a nonnegative multiple of $\frac{1}{2}$. Contradiction! □

We now introduce a homogeneous ideal N of A in a similar fashion as was done in [3]:

(1) $e_i A$ is a Noetherian object in $\mathrm{Gr}(A)$. In particular any ascending chain of submodules of $e_i A$ must stabilize. This allows us to set N_i to be the largest submodule of $e_i A$ of GKdim ≤ 2.

(2) Define N as $\bigoplus_{i \in \mathbb{Z}} N_i$. Then N is a homogeneous two-sided ideal of A. To see why N also has the structure of a left ideal, note that if $a \in A_{i,j}$ then aN_j is a submodule of $e_i A$ of GKdim ≤ 2. This implies that $aN_j \subset N_i$ for otherwise $aN_j + N_i$ would be a strictly larger submodule than N_i but it still has GKdim ≤ 2.

Remark A.4. Recall that A, being a quadric, is 2-periodic [18, Proposition 5.6.1]. i.e. there is an isomorphism $A \cong A(2)$. This isomorphism induces an isomorphism $N \cong N(2)$.

To see this, fix any $i \in \mathbb{Z}$ and let $f_i : e_i A \to e_{i+2} A(2)$ be the induced isomorphism. Then $f_i(e_i N)$ has GKdim $= 2$ in particular, being an $A(2)$-submodule

of $e_{i+2}A(2)$ we must have $f_i(e_iN) \subset e_{i+2}N(2)$. By considering f_i^{-1} we see that this must in fact be an equality.

Lemma A.5. *Let N be as above, then $\bar{A} := A/N$ is a $\mathbb{Z}$-domain.*

Proof. Let $b \in A_{i,j} \setminus N_{i,j}$, we then need to show that the induced morphism

$$e_j\bar{A} \xrightarrow{\bar{b}\cdot} e_i\bar{A}$$

is injective. For this consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(b\cdot) & \longrightarrow & e_jA & \longrightarrow & bA & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\bar{b}\cdot) & \longrightarrow & e_j\bar{A} & \longrightarrow & \bar{b}\bar{A} & \longrightarrow & 0
\end{array}
$$

Now suppose by way of contradiction that $\ker(\bar{b}\cdot) \neq 0$. By construction $e_j\bar{A} = e_jA/N_j$ does not contain submodules of $\mathrm{GKdim} \leq 2$, hence $\mathrm{GKdim}\big(\ker(\bar{b}\cdot)\big) = 3$. This implies that $\mathrm{GKdim}(\ker(b\cdot)) = 3$ as well. By Lemma A.3 we must have $\mathrm{GKdim}(bA) < 3$, hence also $\mathrm{GKdim}\big(\bar{b}\bar{A}\big) < 3$. As $\bar{b}\bar{A} \subset e_i\bar{A}$ and $e_i\bar{A}$ does not contain submodules of $\mathrm{GKdim} \leq 2$ we must have $\bar{b}\bar{A} = 0$, contradicting the fact that $\bar{b} \neq 0$. $\qquad\square$

A.1.2. Dual modules. Next we introduce the notion of dualization of (right-) A-modules. Throughout this section we will use A^{op} to denote the opposite algebra of A. A^{op} is a $\mathbb{Z}$-algebra by setting $(A^{\mathrm{op}})_{i,j} = (A_{-j,-i})^{\mathrm{op}}$. With this $\mathbb{Z}$-algebra structure graded right-A^{op}-modules can be identified with graded left-A-modules; for example e_iA^{op} naturally corresponds to Ae_{-i}. It hence makes sense to let $\mathrm{Gr}(A^{\mathrm{op}})$ denote the categories of graded left A-modules.

Let M be a graded right-A-module, then $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_iA)$ naturally has the structure of a graded left-A-module via:

$$A_{ij} \otimes \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_jA) \to \mathrm{Hom}_{\mathrm{Gr}(A)}(M, e_iA) : x \otimes f \mapsto x \cdot f,$$

where

$$(x \cdot f)(m) := x \cdot f(m).$$

We denote this graded left-A-module by M^* or $\mathrm{Hom}_{\mathrm{Gr}(A)}(M, A)$ and it is called the dual of M. One easily checks that this induces a left-exact functor

$$\mathrm{Hom}_{\mathrm{Gr}(A)}(-, A) = (-)^* : \mathrm{Gr}(A) \to \mathrm{Gr}(A^{\mathrm{op}}).$$

Note that as $\mathrm{Hom}_{\mathrm{Gr}(A)}(e_i A, e_j A) \cong A_{ji}$ we naturally have $\mathrm{Hom}_{\mathrm{Gr}(A)}(e_i A, A) = (e_i A)^* = A e_i$. This allows us to define the right derived functors

$$\mathrm{RHom}_{\mathrm{Gr}(A)}(-, A) \colon D_f^b\big(\mathrm{Gr}(A)\big) \to D_f^b\big(\mathrm{Gr}(A^{\mathrm{op}})\big).$$

If $C^\bullet$ is some object in $D_f^b(\mathrm{Gr}(A))$ which is represented by a bounded exact complex of finitely generated projectives, say

$$0 \to \bigoplus_{i \in \mathbb{Z}} e_i A^{\oplus l_{i,n}} \xrightarrow{\cdot M_n} \bigoplus_{i \in \mathbb{Z}} e_i A^{\oplus l_{i,n-1}} \to \cdots \to \bigoplus_{i \in \mathbb{Z}} e_i A^{\oplus l_{i,m}} \to 0,$$

where M_n is some matrix whose entries are homogeneous elements in A, then $\mathrm{RHom}_{\mathrm{Gr}(A)}(C^\bullet, A)$ is represented by the complex

$$0 \leftarrow \bigoplus_{i \in \mathbb{Z}} A e_i^{\oplus l_{i,n}} \xleftarrow{M_n \cdot} \bigoplus_{i \in \mathbb{Z}} A e_i^{\oplus l_{i,n-1}} \leftarrow \cdots \leftarrow \bigoplus_{i \in \mathbb{Z}} A e_i^{\oplus l_{i,m}} \leftarrow 0$$

(where each term in position j in the original complex gives rise to a term in position $-j$ in the new complex). Similar to the graded case we use the shorthand notation $(C^\bullet)^D := \mathrm{RHom}_{\mathrm{Gr}(A)}(C^\bullet, A)$. If M is some graded right-A-module, then we denote

$$\mathrm{Ext}_{\mathrm{Gr}(A)}^i(M, A) := R^i \mathrm{Hom}_{\mathrm{Gr}(A)}(M, A) = h^i(M^D).$$

Remark A.6. If we introduce

$$\mathrm{RHom}_{\mathrm{Gr}(A^{\mathrm{op}})}(-, A) \colon D_f^b\big(\mathrm{Gr}(A^{\mathrm{op}})\big) \to D_f^b\big(\mathrm{Gr}(A)\big)$$

in an analogous way, then $((-)^D)^D \cong \mathrm{Id}$ holds, giving rise to a biduality spectral sequence as in the graded case.

For a bounded complex $C^\bullet$ of (finitely generated, graded right-) A-modules (or A^{op}-modules) we define the Hilbert series of $C^\bullet$ as

$$h_{C^\bullet}(t) = \sum_{i \in \mathbb{Z}} h_i(C^\bullet) t^i \quad \text{with } h_i(C^\bullet) = \sum_{j \in \mathbb{Z}} (-1)^j \dim_k \big((C^j)_i\big),$$

and we denote $e(C^\bullet)$ to be the leading coefficient of the series expension of $h_{C^\bullet}(t)$ in terms of $(1-t)^{-1}$ and $\mathrm{GKdim}(C^\bullet)$ as the highest power of $(1-t)^{-1}$ in this expansion, i.e. the order of pole of $h_{C^\bullet}(t)$.

We then have the following:

Lemma A.7. *Let $C^\bullet \in D_f^b(\mathrm{Gr}(A))$, then we have the following equality of rational functions:*

$$h_{(C^\bullet)^D}(t) = -t^4 \cdot h_{C^\bullet}(t^{-1}).$$

Proof. By linearity of the definition of $h_{C^\bullet}$, it suffices to prove the equality in case $C^\bullet$ is given by some projective $e_i A$ concentrated in position j. In this case $(C^\bullet)^D$ is given by Ae_i (hence $e_{-i}A^{\mathrm{op}}$) concentrated in position $(-j)$ such that

$$
\begin{aligned}
h_{(C^\bullet)^D}(t) &= (-1)^{-j} \cdot \frac{t^{-i}}{(1-t^2)(1-t)^2} \\
&= (-1)^j \cdot \frac{(t^{-1})^i}{-t^4\big(1-(t^{-1})^2\big)(1-t^{-1})^2} \\
&= -t^{-4} \cdot \frac{(t^{-1})^i}{\big(1-(t^{-1})^2\big)(1-t^{-1})^2} \\
&= -t^{-4} \cdot h_{C^\bullet}(t^{-1}). \qquad \qquad \square
\end{aligned}
$$

Corollary A.8. *Let $C^\bullet$ be a bounded complex of (finitely generated) right A-modules. Let $m = \mathrm{GKdim}(C^\bullet)$, then:*

(1) $\mathrm{GKdim}((C^\bullet)^D) = m$;

(2) $e((C^\bullet)^D) = (-1)^{m+1}e(C^\bullet)$.

Proof. Suppose

$$
h_{C^\bullet}(t) = \frac{\sum_{i=0}^{\infty} \alpha_i (1-t)^i}{(1-t)^m}
$$

with $\alpha_0 = e(C^\bullet) \neq 0$. Then we need to show that

$$
h_{(C^\bullet)^D}(t) = \frac{\sum_{i=0}^{\infty} \widetilde{\alpha}_i (1-t)^i}{(1-t)^m}
$$

with $\widetilde{\alpha}_0 = (-1)^{m+1}\alpha_0$.

First note that for each $n \in \mathbb{Z}$ we can write:

$$
t^n = 1 + \sum_{j=1}^{\infty} \beta_{n,j}(1-t)^j
$$

with $\beta_{n,j} \in k^*$. Then by Lemma A.7 we have

$$
\begin{aligned}
h_{(C^\bullet)^D}(t) &= \frac{\sum_{i=0}^{\infty} \alpha_i (-t^{-4})(1-t^{-1})^i}{(1-t^{-1})^m} \\
&= \frac{\sum_{i=0}^{\infty} -\alpha_i (-1)^i t^{-i-4}(1-t)^i}{(-1)^m t^{-m}(1-t)^m} \\
&= \frac{\sum_{i=0}^{\infty} \alpha_i (-1)^{m+i+1} t^{m-i-4}(1-t)^i}{(1-t)^m} \\
&= \frac{\sum_{i=0}^{\infty} \widetilde{\alpha}_i (1-t)^i}{(1-t)^m},
\end{aligned}
$$

where

$$\widetilde{\alpha_i} = (-1)^{m+i+1}\left(\alpha_i + \sum_{j=1}^{i}(-1)^j \beta_{m-i+j-4,j}\alpha_{i-j}\right). \qquad \square$$

Lemma A.9. *Let M be a finitely generated right-A-module, then $\mathrm{Ext}^3_{\mathrm{Gr}(A)}(M, A)$ is a finite dimensional k-vectorspace.*

Proof. This is an immediate generalization of [3, Proposition 2.46(ii)]. $\qquad \square$

We now prove some more results on the homogeneous ideal N as above:

Lemma A.10. *For each $i \in \mathbb{Z}$ we have $\mathrm{GKdim}(Ne_i) \leq 2$.*

Proof. By construction we know that for each i we have $\mathrm{GKdim}(e_i N) \leq 2$. In particular there is for each i a degree 2 polynomial P_i such that $\dim_k(N_{i,i+l}) \leq P_i(l)$ holds for all l sufficiently large. Now fix some $i \in \mathbb{Z}$, we must show that there is a degree 2 polynomial Q such that $\dim_k(N_{i-l,i}) \leq Q(l)$. For this recall that the 2-periodicity of A descends to N (see Remark A.4). In particular we have

$$\dim_k(N_{i-l,i}) = \begin{cases} \dim_k(N_{i,i+l}), & \text{if } l \text{ is even,} \\ \dim_k(N_{i+1,i+l+1}), & \text{if } l \text{ is odd.} \end{cases}$$

Without loss of generality we can now assume that $P_i(l) \geq P_{i+1}(l)$ holds for all l sufficiently large. We can finish the proof by setting $Q = P_i$. $\qquad \square$

Lemma A.11. *Fix some $i \in \mathbb{Z}$ and let $I \subset Ae_i$ be the left-annihilator of $e_i N$, then $\mathrm{GKdim}(I) = 3$.*

Proof. As $e_i N$ is a submodule of the noetherian right A-module $e_i A$, it is finitely generated. i.e. there are elements $x_{i,i} \in A_{i,i}, x_{i,i+1} \in A_{i,i+1}, \ldots, x_{i,n} \in A_{i,n}$ such that

$$e_i N = \sum_{j=i}^{n} x_{i,j} A = \sum_{j=i}^{n} x_{i,j} e_j A.$$

Let I_j be the left annihilator of $x_{i,j}$, i.e.

$$I_j = \{a \in Ae_i \mid ax_{i,j} = 0.$$

Then there is an exact sequence of left A-modules

$$0 \to I_j \to Ae_i \xrightarrow{\cdot x_{i,j}} Ae_j$$

such that $Ae_i/I_j \cong Ax_{i,j}$. Moreover, $I = \bigcap_j I_j$ such that

$$\mathrm{GKdim}(A/I) \leq \max_j \mathrm{GKdim}(Ax_{i,j}) \leq \max_j \mathrm{GKdim}(Ae_j) = 2.$$

The result now follows from Lemma A.3. $\qquad \square$

Lemma A.12. *For each $i \in \mathbb{Z}$ we have:*

(i) $e_i N$ *is a second syzygy;*

(ii) $pd(e_i N) \leq 1$.

Proof. (ii) obviously follows from (i), so we only need to prove $e_i N$ is a second syzygy. By Lemmas A.11 and A.10 we know there exists an element $b \in A_{ji}$ such that $bN = 0$ while $b \notin Ne_i$. Hence, $e_i N \subset \ker(b\cdot)$ while $e_i \bar{A} \xrightarrow{b\cdot} e_j \bar{A}$ is injective by Lemma A.5. This implies that we have a left exact sequence

$$0 \to e_i N \to e_i A \xrightarrow{b\cdot} e_j A$$

finishing the proof. □

A.2. Proof of Theorem 2.8. Let N be as above. By Lemma A.5 it suffices to prove that $N = 0$. Suppose by way of contradiction that this is not the case. Without loss of generality we can assume $e_0 N \neq 0$. Then, by Lemma A.12, $pd(e_0 N) \leq 1$, which by Lemma A.1 implies GKdim$(e_0 N) \geq 2$. As by construction GKdim$(e_0 N) \leq 2$, we have GKdim$(e_0 N) = 2$. Let $(e_0 N)^D = \text{RHom}(e_0 N, A)$ denote the dual complex as above, by the projective dimension of $e_0 N$, this complex only has homology at position 0 and 1. By Lemma A.12 $e_0 N$ is a second syzygy and hence we have

$$h^1\big((e_0 N)^D\big) = \text{Ext}^1(e_0 N, A) \cong \text{Ext}^3(M, A)$$

for some module M.

Lemma A.9 then implies that $h^1((e_0 N)^D)$ is finite dimensional. In particular the Gelfand–Kirillov dimension and multiplicity of $(e_0 N)^D$ are solely determined by $(e_0 N)^*$. Corollary A.8 gives

$$e\big((e_0 N)^*\big) = e\big((e_0 N)^D\big) = -e(e_0 N).$$

A contradiction!

A.3. Proof of Theorem 2.12. By Theorem 2.8 and Proposition 2.11 it suffices to prove that all $e_i A$ are uniform modules. For this fix any i and nonzero $M \subset e_i A$. Then GKdim$(M) = 3$. To see this let x be any nonzero element in $M_j \subset A_{i,j}$, then by Theorem 2.8 we have

$$3 = \text{GKdim}(e_j A) = \text{GKdim}(x e_j A) \leq \text{GKdim}(M) \leq \text{GKdim}(e_i A) = 3.$$

Now let N be any other nonzero submodule of $e_i A$. Then obviously GKdim$(N) = 3$ as well. Suppose by way of contradiction that $M \cap N = 0$, then the following composition is a monomorphism:

$$N \hookrightarrow e_i A \to e_i A / M$$

such that $\text{GKdim}(e_i A/M) = 3$. This gives a contradiction with Lemma A.3. Hence for any nonzero $M, N \subset e_A$ we must have $M \cap N \neq 0$, s that $e_i A$ is a uniform module.

B. I-bases for quadratic Sklyanin algebras and Lemma 7.3

Throughout this section we assume $A = A(Y, \mathcal{L}, \psi)$ is a quadratic Sklyanin algebra with Hilbert series h. p and q are points lying in different τ-orbits with $\tau = \psi^3$. Our goal is to prove that for $a \leq -2$:

$$\dim_k \left(\text{Hom} \left(\mathcal{O}_X(-i - a - j - b), \mathcal{O}_X(-i - j) \otimes m_{d_j} \ldots m_{d_{j+b-1}} \right) \right)$$
$$= h'(2a + b) = \begin{cases} (n+1)^2, & \text{if } 2a + b = 2n \geq 0, \\ (n+1)(n+2), & \text{if } 2a + b = 2n + 1 > 0, \\ 0, & \text{if } 2a + b < 0, \end{cases} \quad \text{(B.1)}$$

where d_i is as in (4.7). Using

$$\left(\mathcal{O}_X(n) \otimes m_p \right)(m) = \left(\mathcal{O}_X \otimes m_{\psi^{-n} p} \right)(n + m)$$

(see for example [11, §6]) and replacing p and q by $\psi^x p$, $\psi^y q$ for the appropriate values of x and y this is equivalent to proving

$$\dim_k \left(\text{Hom} \left(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{d_0} \ldots m_{d_{b-1}})(a + b) \right) \right) = h'(2a + b). \quad \text{(B.2)}$$

We will prove this using I-bases.

B.1. I-bases. In this subsection we recall the definition and construction of an I-basis for a quadratic Sklyanin algebra. For a more thorough introduction to I-bases we refer the reader to [16].

Definition B.1. Let $A = A(Y, \mathcal{L}, \psi)$ be a quadratic Sklyanin algebra and let G denote the monoid of monomials in x, y, z. Let G_n denote the subset of all degree n monomials. An I-basis for A is then given by a map $v: G \to A$ satisfying the following properties:

(i) $v(G_n)$ is a k-basis for A_n;

(ii) for any $g \in G$ there are elements $x_g, y_g, z_g \in A_1$ such that

$$v(gx) = v(g)x_g, \quad v(gy) = v(g)y_g, \quad \text{and} \quad v(gz) = v(g)z_g.$$

Remark B.2. Note that $v(x) = x_1, v(y) = y_1, v(z) = z_1$. An I basis can hence alternatively be given by a collection of $\{x_g, y_g, z_g\}_{g \in G}$ satisfying $x_g y_{xg} = y_g x_{yg}$, $x_g z_{xg} = z_g x_{zg}, y_g z_{yg} = z_g y_{zg}$.

In [16, §4] Tate and Van den Bergh give a construction for an I-basis for a Sklyanin algebra. In the case of a quadratic Sklyanin algebra this construction depends on the choice of a rational point $\overline{o} = (o_1, o_2, o_3) \in Y^3$. For each $g \in G$ one defines $\overline{og}$ by setting

$$\overline{ox} = (\psi o_1, \psi^{-2} o_2, \psi^{-2} o_3),$$
$$\overline{oy} = (\psi^{-2} o_1, \sigma o_2, \psi^{-2} o_3),$$
$$\overline{oz} = (\psi^{-2} o_1, \psi^{-2} o_2, \psi o_3),$$

such that if $g = x^\alpha y^\beta z^\lambda$ then $\overline{og} = (\psi^{\alpha - 2\beta - 2\lambda} o_1, \psi^{\beta - 2\alpha - 2\lambda} o_2, \psi^{\lambda - 2\alpha - 2\beta} o_3)$.

We then define $x_g, y_g, z_g \in A_1 = \Gamma(Y, \mathcal{L})$ (up to a scalar multiple) by setting

$$x_g\big((\overline{og})_1\big) \neq 0, \quad x_g\big((\overline{og})_2\big) = 0, \quad x_g\big((\overline{og})_3\big) = 0,$$
$$y_g\big((\overline{og})_1\big) = 0, \quad y_g\big((\overline{og})_2\big) \neq 0, \quad y_g\big((\overline{og})_3\big) = 0,$$
$$z_g\big((\overline{og})_1\big) = 0, \quad z_g\big((\overline{og})_2\big) = 0, \quad z_g\big((\overline{og})_3\big) \neq 0,$$

(the scalar multiples are then chosen such that the relations in Remark B.2 hold).

In particular,

$$v(x) = x_1 \in \Gamma\big(Y, \mathcal{L}(-o_2 - o_3)\big) = \mathrm{Hom}_X\big(\mathcal{O}_X, \mathcal{O}_X(1) \otimes m_{o_2 + o_3}\big)$$
$$= \mathrm{Hom}_X\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\sigma^{-1} o_2 + \sigma^{-1} o_3})(1)\big),$$

and analogously

$$v(y) \in \mathrm{Hom}_X\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} o_1 + \psi^{-1} o_3})(1)\big)$$
$$v(z) \in \mathrm{Hom}_X\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} o_1 + \psi^{-1} o_2})(1)\big).$$

Similar computations are possible for monomials of higher degree, for example: $v(xy) = x_1 \cdot y_x$ lies in the image of

$$\mathrm{Hom}_X\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} o_2 + \psi^{-1} o_3})(1)\big)$$
$$\otimes \mathrm{Hom}_X\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} \overline{ox}_1 + \psi^{-1} \overline{ox}_3})(1)\big) \hookrightarrow A_1 \otimes A_1 \to A_2.$$

This image is given by

$$\mathrm{Hom}\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} o_2 + \psi^{-1} o_3} m_{\psi^{-1} o_1 + \psi^{-4} o_3})(2)\big)$$
$$\cong \mathrm{Hom}\big(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{\psi^{-1} o_1 + \psi^{-1} o_2 + \psi^{-1} o_3} m_{\tau^{-1} \psi^{-1} o_3})(2)\big),$$

where we used the fact that o_1, o_2, o_3 lie in different τ-orbits. Inspired by the above results we make the following choices for o_1, o_2, o_3:

$$o_1 = \sigma p, \quad o_2 = \sigma q, \quad \text{and} \quad o_3 = \sigma r, \tag{B.3}$$

where r is some point on Y lying in a different τ-orbit than p and q. (Recall that we required p and q to lie in different τ-orbits.)

We can can inductively show the following: assume α, β, λ are nonnegative integers then:

$$v(x^\alpha y^\beta z^\lambda) \in \mathrm{Hom}\left(\mathcal{O}_X, \left(\mathcal{O}_X \otimes m_p m_{\tau-1 \, p} \ldots m_{\tau-\beta-\gamma+1 \, p}\right.\right. \tag{B.4}$$
$$m_q m_{\tau-1 \, q} \ldots m_{\tau-\alpha-\gamma+1 \, q}$$
$$\left.\left. m_r m_{\tau-1 \, r} \ldots m_{\tau-\alpha-\beta+1 \, r}\right)(\alpha + \beta + \gamma)\right).$$

B.2. Proof of Lemma 7.3. Using the above language of I-bases the following proposition reduces the proof of (B.2) to a combinatorial problem.

Proposition B.3. *Let* $\alpha, \beta, \gamma \geq 0$ *then*

$$v(x^\alpha y^\beta z^\lambda) \in \mathrm{Hom}\left(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{d_0} \ldots m_{d_{h-1}})(\alpha + \beta + \gamma)\right)$$
$$\Updownarrow \tag{B.5}$$
$$\left\lceil \frac{h}{2} \right\rceil \leq \beta + \gamma \quad and \quad \left\lfloor \frac{h}{2} \right\rfloor \leq \alpha + \gamma.$$

Proof. $\Uparrow$ follows from (B.4). For $\Downarrow$ we need some more computations... $\qquad\square$

Fix $h \in \mathbb{N}$ and define the following right submodules M and M' of A_A:

$$M_n = \mathrm{Hom}_X\left(\mathcal{O}_X, (\mathcal{O}_X \otimes m_{d_0} \ldots m_{d_{h-1}})(n)\right),$$
$$M'_n = \mathrm{Span}\{v(x^\alpha y^\beta z^\lambda) \mid \alpha + \beta + \lambda = n, \ \lceil h/2 \rceil \leq \beta + \gamma \text{ and } \lfloor h/2 \rfloor \leq \alpha + \gamma\}.$$

To see that M' is a right A-module, recall that for each $g = x^\alpha y^\beta z^\lambda$, the elements x_g, y_g, z_g give a k-basis for A_1 hence the image of

$$k \cdot v(x^\alpha y^\beta z^\lambda) \otimes A_1 \to A_n \otimes A_1 \to A_{n+1}$$

lies inside $\mathrm{Span}\{v(x^{\alpha+1} y^\beta z^\lambda), v(x^\alpha y^{\beta+1} z^\lambda), v(x^\alpha y^\beta z^{\lambda+1})\}$.

We then have the following lemmas:

Lemma B.4. *Let* M *and* M' *be as above then for all n sufficiently large we have*

$$\dim_k(M_n) = \dim_k(M'_n).$$

Proof. We prove this for h even. The case h odd is completely similar. Let $h = 2a$, then we have for $n \geq 2a - 1$ we have

$$\dim_k(M_n) = \frac{(n+1)(n+2)}{2} - 2 \cdot \frac{a(a+1)}{2} = \dim_k(M'_n)$$

for M this follows from Lemma 4.10 and [17, Corollary 5.2.4]. For M' this follows from the fact that for $n \geq 2a-1$ at most one of the inequalities $\alpha + \lambda \geq a, \beta + \lambda \geq a\}$ can fail when $\alpha + \beta + \lambda = n$. $\qquad\square$

Lemma B.5. *Let N be a right A-modules such that $M' \subset N \subset A$ and suppose there is an $n_0 \in \mathbb{N}$ such that $M'_{n_0} \subsetneq N_{n_0}$ then we have $M'_n \subsetneq N_n$ for all $n \geq n_0$.*

Proof. By induction it suffices to show $M'_{n_0+1} \subsetneq N_{n_0+1}$. For this choose some nonzero element in $N_{n_0} \setminus M'_{n_0}$. This element can be written as

$$\sum_{\substack{\alpha,\beta,\lambda \\ \alpha+\beta+\lambda=n}} t_{\alpha,\beta,\lambda} v(x^\alpha y^\beta z^\lambda).$$

Without loss of generality we can assume there is a $t_{\alpha_0,\beta_0,\lambda_0} \neq 0$ with $\beta_0 + \lambda_0 < \lceil \frac{h}{2} \rceil$. Choose such a t with α_0 maximal and let $g = x^{\alpha_0} y^{\beta_0} z^{\lambda_0}$ then

$$\left(\sum_{\substack{\alpha,\beta,\lambda \\ \alpha+\beta+\lambda=n}} t_{\alpha,\beta,\lambda} v(x^\alpha y^\beta z^\lambda) \right) x_g = t_{\alpha_0,\beta_0,\lambda_0} v(x^{\alpha_0+1} y^{\beta_0} z^{\lambda_0})$$

$$+ \sum_{\substack{\alpha',\beta',\lambda' \\ \alpha'+\beta'+\lambda'=n+1 \\ \alpha' \leq \alpha_0}} t'_{\alpha',\beta',\lambda'} v(x^{\alpha'} y^{\beta'} z^{\lambda'})$$

is a nonzero element in $N_{n_0+1} \setminus M'_{n_0+1}$. □

Continuation of the proof of Proposition B.3. We have already proven $\Downarrow$ which is equivalent to $M' \subset M$. It then immediately follows from Lemmas B.4 and B.5 that $M = M'$, finishing the proof of the proposition. □

We can now prove (*B.2*). By Proposition B.3 it suffices to count the number of triples of natural numbers α, β, λ satisfying $\alpha + \beta + \lambda = a + b$ and $\lceil \frac{b}{2} \rceil \leq \beta + \gamma$ and $\lfloor \frac{b}{2} \rfloor \leq \alpha + \gamma$. The latter is equivalent to $\alpha \leq a + b - \lceil \frac{b}{2} \rceil$, $\beta \leq a + b - \lfloor \frac{b}{2} \rfloor$ such that our problem has turned into a combinatorial problem: we need to show

Lemma B.6.

$$\#\Big\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b,$$

$$\alpha \leq a + b - \left\lceil \frac{b}{2} \right\rceil, \ \beta \leq a + b - \left\lfloor \frac{b}{2} \right\rfloor \Big\} = h'(2a + b). \quad \text{(B.6)}$$

We first show that the left hand side equals zero when $2a + b < 0$. Note that in this case $a + b < \frac{b}{2} \leq \lceil \frac{b}{2} \rceil$, hence the condition $\alpha \leq a + b - \lceil \frac{b}{2} \rceil$ contradicts $\alpha \in \mathbb{N}$ such that the left hand side of (B.6) is zero. Hence from now on we can assume $2a + b \geq 0$.

This combinatorial problem has a graphical interpretation: it asks for counting the number of dots in Figure 3 whose coefficients (α, β, λ) satisfy the above inequalities.

$$(a+b, 0, 0)$$

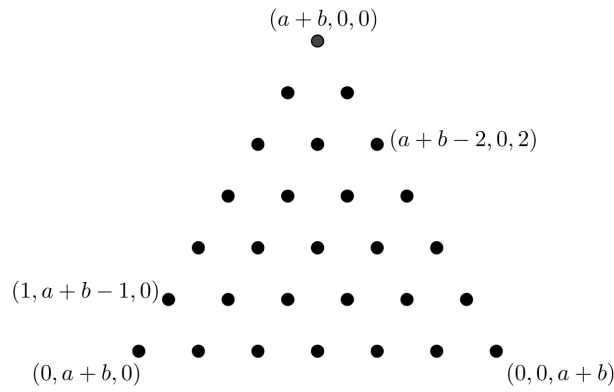$$(a+b-2, 0, 2)$$

$$(1, a+b-1, 0)$$

$$(0, a+b, 0) \qquad\qquad (0, 0, a+b)$$

Figure 3. Combinatorial problem.

In order to compute this number of dots we can rewrite (B.6) as:

$$
\#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b,\ \alpha \leq a + b - \left\lceil \frac{b}{2} \right\rceil,\ \beta \leq a + b - \left\lfloor \frac{b}{2} \right\rfloor \right\}
$$

$$
= \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b \right\}
$$

$$
\color{green}{- \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b,\ \alpha \geq a + b + 1 - \left\lceil \frac{b}{2} \right\rceil \right\}}
$$

$$
\color{red}{- \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b,\ \beta \geq a + b + 1 - \left\lfloor \frac{b}{2} \right\rfloor \right\}} \qquad \text{(B.7)}
$$

$$
\color{blue}{+ \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b,\ \alpha \geq a + b + 1 - \left\lceil \frac{b}{2} \right\rceil, \right.}
$$

$$
\color{blue}{\left. \beta \geq a + b + 1 - \left\lfloor \frac{b}{2} \right\rfloor \right\}.}
$$

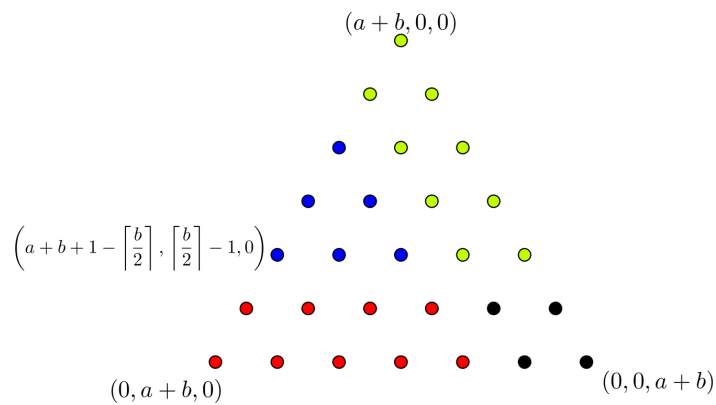The green, red and blue numbers can be visualized as in Figure 4.

Figure 4. Visualizing formula (B.7).

The reason for writing our combinatorial problem as in (B.7) is the existence of the following bijection:

$$\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = n_1, \alpha \geq n_2\}$$
$$\rightarrow \{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = n_1 - n_2\} :$$
$$(\alpha, \beta, \lambda) \mapsto (\alpha - n_2, \beta, \lambda).$$

Using similar bijections for the other sets we can write (B.7) as

$$\#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b, \alpha \leq a + b - \left\lceil \frac{b}{2} \right\rceil, \beta \leq a + b - \left\lfloor \frac{b}{2} \right\rfloor\right\}$$
$$= \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b\right\}$$
$$- \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = \left\lceil \frac{b}{2} \right\rceil - 1\right\}$$
$$- \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = \left\lfloor \frac{b}{2} \right\rfloor - 1\right\}$$
$$+ \#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = \left\lceil \frac{b}{2} \right\rceil + \left\lfloor \frac{b}{2} \right\rfloor - a - b - 2 = -a - 2\right\}.$$
$$\tag{B.8}$$

Now we can use the following: For all $n \geq 0$:

$$\#\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = n\} = \sum_{i=0}^{n} \#\{(\alpha, \beta) \in \mathbb{N}^2 \mid \alpha + \beta = i\}$$

$$= \sum_{i=0}^{n} (i+1) = \frac{(n+2)(n+1)}{2}. \tag{B.9}$$

(Recall that we assumed $a \leq -2$ and $a+b \geq 0$ such that $-a-2, \lceil \frac{b}{2} \rceil - 1, \lfloor \frac{b}{2} \rfloor - 1 \geq 0$.)
Hence, combining (B.8) and (B.9), we find that

$$\#\left\{(\alpha, \beta, \lambda) \in \mathbb{N}^3 \mid \alpha + \beta + \lambda = a + b, \ \alpha \leq a + b - \left\lceil \frac{b}{2} \right\rceil, \ \beta \leq a + b - \left\lfloor \frac{b}{2} \right\rfloor \right\}$$

equals

$$\frac{(a+b+2)(a+b+1)}{2} - \frac{\lceil \frac{b}{2} \rceil (\lceil \frac{b}{2} \rceil + 1)}{2} - \frac{\lfloor \frac{b}{2} \rfloor (\lfloor \frac{b}{2} \rfloor + 1)}{2} + \frac{(-1-a)(-a)}{2}$$

$$= \frac{a^2 + 2ab + b^2 + 3a + 3b + 2 - \lceil \frac{b}{2} \rceil^2 - \lfloor \frac{b}{2} \rfloor^2 - \lceil \frac{b}{2} \rceil - \lfloor \frac{b}{2} \rfloor + a^2 + a}{2}$$

$$= \frac{2a^2 + 2ab + b^2 + 4a + 2b + 2 - \lceil \frac{b}{2} \rceil^2 - \lfloor \frac{b}{2} \rfloor^2}{2}. \tag{B.10}$$

We now treat the cases b even and b odd separately. First assume $b = 2r$ for some $r \in \mathbb{N}$. Then (B.10) equals

$$\frac{2a^2 + 4ar + 4r^2 + 4a + 4r + 2 - 2r^2}{2} = a^2 + 2ar + r^2 + 2a + 2r + 1$$

$$= (a + r + 1)^2.$$

Next assume $b = 2r + 1$. Then (B.10) equals

$$\frac{2a^2 + 4ar + 2a + 4r^2 + 4r + 1 + 4a + 4r + 2 + 2 - (r+1)^2 - r^2}{2}$$

$$= \frac{2a^2 + 4ar + 2r^2 + 6r + 6a + 4}{2}$$

$$= a^2 + 2ar + r^2 + 3r + 3a + 2$$

$$= (a + r + 1)(a + r + 2).$$

Finally letting $n = a + r$ we see that this agrees with (B.1).

References

[1] M. Artin and W. F. Schelter, Graded algebras of global dimension 3, *Adv. Math.*, **66** (1987), 171–216. Zbl 0633.16001 MR 917738

[2] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in *The Grothendieck Festschrift*, P. Cartier et al., (eds.), 33–85, Modern Birkhäuser Classics, 1, Birkhäuser Boston, 1990. Zbl 0744.14024 MR 1086882

[3] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.*, **106** (1991), no. 1, 335–388. Zbl 0763.14001 MR 1128218

[4] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, *J. Algebra*, **133** (1990), no. 2, 249–271. Zbl 0717.14001 MR 1067406

[5] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.*, **109** (1994), no. 2, 228–287. Zbl 0833.14002 MR 1304753

[6] A. Bondal and A. Polishchuk, Homological properties of associative algebras: the method of helices, *Russian Acad. Sci. Izv. Math.*, **42** (1994), 219–260. Zbl 0847.16010 MR 1230966

[7] D. Chan and A. Nyman, Species and non-commutative $\mathbb{P}^1$'s over non-algebraic bimodules, *J. Algebra*, **460** (2016), 143–180. Zbl 1379.16008 MR 3510397

[8] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 35, Springer, 1967. Zbl 0186.56802 MR 210125

[9] R. Hartshorne, *Algebraic geometry*, eighth edition, Graduate Texts in Mathematics, Springer-Verlag, 1997. Zbl 0367.14001 MR 463157

[10] D. Presotto, $\mathbb{Z}^2$-algebras as noncommutative blow-ups, 2017. arXiv:1701.00413

[11] D. Presotto and M. Van den Bergh, Noncommutative versions of some classical birational transformations, *J. Noncommut. Geom.*, **10** (2016), no. 1, 221–244. Zbl 1371.14005 MR 3500820

[12] D. Rogalski, S. J. Sierra, and J. T. Stafford, Noncommutative blowups of elliptic algebras, *Algebr. Represent. Theory*, **18** (2015), no. 2, 491–529. Zbl 1331.14004 MR 3336351

[13] S. J. Sierra, *G*-algebras, twistings, and equivalences of graded categories, *Algebr. Represent. Theory*, **14** (2011), no. 2, 377–390. Zbl 1258.16047 MR 2776790

[14] S. J. Sierra, *Talk: Ring-theoretic blowing down (joint work with D. Rogalski and J. T. Stafford)*, Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra, 2014.

[15] J. T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc. (N.S.)*, **38** (2001), no. 2, 171–216. Zbl 1042.16016 MR 1816070

[16] J. Tate and M. Van den Bergh, Homological properties of sklyanin algebras, *Invent. Math.*, **124** (1996), no. 1, 619–647. Zbl 0876.17010 MR 1369430

[17] M. Van den Bergh, Blowing up non-commutative smooth surfaces, *Mem. Amer. Math. Soc.*, **154** (2001), no. 734, x+140pp. Zbl 0998.14002 MR 1846352

[18] M. Van den Bergh, Noncommutative quadrics, *Int. Math. Res. Not. IMRN*, (2011), no. 17, 3983–4026. Zbl 1311.14003 MR 2836401

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