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A categorical perspective on the Atiyah–Segal completion theorem in KK-theory

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Abstract. We investigate the homological ideal $\mathfrak{J}_G^H$, the kernel of the restriction functors in compact Lie group equivariant Kasparov categories. Applying the relative homological algebra developed by Meyer and Nest, we relate the Atiyah–Segal completion theorem with the comparison of $\mathfrak{J}_G^H$ with the augmentation ideal of the representation ring.

In relation to it, we study on the Atiyah–Segal completion theorem for groupoid equivariant KK-theory, McClure's restriction map theorem and permanence property of the Baum–Connes conjecture under extensions of groups.

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1. Introduction

Equivariant KK-theory is one of the main subjects in the noncommutative topology, which deals with topological properties of C*-algebras. The main subject of this paper is the homological ideal

$$\mathfrak{J}_G^H(A, B) := \operatorname{Ker}\left(\operatorname{Res}_G^H \colon \mathrm{KK}^G(A, B) \to \mathrm{KK}^H(A, B)\right)$$

of the Kasparov category $\mathfrak{KK}^G$, whose objects are separable G-C*-algebras, morphisms are equivariant KK-groups and composition is given by the Kasparov product.

In [31], Meyer and Nest introduced a new approach to study the homological algebra of the Kasparov category. They observed that the Kasparov category has a canonical structure of the triangulated category. Moreover, they applied the Verdier localization for $\mathfrak{KK}^G$ in order to give a categorical formulation of the Baum–Connes assembly map. Actually they prove that an analogue of the simplicial approximation in the Kasparov category is naturally isomorphic to the assembly map. Their argument is refined in [29] in terms of relative homological algebra of the projective class developed by Christensen [11]. Moreover it is proved that the ABC spectral sequence (a generalization of Adams spectral sequence in relative homological algebra) for the functor $\mathrm{K}_*(G \ltimes \sqcup)$ and an object A converges to the domain of the assembly map.

These results are essentially based on the fact that the induction functor Ind_H^G is the left adjoint of the restriction functor Res_G^H when $H \leq G$ is an open subgroup. On the other hand, it is also known that when $H \leq G$ is a cocompact subgroup, Ind_H^G is the right adjoint of Res_G^H. This relation enables us to apply the homological algebra of the injective class for KK-theory. It should be noted that the category of separable G-C^*-algebras is not closed under countable direct product although the fact that $\mathfrak{KK}^G$ have countable direct sums plays an essential role in [29, 31, 32]. Therefore, we replace the category G-$\mathfrak{C}^*\mathfrak{sep}$ of separable G-C^*-algebras with its (countable) pro-category. Actually, the category $\mathrm{Pro}_{\mathbb{Z}_{>0}} G$-$\mathfrak{C}^*\mathfrak{sep}$ is naturally equivalent to the category σG-$\mathfrak{C}^*\mathfrak{sep}$ of σ-G-C^*-algebras, which is dealt with by Phillips in his study of the Atiyah–Segal completion theorem. Fortunately, KK-theory for (non-equivariant) σ-C^*-algebras are investigated by Bonkat [7]. We check that his definition is generalized for equivariant KK-theory and obtain the following theorem.

Theorem A.16 and Theorem 3.4. *For a compact group G, the equivariant Kasparov category $\sigma\mathfrak{KK}^G$ of σ-G-C^*-algebras has a structure of the triangulated category. Moreover, for a family $\mathcal{F}$ of G, the pair of thick subcategories $(\mathcal{F}\mathfrak{C}, \langle \mathcal{F}\mathfrak{I} \rangle^{\mathrm{loc}})$ is complementary. Here $\mathcal{F}\mathfrak{C}$ is the full subcategory of $\mathcal{F}$-contractible objects and $\mathcal{F}\mathfrak{I}$ is the class of $\mathcal{F}$-induced objects (see Definition 3.3).*

Next, we observe that this semi-orthogonal decomposition is related to a classical idea in equivariant K-theory called the Atiyah–Segal completion. In the theory of equivariant cohomology, there is a canonical way to construct an equivariant general cohomology theory from a non-equivariant cohomology theory. Actually, for a compact Lie group G and a G-CW-complex X, the general cohomology group of the new space given by the Borel construction $X \times_G EG$ is regarded as the equivariant version of the given cohomology group of X. On the other hand, equivariant K-theory is defined in terms of equivariant vector bundles by Atiyah and Segal in [3, 47]. This group has a structure of modules over the representation ring $R(G)$ and hence is related to the representation theory of compact Lie groups. In 1969, Atiyah and Segal discovered a beautiful relation between them [4]. When the equivariant K-group $\mathrm{K}_G^*(X)$ of a compact G-space is finitely generated as an $R(G)$-module, then the completion of the equivariant K-group by the augmentation ideal is actually isomorphic to the (representable) K-group of the Borel construction of X.

This theorem is generalized in [1] for families of subgroups. The completion of $\mathrm{K}_G^*(X)$ by the family of ideals I_G^H $(H \in \mathcal{F})$ is isomorphic to the equivariant K-group $\mathrm{K}_G(X \times E_{\mathcal{F}}G)$ where $E_{\mathcal{F}}G$ is the universal $\mathcal{F}$-free G-space. On the other hand, Phillips [41] generalizes it for K-theory of C^*-algebras. In order to formulate the statement, he generalizes operator K-theory for σ-C^*-algebras in [42]. Actually, this contains the Atiyah–Segal completion theorem for twisted K-theory because the twisted equivariant K-group is isomorphic to the K-group of certain C*-algebra bundles with (twisted) group actions.

The Atiyah–Segal completion theorem is generalized for equivariant KK-theory by Uuye [56]. Here he assumes that $\mathrm{KK}_*^H(A, B)$ are finitely generated for all subgroups H of G in order to regard the correspondence $X \mapsto \mathrm{KK}^G(A, B \otimes C(X))$ as an equivariant cohomology theory of finite type. We prove the categorical counterpart of the Atiyah–Segal completion theorem under weaker assumptions.

Theorem 3.15. *Let G be a compact Lie group and let A, B be σ-C*-algebras such that $\mathrm{KK}_*^G(A, B)$ are finitely generated for $* = 0, 1$. Then the filtrations $(\mathfrak{J}_G^{\mathcal{F}})^*(A, B)$ and $(I_G^{\mathcal{F}})^*\mathrm{KK}^G(A, B)$ are equivalent.*

Applying Theorem 3.15 for the relative homological algebra of the injective class, we obtain the following generalization of the Atiyah–Segal completion theorem.

Theorem 3.21. *When $\mathrm{KK}^G(A, B)$ are finitely generated $R(G)$-modules for $* = 0, 1$, the following $R(G)$-modules are canonically isomorphic,*

$$\mathrm{KK}^G(A, B)^\wedge_{I_G^{\mathcal{F}}} \cong \mathrm{KK}^G(A, \widetilde{B}) \cong \mathrm{RKK}^G(E_{\mathcal{F}} G; A, B) \cong \sigma\mathfrak{KK}^G/\mathcal{F}\mathcal{C}(A, B).$$

Note that in some special cases we need not to assume that $\mathrm{KK}_*^G(A, B)$ are finitely generated. In particular, we obtain the following.

Corollary 3.13. *Let $\mathcal{Z}$ be the family generated by all cyclic subgroups of G. Then, there is $n > 0$ such that $(\mathfrak{J}_G^{\mathcal{Z}})^n = 0$.*

It immediately follows from Corollary 3.13 that if $\mathrm{Res}_G^H A$ is KK^H-contractible for any cyclic subgroup H of G, then A is KK^G-contractible. This is a variation of McClure's restriction map theorem [27] which is generalized by Uuye [56] for equivariant KK-theory. Since we improve the Atiyah–Segal completion theorem, the assumption in Theorem 0.1 of [56] is also weakened (Corollary 3.23).

Moreover, the Atiyah–Segal completion theorem for proper actions and groupoids are studied in [8, 23]. We generalize Theorem 3.21 for groupoid equivariant KK-theory (Theorem 4.7) and equivariant KK-theory for proper G-C*-algebras (Theorem 4.8) under certain assumptions.

Next we apply Corollary 3.13 for the study of the complementary pair $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$ of the Kasparov category $\sigma\mathfrak{KK}^G$ and the Baum–Connes conjecture (BCC). Our main interest here is permanence property of the BCC under group extensions, which is studied by Chabert, Echterhoff and Oyono-Oyono in [9, 10, 35] with the use of the partial assembly map. Let $1 \to N \to G \xrightarrow{\pi} G/N \to 1$ be an extension of groups. It is proved in Corollary 3.4 of [9] and Theorem 10.5 of [31] that if G/N and $\pi^{-1}(F)$ for any compact subgroup F of G/N satisfy the (resp., strong) BCC, then so does G. Here, the assumption that $\pi^{-1}(F)$ satisfy the BCC is related to the fact that the assembly map is defined in terms of the complementary pair $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$ (this assumption is refined by Schick [45] when G is discrete, H is cohomologically complete and has enough torsion-free amenable quotients by group-theoretic arguments). On the other hand, Corollary 3.13 implies that the subcategories $\mathcal{CC}$ and $\mathcal{CZC}$ coincide in $\sigma\mathfrak{KK}^G$. As a consequence we refine their results as following.

Theorem 5.4. *Let $1 \to N \to G \to G/N \to 1$ be an extension of second countable groups such that all compact subgroups of G/N are Lie groups and let A be a G-C^*-algebra. Then the following holds:*

(1) *If $\pi^{-1}(H)$ satisfies the (resp., strong) BCC for A for any compact cyclic subgroup H of G/N, then G satisfies the (resp., strong) BCC for A if and only if G/N satisfies the (resp., strong) BCC for $N \ltimes_r^{\mathrm{PR}} A$.*

(2) *If $\pi^{-1}(H)$ and G/N have the γ-element for any compact cyclic subgroup H of G/N, then so does G. Moreover, in that case $\gamma_{\pi^{-1}(H)} = 1$ and $\gamma_{G/N} = 1$ if and only if $\gamma_G = 1$.*

This paper is organized as follows. In Section 2, we briefly summarize terminologies and basic facts on the relative homological algebra of triangulated categories. In Section 3, we study the relative homological algebra of the injective class in the Kasparov category and prove the Atiyah–Segal completion theorem in KK-theory. In Section 4 we generalize the Atiyah–Segal completion theorem for groupoid equivariant case. In Section 5, we discuss on permanence property of the Baum–Connes conjecture under extensions of groups. In Appendix A, we survey definitions and some basic properties of equivariant KK-theory for σ-C^*-algebras.

2. Preliminaries in the relative homological algebra

In this section we briefly summarize some terminologies and basic facts on the relative homological algebra of triangulated categories. The readers can find more details in [32] and [29]. We modify a part of the theory in order to deal with the relative homological algebra of the injective class for countable families of homological ideals.

A triangulated category is an additive category together with the category automorphism Σ called the suspension and the class of triangles (a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

such that $g \circ f = h \circ g = \Sigma f \circ h = 0$) which satisfies axioms [TR0]–[TR4] (see [34, Chapter 1]). We often write an exact triangle $A \to B \to C \to \Sigma A$ as

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \nwarrow_{\circ} \quad \swarrow & \\ & C. & \end{array}$$

Here the symbol $A \multimap B$ represents a morphism from A to ΣB.

Let $\mathfrak{T}$ be a triangulated category. An *ideal* $\mathfrak{J}$ of $\mathfrak{T}$ is a family of subgroups $\mathfrak{J}(A, B)$ of $\mathfrak{T}(A, B)$ such that

$$\mathfrak{T}(A, B) \circ \mathfrak{J}(B, C) \circ \mathfrak{T}(C, D) \subset \mathfrak{J}(A, D).$$

A typical example is the kernel of an additive functor $F: \mathfrak{T} \to \mathfrak{A}$. We say that an ideal is a *homological ideal* if it is the kernel of a stable homological functor from $\mathfrak{T}$ to an abelian category $\mathfrak{A}$ with the suspension automorphism. Here a covariant functor F is homological if $F(A) \to F(B) \to F(C)$ is exact for any exact triangle

$$A \to B \to C \to \Sigma A$$

and stable if $F \circ \Sigma = \Sigma \circ F$. Note that the kernel of an exact functor between triangulated categories is a homological ideal by Proposition 20 of [32].

For a homological ideal $\mathfrak{J}$ of $\mathfrak{T}$, an object A is $\mathfrak{J}$-*contractible* if id_A is in $\mathfrak{J}$ and is $\mathfrak{J}$-*injective* if $f^*: \mathfrak{T}(D, A) \to \mathfrak{T}(B, A)$ is zero for any $f \in \mathfrak{J}(B, D)$. The triangulated category $\mathfrak{T}$ *has enough* $\mathfrak{J}$-*injectives* if for any object $A \in \mathrm{Obj}\, \mathfrak{T}$ there is a $\mathfrak{J}$-injective object I and a $\mathfrak{J}$-monic morphism $A \to I$ i.e. the morphism ι in the exact triangle

$$N \xrightarrow{\iota} A \to I \to \Sigma N$$

is in $\mathfrak{J}$. Note that the morphism ι is $\mathfrak{J}$-*coversal*, that is, an arbitrary morphism $f: B \to A$ in $\mathfrak{J}$ factors through ι (see [29, Lemma 3.5]).

More generally, we consider the above homological algebra for a countable family $\mathfrak{J} = \{\mathfrak{J}_k\}_{k \in \mathbb{Z}_{>0}}$ of homological ideals of $\mathfrak{T}$. For example, we say an object A is $\mathfrak{J}$-contractible if A is $\mathfrak{J}_k$-contractible for any $k \in \mathbb{Z}_{>0}$.

Definition 2.1. A *filtration* associated to $\mathfrak{J}$ is a filtration of the morphism sets of $\mathfrak{T}$ coming from the composition of ideals $\{\mathfrak{J}_{i_1} \circ \mathfrak{J}_{i_2} \circ \cdots \circ \mathfrak{J}_{i_r}\}_{r \in \mathbb{Z}_{>0}}$ where $\{i_1, i_2, \dots\}$ is a sequence of positive integers such that each $k \in \mathbb{Z}_{>0}$ arises infinitely many times.

Note that two filtrations associated to $\mathfrak{J}$ are equivalent (here, we say that two filtrations A_* and A'_* of an abelian group A are equivalent if for any $n \in \mathbb{Z}_{>0}$ there is $m \in \mathbb{Z}_{>0}$ such that $A_m \subset A'_n$ and $A'_m \subset A_n$). For simplicity of notation, we use the notation $\mathfrak{J}^r$ for the r-th component of a (fixed) filtration associated to $\mathfrak{J}$ unless otherwise noted.

The relative homological algebra is related to the complementary pairs (or semi-orthogonal decompositions) of the triangulated categories. For a thick triangulated subcategory $\mathfrak{C}$ of $\mathfrak{T}$ ([34, Definitions 1.5.1 and 2.1.6]), there is a natural way to obtain a new triangulated category $\mathfrak{T}/\mathfrak{C}$ called the Verdier localization (see [34, Section 2.1]). A pair $(\mathfrak{N}, \mathfrak{I})$ is a *complementary pair* if $\mathfrak{T}(N, I) = 0$ for any $N \in \mathrm{Obj}\, \mathfrak{N}$, $I \in \mathrm{Obj}\, \mathfrak{I}$ and for any $A \in \mathrm{Obj}\, \mathfrak{T}$ there is an exact triangle

$$\mathsf{N}_A \to A \to \mathsf{I}_A \to \Sigma \mathsf{N}_A$$

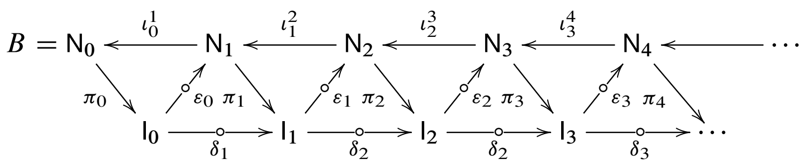
such that $\mathsf{N}_A \in \operatorname{Obj} \mathfrak{N}$ and $\mathsf{I}_A \in \operatorname{Obj} \mathfrak{I}$. Actually, such an exact triangle is unique up to isomorphism for each A and there are functors $N : \mathfrak{T} \to \mathfrak{N}$ and $I : \mathfrak{T} \to \mathfrak{I}$ that maps A to N_A and I_A respectively. We say that N (resp., I) the *left* (resp., *right*) *approximation functor* with respect to the complementary pair $(\mathfrak{N}, \mathfrak{I})$. These functors induces the category equivalence $\mathsf{I} : \mathfrak{T}/\mathfrak{N} \to \mathfrak{I}$ and $\mathsf{N} : \mathfrak{T}/\mathfrak{I} \to \mathfrak{N}$.

Moreover we assume that a triangulated category $\mathfrak{T}$ admits countable direct sums and direct products. A thick triangulated subcategory of $\mathfrak{T}$ is *colocalizing* (resp., *localizing*) if it is closed under countable direct products (resp., direct sums). For a class $\mathcal{C}$ of objects in $\mathfrak{T}$, let $\langle \mathcal{C} \rangle^{\mathrm{loc}}$ (resp., $\langle \mathcal{C} \rangle_{\mathrm{loc}}$) denote the smallest colocalizing (resp., localizing) thick triangulated subcategory which includes all objects in $\mathcal{C}$. We say that an ideal $\mathfrak{J}$ is *compatible with countable direct products* if the canonical isomorphism $\mathfrak{T}(A, \prod B_n) \cong \prod \mathfrak{T}(A, B_n)$ restricts to $\mathfrak{J}(A, \prod B_n) \cong \prod \mathfrak{J}(A, B_n)$.

We write $\mathfrak{N}_{\mathfrak{J}}$ for the thick subcategory of objects which is $\mathfrak{J}_k$-contractible for any k. If each $\mathfrak{J}_k$ is compatible with countable direct products, $\mathfrak{N}_{\mathfrak{J}}$ is colocalizing. We write $\mathfrak{I}_{\mathfrak{J}}$ for the class of $\mathfrak{J}_k$-injective objects for some k.

Theorem 2.2 ([29, Theorem 3.21]). *Let $\mathfrak{T}$ be a triangulated category with countable direct product and let $\mathfrak{J} = \{\mathfrak{J}_i\}$ be a family of homological ideals with enough $\mathfrak{J}_i$-injective objects which are compatible with countable direct products. Then, the pair $(\mathfrak{N}_{\mathfrak{J}}, \langle \mathfrak{I}_{\mathfrak{J}} \rangle^{\mathrm{loc}})$ is complementary.*

We review the explicit construction of the left and right approximation in Theorem 3.21 of [29]. We start with the following diagram called the *phantom tower* for B:



where ι_k^{k+1} are in $\mathfrak{J}_{i_k}$ and I_k are $\mathfrak{J}_{i_k}$-injective (here $\{i_k\}_{k \in \mathbb{Z}_{>0}}$ is the same as in Definition 2.1). There exists such a diagram for any B since $\mathfrak{T}$ has enough $\mathfrak{J}$-injectives. We write ι_k^l for the composition

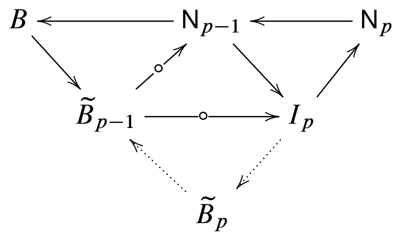
$$\iota_{l-1}^{l} \circ \iota_{l-2}^{l-1} \circ \cdots \circ \iota_k^{k+1}.$$

Since each ι_k^{k+1} is $\mathfrak{J}_{i_k}$-coversal, we obtain $\mathfrak{J}^p(A, B) = \operatorname{Im}(\iota_0^p)_*$ for any A.

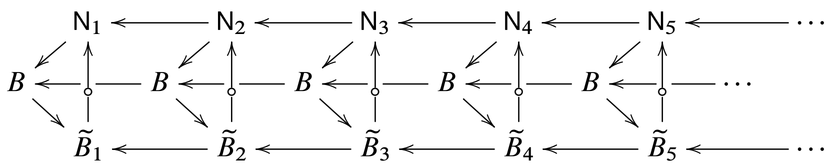
Next we extend this diagram to the *phantom castle*. Due to the axiom [TR1], there is a (unique) object $\widetilde{B}_p$ in $\mathfrak{T}$ and an exact triangle

$$\mathsf{N}_p \to B \to \widetilde{B}_p \to \Sigma \mathsf{N}_p$$

for each p. By the axiom [TR4], we can complete the following diagram by dotted morphisms



and hence $\widetilde{B}_p$ is $\mathfrak{J}^p$-injective. Moreover, we obtain a projective system



of exact triangles. Now we take the homotopy projective limit

$$\mathsf{I}_B := \text{ho-}\varprojlim_p \widetilde{B}_p$$

(we also use the symbol $\widetilde{B}$ for this object) and

$$\mathsf{N}_B := \text{ho-}\varprojlim \mathsf{N}_p.$$

Here the homotopy projective limit of a projective system (B_p, φ_p^{p+1}) is the second part of the exact triangle

$$\Sigma^{-1} \prod B_p \to \text{ho-}\varprojlim B_p \to \prod B_p \xrightarrow{\text{id}-S} \prod B_p,$$

where $S := \prod \varphi_m^{m+1}$. Then, the axiom [TR4] implies that the homotopy projective limit

$$\mathsf{N}_B \to B \to \mathsf{I}_B \to \Sigma \mathsf{N}_B$$

of the projective system of exact triangles is also exact. In fact, it can be checked that I_B is in $\langle \mathfrak{I}_{\mathfrak{J}} \rangle^{\text{loc}}$ and N_B is in $\mathfrak{N}_{\mathfrak{J}}$ and hence N_B and I_B gives the left and right approximation of B.

At the end of this section, we review the ABC spectral sequence, introduced in [29] and named after Adams, Brinkmann, and Christensen. Let B be an object in $\mathfrak{T}$, let $\mathfrak{J}$ be a countable family of homological ideals with a fixed filtration and let $F\colon \mathfrak{T} \to \mathfrak{Ab}$ be a homological functor. Set

$$\begin{cases} D = \bigoplus D^{p,q}, & D^{p,q} := F_{p+q+1}(\mathsf{N}_{p+1}), \\ E = \bigoplus E^{p,q}, & E^{p,q} := F_{p+q+1}(\mathsf{I}_{p+1}), \end{cases}$$

$$\begin{cases} i_{p,q} := (\iota_{p+1}^{p+2})^* & : D^{p,q} \to D^{p+1,q-1}, \\ j_{p,q} := (\varepsilon_{p+1})^* & : D^{p,q} \to E^{p-1,q+1}, \\ k_{p,q} := (\pi_p)^* & : E^{p,q} \to D^{p,q+1}, \end{cases}$$

where $\mathsf{N}_p = A$ and $\mathsf{I}_p = 0$ for $p < 0$. Then the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array}$$

forms an exact couple. We call the associated spectral sequence is the *ABC spectral sequence* for B and F.

Proposition 2.3 ([29, Proposition 4.3]). *Let B be an object in $\mathfrak{T}$ and let F be a homological functor. Set*

$$D_r^{pq} = D_r^{pq}(B) := i^{r-1}(D^{p-r+1,p+r-1})$$

and

$$E_r^{pq} = E_r^{pq}(B) := k^{-1}(D_r^{p,q})/j(\operatorname{Ker} i^r).$$

Then the following hold:

(1)
$$D_r^{p-1,q} = \begin{cases} \mathfrak{J}^{r-1} F_{p+q+1}(\mathsf{N}_p), & \text{if } p \geq 0, \\ \mathfrak{J}^{p+r-1} F_{p+q+1}(B), & \text{if } -r+1 \leq p \leq 0, \\ F_{p+q+1}(B), & \text{if } p \leq -r+1, \end{cases}$$

where $\mathfrak{J}^p F(B)$ denotes the subgroup

$$\{f_*\xi \mid \xi \in F(A),\ f \in \mathfrak{J}^p(A, B)\}$$

of $F(B)$.

(2) *The E_2-page E_2^{pq} is isomorphic to the right derived functor*

$$\mathbb{R}^p F^q(B) := H_p\big(F_q(\mathsf{I}_*), (\delta_i)_*\big).$$

(3) *There is an exact sequence*

$$0 \to \frac{\mathfrak{J}^p F_{p+q+1}(B)}{\mathfrak{J}^{p+1} F_{p+q+1}(B)} \to E_\infty^{p,q} \to \mathsf{Bad}^{p+1,p+q+1} \xrightarrow{i} \mathsf{Bad}^{p,p+q+1},$$

where $\mathsf{Bad}^{p,q}(B) = \mathsf{Bad}^{p,q} := \mathfrak{J}^\infty F_q(\mathsf{N}_p)$.

Lemma 2.4. *Assume that*

$$i\colon \mathsf{Bad}^{p+1,p+q+1}(B) \to \mathsf{Bad}^{p,p+q+1}(B)$$

is injective. Then, the ABC spectral sequence E^r_{pq} converges to $F(B)$ with the filtration $\mathfrak{J}^* F(B)$. Moreover, $\alpha_*\colon F(B) \to F(\widetilde{B})$ induces an isomorphism of graded quotients with respect to the filtration $\mathfrak{J}^* F$.

Proof. The convergence of the ABC spectral sequence follows from Proposition 2.3(3). By (the dual of) Proposition 3.27 of [29], we have the morphism between exact couples and hence the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathfrak{J}^p F_{p+q+1}(B)}{\mathfrak{J}^{p+1} F_{p+q+1}(B)} & \longrightarrow & E^{pq}_\infty(B) & \xrightarrow{\chi} & \mathsf{Bad}^{p+q+1,p}(B) & \xrightarrow{i} & \mathsf{Bad}^{p,q}(B) \\
 & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\
0 & \longrightarrow & \dfrac{\mathfrak{J}^p F_{p+q+1}(\widetilde{B})}{\mathfrak{J}^{p+1} F_{p+q+1}(\widetilde{B})} & \longrightarrow & E^{pq}_\infty(\widetilde{B}) & \xrightarrow{\widetilde{\chi}} & \mathsf{Bad}^{p+q+1,p}(\widetilde{B}) & \xrightarrow{i} & \mathsf{Bad}^{p,q}(\widetilde{B}).
\end{array}$$

Now, by Proposition 2.3(2), the map $\alpha_*\colon E^{pq}_2(B) \to E^{pq}_2(\widetilde{B})$ is an isomorphism and hence so is $\alpha_*\colon E^{pq}_\infty(B) \to E^{pq}_\infty(\widetilde{B})$. Therefore, injectivity of

$$i\colon \mathsf{Bad}^{p+1,p+q+1}(B) \to \mathsf{Bad}^{p,p+q+1}(B)$$

implies $\chi = 0$. Consequently we get $\widetilde{\chi} = 0$, which gives the conclusion. □

3. The Atiyah–Segal completion theorem

In this section we apply the relative homological algebra of the injective class introduced in Section 2 for equivariant KK-theory and relate it with the Atiyah–Segal completion theorem. We deal with the Kasparov category $\sigma\mathfrak{KK}^G$ of σ-G-C*-algebras, which is closed under countably infinite direct products. The definition and the basic properties of equivariant KK-theory for σ-G-C*-algebras are summarized in Appendix A. In most part of this section we assume that G is a compact Lie group. We need not to assume that G is either connected or simply connected.

For a subgroup $H \leq G$, consider the homological ideal $\mathfrak{J}^H_G := \operatorname{Ker}\operatorname{Res}^H_G$ of $\sigma\mathfrak{KK}^G$. There are only countably many homological ideals of the form $\mathfrak{J}^H_G$ since $\mathfrak{J}^{H_1}_G = \mathfrak{J}^{H_2}_G$ when H_1 and H_2 are conjugate and the set of conjugacy classes of subgroups of a compact Lie group G is countable ([36, Corollary 1.7.27]).

Definition 3.1. Let $\mathcal{F}$ be a *family*, that is, a set of closed subgroups of a compact group G that is closed under subconjugacy. We write $\mathfrak{J}^{\mathcal{F}}_G$ for the countable family of homological ideals $\{\mathfrak{J}^H_G \mid H \in \mathcal{F}\}$.

In particular, we say that the family $\mathcal{T}$ consisting of the trivial subgroup $\{e\}$ is the trivial family.

Let us recall that the induction functor

$$\operatorname{Ind}_H^G : \sigma H\text{-}\mathfrak{C}^*\mathfrak{sep} \to \sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$$

is given by

$$\operatorname{Ind}_H^G A := \{ f \in C(G, A) \mid \alpha_h(f(g \cdot h)) = f(g) \}$$

with the left regular G-action $\lambda_g(f)(g') = f(g^{-1}g')$ when H is a cocompact subgroup of G. By the universal property of the Kasparov category (Theorem A.15), it induces the functor between Kasparov categories. An important property of this functor is the following Frobenius reciprocity.

Proposition 3.2 ([31, Section 3.2]). *Let G be a locally compact group and $H \leq G$ be a cocompact subgroup. Then the induction functor Ind_H^G is the right adjoint of the restriction functor Res_G^H. That is, for any σ-G-C^*-algebra A and σ-H-C^*-algebra B we have*

$$\mathrm{KK}^G\big(A, \operatorname{Ind}_H^G B\big) \cong \mathrm{KK}^H\big(\operatorname{Res}_G^H A, B\big).$$

Proof. The equivariant KK-cycles induced from the $*$-homomorphisms

$$\varepsilon_A : \operatorname{Res}_G^H \operatorname{Ind}_H^G A \cong C(G, A)^H \to A, \qquad f \mapsto f(e),$$
$$\eta_B : B \to \operatorname{Ind}_H^G \operatorname{Res}_G^H B \cong C(G/H) \otimes B, \quad a \mapsto a \otimes 1_{G/H},$$

form a counit and a unit of an adjunction between Ind_H^G and Res_G^H. Actually, it directly follows from the definition that the compositions

$$\operatorname{Res}_G^H A \xrightarrow{\operatorname{Res}_G^H \eta_A} \operatorname{Res}_G^H \operatorname{Ind}_H^G \operatorname{Res}_G^H A \xrightarrow{\varepsilon_{\operatorname{Res}_G^H A}} \operatorname{Res}_G^H A,$$
$$\operatorname{Ind}_H^G B \xrightarrow{\eta_{\operatorname{Ind}_H^G B}} \operatorname{Ind}_H^G \operatorname{Res}_G^H \operatorname{Ind}_H^G B \xrightarrow{\operatorname{Ind}_H^G \varepsilon_B} \operatorname{Ind}_H^G B$$

are identities in $\sigma\mathfrak{KK}^G$. □

Definition 3.3. Let G be a compact group and let $\mathcal{F}$ be a family of G.

(1) A separable σ-G-C^*-algebra A is $\mathcal{F}$*-induced* if A is isomorphic to the inductions $\operatorname{Ind}_H^G A_0$ where A_0 is a separable σ-H-C^*-algebra and $H \in \mathcal{F}$. We write $\mathcal{F}\mathcal{I}$ for the class of $\mathcal{F}$-induced objects.

(2) A separable σ-G-C^*-algebra A is $\mathcal{F}$*-contractible* if $\operatorname{Res}_G^H A$ is KK^H-contractible for any $H \in \mathcal{F}$. We write $\mathcal{F}\mathcal{C}$ for the class of $\mathcal{F}$-contractible objects.

In particular, when $\mathcal{F} = \mathcal{T}$ we say that A is trivially induced and trivially contractible, respectively.

Theorem 3.4. *Let G be a compact group and let $\mathcal{F}$ be a family G. The pair $(\mathcal{F}\mathcal{C}, \langle \mathcal{F}\mathcal{I} \rangle^{\mathrm{loc}})$ is complementary in $\sigma\mathfrak{KK}^G$.*

Proof. This is proved in the same way as Proposition 3.21 of [29].

Note that

$$\mathcal{F}\mathcal{C} = \mathfrak{N}_{\mathfrak{J}_G^{\mathcal{F}}} \quad \text{and} \quad \mathcal{F}\mathcal{I} \subset \mathfrak{I}_{\mathfrak{J}_G^{\mathcal{F}}}.$$

By Theorem 2.2, it suffices to show that $\sigma\mathfrak{KK}^G$ has enough $\mathfrak{J}_G^{\mathcal{F}}$-injectives and all $\mathfrak{J}_G^{\mathcal{F}}$-injective objects are in $\langle \mathcal{F}\mathcal{I}\rangle^{\mathrm{loc}}$. The first assertion follows from the existence of the right adjoint functor of Res_G^H. Actually, for any $H \in \mathcal{F}$, the morphism

$$A \to I_1 := \mathrm{Ind}_H^G \mathrm{Res}_G^H A$$

is $\mathfrak{J}_G^H$-monic and I_1 is $\mathfrak{J}_G^H$-injective. Moreover, the morphism A is a direct summand of I_1 when A is $\mathfrak{J}_G^H$-injective. This implies the second assertion. □

In particular, applying Theorem 3.4 for the case of $\mathcal{F} = \mathcal{T}$, we immediately get the following simple but non-trivial application.

Corollary 3.5. *Let A be a separable σ-C*-algebra and let $\{\alpha_t\}_{t\in[0,1]}$ be a homotopy of G-actions on A. We write A_t for the σ-G-C*-algebra (A, α_t). Then, A_0 and A_1 are equivalent in $\sigma\mathfrak{KK}^G/\mathcal{T}\mathcal{C}$. In particular, if A_0 and A_1 are in $\langle \mathcal{T}\mathcal{I}\rangle^{\mathrm{loc}}$, then they are KK^G-equivalent.*

Corollary 3.5 is applied for the study of C*-dynamical systems in [2]. Actually, it follows from Thomsen's description of KK-groups using completely positive asymptotic morphisms [52] that a unital G-C*-algebra with the continuous Rokhlin property (or more generally finite continuous Rokhlin dimension with commuting towers) is contained in the subcategory $\langle \mathcal{T}\mathcal{I}\rangle^{\mathrm{loc}}$.

Proof. Consider the σ-G-C*-algebra

$$\tilde{A} := \big(A \otimes C[0,1], \tilde{\alpha}\big),$$

where $\tilde{\alpha}(a)(t) = \alpha_t(a(t))$. Since the evaluation maps $\mathrm{ev}_t\colon \tilde{A} \to A_t$ are non-equivariantly homotopy equivalent, they induce equivalences in $\sigma\mathfrak{KK}^G/\mathcal{T}\mathcal{C}$. Consequently,

$$\mathrm{ev}_1 \circ (\mathrm{ev}_0)^{-1}\colon A_0 \to A_1$$

is an equivalence in $\sigma\mathfrak{KK}^G/\mathcal{T}\mathcal{C}$. The second assertion is obvious. □

Next we study a canonical model of phantom towers and phantom castles. Actually, we observe that the cellular approximation tower obtained in the proof of Theorem 3.4 is nothing but the Milnor construction of the universal $\mathcal{F}$-free (i.e. every stabilizer subgroups are in $\mathcal{F}$) proper (in the sense of [37]) G-space (see [25]). Hereafter, for a compact G-space X, we write $\mathcal{C}_X$ for the mapping cone

$$\big\{ f \in C_0\big([0,\infty), C(X)\big) \mid f(0) = \mathbb{C}\cdot 1_X \big\}$$

of the $*$-homomorphism $\mathbb{C} \to C(X)$ induced from the collapsing map $X \to \mathrm{pt}$.

Definition 3.6. Let $\{H_p\}_{p\in\mathbb{Z}_{>0}}$ be a countable family of subgroups in $\mathcal{F}$ such that any $L \in \mathcal{F}$ are contained infinitely many H_p's. We call the phantom tower and the phantom castle determined inductively by

$$\mathsf{I}_p(B) := \mathrm{Ind}_{H_p}^G \mathrm{Res}_G^{H_p} \mathsf{N}_p(B) \cong \mathsf{N}_p(B) \otimes C(G/H_p)$$

is the *Milnor phantom tower* and the *Milnor phantom castle* (associated to $\{H_p\}$), respectively.

By definition, I_k and N_k in the Milnor phantom tower are explicitly of the form

$$\mathsf{N}_k \cong A \otimes \mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_k},$$
$$\mathsf{I}_k \cong A \otimes \mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_{k-1}} \otimes C(G/H_k),$$

and ι_k^{k+1} is induced from the restriction (evaluation) $*$-homomorphism $\mathrm{ev}_0\colon \mathcal{C}_{G/H_k} \to \mathbb{C}$ given by $f \mapsto f(0)$.

For G-spaces $X_1, \ldots, X_n$, the join $\mathop{\ast}_{k=1}^n X_k$ is defined to be the quotient of $\Delta^n \times (\prod X_k)$, where

$$\Delta^n := \Big\{(t_1, \ldots, t_n) \in [0,1]^n \mid \sum t_i = 1\Big\},$$

with the relation

$$(t_1, \ldots, t_n, x_1, \ldots, x_n) \sim (t_1, \ldots, t_n, y_1, \ldots, y_n)$$

if $x_k = y_k$ for any k such that $t_k \neq 0$. It is equipped with the G-action induced from the diagonal action on $\Delta^n \times \prod X_k$ (where G acts on Δ^n trivially). For $n \in \mathbb{Z}_{>0}$, $E_{\mathcal{F},n}G$ denotes the n-th step of the Milnor construction $\mathop{\ast}_{k=1}^n G/H_k$.

Lemma 3.7. *The nth step of the cellular approximation $\widetilde{\mathbb{C}}_n$ of $\mathbb{C}$ is isomorphic to $C(E_{\mathcal{F},n}G)$.*

Proof. More generally, let X be a $\mathcal{F}$-free finite G-CW-complex containing a point x whose stabilizer subgroup is H. By Proposition 2.2 of [29], there is $n > 0$ such that $C(X)$ is $(\mathfrak{J}_G^{\mathcal{F}})^n$-injective. Moreover, the morphism $\mathrm{ev}_0\colon \mathcal{C}_X \to \mathbb{C}$ is in $\mathfrak{J}_G^H$ since the path of H-equivariant $*$-homomorphisms $\mathrm{ev}_{(t,x)}\colon \mathcal{C}_X \to \mathbb{C}$ connects ev_0 and zero. Let $\{X_i\}$ be a family of $\mathcal{F}$-free compact G-CW-complexes such that for any $H \in \mathcal{F}$ there are infinitely many X_i's such that $X_i^H \neq \emptyset$. Then, in the same way as Theorem 2.2, the exact triangle

$$SC\Big(\mathop{\ast}_{i=1}^{\infty} X_i\Big) \to \varprojlim_{n\to\infty} \bigotimes_{i=1}^{n} \mathcal{C}_{X_i} \to \mathbb{C} \to C\Big(\mathop{\ast}_{i=1}^{\infty} X_i\Big)$$

gives the approximations of $\mathbb{C}$ with respect to the complementary pair $(\mathcal{F}\mathcal{C}, \langle \mathcal{F}\mathcal{I}\rangle^{\mathrm{loc}})$.

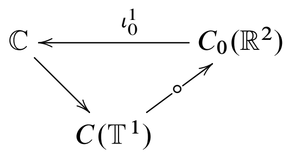
Now we compare the filtration $(\mathfrak{J}_G^{\mathcal{F}})^*(A, B)$ with another one:

$$(I_G^{\mathcal{F}})^n \mathrm{KK}^G(A, B) := \Big\{ \sum_i \gamma_i^1 \cdots \gamma_i^n \xi_i \mid \gamma_i^k \in I_G^{H_k},\ \xi_i \in \mathrm{KK}^G(A, B) \Big\},$$

where I_G^H are the augmentation ideals $\mathrm{Ker}\,\mathrm{Res}_G^H$ of $R(G)$ and $\{H_i\}$ is the same as Definition 3.6. Obviously its equivalence class is independent of the choice of such $\{H_i\}$. We also remark that

$$(I_G^{\mathcal{F}})^n \mathrm{KK}^G(A, B) \subset (\mathfrak{J}_G^{\mathcal{F}})^n(A, B). \qquad \square$$

Example 3.8. We consider the case that $G = \mathbb{T}^1$ and $\mathcal{F} = \mathcal{T}$. The first triangle in the Milnor phantom tower is



where $\mathbb{T}^1 = U(1)$ acts on $\mathbb{R}^2 = \mathbb{C}$ canonically. By the Bott periodicity, $\mathrm{KK}^G(\mathsf{N}_1, \mathbb{C})$ is freely generated by the Bott generator $\beta \in \mathrm{KK}^G(\mathsf{N}_1, \mathbb{C})$ and $\mathfrak{J}_G(\mathsf{N}_1, \mathbb{C}) = I_G \cdot \beta$. Consequently, ι_0^1 is in $I_G \mathrm{KK}^G(A, B)$. More explicitly, $\iota_0^1 = \lambda \cdot \beta$ where $\lambda := [\Lambda^0 \mathbb{C}] - [\Lambda^1 \mathbb{C}]$. Since ι_0^1 is $\mathfrak{J}_G$-coversal, $\mathfrak{J}_G(A, B) = I_G \mathrm{KK}^G(A, B)$ holds for any A and B.

Example 3.9. Let G be a compact connected Lie group such that $\pi_1(G)$ has no 2-torsion element and let T be a maximal torus of G. In this case the following lemma shows that $\iota_0^1 = 0$ and hence $\mathfrak{J}_G^T = I_G^T \mathrm{KK}^G = 0$.

Lemma 3.10. *The morphism $\pi_0 \in \mathrm{KK}^G(\mathbb{C}, C(G/T))$ in the Milnor phantom tower has a left inverse.*

Proof. Let us fix a choice of positive roots $P \subset \Delta$ and $\rho := \sum_{\alpha \in P} \alpha/2$. By the assumption about 2-torsion elements of $\pi_1(G)$, the weight $i\rho$ is analytically integral, that is, $\langle X, i\rho \rangle \in 2\pi\mathbb{Z}$ for any $X \in i\mathfrak{t}$ such that $e^X = 1 \in T$ (in terms of bundles, this means that the flag manifold G/T has a homogeneous Spin^c-structure).

Let $\lambda \in i\mathfrak{t}^*$ be an analytically integral weight such that $\lambda + \rho$ is regular i.e. $\langle \alpha, \lambda + \rho \rangle \neq 0$ for all $\alpha \in P$. The Borel–Weil–Bott theorem (see, for example, [5, Theorem 8.7]) says that the equivariant index of the twisted Dirac operator D_λ on G/T twisted by λ is the highest weight module $[V_\lambda] \in R(G)$. In particular, when $\lambda = 0$, the index of the (untwisted) Dirac operator is $1 \in R(G)$. Therefore, the corresponding K-homology cycle $[D] \in \mathrm{KK}^G(C(G/T), \mathbb{C})$ satisfies

$$[\pi_0] \otimes_{C(G/T)} [D] = [\mathrm{Ind}\, D] = 1 \in R(G) \cong \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$$

since π_0 is induced from the $*$-homomorphism mapping $1 \in \mathbb{C}$ to the identity element in $C(G/T)$. $\square$

For a group homomorphism $L \to G$ and a family $\mathcal{F}$ of G, define the pull-back to be

$$\varphi^*\mathcal{F} := \{\varphi^{-1}(H) \mid H \in \mathcal{F}\}.$$

Then, the functor $\varphi^*\colon \sigma\mathfrak{KK}^G \to \sigma\mathfrak{KK}^L$ maps $\langle \mathcal{F}\mathcal{I}\rangle$ and $\mathcal{F}\mathcal{C}$ to $\langle \varphi^*\mathcal{F}\mathcal{I}\rangle$ and $\varphi^*\mathcal{F}\mathcal{C}$, respectively. For a subgroup $H \leq G$, set

$$\mathcal{F}_H := \{gHg^{-1} \mid g \in G\}.$$

Theorem 3.11. *Let $H \leq G$ be compact connected Lie groups without any 2-torsion in their fundamental groups and* $\operatorname{rank} G - \operatorname{rank} H \leq 1$. *For a group homomorphism $\varphi\colon L \to G$, let $\mathcal{F} := \varphi^*\mathcal{F}_H$. Then, for any $r \in \mathbb{Z}_{>0}$ there is $k \in \mathbb{Z}_{>0}$ such that*

$$(\mathfrak{I}_L^{\mathcal{F}})^k(A, B) \subset (I_L^{\mathcal{F}})^r \mathrm{KK}^L(A, B)$$

for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^L$.

Proof. It suffices to find a compact $\mathcal{F}$-free proper L-space X such that the exact triangle

$$\mathcal{C}_X \xrightarrow{\iota} \mathbb{C} \xrightarrow{\pi} C(X) \to \Sigma\mathcal{C}_X$$

in $\sigma\mathfrak{KK}^L$ satisfies $\iota \in (I_L^{\mathcal{F}})^r \mathrm{KK}^L(\mathcal{C}_X, \mathbb{C})$ because

$$\mathrm{Im}(\iota \otimes \mathrm{id}_B)_* = \mathrm{Ker}(\pi \otimes \mathrm{id}_B)_* \supset (\mathfrak{I}_G^{\mathcal{F}})^k(A, B),$$

for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^L$ and $k \in \mathbb{Z}_{>0}$ such that $X \subset E_{\mathcal{F},k}L$. Since

$$\varphi^* I_G^H \subset I_L^M$$

for any $M \in \mathcal{F}$, we can reduce the problem for the case that $\varphi = \mathrm{id}$.

When $\operatorname{rank} G = \operatorname{rank} H$, it immediately follows from Example 3.9 (note that in this case $(\mathfrak{I}_G^{\mathcal{F}})^k = 0$ for some $k > 0$). To see the case that $\operatorname{rank} G - \operatorname{rank} H = 1$, choose an inclusion of maximal tori $T_H \subset T_G$. Consider the exact triangle

$$SC(T_G/T_H) \to \mathcal{C}_{T_G/T_H} \to \mathbb{C} \to C(T_G/T_H).$$

By Example 3.8, $\mathrm{Res}_G^{T_G} \iota_0^1$ is in $I_{T_G}^{T_H} \mathrm{KK}^{T_G}(\mathsf{N}_1, \mathbb{C})$. Since $(I_{T_G}^{T_H})^n \subset I_G^{T_H} R(T_G)$ for sufficiently large $n > 0$ ([1, Lemma 3.4]), for any $l > 0$ there is $k > 0$ such that

$$\iota_0^k = \iota_0^1 \otimes \cdots \otimes \iota_0^1 \in (I_G^H)^l \mathrm{KK}^{T_G}(\mathsf{N}_k, \mathbb{C})$$

(note that $I_G^{T_H} = I_G^H$). Moreover, ι_0^k is actually in $(I_G^H)^l \mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ since $\mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ is a direct summand of $\mathrm{KK}^{T_G}(\mathsf{N}_k, \mathbb{C})$ by Example 3.9. Now $E_{\mathcal{F},k}G$ is the desired X (recall that $C(E_{\mathcal{F},k}G) \cong \widetilde{\mathbb{C}}_k$). □

As a corollary, we obtain a generalization of Corollary 1.3 of [1]. For a family $\mathcal{F}$ of G, we write $\mathcal{F}_{\mathrm{cyc}}$ for the family generated by (topologically) cyclic subgroups in $\mathcal{F}$. In particular, let $\mathcal{Z}$ denote the family generated by all cyclic subgroups. Here, we say that $T \le G$ is a cyclic subgroup of G if there is an element $g \in T$ such that $\overline{\{g^n\}} = T$. Note that T is cyclic if and only if $T \cong \mathbb{T}^m \times \mathbb{Z}/l\mathbb{Z}$.

Lemma 3.12. *Let $\mathcal{F} \subset \mathcal{F}'$ be families of G. If for any $H \in \mathcal{F}'$ there is $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{I}_H^{\mathcal{F}|_H})^k = 0$, then two filtrations $\mathfrak{I}_G^{\mathcal{F}}$ and $\mathfrak{I}_G^{\mathcal{F}'}$ are equivalent uniformly, that is, for any $k > 0$ there is $n > 0$ (independent of A and B) such that*

$$(\mathfrak{I}_G^{\mathcal{F}})^n(A,B) \subset (\mathfrak{I}_G^{\mathcal{F}'})^k(A,B)$$

for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^G$.

Proof. Pick

$$H_1, \dots, H_k \in \mathcal{F}'.$$

By assumption, we can choose $L_{i,1}, \dots, L_{i,j_i}$ $(i = 1, \dots, k)$ such that

$$\mathfrak{I}_{H_i}^{L_{i,1}} \circ \cdots \circ \mathfrak{I}_{H_i}^{L_{i,j_i}} = 0.$$

Then, by definition

$$\big(\mathfrak{I}_G^{L_{1,1}} \circ \cdots \circ \mathfrak{I}_G^{L_{1,j_1}}\big) \circ \cdots \circ \big(\mathfrak{I}_G^{L_{i,1}} \circ \cdots \circ \mathfrak{I}_G^{L_{k,j_k}}\big) \subset \mathfrak{I}_G^{H_1} \circ \cdots \circ \mathfrak{I}_G^{H_k},$$

which is the conclusion. □

Corollary 3.13. *For a compact Lie group G, the following hold:*

(1) *There is $n > 0$ such that $(\mathfrak{I}_G^{\mathcal{Z}})^n = 0$. In particular, the subcategory $\mathcal{Z}\mathcal{C}$ is zero in $\sigma\mathfrak{K}\mathfrak{K}^G$.*

(2) *For any family $\mathcal{F}$ of G, the filtrations $(\mathfrak{I}_G^{\mathcal{F}})^*$ and $(\mathfrak{I}_G^{\mathcal{F}_{\mathrm{cyc}}})^*$ are equivalent. Moreover, $\mathcal{F}\mathcal{C} = \mathcal{F}_{\mathrm{cyc}}\mathcal{C}$ in $\sigma\mathfrak{K}\mathfrak{K}^G$.*

Note that the second assertion means that for any $n > 0$ we obtain $k > 0$ (which does not depend on A and B) such that

$$\big(\mathfrak{I}_G^{\mathcal{F}}\big)^k(A,B) \subset \big(\mathfrak{I}_G^{\mathcal{F}_{\mathrm{cyc}}}\big)^n(A,B).$$

Proof. First, we prove when G is abelian by induction with respect to the order of G/G^0, where G^0 is the identity component of G. When G/G^0 is cyclic, then the assertion holds because G is also cyclic. Now we assume that G/G^0 is not cyclic (and hence any element in G/G^0 is contained in a proper subgroup). Let $\mathcal{P}$ be the family of G generated by pull-backs of proper subgroups of G/G^0. By the induction hypothesis and Lemma 3.12, it suffices to show that there is a large $n > 0$ such that $(\mathfrak{I}_G^{\mathcal{P}})^n = 0$. Because G is covered by finitely many subgroups in $\mathcal{P}$, we obtain a large $m > 0$ such that $(I_G^{\mathcal{P}})^m = 0$. Since G/G^0 is a direct product of finite cyclic groups, there is a nontrivial group homomorphism $f: G/G^0 \to \mathbb{T}^1$. Applying Theorem 3.11 for compositions of the quotient $G \to G/G^0$ and f, we get $n > 0$ such that $(\mathfrak{I}_G^{\mathcal{P}})^n \subset (I_G^{\mathcal{P}})^m \mathrm{KK}^G = 0$.

Fot general G, let $\pi: G \to U(n)$ be a faithful representation of G. As is pointed out in the proof of Theorem 3.11 for $T_{U(n)} \le U(n)$ and π (in this case $\mathcal{F}$ is equal to the family of all abelian subgroups $\mathcal{AB}$ of G), Example 3.9 implies that there is $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{I}_G^{\mathcal{AB}})^k = 0$. Now, we get the conclusion by Lemma 3.12 for $\mathcal{Z} \subset \mathcal{AB}$.

Now, the assertion (2) immediately follows from (1) and Lemma 3.12. □

Remark 3.14. Unfortunately, in contrast to Theorem 3.11, $\iota_0^k \in I_G^{\mathcal{F}} \mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ does not hold for general compact Lie groups and families. For example, consider the case that $G = \mathbb{T}^2$ and $\mathcal{F} = \mathcal{T}$. Computing the six-term exact sequence of the equivariant K-homology groups associated to the exact triangle

$$SC\big(S^{2n-1} \times S^{2n-1}\big) \to \mathscr{C}_{S^{2n-1} \times S^{2n-1}} \to \mathbb{C} \to C\big(S^{2n-1} \times S^{2n-1}\big),$$

we obtain

$$\mathrm{KK}^G(\mathscr{C}_{S^{2n-1} \times S^{2n-1}}, \mathbb{C}) \cong R(G) \cdot \iota_0^k$$

(Note that

$$\mathrm{KK}_1^G\big(C(S^{2n-1} \times S^{2n-1}), \mathbb{C}\big) \cong \mathrm{K}_1\big(\mathbb{C}P^n \times \mathbb{C}P^n\big) = 0$$

by Poincaré duality.) By Theorem A.12(3), we obtain

$$\mathrm{KK}^G(\mathsf{N}_{\mathbb{C}}, \mathbb{C}) \cong R(G) \cdot \iota_0^\infty$$

and hence ι_0^∞ is not in $I_G \mathrm{KK}^G(\mathsf{N}_{\mathbb{C}}, \mathbb{C})$.

Instead of Theorem 3.11, the following theorem holds for general compact Lie groups and families.

Theorem 3.15. *Let G be a compact Lie group and let A, B be σ-C^*-algebras such that $\mathrm{KK}_*^G(A, B)$ is finitely generated for $* = 0, 1$. Then the filtrations $(\mathfrak{I}_G^{\mathcal{F}})^*(A, B)$ and $(I_G^{\mathcal{F}})^* \mathrm{KK}^G(A, B)$ are equivalent.*

Note that this is a direct consequence of Lemma 3.7 and Corollary 2.5 of [56] when $\mathrm{KK}_*^H(A, B)$ are finitely generated for any $H \le G$ and $* = 0, 1$.

In order to show Theorem 3.15, we prepare some lemmas.

Lemma 3.16. *Let G be a compact Lie group, let X be a compact G-space and let A, B be σ-$G \ltimes X$-C^*-algebras. We assume that $\mathrm{KK}_*^{G \ltimes X}(A, B)$ are finitely generated for $* = 0, 1$. Then, the following holds:*

(1) *Assume that G satisfies Hodgkin condition and let T be a maximal torus of G. Then $\mathrm{KK}_*^{T \ltimes X}(A, B)$ are finitely generated for $* = 0, 1$.*

(2) *When* $G = \mathbb{T}^n$, $\mathrm{KK}_*^{H \ltimes X}(A, B)$ *are finitely generated for any* $H \leq \mathbb{T}^n$.

(3) *For any cyclic subgroup* H *of* G, *there is a* G*-space* Y *such that* $C(Y)$ *is* $(\mathfrak{J}_G^H)^k$*-injective for some* $k > 0$ *and* $\mathrm{KK}_*^{G \ltimes X}(A, B \otimes C(Y))$ *are finitely generated for* $* = 0, 1$.

Proof. First, (1) follows from the fact that $C(G/T)$ is KK^G-equivalent to $\mathbb{C}^{|W_G|}$ (which is essentially proved in [44, p. 31]). To see (2), first we consider the case that $\mathbb{T}^n/H$ is isomorphic to $\mathbb{T}$. Then, the assertion follows from the six-term exact sequence of the functor $\mathrm{KK}^{\mathbb{T}^n \ltimes X}(A, B \otimes \sqcup)$ associated to the exact triangle

$$SC(\mathbb{T}^1) \to C_0(\mathbb{R}^2) \to \mathbb{C} \to C(\mathbb{T}^1).$$

In general $\mathbb{T}^n/H$ is isomorphic to $\mathbb{T}^m$. By iterating this argument m times, we immediately obtain the conclusion.

Finally we show (3). Since the space of conjugacy classes of G is homeomorphic to the quotient of a finite copies of the maximal torus T of G^0 by a finite group, there is a finite family of class functions separating conjugacy classes of G. A moment of thought will give you a finite faithful family of representations $\{\pi_i : G \to U(n_i)\}$ such that $\{\chi(\pi_i)\}$ separates the conjugacy classes of G. Then, two elements g_1, g_2 in G are conjugate in G if and only if so are in $U := \prod U(n_i)$ (here G is regarded as a subgroup of U by $\prod \pi_i$). Set

$$\mathcal{F} := \left\{ L \leq G \cap gHg^{-1} \mid g \in U \right\}.$$

Then G acts on U/H $\mathcal{F}$-freely and every subgroup in $\mathcal{F}_{\mathrm{cyc}}$ is contained in a conjugate of H. By Corollary 3.13(2), $C(U/H)$ is $(\mathfrak{J}_G^H)^k$-injective for some $k > 0$. Moreover, $\mathrm{KK}_*^G(A, B \otimes C(U/H))$ are finitely generated $R(G)$-modules. To see this, choose a maximal torus T of U containing H. Then U/H is a principal T/H-bundle over U/T and we can apply the same argument as (2). □

Lemma 3.17. *Let* X *be a compact* G*-space and let* $X_1, \dots, X_n$ *be closed* G*-subsets of* X *such that*

$$X_1 \cup \cdots \cup X_n = X.$$

Then, in the category $\sigma\mathfrak{KK}^{G \ltimes X}$, *the filtration associated to the family of ideals*

$$\mathfrak{J}_{X_1, \dots, X_n} := \left\{ \operatorname{Ker} \operatorname{Res}_{G \ltimes X}^{G \ltimes X_i} \right\}$$

is trivial (i.e. there is $k > 0$ *such that* $(\mathfrak{J}_{X_1, \dots, X_n})^k = 0$*).*

Proof. It suffices to show the following: Let X be a compact G-space and X_1, X_2 be a closed G-subspaces such that $X = X_1 \cup X_2$. For separable σ-$G \ltimes X$-C*-algebras A, B, D and $\xi_1 \in \mathrm{KK}^{G \ltimes X}(A, B)$, $\xi_2 \in \mathrm{KK}^{G \ltimes X}(B, D)$ such that

$$\operatorname{Res}_{G \ltimes X}^{G \ltimes X_1} \xi_1 = 0 \quad \text{and} \quad \operatorname{Res}_{G \ltimes X}^{G \ltimes X_2} \xi_2 = 0$$

holds, we have $\xi_2 \circ \xi_1 = 0$.

To see this, we use the Cuntz picture. Let $\mathbb{K}_G := \mathbb{K}(L^2(G)^\infty)$ and let $q_{s,X}A$ be the kernel of the canonical $*$-homomorphism

$$\big((A \otimes \mathbb{K}_G) *_X (A \otimes \mathbb{K}_G)\big) \otimes \mathbb{K}_G \to (A \otimes \mathbb{K}_G) \otimes \mathbb{K}_G$$

for a $G \ltimes X$-C*-algebra A. Then, $\mathrm{KK}^{G\ltimes X}(A, B)$ is isomorphic to the set of homotopy classes of $G \ltimes X$-equivariant $*$-homomorphisms from $q_{s,X}A$ to $q_{s,X}B$ and the Kasparov product is given by the composition.

Let X' be the G-space

$$X_1 \times \{0\} \cup (X_1 \cap X_2) \times [0,1] \cup X_2 \times \{1\} \subset X \times [0,1]$$

and let $p \colon X' \to X$ be the projection. Note that p is a homotopy equivalence. Let $\varphi_1 \colon q_{s,X}A \to q_{s,X}B$ be a $G \ltimes X$-equivariant $*$-homomorphism such that $[\varphi_1] = \xi_1$. By using a homotopy trivializing $\varphi_1|_{X_1}$, we obtain a $G \ltimes X'$-equivariant $*$-homomorphism

$$\varphi_1' \colon q_{s,X'}p^*A \to q_{s,X'}p^*B$$

such that $[\varphi_1'] = \xi_1$ under the isomorphism

$$\mathrm{KK}^{G\ltimes X}(A, B) \cong \mathrm{KK}^{G\ltimes X'}(p^*A, p^*B)$$

and $\varphi_1' = 0$ on $X' \cap X \times [0, 1/2]$. Similarly, we get

$$\varphi_2' \colon p^*q_s B \to p^*q_s D$$

such that $[\varphi_2'] = \xi_2$ and $\varphi_2' = 0$ on $X' \cap X \times [1/2, 1]$. Then,

$$\xi_2 \circ \xi_1 = [\varphi_2' \circ \varphi_1'] = 0. \qquad \square$$

Proof of Theorem 3.15. By Corollary 3.13, it suffices to show the theorem for $\mathcal{F}_{\mathrm{cyc}}$. Hence we may assume that $\mathcal{F} = \mathcal{F}_{\mathrm{cyc}}$ without loss of generality. When $G = \mathbb{T}^n$, the conclusion follows from Lemma 3.16(2) and Corollary 2.5 of [56].

For general G, let U be the Lie group as in the proof of Lemma 3.16(3) and let T be a maximal torus of U. Consider the inclusion

$$\begin{aligned}\mathrm{KK}^G(A, B) &\cong \mathrm{KK}^{U\ltimes U/G}\big(\mathrm{Ind}_G^U A, \mathrm{Ind}_G^U B\big)\\ &\subset KK^{T\ltimes U/G}\big(\mathrm{Ind}_G^U A, \mathrm{Ind}_G^U B\big).\end{aligned}$$

Set $\tilde{\mathcal{F}}$ and $\mathcal{F}'$ the family of G and T respectively given by

$$\begin{aligned}\tilde{\mathcal{F}} &:= \{L \le G \cap gHg^{-1} \mid H \in \mathcal{F}, g \in U\},\\ \mathcal{F}' &:= \{L \le T \cap gHg^{-1} \mid H \in \mathcal{F}, g \in U\}.\end{aligned}$$

Note that Corollary 3.13 implies that the filtration $(\mathfrak{I}_G^{\tilde{\mathcal{F}}})^*$ is equivalent to $(\mathfrak{I}_G^{\mathcal{F}})^*$ since $\mathcal{F}_{\mathrm{cyc}} = \tilde{\mathcal{F}}_{\mathrm{cyc}}$.

Consider the family of homological ideals

$$\mathfrak{I}^{\mathcal{F}'}_{T\ltimes U/G} := \{\operatorname{Ker}\operatorname{Res}^{H\ltimes U/G}_{T\ltimes U/G} \mid H \in \mathcal{F}'\}.$$

We claim that the restriction of the filtration

$$(\mathfrak{I}^{\mathcal{F}'}_{T\ltimes U/G})^*(\operatorname{Ind}^U_G A, \operatorname{Ind}^U_G B)$$

on $\mathrm{KK}^G(A,B)$ is equivalent to $(\mathfrak{I}^{\mathcal{F}}_G)^*(A,B)$.

Pick $L \in \mathcal{F}'$. The slice theorem ([58, Theorem 2.4]) implies that there is a family of closed L-subspaces $X_1,\dots,X_n$ of U/G and $x_i \in X_i$ such that $\bigcup X_i = U/G$ and the inclusions $Lx_i \to X_i$ are L-equivariant homotopy equivalences. Now we have canonical isomorphisms

$$\begin{aligned}\mathrm{KK}^{L\ltimes X_i}(\operatorname{Ind}^U_G A|_{X_i}, \operatorname{Ind}^U_G B|_{X_i}) \xrightarrow{\operatorname{Res}^{Lx_i}_{X_i}} \mathrm{KK}^{L\ltimes Lx_i}(\operatorname{Ind}^U_G A|_{Lx_i}, \operatorname{Ind}^U_G B|_{Lx_i})\\ \to \mathrm{KK}^{gLg^{-1}\cap G}(A,B)\end{aligned}$$

such that

$$\operatorname{Res}^{gLg^{-1}\cap G}_G = \operatorname{Res}^{L\ltimes X_i}_{U\ltimes U/G}$$

under these identifications (here $g \in U$ such that $gL = x_i \in U/L$). Now, we have $gLg^{-1} \cap G \in \tilde{\mathcal{F}}$. Therefore, by Lemma 3.17, we obtain $(\mathfrak{I}^{\tilde{\mathcal{F}}}_G)^k \subset \mathfrak{I}^{\mathcal{F}'}_{T\ltimes U/G}$ for some $k > 0$. Conversely, since $\mathcal{F} = \mathcal{F}_{\mathrm{cyc}}$, for any $L \in \tilde{\mathcal{F}}$, we can take $g \in U$ such that $gLg^{-1} \in \mathcal{F}'$. Hence,

$$\mathrm{KK}^G(A,B) \cap \mathfrak{I}^{\mathcal{F}'}_{T\rtimes U/G}(A,B) \subset \mathfrak{I}^{\tilde{\mathcal{F}}}_G(A,B).$$

Similarly, the filtration $(I^{\mathcal{F}}_G)^*\mathrm{KK}^G(A,B)$ is equivalent to the restriction of $(I^{\mathcal{F}'}_T)^*\mathrm{KK}^{T\ltimes U/G}(\operatorname{Ind}^U_G A, \operatorname{Ind}^U_G B)$. Actually, by Lemma 3.4 of [1], the $I^{\mathcal{F}}_G$-adic and $I^{\mathcal{F}''}_U$-adic topologies on $\mathrm{KK}^G(A,B)$ (here $\mathcal{F}''$ is the smallest family of U containing $\mathcal{F}'$) coincide and so do the $I^{\mathcal{F}'}_U$-adic and $I^{\mathcal{F}'}_T$-adic topologies on

$$\mathrm{KK}^{T\ltimes U/G}(\operatorname{Ind}^U_G A, \operatorname{Ind}^U_G B).$$

Finally, the assertion is reduced to the case of $G = \mathbb{T}^n$. □

Theorem 3.15 can be regarded as a categorical counterpart of the Atiyah–Segal completion theorem. Since Theorem 3.15 holds without assuming that $\mathrm{KK}^H_*(A,B)$ are finitely generated for every $H \le G$, we also obtain a refinement of the Atiyah–Segal theorem ([56, Corollary 2.5]).

Lemma 3.18. *Let A, B be separable σ-G-C^*-algebras such that $\mathrm{KK}^G_*(A,B)$ are finitely generated for $* = 0, 1$. Then there is a pro-isomorphism*

$$\{\mathrm{KK}^G(A,B)/(\mathfrak{I}^{\mathcal{F}}_G)^p(A,B)\}_{p\in\mathbb{Z}_{>0}} \to \{\mathrm{KK}^G(A,\tilde{B}_p)\}_{p\in\mathbb{Z}_{>0}}.$$

Proof. By Lemma 3.16(3), there are compact G-spaces $\{X_k\}_{k\in\mathbb{Z}_{>0}}$ such that

$$\mathrm{KK}_*^G\big(A, B\otimes C(X_k)\big)$$

are finitely generated for $* = 0, 1$, each $C(X_i)$ is $(\mathfrak{I}_G^{\mathcal{F}})^r$-injective for some $r > 0$ and for any $H \in \mathcal{F}$ there are infinitely many X_k's such that $X_k^H \neq \emptyset$. Set

$$\mathsf{N}'_p := B\otimes \bigotimes_{i=1}^{p} \mathcal{C}_{X_i}, \quad \mathsf{I}'_p := \mathsf{N}'_{p-1}\otimes C(X_p), \quad \widetilde{B}'_p := B\otimes C\big(\mathop{\ast}_{i=1}^{p} X_i\big)$$

and

$$\mathsf{N}'_B := \text{ho-}\varprojlim \mathsf{N}'_p, \quad \widetilde{B}' := \text{ho-}\varprojlim \widetilde{B}'_p.$$

By the same argument as Theorem 2.2, we obtain that

$$S\widetilde{B}' \to \mathsf{N}'_B \to B \to \widetilde{B}'$$

is the approximation of B with respect to $(\mathcal{F}\mathcal{C}, \langle \mathcal{F}\mathcal{I}\rangle^{\mathrm{loc}})$. Moreover, by the six-term exact sequence, we obtain that $\mathrm{KK}_*^G(A, \widetilde{B}'_p)$ are finitely generated $R(G)$-modules.

Consider the long exact sequence of projective systems

$$\big\{\mathrm{KK}_*^G(A, S\widetilde{B}'_p)\big\}_p \xrightarrow{\partial_p} \big\{\mathrm{KK}_*^G(A, \mathsf{N}'_p)\big\}_p \xrightarrow{(\iota_0^p)_*} \big\{\mathrm{KK}_*^G(A, B)\big\} \xrightarrow{(\alpha_0^p)_*} \big\{\mathrm{KK}_*^G(A, \widetilde{B}'_p)\big\}_p.$$

Then, $\{\mathrm{Im}(\iota_0^p)_*\}_p = \{\mathrm{Ker}(\alpha_0^p)_*\}_p$ is pro-isomorphic to $(\mathfrak{I}_G^{\mathcal{F}})^*(A, B)$. Actually, for any $p > 0$ there is $r > 0$ such that

$$\big(\mathfrak{I}_G^{\mathcal{F}}\big)^r(A, B) \subset \mathrm{Ker}(\alpha_0^p)_* = \mathrm{Im}(\iota_0^p)_* \subset \big(\mathfrak{I}_G^{\mathcal{F}}\big)^p(A, B)$$

since $\widetilde{B}'_p$ is $(\mathfrak{I}_G^{\mathcal{F}})^r$-injective for some $r > 0$.

Therefore, it suffices to show that the boundary map $\{\partial_p\}$ is pro-zero. Apply Theorem 3.15 and the Artin–Rees lemma for finitely generated $R(G)$-modules

$$M := \mathrm{KK}^G\big(A, \mathsf{N}'_p\big) \quad \text{and} \quad L := \mathrm{Im}\,\partial_p.$$

Since $\widetilde{B}'_p$ is $(\mathfrak{I}_G^{\mathcal{F}})^r$-injective for some $r > 0$, there is $k > 0$ and $l > 0$ such that

$$\mathrm{Im}\big(\iota_p^{p+l}\big)_* \cap L = \big(\mathfrak{I}_G^{\mathcal{F}}\big)^l\big(A, \mathsf{N}'_p\big) \cap L \subset \big(I_G^{\mathcal{F}}\big)^k M \cap L \subset \big(I_G^{\mathcal{F}}\big)^r L = 0.$$

Consequently, for any $p > 0$ there is $l > 0$ such that $\mathrm{Im}\,\iota_p^{p+l} \circ \partial_{p+l} = 0$. □

Remark 3.19. It is also essential for Lemma 3.18 to assume that $\mathrm{KK}_*^G(A, B)$ are finitely generated. Actually, by Theorem 3.11, the pro-isomorphism in Lemma 3.18

implies the completion theorem when $G = \mathbb{T}^1$ and $\mathcal{F} = \mathcal{T}$. On the other hand, since the completion functor is not exact in general, there is a σ-C*-algebra A such that the completion theorem fails for $\mathrm{K}_*^G(A)$. For example, let A be the mapping cone of

$$\bigoplus_{n=0}^{\infty} \lambda^n : \bigoplus^{\infty} \mathbb{C} \to \bigoplus^{\infty} \mathbb{C}.$$

Then, the completion functor for the exact sequence

$$0 \to R(G)^{\infty} \to R(G)^{\infty} \to K_0^G(A) \to 0$$

is not exact in the middle (cf. Example 8 of [50, Chapter 86]).

Lemma 3.20. *Let A, B be separable σ-G-C*-algebras such that $\mathrm{KK}_*^G(A, B)$ are finitely generated for $* = 0, 1$. Then, the ABC spectral sequence for $\mathrm{KK}^G(A, \textvisiblespace)$ and B converges toward $\mathrm{KK}^G(A, B)$ with the filtration $(\mathfrak{I}_G^{\mathcal{F}})^*(A, B)$.*

Proof. According to Lemma 2.4, it suffices to show that

$$i : \mathsf{Bad}^{p+1, p+q+1} \to \mathsf{Bad}^{p, p+q+1}$$

is injective. As is proved in Lemma 3.18, the boundary map ∂_p is pro-zero homomorphism and hence the projective system $\{\mathrm{Ker}\, \iota_0^p\} = \{\mathrm{Im}\, \partial_p\}$ is pro-zero. Therefore, for any $p > 0$ there is a large $q > 0$ such that

$$\mathrm{Ker}\, \iota_0^1 \cap \left(\mathfrak{I}_G^{\mathcal{F}}\right)^{\infty}\left(A, \mathsf{N}_p\right) \subset \mathrm{Ker}\, \iota_0^p \cap \left(\mathfrak{I}_G^{\mathcal{F}}\right)^{q}\left(A, \mathsf{N}_p\right) = \mathrm{Ker}\, \iota_0^p \cap \mathrm{Im}\, \iota_p^{p+q} = 0. \quad \square$$

Theorem 3.21. *Let A and B be separable σ-G-C*-algebras such that $\mathrm{KK}_*^G(A, B)$ are finitely generated $R(G)$-modules ($* = 0, 1$). Then, the morphisms*

- $\mathrm{KK}^G(A, B) \to \mathrm{KK}^G(A, \widetilde{B})$,
- $\mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(E_{\mathcal{F}} G; A, B)$,
- $\mathrm{KK}^G(A, B) \to \sigma\mathfrak{KK}^G/\mathcal{F}\mathfrak{C}(A, B)$,

induce the isomorphism of graded quotients with respect to the filtration $(\mathfrak{I}_G^{\mathcal{F}})^*(A, B)$. In particular, we obtain isomorphisms

$$\mathrm{KK}^G(A, B)^{\wedge}_{I_G^{\mathcal{F}}} \cong \mathrm{KK}^G(A, \widetilde{B}) \cong \mathrm{RKK}^G\big(E_{\mathcal{F}} G; A, B\big) \cong \sigma\mathfrak{KK}^G/\mathcal{F}\mathfrak{C}(A, B).$$

Proof. This is a direct consequence of Lemma 3.18 and Lemma 3.20. Note that Lemma 3.18 implies that the projective system $\{\mathrm{KK}^G(A, \widetilde{B}_p)\}$ satisfies the Mittag-Leffler condition and hence the $\varprojlim^1$-term vanishes. $\square$

Corollary 3.22. *Let A be a separable σ-C*-algebra and let β_t be a homotopy of continuous actions of a compact Lie group G on a σ-C*-algebra B. We write B_t for σ-G-C*-algebras (B, β_t). If $\mathrm{KK}_*^G(A, B_0)$ and $\mathrm{KK}_*^G(A, B_1)$ are finitely generated for $* = 0, 1$, there is an isomorphism*

$$\mathrm{KK}^G(A, B_0)^{\wedge}_{I_G^{\mathcal{T}}} \to \mathrm{KK}^G(A, B_1)^{\wedge}_{I_G^{\mathcal{T}}}.$$

We also weaken the assumption of Theorem 0.1 of Uuye [56], a generalization of McClure's restriction map theorem ([27, Theorem A and Corollary C]) for KK-theory.

Corollary 3.23. *Let G be a compact Lie group and let A and B separable G-C^*-algebras. We assume that $\mathrm{KK}^G_*(A, B)$ are finitely generated for $* = 0, 1$. Then the following hold:*

(1) *If $\mathrm{KK}^H(A, B) = 0$ holds for any finite cyclic subgroup H of G, then $\mathrm{KK}^G(A, B) = 0$.*

(2) *If $\xi \in \mathrm{KK}^G(A, B)$ satisfies $\mathrm{Res}^H_G\, \xi = 0$ for any elementary finite subgroup H of G, then $\xi = 0$.*

Proof. It is proved in Theorem 0.1 of [56] under a stronger assumption that $\mathrm{KK}^H(A, B)$ are finitely generated $R(G)$-modules for any closed subgroup $H \leq G$. Applying Theorem 3.21, the same proof shows the conclusion. □

4. Generalization for groupoids and proper actions

In this section, we generalize the Atiyah–Segal completion theorem for equivariant KK-theory of certain proper topological groupoids. Groupoid equivariant K-theory and KK-theory are studied, for example, in [22] and [54].

First, we recall some conventions on topological groupoids. Let $\mathcal{G} = (\mathcal{G}^1, \mathcal{G}^0, s, r)$ be a second countable locally compact Hausdorff topological groupoid with a haar system. We assume that $\mathcal{G}$ is proper, that is, the combination of the source and the range maps $(s, r): \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0$ is proper. We write $[\mathcal{G}]$ for the orbit space $\mathcal{G}^0/\mathcal{G}$ of $\mathcal{G}$ and $\pi: \mathcal{G}^0 \to [\mathcal{G}]$ for the canonical projection. For a closed subset $S \subset \mathcal{G}^0$, let $\mathcal{G}_S$ denote the full subgroupoid given by

$$\mathcal{G}^1_S := \left\{g \in \mathcal{G}^1 \mid s(g), r(g) \in S\right\} \quad \text{and} \quad \mathcal{G}^0_S := S.$$

Hereafter we deal with proper groupoids satisfying the following two conditions:

For any $x \in \mathcal{G}^0$, there is an open neighborhood U of x, a compact $\mathcal{G}^x_x$-space S_x with a $\mathcal{G}^x_x$-fixed base point x_0 and a groupoid homomorphism $\varphi_x: \mathcal{G}^x_x \ltimes S_x \to \mathcal{G}_{\bar{U}}$ such that (4.1)

- the inclusion $\{x_0\} \to S_x$ is a $\mathcal{G}^x_x$-homotopy equivalence,
- the homomorphism φ_x is injective and a local equivalence ([17, Definition A.4]) such that $\varphi_x(x_0) = x$ and $\varphi_x|_{\mathcal{G}^x_x \ltimes \{x_0\}} = \mathrm{id}_{\mathcal{G}^x_x}$,

The groupoid $\mathcal{G}$ admits a finite dimensional unitary representation whose restriction on $\mathcal{G}^x_x$ is faithful for each $x \in \mathcal{G}^0$. (4.2)

We say that a triple $(\bar{U}, S_x, \varphi_x)$ as in (4.1) is a slice of $\mathcal{G}$ at x.

Example 4.1. The slice theorem for G-CW-complexes ([24, Theorem 7.1; see also Lemma 4.4(ii)]) implies that (4.1) holds for $\mathcal{G}$ such that for any $x \in \mathcal{G}$ there is a saturated neighborhood U of x and a local equivalence $G \ltimes X \to \mathcal{G}_U$ where G are Lie groups and X are G-CW-complexes.

Example 4.2. All proper Lie groupoid satisfies (4.1). Actually, the slice theorem for proper Lie groupoids ([58, Theorem 4.1]) implies that for any orbit $\mathcal{O}$ of $\mathcal{G}$ there isa tubular neighborhood U of $\mathcal{O}$ and a local equivalence $\mathcal{G}_x^x \ltimes N_x\mathcal{O} \to \mathcal{G}_U$ where $x \in \mathcal{O}$ and $N\mathcal{O}$ is the normal bundle of $\mathcal{O}$. On the other hand, a proper Lie groupoid does not satisfies (4.2) in general even if it is an action groupoid. Actually, let G be the group as Section 5 of [23]. Then, the groupoid $\mathcal{G} := G \ltimes \mathbb{R}$ is actually a counterexample. To see this, compare Lemma 4.4(2) below with the fact that

$$\operatorname{Im}\big(R(\mathcal{G}) \to R(\mathcal{G}_x^x) \cong R(T)\big) = R(T/K)$$

(see [23, p. 615]).

Example 4.3. By Lemma 4.4 below and Theorem 6.15 of [16], an action groupoid $G \ltimes X$ satisfies (4.2) if

- G is a closed subgroup of an almost connected group H; or
- G is discrete, X/G has finite covering dimension and all finite subgroups of G have order at most k for some $k \in \mathbb{Z}_{>0}$.

Lemma 4.4. *Let $\mathcal{G}$ be a proper groupoid whose orbit space is compact.*

(1) *If the Hilbert $\mathcal{G}$-bundle $L^2\mathcal{G}$ is AFGP ([55, Definition 5.14]), then $\mathcal{G}$ satisfies* (4.2)*.*

(2) *If $\mathcal{G}$ satisfies* (4.2)*, the representation ring $R(\mathcal{G}_x^x)$ is a noetherian module over $R(\mathcal{G}) := \mathrm{KK}^{\mathcal{G}}(\mathbb{C}, \mathbb{C})$ for any $x \in \mathcal{G}^0$.*

(3) *If $\mathcal{G}$ satisfies* (4.1) *and* (4.2)*, then $R(\mathcal{G})$ is a noetherian ring.*

Proof. First we check (1). Let $(\mathcal{H}_n, \pi_n)$ be an increasing sequence of finite dimensional subrepresentations of $L^2\mathcal{G}$ whose union is dense. For any $x \in \mathcal{G}^0$, there is $n > 0$ such that $\pi_n|_{\mathcal{G}_x^x}$ is faithful. By continuity, there is a saturated neighborhood U of x such that $\pi_n|_{\mathcal{G}_y^y}$ is faithful for any $y \in U$. We obtain the conclusion since $[\mathcal{G}]$ is compact.

To see (2), take an n-dimensional unitary representation $\mathcal{H}$ of $\mathcal{G}$ and let $U(\mathcal{H})$ be the corresponding principal $U(n)$-bundle. Then we have the ring homomorphism

$$R\big(U(n)\big) \to R(\mathcal{G}); \quad [V] \mapsto \big[U(\mathcal{H}) \times_{U(n)} V\big].$$

Now, the composition $R(U(n)) \to R(\mathcal{G}) \to R(\mathcal{G}_x^x)$ is actually induced from a group homomorphism $\mathcal{G}_x^x \to U(n)$ which is injective by assumption. By Proposition 3.2 of [48], $R(\mathcal{G}_x^x)$ is a finitely generated (and hence noetherian) module over $R(U(n))$. Consequently, we obtain that $R(\mathcal{G}_x^x)$ is noetherian as an $R(\mathcal{G})$-module.

If $\mathcal{G}$ satisfies (4.1) in addition, there is an open covering $\{U_i\}$ and $x_i \in U_i$ such that $R(\mathcal{G}_{\bar{U}_i})$ is isomorphic to $R(\mathcal{G}_{x_i}^{x_i})$ and in particular is a noetherian $R(\mathcal{G})$-module. By a Mayer–Vietoris argument, we obtain that $R(\mathcal{G})$ itself is a noetherian $R(\mathcal{G})$-module. □

The induction for groupoid C*-algebras is given in Definition 4.18 of [38]. Let $\mathcal{G}$ be a second countable locally compact groupoid and $\mathcal{H}$ be a subgroupoid. Let (Ω, σ, ρ) be a Hilsum–Skandalis morphism [19] from $\mathcal{G}$ to $\mathcal{H}$ given by

$$\Omega := \{g \in \mathcal{G}^1 \mid s(g) \in \mathcal{H}^0\}, \quad \sigma := s : \Omega \to \mathcal{H}^0, \quad \rho := r : \Omega \to \mathcal{G}^0$$

together with the left $\mathcal{G}$-action and the right $\mathcal{H}$-action given by the composition. The induction functor $\sigma\mathcal{H}\text{-}\mathfrak{C}^*\mathfrak{sep} \to \sigma\mathcal{G}\text{-}\mathfrak{C}^*\mathfrak{sep}$ is given by

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} A = \Omega^* A := \big(C_b(\Omega) \otimes_{\mathcal{H}^0} A\big)^{\mathcal{H}}.$$

In the same way as the case of groups, it induces the functor between Kasparov categories.

Proposition 4.5. *Let $\mathcal{G}$ be a proper groupoid and let $\mathcal{H}$ be a closed subgroupoid. Then, the induction functor $\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}}$ is the right adjoint of the restriction functor $\operatorname{Res}_{\mathcal{G}}^{\mathcal{H}}$, that is,*

$$\mathrm{KK}^{\mathcal{G}}\big(A, \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} B\big) \cong \mathrm{KK}^{\mathcal{H}}\big(\operatorname{Res}_{\mathcal{G}}^{\mathcal{H}} A, B\big).$$

Proof. We have the isomorphism

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} \operatorname{Res}_{\mathcal{G}}^{\mathcal{H}} A = \big(C_b(\Omega) \otimes_{\mathcal{H}^0} A\big)^{\mathcal{H}} \xrightarrow{\cong} C_b(\Omega/\mathcal{H}) \otimes_{\mathcal{G}^0} A; \quad a(\gamma) \mapsto \alpha_{\gamma^{-1}}\big(a(\gamma)\big).$$

Let Δ be the subspace of Ω consisting of all identity morphisms in $\mathcal{H}$. The same argument as Proposition 3.2 we can observe that the following $*$-homomorphisms

$$\varepsilon_A : \operatorname{Res}_{\mathcal{G}}^{\mathcal{H}} \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} A \cong (C(\Omega) \otimes_X A)^{\mathcal{H}} \to A; \quad f \mapsto f|_{\Delta},$$
$$\eta_B : B \to \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} \operatorname{Res}_{\mathcal{G}}^{\mathcal{H}} B \cong C(\Omega/\mathcal{H}) \otimes_X B; \quad a \mapsto a \otimes 1_{\Omega/\mathcal{H}}$$

gives the unit and counit of the adjunctions. □

Now we introduce two generalizations of Theorem 3.21. First we consider a proper groupoid $\mathcal{G}$ satisfying (4.1) and (4.2). For simplicity, we assume that $[\mathcal{G}]$ is connected. Then we have a ring homomorphism $\dim : R(\mathcal{G}) \to \mathbb{Z}$. Set $I_{\mathcal{G}} := \operatorname{Ker} \dim$ be the augmentation ideal. We regard a closed subspace $S \subset \mathcal{G}^0$ as a subgroupoid consisting of all identity morphisms on $x \in S$. We write $\mathfrak{J}_{\mathcal{G}}^S$ for the homological ideal $\operatorname{Ker} \operatorname{Res}_{\mathcal{G}}^S$ of $\sigma\mathfrak{KK}^{\mathcal{G}}$ and in particular set $\mathfrak{J}_{\mathcal{G}} := \mathfrak{J}_{\mathcal{G}}^{\mathcal{G}^0}$. We say that σ-$\mathcal{G}$-C*-algebras of the form $A = \operatorname{Ind}_{\mathcal{G}^0}^{\mathcal{G}} A_0$ is trivially induced and we write $\mathcal{TI}$ for the class of trivially induced objects. Similarly, we say that σ-$\mathcal{G}$-C*-algebras B such that $\operatorname{Res}_{\mathcal{G}}^{\mathcal{G}^0} B$ is $\mathrm{KK}^{\mathcal{G}^0}$-contractible is trivially contractible and we write $\mathcal{TC}$ for the class of trivially contractible objects.

Lemma 4.6. *Let $(\bar{U}, S, \varphi)$ be a slice of $\mathcal{G}$ at $x \in \mathcal{G}^0$ and let V be the smallest saturated closed subspace of $\mathcal{G}^0$ containing $\varphi(S)$.*

(1) *Let A be a σ-$\mathcal{G}$-C*-algebra. If $\mathrm{Res}^S_{\mathcal{G}} A$ is KK^S-contractible, then $\mathrm{Res}^V_{\mathcal{G}} A$ is KK^V-contractible.*

(2) *If V is compact, the filtrations $\mathfrak{I}^*_{\mathcal{G}_S}$ and $\mathfrak{I}^*_{\mathcal{G}_V}$ are equivalent under the isomorphism $\sigma\mathfrak{KK}^{\mathcal{G}_S} \cong \sigma\mathfrak{KK}^{\mathcal{G}_V}$.*

Proof. Since the homomorphism $\varphi: \mathcal{G}^x_x \ltimes S \to \mathcal{G}$ is a local equivalence, for any $y \in \mathcal{G}^0_V$ we have a closed subspace W of $\mathcal{G}^0_V$ containing y in its interior and a continuous map $f: W \to \mathcal{G}^1$ such that $s \circ f = \mathrm{id}$ and $r \circ f(W) \subset S$, which induces a group homomorphism

$$\{\mathrm{Ad}\, f(w)\}_{w \in W}: \mathrm{KK}^S\big(\mathrm{Res}^S_{\mathcal{G}} A, \mathrm{Res}^S_{\mathcal{G}} B\big) \to \mathrm{KK}^W\big(\mathrm{Res}^W_{\mathcal{G}} A, \mathrm{Res}^W_{\mathcal{G}} B\big).$$

Since $\mathrm{Res}^W_{\mathcal{G}_V} = \mathrm{Ad}\, f(u) \circ \mathrm{Res}^S_{\mathcal{G}_V}$, we obtain $\mathfrak{I}^W_{\mathcal{G}_V} \subset \mathfrak{I}^S_{\mathcal{G}_V}$.

In particular, if $\mathrm{Res}^S_{\mathcal{G}} A$ is KK^S-contractible, then $\mathrm{Res}^W_{\mathcal{G}} A$ is KK^W-contractible. We obtain (1) because any locally contractible X-C*-algebra is globally contractible (which follows from a Mayer–Vietoris argument).

To see (2), let $\{W_i\}$ be a finite family of closed subspaces of $\mathcal{G}^0_V$ obtained as above such that $\bigcup W_i = \mathcal{G}^0_V$. Then, in the same way as Lemma 3.17, we obtain

$$(\mathfrak{I}^S_{\mathcal{G}_V})^n \subset \mathfrak{I}^{W_1}_{\mathcal{G}_V} \circ \cdots \circ \mathfrak{I}^{W_n}_{\mathcal{G}_V} \subset \mathfrak{I}^V_{\mathcal{G}_V}. \qquad \square$$

Consider the following assumption for a pair (A, B) of σ-$\mathcal{G}$-C*-algebras corresponding to the assumption that $\mathrm{KK}^G_*(A, B)$ are finitely generated $R(G)$-modules in Theorem 3.21:

There is a basis $\{U_i\}$ of the topology of $\mathcal{G}$ such that $R(\mathcal{G})$-modules

$$\mathrm{KK}^{\mathcal{G}_{\bar{U}_i}}_*\big(\mathrm{Res}^{\mathcal{G}_{\bar{U}_i}}_{\mathcal{G}}, \mathrm{Res}^{\mathcal{G}_{\bar{U}_i}}_{\mathcal{G}} B\big) \tag{4.3}$$

are finitely generated.

Theorem 4.7. *Let $\mathcal{G}$ be a proper groupoid satisfying (4.1) and (4.2) whose orbit space is compact. Then the following holds:*

(1) *A pair $(\mathcal{TC}, \langle \mathcal{TI} \rangle^{\mathrm{loc}})$ is complementary in $\sigma\mathfrak{KK}^{\mathcal{G}}$.*

(2) *For any pair of σ-G-C*-algebras (A, B) satisfying (4.3), there are isomorphisms of $R(\mathcal{G})$-modules*

$$\mathrm{KK}^{\mathcal{G}}(A, B)^{\wedge}_{I_{\mathcal{G}}} \cong \mathrm{KK}^{\mathcal{G}}(A, \widetilde{B}) \cong \mathrm{RKK}^{\mathcal{G}}\big(E\mathcal{G}; A, B\big) \cong \sigma\mathfrak{KK}^{\mathcal{G}}/\mathcal{TC}(A, B).$$

Proof. Assertion (1) can be shown in the same way as Theorem 3.4.

To see (2), take slices $\{(X_i, S_i, \varphi_i)\}_{i\in I}$ such that

$$\mathrm{KK}_*^{\mathcal{G}_{X_i}}\big(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_{X_i}} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_{X_i}} B\big)$$

are finitely generated and $\bigcup \pi(X_i) = [\mathcal{G}]$. Consider the groupoid

$$\widetilde{\mathcal{G}}^0 := \bigsqcup S_i, \quad \widetilde{\mathcal{G}}^1 := \big\{(g,i,j) \in \mathcal{G} \times I \times I \mid s(g) \in \varphi_i(S_i),\ r(g) \in \varphi_j(S_j)\big\}$$

with

$$s(g,i,j) = s(g) \in S_i, \quad r(g,i,j) = r(g) \in S_j,$$

and

$$(h,j,k) \circ (g,i,j) = (g \circ h, i, k).$$

Then, $\widetilde{\mathcal{G}}$ is Morita equivalent to $\mathcal{G}$ and we have the family of closed full subgroupoids

$$\big\{\mathcal{G}_i := \mathcal{G}|_{\pi^{-1}(\pi(S_i))}\big\}_{i\in I}$$

such that $\widetilde{\mathcal{G}} = \bigcup \mathcal{G}_i$ and the pair $(A|_{\mathcal{G}_i^0}, B|_{\mathcal{G}_i^0})$ of σ-$\mathcal{G}_i$-C*-algebras satisfies (2).

Let $\mathcal{H}$ be a proper groupoid which admits a local equivalence $\varphi\colon G \ltimes X \to \mathcal{H}$ where G is a compact Lie group and X is a compact G-CW-complex (such as $\mathcal{G}_i$ or $\mathcal{G}_i \cap \mathcal{G}_j$). Then, by Lemma 4.6, $I_{\mathcal{H}}$-adic topology and I_G-adic topology on

$$\mathrm{KK}^{\mathcal{H}}(A,B) \cong \mathrm{KK}^{G\ltimes X}\big(\varphi^*A, \varphi^*B\big)$$

coincide. Moreover φ^* preserves $\mathcal{TC}$ and $\langle \mathcal{TI}\rangle^{\mathrm{loc}}$. Hence, (2) holds for $\mathcal{H}$ by Theorem 3.21.

By Lemma 3.4 of [1] and the proof of Lemma 4.4, $I_{\mathcal{G}}$-adic and $I_{\mathcal{G}_i}$-adic topologies coincide on $\mathrm{KK}^{\mathcal{G}_i}(A|_{\mathcal{G}_i^0}, B|_{\mathcal{G}_i^0})$. Moreover, $\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_i}$ preserves $\mathcal{TC}$ and $\langle \mathcal{TI}\rangle^{\mathrm{loc}}$. Finally we obtain (2) for $\widetilde{\mathcal{G}}$ by using the Mayer–Vietoris exact sequence

$$\begin{array}{ccccccc}
\cdots \to & \mathrm{KK}^{\mathcal{G}}(A,B)^{\wedge}_{I_{\mathcal{G}}} & \to & \begin{matrix}\mathrm{KK}^{\mathcal{G}_1}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_1} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_1} B)^{\wedge}_{I_{\mathcal{G}}} \\ \oplus \\ \mathrm{KK}^{\mathcal{G}_2}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_2} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_2} B)^{\wedge}_{I_{\mathcal{G}}}\end{matrix} & \to & \mathrm{KK}^{\mathcal{G}_0}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_0} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_0} B)^{\wedge}_{I_{\mathcal{G}}} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{KK}^{\mathcal{G}}(A,\widetilde{B}) & \to & \begin{matrix}\mathrm{KK}^{\mathcal{G}_1}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_1} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_1} \widetilde{B}) \\ \oplus \\ \mathrm{KK}^{\mathcal{G}_2}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_2}, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_2} \widetilde{B})\end{matrix} & \to & \mathrm{KK}^{\mathcal{G}_0}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_0} A, \mathrm{Res}_{\mathcal{G}}^{\mathcal{G}_0} \widetilde{B}) & \to \cdots
\end{array}$$

(for $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, $\mathcal{G}_0 := \mathcal{G}_1 \cap \mathcal{G}_2$) and the five lemma recursively. Note that the first row is exact because the completion functor is exact when modules are finitely generated. Since the augmentation ideal $I_{\mathcal{G}}$ and the complementary pair $(\mathcal{TC}, \langle \mathcal{TI}\rangle^{\mathrm{loc}})$ are preserved under Morita equivalence, we obtain the consequence. □

Second generalization is the Atiyah–Segal completion theorem for proper actions. Let G be one of:

- a countable discrete group such that and all finite subgroups of G have order at most k for some $k \in \mathbb{Z}_{>0}$ and has a model of the universal proper G-space $E_{\mathcal{C}}G$ which is G-compact and finite covering dimension; or
- a cocompact subgroup of an almost connected second countable group,

and let $\mathcal{F}$ be a family of G consisting of compact subgroups. Set $\mathcal{G} := G \ltimes E_{\mathcal{C}}G$. According to Section 7 of [31], the category $\sigma\mathfrak{KK}^{\mathcal{G}}$ is identified with the subcategory $\langle \mathcal{CI} \rangle_{\mathrm{loc}}$ of $\sigma\mathfrak{KK}^G$ by the natural isomorphism

$$p^*_{E_{\mathcal{C}}G} \colon \mathrm{KK}^G(A,B) \xrightarrow{\cong} \mathrm{KK}^{G \ltimes E_{\mathcal{C}}G}\big(A \otimes C(E_{\mathcal{C}}G), B \otimes E_{\mathcal{C}}G\big)$$

since G has a Dirac element coming from a proper σ-G-C*-algebra when G is discrete ([53, Theorem 2.1]) or a closed subgroup of an almost connected second countable group H ([21, Theorem 4.8]).

Theorem 4.8. *Let G and $\mathcal{F}$ be as above. Then, the following holds:*

(1) *A pair $(\mathcal{FC}, \langle \mathcal{FI} \rangle^{\mathrm{loc}})$ is complementary in $\langle \mathcal{CI} \rangle_{\mathrm{loc}} \subset \sigma\mathfrak{KK}^G$.*

(2) *For any pair of proper σ-G-C*-algebras A, B such that $\mathrm{KK}^H_*(A,B)$ are finitely generated for any compact subgroup H of G, there are isomorphisms of $R(\mathcal{G})$-modules*

$$\mathrm{KK}^G(A,B)^{\wedge}_{I^{\mathcal{F}}_{\mathcal{G}}} \cong \mathrm{KK}^G(A,\widetilde{B}) \cong \mathrm{RKK}^G\big(E_{\mathcal{F}}\mathcal{G}; A, B\big) \cong \sigma\mathfrak{KK}^G/\mathcal{FC}(A,B).$$

Proof. The proof is given in the same way as Theorem 4.7. Note that $\mathfrak{I}^H_{\mathcal{G}} = \mathfrak{I}^{H \ltimes X}_{\mathcal{G}}$ for any H-subspace X of $E_{\mathcal{C}}G$ (even if X is not compact) since the composition

$$\sigma\mathfrak{KK}^{H \ltimes E_{\mathcal{C}}G} \to \sigma\mathfrak{KK}^{H \ltimes X} \to \sigma\mathfrak{KK}^H$$

is identity. □

5. The Baum–Connes conjecture for group extensions

In this section we apply Corollary 3.13 for the study of the complementary pair $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$ of the Kasparov category $\sigma\mathfrak{KK}^G$ when G is a Lie group. As a consequence, we refine the theory of Chabert, Echterhoff and Oyono-Oyono [9,10,35] on permanence property of the Baum–Connes conjecture under extensions of groups.

Let G be a second countable locally compact group such that any compact subgroup of G is a Lie group. We bear the case that G is a real Lie group in mind. We write $\mathcal{C}$ and $\mathcal{CZ}$ for the family of compact and compact cyclic subgroups of G, respectively.

Corollary 5.1. *We have* $\mathcal{CC} = \mathcal{CZC}$ *and* $\langle \mathcal{CI} \rangle_{\mathrm{loc}} = \langle \mathcal{CZI} \rangle_{\mathrm{loc}}$.

Proof. Since $\mathcal{CZ} \subset \mathcal{C}$, we have $\mathcal{CZI} \subset \mathcal{CI}$ and $\mathcal{CC} \subset \mathcal{CZC}$. Hence it suffices to show $\mathcal{CC} = \mathcal{CZC}$, which immediately follows from Corollary 3.13(2). □

Corollary 5.2 (cf. [26, Theorem 1.1]). *The canonical map* $f \colon E_{\mathcal{CZ}}G \to E_{\mathcal{C}}G$ *induces the* KK^G*-equivalence* $f^* \colon C(E_{\mathcal{CZ}}G) \to C(E_{\mathcal{C}}G)$.

Note that the topological K-homology group $\mathrm{K}_*^{\mathrm{top}}(G; A)$ is isomorphic to the KK-group $\mathrm{KK}^G(C(E_{\mathcal{C}}G), A)$ of σ-C*-algebras for any G-C*-algebra A.

Proof. Since f is a T-equivariant homotopy equivalence between $E_{\mathcal{C}}G$ and $E_{\mathcal{CZ}}G$ for any $T \in \mathcal{CZ}$, f^* is an equivalence in $\sigma\mathfrak{KK}^G/\mathcal{CZC}$. The conclusion follows from Corollary 5.1 because $C(E_{\mathcal{CZ}}G)$ and $C(E_{\mathcal{C}}G)$ are in $\langle \mathcal{CI} \rangle_{\mathrm{loc}} = \langle \mathcal{CZI} \rangle_{\mathrm{loc}}$. □

Next we review the Baum–Connes conjecture for extensions of groups. Let

$$1 \to N \to G \to G/N \to 1$$

be an extension of second countable locally compact groups. We assume that any compact subgroup of G/N is a Lie group. As in Subsection 5.2 of [15], we say that a subgroup H of G is N*-compact* if $\pi(H)$ is compact in G/N. We write $\mathcal{C}_N$ for the family of N-compact subgroups of G. Then, we have the complementary pair $(\langle \mathcal{C}_N \mathcal{I} \rangle_{\mathrm{loc}}, \mathcal{C}_N\mathcal{C})$. It is checked as following. First, in the same way as Lemma 3.3 of [31], for a large compact subgroup H of G/N we have

$$\mathrm{KK}^G\big(\mathrm{Ind}_{\tilde{H}}^G A, B\big) \cong \mathrm{KK}^{\tilde{H}}\big(\mathrm{Res}_{\tilde{U}_H}^{\tilde{H}} \mathrm{Ind}_{\tilde{H}}^{\tilde{U}_H} A, \mathrm{Res}_G^{\tilde{H}} B\big),$$

where $\tilde{H} := \pi^{-1}(H)$ for any $H \le G/N$ and U_H is as Section 3 of [31]. Hence, $\mathrm{KK}^G(Q, M) = 0$ for any $Q \in \mathcal{C}_N\mathcal{I}$ and $M \in \mathcal{C}_N\mathcal{C}$. Let

$$S\mathsf{M} \to \mathsf{Q} \to \mathbb{C} \to \mathsf{M}$$

be the approximation exact triangle of $\mathbb{C}$ in $\sigma\mathfrak{KK}^{G/N}$ with respect to $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$. Since the functor $\pi^* \colon \sigma\mathfrak{KK}^{G/N} \to \sigma\mathfrak{KK}^G$ maps $\mathcal{CI}$ to $\mathcal{C}_N\mathcal{I}$ and $\mathcal{CC}$ to $\mathcal{C}_N\mathcal{C}$ respectively,

$$S\pi^*\mathsf{M} \to \pi^*\mathsf{Q} \to \mathbb{C} \to \pi^*\mathsf{M}$$

gives the approximation of $\mathbb{C}$ in $\sigma\mathfrak{KK}^G$ with respect to $(\langle \mathcal{C}_N\mathcal{I} \rangle_{\mathrm{loc}}, \mathcal{C}_N\mathcal{C})$. Hereafter, for simplicity of notations we omit π^* for σ-(G/N)-C*-algebras which are regarded as σ-G-C*-algebras.

Since $\mathcal{CI} \subset \mathcal{C}_N\mathcal{I}$ and $\mathcal{C}_N\mathcal{C} \subset \mathcal{CC}$, we obtain the diagram of semi-orthogonal decompositions

$$
\begin{array}{ccccc}
\langle \mathcal{CI} \rangle_{\mathrm{loc}} & = & \langle \mathcal{CI} \rangle_{\mathrm{loc}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\langle \mathcal{C}_N\mathcal{I} \rangle_{\mathrm{loc}} & \to & \mathfrak{KK}^G & \to & \mathcal{C}_N\mathcal{C} \\
\downarrow & & \downarrow & & \| \\
\langle \mathcal{C}_N\mathcal{I} \rangle_{\mathrm{loc}} \cap \mathcal{CC} & \to & \mathcal{CC} & \to & \mathcal{C}_N\mathcal{C},
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{P} & = & \mathsf{P} & \to & 0 \\
\downarrow{\scriptstyle \mathsf{D}_G^{G/N}} & & \downarrow{\scriptstyle \mathsf{D}_G} & & \downarrow \\
\mathsf{Q} & \xrightarrow{\mathsf{D}_{G/N}} & \mathbb{C} & \to & \mathsf{M} \\
\downarrow & & \downarrow & & \| \\
\mathsf{Q} \otimes \mathsf{N} & \to & \mathsf{N} & \to & \mathsf{M}.
\end{array}
\tag{5.1}
$$

For a σ-G-C*-algebra A, the (full or reduced) crossed product $N \ltimes A$ is a twisted σ-G/N-C*-algebra ([43, Definition 2.1]). By the Packer–Raeburn stabilization trick ([14, Theorem 1]), it is Morita equivalent to the untwisted G/N-C*-algebra

$$N \ltimes^{\mathrm{PR}} A := C_0\big(G/N, N \ltimes A\big) \rtimes_{\tilde{\alpha},\tilde{\tau}} (G/N),$$

where $\tilde{\alpha}$ and $\tilde{\tau}$ are induced from the canonical G-action on $C_0(G/N, N \ltimes A)$. The *Packer–Raeburn crossed product* $N \ltimes^{\mathrm{PR}} \sqcup$ is a functor from G-$\mathfrak{C}^*\mathfrak{sep}$ to G/N-$\mathfrak{C}^*\mathfrak{sep}$, which induces the *partial descent functor* ([10, Section 4])

$$j_G^{G/N} : \sigma\mathfrak{KK}^G \to \sigma\mathfrak{KK}^{G/N}$$

by universality of $\sigma\mathfrak{KK}^G$ (Theorem A.15).

Lemma 5.3. *The functor* $j_G^{G/N}$ *maps* $\langle \mathcal{C}_N\mathcal{I} \rangle_{\mathrm{loc}}$ *to* $\langle \mathcal{CI} \rangle_{\mathrm{loc}}$ *and* $\mathcal{C}_N\mathcal{C}$ *to* $\mathcal{CC}$.

Proof. Let H be a N-compact subgroup of G and let A be a σ-H-C*-algebra. Then, $N \ltimes^{\mathrm{PR}} \mathrm{Ind}_H^G A$ admits a canonical σ-$G/N \ltimes ((G/N \times H\backslash G)/G)$-C*-algebra structure. Since the G/N-action on $(G/N \times H\backslash G)/G$ is proper, $N \ltimes^{\mathrm{PR}} \mathrm{Ind}_H^G A$ is in $\langle \mathcal{CI} \rangle_{\mathrm{loc}}$. Consequently we obtain

$$j_G^Q(\langle \mathcal{C}_N\mathcal{I} \rangle_{\mathrm{loc}}) \subset \langle \mathcal{CI} \rangle_{\mathrm{loc}}.$$

Let A be a $\mathcal{C}_N$-contractible σ-C*-algebra. Then, for any compact subgroup H of G/N,

$$\mathrm{Res}_{G/N}^H \big(N \ltimes^{\mathrm{PR}} A\big) = N \ltimes \mathrm{Res}_G^{\pi^{-1}(K)} A$$

is KK^H-contractible. Hence we obtain $j_G^{G/N}(\mathcal{C}_N\mathcal{C}) \subset \mathcal{CC}$. □

Consider the partial assembly map

$$\mu_{G,A}^{G/N} : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*^{\mathrm{top}}\big(G/N; N \ltimes A\big)$$

constructed in Definition 5.14 of [9]. Then, in the same way as Theorem 5.2 of [30], we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{K}_*^{\mathrm{top}}(G;\mathsf{P}\otimes A) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G;\mathsf{Q}\otimes A) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G;A) \\
\downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle \mu_{G,A}^{G/N}} \\
\mathrm{K}_*^{\mathrm{top}}(G/N;N\ltimes^{\mathrm{PR}}(\mathsf{P}\otimes A)) & \longrightarrow & \mathrm{K}_*^{\mathrm{top}}(Q;N\ltimes^{\mathrm{PR}}(\mathsf{Q}\otimes A)) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G/N;N\ltimes^{\mathrm{PR}}A) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle \mu_{G/N,N\ltimes^{\mathrm{PR}}A}} \\
\mathrm{K}_*(G\ltimes(\mathsf{P}\otimes A)) & \xrightarrow{j_G(\mathsf{D}_G^{G/N})} & \mathrm{K}_*(G\ltimes(\mathsf{Q}\otimes A)) & \xrightarrow{j_G(\mathsf{D}_{G/N})} & \mathrm{K}_*(G\ltimes A)
\end{array}$$

and hence the composition of partial assembly maps

$$\mu_{G,A}=\mu_{G/N,N\ltimes^{\mathrm{PR}}A}\circ\mu_{G,A}^{G/N}\colon \mathrm{K}_*^{\mathrm{top}}(G;A)\to\mathrm{K}_*^{\mathrm{top}}\big(G/N;N\ltimes^{\mathrm{PR}}A\big)\to\mathrm{K}_*(G\ltimes A)$$

is isomorphic to the canonical map

$$\mathrm{K}_*\big(G\ltimes(\mathsf{P}\otimes A)\big)\to\mathrm{K}_*\big(G\ltimes(\mathsf{Q}\otimes A)\big)\to\mathrm{K}_*(G\ltimes A).$$

In other words, the partial assembly map $\mu_{G,A}^{G/N}$ is isomorphic to the assembly map $\mu_{G,\mathsf{Q}\otimes A}$ for $\mathsf{Q}\otimes A$.

We say that a separable σ-G-C*-algebra A satisfies the (resp., strong) Baum–Connes conjecture (BCC) if $j_G(\mathsf{D}_G)$ induces the isomorphism of K-groups (resp., the KK-equivalence).

Theorem 5.4. *Let $1\to N\to G\to G/N\to 1$ be an extension of second countable groups such that all compact subgroups of G/N are Lie groups and let A be a separable σ-G-C*-algebra. Then the following hold:*

(1) *If $\pi^{-1}(H)$ satisfies the (resp., strong) BCC for A for any $H\in\mathcal{CZ}$, then G satisfies the (resp., strong) BCC for A if and only if G/N satisfies the (resp., strong) BCC for $N\ltimes_r^{\mathrm{PR}}A$.*

(2) *If $\pi^{-1}(H)$ for any $H\in\mathcal{CZ}$ and G/N have the γ-element, then so does G. Moreover, in that case $\gamma_{\pi^{-1}(H)}=1$ for any $H\in\mathcal{CZ}$ and $\gamma_{G/N}=1$ if and only if $\gamma_G=1$.*

Proof. To see (1), it suffices to show that G satisfies the (resp., strong) BCC for $\mathsf{Q}\otimes A$. Consider the full subcategory $\mathfrak{N}$ of $\sigma\mathfrak{KK}^G$ consisting of objects D such that G satisfies the (resp., strong) BCC for $D\otimes A$. Set $\mathcal{CZI}_1$ be the family of all G-C*-algebras of the form $C_0((G/N)/H)$ for $H\in\mathcal{CZ}$. By assumption, $\mathfrak{N}$ contains $\pi^*\mathcal{CZI}_1$. Since $\mathfrak{N}$ is localizing and colocalizing, $\mathfrak{N}$ contains $\pi^*\langle\mathcal{CZI}_1\rangle_{\mathrm{loc}}^{\mathrm{loc}}$, which is equal to $\pi^*\langle\mathcal{CI}_1\rangle_{\mathrm{loc}}^{\mathrm{loc}}$ because $C_0(G/N)/H)$ are KK^G-equivalent to

$$C_0\big((G/N)/H\big)\otimes C\big(E_{\mathcal{CZ}}H\big)\in\pi^*\langle\mathcal{CZI}_1\rangle^{\mathrm{loc}}.$$

By Proposition 9.2 of [31], we obtain $\mathsf{Q}\in\mathfrak{N}$.

Assertion (2) is proved in the same way as Theorem 33 of [15]. Actually, since we may assume without loss of generality that G/N is totally disconnected by Corollary 34 of [15], the homomorphism

$$\mathsf{D}_G^*\colon \mathrm{KK}^G(A,\mathsf{P}) \to \mathrm{KK}^G(\mathsf{P}\otimes A,\mathsf{P})$$

is an isomorphism if $A \in \pi^*\langle\mathcal{CZI}\rangle_{\mathrm{loc}}$ and in particular when $A = \mathsf{Q}$ (note that any compact subgroup is contained in an open compact subgroup which is also a Lie group by assumption). Consequently we obtain a left inverse

$$\eta_G^{G/N}\colon \mathsf{Q} \to \mathsf{P}$$

of $\mathsf{D}_G^{G/N}$. Now, the composition

$$\eta_G^{G/N} \circ \pi^*\eta_{G/N}\colon \mathbb{C} \to \mathsf{P}$$

is a dual Dirac morphism of G. Of course, $\eta_G \circ \mathsf{D}_G = \mathrm{id}_{\mathbb{C}}$ if $\eta_G^{G/N} \circ \mathsf{D}_G^{G/N} = \mathrm{id}_{\mathsf{Q}}$ and $\eta_{G/N} \circ \mathsf{D}_{G/N} = \mathrm{id}_{\mathbb{C}}$. □

A. Equivariant KK-theory for σ-C*-algebras

In this appendix we summarize basic properties of equivariant KK-theory for σ-C^*-algebras for the convenience of readers. Most of them are obvious generalizations of equivariant KK-theory for C*-algebras (a basic reference is [6]) and non-equivariant KK-theory for σ-C*-algebras by Bonkat [7]. Throughout this section we assume that G is a second countable locally compact topological group.

A.1. Generalized operator algebras and Hilbert C*-modules. Topological properties of inverse limits of C*-algebras was studied by Phillips in [39–42]. He introduced the notion of representable K-theory for σ-C*-algebras in order to formulate the Atiyah–Segal completion theorem for C*-algebras.

Definition A.1. A *pro-G-C*-algebra* is a complete locally convex $*$-algebra with continuous G-action whose topology is determined by its G-invariant continuous C*-seminorms. A pro-G-C*-algebra is a *σ-G-C*-algebra* if its topology is generated by countably many G-invariant C*-seminorms.

In other words, a pro-G-C*-algebra is a projective limit of G-C*-algebras. Actually, a pro-G-C*-algebra A is isomorphic to $\varprojlim_{p\in\mathcal{S}(A)} A_p$, where $\mathcal{S}(A)$ is the net of G-invariant continuous seminorms and

$$A_p := A/\{x \in A \mid p(x^*x) = 0\}$$

is the completion of A by the seminorm $p \in \mathcal{S}(A)$. A pro-G-C*-algebra is *separable* if A_p are separable for any $p \in \mathcal{S}(A)$. If A is a separable σ-G-C*-algebra, then it is separable as a topological space. Basic operations (full and reduced tensor products, free products and crossed products) are also well-defined for pro-C*-algebras. When G is compact, any σ-C*-algebras with continuous G-action are actually σ-G-C*-algebras.

We write $\sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$ for the category of separable σ-G-C*-algebras and equivariant $*$-homomorphisms. Then we have the category equivalence

$$\varprojlim\colon \mathrm{Pro}_{\mathbb{Z}_{>0}} G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \sigma G\text{-}\mathfrak{C}^*\mathfrak{sep},$$

where $\mathrm{Pro}_{\mathbb{Z}_{>0}} G\text{-}\mathfrak{C}^*\mathfrak{sep}$ is the category of surjective projective systems of separable G-C*-algebras indexed by $\mathbb{Z}_{>0}$ with the morphism set

$$\mathrm{Hom}\big(\{A_n\},\{B_m\}\big) := \varprojlim_n \varinjlim_m \mathrm{Hom}(A_n, B_m).$$

Actually, a $*$-homomorphism $\varphi\colon A \to B$ induces a morphism between projective systems since each composition $A \xrightarrow{\varphi} B \to B_p$ factors through some A_q.

Next we introduce the notion of Hilbert module over pro-C*-algebras.

Definition A.2. A *G-equivariant pre-Hilbert B-module* is a locally convex B-module together with the B-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to B$ and the continuous G-action such that

$$\langle e_1, e_2 b\rangle = \langle e_1, e_2\rangle\, b, \quad \langle e_1, e_2\rangle^* = \langle e_2, e_1\rangle\,,$$
$$g(\langle e_1, e_2\rangle) = \langle g(e_1), g(e_2)\rangle\,, \quad g(eb) = g(e)g(b),$$

and the topology of E is induced by seminorms $p_E(e) := p(\langle e, e\rangle)^{1/2}$ for $p \in \mathcal{S}(B)$. A G-equivariant pre-Hilbert B-module is a *G-equivariant Hilbert B-module* if it is complete with respect to these seminorms.

Basic operations (direct sums, interior and exterior tensor products and crossed products) are also well-defined (see [46, Section 1]).

As a locally convex space, E is isomorphic to the projective limit $\varprojlim_{p\in\mathcal{S}(B)} E_p$ where

$$E_p := E/\big\{e \in E \mid p(\langle e, e\rangle) = 0\big\}.$$

A G-equivariant Hilbert B-module E is *countably generated* if E_p are countably generated for any $p \in \mathcal{S}(B)$.

Let $\mathbb{L}(E)$ and $\mathbb{K}(E)$ be the algebra of adjointable bounded and compact operators on E respectively. They are actually pro-G-C*-algebras since we have isomorphisms

$$\mathbb{L}(E) \cong \varprojlim_{p\in\mathcal{S}(B)} \mathbb{L}(E_p), \quad \mathbb{K}(E) \cong \varprojlim_{p\in\mathcal{S}(B)} \mathbb{K}(E_p).$$

In particular, $\mathbb{L}(E)$ and $\mathbb{K}(E)$ are σ-G-C*-algebra if so is B. Note that $\mathbb{L}(E)$ is not separable and the canonical G-action on $\mathbb{L}(E)$ is not continuous in norm topology.

Kasparov's stabilization theorem is originally introduced in [20] and generalized by Mingo–Phillips [33] and Meyer [28] for equivariant cases. Bonkat [7] also gives a generalization for σ-C^*-algebras. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and we write $\mathcal{H}_B$, $\mathcal{H}_{G,B}$ and $\mathbb{K}_G$ for $\mathcal{H} \otimes B$, $\mathcal{H} \otimes L^2(G) \otimes B$, and $\mathbb{K}(L^2 G \otimes \mathcal{H})$, respectively.

Theorem A.3. *Let B be a σ-unital σ-G-C*-algebra and let E be a countably generated G-equivariant Hilbert B-module together with an essential G-equivariant $*$-homomorphism $\varphi\colon \mathbb{K}_G \to \mathbb{L}(E)$. Then there is an isomorphism*

$$E \oplus \mathcal{H}_{G,B} \cong \mathcal{H}_{G,B}$$

as G-equivariant Hilbert B-modules.

Proof. In the non-equivariant case, the proof is given in Section 1.3 of [7]. In fact, we have a sequence $\{e^i\}$ in E such that $\sup_n \|e^i_n\| \le 1$ and $\{\pi(e^i)\}$ generates E_p for any $p \in \mathcal{S}(B)$ since the projection $(E_p)_1 \to (E_q)_1$ between unit balls is surjective for any $p \ge q$. Now we obtain the desired unitary U as the unitary factor in the polar decomposition of the compact operator

$$T\colon \mathcal{H}_B \to E \oplus \mathcal{H}_B; \quad T(\xi^i) = 2^{-i} e^i \oplus 4^{-i} \xi^i,$$

where $\{\xi^i\}$ is a basis of $\mathcal{H}_B$. Actually the range of $|T|$ is dense because

$$T^* T = \operatorname{diag}\left(4^{-2}, 4^{-4}, \dots\right) + \left(2^{-i-j} \langle e_i, e_j \rangle\right)_{ij}$$

is strictly positive.

In the equivariant case, we identify E with $L^2(G, \mathcal{H}) \otimes (L^2(G, \mathcal{H})^* \otimes_{\mathbb{K}_G} E)$ and set

$$E_0 := \mathcal{H} \otimes_{\mathbb{C}} \left(L^2(G, \mathcal{H})^* \otimes_{\mathbb{K}_G} E\right).$$

Let U be the (possibly non-equivariant) unitary from $\mathcal{H}_B$ to $E_0 \oplus \mathcal{H}_B$ as above. Then we obtain

$$\tilde{U}(g) := g(U)\colon C_c(G, \mathcal{H}_B) \to C_c(G, E_0 \oplus \mathcal{H}_B),$$

which extends to a G-equivariant unitary

$$\tilde{U}\colon \mathcal{H}_{G,B} \cong L^2\big(G, \mathcal{H}_B\big) \to L^2\big(G, E_0 \oplus \mathcal{H}_B\big) \cong E \oplus \mathcal{H}_{G,B}.$$

More detail is found in Section 3 of [28]. □

A pro-C*-algebra is *σ-unital* if there is a strictly positive element $h \in A$. Here, we say that an element $h \in A$ is strictly positive if $\overline{hA} = \overline{Ah} = A$. A pro-C*-algebra A is σ-unital if and only if it has a countable approximate unit. A separable σ-C*-algebra is σ-unital and moreover has a countable increasing approximate unit ([18, Lemma 5]).

Lemma A.4. *Let B be a σ-C*-algebra with G-action, $A \subset B$ a σ-G-C*-algebra, Y a σ-compact locally compact space, $\varphi\colon Y \to B$ a function such that $y \mapsto [\varphi(y), a]$ are continuous functions which take values in A. Then there is a countable approximate unit $\{u_i\}$ for A that is quasi-central for $\varphi(Y)$ and quasi-invariant, that is, the sequences $[u_i, \varphi(y)]$ ($y \in Y$) and $g(u_i) - u_i$ converge to zero.*

Proof. Let $\{p_n\}_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of invariant C*-seminorms on B generating the topology of B and let $\{v_m\}$ be a countable increasing approximate unit for A and $h := \sum 2^{-k} v_k$. By induction, we can choose an increasing sequence $\{u_n\}$ given by convex combinations of v_i's such that:

(1) $p_n(u_n h - h) \leq 1/n$;

(2) $p_n\big([u_n, \varphi(y)]\big) \leq 1/n$ for any $y \in \overline{Y_n}$;

(3) $p_n(g(u_n) - u_n) \leq 1/n$ for any $g \in \overline{X_n}$.

Each induction step is the same as in Section 1.4 of Kasparov [21]. □

Theorem A.5. *Let J be a σ-G-C*-algebra, A_1 and A_2 σ-unital closed subalgebras of $M(J)$ where G acts continuously on A_1, Δ a separable subset of $M(J)$ such that $[\Delta, A_1] \subset A_1$ and $\varphi\colon G \to M(J)$ a function such that*

$$\sup_{g\in G,\, p\in\mathcal{S}(M(J))} p\big(\varphi(g)\big)$$

is bounded. Moreover we assume that $A_1 \cdot A_2$, $A_1 \cdot \varphi(G)$ and $\varphi(G) \cdot A_1$ are in J and $g \mapsto \varphi(g)a$ are continuous functions on G for any $a \in A_1 + J$. Then, there are G-continuous even positive elements $M_1, M_2 \in M(J)$ such that:

- *$M_1 + M_2 = 1$,*
- *$M_i a_i$, $[M_i, d]$, $M_2\varphi(g)$, $\varphi(g)M_2$, $g(M_i) - M_i$ are in J for any $a_i \in A_i$, $d \in \Delta$, $g \in G$,*
- *$g \mapsto M_2\varphi(g)$ and $g \mapsto \varphi(g)M_2$ are continuous.*

Proof. The proof is given by the combination of arguments on p. 151 of [21] and in Theorem 10 of [18]. Actually, by Lemma A.4 we get an approximate unit $\{u_n\}$ for A_1 and $\{v_n\}$ for J such that:

(1) $p_n(u_n h_1 - h_1) \leq 2^{-n}$,

(2) $p_n\big([u_n, y]\big) \leq 2^{-n}$ for any $y \in Y$,

(3) $p_n\big(g(u_n) - u_n\big) \leq 2^{-n}$ for any $g \in X_n$,

(4) $p_n(v_n w - w) \leq 2^{-2n}$ for any $w \in W_n$,

(5) $p_n\big([v_n, z]\big)$ is small enough to $p_n\big([b_n, z]\big) \leq 2^{-n}$ for any $z \in \{h_1, h_2\} \cup Y \cup \varphi(\bar{X}_n)$,

(6) $p_n\big(g(b_n) - b_n\big) \leq 2^{-n}$ for any $g \in \bar{X}_n$,

where h_1, h_2, k are strictly positive element in A_1, A_2 and J respectively such that

$$p_n(h_1), p_n(h_2), p_n(k) \leq 1$$

for any n, $Y \subset \Delta$ is a compact subset whose linear span is dense in Δ, X_n is a increasing sequence of relatively compact open subsets of G whose union is dense in G,

$$W_n := \{k, u_n h_2, u_{n+1} h_2\} \cup u_n \varphi(\bar{X}_n) \cup u_{n+1} \varphi(\bar{X}_{n+1}) \cup \varphi(\bar{X}_n) u_n \cup \varphi(\bar{X}_{n+1}) u_{n+1}$$

and

$$b_n := (v_n - v_{n-1})^{1/2}.$$

Now, it can be checked that the finite sum $\sum b_n u_n b_n$ converges in the strict topology to the desired element $M_2 \in M(J)$. □

A.2. Equivariant KK-groups. A generalization of KK-theory for pro-C*-algebras was first defined by Weidner [57] and was generalized for equivariant case by Schochet [46]. Here the notion of coherent A-B bimodule is introduced in order to avoid Kasparov's technical theorem for pro-C*-algebras. On the other hand, Bonkat [7] introduced a new definition of KK-theory for σ-C*-algebras applying the technical theorem A.5 for σ-C*-algebras. In this paper we adopt the latter definition.

Definition A.6. Let A and B be σ-unital $\mathbb{Z}/2$-graded σ-G-C*-algebras. A G-equivariant Kasparov A-B bimodule is a triplet (E, φ, F) where

- E is a $\mathbb{Z}/2$-graded countably generated G-equivariant Hilbert B-module;
- $\varphi\colon A \to \mathbb{L}(E)$ is a graded G-equivariant $*$-homomorphism;
- $F \in \mathbb{L}(E)^{\mathrm{odd}}_{\mathrm{s.a.}}$ such that $[F, \varphi(A)]$, $\varphi(A)(F^2 - 1)$, $\varphi(A)(g(F) - F) \in \mathbb{K}(E)$, and $\varphi(a)F$, $F\varphi(a)$ are G-continuous.

Two G-equivariant Kasparov A-B bimodules (E_1, φ_1, F_1) and (E_2, φ_2, F_2) are *unitarily equivalent* if there is a unitary $u \in \mathbb{L}(E_1, E_2)$ such that $u\varphi_1 u^* = \varphi_2$ and $uF_1u^* = F_2$. Two G-equivariant Kasparov A-B bimodules (E_1, φ_1, F_1) and (E_2, φ_2, F_2) are *homotopic* if there is a Kasparov G-equivariant A-IB bimodule (E, φ, F) such that $(\mathrm{ev}_i)_*(E, \varphi, F)$ are unitarily equivalent to (E_i, φ_i, F_i) for $i = 0, 1$.

Definition A.7. Let A and B be σ-unital $\mathbb{Z}/2$-graded σ-G-C*-algebras. The KK-group $\mathrm{KK}^G(A, B)$ is the set of homotopy equivalence classes of G-equivariant Kasparov A-B bimodules.

It immediately follows from the definition that $\mathrm{KK}^G(\mathbb{C}, A)$ is canonically isomorphic to the representable equivariant K-group $\mathcal{R}K_0^G(A)$ introduced in [42].

Definition A.8. Let (E_1, φ_1, F_1) be a G-equivariant Kasparov A-B bimodule and (E_2, φ_2, F_2) a G-equivariant Kasparov B-C bimodule. A Kasparov product of (E_1, φ_1, F_1) and (E_2, φ_2, F_2) is a G-equivariant Kasparov A-C bimodule $(E_1 \otimes_B E_2, \varphi, F)$ that satisfies the following:

(1) The operator $F \in \mathbb{L}(E_1 \otimes_B E_2)$ is an F_2-*connection*. That is,

$$T_x \circ F_2 - (-1)^{\deg x \cdot \deg F_2} F \circ T_x \quad \text{and} \quad F_2 \circ T_x^* - (-1)^{\deg x \cdot \deg F_2} T_x^* \circ F$$

are compact for any $x \in E_1$;

(2) $\varphi(a)[F_1 \otimes 1, F]\varphi(a)^* \geq 0 \bmod \mathbb{K}(E)$.

Theorem A.9. *Let A, B, C and D be σ-unital σ-G-C^*-algebras. Moreover we assume that A is separable. The Kasparov product gives a well-defined group homomorphism*

$$\mathrm{KK}^G(A, B) \otimes \mathrm{KK}^G(B, C) \to \mathrm{KK}^G(A, C),$$

which is associative, that is,

$$(x \otimes_B y) \otimes_C z = x \otimes_B (y \otimes_C z)$$

for any $x \in \mathrm{KK}^G(A, B)$, $y \in \mathrm{KK}^G(B, C)$ and $z \in \mathrm{KK}^G(C, D)$ when B is also separable.

Proof. What we have to show is existence, uniqueness up to homotopy, well-definedness of maps between KK-groups and associativity of the Kasparov product. All of them are proved in the same way as in Theorem 12 and Theorem 21 of [49] or Theorem 2.11 and Theorem 2.14 of [21]. Note that we can apply the Kasparov technical theorem A.5 since we may assume that $\sup_{p \in \mathcal{S}(\mathbb{L}(E))} p(F) \leq 1$ by a functional calculus and a separable σ-C^*-algebra is separable as a topological algebra (see also [6, Sections 18.3–18.6]). □

Moreover, we obtain the Puppe exact sequence (as [6, Theorem 19.4.3]) for a $*$-homomorphism between σ-C^*-algebras and the six term exact sequences ([6, Theorem 19.5.7]) for a semisplit exact sequence of σ-C^*-algebras by the same proofs.

Next we deal with the Cuntz picture [12] (see also [28]) of KK-theory for σ-G-C^*-algebras.

Definition A.10 ([12, Definition 2.2]). We say that $(\varphi_0, \varphi_1) \colon A \rightrightarrows D \rhd J \to B$ is an equivariant *prequasihomomorphism* from A to B if D is a σ-unital σ-C^*-algebra with G-action, φ_0 and φ_1 are equivariant $*$-homomorphisms from A to D such that $\varphi_0(a) - \varphi_1(a)$ are in a separable G-invariant ideal J of D such that the restriction of the G-action on J is continuous, and $J \to B$ is an equivariant $*$-homomorphism. Moreover, we say that (φ_0, φ_1) is *quasihomomorphism* if D is generated by $\varphi_0(A)$ and $\varphi_1(A)$, J is generated by $\{\varphi_0(a) - \varphi_1(a) \mid a \in A\}$ and $J \to B$ is injective.

The idea given in [13] is also generalized for σ-G-C*-algebras.

Definition A.11. Let A and B be σ-G-C*-algebras. The full free product $A * B$ is the σ-G-C*-algebra given by the completion of the algebraic free product $A *_{\mathrm{alg}} B$ by seminorms

$$p_{\pi_A,\pi_B}(a_1b_1\dots a_nb_n) = \|\pi_A(a_1)\pi_B(b_1)\dots\pi_A(a_n)\pi_B(b_n)\|,$$

where π_A and π_B are $*$-representations of A and B on the same Hilbert space. In other words, when $A = \varprojlim A_n$ and $B = \varprojlim B_m$, the free product $A * B$ is the projective limit

$$\varprojlim(A_n * B_m).$$

By definition, any $*$-homomorphisms $\varphi_A\colon A \to D$ and $\varphi_B\colon B \to D$ are uniquely extended to $\varphi_A * \varphi_B\colon A * B \to D$. We denote by QA the free product $A * A$ and by qA the kernel of the $*$-homomorphism $\mathrm{id}_A * \mathrm{id}_A\colon QA \to A$.

Since we have the stabilization Theorem A.3 and the technical Theorem A.5 for σ-G-C*-algebras, the following properties of quasihomomorphisms and KK-theory is proved in the same way. We only enumerate their statements and references for the proofs. Here we write q_sA for the G-C*-algebra $q(A \otimes \mathbb{K}_G)$.

- The set of homotopy classes of G-equivariant quasihomomorphisms from $A \otimes \mathbb{K}_G$ to $B \otimes \mathbb{K}_G$ is isomorphic to $\mathrm{KK}^G(A, B)$ ([12, Section 5]).
- The functor $\mathrm{KK}^G\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \times G\text{-}\mathfrak{C}^*\mathfrak{sep} \to R(G)\text{-}\mathbf{Mod}$ is stable and split exact in both variables ([13, Proposition 2.1]).
- For any σ-G-C*-algebras A and B, $A * B$ and $A \oplus B$ are KK^G-equivalent ([13, Proof of Proposition 3.1]).
- The element $\pi_A := [\pi_0]$ in $\mathrm{KK}^G(qA, A)$ where $\pi_0 := (\mathrm{id}_A * 0)|_{qA}\colon qA \to A$ is the KK^G-equivalence ([13, Proposition 3.1]).
- There is a one-to-one correspondence between G-equivariant quasihomomorphisms from $A \otimes \mathbb{K}_G$ to $B \otimes \mathbb{K}_G$ and G-equivariant $*$-homomorphisms from q_sA to $B \otimes \mathbb{K}_G$ ([28, Theorem 5.5]).
- There is a canonical isomorphism $\mathrm{KK}^G(A, B) \cong [q_sA, B \otimes \mathbb{K}_G]^G$ (the stabilization Theorem A.3 and [13, Proposition 1.1]).
- The correspondence

$$\begin{aligned} [q_sA \otimes \mathbb{K}_G, q_sB \otimes \mathbb{K}_G]^G &\to \mathrm{KK}^G(A, B) \\ \varphi &\mapsto \pi_B \circ \varphi \circ (\pi_A)^{-1} \end{aligned}$$

  induces the natural isomorphism ([28, Theorem 6.5]).

For a projective system $\{A_n, \pi_n\}$ of σ-C*-algebras, the homotopy projective limit ho-$\varprojlim A_n$ is actually isomorphic to the mapping telescope

$$\operatorname{Tel} A_n := \Big\{ f \in \prod C\big([0,1], A_n\big) \mid f_n(1) = \pi_n\big(f(0)\big) \Big\}.$$

The following theorem follows from the fact that the functor $\mathrm{KK}^G(A, \sqcup)$ and $\mathrm{KK}^G(\sqcup, B)$ is compatible with direct products when B is a G-C*-algebra.

Theorem A.12. *The following holds:*

(1) *Let $\{A_n\}_{n \in \mathbb{Z}_{>0}}$ be a inductive system of σ-G-C*-algebras and $A :=$ ho-$\varinjlim A_n$. For a σ-G-C*-algebra B, there is an exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{KK}^G_{*+1}(A_n, B) \to \mathrm{KK}^G(A, B) \to \mathrm{KK}^G_*(A_n, B) \to 0.$$

(2) *Let $\{B_n\}_{n \in \mathbb{Z}_{>0}}$ be a projective system of σ-G-C*-algebras and $B :=$ ho-$\varprojlim B_n$. For a σ-G-C*-algebra B, there is an exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{KK}^G_{*+1}(A, B_n) \to \mathrm{KK}^G(A, B) \to \varprojlim \mathrm{KK}^G_*(A, B_n) \to 0.$$

(3) *Let $\{A_n\}_{n \in \mathbb{Z}_{>0}}$ be a projective system of σ-G-C*-algebras and $A :=$ ho-$\varprojlim A_n$. For a G-C*-algebra B, there is an isomorphism*

$$\mathrm{KK}^G(A, B) \cong \varinjlim \mathrm{KK}^G(A_n, B).$$

Corollary A.13. *Let $A =$ ho-$\varprojlim A_n$ and $B =$ ho-$\varprojlim B_m$ be homotopy projective limits of* C*-*algebras. There is an exact sequence*

$$0 \to \varprojlim_m{}^1 \varinjlim_n \mathrm{KK}^G_{*+1}(A_n, B_m) \to \mathrm{KK}^G_*(A, B) \to \varprojlim_m \varinjlim_n \mathrm{KK}^G(A_n, B_m) \to 0.$$

In particular, if two homotopy projective limits

$$A = \text{ho-}\varprojlim A_n \quad \text{and} \quad B = \text{ho-}\varprojlim B_m$$

of G-C*-algebras are KK^G-equivalent, then we get a pro-isomorphism of projecctive systems $\{A_n\}_n \to \{B_m\}_m$ in $\mathfrak{KK}^G$.

A.3. The Kasparov category.

Definition A.14. We write $\sigma\mathfrak{KK}^G$ for the Kasparov category of σ-G-C*-algebras i.e. the additive category whose objects are separable σ-G-C*-algebras, morphisms from A to B are $\mathrm{KK}^G(A, B)$ and composition is given by the Kasparov product.

Note that the inclusion G-$\mathfrak{C}^*\mathfrak{sep} \subset \sigma G$-$\mathfrak{C}^*\mathfrak{sep}$ induces a full embedding $\mathfrak{KK}^G$ in $\sigma\mathfrak{KK}^G$. Additional structures of $\mathfrak{KK}^G$ such as tensor products, crossed products and countable direct sums are extended on $\sigma\mathfrak{KK}^G$. Moreover the category $\mathfrak{KK}^G$ has countably infinite direct products.

Theorem A.15 ([51, Theorem 2.2]; [7, Satz 3.5.10]). *The category $\sigma\mathfrak{KK}^G$ is an additive category that has the following universal property: there is the canonical functor*

$$\mathrm{KK}^G\colon \sigma\mathfrak{C}^*\mathfrak{sep} \to \sigma\mathfrak{KK}^G$$

such that for any C^**-stable, split-exact, and homotopy invariant functor* $F\colon \sigma\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{A}$ *there is a unique functor* $\tilde{F}$ *such that the following diagram*

$$\begin{array}{ccc} \sigma\mathfrak{C}^*\mathfrak{sep}^G & \longrightarrow & \mathfrak{A} \\ \big\downarrow & \nearrow & \\ \sigma\mathfrak{KK}^G & & \end{array}$$

commutes.

This follows from the Cuntz picture introduced in the previous subsection.

A structure of the triangulated category on $\mathfrak{KK}^G$ is introduced in [31]. Let S be the suspension functor $SA := C_0(\mathbb{R}) \otimes A$ of C^*-algebras. Roughly speaking, the inverse $\Sigma := S^{-1}$ and the mapping cone exact sequence

$$\Sigma B \to \operatorname{cone}(f) \to A \xrightarrow{f} B$$

determines a triangulated category structure of $\mathfrak{KK}^G$. More precisely we need to replace the category $\mathfrak{KK}^G$ with another one that is equivalent to $\mathfrak{KK}^G$, whose objects are pair (A, n) where A is a separable σ-G-C^*-algebra and $n \in \mathbb{Z}$, morphisms from (A, n) to (B, m) are $\mathrm{KK}_{n-m}(A, B)$ and composition is given by the Kasparov product. In this category the functor $\Sigma\colon (A, n) \mapsto (A, n+1)$ is an category isomorphism (not only an equivalence) and $S \circ \Sigma = \Sigma \circ S$ are natural equivalent with the identity functor. A triangle

$$\Sigma(B, m) \to (C, l) \to (A, n) \to (B, m)$$

is exact if there is a $*$-homomorphism from A' to B' and the isomorphism α, β, and γ such that the following diagram

$$\begin{array}{ccccccc} \Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\ \cong\big\downarrow \Sigma\beta & & \cong\big\downarrow \gamma & & \cong\big\downarrow \alpha & & \cong\big\downarrow \beta \\ \Sigma B' & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & A' & \xrightarrow{f} & B'. \end{array}$$

commutes. For simplicity of notation we use the same letter $\mathfrak{KK}^G$ for this category.

Theorem A.16. *The category $\sigma\mathfrak{KK}^G$, with the suspension Σ and exact triangles as above, is a triangulated category.*

We omit the proof. Actually, the same proof as for $\mathfrak{KK}^G$ given in Appendix 1 of [31] works since we have the Cuntz picture of equivariant KK-theory introduced in the previous subsection.

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