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The prime spectrum of the algebra $\mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$ and a classification of simple weight modules

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Abstract. For the algebra A in the title, it is shown that its centre is generated by an explicit quartic element. Explicit descriptions are given of the prime, primitive and maximal spectra of the algebra A. A classification of simple weight A-modules is obtained. The classification is based on a classification of (all) simple modules of the centralizer $C_A(K)$ of the quantum Cartan element K which is given in the paper. Explicit generators and defining relations are found for the algebra $C_A(K)$ (it is generated by 5 elements subject to the defining relations two of which are quadratic and one is cubic).

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1. Introduction

In this paper, module means a left module, $\mathbb{K}$ is a field, $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$, an element $q \in \mathbb{K}^*$ is not a root of unity, algebra means a unital $\mathbb{K}$-algebra, $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$.

For a Hopf algebra and its module one can form a *smash product algebra* (see [22, 4.1.3] for detail). The algebras obtained have rich structure. However, little is known about smash product algebras; in particular, about their prime, primitive and maximal spectra and simple modules. One of the classical objects in this area is the smash product algebra $A := \mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$, where $\mathbb{K}_q[X, Y] := \mathbb{K}\langle X, Y \mid XY = qYX\rangle$ is the *quantum plane* and $q \in \mathbb{K}^*$ is not a root of unity. As an abstract algebra, the algebra A is generated over $\mathbb{K}$ by elements E, F, K, K^{-1}, X, and Y subject to the defining relations (where K^{-1} is the inverse of K):

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$EX = qXE, \quad EY = X + q^{-1}YE, \quad FX = YK^{-1} + XF, \quad FY = YF,$$
$$KXK^{-1} = qX, \quad KYK^{-1} = q^{-1}Y, \quad qYX = XY.$$

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The algebra A admits a PBW basis and the ordering of the generators can be arbitrary. The study of semidirect product algebras has recently gained momentum: An important class of algebras — the *symplectic reflection algebras* — was introduced by Etingof and Ginzburg, [13]. This led to study of infinitesimal and continuous Hecke algebras by Etingof, Gan and Ginzburg, [14] (see also papers of Ding, Khare, Losev, Tikaradze and Tsymbaliuk and others in this direction).

The centre of the algebra A. A PBW deformation of this algebra, the *quantized symplectic oscillator algebra of rank one*, was studied by Gan and Khare [15] and some representations were considered. They showed that the centre of the deformed algebra is $\mathbb{K}$. In this paper, we show that the centre of A is a polynomial algebra $\mathbb{K}[C]$ (Theorem 2.10) and the generator C has degree 4:

$$C = (1-q^2)FYXE + FX^2 - Y^2K^{-1}E - \frac{1}{1-q^2}YK^{-1}X + \frac{q^2}{1-q^2}YKX.$$

The method we use in finding the central element C of A can be summarized as follows. The algebra A is "covered" by a chain of large subalgebras. They turn out to be generalized Weyl algebras. Their central/normal elements can be determined by applying Proposition 2.4. At each step generators of the covering subalgebras are getting more complicated but their relations become simpler. At the final step, we find a central element of a large subalgebra $\mathbb{A}$ of A which turns out to be the central element C of the algebra A.

The prime, primitive and maximal spectra of A. In Section 3, we classify the prime, primitive and maximal ideals of the algebra A (Theorem 3.7, Theorem 3.11 and Corollary 3.9, respectively). It is shown that every nonzero ideal has nonzero intersection with the centre of the algebra A (Corollary 3.8). In classifying prime ideals certain localizations of the algebra A are used. The set of completely prime ideals is also described (Corollary 3.12).

A classification of simple weight A-modules. An A-module M is called a *weight module* if $M = \bigoplus_{\mu\in\mathbb{K}^*} M_\mu$ where $M_\mu = \{m \in M \mid Km = \mu m\}$. In Section 6, a classification of simple weight A-modules is given. It is too technical to describe the result in the Introduction but we give a flavour and explain main ideas. The set of isomorphism classes of simple weight A-modules are partitioned into several subclasses, and each of them requires different techniques to deal with. The key point is that each weight component of a simple weight A-module is a *simple* module over the centralizer $C_A(K)$ of the quantum Cartan element K and this simple $C_A(K)$-module can be an arbitrary simple $C_A(K)$-module. Therefore, first we study the algebra $C_A(K)$, classify its simple modules and using this classification we classify simple weight A-modules. There are plenty of them and a "generic/typical" simple weight A-module depends on arbitrary many independent parameters (the number of which is finite but can be arbitrary large).

The centralizer $C_A(K)$ and a classification of its simple modules. The algebra $C_A(K)$ is generated by (explicit) elements $K^{\pm 1}$, C, Θ, t, and u subject to the defining relations, Theorem 4.6 ($K^{\pm 1}$ and C are central elements):

$$\Theta \cdot t = q^2 t \cdot \Theta + (q + q^{-1})u + (1 - q^2)C,$$
$$\Theta \cdot u = q^{-2} u \cdot \Theta - q(1 + q^2)t + (1 - q^2)K^{-1}C,$$
$$t \cdot u = q^2 u \cdot t, \quad \Theta \cdot t \cdot u - \frac{1}{q(1-q^2)}u^2 - C \cdot u = \frac{q^7}{1-q^2}t^2 - q^4 K^{-1} C \cdot t.$$

It is proved that the centre of the algebra $C_A(K)$ is $\mathbb{K}[C, K^{\pm 1}]$. The problem of classification of simple $C_A(K)$-modules is reduced to the one for the factor algebras $\mathcal{C}^{\lambda,\mu} := C_A(K)/C_A(K)(C-\lambda, K-\mu)$ where $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The algebra $\mathcal{C}^{\lambda,\mu}$ is a domain (Theorem 4.11.(2)). The algebra $\mathcal{C}^{\lambda,\mu}$ is simple iff $\lambda \neq 0$ (Theorem 4.11.(1)). A classification of simple $\mathcal{C}^{\lambda,\mu}$-modules is given in Section 5. One of the key observations is that the localization $\mathcal{C}_t^{\lambda,\mu}$ of the algebra $\mathcal{C}^{\lambda,\mu}$ at the powers of the element $t = YX$ is a central, simple, generalized Weyl algebra (Proposition 4.9). The other one is that, for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we can embed the algebra $\mathcal{C}^{\lambda,\mu}$ into a generalized Weyl algebra $\mathcal{A}$ (which is also a central simple algebra), see Proposition 5.3. These two facts enable us to give a complete classification of simple $C_A(K)$-modules. The problem of classifying simple $\mathcal{C}^{\lambda,\mu}$-modules splits into two distinct cases, namely the case when $\lambda = 0$ and the case when $\lambda \neq 0$. In the case $\lambda = 0$, we embed the algebra $\mathcal{C}^{0,\mu}$ into a skew polynomial algebra $\mathcal{R} = \mathbb{K}[h^{\pm 1}][t;\sigma]$ where $\sigma(h) = q^2 h$ (it is a subalgebra of the algebra $\mathcal{A}$) for which the classification of simple modules is known. In the case $\lambda \neq 0$, we use a close relation of $\mathcal{C}^{\lambda,\mu}$ with the localization $\mathcal{C}_t^{\lambda,\mu}$, and the arguments are more complicated.

The algebra A can be seen as a quantum analogue of another classical algebra, the enveloping algebra $U(V_2 \rtimes \mathfrak{sl}_2)$ of the semidirect product Lie algebra $V_2 \rtimes \mathfrak{sl}_2$ (where V_2 is the 2-dimensional simple $\mathfrak{sl}_2$-module) which was studied in [9]. These two algebras are similar in many ways. For example, the prime spectra of these two algebras have similar structures; the representation theory of A has many parallels with that of $U(V_2 \rtimes \mathfrak{sl}_2)$; the *quartic* Casimir element C of A degenerates to the *cubic* Casimir element of $U(V_2 \rtimes \mathfrak{sl}_2)$ as "$q \to 1$". The centre of $U(V_2 \rtimes \mathfrak{sl}_2)$ is generated by the cubic Casimir element, [24]. The study of quantum algebras usually requires more computations and the methods of this paper and [9] are quite different. Much work has been done on quantized enveloping algebras of semisimple Lie algebras (see, e.g., [17, 18]). In the contrast, only few examples can be found in the literature on the quantized algebras of enveloping algebras of non-semisimple Lie algebras.

2. The centre of the algebra A

In this section, it is proved that the centre $Z(A)$ of the algebra A is a polynomial algebra $\mathbb{K}[C]$ (Theorem 2.10) and the element C is given explicitly, (2.14)–(2.17). Several important subalgebras and localizations of the algebra A are introduced, they are instrumental in finding the centre of A. We also show that the quantum Gelfand–Kirillov conjecture holds for the algebra A.

The algebra A. In this paper, $\mathbb{K}$ is a field and an element $q \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$ is not a root of unity. Recall that the *quantized enveloping algebra* of $\mathfrak{sl}_2$ is the $\mathbb{K}$-algebra $U_q(\mathfrak{sl}_2)$ with generators E, F, K, K^{-1} subject to the defining relations (see [17]):

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The centre of $U_q(\mathfrak{sl}_2)$ is a polynomial algebra $Z(U_q(\mathfrak{sl}_2)) = \mathbb{K}[\Omega]$ where $\Omega := FE + (qK + q^{-1}K^{-1})/(q - q^{-1})^2$. A Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ is defined as follows:

$$\begin{aligned} &\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1}, \\ &\Delta(E) = E \otimes 1 + K \otimes E, && \varepsilon(E) = 0, && S(E) = -K^{-1}E, \\ &\Delta(F) = F \otimes K^{-1} + 1 \otimes F, && \varepsilon(F) = 0, && S(F) = -FK, \end{aligned}$$

where Δ is the comultiplication on $U_q(\mathfrak{sl}_2)$, ε is the counit and S is the antipode of $U_q(\mathfrak{sl}_2)$. Note that the Hopf algebra $U_q(\mathfrak{sl}_2)$ is neither cocommutative nor commutative. The *quantum plane* $\mathbb{K}_q[X, Y] := \mathbb{K}\langle X, Y \mid XY = qYX\rangle$ is a $U_q(\mathfrak{sl}_2)$-module algebra where

$$\begin{aligned} &K \cdot X = qX, && E \cdot X = 0, && F \cdot X = Y, \\ &K \cdot Y = q^{-1}Y, && E \cdot Y = X, && F \cdot Y = 0. \end{aligned}$$

Then one can form the smash product algebra $A := \mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$. For details about smash product algebras, see [22]. The generators and defining relations for this algebra are given in the Introduction.

Generalized Weyl algebras.

Definition 2.1 ([1–3]). Let D be a ring, σ be an automorphism of D and a is an element of the centre of D. *The generalized Weyl algebra* $A := D(\sigma, a) := D[X, Y; \sigma, a]$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \ \text{ and } \ Y\alpha = \sigma^{-1}(\alpha)Y \text{ for all } \alpha \in D, \quad YX = a \quad \text{and} \quad XY = \sigma(a).$$

The algebra $A = \oplus_{n \in \mathbb{Z}} A_n$ is $\mathbb{Z}$-graded where $A_n = Dv_n$, $v_n = X^n$ for $n > 0$, and $v_n = Y^{-n}$ for $n < 0$ and $v_0 = 1$.

Definition 2.2 ([6]). Let D be a ring and σ be its automorphism. Suppose that elements b and ρ belong to the centre of the ring D, ρ is invertible and $\sigma(\rho) = \rho$. Then $E := D[X, Y; \sigma, b, \rho]$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y \text{ for all } \alpha \in D, \quad \text{and } XY - \rho YX = b.$$

An element d of a ring D is *normal* if $dD = Dd$. The next proposition shows that the rings E are GWAs and under a (mild) condition they have a "canonical" normal element.

Proposition 2.3. *Let $E = D[X, Y; \sigma, b, \rho]$. Then*

(1) [6, Lemma 1.3] *The following statements are equivalent:*

(a) [6, Corollary 1.4] $C = \rho(YX + \alpha) = XY + \sigma(\alpha)$ *is a normal element in E for some central element $\alpha \in D$,*

(b) $\rho\alpha - \sigma(\alpha) = b$ *for some central element $\alpha \in D$.*

(2) [6, Corollary 1.4] *If one of the equivalent conditions of statement 1 holds then the ring $E = D[C][X, Y; \sigma, a = \rho^{-1}C - \alpha]$ is a GWA where $\sigma(C) = \rho C$.*

The next proposition is a corollary of Proposition 2.3 when $\rho = 1$. The rings E with $\rho = 1$ admit a "canonical" central element (under a mild condition).

Proposition 2.4. *Let $E = D[X, Y; \sigma, b, \rho = 1]$. Then*

(1) [6, Lemma 1.5] *The following statements are equivalent:*

(a) $C = YX + \alpha = XY + \sigma(\alpha)$ *is a central element in E for some central element $\alpha \in D$,*

(b) $\alpha - \sigma(\alpha) = b$ *for some central element $\alpha \in D$.*

(2) [6, Corollary 1.6] *If one of the equivalent conditions of statement 1 holds then the ring $E = D[C][X, Y; \sigma, a = C - \alpha]$ is a GWA where $\sigma(C) = C$.*

An involution τ of A. The algebra A admits the following involution τ (see [15, p. 693]):

$$\begin{gathered}\tau(E) = -FK, \quad \tau(F) = -K^{-1}E, \quad \tau(K) = K, \quad \tau(K^{-1}) = K^{-1}, \\ \tau(X) = Y, \quad \tau(Y) = X.\end{gathered} \tag{2.1}$$

For an algebra B, we denote by $Z(B)$ its centre.

The algebra $\mathbb{E}$ is a GWA. Let $\mathbb{E}$ be the subalgebra of A which is generated by the elements E, X, and Y. The elements E, X, and Y satisfying the defining relations

$$EX = qXE, \quad YX = q^{-1}XY, \quad \text{and} \quad EY - q^{-1}YE = X.$$

Therefore, $\mathbb{E} = \mathbb{K}[X][E, Y; \sigma, b = X, \rho = q^{-1}]$ where $\sigma(X) = qX$. The polynomial $\alpha = (q/(1-q^2))X$ is a solution to the equation $q^{-1}\alpha - \sigma(\alpha) = X$. Hence, by Proposition 2.3, the element

$$\widetilde{C} = q^{-1}\Big(YE + \frac{q}{1-q^2}X\Big) = EY + \frac{q^2}{1-q^2}X$$

is a normal element of $\mathbb{E}$ and the algebra $\mathbb{E}$ is a GWA

$$\mathbb{E} = \mathbb{K}[\widetilde{C}, X]\Big[E, Y; \sigma, a := q\widetilde{C} - \frac{q}{1-q^2}X\Big],$$

where $\sigma(\widetilde{C}) = q^{-1}\widetilde{C}$, $\sigma(X) = qX$.

Let

$$\varphi := (1-q^2)\widetilde{C}. \tag{2.2}$$

Then $\varphi = X + (q^{-1} - q)YE = (1-q^2)EY + q^2X$. Hence,

$$\mathbb{E} = \mathbb{K}[\varphi, X]\Big[E, Y; \sigma, a = \frac{\varphi - X}{q^{-1} - q}\Big], \tag{2.3}$$

where $\sigma(\varphi) = q^{-1}\varphi$ and $\sigma(X) = qX$. Using the defining relations of the GWA $\mathbb{E}$, we see that the set $\{Y^i \mid i \in \mathbb{N}\}$ is a left and right Ore set in $\mathbb{E}$. The localization of the algebra $\mathbb{E}$ at this set, $\mathbb{E}_Y := \mathbb{K}[\varphi, X][Y^{\pm 1}; \sigma]$ is the skew Laurent polynomial ring. Similarly, the set $\{X^i \mid i \in \mathbb{N}\}$ is a left and right Ore set in $\mathbb{E}_Y$ and the algebra

$$\mathbb{E}_{Y,X} = \mathbb{K}[\varphi, X^{\pm 1}][Y^{\pm 1}; \sigma] = \mathbb{K}[\Phi] \otimes \mathbb{K}[X^{\pm 1}][Y^{\pm 1}; \sigma] \tag{2.4}$$

is the tensor product of the polynomial algebra $\mathbb{K}[\Phi]$ where $\Phi = X\varphi$ and the Laurent polynomial algebra $\mathbb{K}[X^{\pm 1}][Y^{\pm 1}; \sigma]$ which is a central simple algebra. In particular, $Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$. So, we have the inclusion of algebras $\mathbb{E} \subseteq \mathbb{E}_Y \subseteq \mathbb{E}_{Y,X}$.

The next lemma describes the centre of the algebras $\mathbb{E}$, $\mathbb{E}_Y$ and $\mathbb{E}_{Y,X}$.

Lemma 2.5. $Z(\mathbb{E}) = Z(\mathbb{E}_Y) = Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$ *is a polynomial algebra where* $\Phi := X\varphi$.

Proof. By (2.4), $\mathbb{K}[\Phi] \subseteq Z(\mathbb{E}) \subseteq Z(\mathbb{E}_Y) \subseteq Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$, and the result follows. □

We have the following commutation relations:

$$X\varphi = \varphi X, \quad Y\varphi = q\varphi Y, \quad E\varphi = q^{-1}\varphi E, \quad K\varphi = q\varphi K. \tag{2.5}$$

$$X\Phi = \Phi X, \quad Y\Phi = \Phi Y, \quad E\Phi = \Phi E, \quad K\Phi = q^2\Phi K. \tag{2.6}$$

Lemma 2.6. (1) $[F,\varphi] = YK$.

(2) *The powers of φ form a left and right Ore set in A.*

(3) *The powers of X form a left and right Ore set in A.*

(4) *The powers of Y form a left and right Ore set in A.*

Proof. (1) $[F,\varphi] = [F, X + (q^{-1} - q)YE]$

$$= YK^{-1} + (q^{-1} - q)Y\left(-\frac{K - K^{-1}}{q - q^{-1}}\right) = YK.$$

(2) Statement 2 follows at once from the equalities (2.5) and statement 1.

(3) The statement follows at once from the defining relations of the algebra A where X is involved.

(4) The statement follows at once from the defining relations of the algebra A where Y is involved. □

The algebra $\mathbb{F}$ is a GWA. Let $\mathbb{F}$ be the subalgebra of A which is generated by the elements F, X, and $Y' := YK^{-1}$. The elements F, X and Y' satisfy the defining relations

$$FY' = q^{-2}Y'F, \quad XY' = q^2Y'X, \quad \text{and} \quad FX - XF = Y'.$$

Therefore, the algebra $\mathbb{F} = \mathbb{K}[Y'][F, X; \sigma, b = Y', \rho = 1]$ where $\sigma(Y') = q^{-2}Y'$. The polynomial $\alpha = (1/(1-q^{-2}))Y' \in \mathbb{K}[Y']$ is a solution to the equation $\alpha - \sigma(\alpha) = Y'$. By Proposition 2.4, the element

$$C' := XF + \frac{1}{1-q^{-2}}Y' = FX + \frac{1}{q^2-1}Y'$$

belongs to the centre of the GWA

$$\mathbb{F} = \mathbb{K}[C', Y']\left[F, X; \sigma, a = C' - \frac{1}{1-q^{-2}}Y'\right].$$

Let

$$\psi := (1-q^2)C'. \tag{2.7}$$

Then $\psi = (1-q^2)FX - Y' = (1-q^2)XF - q^2Y' \in Z(\mathbb{F})$ and

$$\mathbb{F} = \mathbb{K}[\psi, Y']\left[F, X; \sigma, a = \frac{\psi + q^2Y'}{1-q^2}\right], \tag{2.8}$$

where $\sigma(\psi) = \psi$ and $\sigma(Y') = q^{-2}Y'$. Similar to the algebra $\mathbb{E}$, the localization of the algebra $\mathbb{F}$ at the powers of the element X is equal to

$$\mathbb{F}_X := \mathbb{K}[\psi, Y'][X^{\pm 1}; \sigma^{-1}] = \mathbb{K}[\psi] \otimes \mathbb{K}[Y'][X^{\pm 1}; \sigma^{-1}],$$

where σ is defined in (2.8). The centre of the algebra $\mathbb{K}[Y'][X^{\pm 1}; \sigma^{-1}]$ is $\mathbb{K}$. Hence, $Z(\mathbb{F}_X) = \mathbb{K}[\psi]$.

Lemma 2.7. $Z(\mathbb{F}) = Z(\mathbb{F}_X) = \mathbb{K}[\psi]$.

Proof. The result follows from the inclusions $\mathbb{K}[\psi] \subseteq Z(\mathbb{F}) \subseteq Z(\mathbb{F}_X) = \mathbb{K}[\psi]$. □

The GWA $\mathbb{A}$. Let T be the subalgebra of A generated by the elements $K^{\pm 1}$, X, and Y. Clearly,

$$T := \Lambda[K^{\pm 1}; \tau], \tag{2.9}$$

where $\Lambda := \mathbb{K}\langle X, Y \mid XY = qYX\rangle$, and $\tau(X) = qX$ and $\tau(Y) = q^{-1}Y$. It is easy to determine the centre of the algebra T.

Lemma 2.8. $Z(T) = \mathbb{K}[z]$ *where* $z := KYX$.

Proof. Clearly, the element $z = KYX$ belongs to the centre of the algebra T. The centralizer $C_T(K)$ is equal to $\mathbb{K}[K^{\pm 1}, YX]$. Then the centralizer $C_T(K, X)$ is equal to $\mathbb{K}[z]$, hence $Z(T) = \mathbb{K}[z]$. □

Let $\mathbb{A}$ be the subalgebra of A generated by the algebra T and the elements φ and ψ. The generators $K^{\pm 1}$, X, Y, φ, and ψ satisfy the following relations:

$$\varphi X = X\varphi, \quad \varphi Y = q^{-1}Y\varphi, \quad \varphi K = q^{-1}K\varphi,$$
$$\psi X = X\psi, \quad \psi Y = qY\psi, \quad \psi K = qK\psi, \quad \varphi\psi - \psi\varphi = -q(1-q^2)z.$$

These relations together with the defining relations of the algebra T are defining relations of the algebra $\mathbb{A}$. In more detail, let, for a moment, $\mathbb{A}'$ be the algebra generated by the defining relations as above. We will see $\mathbb{A}' = \mathbb{A}$. Indeed,

$$\mathbb{A}' = T[\varphi, \psi; \sigma, b = -q(1-q^2)z, \rho = 1].$$

Hence, the set of elements $\{K^i X^j Y^k \varphi^l \psi^m \mid i \in \mathbb{Z},\ j, k, l, m \in \mathbb{N}\}$ is a basis of the algebra $\mathbb{A}'$. This set is also a basis for the algebra $\mathbb{A}$. This follows from the explicit expressions for the elements $\varphi = (q^{-1}-q)YE + X$ and $\psi = (1-q^2)XF - q^2YK^{-1}$. In particular, the leading terms of φ and ψ are equal to $(q^{-1}-q)YE$ and $(1-q^2)XF$, respectively ($\deg(K^{\pm 1}) = 0$). So, $\mathbb{A} = \mathbb{A}'$, i.e.,

$$\mathbb{A} = T[\varphi, \psi; \sigma, b = -q(1-q^2)z, \rho = 1],$$

where $\sigma(X) = X$, $\sigma(Y) = q^{-1}Y$, and $\sigma(K) = q^{-1}K$. Recall that the element b belongs to the centre of the algebra T (Lemma 2.8). The element $\alpha = q^3z$ is a solution to the equation $\alpha - \sigma(\alpha) = b$. Then, by Proposition 2.4, the element

$$C'' = \psi\varphi + q^3z = \varphi\psi + qz$$

is a central element of the algebra $\mathbb{A}$ (since $\sigma(z) = q^{-2}z$) which is the GWA

$$\mathbb{A} = T[C''][\varphi, \psi; \sigma, a = C'' - q^3z],$$

where $\sigma(C'') = C''$, $\sigma(X) = X$, $\sigma(Y) = q^{-1}Y$, $\sigma(K) = q^{-1}K$.

Let $C := C''/(1-q^2)$. Then

$$C = (1-q^2)^{-1}(\psi\varphi + q^3 z) = (1-q^2)^{-1}(\varphi\psi + qz), \tag{2.10}$$

is a central element of the GWA

$$\mathbb{A} = T[C][\varphi, \psi; \sigma, a = (1-q^2)C - q^3 z], \tag{2.11}$$

where $\sigma(C) = C$, $\sigma(X) = X$, $\sigma(Y) = q^{-1}Y$, and $\sigma(K) = q^{-1}K$. Using expressions of the elements $\varphi = X + (q^{-1}-1)YE$ and $\psi = (1-q^2)XF - q^2YK^{-1}$, we see that

$$\mathbb{A}_{X,Y} = A_{X,Y}. \tag{2.12}$$

Hence, $C \in Z(A)$. We now show our first main result: $Z(A) = \mathbb{K}[C]$ (Theorem 2.10). In order to show this fact we need to consider the localization $A_{X,Y,\varphi}$. Let $\mathbb{T} := T_{X,Y} = \Lambda_{X,Y}[K^{\pm 1}; \tau]$ where τ is defined in (2.9) and $\Lambda_{X,Y}$ is the localization of the algebra Λ at the powers of the elements X and Y. By (2.12) and (2.11),

$$A_{X,Y,\varphi} = \mathbb{A}_{X,Y,\varphi} = T_{X,Y}[C][\varphi^{\pm 1}; \sigma] = \mathbb{K}[C] \otimes \mathbb{T}[\varphi^{\pm 1}; \sigma] = \mathbb{K}[C] \otimes \Lambda', \tag{2.13}$$

where $\Lambda' = \mathbb{T}[\varphi^{\pm 1}; \sigma]$ and σ is as in (2.11).

Lemma 2.9. (1) $Z(\Lambda') = \mathbb{K}$. (2) *The algebra Λ' is a simple algebra.*

Proof. (1) Let $u = \sum \lambda_{i,j,k,l} K^i X^j Y^k \varphi^l \in Z(\Lambda)$, where $\lambda_{i,j,k,l} \in \mathbb{K}$. Since $[K, u] = 0$, we have $j - k + l = 0$. Similarly, since $[X, u] = [Y, u] = [\varphi, u] = 0$, we have the following equations: $-i + k = 0, i - j + l = 0, -i - k = 0$, respectively. These equations imply that $i = j = k = l = 0$. Thus $Z(\Lambda) = \mathbb{K}$.

(2) Since the algebra Λ' is central, it is a simple algebra, by [16, Corollary 1.5.(a)]. □

Theorem 2.10. *The centre $Z(A)$ of the algebra A is the polynomial algebra in one variable $\mathbb{K}[C]$.*

Proof. By (2.13) and Lemma 2.9.(1), $Z(A_{X,Y,\varphi}) = \mathbb{K}[C]$. Hence, $Z(A) = \mathbb{K}[C]$. □

Using the defining relations of the algebra A, we can rewrite the central element C as follows:

$$C = (1-q^2)FYXE + FX^2 - Y^2K^{-1}E - \frac{1}{1-q^2}YK^{-1}X + \frac{q^2}{1-q^2}YKX. \tag{2.14}$$

$$C = (FE - q^2EF)YX + q^2FX^2 - K^{-1}EY^2. \tag{2.15}$$

$$C = FX(EY - qYE) - K^{-1}EY^2 + \frac{q^3}{1-q^2}(K - K^{-1})YX. \tag{2.16}$$

$$C = (1-q^2)FEYX + \frac{q^3}{1-q^2}(K - K^{-1})YX + q^2FX^2 - K^{-1}EY^2. \tag{2.17}$$

The subalgebra $\mathcal{A}$ of A. Let $\mathcal{A}$ be the subalgebra of A generated by the elements $K^{\pm 1}$, E, X, and Y. The properties of this algebra were studied in [8] where the prime, maximal and primitive spectrum of $\mathcal{A}$ were found. In particular, the algebra

$$\mathcal{A} = \mathbb{E}[K^{\pm 1}; \tau] \tag{2.18}$$

is a skew Laurent polynomial algebra where $\tau(E) = q^2 E$, $\tau(X) = qX$, and $\tau(Y) = q^{-1}Y$. The elements X, $\varphi \in \mathcal{A}$ are normal elements of the algebra $\mathcal{A}$. The set $\mathcal{S}_{X,\varphi} := \{X^i \varphi^j \mid i, j \in \mathbb{N}\}$ is a left and right denominator set of the algebras A and $\mathcal{A}$. Clearly $\mathcal{A}_{X,\varphi} := \mathcal{S}_{X,\varphi}^{-1}\mathcal{A} \subseteq A_{X,\varphi} := \mathcal{S}_{X,\varphi}^{-1}A$.

Lemma 2.11 ([8]). *The algebra $\mathcal{A}_{X,\varphi}$ is a central simple algebra.*

Using the defining relations of the algebra A, the algebra A is a skew polynomial algebra

$$A = \mathcal{A}[F; \sigma, \delta] \tag{2.19}$$

where σ is an automorphism of $\mathcal{A}$ such that $\sigma(K) = q^2 K, \sigma(E) = E, \sigma(X) = X$, $\sigma(Y) = Y$; and δ is a σ-derivation of the algebra $\mathcal{A}$ such that $\delta(K) = 0$, $\delta(E) = (K - K^{-1})/(q - q^{-1})$, $\delta(X) = YK^{-1}$, and $\delta(Y) = 0$. For an element $a \in A$, let $\deg_F(a)$ be its F-degree. Since the algebra A is a domain,

$$\deg_F(ab) = \deg_F(a) + \deg_F(b)$$

for all elements $a, b \in A$.

Lemma 2.12. *The algebra $A_{X,\varphi} = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}$ is a tensor product of algebras.*

Proof. Recall that $\varphi = EY - qYE$. Then the equality (2.16) can be written as $C = FX\varphi - K^{-1}EY^2 + (q^3/(1-q^2))(K - K^{-1})YX$. The element $X\varphi$ is invertible in $A_{X,\varphi}$. Now, using (2.19), we see that

$$A_{X,\varphi} = \mathcal{A}_{X,\varphi}[F; \sigma, \delta] = \mathcal{A}_{X,\varphi}[C] = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}. \qquad \square$$

Quantum Gelfand–Kirillov conjecture for A. If we view A as the quantum analogue of the enveloping algebra $U(V_2 \rtimes \mathfrak{sl}_2)$, a natural question is whether A satisfies the quantum Gelfand–Kirillov conjecture. Recall that a *quantum Weyl field* over $\mathbb{K}$ is the field of fractions of a quantum affine space. We say that a $\mathbb{K}$-algebra A admitting a skew field of fractions $\operatorname{Frac}(A)$ satisfies the *quantum Gelfand–Kirillov conjecture* if $\operatorname{Frac}(A)$ is isomorphic to a quantum Weyl field over a purely transcendental field extension of $\mathbb{K}$; see [11, II.10, p. 230].

Theorem 2.13. *The quantum Gelfand–Kirillov conjecture holds for the algebra A.*

Proof. This follows immediately from (2.13). $\square$

3. Prime, primitive and maximal spectra of A

The aim of this section is to give classifications of prime, primitive and maximal ideals of the algebra A (Theorem 3.7, Theorem 3.11 and Corollary 3.9). It is proved that every nonzero ideal of the algebra A has nonzero intersection with the centre of A (Corollary 3.8). The set of completely prime ideals of the algebra A is described in Corollary 3.12. Our goal is a description of the prime spectrum of the algebra A together with their inclusions. Next several results are steps in this direction, they are interesting in their own right.

Lemma 3.1. *The following identities hold in the algebra A.*

(1) $FX^i = X^iF + ((1-q^{2i})/(1-q^2))YK^{-1}X^{i-1}$.

(2) $XF^i = F^iX - ((1-q^{2i})/(1-q^2))YF^{i-1}K^{-1}$.

Proof. By induction on i and using the defining relations of A. □

Let R be a ring. For an element $r \in R$, we denote by (r) the (two-sided) ideal of R generated by the element r.

Lemma 3.2. (1) *In the algebra A, $(X) = (Y) = (\varphi) = AX + AY$.*

(2) $A/(X) \simeq U_q(\mathfrak{sl}_2)$.

Proof. (1) The equality $(X) = (Y)$ follows from the equalities $FX = YK^{-1} + XF$ and $EY = X + q^{-1}YE$. The inclusion $(\varphi) \subseteq (Y)$ follows from the equality $\varphi = EY - qYE$. The reverse inclusion $(\varphi) \supseteq (Y)$ follows from $Y = [F,\varphi]K^{-1}$ (Lemma 2.6). Let us show that $XA \subseteq AX + AY$. Recall that X is a normal element of $\mathcal{A}$. Then by (2.19),

$$XA = \sum_{k\geqslant 0} \mathcal{A}XF^k = \mathcal{A}X + \sum_{k\geqslant 1} \mathcal{A}XF^k \subseteq AX + AY$$

(the inclusion follows from Lemma 3.1.(2)). Then

$$(X) = AXA \subseteq AX + AY \subseteq (X,Y) = (X),$$

i.e., $(X) = AX + AY$.

(2) By statement 1, $A/(X) = A/(X,Y) \simeq U_q(\mathfrak{sl}_2)$. □

The next result shows that the elements X and φ are rather special.

Lemma 3.3. (1) *For all $i \geqslant 1$, $(X^i) = (X)^i$.*

(2) *For all $i \geqslant 1$, $(\varphi^i)_X = (\varphi)^i_X = A_X$.*

Proof. (1) To prove the statement we use induction on i. The case $i = 1$ is obvious. Suppose that $i > 1$ and the equality $(X^j) = (X)^j$ holds for all $1 \leqslant j \leqslant i-1$. By Lemma 3.1.(1), the element $YX^{i-1} \in (X^i)$. Now,

$$(X)^i = (X)(X)^{i-1} = (X)(X^{i-1}) = AXAX^{i-1}A \subseteq (X^i) + AYX^{i-1}A \subseteq (X^i).$$

Therefore, $(X)^i = (X^i)$.

(2) It suffices to show that $(\varphi^i)_X = A_X$ for all $i \geqslant 1$. The case $i = 1$ follows from the equality of ideals $(\varphi) = (X)$ in the algebra A (Lemma 3.2). We use induction on i. Suppose that the equality is true for all $i' < i$. By Lemma 2.6.(1), $[F, \varphi^i] = ((1-q^{-2i})/(1-q^{-2}))YK\varphi^{i-1}$, hence $Y\varphi^{i-1} \in (\varphi^i)$. Using the equalities $EY - q^{-1}YE = X$ and $E\varphi = q^{-1}\varphi E$, we see that

$$EY\varphi^{i-1} - q^{-i}Y\varphi^{i-1}E = (EY - q^{-1}YE)\varphi^{i-1} = X\varphi^{i-1}.$$

Now, $(\varphi^i)_X \supseteq (\varphi^{i-1})_X = A_X$, by induction. Therefore, $(\varphi^i)_X = A_X$ for all i. □

One of the most difficult steps in classification of the prime ideals of the algebra A is to show that each maximal ideal $\mathfrak{q}$ of the centre $Z(A) = \mathbb{K}[C]$ generates the prime ideal $A\mathfrak{q}$ of the algebra A. There are two distinct cases: $\mathfrak{q} \neq (C)$ and $\mathfrak{q} = (C)$. The next theorem deals with the first case.

Theorem 3.4. *Let* $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}$. *Then*

(1) *The ideal* $(\mathfrak{q}) := A\mathfrak{q}$ *of* A *is a maximal, completely prime ideal.*

(2) *The factor algebra* $A/(\mathfrak{q})$ *is a simple algebra.*

Proof. Notice that $\mathfrak{q} = \mathbb{K}[C]q'$ where $q' = q'(C)$ is an irreducible monic polynomial such that $q'(0) \in \mathbb{K}^*$.

(i) *The factor algebra* $A/(\mathfrak{q})$ *is a simple algebra, i.e.,* $(\mathfrak{q})$ *is a maximal ideal of* A: Consider the chain of localizations

$$A/(\mathfrak{q}) \longrightarrow \frac{A_X}{(\mathfrak{q})_X} \longrightarrow \frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}}.$$

By Lemma 2.12, $A_{X,\varphi}/(\mathfrak{q})_{X,\varphi} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$ where $L_{\mathfrak{q}} := \mathbb{K}[C]/\mathfrak{q}$ is a finite field extension of $\mathbb{K}$. By Lemma 2.11, the algebra $\mathcal{A}_{X,\varphi}$ is a central simple algebra. Hence, the algebra $A_X/(\mathfrak{q})_X$ is simple iff $(\varphi^i, \mathfrak{q})_X = A_X$ for all $i \geqslant 1$. By Lemma 3.3.(2), $(\varphi^i)_X = A_X$ for all $i \geqslant 1$. Therefore, the algebra $A_X/(\mathfrak{q})_X$ is simple. Hence, the algebra $A/(\mathfrak{q})$ is simple iff $(X^i, \mathfrak{q}) = A$ for all $i \geqslant 1$.

By Lemma 3.3.(1), $(X^i) = (X)^i$ for all $i \geqslant 1$. Therefore, $(X^i, \mathfrak{q}) = (X)^i + (\mathfrak{q})$ for all $i \geqslant 1$. It remains to show that $(X)^i + (\mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 3.2.(1), $(X) = (X, Y)$. If $i = 1$ then $(X) + (\mathfrak{q}) = (X, Y, \mathfrak{q}) = (X, Y, q'(0)) = A$, by (2.14) and $q'(0) \in \mathbb{K}^*$. Now,

$$A = A^i = ((X) + (\mathfrak{q}))^i \subseteq (X)^i + (\mathfrak{q}) \subseteq A,$$

i.e., $(X)^i + (\mathfrak{q}) = A$, as required.

(ii) $(\mathfrak{q})$ *is a completely prime ideal of* A: The set $\mathcal{S} = \{X^i\varphi^j \mid i,j \in \mathbb{N}\}$ is a denominator set of the algebra A. Since $A_{X,\varphi}/(\mathfrak{q})_{X,\varphi} \simeq \mathcal{S}^{-1}(A/(\mathfrak{q}))$ is a (nonzero) algebra and $(\mathfrak{q})$ is a maximal ideal of the algebra A, we have that $\mathrm{tor}_{\mathcal{S}}(A/(\mathfrak{q}))$ is an ideal of the algebra $A/(\mathfrak{q})$ distinct from $A/(\mathfrak{q})$, hence $\mathrm{tor}_{\mathcal{S}}(A/(\mathfrak{q})) = 0$. This means that the algebra $A/(\mathfrak{q})$ is a subalgebra of the algebra $A_{X,\varphi}/(\mathfrak{q})_{X,\varphi} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$, which is a domain. Therefore, the ideal $(\mathfrak{q})$ of A is a completely prime ideal.

(iii) $Z(A/(\mathfrak{q})) = L_{\mathfrak{q}}$: By Lemma 2.11, $Z(\mathcal{A}_{X,\varphi}) = \mathbb{K}$, and $A/(\mathfrak{q}) \subseteq A_{X,\varphi}/(\mathfrak{q})_{X,\varphi} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$, hence $Z(A/(\mathfrak{q})) = L_{\mathfrak{q}}$. □

The case where $\mathfrak{q} = (C)$ is dealt with in the next proposition.

Proposition 3.5. $A \cap (C)_{X,\varphi} = (C)$ *and the ideal* (C) *of* A *is a completely prime ideal.*

Proof. Recall that $A = \mathcal{A}[F;\sigma,\delta]$ (see (2.19)), $\Phi = X\varphi \in \mathcal{A}$ is a product of normal elements X and φ in $\mathcal{A}$ and, by (2.16), the central element C can be written as $C = \Phi F + s$ where

$$\widetilde{y} := \frac{q^4}{1-q^2}YK^{-1} - \frac{1}{1-q^2}YK \quad \text{and} \quad s = -q^2K^{-1}EY^2 - X\widetilde{y}.$$

(i) *If* $Xf \in (C)$ *for some* $f \in A$ *then* $f \in (C)$: Notice that $Xf = Cg$ for some $g \in A$. To prove the statement (i), we use induction on the degree $m = \deg_F(f)$ of the element $f \in A$. Notice that A is a domain and $\deg_F(fg) = \deg_F(f)+\deg_F(g)$ for all $f,g \in A$. The case when $m \leqslant 0$ i.e., $f \in \mathcal{A}$, is obvious since the equality $Xf = Cg$ holds iff $f = g = 0$ (since $\deg_F(Xf) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So, we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. The equality $Xf = Cg$ implies that $\deg_F(g) = \deg_F(Xf) - \deg_F(C) = m-1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ for some $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then (where δ is defined in (2.19))

$$\begin{aligned}
&Xf_0 + Xf_1F + \cdots + Xf_mF^m \\
&= (\Phi F + s)\big(g_0 + g_1F + \cdots + g_{m-1}F^{m-1}\big) \\
&= \Phi\Big(\sigma(g_0)F + \delta(g_0)\Big) + \Phi\big(\sigma(g_1)F + \delta(g_1)\big)F + \cdots + \Phi\big(\sigma(g_{m-1})F \\
&\qquad\qquad + \delta(g_{m-1})\big)F^{m-1} + sg_0 + sg_1F + \cdots + sg_{m-1}F^{m-1} \\
&= \Phi\delta(g_0) + sg_0 + \big(\Phi\sigma(g_0) + \Phi\delta(g_1) + sg_1\big)F + \cdots + \Phi\sigma(g_{m-1})F^m.
\end{aligned} \tag{3.1}$$

Comparing the terms of degree zero we have the equality

$$Xf_0 = \Phi\delta(g_0) + sg_0 = X\varphi\delta(g_0) + (-q^2K^{-1}EY^2 - X\widetilde{y})g_0,$$

i.e., $X(f_0 - \varphi\delta(g_0) + \widetilde{y}g_0) = -q^2K^{-1}EY^2g_0$. All the terms in this equality belong to the algebra $\mathcal{A}$. Recall that X is a normal element in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{A}X$ is a domain (see [8]) and the element $K^{-1}EY^2$ does not belong to the ideal $\mathcal{A}X$. Hence we have $g_0 \in \mathcal{A}X$, i.e., $g_0 = Xh_0$ for some $h_0 \in \mathcal{A}$. Now the element g can be written as $g = Xh_0 + g'F$, where $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2 = \deg_F(g) - 1$ if $m \geqslant 2$. Now, $Xf = C(Xh_0 + g'F)$ and so $X(f - Ch_0) = Cg'F$. Notice that $Cg'F$ has zero constant term as a noncommutative polynomial in F (where the coefficients are written on the left). Therefore, the element $f - Ch_0$ has zero constant term, and hence can be written as $f - Ch_0 = f'F$ for some $f' \in A$ with

$$\begin{aligned}\deg_F(f') + \deg_F(F) &= \deg_F(f'F) = \deg_F(f') + 1 \\ &= \deg_F(f - Ch_0) \leqslant \max\big(\deg_F(f), \deg_F(Ch_0)\big) = m.\end{aligned}$$

Notice that, $\deg_F(f') < \deg_F(f)$. Now, $Cg'F = X(f - Ch_0) = Xf'F$, hence $Xf' = Cg' \in (C)$ (by deleting F). By induction, $f' \in (C)$, and then

$$f = Ch_0 + f'F \in (C),$$

as required.

(ii) *If* $\varphi f \in (C)$ *for some* $f \in A$ *then* $f \in (C)$: Notice that $\varphi f = Cg$ for some $g \in A$. To prove the statement (ii) we use similar arguments to the ones given in the proof of the statement (i). We use induction on $m = \deg_F(f)$. The case where $m \leqslant 0$, i.e., $f \in \mathcal{A}$ is obvious since the equality $\varphi f = Cg$ holds iff $f = g = 0$ (since $\deg_F(\varphi f) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. Then the equality $\varphi f = Cg$ implies that $\deg_F(g) = \deg_F(\varphi f) - \deg_F(C) = m - 1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ where $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then replacing X by φ in (3.1), we have the equality

$$\varphi f_0 + \varphi f_1 F + \cdots + \varphi f_m F^m = \Phi\delta(g_0) + sg_0 + \cdots + \Phi\sigma(g_{m-1})F^m. \tag{3.2}$$

The element s can be written as a sum $s = ((-q/(1-q^2))\varphi K^{-1} + (1/(1-q^2))KX)Y$. Then equating the constant terms of the equality (3.2) and then collecting terms that are multiple of φ we obtain the equality in the algebra $\mathcal{A}$:

$$\varphi\Big(f_0 - X\delta(g_0) + \frac{q}{1-q^2}K^{-1}Yg_0\Big) = \frac{1}{1-q^2}KXYg_0.$$

The element $\varphi \in \mathcal{A}$ is a normal element such that the factor algebra $\mathcal{A}/\mathcal{A}\varphi$ is a domain (see [8]) and the element KXY does not belong to the ideal $\mathcal{A}\varphi$. Therefore, $g_0 \in \mathcal{A}\varphi$, i.e., $g_0 = \varphi h_0$ for some element $h_0 \in \mathcal{A}$. Recall that $\deg_F(g) = m - 1$. Now, $g = \varphi h_0 + g'F$, where $g' \in A$ and $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2 = \deg_F(g) - 1$ if $m \geqslant 2$. So, $\varphi f = Cg = C(\varphi h_0 + g'F)$. Hence,

$$\varphi(f - Ch_0) = Cg'F,$$

and so $f - Ch_0 = f'F$ for some $f' \in A$ with

$$\begin{aligned}\deg_F(f') + \deg_F(F) &= \deg_F(f'F) = \deg_F(f') + 1\\ &= \deg_F(f - Ch_0) \leqslant \max\big(\deg_F(f), \deg_F(Ch_0)\big) = m.\end{aligned}$$

Notice that, $\deg_F(f') < \deg_F(f)$. Now, $Cg'F = \varphi(f - Ch_0) = \varphi f'F$, hence $\varphi f' = Cg' \in (C)$ (by deleting F). Now, by induction, $f' \in (C)$, and then

$$f = Ch_0 + f'F \in (C),$$

as required.

(iii) $A \cap (C)_{X,\varphi} = (C)$: Let $u \in A \cap (C)_{X,\varphi}$. Then $X^i\varphi^j u \in (C)$ for some $i, j \in \mathbb{N}$. It remains to show that $u \in (C)$. By the statement (i), $\varphi^j u \in (C)$, and then by the statement (ii), $u \in (C)$.

(iv) *The ideal (C) of A is a completely prime ideal*: By Lemma 2.12, $A_{X,\varphi}/(C)_{X,\varphi} \simeq \mathcal{A}_{X,\varphi}$, in particular, $A_{X,\varphi}/(C)_{X,\varphi}$ is a domain. By the statement (iii), the algebra $A/(C)$ is a subalgebra of $A_{X,\varphi}/(C)_{X,\varphi}$, so $A/(C)$ is a domain. This means that the ideal (C) is a completely prime ideal of A. □

Let R be a ring. Then each element $r \in R$ determines two maps from R to R, $r\cdot: x \mapsto rx$ and $\cdot r: x \mapsto xr$ where $x \in R$. The next proposition is used in the proof of one of the main results of the paper, Theorem 3.7. It explains why the elements (like X and φ) that satisfy the property of Lemma 3.3 are important in description of prime ideals.

Proposition 3.6 ([8]). *Let R be a Noetherian ring and s be an element of R such that $\mathcal{S}_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s) \subseteq \ker(s\cdot)$). Then,*

$$\mathrm{Spec}\,(R) = \mathrm{Spec}\,(R, s) \sqcup \mathrm{Spec}_s(R),$$

where $\mathrm{Spec}(R, s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) := \{\mathfrak{q} \in \mathrm{Spec}\,(R) \mid s \notin \mathfrak{q}\}$ *and*

(a) *the map* $\mathrm{Spec}(R, s) \mapsto \mathrm{Spec}\,(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, *is a bijection with the inverse* $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ *where* $\pi: R \to R/(s), r \mapsto r + (s)$,

(b) *the map* $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, *is a bijection with the inverse* $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$, *where* $\sigma: R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto r/1$.

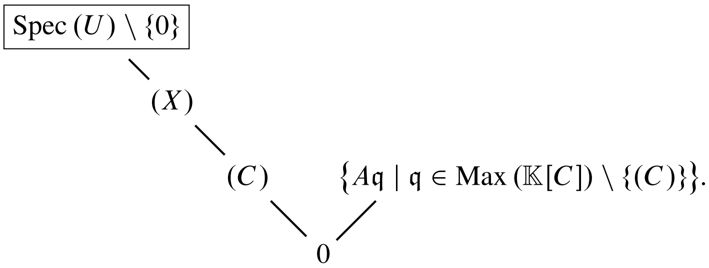
(c) *For all* $\mathfrak{p} \in \mathrm{Spec}\,(R, s)$ *and* $\mathfrak{q} \in \mathrm{Spec}_s(R)$, $\mathfrak{p} \not\subseteq \mathfrak{q}$.

The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(A), \subseteq)$.

Theorem 3.7. *Let $U := U_q(\mathfrak{sl}_2)$. The prime spectrum of the algebra A is a disjoint union*

$$\begin{aligned}\mathrm{Spec}(A) &= \mathrm{Spec}(U) \sqcup \mathrm{Spec}(A_{X,\varphi})\\ &= \big\{(X, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(U)\big\} \sqcup \big\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,\big(\mathbb{K}[C]\big)\big\}.\end{aligned} \tag{3.3}$$

Furthermore,



(3.4)

Proof. By Lemma 3.2.(2), $A/(X) \simeq U$. By Lemma 3.3.(1) and Proposition 3.6, $\mathrm{Spec}\,(A) = \mathrm{Spec}\,(A, X) \sqcup \mathrm{Spec}\,(A_X)$. By Lemma 3.3.(2) and Proposition 3.6, $\mathrm{Spec}\,(A_X) = \mathrm{Spec}\,(A_{X,\varphi})$. Therefore,

$$\mathrm{Spec}\,(A) = \big\{(X, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(U)\big\} \sqcup \big\{A \cap A_{X,\varphi}\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,\big(\mathbb{K}[C]\big)\big\}.$$

Finally, by Theorem 3.4.(1), $A \cap A_{X,\varphi}\mathfrak{q} = (\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$. By Proposition 3.5, $A \cap A_{X,\varphi}C = (C)$. Therefore, (3.3) holds. For all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$, the ideals $A\mathfrak{q}$ of A are maximal. By (2.14), $AC \subseteq (X)$. Therefore, (3.4) holds. □

The next corollary shows that every nonzero ideal of the algebra A meets the centre of A.

Corollary 3.8. *If I is a nonzero ideal of the algebra A then $I \cap \mathbb{K}[C] \neq 0$.*

Proof. Suppose that the result is not true, let us choose an ideal $J \neq 0$ maximal such that $J \cap \mathbb{K}[C] = 0$. We claim that J is a prime ideal. Otherwise, suppose that J is not prime, then there exist ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $J \subsetneqq \mathfrak{p}$, $J \subsetneqq \mathfrak{q}$ and $\mathfrak{p}\mathfrak{q} \subseteq J$. By the maximality of J, $\mathfrak{p} \cap \mathbb{K}[C] \neq 0$ and $\mathfrak{q} \cap \mathbb{K}[C] \neq 0$. Then

$$J \cap \mathbb{K}[C] \supseteq \mathfrak{p}\mathfrak{q} \cap \mathbb{K}[C] \neq 0,$$

a contradiction. So, J is a prime ideal, but by Theorem 3.7 for all nonzero primes P, $P \cap \mathbb{K}[C] \neq 0$, a contradiction. Therefore, for any nonzero ideal I, $I \cap \mathbb{K}[C] \neq 0$. □

The next result is an explicit description of the set of maximal ideals of the algebra A.

Corollary 3.9. $\mathrm{Max}\,(A) = \mathrm{Max}\,(U) \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\}$.

Proof. It is clear by (3.4). □

In the following lemma, we define a family of left A-modules that has bearing of Whittaker modules. It shows that these modules are simple A-modules and their annihilators are equal to (C).

Lemma 3.10. *For $\lambda \in \mathbb{K}^*$, we define the left A-module $W(\lambda) := A/A(X-\lambda, Y, F)$. Then:*

(1) *The module $W(\lambda)$ is a simple A-module.*

(2) $\mathrm{ann}_A(W(\lambda)) = (C)$.

Proof. (1) Let $\bar{1} = 1 + A(X-\lambda, Y, F)$ be the canonical generator of the A-module $W(\lambda)$. Then, $W(\lambda) = \sum_{i\in\mathbb{N}} E^i \mathbb{K}[K^{\pm 1}]\,\bar{1}$. Suppose that V is a nonzero submodule of $W(\lambda)$, we have to show that $V = W(\lambda)$. Let $v = \sum_{i=0}^{n} E^i f_i \bar{1}$ be a nonzero element of the module V where $f_i \in \mathbb{K}[K^{\pm 1}]$ and $f_n \neq 0$. Then,

$$Yv = \sum_{i=1}^{n} \Big(q^i E^i Y - \frac{q(1-q^{2i})}{1-q^2} X E^{i-1} \Big) f_i \bar{1} = \sum_{i=1}^{n} - \frac{q(1-q^{2i})}{1-q^2} X E^{i-1} f_i \bar{1}.$$

By induction, we see that $Y^n v = P\bar{1} \in V$ where P is a nonzero Laurent polynomial in $\mathbb{K}[K^{\pm 1}]$. Then it follows that $\bar{1} \in V$, and so $V = W(\lambda)$.

(2) It is clear that $\mathrm{ann}_A(W(\lambda)) \supseteq (C)$ and $X \notin \mathrm{ann}_A(W(\lambda))$. By (3.4),

$$\mathrm{ann}_A(W(\lambda)) = (C). \qquad \square$$

The next theorem is a description of the set of primitive ideals of the algebra A.

Theorem 3.11. $\mathrm{Prim}\,(A) = \mathrm{Prim}\,(U) \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}(\mathbb{K}[C]) \setminus \{0\}\}$.

Proof. Clearly, $\mathrm{Prim}\,(U) \subseteq \mathrm{Prim}\,(A)$ and

$$\big\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Max}\,\big(\mathbb{K}[C]\big) \setminus \{C\mathbb{K}[C]\}\big\} \subseteq \mathrm{Prim}\,(A)$$

since $A\mathfrak{q}$ is a maximal ideal (Corollary 3.9). By Corollary 3.8, 0 is not a primitive ideal. In view of (3.4) it suffices to show that $(C) \in \mathrm{Prim}\,(A)$. But this follows from Lemma 3.10. $\square$

The next corollary is a description of the set $\mathrm{Spec}_c(A)$ of completely prime ideals of the algebra A.

Corollary 3.12. *The set $\mathrm{Spec}_c(A)$ of completely prime ideals of A is equal to*

$$\begin{aligned}\mathrm{Spec}_c(A) &= \mathrm{Spec}_c(U) \sqcup \big\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,\big(\mathbb{K}[C]\big)\big\} \\ &= \big\{(X, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(U),\, \mathfrak{p} \neq \mathrm{ann}_U(M) \\ &\qquad \textit{for some simple finite dimensional } U\textit{-module } M \\ &\qquad\qquad \textit{of } \dim_{\mathbb{K}}(M) \geqslant 2\big\} \sqcup \big\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,\big(\mathbb{K}[C]\big)\big\}.\end{aligned}$$

Proof. The result follows from Theorem 3.4.(1) and Proposition 3.5. $\square$

4. The centralizer $C_A(K)$ of the element K in the algebra A

In this section, we find the explicit generators and defining relations of the centralizer $C_A(K)$ of the element K in the algebra A.

Proposition 4.1. *The algebra $C_A(K) = \mathbb{K}\langle K^{\pm 1}, FE, YX, EY^2, FX^2\rangle$ is a Noetherian domain.*

Proof. Since A is a domain, then so is its subalgebra $C_A(K)$. Notice that the algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded Noetherian algebra, where $A_i = \{a \in A \mid KaK^{-1} = q^i a\}$. Then the algebra $A_0 = C_A(K)$ is a Noetherian algebra.

The algebra $U_q(\mathfrak{sl}_2)$ is a GWA:

$$U_q(\mathfrak{sl}_2) \simeq \mathbb{K}[K^{\pm 1}, \Omega]\left[E, F; \sigma, a := \Omega - \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}\right],$$

where $\Omega = FE + (qK + q^{-1}K^{-1})/(q-q^{-1})^2$, $\sigma(K) = q^{-2}K$, and $\sigma(\Omega) = \Omega$. In particular, $U_q(\mathfrak{sl}_2)$ is a $\mathbb{Z}$-graded algebra $U_q(\mathfrak{sl}_2) = \bigoplus_{i\in\mathbb{Z}} Dv_i$, where $D := \mathbb{K}[K^{\pm 1}, \Omega] = \mathbb{K}[K^{\pm 1}, FE]$, $v_i = E^i$ if $i \geqslant 1$, $v_i = F^{|i|}$ if $i \leqslant -1$ and $v_0 = 1$. The quantum plane $\mathbb{K}_q[X, Y]$ is also a GWA:

$$\mathbb{K}_q[X, Y] \simeq \mathbb{K}[t][X, Y; \sigma, t], \quad \text{where } t := YX \text{ and } \sigma(t) = qt.$$

Therefore, the quantum plane is a $\mathbb{Z}$-graded algebra $\mathbb{K}_q[X, Y] = \bigoplus_{j\in\mathbb{Z}} \mathbb{K}[t]w_j$, where $w_j = X^j$ if $j \geqslant 1$, $w_j = Y^{|j|}$ if $j \leqslant -1$ and $w_0 = 1$. Since $A = U_q(\mathfrak{sl}_2) \otimes \mathbb{K}_q[X, Y]$ (tensor product of vector spaces), and notice that $Et = tE + X^2$, $Ft = tF + q^{-2}K^{-1}Y^2$, we have

$$A = U_q(\mathfrak{sl}_2) \otimes \mathbb{K}_q[X, Y] = \bigoplus_{i\in\mathbb{Z}} Dv_i \otimes \bigoplus_{j\in\mathbb{Z}} \mathbb{K}[t]w_j = \bigoplus_{i,j\in\mathbb{Z}} D[t]v_i w_j. \tag{4.1}$$

By (4.1), for each $k \in \mathbb{Z}$,

$$A_k = \bigoplus_{i,j\in\mathbb{Z},\, 2i+j=k} D[t]v_i w_j = \bigoplus_{i\in\mathbb{Z}} D[t]v_i w_{k-2i}.$$

Then,

$$C_A(K) = A_0 = \bigoplus_{i\geqslant 0} D[t]E^i Y^{2i} \oplus \bigoplus_{j\geqslant 1} D[t]F^j X^{2j}.$$

Notice that $EY^2 \cdot t = q^{-2}t \cdot EY^2 + qt^2$ and $FX^2 \cdot t = q^2 t \cdot FX^2 + q^{-1}K^{-1}t^2$. By induction, one sees that for all $i, j \geqslant 0$,

$$E^i Y^{2i} \in \bigoplus_{n\in\mathbb{N}} \mathbb{K}[t](EY^2)^n \quad \text{and} \quad F^j X^{2j} \in \bigoplus_{n\in\mathbb{N}} \mathbb{K}[K^{\pm 1}, t](FX^2)^n.$$

Hence,

$$C_A(K) = A_0 = \bigoplus_{i\geqslant 0} D[t](EY^2)^i \oplus \bigoplus_{j\geqslant 1} D[t](FX^2)^j.$$

In particular, the centralizer $C_A(K) = \mathbb{K}\langle K^{\pm 1}, FE, YX, EY^2, FX^2\rangle$. □

Lemma 4.2. (1) $C_{A_{X,Y,\varphi}}(K) = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[(YX)^{\pm 1}, (Y\varphi)^{\pm 1}]$ *is a tensor product of algebras, where* $\mathbb{K}_{q^2}[(YX)^{\pm 1}, (Y\varphi)^{\pm 1}]$ *is a central, simple, quantum torus with* $YX \cdot Y\varphi = q^2 Y\varphi \cdot YX$.

(2) $\mathrm{GK}(C_{A_{X,Y,\varphi}}(K)) = 4$.

(3) $\mathrm{GK}(C_A(K)) = 4$.

(4) $A_{X,Y,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{A_{X,\varphi,Y}}(K)Y^i$.

Proof. (1) By (2.13), $A_{X,Y,\varphi} = \mathbb{K}[C] \otimes \Lambda'$ where Λ' is a quantum torus. Then, $C_{A_{X,Y,\varphi}}(K) = \mathbb{K}[C] \otimes C_{\Lambda'}(K)$. Since Λ' is a quantum torus, it is easy to see that

$$C_{\Lambda'}(K) = \bigoplus_{i,j,k\in\mathbb{Z}} K^i(YX)^j(Y\varphi)^k,$$

i.e., $C_{\Lambda'}(K) = \mathbb{K}[K^{\pm 1}] \otimes \mathbb{K}_{q^2}[(YX)^{\pm 1}, (Y\varphi)^{\pm 1}]$. Then statement 1 follows.

(2) Statement 2 follows from statement 1.

(3) Let R be the subalgebra of $C_A(K)$ generated by the elements C, $K^{\pm 1}$, YX, and $Y\varphi$. Then, $R = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[YX, Y\varphi]$ is a tensor product of algebras. Clearly R is a Noetherian algebra of Gelfand–Kirillov dimension 4. So,

$$\mathrm{GK}(C_A(K)) \geqslant \mathrm{GK}(R) = 4.$$

By statement 2,

$$\mathrm{GK}(C_A(K)) \leqslant \mathrm{GK}(C_{A_{X,Y,\varphi}}(K)) = 4.$$

Hence, $\mathrm{GK}(C_A(K)) = 4$.

(4) Statement 4 follows from statement 1 and (2.13). □

Proposition 4.3. *Let* $h := \varphi X^{-1}$, $e := EX^{-2}$, *and* $t := YX$. *Then:*

(1) $C_{A_{X,\varphi}}(K) = \mathbb{K}[C, K^{\pm 1}] \otimes \mathcal{A}$ *is a tensor product of algebras, where*

$$\mathcal{A} := \mathbb{K}[h^{\pm 1}]\left[t, e; \sigma, a = \frac{q^{-2}h - 1}{1 - q^2}\right]$$

is a central simple GWA (where $\sigma(h) = q^2 h$*).*

(2) $\mathrm{GK}(C_{A_{X,\varphi}}(K)) = 4$.

(3) $A_{X,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{A_{X,\varphi}}(K)X^i$.

Proof. (1) Let $\mathcal{A}$ be the subalgebra of $C_{A_{X,\varphi}}(K)$ generated by the elements $h^{\pm 1}$, e, and t.

(i) *$\mathcal{A}$ is a central simple GWA*: The elements $h^{\pm 1}$, e and t satisfy the following relations

$$hh^{-1} = h^{-1}h = 1, \quad th = q^2 ht, \quad eh = q^{-2}he,$$
$$et = \frac{q^{-2}h - 1}{1 - q^2}, \quad te = \frac{h - 1}{1 - q^2}. \tag{4.2}$$

Hence, $\mathcal{A}$ is an epimorphic image of the GWA

$$\mathcal{A}' = \mathbb{K}[h^{\pm 1}]\left[t, e; \sigma, a = \frac{q^{-2}h - 1}{1 - q^2}\right],$$

where $\sigma(h) = q^2 h$. Now, we prove that $\mathcal{A}'$ is a central simple algebra. Let $\mathcal{A}'_e$ be the localization of $\mathcal{A}'$ at the powers of the element e. Then $\mathcal{A}'_e = \mathbb{K}[h^{\pm 1}][e^{\pm 1}; \sigma']$, where $\sigma'(h) = q^{-2}h$. Clearly, $Z(\mathcal{A}'_e) = \mathbb{K}$ and $\mathcal{A}'_e$ is a simple algebra. So, $Z(\mathcal{A}') = Z(\mathcal{A}'_e) \cap \mathcal{A}' = \mathbb{K}$. To show that $\mathcal{A}'$ is simple, it suffices to prove that $\mathcal{A}' e^i \mathcal{A}' = \mathcal{A}'$ for any $i \in \mathbb{N}$. The case $i = 1$ is obvious, since $1 = q^2 et - te \in \mathcal{A}' e \mathcal{A}'$. By induction, for $i > 1$, it suffices to show that $e^{i-1} \in \mathcal{A}' e^i \mathcal{A}'$. This follows from the equality

$$te^i = q^{2i} e^i t - \frac{1 - q^{2i}}{1 - q^2} e^{i-1}.$$

So, $\mathcal{A}'$ is a simple algebra. Now, the epimorphism of algebras $\mathcal{A}' \longrightarrow\!\!\!\!\rightarrow \mathcal{A}$ is an isomorphism. Hence, $\mathcal{A} \simeq \mathcal{A}'$ is a central simple GWA.

(ii) $C_{A_{X,\varphi}}(K) = \mathbb{K}[C, K^{\pm 1}] \otimes \mathcal{A}$: By Lemma 2.12, $A_{X,\varphi} = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}$. So,

$$C_{A_{X,\varphi}}(K) = \mathbb{K}[C] \otimes C_{\mathcal{A}_{X,\varphi}}(K).$$

By (2.18), $\mathcal{A}_{X,\varphi} = \mathbb{E}_{X,\varphi}[K^{\pm 1}; \tau]$, where $\tau(E) = q^2 E$, $\tau(X) = qX$, $\tau(Y) = q^{-1}Y$, and $\tau(\varphi) = q\varphi$. Then,

$$C_{\mathcal{A}_{X,\varphi}}(K) = \mathbb{K}[K^{\pm 1}] \otimes \mathbb{E}^{\tau}_{X,\varphi}.$$

To finish the proof of statement (ii), it suffices to show that $\mathbb{E}^{\tau}_{X,\varphi} = \mathcal{A}$. By (2.3),

$$\mathbb{E}_{X,\varphi} = \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}]\left[E, Y; \sigma, a = \frac{\varphi - X}{q^{-1} - q}\right]$$

is a GWA. Then,

$$\mathbb{E}_{X,\varphi} = \bigoplus_{i \geqslant 0} \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}] E^i \oplus \bigoplus_{j \geqslant 1} \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}] Y^j$$
$$= \bigoplus_{i \geqslant 0, k \in \mathbb{Z}} \mathbb{K}[h^{\pm 1}] E^i X^k \oplus \bigoplus_{j \geqslant 1, k \in \mathbb{Z}} \mathbb{K}[h^{\pm 1}] Y^j X^k.$$

Now, it is clear that $\mathbb{E}^{\tau}_{X,\varphi} = \bigoplus_{i \geqslant 0} \mathbb{K}[h^{\pm 1}] e^i \oplus \bigoplus_{j \geqslant 1} \mathbb{K}[h^{\pm 1}] t^j = \mathcal{A}$.

(2) Notice that $\mathrm{GK}(\mathcal{A}) = 2$, statement 2 follows from statement 1.

(3) Notice that $\mathcal{A}_{X,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{\mathcal{A}_{X,\varphi}}(K)X^i$, statement 3 then follows from Lemma 2.12. □

Defining relations of the algebra $C_A(K)$. We have to select appropriate generators of the algebra $C_A(K)$ to make the corresponding defining relations simpler.

Lemma 4.4. *We have the following relations:*

(1) $YX \cdot Y\varphi = q^2 Y\varphi \cdot YX$.

(2) $FE \cdot YX = q^2 YX \cdot FE + \dfrac{q+q^{-1}}{1-q^2}K^{-1}Y\varphi - \dfrac{q^2(qK+q^{-1}K^{-1})}{1-q^2}YX + C$.

(3) $FE \cdot Y\varphi = q^{-2}Y\varphi \cdot FE + \dfrac{qK+q^{-1}K^{-1}}{1-q^2}Y\varphi - \dfrac{q(1+q^2)}{1-q^2}KYX + C$.

Proof. (1) Obvious.

(2) Using the defining relations of A, the expression (2.14) of C, and $Y\varphi = q^4YX + q(1-q^2)EY^2$,

$$\begin{aligned}
FE\cdot YX &= F(X+q^{-1}YE)X \\
&= FX^2 + YFXE = FX^2 + Y(YK^{-1}+XF)E \\
&= FX^2 + q^{-2}K^{-1}Y^2E + YXFE \\
&= q^2(YX)(FE) + (1+q^2)K^{-1}EY^2 - \frac{q^3K + (q-q^3-q^5)K^{-1}}{1-q^2}YX + C \\
&= q^2YX\cdot FE + \frac{q+q^{-1}}{1-q^2}K^{-1}Y\varphi - \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}YX + C.
\end{aligned}$$

(3) $$\begin{aligned}
FE\cdot Y\varphi &= F(X+q^{-1}YE)\varphi \\
&= FX\varphi + q^{-2}YF\varphi E = FX\varphi + q^{-2}Y(\varphi F + YK)E \\
&= q^{-2}Y\varphi FE + (q^2K+K^{-1})EY^2 - \Big(\frac{q^3(K-K^{-1})}{1-q^2} + q(1+q^2)K\Big)YX + C \\
&= q^{-2}Y\varphi\cdot FE + \frac{qK+q^{-1}K^{-1}}{1-q^2}Y\varphi - \frac{q(1+q^2)}{1-q^2}KYX + C.
\end{aligned}$$ □

Let $\Theta := (1-q^2)\Omega = (1-q^2)FE + q^2(qK+q^{-1}K^{-1})/(1-q^2) \in Z(U_q(\mathfrak{sl}_2))$. By (2.15), we have

$$C = \Big(\Theta - \frac{qK^{-1}}{1-q^2}\Big)YX + q^2FX^2 - \frac{1}{q(1-q^2)}K^{-1}Y\varphi. \tag{4.3}$$

By Lemma 4.4.(2), (3), we have

$$\Theta \cdot YX = q^2 YX \cdot \Theta + (q + q^{-1}) K^{-1} Y\varphi + (1 - q^2)C, \tag{4.4}$$
$$\Theta \cdot Y\varphi = q^{-2} Y\varphi \cdot \Theta - q(1 + q^2) KYX + (1 - q^2)C. \tag{4.5}$$

Lemma 4.5. *In the algebra $C_A(K)$, the following relation holds*

$$\Theta \cdot YX \cdot Y\varphi - \frac{1}{q(1-q^2)} K^{-1} (Y\varphi)^2 - C \cdot Y\varphi = \frac{q^7}{1-q^2} K(YX)^2 - q^4 C \cdot YX.$$

Proof. By (4.3),

$$\Theta \cdot YX = C + \frac{q}{1-q^2} K^{-1} YX - q^2 FX^2 + \frac{1}{q(1-q^2)} K^{-1} Y\varphi.$$

So,

$$\Theta \cdot YX \cdot Y\varphi = C \cdot Y\varphi + \frac{q}{1-q^2} K^{-1} YX \cdot Y\varphi - q^2 FX^2 \cdot Y\varphi + \frac{1}{q(1-q^2)} K^{-1} (Y\varphi)^2.$$

Then,

$$\Theta \cdot YX \cdot Y\varphi - \frac{1}{q(1-q^2)} K^{-1} (Y\varphi)^2 - C \cdot Y\varphi = \frac{q}{1-q^2} K^{-1} YX \cdot Y\varphi - q^2 FX^2 \cdot Y\varphi.$$

We have that $YX \cdot Y\varphi = q^4 (YX)^2 + q(1-q^2) YX \cdot EY^2$, $FX^2 \cdot Y\varphi = q^2 FX\varphi \cdot YX$, and $EY^2 \cdot YX = q(YX)^2 + q^{-2} YX \cdot EY^2$. Then by (2.16), we obtain the identity as desired. □

Theorem 4.6. *Let $u := K^{-1} Y\varphi$ and recall that $t = YX$, $\Theta = (1 - q^2)FE + q^2 (qK + q^{-1} K^{-1})/(1 - q^2)$. Then the algebra $C_A(K)$ is generated by the elements $K^{\pm 1}$, C, Θ, t, and u subject to the following defining relations:*

$$t \cdot u = q^2 u \cdot t, \tag{4.6}$$
$$\Theta \cdot t = q^2 t \cdot \Theta + (q + q^{-1})u + (1 - q^2)C, \tag{4.7}$$
$$\Theta \cdot u = q^{-2} u \cdot \Theta - q(1 + q^2)t + (1 - q^2) K^{-1} C, \tag{4.8}$$
$$\Theta \cdot t \cdot u - \frac{1}{q(1-q^2)} u^2 - C \cdot u = \frac{q^7}{1-q^2} t^2 - q^4 K^{-1} C \cdot t, \tag{4.9}$$
$$[K^{\pm 1}, \cdot] = 0, \quad \text{and} \quad [C, \cdot] = 0, \tag{4.10}$$

where (4.10) means that the elements $K^{\pm 1}$ and C are central in $C_A(K)$. Furthermore, $Z(C_A(K)) = \mathbb{K}[C, K^{\pm 1}]$.

Proof. (i) *Generators of* $C_A(K)$: Notice that $Y\varphi = q^4YX + q(1-q^2)EY^2$. Then by Proposition 4.1 and (4.3), the algebra $C_A(K)$ is generated by the elements C, $K^{\pm 1}$, Θ, t, and u. By (4.4), (4.5) and Lemma 4.5, the elements C, $K^{\pm 1}$, Θ, t, and u satisfy the relations (4.6)–(4.10). It remains to show that these relations are defining relations.

Let $\mathscr{C}$ be the $\mathbb{K}$-algebra generated by the symbols C, $K^{\pm 1}$, Θ, t and u subject to the defining relations (4.6)–(4.10). Then there is a natural epimorphism of algebras $f : \mathscr{C} \twoheadrightarrow C_A(K)$. Our aim is to prove that f is an algebra isomorphism.

(ii) $\mathrm{GK}(\mathscr{C}) = 4$ *and* $Z(\mathscr{C}) = \mathbb{K}[C, K^{\pm 1}]$: Let R be the subalgebra of $\mathscr{C}$ generated by the elements C, $K^{\pm 1}$, t and u. Then $R = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t,u]$ is a tensor product of algebra where $\mathbb{K}_{q^2}[t,u] := \mathbb{K}\langle t, u \mid tu = q^2ut\rangle$ is a quantum plane. Clearly, R is a Noetherian algebra of Gelfand–Kirillov dimension 4. Let $\mathscr{C}_{t,u}$ be the localization of $\mathscr{C}$ at the powers of the elements t and u. Then,

$$\mathscr{C}_{t,u} = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}] = R_{t,u}.$$

So, $\mathrm{GK}(\mathscr{C}_{t,u}) = 4$. Now, the inclusions $R \subseteq \mathscr{C} \subseteq \mathscr{C}_{t,u}$ yield that

$$4 = \mathrm{GK}(R) \leqslant \mathrm{GK}(\mathscr{C}) \leqslant \mathrm{GK}(\mathscr{C}_{t,u}) = 4,$$

i.e., $\mathrm{GK}(\mathscr{C}) = 4$. Moreover, since $\mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a central simple algebra,

$$Z(\mathscr{C}_{t,u}) = \mathbb{K}[C, K^{\pm 1}].$$

Hence, $Z(\mathscr{C}) = \mathbb{K}[C, K^{\pm 1}]$.

By Lemma 4.2.(3), $\mathrm{GK}(\mathscr{C}) = \mathrm{GK}(C_A(K)) = 4$. In view of [20, Proposition 3.15], to show that the epimorphism $f : \mathscr{C} \twoheadrightarrow C_A(K)$ is an isomorphism it suffices to prove that $\mathscr{C}$ is a domain.

Let $\mathscr{D}$ be the algebra generated by the symbols C, $K^{\pm 1}$, Θ, t, and u subject to the defining relations (4.6)–(4.8) and (4.10). Then $\mathscr{D}$ is an Ore extension

$$\mathscr{D} = R[\Theta; \sigma, \delta],$$

where $R = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t,u]$ is a Noetherian domain; $\sigma(C) = C$, $\sigma(K^{\pm 1}) = K^{\pm 1}$, $\sigma(t) = q^2 t$, $\sigma(u) = q^{-2}u$; δ is a σ-derivation of R given by the rule

$$\delta(C) = \delta(K^{\pm 1}) = 0,$$
$$\delta(t) = (q+q^{-1})u + (1-q^2)C, \quad \text{and} \quad \delta(u) = -q(1+q^2)t + (1-q^2)K^{-1}C.$$

In particular, $\mathscr{D}$ is a Noetherian domain. Let

$$\begin{aligned} Z &:= \Theta tu - \frac{1}{q(1-q^2)}u^2 - Cu - \frac{q^7}{1-q^2}t^2 + q^4K^{-1}Ct \\ &= tu\Theta - \widehat{q}(u^2+t^2) - q^2C(u - K^{-1}t) \in \mathscr{D}, \end{aligned}$$

where $\widehat{q} = q^3/(1-q^2)$. Then Z is a central element of $\mathcal{D}$ and $\mathcal{C} \simeq \mathcal{D}/(Z)$. To prove that $\mathcal{C}$ is a domain, it suffices to show that (Z) is a completely prime ideal of $\mathcal{D}$. Notice that $\mathcal{D}_{t,u} = \mathbb{K}[C, K^{\pm 1}, Z] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a tensor product of algebras. Then,

$$\mathcal{C}_{t,u} \simeq \mathcal{D}_{t,u}/(Z)_{t,u} \simeq \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}] \simeq R_{t,u}.$$

In particular, $\mathcal{C}_{t,u}$ is a domain and $(Z)_{t,u}$ is a completely prime ideal of $\mathcal{D}_{t,u}$.

(iii) *If $tx \in (Z)$ for some element $x \in \mathcal{D}$ then $x \in (Z)$*: Since Z is central in $\mathcal{D}$, $tx = Zd$ for some element $d \in \mathcal{D}$. We prove statement (iii) by induction on the degree $\deg_\Theta(x)$ of the element x. Since $\mathcal{D}$ is a domain, $\deg_\Theta(dd') = \deg_\Theta(d) + \deg_\Theta(d')$ for all elements $d, d' \in \mathcal{D}$. Notice that $\deg_\Theta(Z) = 1$, the case $x \in R$ is trivial. So we may assume that $m = \deg_\Theta(x) \geqslant 1$ and then the element x can be written as $x = a_0 + a_1\Theta + \cdots + a_m\Theta^m$ where $a_i \in R$ and $a_m \neq 0$. The equality $tx = Zd$ yields that $\deg_\Theta(d) = m-1$ since $\deg_\Theta(Z) = 1$. Hence,

$$d = d_0 + d_1\Theta + \cdots + d_{m-1}\Theta^{m-1}$$

for some $d_i \in R$ and $d_{m-1} \neq 0$. Now, the equality $tx = Zd$ can be written as follows:

$$\begin{aligned} &t(a_0 + a_1\Theta + \cdots + a_m\Theta^m) \\ &\quad = \big(tu\Theta - \widehat{q}(u^2+t^2) - q^2C(u - K^{-1}t)\big)\big(d_0 + d_1\Theta + \cdots + d_{m-1}\Theta^{m-1}\big). \end{aligned}$$

Comparing the terms of degree zero in the equality we have

$$ta_0 = tu\delta(d_0) - \big(\widehat{q}(u^2+t^2) + q^2C(u - K^{-1}t)\big)d_0,$$

i.e., $t(a_0 - u\delta(d_0) + \widehat{q}td_0 - q^2CK^{-1}d_0) = -u(\widehat{q}u + q^2C)d_0$. All terms in this equality are in the algebra R. Notice that t is a normal element of R, the elements $u \notin tR$ and $\widehat{q}u + q^2C \notin tR$, we have $d_0 \in tR$. So $d_0 = tr$ for some element $r \in R$. Then $d = tr + w\Theta$, where $w = d_1 + \cdots + d_{m-1}\Theta^{m-2}$ if $m \geqslant 2$ and $w = 0$ if $m = 1$. If $m = 1$ then $d = tr$ and the equality $tx = Zd$ yields that $tx = tZr$, i.e., $x = Zr \in (Z)$ (by deleting t), we are done. So we may assume that $m \geqslant 2$. Now, the equality $tx = Zd$ can be written as $tx = Z(tr + w\Theta)$, i.e., $t(x - Zr) = Zw\Theta$. This implies that $x - Zr = x'\Theta$ for some $x' \in \mathcal{D}$, where $\deg_\Theta(x') < \deg_\Theta(x)$. Now, $tx'\Theta = Zw\Theta$ and hence, $tx' = Zw$ (by deleting Θ). By induction $x' \in (Z)$. Then $x = x' + Zr \in (Z)$.

(iv) *If $ux \in (Z)$ for some element $x \in \mathcal{D}$ then $x \in (Z)$*: Notice that the elements u and t are "symmetric" in the algebra $\mathcal{D}$, statement (iv) can be proved similarly as that of statement (iii).

(v) $\mathcal{D} \cap (Z)_{t,u} = (Z)$: The inclusion $(Z) \subseteq \mathcal{D} \cap (Z)_{t,u}$ is obvious. Let $x \in \mathcal{D} \cap (Z)_{t,u}$. Then, $t^iu^jx \in (Z)$ for some $i, j \in \mathbb{N}$. By statement (iii) and statement (iv), $x \in (Z)$. Hence, $\mathcal{D} \cap (Z)_{t,u} = (Z)$.

By statement (v), the algebra $\mathcal{D}/(Z)$ is a subalgebra of $\mathcal{D}_{t,u}/(Z)_{t,u}$. Hence, $\mathcal{D}/(Z)$ is a domain. This completes the proof. □

The next proposition gives a $\mathbb{K}$-basis for the algebra $\mathcal{C} := C_A(K)$.

Proposition 4.7.

$$\mathcal{C} = \mathbb{K}[C, K^{\pm 1}] \otimes_{\mathbb{K}} \Big(\bigoplus_{i,j \geqslant 1} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{k \geqslant 1} \mathbb{K}\Theta^k \oplus \bigoplus_{l,m \geqslant 1} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{a,b \geqslant 0} \mathbb{K} u^a t^b \Big).$$

Proof. The relations (4.6)–(4.9) can be written in the following equivalent form,

$$u \cdot t = q^{-2} t \cdot u, \quad \Theta \cdot t \cdot u = \frac{1}{q(1-q^2)} u^2 + C \cdot u + \frac{q^7}{1-q^2} t^2 - q^4 K^{-1} C \cdot t,$$
$$u \cdot \Theta = q^2 \Theta \cdot u + q^3(1+q^2) t - q^2(1-q^2) K^{-1} C,$$
$$t \cdot \Theta = q^{-2} \Theta \cdot t - q^{-2}(q+q^{-1}) u - q^{-2}(1-q^2) C.$$

On the free monoid W generated by C, K, K', Θ, t, and u (where K' plays the role of K^{-1}), we introduce the length-lexicographic ordering such that $K' < K < C < \Theta < t < u$. With respect to this ordering the Diamond lemma (see [10], [11, I.11]) can be applied to $\mathcal{C}$ as there is only one ambiguity which is the overlap ambiguity $ut\Theta$ and it is resolvable as the following computations show:

$$\begin{aligned}
(ut)\Theta &\to q^{-2} t u \Theta \\
&\to q^{-2} t \big(q^2 \Theta u + q^3(1+q^2) t - q^2(1-q^2) K' C \big) \\
&\to t \Theta u + q(1+q^2) t^2 - (1-q^2) K' C t \\
&\to \big(q^{-2} \Theta t - q^{-2}(q+q^{-1}) u - q^{-2}(1-q^2) C \big) u \\
&\qquad\qquad + q(1+q^2) t^2 - (1-q^2) K' C t \\
&\to q^{-2} \Theta t u - q^{-2}(q+q^{-1}) u^2 - q^{-2}(1-q^2) C u \\
&\qquad\qquad + q(1+q^2) t^2 - (1-q^2) K' C t \\
&\to \frac{q}{1-q^2} u^2 + C u + \frac{q}{1-q^2} t^2 - K' C t, \\
u(t\Theta) &\to u \big(q^{-2} \Theta t - q^{-2}(q+q^{-1}) u - q^{-2}(1-q^2) C \big) \\
&\to q^{-2} u \Theta t - q^{-2}(q+q^{-1}) u^2 - q^{-2}(1-q^2) C u \\
&\to q^{-2} \big(q^2 \Theta u + q^3(1+q^2) t - q^2(1-q^2) K' C \big) t - q^{-2}(q+q^{-1}) u^2 \\
&\qquad\qquad - q^{-2}(1-q^2) C u \\
&\to \Theta u t + q(1+q^2) t^2 - (1-q^2) K' C t - q^{-2}(q+q^{-1}) u^2 \\
&\qquad\qquad - q^{-2}(1-q^2) C u
\end{aligned}$$

$$\to q^{-2}\Theta tu + q(1+q^2)t^2 - (1-q^2)K'Ct - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu$$

$$\to \frac{q}{1-q^2}u^2 + Cu + \frac{q}{1-q^2}t^2 - K'Ct.$$

So, by the Diamond lemma, the result is proved. □

The algebra $\mathcal{C}^{\lambda,\mu}$. For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, let $\mathcal{C}^{\lambda,\mu} := \mathcal{C}/(C-\lambda, K-\mu)$. By Theorem 4.6, the algebra $\mathcal{C}^{\lambda,\mu}$ is generated by the images of the elements Θ, t, and u in $\mathcal{C}^{\lambda,\mu}$. For simplicity, we denote by the same letters their images.

Corollary 4.8. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. Then:*

(1) *The algebra $\mathcal{C}^{\lambda,\mu}$ is generated by the elements Θ, t and u subject to the following defining relations*

$$t \cdot u = q^2 u \cdot t, \tag{4.11}$$

$$\Theta \cdot t = q^2 t \cdot \Theta + (q+q^{-1})u + (1-q^2)\lambda, \tag{4.12}$$

$$\Theta \cdot u = q^{-2} u \cdot \Theta - q(1+q^2)t + (1-q^2)\mu^{-1}\lambda, \tag{4.13}$$

$$\Theta \cdot t \cdot u = \frac{1}{q(1-q^2)}u^2 + \lambda u + \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t. \tag{4.14}$$

(2) $\mathcal{C}^{\lambda,\mu} = \bigoplus_{i,j\geqslant 1} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{k\geqslant 1} \mathbb{K}\Theta^k \oplus \bigoplus_{l,m\geqslant 1} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{a,b\geqslant 0} \mathbb{K}u^a t^b.$

Proof. (1) Statement 1 follows from Theorem 4.6.

(2) Statement 2 follows from Proposition 4.7. □

Let $\mathcal{C}_t$ (resp., $\mathcal{C}_t^{\lambda,\mu}$) be the localization of the algebra $\mathcal{C}$ (resp., $\mathcal{C}^{\lambda,\mu}$) at the powers of the element $t = YX$. The next proposition shows that $\mathcal{C}_t$ and $\mathcal{C}_t^{\lambda,\mu}$ are GWAs.

Proposition 4.9. (1) *Let $v := \Theta t - (1/q(1-q^2))u - C$. The algebra*

$$\mathcal{C}_t = \mathbb{K}[C, K^{\pm 1}, t^{\pm 1}][u, v; \sigma, a]$$

is a GWA of Gelfand–Kirillov dimension 4, where $a = (q^7/(1-q^2))t^2 - q^4K^{-1}Ct$ and σ is the automorphism of the algebra $\mathbb{K}[C, K^{\pm 1}, t^{\pm 1}]$ defined by the rule:

$$\sigma(C) = C, \quad \sigma(K^{\pm 1}) = K^{\pm 1}, \quad \text{and} \quad \sigma(t) = q^{-2}t.$$

(2) *Let $\lambda \in \mathbb{K}$, $\mu \in \mathbb{K}^*$, and $v := \Theta t - (1/q(1-q^2))u - \lambda$. Then the algebra*

$$\mathcal{C}_t^{\lambda,\mu} = \mathbb{K}[t^{\pm 1}][u, v; \sigma, a]$$

is a GWA of Gelfand–Kirillov dimension 2 where $a = (q^7/(1-q^2))t^2 - q^4\mu^{-1}\lambda t$ and σ is the automorphism of the algebra $\mathbb{K}[t^{\pm 1}]$ defined by $\sigma(t) = q^{-2}t$.

(3) *For any* $\lambda \in \mathbb{K}$ *and* $\mu \in \mathbb{K}^*$, *the algebra* $\mathcal{C}_t^{\lambda,\mu}$ *is a central simple algebra.*
(4) $Z(\mathcal{C}^{\lambda,\mu}) = \mathbb{K}$ *and* $\mathrm{GK}(\mathcal{C}^{\lambda,\mu}) = 2$.

Proof. (1) By Theorem 4.6, the algebra $\mathcal{C}_t$ is generated by the elements C, $K^{\pm 1}$, v, $t^{\pm 1}$, and u. Note that the element v can be written as

$$v = -\frac{q^2}{1-q^2}\psi X = \frac{q}{1-q^2}\tau(u),$$

where τ is the involution (2.1). It is straightforward to verify that the following relations hold in the algebra $\mathcal{C}_t$

$$ut = q^{-2}tu, \quad vt = q^2 tv,$$
$$vu = \frac{q^7}{1-q^2}t^2 - q^4K^{-1}Ct, \quad uv = \frac{q^3}{1-q^2}t^2 - q^2K^{-1}Ct.$$

Then $\mathcal{C}_t$ is an epimorphic image of the GWA $T := \mathbb{K}[C, K^{\pm 1}, t^{\pm 1}][u, v; \sigma, a]$. Notice that T is a Noetherian domain of Gelfand–Kirillov dimension 4. The inclusions $\mathcal{C} \subseteq \mathcal{C}_t \subseteq \mathcal{C}_{t,u}$ yield that $4 = \mathrm{GK}(\mathcal{C}) \leqslant \mathrm{GK}(\mathcal{C}_t) \leqslant \mathcal{C}_{t,u} = 4$ (see Lemma 4.2.(3)), i.e., $\mathrm{GK}(\mathcal{C}_t) = 4$. So, $\mathrm{GK}(T) = \mathrm{GK}(\mathcal{C}_t)$. By [20, Proposition 3.15], the epimorphism of algebras $T \twoheadrightarrow \mathcal{C}_t$ is an isomorphism.

(2) Statement 2 follows from statement 1.

(3) Let $\mathcal{C}_{t,u}^{\lambda,\mu}$ be the localization of $\mathcal{C}_t^{\lambda,\mu}$ at the powers of the element u. Then, by statement 2, $\mathcal{C}_{t,u}^{\lambda,\mu} = \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a central, simple quantum torus. So,

$$Z(\mathcal{C}_t^{\lambda,\mu}) = Z(\mathcal{C}_{t,u}^{\lambda,\mu}) \cap \mathcal{C}_t^{\lambda,\mu} = \mathbb{K}.$$

For any nonzero ideal $\mathfrak{a}$ of the algebra $\mathcal{C}_t^{\lambda,\mu}$, $u^i \in \mathfrak{a}$ for some $i \in \mathbb{N}$ since $\mathcal{C}_{t,u}^{\lambda,\mu}$ is a simple Noetherian algebra. Therefore, to prove that $\mathcal{C}_t^{\lambda,\mu}$ is a simple algebra, it suffices to show that $\mathcal{C}_t^{\lambda,\mu}u^i\mathcal{C}_t^{\lambda,\mu} = \mathcal{C}_t^{\lambda,\mu}$ for any $i \in \mathbb{N}$. The case $i = 1$ follows from the equality $vu = q^2uv - q^5t^2$. By induction, for $i > 1$, it suffices to show that $u^{i-1} \in \mathcal{C}_t^{\lambda,\mu}u^i\mathcal{C}_t^{\lambda,\mu}$. This follows from the equality

$$vu^i = q^{2i}u^iv + \frac{q^7(1-q^{-2i})}{1-q^2}t^2u^{i-1}.$$

Hence, $\mathcal{C}_t^{\lambda,\mu}$ is a simple algebra.

(4) Since $\mathbb{K} \subseteq Z(\mathcal{C}^{\lambda,\mu}) \subseteq Z(\mathcal{C}_t^{\lambda,\mu}) \cap \mathcal{C}^{\lambda,\mu} = \mathbb{K}$, we have $Z(\mathcal{C}^{\lambda,\mu}) = \mathbb{K}$. It is clear that $\mathrm{GK}(\mathcal{C}^{\lambda,\mu}) = 2$. □

Lemma 4.10. *In the algebra* $\mathcal{C}^{\lambda,\mu}$ *where* $\lambda \in \mathbb{K}$ *and* $\mu \in \mathbb{K}^*$, *the following equality holds*

$$\Theta t^i = q^{2i}t^i\Theta + \frac{q^{-2i+1} - q^{2i+1}}{1-q^2}t^{i-1}u + (1-q^{2i})\lambda t^{i-1}.$$

Proof. By induction on i and using the equality (4.12). □

Theorem 4.11. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$.*

(1) *The algebra $\mathcal{C}^{\lambda,\mu}$ is a simple algebra iff $\lambda \neq 0$.*

(2) *The algebra $\mathcal{C}^{\lambda,\mu}$ is a domain.*

Proof. (1) If $\lambda = 0$ then the ideal (t) is a proper ideal of the algebra $\mathcal{C}^{0,\mu}$. Hence, $\mathcal{C}^{0,\mu}$ is not a simple algebra. Now, suppose that $\lambda \neq 0$, we have to prove that $\mathcal{C}^{\lambda,\mu}$ is a simple algebra. By Proposition 4.9.(3), $\mathcal{C}_t^{\lambda,\mu}$ is a simple algebra. Hence, it suffices to show that $\mathcal{C}^{\lambda,\mu}t^i\mathcal{C}^{\lambda,\mu} = \mathcal{C}^{\lambda,\mu}$ for all $i \in \mathbb{N}$. We prove this by induction on i.

Firstly, we prove the case for $i = 1$, i.e., $\mathfrak{a} := \mathcal{C}^{\lambda,\mu}t\mathcal{C}^{\lambda,\mu} = \mathcal{C}^{\lambda,\mu}$. By (4.12), the element $(q+q^{-1})u + (1-q^2)\lambda \in \mathfrak{a}$, so, $u \equiv ((q^2-1)/(q+q^{-1}))\lambda \bmod \mathfrak{a}$. By (4.14), $(1/q(1-q^2))u^2 + \lambda u \in \mathfrak{a}$. Hence,

$$\frac{1}{q(1-q^2)}\Big(\frac{q^2-1}{q+q^{-1}}\lambda\Big)^2 + \lambda\Big(\frac{q^2-1}{q+q^{-1}}\lambda\Big) \equiv 0 \bmod \mathfrak{a},$$

i.e., $q^2(q^2-1)\lambda^2/(q^2+1) \equiv 0 \bmod \mathfrak{a}$. Since $\lambda \neq 0$, this implies that $1 \in \mathfrak{a}$, thus, $\mathfrak{a} = \mathcal{C}^{\lambda,\mu}$.

Let us now prove that $\mathfrak{b} := \mathcal{C}^{\lambda,\mu}t^i\mathcal{C}^{\lambda,\mu} = \mathcal{C}^{\lambda,\mu}$ for any $i \in \mathbb{N}$. By induction, for $i > 1$, it suffices to show that $t^{i-1} \in \mathfrak{b}$. By Lemma 4.10, the element

$$\mathbf{u} := \frac{q^{-2i+1} - q^{2i+1}}{1-q^2}t^{i-1}u + (1-q^{2i})\lambda t^{i-1} \in \mathfrak{b}.$$

Then $v\mathbf{u} \in \mathfrak{b}$, where

$$v = \Theta t - \frac{1}{q(1-q^2)}u - \lambda,$$

see Proposition 4.9.(2). This implies that $(1-q^{2i})\lambda v t^{i-1} \in \mathfrak{b}$ and so, $vt^{i-1} \in \mathfrak{b}$. But then the inclusion $vt^{i-1} = (\Theta t - (1/q(1-q^2))u - \lambda)t^{i-1} \in \mathfrak{b}$ yields that the element

$$\mathbf{v} := \frac{q^{-2i+1}}{1-q^2}t^{i-1}u + \lambda t^{i-1} \in \mathfrak{b}.$$

By the expressions of the elements $\mathbf{u}$ and $\mathbf{v}$ we see that $t^{i-1} \in \mathfrak{b}$, as required.

(2) By Proposition 4.9.(2), the GWA $\mathcal{C}_t^{\lambda,\mu} \simeq \mathcal{C}_t/\mathcal{C}_t(C-\lambda, K-\mu)$ is a domain. Let

$$\mathfrak{a} = \mathcal{C}(C-\lambda, K-\mu) \quad \text{and} \quad \mathfrak{a}' = \mathcal{C} \cap \mathcal{C}_t(C-\lambda, K-\mu).$$

To prove that $\mathcal{C}^{\lambda,\mu}$ is a domain, it suffices to show that $\mathfrak{a} = \mathfrak{a}'$. The inclusion $\mathfrak{a} \subseteq \mathfrak{a}'$ is obvious. If $\lambda \neq 0$ then, by statement 1, the algebra $\mathcal{C}^{\lambda,\mu}$ is a simple algebra, so the ideal $\mathfrak{a}$ is a maximal ideal of $\mathcal{C}$. Then we must have $\mathfrak{a} = \mathfrak{a}'$. Suppose that $\lambda = 0$ and $\mathfrak{a} \subsetneq \mathfrak{a}'$, we seek a contradiction. Notice that the ideal $\mathfrak{a}'$ is a prime ideal of $\mathcal{C}$.

Hence, $\mathfrak{a}'/\mathfrak{a}$ is a nonzero prime ideal of the algebra $\mathcal{C}^{0,\mu}$. By Proposition 4.9.(3), the algebra $\mathcal{C}_t^{0,\mu}$ is a simple algebra, so, $t^i \in \mathfrak{a}'/\mathfrak{a}$ for some $i \in \mathbb{N}$. Then $(\mathfrak{a}'/\mathfrak{a})_t = \mathcal{C}_t^{\lambda,\mu}$. But $(\mathfrak{a}'/\mathfrak{a})_t = \mathfrak{a}'_t/\mathfrak{a}_t = 0$, a contradiction. □

Proposition 4.12. (1) *In the algebra* $\mathcal{C}^{0,\mu}$, $(t) = (u) = (t,u) = \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u$.
(2) $\mathcal{C}^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$.
(3) *In the algebra* $\mathcal{C}^{0,\mu}$, $(t^i) = (t)^i$ *for all* $i \geqslant 1$.
(4) $\mathrm{Spec}\,(\mathcal{C}^{0,\mu}) = \{0,\ (t),\ (t,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[\Theta])\}$.

Proof. (1) The equality $(t) = (u)$ follows from (4.12) and (4.13). The second equality then is obvious. To prove the third equality let us first show that

$$t\mathcal{C}^{0,\mu} \subseteq \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u.$$

In view of Corollary 4.8.(2), it suffices to prove that $t\Theta^i \in \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u$ for all $i \geqslant 1$. This can be proved by induction on i. The case $i = 1$ follows from (4.12). Suppose that the inclusion holds for all $i' < i$. Then

$$\begin{aligned} t\Theta^i &= t\Theta^{i-1}\Theta \in (\mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u)\Theta \\ &= \mathcal{C}^{0,\mu}\big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u\big) + \mathcal{C}^{0,\mu}\big(q^2\Theta u + q^3(1+q^2)t\big) \\ &\subseteq \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u. \end{aligned}$$

Hence, we proved that

$$t\mathcal{C}^{0,\mu} \subseteq \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u.$$

Now, the inclusions $(t) \subseteq \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u \subseteq (t,u) = (t)$ yield that

$$(t) = \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u.$$

(2) By statement 1, $\mathcal{C}^{0,\mu}/(t) = \mathcal{C}^{0,\mu}/(t,u) \simeq \mathbb{K}[\Theta]$.

(3) The inclusion $(t^i) \subseteq (t)^i$ is obvious. We prove the reverse inclusion $(t)^i \subseteq (t^i)$ by induction on i. The case $i = 1$ is trivial. Suppose that the inclusion holds for all $i' < i$. Then,

$$(t)^i = (t)(t)^{i-1} = (t)(t^{i-1}) = \mathcal{C}^{0,\mu}t\mathcal{C}^{0,\mu}t^{i-1}\mathcal{C}^{0,\mu} \subseteq (t^i) + (t^{i-1}u)$$

since $t\mathcal{C}^{0,\mu} \subseteq \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u$ (see statement 1). By Lemma 4.10, the element $t^{i-1}u$ belongs to the ideal (t^i) of $\mathcal{C}^{0,\mu}$. Hence, $(t)^i \subseteq (t^i)$, as required.

(4) By Proposition 3.6 and statement 3,

$$\mathrm{Spec}\,(\mathcal{C}^{0,\mu}) = \mathrm{Spec}\,(\mathcal{C}^{0,\mu}, t) \sqcup \mathrm{Spec}_t(\mathcal{C}^{0,\mu}).$$

Notice that $\mathcal{C}_t^{0,\mu}$ is a simple algebra (see Proposition 4.9.(3)) and $\mathcal{C}^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$ (see statement 2). Then,

$$\mathrm{Spec}\,(\mathcal{C}^{0,\mu}) = \{0\} \sqcup \mathrm{Spec}\,\big(\mathbb{K}[\Theta]\big) = \big\{0,\ (t),\ (t,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,\big(\mathbb{K}[\Theta]\big)\big\}.$$ □

5. Classification of simple $C_A(K)$-modules

In this section, $\mathbb{K}$ is an algebraically closed field. A classification of simple $C_A(K)$-modules is given in Theorem 5.2, Theorem 5.6 and Theorem 5.11. For an algebra B, we denote by $\widehat{B}$ the set of isomorphism classes of simple B-modules. If $\mathscr{P}$ is an isomorphism invariant property on simple B-modules then $\widehat{B}\,(\mathscr{P})$ is the set of isomorphism classes of B-modules that satisfy the property $\mathscr{P}$. The set $\widehat{C_A(K)}$ of isomorphism classes of simple $C_A(K)$-modules is partitioned (according to the central character) as follows:

$$\widehat{C_A(K)} = \bigsqcup_{\lambda\in\mathbb{K},\,\mu\in\mathbb{K}^*} \widehat{\mathcal{C}^{\lambda,\mu}}. \tag{5.1}$$

Given $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, the set $\widehat{\mathcal{C}^{\lambda,\mu}}$ can be partitioned further into disjoint union of two subsets consisting of t-torsion modules and t-torsionfree modules, respectively,

$$\widehat{\mathcal{C}^{\lambda,\mu}} = \widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsion}) \sqcup \widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsionfree}). \tag{5.2}$$

The set $\widehat{\mathcal{C}^{\lambda,\mu}}$ (t-torsion). An explicit description of the set $\widehat{\mathcal{C}^{\lambda,\mu}}$ (t-torsion) is given in Theorem 5.2. For $\lambda,\,\mu \in \mathbb{K}^*$, we define the left $\mathcal{C}^{\lambda,\mu}$-modules

$$\mathsf{t}^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t,u) \quad \text{and} \quad \mathsf{T}^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t,u-\widehat{\lambda}),$$

where $\widehat{\lambda} := q(q^2-1)\lambda$. By Corollary 4.8.(2),

$$\mathsf{t}^{\lambda,\mu} = \mathbb{K}[\Theta]\,\bar{1} \simeq {}_{\mathbb{K}[\Theta]}\mathbb{K}[\Theta]$$

is a free $\mathbb{K}[\Theta]$-module, where $\bar{1} = 1 + \mathcal{C}^{\lambda,\mu}(t,u)$, and

$$\mathsf{T}^{\lambda,\mu} = \mathbb{K}[\Theta]\,\tilde{1} \simeq {}_{\mathbb{K}[\Theta]}\mathbb{K}[\Theta]$$

is a free $\mathbb{K}[\Theta]$-module, where $\tilde{1} = 1 + \mathcal{C}^{\lambda,\mu}(t,u-\widehat{\lambda})$. Clearly, the modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are of Gelfand–Kirillov dimension 1. The concept of $\deg_\Theta$ of the elements of $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ is well-defined ($\deg_\Theta(\Theta^i\,\bar{1}) = i$ and $\deg_\Theta(\Theta^i\,\tilde{1}) = i$ for all $i \geqslant 0$).

Lemma 5.1. *Let $\lambda,\,\mu \in \mathbb{K}^*$. Then:*

(1) *The $\mathcal{C}^{\lambda,\mu}$-module $\mathsf{t}^{\lambda,\mu}$ is a simple module.*

(2) *The $\mathcal{C}^{\lambda,\mu}$-module $\mathsf{T}^{\lambda,\mu}$ is a simple module.*

(3) *The modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic.*

Proof. (1) Let us show that for all $i \geqslant 1$,

$$t \cdot \Theta^i \,\bar{1} = (1-q^{-2i})\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots, \tag{5.3}$$

$$u \cdot \Theta^i \,\bar{1} = -q^2(1-q^{2i})\mu^{-1}\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots, \tag{5.4}$$

where the three dots means terms of $\deg_\Theta < i-1$. We prove the equalities by induction on i. By (4.12),

$$t\Theta\,\bar{1} = (1-q^{-2})\lambda\,\bar{1},$$

and by (4.13),

$$u\Theta\,\bar{1} = -q^2(1-q^2)\mu^{-1}\lambda\,\bar{1}.$$

So, the equalities (5.3) and (5.4) hold for $i = 1$. Suppose that the equalities hold for all integers $i' < i$. Then,

$$\begin{aligned} t \cdot \Theta^i\,\bar{1} &= \big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)\lambda\big)\Theta^{i-1}\,\bar{1}\\ &= q^{-2}(1-q^{-2(i-1)})\lambda\Theta^{i-1}\,\bar{1} - q^{-2}(1-q^2)\lambda\Theta^{i-1}\,\bar{1} + \cdots\\ &= (1-q^{-2i})\lambda\cdot\Theta^{i-1}\,\bar{1} + \cdots,\\ u\cdot\Theta^i\,\bar{1} &= \big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)\mu^{-1}\lambda\big)\Theta^{i-1}\,\bar{1}\\ &= -q^4(1-q^{2(i-1)})\mu^{-1}\lambda\Theta^{i-1}\,\bar{1} - q^2(1-q^2)\mu^{-1}\lambda\Theta^{i-1}\,\bar{1} + \cdots\\ &= -q^2(1-q^{2i})\mu^{-1}\lambda\cdot\Theta^{i-1}\,\bar{1} + \cdots. \end{aligned}$$

The simplicity of the module $\mathsf{t}^{\lambda,\mu}$ follows from the equality (5.3) (or the equality (5.4)).

(2) Let us show that for all $i \geqslant 1$,

$$t \cdot \Theta^i\,\tilde{1} = (1-q^{2i})\lambda\cdot\Theta^{i-1}\,\tilde{1} + \cdots, \tag{5.5}$$

$$u\cdot\Theta^i\,\tilde{1} = q^{2i}\hat{\lambda}\cdot\Theta^i\,\tilde{1} - q^2(1-q^{2i})\mu^{-1}\lambda\cdot\Theta^{i-1}\,\tilde{1} + \cdots, \tag{5.6}$$

where the three dots means terms of smaller degrees. We prove the equalities by induction on i. The case $i = 1$ follows from (4.12) and (4.13). Suppose that the equalities (5.5) and (5.6) hold for all integers $i' < i$. Then,

$$\begin{aligned} t\cdot\Theta^i\,\tilde{1} &= \big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)\lambda\big)\Theta^{i-1}\,\tilde{1}\\ &= q^{-2}(1-q^{2(i-1)})\lambda\Theta^{i-1}\,\tilde{1} - q^{-2}(q+q^{-1})q^{2(i-1)}\hat{\lambda}\Theta^{i-1}\,\tilde{1}\\ &\qquad - q^{-2}(1-q^2)\lambda\Theta^{i-1}\tilde{1} + \cdots\\ &= (1-q^{2i})\lambda\cdot\Theta^{i-1}\,\tilde{1} + \cdots,\\ u\cdot\Theta^i\,\tilde{1} &= \big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)\mu^{-1}\lambda\big)\Theta^{i-1}\,\tilde{1}\\ &= q^2\big(q^{2(i-1)}\hat{\lambda}\Theta^i\,\tilde{1} - q^2(1-q^{2(i-1)})\mu^{-1}\lambda\Theta^{i-1}\,\tilde{1}\big)\\ &\qquad - q^2(1-q^2)\mu^{-1}\lambda\Theta^{i-1}\,\tilde{1} + \cdots\\ &= q^{2i}\hat{\lambda}\cdot\Theta^i\,\tilde{1} - q^2(1-q^{2i})\mu^{-1}\lambda\cdot\Theta^{i-1}\,\tilde{1} + \cdots. \end{aligned}$$

The simplicity of the module $\mathsf{T}^{\lambda,\mu}$ follows from the equality (5.5).

(3) By (5.4), the element u acts locally nilpotently on the module $\mathsf{t}^{\lambda,\mu}$. But, by (5.6), the action of the element u on the module $\mathsf{T}^{\lambda,\mu}$ is not locally nilpotent. Hence, the modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic. □

Theorem 5.2.

(1) $\widehat{\mathcal{C}^{0,\mu}}\,(t\text{-torsion}) = \{[\mathcal{C}^{0,\mu}/\mathcal{C}^{0,\mu}(t,u,\Theta-\alpha) \simeq \mathbb{K}[\Theta]/(\Theta-\alpha)] \mid \alpha \in \mathbb{K}\}$.

(2) *Let* $\lambda,\ \mu \in \mathbb{K}^*$. *Then* $\widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsion}) = \{[\mathsf{t}^{\lambda,\mu}], [\mathsf{T}^{\lambda,\mu}]\}$.

Proof. (1) We claim that $\mathrm{ann}_{\mathcal{C}^{0,\mu}}(M) \supseteq (t)$ for all $M \in \widehat{\mathcal{C}^{0,\mu}}\,(t\text{-torsion})$: In view of Proposition 4.12.(1), it suffices to show that there exists a nonzero element $m \in M$ such that $tm = 0$ and $um = 0$. Since M is t-torsion, there exists a nonzero element $m' \in M$ such that $tm' = 0$. Then, by the equality (4.14) (where $\lambda = 0$), we have $u^2m' = 0$. If $um' = 0$, we are done. Otherwise, the element $m := um'$ is a nonzero element of M such that $tm = um = 0$ (since $tu = q^2ut$). Now, statement 1 follows from the claim immediately.

(2) Let $M \in \widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsion})$. Then there exists a nonzero element $m \in M$ such that $tm = 0$. By (4.14), we have $(u-\widehat{\lambda})um = 0$. Therefore, either $um = 0$ or otherwise the element $m' := um \in M$ is nonzero and $(u-\widehat{\lambda})m' = 0$. If $um = 0$ then the module M is an epimorphic image of the module $\mathsf{t}^{\lambda,\mu}$. By Lemma 5.1.(1), $\mathsf{t}^{\lambda,\mu}$ is a simple $\mathcal{C}^{\lambda,\mu}$-module. Hence, $M \simeq \mathsf{t}^{\lambda,\mu}$. If $m' = um \neq 0$, then $tm' = 0$ and $(u-\widehat{\lambda})m' = 0$. So, the $\mathcal{C}^{\lambda,\mu}$-module M is an epimorphic image of the module $\mathsf{T}^{\lambda,\mu}$. By Lemma 5.1.(2), $\mathsf{T}^{\lambda,\mu}$ is a simple $\mathcal{C}^{\lambda,\mu}$-module. Then $M \simeq \mathsf{T}^{\lambda,\mu}$. By Lemma 5.1.(3), the two modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic, this completes the proof. □

Recall that the algebra

$$C_{A_{X,\varphi}}(K) = \mathbb{K}[C, K^{\pm 1}] \otimes \mathcal{A},$$

where $\mathcal{A}$ is a central simple GWA, see Proposition 4.3. The algebra $C_A(K)$ is a subalgebra of the algebra $C_{A_{X,\varphi}}(K)$, where

$$u = K^{-1}Y\varphi = K^{-1}\cdot YX\cdot \varphi X^{-1} = K^{-1}th, \tag{5.7}$$

$$\Theta = (1-q^2)Ceh^{-1} + \frac{qK^{-1}}{1-q^2}h + \frac{q^3K}{1-q^2}h^{-1}. \tag{5.8}$$

In more detail: by (2.16),

$$F = \Big(C + K^{-1}EY^2 - \frac{q^3}{1-q^2}(K-K^{-1})YX\Big)X^{-1}\varphi^{-1}.$$

Then the element FE can be written as

$$\begin{aligned}
FE &= CEX^{-1}\varphi^{-1} + K^{-1}EY^2EX^{-1}\varphi^{-1} - \frac{q^2}{1-q^2}(K-K^{-1})YE\varphi^{-1} \\
&= C \cdot EX^{-2} \cdot X\varphi^{-1} + K^{-1} \cdot EX^{-2} \cdot q^3(YX)^2 \cdot EX^{-2} \cdot X\varphi^{-1} \\
&\qquad\qquad - \frac{q^3(K-K^{-1})}{1-q^2} \cdot YX \cdot EX^{-2} \cdot X\varphi^{-1} \\
&= Ceh^{-1} + q^3K^{-1}et^2eh^{-1} - \frac{q^3(K-K^{-1})}{1-q^2}teh^{-1} \\
&= Ceh^{-1} + \frac{qK^{-1}}{(1-q^2)^2}h + \frac{q^3K}{(1-q^2)^2}h^{-1} - \frac{q^2(qK+q^{-1}K^{-1})}{(1-q^2)^2},
\end{aligned}$$

where the last equality follows from (4.2). Then the equality (5.8) follows immediately since

$$\Theta = (1-q^2)FE + \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}.$$

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, let

$$\mathcal{C}^{\lambda,\mu}_{A_{X,\varphi}} := C_{A_{X,\varphi}}(K)/(C-\lambda, K-\mu).$$

Then by Proposition 4.3.(1), $\mathcal{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathcal{A}$ is a central simple GWA. So, there is a natural algebra homomorphism

$$\mathcal{C}^{\lambda,\mu} \to \mathcal{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathcal{A}.$$

The next proposition shows that this homomorphism is a monomorphism.

Proposition 5.3. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The following map is an algebra homomorphism*

$$\rho : \mathcal{C}^{\lambda,\mu} \longrightarrow \mathcal{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathcal{A},$$

$$t \mapsto t, \quad u \mapsto \mu^{-1}th, \quad \Theta \mapsto (1-q^2)\lambda eh^{-1} + \frac{q\mu^{-1}}{1-q^2}h + \frac{q^3\mu}{1-q^2}h^{-1}.$$

Moreover, the homomorphism ρ is a monomorphism.

Proof. The fact that the map ρ is an algebra homomorphism follows from (5.7) and (5.8). Now, we prove that ρ is an injection. If $\lambda \neq 0$ then by Theorem 4.11.(1), the algebra $\mathcal{C}^{\lambda,\mu}$ is a simple algebra. Hence, the kernel ker ρ of the homomorphism ρ must be zero, i.e., ρ is an injection. If $\lambda = 0$ and suppose that ker ρ is nonzero, we seek a contradiction. Then $t^i \in \ker \rho$ for some $i \in \mathbb{N}$. But $\rho(t^i) = t^i \neq 0$, a contradiction. □

Let $\mathcal{A}_t$ be the localization of the algebra $\mathcal{A}$ at the powers of the element t. Then $\mathcal{A}_t = \mathbb{K}[h^{\pm 1}][t^{\pm 1}; \sigma]$ is a central simple quantum torus, where $\sigma(h) = q^2 h$. It is clear that $\mathcal{C}_{t,u}^{\lambda,\mu} \simeq \mathcal{A}_t$. Let $\mathcal{B}$ be the localization of $\mathcal{A}$ at the set $S = \mathbb{K}[h^{\pm 1}] \setminus \{0\}$. Then $\mathcal{B} = S^{-1}\mathcal{A} = \mathbb{K}(h)[t^{\pm 1}; \sigma]$ is a skew Laurent polynomial algebra where $\mathbb{K}(h)$ is the field of rational functions in h and $\sigma(h) = q^2 h$. The algebra $\mathcal{B}$ is a Euclidean ring with left and right division algorithms. In particular, $\mathcal{B}$ is a principle left and right ideal domain. For all $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we have the following inclusions of algebras

$$\begin{array}{ccccc} \mathcal{C}^{\lambda,\mu} & \xrightarrow{\rho} & \mathcal{A} & & \\ \downarrow & & \downarrow & & \\ \mathcal{C}_t^{\lambda,\mu} & \longrightarrow & \mathcal{C}_{t,u}^{\lambda,\mu} = \mathcal{A}_t & \longrightarrow & \mathcal{B}. \end{array}$$

The set $\widehat{\mathcal{C}^{0,\mu}}$ (t-torsionfree). An explicit description of the set $\widehat{\mathcal{C}^{0,\mu}}$ (t-torsionfree) is given in Theorem 5.6. The idea is to embed the algebra $\mathcal{C}^{0,\mu}$ in a skew polynomial algebra $\mathcal{R}$ for which the simple modules are classified. The simple modules over these two algebras are closely related. It will be shown that

$$\widehat{\mathcal{C}^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{\mathcal{R}}\,(t\text{-torsionfree}).$$

Let $\mathcal{R}$ be the subalgebra of $\mathcal{A}$ generated by the elements $h^{\pm 1}$ and t. Then $\mathcal{R} = \mathbb{K}[h^{\pm 1}][t; \sigma]$ is a skew polynomial algebra where $\sigma(h) = q^2 h$. By Proposition 5.3, the algebra $\mathcal{C}^{0,\mu}$ is a subalgebra of $\mathcal{R}$. Hence, we have the inclusions of algebras

$$\mathcal{C}^{0,\mu} \subset \mathcal{R} \subset \mathcal{A} \subset \mathcal{R}_t = \mathcal{A}_t \subset \mathcal{B}.$$

We identify the algebra $\mathcal{C}^{0,\mu}$ with its image in the algebra $\mathcal{R}$.

Lemma 5.4. *Let $\mu \in \mathbb{K}^*$. Then:*

(1) $\mathcal{C}^{0,\mu} = \bigoplus_{i \geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta]$.

(2) $\mathcal{R} = \mathcal{C}^{0,\mu} \oplus \mathbb{K}[\Theta]h$.

(3) $(t) = \bigoplus_{i \geqslant 1} \mathbb{K}[h^{\pm 1}]t^i = \mathcal{R}t$, *where (t) is the ideal of $\mathcal{C}^{0,\mu}$ generated by the element t.*

Proof. (1) and (2) Notice that $\mathbb{K}[\Theta] \subset \mathbb{K}[h^{\pm 1}]$ and $\mathbb{K}[h^{\pm 1}] = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h$. Multiplying this equality on the right by the element t yields that

$$\mathbb{K}[h^{\pm 1}]t = \mathbb{K}[\Theta]t \oplus \mathbb{K}[\Theta]u \subseteq \mathcal{C}^{0,\mu}.$$

Then for all $i \geqslant 1$,

$$\mathbb{K}[h^{\pm 1}]t^i = \mathbb{K}[h^{\pm 1}]t \cdot t^{i-1} \subseteq \mathcal{C}^{0,\mu}t^{i-1} \subseteq \mathcal{C}^{0,\mu}.$$

Notice that

$$\begin{aligned}\mathcal{R} = \bigoplus_{i\geqslant 0} \mathbb{K}[h^{\pm 1}]t^i &= \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[h^{\pm 1}] \\ &= \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h. \end{aligned} \tag{5.9}$$

Then,

$$\mathcal{C}^{0,\mu} = \mathcal{C}^{0,\mu} \cap \mathcal{R} = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta]$$

since $\mathcal{C}^{0,\mu} \cap \mathbb{K}[\Theta]h = 0$. The statement 2 then follows from (5.9).

(3) By Proposition 4.12.(1), $(t) = \mathcal{C}^{0,\mu}t + \mathcal{C}^{0,\mu}u$. Then the first equality follows from statement 1. The second equality is obvious. □

The set $\mathbb{K}[h^{\pm 1}] \setminus \{0\}$ is an Ore set of the ring $\mathcal{R}$. Abusing the language, we say $\mathbb{K}[h^{\pm 1}]$-torsion meaning $\mathbb{K}[h^{\pm 1}] \setminus \{0\}$-torsion. In particular, we denote by $\widehat{\mathcal{R}}\,(\mathbb{K}[h]\text{-torsion})$ the set of isomorphism classes of $\mathbb{K}[h]$-torsion simple $\mathcal{R}$-modules.

Proposition 5.5. *Let* $\operatorname{Irr}(\mathcal{B})$ *be the set of irreducible elements of the algebra* $\mathcal{B}$.

(1) $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion}) = \widehat{\mathcal{R}}\,(t\text{-torsion})$

$$= \widehat{\mathcal{R}/(t)} = \{[\mathcal{R}/\mathcal{R}(h-\alpha, t)] \mid \alpha \in \mathbb{K}^*\}.$$

(2) $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsionfree}) = \widehat{\mathcal{R}}\,(t\text{-torsionfree})$

$$= \{[M_b] \mid b \in \operatorname{Irr}(\mathcal{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathcal{B}b\},$$

where $M_b := \mathcal{R}/\mathcal{R} \cap \mathcal{B}b$; $M_b \simeq M_{b'}$ *iff the elements* b *and* b' *are similar (iff* $\mathcal{B}/\mathcal{B}b \simeq \mathcal{B}/\mathcal{B}b'$ *as* $\mathcal{B}$*-modules).*

Proof. (1) The last two equalities are obvious, since t is a normal element of the algebra $\mathcal{R}$. Then it is clear that $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion}) \supseteq \widehat{\mathcal{R}}\,(t\text{-torsion})$. Now, we show the reverse inclusion holds. Let $M \in \widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion})$. Then M is an epimorphic image of the $\mathcal{R}$-module $\mathcal{R}/\mathcal{R}(h-\alpha) = \mathbb{K}[t]\bar{1}$ for some $\alpha \in \mathbb{K}^*$, where $\bar{1} = 1 + \mathcal{R}(h-\alpha)$. Notice that $t\mathbb{K}[t]\bar{1}$ is the only maximal $\mathcal{R}$-submodule of $\mathcal{R}/\mathcal{R}(h-\alpha)$. Then $M \simeq \mathcal{R}/\mathcal{R}(h-\alpha, t) \in \widehat{\mathcal{R}}\,(t\text{-torsion})$, as required.

(2) The first equality follows from the first equality in statement 1. By [7, Theorem 1.3]

$$\widehat{\mathcal{R}}\left(\mathbb{K}[h^{\pm 1}]\text{-torsionfree}\right) = \left\{[M_b] \mid b \in \operatorname{Irr}(\mathcal{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathcal{B}b\right\}$$

(the condition (LO) of [7, Theorem 1.3] is equivalent to the condition $\mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathcal{B}b$). □

Theorem 5.6.

$$\begin{aligned}\widehat{\mathcal{C}^{0,\mu}}\,(t\text{-torsionfree}) &= \widehat{\mathcal{R}}\,(t\text{-torsionfree})\\ &= \widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsionfree})\\ &= \{[M_b = \mathcal{R}/\mathcal{R}\cap\mathcal{B}b] \mid b\in \mathrm{Irr}(\mathcal{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R}\cap\mathcal{B}b\}\end{aligned}$$

(see Proposition 5.5).

Proof. In view of Proposition 5.5.(2), it remains to show that the first equality holds. Let $[M] \in \widehat{\mathcal{C}^{0,\mu}}\,(t\text{-torsionfree})$. Then $M = (t)M = \mathcal{R}tM \in \widehat{\mathcal{R}}\,(t\text{-torsionfree})$. Given $[N] \in \widehat{\mathcal{R}}\,(t\text{-torsionfree})$. To finish the proof of statement 2, it suffices to show that N is a simple $\mathcal{C}^{0,\mu}$-module. If L is a nonzero $\mathcal{C}^{0,\mu}$-submodule of N then $N \supseteq L \supseteq (t)L \neq 0$, since N is t-torsionfree. Then $(t)L = \mathcal{R}tL = N$, since N is a simple $\mathcal{R}$-module. Hence, $L = N$, i.e., N is a simple $\mathcal{C}^{0,\mu}$-module, as required. □

The set $\widehat{\mathcal{C}^{\lambda,\mu}}$ (t-torsionfree) where $\lambda \in \mathbb{K}^*$. An explicit description of the set $\widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsionfree})$, where $\lambda \in \mathbb{K}^*$ is given in Theorem 5.11. Recall that the algebra

$$\mathcal{C}_t^{\lambda,\mu} = \mathbb{K}[t^{\pm 1}][u, v; \sigma, a]$$

is a GWA where $a = (q^7/(1-q^2))t^2 - q^4\mu^{-1}\lambda t$ and σ is the automorphism of the algebra $\mathbb{K}[t^{\pm 1}]$ defined by $\sigma(t) = q^{-2}t$ (Proposition 4.9.(2)). Clearly,

$$\begin{aligned}&\widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsionfree})\\ &\quad = \widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsionfree},\ \mathbb{K}[t]\text{-torsion}) \sqcup \widehat{\mathcal{C}^{\lambda,\mu}}\,(\mathbb{K}[t]\text{-torsionfree}).\end{aligned} \quad (5.10)$$

Lemma 5.7. *Let $\lambda, \mu \in \mathbb{K}^*$ and $\nu := q^{-3}(1-q^2)\mu^{-1}\lambda$. Then*

(1) *The module $\mathfrak{f}^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t-\nu, u)$ is a simple $\mathcal{C}^{\lambda,\mu}$-module.*

(2) *The module $\mathsf{F}^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t-q^2\nu, v)$ is a simple $\mathcal{C}^{\lambda,\mu}$-module.*

(3) *Let $\gamma, \gamma' \in \mathbb{K}^* \setminus \{q^{2i}\nu \mid i \in \mathbb{Z}\}$. The module $\mathcal{F}_\gamma^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t-\gamma)$ is a simple $\mathcal{C}^{\lambda,\mu}$-module. The simple modules $\mathcal{F}_\gamma^{\lambda,\mu} \simeq \mathcal{F}_{\gamma'}^{\lambda,\mu}$ iff $\gamma = q^{2i}\gamma'$ for some $i \in \mathbb{Z}$.*

Proof. (1) Note that $a = (q^7/(1-q^2))(t-\nu)t$ and $\sigma(a) = (q^3/(1-q^2))(t-q^2\nu)t$. By Corollary 4.8.(2) and the expression of the element v,

$$\mathfrak{f}^{\lambda,\mu} = \mathbb{K}[\Theta]\bar{1} = \mathbb{K}[v]\bar{1},$$

where $\bar{1} = 1 + \mathcal{C}^{\lambda,\mu}(t-\nu, u)$. The simplicity of the module $\mathfrak{f}^{\lambda,\mu}$ follows from the equality:

$$uv^i\bar{1} = v^{i-1}\sigma^i(a)\bar{1} \in \mathbb{K}^* v^{i-1}\bar{1}$$

for all $i \geqslant 1$.

(2) Notice that $\mathsf{F}^{\lambda,\mu} = \mathbb{K}[u]\bar{1}$, where $\bar{1} = 1 + \mathcal{C}^{\lambda,\mu}(t - q^2\nu, v)$. The simplicity of the module $\mathsf{F}^{\lambda,\mu}$ follows from the equality:

$$vu^i\bar{1} = u^{i-1}\sigma^{-i+1}(a)\bar{1} \in \mathbb{K}^* u^{i-1}\bar{1}$$

for all $i \geqslant 1$.

(3) Notice that

$$\mathcal{F}_\gamma^{\lambda,\mu} = \sum_{i,j\geqslant 0} \mathbb{K}u^i\Theta^j\tilde{1} = \sum_{i,j\geqslant 0} \mathbb{K}u^iv^j\tilde{1} = \mathbb{K}[u]\tilde{1} + \mathbb{K}[v]\tilde{1},$$

where $\tilde{1} = 1 + \mathcal{C}^{\lambda,\mu}(t-\gamma)$. Since $\gamma \in \mathbb{K}^* \setminus \{q^{2i}\nu \mid i \in \mathbb{Z}\}$, $\sigma^i(a)\bar{1} \in \mathbb{K}^*\bar{1}$ for all $i \in \mathbb{Z}$. Then the simplicity of the module $\mathcal{F}_\gamma^{\lambda,\mu}$ follows from the equalities in the proof of statements 1 and 2. The set of eigenvalues of the element $t_{\mathcal{F}_\gamma^{\lambda,\mu}}$ is

$$\mathrm{Ev}_{\mathcal{F}_\gamma^{\lambda,\mu}}(t) = \{q^{2i}\gamma \mid i \in \mathbb{Z}\}.$$

If $\mathcal{F}_\gamma^{\lambda,\mu} \simeq \mathcal{F}_{\gamma'}^{\lambda,\mu}$, then $\mathrm{Ev}_{\mathcal{F}_\gamma^{\lambda,\mu}}(t) = \mathrm{Ev}_{\mathcal{F}_{\gamma'}^{\lambda,\mu}}(t)$, so

$$\gamma = q^{2i}\gamma'$$

for some $i \in \mathbb{Z}$. Conversely, suppose that $\gamma = q^{2i}\gamma'$ for some $i \in \mathbb{Z}$. Let $\tilde{1}$ and $\tilde{1}'$ be the canonical generators of the modules $\mathcal{F}_\gamma^{\lambda,\mu}$ and $\mathcal{F}_{\gamma'}^{\lambda,\mu}$, respectively. The map

$$\mathcal{F}_\gamma^{\lambda,\mu} \to \mathcal{F}_{\gamma'}^{\lambda,\mu}, \quad \tilde{1} \mapsto u^i\tilde{1}'$$

defines an isomorphism of $\mathcal{C}^{\lambda,\mu}$-modules if $i \geqslant 0$, and the map

$$\mathcal{F}_\gamma^{\lambda,\mu} \to \mathcal{F}_{\gamma'}^{\lambda,\mu}, \quad \tilde{1} \mapsto v^i\tilde{1}'$$

defines an isomorphism of $\mathcal{C}^{\lambda,\mu}$-modules if $i < 0$. □

Definition 5.8 ([4], l-normal elements of the algebra $\mathcal{C}_t^{\lambda,\mu}$). (1) Let α and β be nonzero elements of the Laurent polynomial algebra $\mathbb{K}[t^{\pm 1}]$. We say that $\alpha < \beta$ if there are no roots λ and μ of the polynomials α and β, respectively, such that, $\lambda = q^{2i}\mu$ for some $i \geqslant 0$.

(2) An element $b = v^m\beta_m + v^{m-1}\beta_{m-1} + \cdots + \beta_0 \in \mathcal{C}_t^{\lambda,\mu}$, where $m > 0$, $\beta_i \in \mathbb{K}[t^{\pm 1}]$, and $\beta_0, \beta_m \neq 0$ is called *l-normal* if

$$\beta_0 < \beta_m \quad \text{and} \quad \beta_0 < \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t.$$

Theorem 5.9 ([2,3]). *Let $\lambda, \mu \in \mathbb{K}^*$. Then*

$$\widehat{\mathcal{C}_t^{\lambda,\mu}}\big(\mathbb{K}[t]\text{-torsionfree}\big) = \big\{\big[N_b := \mathcal{C}_t^{\lambda,\mu}/\mathcal{C}_t^{\lambda,\mu}\cap\mathcal{B}b\big] \mid b \text{ is } l\text{-normal}, b \in \mathrm{Irr}(\mathcal{B})\big\}.$$

Simple $\mathcal{C}_t^{\lambda,\mu}$-modules N_b and $N_{b'}$ are isomorphic iff the elements b and b' are similar.

Recall that, the algebra $\mathcal{C}^{\lambda,\mu}$ is generated by the canonical generators t, u, and Θ. Let $\mathcal{F} = \{\mathcal{F}_n\}_{n\geqslant 0}$ be the standard filtration associated with the canonical generators. By Corollary 4.8, for $n \geqslant 0$,

$$\mathcal{F}_n = \bigoplus_{\substack{i,j\geqslant 1,\\ i+j\leqslant n}} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{1\leqslant k\leqslant n} \mathbb{K}\Theta^k \oplus \bigoplus_{\substack{l,m\geqslant 1,\\ l+m\leqslant n}} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{\substack{a,b\geqslant 0,\\ a+b\leqslant n}} \mathbb{K}u^a t^b.$$

For all $n \geqslant 1$,

$$\dim \mathcal{F}_n = \frac{3}{2}n^2 + \frac{3}{2}n + 1 = f(n),$$

where $f(s) = \frac{3}{2}s^2 + \frac{3}{2}s + 1 \in \mathbb{Q}[s]$. For each nonzero element $a \in \mathcal{C}^{\lambda,\mu}$, the unique natural number n such that $a \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ is called the *total degree* of the element a, denoted by $\deg(a)$. Set $\deg(0) := -\infty$. Then

$$\deg(ab) \leqslant \deg(a) + \deg(b)$$

for all elements $a, b \in \mathcal{C}^{\lambda,\mu}$.

For an R-module M, we denote by $l_R(M)$ the *length* of the R-module M. The next proposition shows that $l_{\mathcal{C}^{\lambda,\mu}}(\mathcal{C}^{\lambda,\mu}/I) < \infty$ for all left ideals I of the algebra $\mathcal{C}^{\lambda,\mu}$.

Proposition 5.10. *Let $\lambda, \mu \in \mathbb{K}^*$. For each element nonzero element $a \in \mathcal{C}^{\lambda,\mu}$, the length of the $\mathcal{C}^{\lambda,\mu}$-module $\mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}a$ is finite, more precisely,*

$$l_{\mathcal{C}^{\lambda,\mu}}\big(\mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}a\big) \leqslant 3\deg(a).$$

Proof. Let $M := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}a = \mathcal{C}^{\lambda,\mu}\bar{1} = \bigcup_{i\geqslant 0}\mathcal{F}_i\bar{1}$ be the standard filtration on M where $\bar{1} = 1 + \mathcal{C}^{\lambda,\mu}a$. Then

$$\mathcal{F}_i\bar{1} \simeq \frac{\mathcal{F}_i + \mathcal{C}^{\lambda,\mu}a}{\mathcal{C}^{\lambda,\mu}a} \simeq \frac{\mathcal{F}_i}{\mathcal{F}_i \cap \mathcal{C}^{\lambda,\mu}a}.$$

Let $d := \deg(a)$. Since, for all $i \geqslant 0$, $\mathcal{F}_{i-d}a \subseteq \mathcal{F}_i \cap \mathcal{C}^{\lambda,\mu}a$, we see that

$$\dim(\mathcal{F}_i\bar{1}) \leqslant f(i) - f(i-d) = 3di + \frac{3}{2}d - \frac{3}{2}d^2.$$

Recall that the algebra $\mathcal{C}^{\lambda,\mu}$ is a simple, infinite dimensional algebra since $\lambda \neq 0$ (Theorem 4.11.(1)). So, if $N = \mathcal{C}^{\lambda,\mu}n$ is a nonzero cyclic $\mathcal{C}^{\lambda,\mu}$-module (where $0 \neq n \in N$) and $\{\mathcal{F}_i n\}_{i\geqslant 0}$ is the standard filtration on N then $\dim(\mathcal{F}_i n) \geqslant i + 1$ for all $i \geqslant 0$. This implies that $l_{\mathcal{C}^{\lambda,\mu}}(M) \leqslant 3d$. □

The group $q^{2\mathbb{Z}} = \{q^{2i} \mid i \in \mathbb{Z}\}$ acts on $\mathbb{K}^*$ by multiplication. For each $\gamma \in \mathbb{K}^*$, let $\mathcal{O}(\gamma) = \{q^{2i}\gamma \mid i \in \mathbb{Z}\}$ be the orbit of the element $\gamma \in \mathbb{K}^*$ under the action of the group $q^{2\mathbb{Z}}$. For each orbit $\mathcal{O} \in \mathbb{K}^*/q^{2\mathbb{Z}}$, we fix an element $\gamma_{\mathcal{O}} \in \mathcal{O}(\gamma)$.

Theorem 5.11. *Let* $\lambda, \mu \in \mathbb{K}^*$. *Then*

(1) $\widehat{\mathcal{C}^{\lambda,\mu}}\big(t\text{-torsionfree}, \mathbb{K}[t]\text{-torsion}\big)$
$$= \big\{[\mathfrak{f}^{\lambda,\mu}], [\mathsf{F}^{\lambda,\mu}], [\mathcal{F}^{\lambda,\mu}_{\gamma_{\mathcal{O}}}] \mid \mathcal{O} \in \mathbb{K}^*/q^{2\mathbb{Z}} \setminus \{\mathcal{O}(\nu)\}\big\}.$$

(2) *The map*
$$\widehat{\mathcal{C}^{\lambda,\mu}}\big(\mathbb{K}[t]\text{-torsionfree}\big) \to \widehat{\mathcal{C}^{\lambda,\mu}_t}\big(\mathbb{K}[t]\text{-torsionfree}\big), \quad [M] \mapsto [M_t]$$
is a bijection with the inverse $[N] \mapsto \mathrm{soc}_{\mathcal{C}^{\lambda,\mu}}(N)$.

(3) $\widehat{\mathcal{C}^{\lambda,\mu}}\,(\mathbb{K}[t]\text{-torsionfree})$
$$= \big\{\big[M_b := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu} \cap \mathcal{B}bt^{-i}\big] \mid b \text{ is } l\text{-normal}, b \in \mathrm{Irr}(\mathcal{B}), i \geqslant 3\deg(b)\big\}.$$

Proof. (1) Let $M \in \widehat{\mathcal{C}^{\lambda,\mu}}\,(t\text{-torsionfree}, \mathbb{K}[t]\text{-torsion})$. There exists a nonzero element $m \in M$ such that $tm = \gamma m$ for some $\gamma \in \mathbb{K}^*$. Then M is an epimorphic image of the module $\mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t-\gamma)$. If $\gamma \notin \mathcal{O}(\nu)$, then
$$M \simeq \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu}(t-\gamma) = \mathcal{F}^{\lambda,\mu}_{\gamma}$$
by Lemma 5.7.(3). It remains to consider the case when $\gamma \in \mathcal{O}(\nu)$, i.e., $\gamma = q^{2i}\nu$ for some $i \in \mathbb{Z}$.

(i) If $\gamma = q^{2i}\nu$, where $i \geqslant 1$, then $\sigma^i(a)m = 0$. Notice that
$$u^{i-1}v^{i-1}m = \sigma^{i-1}(a)\cdots\sigma(a)m \neq 0,$$
the element $m' := v^{i-1}m$ is a nonzero element of M. If $vm' = 0$, notice that
$$tm' = tv^{i-1}m = q^2\nu m',$$
then M is an epimorphic image of the simple module $\mathsf{F}^{\lambda,\mu}$. Hence, $M \simeq \mathsf{F}^{\lambda,\mu}$. If $m'' := vm' \neq 0$, notice that
$$tm'' = tv^i m = \nu m'' \quad \text{and} \quad um'' = uv^i m = v^{i-1}\sigma^i(a)m = 0,$$
then M is an epimorphic image of the simple module $\mathfrak{f}^{\lambda,\mu}$. Hence, $M \simeq \mathfrak{f}^{\lambda,\mu}$.

(ii) If $\gamma = q^{-2i}\nu$ where $i \geqslant 0$ then $\sigma^{-i}(a)m = 0$. The element $e := u^i m$ is a nonzero element of M. (The case $i = 0$ is trivial, for $i \geqslant 1$, it follows from the equality $v^i u^i m = \sigma^{-i+1}(a)\cdots\sigma^{-1}(a)am \neq 0$). If $ue = 0$, notice that
$$te = tu^i m = \nu e,$$

then M is an epimorphic image of the simple module $\mathfrak{f}^{\lambda,\mu}$. Hence, $M \simeq \mathfrak{f}^{\lambda,\mu}$. If $e' := ue \neq 0$, notice that

$$te' = tu^{i+1}m = q^2ve' \quad \text{and} \quad ve' = vu^{i+1}m = u^i\sigma^{-i}(a)m = 0,$$

then M is an epimorphic image of the simple module $\mathsf{F}^{\lambda,\mu}$. Hence, $M \simeq \mathsf{F}^{\lambda,\mu}$. This proves statement 1.

(2) The result follows from Proposition 5.10.

(3) Let $[M] \in \widehat{\mathcal{C}^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree). Then $[M_t] \in \widehat{\mathcal{C}_t^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree), and so $M_t \simeq \mathcal{C}_t^{\lambda,\mu}/\mathcal{C}_t^{\lambda,\mu} \cap \mathcal{B}b$, where

$$b = v^m\beta_m + v^{m-1}\beta_{m-1} + \cdots + \beta_0 \in \mathcal{C}^{\lambda,\mu} \quad (\beta_i \in \mathbb{K}[t],\ m > 0 \text{ and } \beta_m,\ \beta_0 \neq 0)$$

is l-normal and irreducible in $\mathcal{B}$. Clearly,

$$0 \neq M_b := \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu} \cap \mathcal{B}b \subseteq M_t$$

and

$$M = \mathrm{soc}_{\mathcal{C}^{\lambda,\mu}}(M_t) = \mathrm{soc}_{\mathcal{C}^{\lambda,\mu}}(M_b),$$

by statement 2. Let $I_b := \mathcal{C}^{\lambda,\mu} \cap \mathcal{B}b$, $J_n = \mathcal{C}^{\lambda,\mu}t^n + I_b$ for all $n \geqslant 0$ and $d = \deg(a)$. By Proposition 5.10, the following descending chain of left ideals of the algebra $\mathcal{C}^{\lambda,\mu}$ stabilizes:

$$\mathcal{C}^{\lambda,\mu} = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n = J_{n+1} = \cdots, \quad n \geqslant 3d.$$

Hence, $\mathrm{soc}_{\mathcal{C}^{\lambda,\mu}}(M_b) = J_n/I_b \simeq \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu} \cap \mathcal{B}bt^{-n}$. □

6. Simple weight A-modules

The aim of this section is to give a classification of simple weight A-modules. The set $\widehat{A}$ (weight) of isomorphism classes of simple weight A-modules is partitioned into the disjoint union of four subsets, see (6.1). We will describe each of them separately.

An A-module M is called a *weight module* provided that $M = \bigoplus_{\mu\in\mathbb{K}^*} M_\mu$, where $M_\mu = \{m \in M \mid Km = \mu m\}$. We denote by $\mathrm{Wt}(M)$ the set of all weights of M, i.e., the set $\{\mu \in \mathbb{K}^* \mid M_\mu \neq 0\}$.

Verma modules and simple highest weight A-modules. For each $\lambda \in \mathbb{K}^*$, we define the Verma module

$$M(\lambda) := A/A(K - \lambda, E, X).$$

Then $M(\lambda) = \mathbb{K}[Y, F]\tilde{1}$, where $\tilde{1} = 1 + A(K - \lambda, E, X)$. If M is an A-module, a *highest weight vector* is any $0 \neq m \in M$ such that m is an eigenvector of K and K^{-1} and $Em = Xm = 0$.

Lemma 6.1. *The set of highest weight vectors of the Verma module $M(\lambda)$ is*

$$\mathsf{H} := \{ kY^n\tilde{1} \mid k \in \mathbb{K}^*, n \in \mathbb{N} \}.$$

Proof. It is clear that any element of H is a highest weight vector. Suppose that $m = \sum \alpha_{ij} Y^i F^j \tilde{1} \in M(\lambda)$ is a highest weight vector of weight μ where $\alpha_{ij} \in \mathbb{K}$. Then

$$Km = \sum \alpha_{ij} \lambda q^{-i-2j} Y^i F^j \tilde{1} = \mu m.$$

This implies that $i + 2j$ is a constant, say $i + 2j = n$. Then m can be written as

$$m = \sum \alpha_j Y^{n-2j} F^j \tilde{1}$$

for some $\alpha_j \in \mathbb{K}$. By Lemma 3.1.(2),

$$Xm = \sum -q^{n-2j} \frac{1 - q^{2j}}{1 - q^2} \alpha_j \lambda^{-1} Y^{n-2j+1} F^{j-1} \tilde{1} = 0.$$

Thus, $\alpha_j = 0$ for all $j \geqslant 1$ and hence, $m \in \mathsf{H}$. □

By Lemma 6.1, there are infinitely many linear independent highest weight vectors. Let $N_n := \mathbb{K}[Y, F]Y^n\tilde{1}$ where $n \in \mathbb{N}$. Then N_n is a Verma A-module with highest weight $q^{-n}\lambda$, i.e., $N_n \simeq M(q^{-n}\lambda)$. Furthermore, $M(\lambda)$ is a submodule of $M(q^n\lambda)$ for all $n \in \mathbb{N}$. Thus, for any $\lambda \in \mathbb{K}^*$, there exists an infinite sequence of Verma modules

$$\cdots \supset M(q^2\lambda) \supset M(q\lambda) \supset M(\lambda) \supset M(q^{-1}\lambda) \supset M(q^{-2}\lambda) \supset \cdots.$$

The following result of Verma $U_q(\mathfrak{sl}_2)$-modules is well-known; see [17, p. 20].

Lemma 6.2 ([17]). *Suppose that q is not a root of unity. Let $V(\lambda)$ be a Verma $U_q(\mathfrak{sl}_2)$-module. Then $V(\lambda)$ is simple if and only if $\lambda \neq \pm q^n$ for all integer $n \geqslant 0$. When $\lambda = q^n$ (resp., $-q^n$) there is a unique simple quotient $L(n, +)$ (resp., $L(n, -)$) of $V(\lambda)$. Each simple $U_q(\mathfrak{sl}_2)$-module of dimension $n + 1$ is isomorphic to $L(n, +)$ or $L(n, -)$.*

Let $V(\lambda) := M(\lambda)/N_1$. Then $V(\lambda) \simeq \mathbb{K}[F]\bar{1}$, where $\bar{1} := 1 + A(K - \lambda, E, X, Y)$.

Theorem 6.3. *Up to isomorphism, the simple A-modules of highest weight λ are as follows:*

(i) *$V(\lambda)$, when $\lambda \neq \pm q^n$ for any $n \in \mathbb{N}$.*

(ii) *$L(n, +)$, when $\lambda = q^n$ for some $n \in \mathbb{N}$.*

(iii) *$L(n, -)$, when $\lambda = -q^n$ for some $n \in \mathbb{N}$.*

In each case, the elements X and Y act trivially on the modules, and these modules are in fact simple highest weight $U_q(\mathfrak{sl}_2)$-modules.

Proof. In view of Lemma 3.2.(1), $\mathrm{ann}_A(V(\lambda)) \supseteq (X)$. So, $V(\lambda) \simeq U/U(K-\lambda, E)$ where $U = U_q(\mathfrak{sl}_2)$. Then the theorem follows immediately from Lemma 6.2. □

Simple weight modules that not highest and lowest weight A-modules. Let $\mathcal{N}$ be *the set of simple weight A-modules M such that $XM \neq 0$ or $YM \neq 0$.* Then $\widehat{A}\,(\mathrm{weight}) = \widehat{U_q(\mathfrak{sl}_2)}\,(\mathrm{weight}) \sqcup \mathcal{N}$.

Lemma 6.4. *Let M be a simple A-module. If $x \in \{X, Y, E, F\}$ annihilates a non-zero element $m \in M$, then x acts locally nilpotently on M.*

Proof. For each element $x \in \{X, Y, E, F\}$, the set $S = \{x^i \mid i \in \mathbb{N}\}$ is an Ore set in the algebra A. Then $\mathrm{tor}_S(M)$ is a nonzero submodule of M. Since M is a simple module, $M = \mathrm{tor}_S(M)$, i.e., the element x acts locally nilpotently on M. □

Theorem 6.5. *Let $M \in \mathcal{N}$, then:*

(1) $\dim M_\lambda = \dim M_\mu$ *for any* $\lambda, \mu \in \mathrm{Wt}\,(M)$.

(2) $\mathrm{Wt}\,(M) = \{q^n\lambda \mid n \in \mathbb{Z}\}$ *for any* $\lambda \in \mathrm{Wt}\,(M)$.

Proof. (1) Suppose that there exists $\lambda \in \mathrm{Wt}\,(M)$ such that $\dim M_\lambda > \dim M_{q\lambda}$. Then the map $X: M_\lambda \to M_{q\lambda}$ is not injective. Hence $Xm = 0$ for some non-zero element $m \in M_\lambda$. By Lemma 6.4, X acts locally nilpotently on M.

If $\dim M_{q^{-1}\lambda} > \dim M_{q\lambda}$, then the linear map $E: M_{q^{-1}\lambda} \to M_{q\lambda}$ is not injective. So $Em' = 0$ for some non-zero element $m' \in M_{q^{-1}\lambda}$. By Lemma 6.4, E acts on M locally nilpotently. Since $EX = qXE$, there exists a non-zero weight vector m'' such that $Xm'' = Em'' = 0$. Therefore, M is a highest weight module. By Theorem 6.3, $XM = YM = 0$, which contradicts to our assumption that $M \in \mathcal{N}$.

If $\dim M_{q^{-1}\lambda} \leqslant \dim M_{q\lambda}$, then $\dim M_{q^{-1}\lambda} < \dim M_\lambda$. Hence the map $Y: M_\lambda \to M_{q^{-1}\lambda}$ is not injective. It follows that $Ym_1 = 0$ for some non-zero element $m_1 \in M_\lambda$. By Lemma 6.4, Y acts on M locally nilpotently. Since $XY = qYX$, there exists some non-zero weight vector $m_2 \in M$ such that $Xm_2 = Ym_2 = 0$. By Lemma 3.2.(1),

$$\mathrm{ann}_A(M) \supseteq (X, Y),$$

a contradiction. Similarly, one can show that there does not exist $\lambda \in \mathrm{Wt}(M)$ such that $\dim M_\lambda < \dim M_{q\lambda}$.

(2) Clearly, $\mathrm{Wt}(M) \subseteq \{q^n\lambda \mid n \in \mathbb{Z}\}$. By the above argument we see that

$$\mathrm{Wt}(M) \supseteq \{q^n\lambda \mid n \in \mathbb{Z}\}.$$

Hence $\mathrm{Wt}(M) = \{q^n\lambda \mid n \in \mathbb{Z}\}$. □

Let M be an A-module and $x \in A$. We say that M is *x-torsion* provided that for each element $m \in M$ there exists some $i \in \mathbb{N}$ such that $x^i m = 0$. We denote by x_M the map $M \to M$, $m \mapsto xm$.

Lemma 6.6. *Let $M \in \mathcal{N}$.*

(1) *If M is X-torsion, then M is (φ, Y)-torsionfree.*

(2) *If M is Y-torsion, then M is (X, φ)-torsionfree.*

(3) *If M is φ-torsion, then M is (X, Y)-torsionfree.*

Proof. (1) Since $M \in \mathcal{N}$ is an X-torsion module, by the proof of Theorem 6.5, Y_M and E_M are injections. Let us show that φ_M is injective. Otherwise, there exists a nonzero element $m \in M$ such that $\varphi m = 0$, i.e., $Xm = (q - q^{-1})YEm$. Since $X^i m = 0$ for some $i \in \mathbb{N}$ and $X(YE) = (YE)X$, we have

$$X^i m = (q - q^{-1})^i (YE)^i m = 0.$$

This contradicts the fact that Y and E are injective maps on M.

(2) Clearly, X_M is an injection. Let us show that φ_M is an injective map. Otherwise, there exists a nonzero element $m \in M$ such that $\varphi m = Ym = 0$ (since $Y\varphi = q\varphi Y$). Then $Xm = 0$ (since $\varphi = (1 - q^2)EY + q^2 X$), a contradiction.

(3) Statement 3 follows from statements 1 and 2. □

By Lemma 6.6,

$$\begin{aligned}\widehat{A}\,(\text{weight}) &= \widehat{U_q(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \mathcal{N} \\ &= \widehat{U_q(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \mathcal{N}\,(X\text{-torsion}) \sqcup \mathcal{N}\,(Y\text{-torsion}) \\ &\qquad \sqcup \mathcal{N}\,((X,Y)\text{-torsionfree}).\end{aligned} \tag{6.1}$$

It is clear that $\mathcal{N}\,((X,Y)\text{-torsionfree}) = \widehat{A}\,(\text{weight}, (X,Y)\text{-torsionfree})$.

Lemma 6.7. *If* $M \in \mathcal{N}\,(X\text{-torsion}) \sqcup \mathcal{N}\,(\varphi\text{-torsion}) \sqcup \mathcal{N}\,(Y\text{-torsion})$ *then* $C_M \neq 0$.

Proof. Suppose that $M \in \mathcal{N}\,(X\text{-torsion})$, and let m be a weight vector such that $Xm = 0$. If $C_M = 0$, then by (2.15),

$$Cm = -K^{-1}EY^2 m = 0$$

i.e., $EY^2 m = 0$. This implies that E_M or Y_M is not injective. By the proof of Theorem 6.5, this is a contradiction. Similarly, one can prove that for $M \in \mathcal{N}\,(Y\text{-torsion})$, $C_M \neq 0$. Now, suppose that $M \in \mathcal{N}\,(\varphi\text{-torsion})$, and let $m \in M_\mu$ be a weight vector such that $\varphi m = 0$. Since $Y\varphi = q(1 - q^2)EY^2 + q^4 YX$, we have

$$Y\varphi m = q(1 - q^2)EY^2 m + q^4 YXm = 0, \tag{6.2}$$

If $C_M = 0$, then by (2.16),

$$Cm = -\mu^{-1}EY^2 m + \frac{q^3}{1 - q^2}(\mu - \mu^{-1})YXm = 0. \tag{6.3}$$

The equalities (6.2) and (6.3) yield that $EY^2 m = 0$ and $YXm = 0$, a contradiction. □

Theorem 6.8. *Let* $M \in \mathcal{N}$. *Then* $\dim M_\mu = \infty$ *for all* $\mu \in \mathrm{Wt}(M)$.

Proof. Since M is a simple A-module, the weight space M_μ of M is a simple $\mathcal{C}^{\lambda,\mu}$-module for some $\lambda \in \mathbb{K}$. If $M \in \mathcal{N}(X\text{-torsion}) \sqcup \mathcal{N}(Y\text{-torsion})$ then by Lemma 6.7, $\lambda = C_M \neq 0$. By Proposition 4.9.(4) and Theorem 4.11.(1), $\mathcal{C}^{\lambda,\mu}$ is an infinite dimensional central simple algebra. Hence, $\dim M_\mu = \infty$. It remains to consider the case where $M \in \mathcal{N}((X,Y)\text{-torsionfree})$. Suppose that there exists a weight space M_ν of M such that $\dim M_\nu = n < \infty$, we seek a contradiction. Then by Theorem 6.5, $\dim M_\mu = n$ for all $\mu \in \mathrm{Wt}(M)$ and $\mathrm{Wt}(M) = \{q^i\nu \mid i \in \mathbb{Z}\}$. Notice that the elements X and Y act injectively on M, then they act bijectively on M (since all the weight spaces are finite dimensional and of the same dimension). In particular, the element $t = YX$ acts bijectively on each weight space M_μ, and so, M_μ is a simple $\mathcal{C}_t^{\lambda,\mu}$-module. By Proposition 4.9.(2,3), the algebra $\mathcal{C}_t^{\lambda,\mu}$ is an infinite dimensional central simple algebra for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. Then, $\dim M_\mu = \infty$, a contradiction. $\square$

Description of the set $\mathcal{N}(X$-torsion$)$. An explicit description of the set $\mathcal{N}(X\text{-torsion})$ is given in Theorem 6.10. It consists of a family of simple modules constructed below (see Proposition 6.9). For each $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{X}^\mu := A/A(K-\mu, X)$. Then

$$\mathbb{X}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^iE^jY^k\,\bar{1},$$

where $\bar{1} = 1 + A(K-\mu, X)$. Let $\lambda \in \mathbb{K}$. By (2.15), we see that the submodule of $\mathbb{X}^\mu$,

$$\begin{aligned}(C-\lambda)\mathbb{X}^\mu &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^iE^jY^k\big(\mu^{-1}EY^2+\lambda\big)\,\bar{1}\\ &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^i\big(\mu^{-1}q^kE^{j+1}Y^{k+2}+\lambda E^jY^k\big)\,\bar{1},\end{aligned} \tag{6.4}$$

is a proper submodule and the map $(C-\lambda)\cdot : \mathbb{X}^\mu \longrightarrow \mathbb{X}^\mu$, $v \mapsto (C-\lambda)v$, is an injection, which is not a bijection. It is obvious that $\mathrm{GK}(\mathbb{X}^\mu) = 3$.

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we define the left A-module

$$\mathbb{X}^{\lambda,\mu} := A/A(C-\lambda,\, K-\mu,\, X).$$

Then,

$$\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^\mu/(C-\lambda)\mathbb{X}^\mu \neq 0. \tag{6.5}$$

We have a short exact sequence of A-modules:

$$0 \longrightarrow \mathbb{X}^\mu \xrightarrow{(C-\lambda)\cdot} \mathbb{X}^\mu \longrightarrow \mathbb{X}^{\lambda,\mu} \longrightarrow 0.$$

The next proposition shows that the module $\mathbb{X}^{\lambda,\mu}$ is a simple module if λ is nonzero. Moreover, the $\mathbb{K}$-basis, the weight space decomposition and the annihilator of the module $\mathbb{X}^{\lambda,\mu}$ are given.

Proposition 6.9. *For $\lambda, \mu \in \mathbb{K}^*$, consider the left A-module*

$$\mathbb{X}^{\lambda,\mu} = A/A(C-\lambda,\ K-\mu,\ X).$$

(1) *The A-module*

$$\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}F^iY^j\,\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}F^iE^k\,\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}YF^iE^k\,\bar{1}$$

is a simple A-module where $\bar{1} = 1 + A(C-\lambda,\ K-\mu,\ X)$.

(2) $$\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}F^iY^j\,\bar{1} \oplus \Big(\bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}F^i\Theta^k\,\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\,\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}E^i\Theta^k\,\bar{1}\Big) \oplus \Big(\bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}YF^i\Theta^k\,\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}Y\Theta^k\,\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}YE^i\Theta^k\,\bar{1}\Big).$$

(3) *The weight subspace $\mathbb{X}^{\lambda,\mu}_{q^s\mu}$ of $\mathbb{X}^{\lambda,\mu}$ that corresponds to the weight $q^s\mu$ is*

$$\mathbb{X}^{\lambda,\mu}_{q^s\mu} = \begin{cases} \mathbb{K}[\Theta]\,\bar{1}, & s=0, \\ E^r\mathbb{K}[\Theta]\,\bar{1}, & s=2r,\ r\geqslant 1, \\ YE^r\mathbb{K}[\Theta]\,\bar{1}, & s=2r-1,\ r\geqslant 1, \\ F^r\mathbb{K}[\Theta]\,\bar{1} \oplus \bigoplus_{\substack{i+j=r,\\ j\geqslant 1}} \mathbb{K}F^iY^{2j}\,\bar{1}, & s=-2r,\ r\geqslant 1, \\ Y\mathbb{K}[\Theta]\,\bar{1}, & s=-1, \\ YF^{r-1}\mathbb{K}[\Theta]\,\bar{1} \oplus \bigoplus_{\substack{2i+j=2r-1,\\ j\geqslant 2}} \mathbb{K}F^iY^j\,\bar{1}, & s=-2(r-1)-1,\ r\geqslant 2. \end{cases}$$

(4) $\mathrm{ann}_A(\mathbb{X}^{\lambda,\mu}) = (C-\lambda)$.

(5) *$\mathbb{X}^{\lambda,\mu}$ is an X-torsion and Y-torsionfree A-module.*

(6) *Let $(\lambda,\mu), (\lambda',\mu') \in \mathbb{K}\times\mathbb{K}^*$. Then $\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^{\lambda',\mu'}$ iff $\lambda=\lambda'$ and $\mu = q^i\mu'$ for some $i\in\mathbb{Z}$.*

Proof. (1) By (6.5), $\mathbb{X}^{\lambda,\mu} \neq 0$ and $\bar{1}\neq 0$. Using the PBW basis for the algebra A, we have

$$\mathbb{X}^{\lambda,\mu} = \sum_{i,j,k\geqslant 0} \mathbb{K}F^iY^jE^k\,\bar{1}.$$

Using (2.15), we have $\lambda\,\bar{1} = C\,\bar{1} = -\mu^{-1}EY^2\,\bar{1}$. Hence $EY^2\,\bar{1} = -\mu\lambda\,\bar{1}$, and then $Y^2E\,\bar{1} = -q^2\mu\lambda\,\bar{1}$. By induction on k, we deduce that

$$E^kY^{2k}\,\bar{1} = (-\mu\lambda)^k q^{-k(k-1)}\,\bar{1} \quad \text{and} \quad Y^{2k}E^k\,\bar{1} = (-q^2\mu\lambda)^k q^{k(k-1)}\,\bar{1}. \tag{6.6}$$

Therefore,

$$\sum_{j,k\geqslant 0} \mathbb{K}Y^jE^k\,\bar{1} = Y^2\mathbb{K}[Y]\,\bar{1} + \mathbb{K}[E]\,\bar{1} + Y\mathbb{K}[E]\,\bar{1},$$

and then

$$\begin{aligned}\mathbb{X}^{\lambda,\mu} &= \sum_{i\geqslant 0, j\geqslant 2} \mathbb{K}F^iY^j\,\bar{1} + \sum_{i,k\geqslant 0} \mathbb{K}F^iE^k\,\bar{1} + \sum_{i,k\geqslant 0} \mathbb{K}YF^iE^k\,\bar{1}\\ &= \mathbb{K}[F]\big(\mathbb{K}[Y]Y^2 + \mathbb{K}[E] + Y\mathbb{K}[E]\big)\,\bar{1}.\end{aligned}$$

So, any element u of $\mathbb{X}^{\lambda,\mu}$ can be written as

$$u = \Big(\sum_{i=0}^{n} F^ia_i\Big)\bar{1},$$

where $a_i \in \Sigma := \mathbb{K}[Y]Y^2 + \mathbb{K}[E] + Y\mathbb{K}[E]$. Statement 1 follows from the following claim: if $a_n \neq 0$, then there is an element $a \in A$ such that $au = \bar{1}$.

(i) $X^nu = a'\,\bar{1}$ for some nonzero element $a' \in \Sigma$: Using Lemma 3.1, we have

$$Xu = \sum_{i=0}^{n-1} F^ib_i\,\bar{1}$$

for some $b_i \in \Sigma$ and $b_{n-1} \neq 0$. Repeating this step $n-1$ times (or using induction on n), we obtain the result as required. So, we may assume that $u = a_0\bar{1}$, where $0 \neq a_0 \in \Sigma$.

(ii) Notice that the element $a_0 \in \Sigma$ can be written as

$$a_0 = pY^2 + \sum_{i=0}^{m} (\lambda_i + \mu_iY)E^i,$$

where $p \in \mathbb{K}[Y]$, λ_i and $\mu_i \in \mathbb{K}$. Then, by (6.6),

$$\begin{aligned}Y^{2m}u = Y^{2m}a_0\,\bar{1} &= \Big(pY^{2m+2} + \sum_{i=0}^{m} (\lambda_i + \mu_iY)Y^{2(m-i)}Y^{2i}E^i\Big)\,\bar{1}\\ &= \Big(pY^{2m+2} + \sum_{i=0}^{m} (\lambda_i + \mu_iY)Y^{2(m-i)}\gamma_i\Big)\,\bar{1} = f\,\bar{1}\end{aligned}$$

for some $\gamma_i \in \mathbb{K}^*$ where f is a nonzero polynomial in $\mathbb{K}[Y]$ (since $a_0 \neq 0$). Hence, we may assume that $u = f\,\bar{1}$ where $0 \neq f \in \mathbb{K}[Y]$.

(iii) Let $f = \sum_{i=0}^{l} \gamma_i Y^i$, where $\gamma_i \in \mathbb{K}$ and $\gamma_l \neq 0$. Since $KY^i\,\bar{1} = \mu q^{-i} Y^i\,\bar{1}$ and all eigenvalues $\{\mu q^{-i} \mid i \geqslant 0\}$ are distinct, there is a polynomial $g \in \mathbb{K}[K]$ such that $gf\,\bar{1} = Y^l\,\bar{1}$. If $l = 0$, we are done. We may assume that $l \geqslant 1$. By multiplying by Y (if necessary) on the equality above we may assume that $l = 2k$ for some natural number k. Then, by (6.6), $\omega_k^{-1} E^k Y^{2k}\,\bar{1} = \bar{1}$, where $\omega_k = (-\mu\lambda)^k q^{-k(k-1)}$, as required.

(2) Recall that the algebra $U_q(\mathfrak{sl}_2)$ is a GWA

$$U_q(\mathfrak{sl}_2) = \mathbb{K}\big[\Theta, K^{\pm 1}\big]\left[E, F; \sigma, a = (1-q^2)^{-1}\Theta - \frac{q^2(qK + q^{-1}K^{-1})}{(1-q^2)^2}\right], \quad (6.7)$$

where $\sigma(\Theta) = \Theta$ and $\sigma(K) = q^{-2}K$. Then for all $i \geqslant 1$,

$$F^i E^i = a\sigma^{-1}(a) \cdots \sigma^{-i+1}(a).$$

Therefore,

$$\bigoplus_{i,k \geqslant 0} \mathbb{K} F^i E^k\,\bar{1} = \bigoplus_{i \geqslant 1, k \geqslant 0} \mathbb{K} F^i \Theta^k\,\bar{1} \oplus \bigoplus_{k \geqslant 0} \mathbb{K}\Theta^k\,\bar{1} \oplus \bigoplus_{i \geqslant 1, k \geqslant 0} \mathbb{K} E^i \Theta^k\,\bar{1}.$$

Then statement 2 follows from statement 1.

(3) Statement 3 follows from statement 2.

(4) Clearly, $(C - \lambda) \subseteq \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$. Since $\lambda \in \mathbb{K}^*$, by Corollary 3.9, the ideal $(C - \lambda)$ is a maximal ideal of A. Then we must have

$$(C - \lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu}).$$

(5) Clearly, $\mathbb{X}^{\lambda,\mu}$ is an X-torsion weight module. Since $\mathbb{X}^{\lambda,\mu}$ is a simple module, then by Lemma 6.6, $\mathbb{X}^{\lambda,\mu}$ is Y-torsionfree.

(6) ($\Rightarrow$) Suppose that $\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^{\lambda',\mu'}$. By statement 4,

$$(C - \lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu}) = \mathrm{ann}_A(\mathbb{X}^{\lambda',\mu'}) = (C - \lambda').$$

Hence, $\lambda = \lambda'$. By Theorem 6.5 (or by statement 3),

$$\{q^i \mu \mid i \in \mathbb{Z}\} = \mathrm{Wt}(\mathbb{X}^{\lambda,\mu}) = \mathrm{Wt}(\mathbb{X}^{\lambda',\mu'}) = \{q^i \mu' \mid i \in \mathbb{Z}\}.$$

Hence, $\mu = q^i \mu'$ for some $i \in \mathbb{Z}$.

($\Leftarrow$) Suppose that $\lambda = \lambda'$ and $\mu = q^i \mu'$ for some $i \in \mathbb{Z}$. Let $\bar{1}$ and $\bar{1}'$ be the canonical generators of the modules $\mathbb{X}^{\lambda,\mu}$ and $\mathbb{X}^{\lambda',\mu'}$, respectively. If $i \leqslant 0$ then the map

$$\mathbb{X}^{\lambda,\mu} \to \mathbb{X}^{\lambda',\mu'}, \quad \bar{1} \mapsto Y^{|i|}\,\bar{1}'$$

defines an isomorphism of A-modules. If $i \geqslant 1$ then the map

$$\mathbb{X}^{\lambda,\mu} \to \mathbb{X}^{\lambda',\mu'}, \quad \bar{1} \mapsto (YE)^i \, \bar{1}'$$

defines an isomorphism of A-modules. □

We define an equivalence relation $\sim$ on the set $\mathbb{K}^*$ as follows: for μ and $\nu \in \mathbb{K}^*$, $\mu \sim \nu$ iff $\mu = q^i \nu$ for some $i \in \mathbb{Z}$. Then the set $\mathbb{K}^*$ is a disjoint union of equivalence classes $\mathcal{O}(\mu) = \{q^i \mu \mid i \in \mathbb{Z}\}$. Let $\mathbb{K}^*/\sim$ be the set of equivalence classes. Clearly, $\mathbb{K}^*/\sim$ can be identified with the factor group $\mathbb{K}^*/\langle q \rangle$ where $\langle q \rangle = \{q^i \mid i \in \mathbb{Z}\}$. For each orbit $\mathcal{O} \in \mathbb{K}^*/\langle q \rangle$, we fix an element $\mu_{\mathcal{O}}$ in the equivalence class $\mathcal{O}$.

Theorem 6.10. $\mathcal{N}(X\text{-torsion}) = \{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*, \ \mathcal{O} \in \mathbb{K}^*/\langle q \rangle\}$.

Proof. Let $M \in \mathcal{N}(X\text{-torsion})$. By Lemma 6.7, the central element C acts on M as a nonzero scalar, say λ. Then M is an epimorphic image of the module $\mathbb{X}^{\lambda,\mu}$ for some $\mu \in \mathbb{K}^*$. By Proposition 6.9.(1), $\mathbb{X}^{\lambda,\mu}$ is a simple A-module, hence $M \simeq \mathbb{X}^{\lambda,\mu}$. Then the theorem follows from Proposition 6.9.(6). □

Lemma 6.11. (1) *For all* $\lambda \in \mathbb{K}$ *and* $\mu \in \mathbb{K}^*$, $\mathrm{GK}(\mathbb{X}^{\lambda,\mu}) = 2$.

(2) $A(C, K - \mu, X) \subsetneq A(K - \mu, X, Y, E) \subsetneq A$.

(3) *For all* $\mu \in \mathbb{K}^*$, *the module* $\mathbb{X}^{0,\mu}$ *is not a simple* A*-module.*

Proof. (1) By [20, Proposition 5.1.(e)],

$$\mathrm{GK}(\mathbb{X}^{\lambda,\mu}) \leqslant \mathrm{GK}(\mathbb{X}^{\mu}) - 1 = 2.$$

If $\lambda \neq 0$ then it follows from Proposition 6.9.(1) that $\mathrm{GK}(\mathbb{X}^{\lambda,\mu}) = 2$. If $\lambda = 0$, then consider the subspace

$$V = \bigoplus_{i,j \geqslant 0} \mathbb{K} F^i E^j \, \bar{1}$$

of the A-module $\mathbb{X}^{\mu}$. By (6.4), we see that $V \cap C\mathbb{X}^{\mu} = 0$. Hence, the vector space V can be seen as a subspace of the A-module $\mathbb{X}^{0,\mu}$. In particular, $\mathrm{GK}(\mathbb{X}^{0,\mu}) \geqslant 2$. Therefore, $\mathrm{GK}(\mathbb{X}^{0,\mu}) = 2$.

(2) Let $\mathfrak{a} = A(C, K - \mu, X)$ and $\mathfrak{b} = A(K - \mu, X, Y, E)$. Since $C \in \mathfrak{b}$ we have the equality $\mathfrak{b} = A(C, K - \mu, X, Y, E)$. Clearly, $\mathfrak{a} \subseteq \mathfrak{b}$. Notice that

$$A/\mathfrak{b} \simeq U/U(K - \mu, E),$$

where $U = U_q(\mathfrak{sl}_2)$. Then $\mathrm{GK}(A/\mathfrak{b}) = 1$, in particular, $\mathfrak{b} \subsetneq A$ is a proper left ideal of A. It follows from statement 1 that,

$$2 = \mathrm{GK}(A/\mathfrak{a}) > \mathrm{GK}(A/\mathfrak{b}),$$

hence the inclusion $\mathfrak{a} \subseteq \mathfrak{b}$ is strict.

(3) By statement 2, the left ideal $A(C, K-\mu, X)$ is not a maximal left ideal. Thus, the A-module $\mathbb{X}^{0,\mu}$ is not a simple module. □

Corollary 6.12. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The A-module $\mathbb{X}^{\lambda,\mu}$ is a simple module iff $\lambda \neq 0$.*

Proof. The result follows from Proposition 6.9.(1) and Lemma 6.11.(3). □

Description of the set $\mathcal{N}(Y$-torsion). An explicit description of the set $\mathcal{N}(Y$-torsion) is given in Theorem 6.14. It consists of a family of simple modules constructed below (see Proposition 6.13). The results and arguments are similar to that of the case for X-torsion modules. But for completeness, we present the results and their proof in detail. For $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{Y}^\mu := A/A(K-\mu, Y)$. Then,

$$\mathbb{Y}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K} E^i F^j X^k \bar{1},$$

where $\bar{1} = 1 + A(K-\mu, Y)$. It is obvious that $\mathrm{GK}(\mathbb{Y}^\mu) = 3$. Let $\lambda \in \mathbb{K}$. By (2.15), we have $(C-\lambda)\,\bar{1} = (q^2 F X^2 - \lambda)\,\bar{1}$. Then using Lemma 3.1, we see that the submodule of $\mathbb{Y}^\mu$,

$$\begin{aligned}(C-\lambda)\,\mathbb{Y}^\mu &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K} E^i F^j X^k (C-\lambda)\,\bar{1} \\ &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K} E^i F^j X^k \big(q^2 F X^2 - \lambda\big)\,\bar{1} \\ &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K} E^i F^j \big(q^2 F X^{k+2} - \lambda X^k\big)\,\bar{1}.\end{aligned} \tag{6.8}$$

Therefore, the submodule $(C-\lambda)\mathbb{Y}^\mu$ of $\mathbb{Y}^\mu$ is a proper submodule, and the map

$$(C-\lambda)\cdot : \mathbb{Y}^\mu \to \mathbb{Y}^\mu, \quad v \mapsto (C-\lambda)v,$$

is an injection, which is not a bijection.

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we define the left A-module

$$\mathbb{Y}^{\lambda,\mu} := A/A(C-\lambda,\, K-\mu,\, Y).$$

Then

$$\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^\mu/(C-\lambda)\mathbb{Y}^\mu \neq 0. \tag{6.9}$$

We have a short exact sequence of A-modules:

$$0 \longrightarrow \mathbb{Y}^\mu \xrightarrow{(C-\lambda)\cdot} \mathbb{Y}^\mu \longrightarrow \mathbb{Y}^{\lambda,\mu} \longrightarrow 0.$$

The next proposition shows that the module $\mathbb{Y}^{\lambda,\mu}$ is a simple module if λ is nonzero. Moreover, the $\mathbb{K}$-basis, the weight space decomposition and the annihilator of the module $\mathbb{Y}^{\lambda,\mu}$ are given.

Proposition 6.13. *For $\lambda, \mu \in \mathbb{K}^*$, consider the left A-module*

$$\mathbb{Y}^{\lambda,\mu} = A/A(C-\lambda,\ K-\mu,\ Y).$$

(1) *The A-module*

$$\mathbb{Y}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}E^i X^j\, \bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}E^i F^k\, \bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}E^i F^k X\, \bar{1}$$

is a simple A-module, where $\bar{1} = 1 + A(C-\lambda,\ K-\mu,\ Y)$.

(2) $$\mathbb{Y}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}E^i X^j\, \bar{1} \oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^k E^i \bar{1} \oplus \bigoplus_{k\geqslant 0} \mathbb{K}\Theta^k\, \bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^k F^i\, \bar{1}\Big)$$
$$\oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^k E^i X \bar{1} \oplus \bigoplus_{k\geqslant 0} \mathbb{K}\Theta^k X\, \bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^k F^i X\, \bar{1}\Big).$$

(3) *The weight subspace $\mathbb{Y}^{\lambda,\mu}_{q^s\mu}$ of $\mathbb{Y}^{\lambda,\mu}$ that corresponds to the weight $q^s\mu$ is*

$$\mathbb{Y}^{\lambda,\mu}_{q^s\mu} = \begin{cases} \mathbb{K}[\Theta]\, \bar{1}, & s = 0, \\ \mathbb{K}[\Theta]E^r\, \bar{1} \oplus \bigoplus_{\substack{i+j=r,\\ j\geqslant 1}} \mathbb{K}E^i X^{2j}\, \bar{1}, & s = 2r,\ r \geqslant 1, \\ \mathbb{K}[\Theta]X\, \bar{1}, & s = 1, \\ \mathbb{K}[\Theta]E^{2r} X\, \bar{1} \oplus \bigoplus_{\substack{2i+j=2r+1,\\ j\geqslant 2}} \mathbb{K}E^i X^j\, \bar{1}, & s = 2r+1,\ r \geqslant 1, \\ \mathbb{K}[\Theta]F^r\, \bar{1}, & s = -2r,\ r \geqslant 1, \\ \mathbb{K}[\Theta]F^r X\, \bar{1}, & s = -2r+1,\ r \geqslant 1. \end{cases}$$

(4) $\mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu}) = (C-\lambda)$.

(5) *$\mathbb{Y}^{\lambda,\mu}$ is a Y-torsion and X-torsionfree A-module.*

(6) *Let $(\lambda,\mu), (\lambda',\mu') \in \mathbb{K}\times\mathbb{K}^*$. Then $\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^{\lambda',\mu'}$, iff $\lambda = \lambda'$ and $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$.*

Proof. (1) Notice that $\mathbb{Y}^{\lambda,\mu} = \sum_{i,j,k\geqslant 0} \mathbb{K}E^i F^j X^k\, \bar{1}$. By (2.15), we have

$$\lambda\, \bar{1} = C\, \bar{1} = q^2 F X^2\, \bar{1},$$

i.e., $FX^2\, \bar{1} = q^{-2}\lambda\, \bar{1}$. By induction on k and using Lemma 3.1.(1), we deduce that

$$F^k X^{2k}\, \bar{1} = (FX^2)^k\, \bar{1} = q^{-2k}\lambda^k\, \bar{1}. \tag{6.10}$$

Therefore,

$$\sum_{j,k\geqslant 0} \mathbb{K}F^j X^k\, \bar{1} = \mathbb{K}[X]X^2\, \bar{1} + \mathbb{K}[F]\, \bar{1} + \mathbb{K}[F]X\, \bar{1},$$

and so

$$\mathbb{Y}^{\lambda,\mu} = \sum_{i\geqslant 0, j\geqslant 2} \mathbb{K}E^i X^j \bar{1} + \sum_{i,k\geqslant 0} \mathbb{K}E^i F^k \bar{1} + \sum_{i,k\geqslant 0} \mathbb{K}E^i F^k X \bar{1}.$$

So, any element u of $\mathbb{Y}^{\lambda,\mu}$ can be written as

$$u = \sum_{i=0}^{n} E^i a_i \bar{1},$$

where $a_i \in \Gamma := \mathbb{K}[X]X^2 + \mathbb{K}[F] + \mathbb{K}[F]X$. Statement 1 follows from the following claim: if $a_n \neq 0$, then there exists an element $a \in A$ such that $au = \bar{1}$.
(i) $Y^n u = a' \bar{1}$ for some nonzero element $a' \in \Gamma$: Notice that

$$Yu = \sum_{i=0}^{n-1} E^i b_i$$

for some $b_i \in \Gamma$ and $b_{n-1} \neq 0$. Repeating this step $n-1$ times, we obtain the result as desired. So, we may assume that $u = a' \bar{1}$ for some nonzero $a' \in \Gamma$.

(ii) Notice that the element a' can be written as

$$a' = pX^2 + \sum_{i=0}^{m} F^i(\lambda_i + \mu_i X),$$

where $p \in \mathbb{K}[X]$, λ_i, and $\mu_i \in \mathbb{K}$. By Lemma 3.1, we see that $F^i X \bar{1} = XF^i \bar{1}$. Then

$$\begin{aligned} X^{2m} u &= \left(pX^{2m+2} + \sum_{i=0}^{m} (\lambda_i + \mu_i X) X^{2m} F^i\right) \bar{1} \\ &= \left(pX^{2m+2} + \sum_{i=0}^{m} (\lambda_i + \mu_i X) X^{2(m-i)} X^{2i} F^i\right) \bar{1} \\ &= \left(pX^{2m+2} + \sum_{i=0}^{m} (\lambda_i + \mu_i X) X^{2(m-i)} \gamma_i\right) \bar{1} = f\, \bar{1} \end{aligned}$$

for some $\gamma_i \in \mathbb{K}^*$ (by (6.10)) and f is a nonzero element in $\mathbb{K}[Y]$. Hence, we may assume that $u = f\, \bar{1}$ where $f \in \mathbb{K}[X] \setminus \{0\}$.

(iii) Let $f = \sum_{i=0}^{l} \alpha_i X^i$ where $\alpha_i \in \mathbb{K}$ and $\alpha_l \neq 0$. Since $KX^i \bar{1} = q^i \mu X^i \bar{1}$ and all eigenvalues $\{q^i \mu \mid i \in \mathbb{N}\}$ are distinct, there is a polynomial $g \in \mathbb{K}[K]$ such that $gf\, \bar{1} = X^l\, \bar{1}$. If $l = 0$, we are done. We may assume that $l \geqslant 1$. By multiplying by X (if necessary) on the equality we may assume that $l = 2k$ for some natural number k. Then, by (6.10), we have $q^{2k} \lambda^{-k} F^k X^{2k}\, \bar{1} = \bar{1}$, as required.

(2) Recall that $U_q(\mathfrak{sl}_2)$ is a generalized Weyl algebra (see (6.7)), then $E^i F^i = \sigma^i(a)\sigma^{i-1}(a)\cdots\sigma(a)$ holds for all $i \geqslant 1$. Hence,

$$\bigoplus_{i,k\geqslant 0} \mathbb{K}F^i E^k \,\bar{1} = \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}\Theta^k E^i \,\bar{1} \oplus \bigoplus_{k\geqslant 0} \mathbb{K}\Theta^k \,\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}\Theta^k F^i \,\bar{1}.$$

Then statement 2 follows from statement 1.

(3) Statement 3 follows from statement 2.

(4) Clearly, $(C-\lambda) \subseteq \mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu})$. Then we must have

$$(C-\lambda) = \mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu})$$

since $(C-\lambda)$ is a maximal ideal of A.

(5) Clearly, $\mathbb{Y}^{\lambda,\mu}$ is Y-torsion. Since $\mathbb{Y}^{\lambda,\mu}$ is a simple module, then by Lemma 6.6, $\mathbb{Y}^{\lambda,\mu}$ is X-torsionfree.

(6) $(\Rightarrow)$ Suppose that $\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^{\lambda',\mu'}$. By statement 4,

$$(C-\lambda) = \mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu}) = \mathrm{ann}_A(\mathbb{Y}^{\lambda',\mu'}) = (C-\lambda').$$

Hence, $\lambda = \lambda'$. By Theorem 6.5 (or by statement 3),

$$\{q^i\mu \mid i \in \mathbb{Z}\} = \mathrm{Wt}(\mathbb{Y}^{\lambda,\mu}) = \mathrm{Wt}(\mathbb{Y}^{\lambda',\mu'}) = \{q^i\mu' \mid i \in \mathbb{Z}\}.$$

Hence, $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$.

$(\Leftarrow)$ Suppose that $\lambda = \lambda'$ and $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$. Let $\bar{1}$ and $\bar{1}'$ be the canonical generators of the modules $\mathbb{Y}^{\lambda,\mu}$ and $\mathbb{Y}^{\lambda',\mu'}$, respectively. If $i \geqslant 0$, then the map

$$\mathbb{Y}^{\lambda,\mu} \to \mathbb{Y}^{\lambda',\mu'}, \quad \bar{1} \mapsto X^i\,\bar{1}'$$

defines an isomorphism of A-modules. If $i \leqslant -1$, then the map

$$\mathbb{Y}^{\lambda,\mu} \to \mathbb{Y}^{\lambda',\mu'}, \quad \bar{1} \mapsto (FX)^i\,\bar{1}'$$

defines an isomorphism of A-modules. □

Theorem 6.14. $\mathcal{N}(Y\text{-torsion}) = \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*,\ \mathcal{O} \in \mathbb{K}^*/\langle q\rangle\}$.

Proof. Let $M \in \mathcal{N}(Y\text{-torsion})$. By Lemma 6.7, the central element C acts on M as a nonzero scalar, say λ. Then M is an epimorphic image of the module $\mathbb{Y}^{\lambda,\mu}$ for some $\mu \in \mathbb{K}^*$. By Proposition 6.13.(1), $\mathbb{Y}^{\lambda,\mu}$ is a simple A-module, hence $M \simeq \mathbb{Y}^{\lambda,\mu}$. Then the theorem follows from Proposition 6.13.(6). □

Lemma 6.15. (1) *For all* $\lambda \in \mathbb{K}$ *and* $\mu \in \mathbb{K}^*$, $\mathrm{GK}(\mathbb{Y}^{\lambda,\mu}) = 2$.

(2) $A(C, K-\mu, Y) \subsetneq A(K-\mu, X, Y, E) \subsetneq A$.

(3) *For all* $\mu \in \mathbb{K}^*$, *the module* $\mathbb{Y}^{0,\mu}$ *is not a simple* A*-module.*

Proof. (1) By [20, Proposition 5.1.(e)],

$$\mathrm{GK}(\mathbb{Y}^{\lambda,\mu}) \leqslant \mathrm{GK}(\mathbb{Y}^{\mu}) - 1 = 2.$$

If $\lambda \neq 0$ then it follows from Proposition 6.13.(1) that $\mathrm{GK}(\mathbb{Y}^{\lambda,\mu}) = 2$. If $\lambda = 0$ then consider the subspace

$$V = \bigoplus_{i,j \geqslant 0} \mathbb{K} E^i F^j \bar{1}$$

of the A-module $\mathbb{Y}^{\mu}$. By (6.8), we see that $V \cap C\mathbb{Y}^{\mu} = 0$. Hence, the vector space V can be seen as a subspace of the A-module $\mathbb{Y}^{0,\mu}$. In particular, $\mathrm{GK}(\mathbb{Y}^{0,\mu}) \geqslant 2$. Therefore, $\mathrm{GK}(\mathbb{Y}^{0,\mu}) = 2$.

(2) Let $\mathfrak{a}' = A(C, K-\mu, Y)$ and $\mathfrak{b} = A(K-\mu, X, Y, E)$. Since $C \in \mathfrak{b}$, we have the equality $\mathfrak{b} = A(C, K-\mu, X, Y, E)$. Clearly, $\mathfrak{a}' \subseteq \mathfrak{b}$. By Lemma 6.11.(2) and its proof, $\mathfrak{b}$ is a proper left ideal of A and $\mathrm{GK}(A/\mathfrak{b}) = 1$. Then it follows from statement 1 that,

$$2 = \mathrm{GK}(A/\mathfrak{a}') > \mathrm{GK}(A/\mathfrak{b}),$$

hence the inclusion $\mathfrak{a}' \subseteq \mathfrak{b}$ is strict.

(3) By statement 2, the left ideal $A(C, K-\mu, Y)$ is not a maximal left ideal. Thus, the A-module $\mathbb{Y}^{0,\mu}$ is not a simple module. □

Corollary 6.16. *Let* $\lambda \in \mathbb{K}$ *and* $\mu \in \mathbb{K}^*$. *The* A*-module* $\mathbb{Y}^{\lambda,\mu}$ *is a simple module iff* $\lambda \neq 0$.

Proof. The result follows from Proposition 6.13.(1) and Lemma 6.15.(3). □

The set $\mathcal{N}((X,Y)$-torsionfree). Theorem 6.18 and Theorem 6.19 give explicit description of the set $\mathcal{N}((X,Y)\text{-torsionfree})$. Recall that

$$\mathcal{N}\big((X,Y)\text{-torsionfree}\big) = \widehat{A}\big(\text{weight}, (X,Y)\text{-torsionfree}\big).$$

Then clearly,

$$\mathcal{N}\big((X,Y)\text{-torsionfree}\big) = \widehat{A(0)}\big(\text{weight},\ (X,Y)\text{-torsionfree}\big) \sqcup \bigsqcup_{\lambda \in \mathbb{K}^*} \widehat{A(\lambda)}\big(\text{weight},\ (X,Y)\text{-torsionfree}\big). \quad (6.11)$$

Let A_t be the localization of the algebra at the powers of the element $t = YX$. Recall that the algebra $\mathcal{C}_t$ is a GWA, see Proposition 4.9.(1).

Lemma 6.17. *$A_t = \mathcal{C}_t[X^{\pm 1}; \iota]$ is a skew polynomial algebra where ι is the automorphism of the algebra $\mathcal{C}_t$ defined by $\iota(C) = C$, $\iota(K^{\pm 1}) = q^{\mp 1}K^{\pm 1}$, $\iota(t) = qt$, $\iota(u) = q^2 u$, and $\iota(v) = v$.*

Proof. Clearly, the algebra $\mathcal{C}_t[X^{\pm 1}; \iota]$ is a subalgebra of A_t. Notice that all the generators of the algebra A_t are contained in the algebra $\mathcal{C}_t[X^{\pm 1}; \iota]$, then

$$A_t \subseteq \mathcal{C}_t[X^{\pm 1}; \iota].$$

Hence, $A_t = \mathcal{C}_t[X^{\pm 1}; \iota]$, as required. □

The set $\widehat{A(0)}$ (weight, (X, Y)-torsionfree). Let $[M] \in \widehat{\mathcal{C}^{0,\mu}}$ (t-torsionfree). By Theorem 5.6, the element t acts *bijectively* on the module M (since t is a normal element of $\mathcal{R}$). Therefore, the $\mathcal{C}$-module M is also a $\mathcal{C}_t$-module. Then by Lemma 6.17, we have the induced A_t-module

$$\widetilde{M} := A_t \otimes_{\mathcal{C}_t} M = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M = \bigoplus_{i\geqslant 1} Y^i \otimes M \oplus \bigoplus_{i\geqslant 0} X^i \otimes M.$$

Clearly, $\widetilde{M}$ is an (X, Y)-torsionfree, weight A-module and

$$\mathrm{Wt}\,(\widetilde{M}) = \{q^i\mu \mid i \in \mathbb{Z}\} = \mathcal{O}(\mu).$$

We claim that $\widetilde{M}$ *is a simple A-module*. Suppose that N is a nonzero A-submodule of $\widetilde{M}$, then $X^i \otimes m \in N$ for some $i \in \mathbb{Z}$ and $m \in M$. If $i = 0$, then $N = Am = \widetilde{M}$. If $i \geqslant 1$, since $Y^i(X^i \otimes m) \in \mathbb{K}^*(1 \otimes t^i m)$, then $1 \otimes tm \in N$ and so $N = \widetilde{M}$. If $i \leqslant -1$, then $X^{|i|}X^i \otimes m = 1 \otimes m \in N$, so $N = \widetilde{M}$. If $M' \in \widehat{\mathcal{C}^{0,\mu'}}$ (t-torsionfree), then the A-modules $\widetilde{M}$ and $\widetilde{M}'$ are isomorphic iff the $\mathcal{C}^{0,\mu}$-modules M and $X^i \otimes M'$ are isomorphic where $\mu = q^i\mu'$ for a unique $i \in \mathbb{Z}$.

Theorem 6.18.

$$\widehat{A(0)}\,(\text{weight, } (X,Y)\text{-torsionfree}) = \left\{[\widetilde{M}] \mid [M] \in \widehat{\mathcal{C}^{0,\mu_{\mathcal{O}}}}\ (t\text{-torsionfree}),\ \mathcal{O} \in \mathbb{K}^*/q^{\mathbb{Z}}\right\}.$$

Proof. Let $V \in \widehat{A(0)}$ (weight, (X, Y)-torsionfree). Then the elements X and Y act injectively on the module V. For any $\mu \in \mathrm{Wt}\,(V)$, the weight space V_μ is a simple t-torsionfree $\mathcal{C}^{0,\mu}$-module. Then,

$$V \supseteq \bigoplus_{i\geqslant 1} Y^i \otimes V_\mu \oplus \bigoplus_{i\geqslant 0} X^i \otimes V_\mu = \widetilde{V}_\mu.$$

Hence, $V = \widetilde{V}_\mu$ since V is a simple module. □

The set $\widehat{A(\lambda)}$ (weight, (X, Y)-torsionfree), where $\lambda \in \mathbb{K}^*$. Below, we use notation and results from Lemma 5.7. Let $M \in \widehat{\mathcal{C}^{\lambda,\mu}}$ (t-torsionfree). Then $M_t \in \widehat{\mathcal{C}_t^{\lambda,\mu}}$. By Lemma 6.17, we have the induced A_t-module

$$M^{\blacklozenge} := A_t \otimes_{\mathcal{C}_t} M_t = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M_t. \tag{6.12}$$

Clearly, $M^{\blacklozenge}$ is a simple weight A_t-module and

$$\mathrm{Wt}\,(M^{\blacklozenge}) = \{q^i\mu \mid i \in \mathbb{Z}\} = \mathcal{O}(\mu).$$

For all $i \in \mathbb{Z}$, the weight space

$$M_i^{\blacklozenge} := X^i \otimes M_t \simeq M_t^{\iota^{-i}}$$

as $\mathcal{C}_t$-modules, where $M_t^{\iota^{-i}}$ is the $\mathcal{C}_t$-module twisted by the automorphism ι^{-i} of the algebra $\mathcal{C}_t$ (the automorphism ι is defined in Lemma 6.17). The set $\widehat{\mathcal{C}^{\lambda,\mu}}$ (t-torsionfree) is described explicitly in Theorem 5.11.(1,3). If $M = \mathfrak{f}^{\lambda,\mu}$, then

$$X^i \otimes \mathfrak{f}_t^{\lambda,\mu} \simeq (\mathfrak{f}_t^{\lambda,\mu})^{\iota^{-i}} \simeq \mathfrak{f}_t^{\lambda,q^i\mu}$$

as $\mathcal{C}_t$-modules. It is clear that $\mathrm{soc}_{\mathcal{C}}(\mathfrak{f}_t^{\lambda,\mu}) = \mathfrak{f}^{\lambda,\mu}$. Hence,

$$\mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathfrak{f}_t^{\lambda,\mu}) = \mathrm{soc}_{\mathcal{C}}(\mathfrak{f}_t^{\lambda,q^i\mu}) = \mathfrak{f}^{\lambda,q^i\mu}.$$

Then the A-module

$$\mathrm{soc}_A\left((\mathfrak{f}^{\lambda,\mu})^{\blacklozenge}\right) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathfrak{f}_t^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathfrak{f}^{\lambda,q^i\mu}. \tag{6.13}$$

Similarly, if $M = \mathsf{F}^{\lambda,\mu}$, then

$$X^i \otimes \mathsf{F}_t^{\lambda,\mu} \simeq (\mathsf{F}_t^{\lambda,\mu})^{\iota^{-i}} \simeq \mathsf{F}_t^{\lambda,q^i\mu}$$

as $\mathcal{C}_t$-modules. It is clear that $\mathrm{soc}_{\mathcal{C}}(\mathsf{F}_t^{\lambda,\mu}) = \mathsf{F}^{\lambda,\mu}$. Hence,

$$\mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathsf{F}_t^{\lambda,\mu}) = \mathrm{soc}_{\mathcal{C}}(\mathsf{F}_t^{\lambda,q^i\mu}) = \mathsf{F}^{\lambda,q^i\mu}.$$

Then the A-module

$$\mathrm{soc}_A\left((\mathsf{F}^{\lambda,\mu})^{\blacklozenge}\right) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathsf{F}_t^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathsf{F}^{\lambda,q^i\mu}. \tag{6.14}$$

If $M = \mathcal{F}_\gamma^{\lambda,\mu}$ where $\gamma \in \mathbb{K}^* \setminus \{q^{2i}\nu \mid i \in \mathbb{Z}\}$, then

$$X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu} \simeq (\mathcal{F}_{\gamma,t}^{\lambda,\mu})^{\iota^{-i}} \simeq \mathcal{F}_{q^{-i}\gamma,t}^{\lambda,q^i\mu}$$

as $\mathcal{C}_t$-modules. It is clear that $\mathrm{soc}_{\mathcal{C}}(\mathcal{F}_{\gamma,t}^{\lambda,\mu}) = \mathcal{F}_{\gamma}^{\lambda,\mu}$. Hence,

$$\mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu}) = \mathcal{F}_{q^{-i}\gamma}^{\lambda,q^i\mu}$$

is a simple $\mathcal{C}$-module. Then the A-module

$$\mathrm{soc}_A\left((\mathcal{F}_{\gamma}^{\lambda,\mu})^{\blacklozenge}\right) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathcal{C}}(X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_{q^{-i}\gamma}^{\lambda,q^i\mu}. \tag{6.15}$$

If $M \in \widehat{\mathcal{C}^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree) then, by Theorem 5.11.(3),

$$M \simeq \mathcal{C}^{\lambda,\mu}/\mathcal{C}^{\lambda,\mu} \cap \mathcal{B}bt^{-n}$$

for some l-normal element $b \in \mathrm{Irr}\,(\mathcal{B})$ and for all $n \gg 0$. For all $i \in \mathbb{Z}$,

$$M_t^{\iota^{-i}} \supseteq \frac{\mathcal{C}_t^{\lambda,q^i\mu}}{\mathcal{C}_t^{\lambda,q^i\mu} \cap \mathcal{B}\iota^i(b)t^{-n}} := \mathcal{M}_{\iota^i(b)t^{-n}}.$$

Then,

$$\mathrm{soc}_{\mathcal{C}}(M_t^{\iota^{-i}}) = \mathrm{soc}_{\mathcal{C}}(\mathcal{M}_{\iota^i(b)t^{-n}}) = \mathcal{M}_{\iota^i(b)t^{-n_i}}$$

for all $n_i \gg 0$. Then the A-module

$$\mathrm{soc}_A(M^{\blacklozenge}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathcal{C}}(X^i \otimes M_t) \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_{\iota^i(b)t^{-n_i}}. \tag{6.16}$$

The next theorem describes the set $\widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree), where $\lambda \in \mathbb{K}^*$.

Theorem 6.19. *Let $\lambda, \mu \in \mathbb{K}^*$. Then*

$$\widehat{A(\lambda)}\,\big(\text{weight, } (X,Y)\text{-torsionfree}\big) = \left\{[\mathrm{soc}_A(M^{\blacklozenge})] \mid [M] \in \widehat{\mathcal{C}^{\lambda,\mu_{\mathcal{O}}}}\ (t\text{-torsionfree}),\ \mathcal{O} \in \mathbb{K}^*/q^{\mathbb{Z}}\right\}$$

and $\mathrm{soc}_A(M^{\blacklozenge})$ is explicitly described in (6.13)*,* (6.14)*,* (6.15)*, and* (6.16)*.*

Proof. Let $\mathcal{M} \in \widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree). Then $\mathrm{Wt}\,(\mathcal{M}) = \mathcal{O}(\mu) \in \mathbb{K}^*/q^{\mathbb{Z}}$ for any $\mu \in \mathrm{Wt}\,(\mathcal{M})$. Then $M := \mathcal{M}_\mu \in \widehat{\mathcal{C}^{\lambda,\mu_{\mathcal{O}}}}$ (t-torsionfree) and $M_t \in \widehat{\mathcal{C}_t^{\lambda,\mu_{\mathcal{O}}}}$. Clearly, $M^{\blacklozenge} = \mathcal{M}_t \supseteq \mathcal{M}$. So, $\mathcal{M} = \mathrm{soc}_A(M^{\blacklozenge})$. □

By (6.1) and (6.11), Theorem 6.10, Theorem 6.14, Theorem 6.18 and Theorem 6.19 give a complete classification of simple weight A-modules.

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List of notations

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