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# Quasimodular Hecke algebras and Hopf actions

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**Abstract.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In this paper, we extend the theory of modular Hecke algebras due to Connes and Moscovici to define the algebra $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$. Then, $\mathcal{Q}(\Gamma)$ carries an action of "the Hopf algebra $\mathcal{H}_1$ of codimension 1 foliations" that also acts on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici. However, in the case of quasimodular forms, we have several new operators acting on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$. Further, for each $\sigma \in SL_2(\mathbb{Z})$, we introduce the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. Then, $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$-module and is endowed with a pairing

$$(\_\_,\_\_)\colon \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma).$$

We show that there is a "Hopf action" of a certain Hopf algebra $\mathfrak{h}_1$ on the pairing on $\mathcal{Q}_\sigma(\Gamma)$. Finally, for any $\sigma \in SL_2(\mathbb{Z})$, we consider operators acting between the levels of the graded module $\mathbb{Q}_\sigma(\Gamma) = \oplus_{m\in\mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$, where

$$\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$$

for any $m \in \mathbb{Z}$. The pairing on $\mathcal{Q}_\sigma(\Gamma)$ can be extended to a graded pairing on $\mathbb{Q}_\sigma(\Gamma)$ and we show that there is a Hopf action of a larger Hopf algebra $\mathfrak{h}_{\mathbb{Z}} \supseteq \mathfrak{h}_1$ on the pairing on $\mathbb{Q}_\sigma(\Gamma)$.

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## 1. Introduction

Let $N \geq 1$ be an integer and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In [6, 7], Connes and Moscovici have introduced the "modular Hecke algebra" $\mathcal{A}(\Gamma)$ that combines the pointwise product on modular forms with the action of Hecke operators. Further, Connes and Moscovici have shown that the modular Hecke algebra $\mathcal{A}(\Gamma)$ carries an action of "the Hopf algebra $\mathcal{H}_1$ of codimension 1 foliations". The Hopf algebra $\mathcal{H}_1$ is part of a larger family of Hopf algebras $\{\mathcal{H}_n | n \geq 1\}$ defined by them in [5], with the Hopf algebra $\mathcal{H}_n$ acting on $C^*$-algebras coming from foliations of codimension $n$. Then, the discovery by Connes and Moscovici [6] of the $\mathcal{H}_1$-action on the modular Hecke algebra $\mathcal{A}(\Gamma)$ reveals deep connections between noncommutative geometry and number theory. For further work on this Hopf algebra $\mathcal{H}_1$, we refer the reader, for instance, to [4, 13].

In [1], we showed that the action of the Hopf algebra $\mathcal{H}_1$ is associated with Frobenius and monodromy operators in arithmetic geometry. In fact, the Hopf algebra $\mathcal{H}_1$ acts on a complex in [1] that is obtained by modifying a certain bi-complex introduced by Consani [8, § 4] for computing the cohomology of the "fiber at infinity" of an arithmetic variety. The bi-complex of Consani [8] is the arithmetic analogue of the 'nearby cycles complex' in algebraic geometry (see, for instance, [9, § 2]). By considering modular forms as sections of line bundles, we also developed in [2] an $\mathcal{H}_1$-action on an algebra of Hecke operators lifted to line bundles over modular curves. The lifting of Hecke operators to the level of line bundles in [2] also leads to additional operators that are obtained by modifying the $\mathcal{H}_1$-action.

The objective of this paper is to introduce and study quasimodular Hecke algebras $\mathcal{Q}(\Gamma)$ that combine the pointwise product on quasimodular forms with the action of Hecke operators. Further, we will also study the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators twisted by some $\sigma \in SL_2(\mathbb{Z})$. The latter is an extension of our theory of twisted modular Hecke operators introduced in [3]. We recall that quasimodular forms can be interpreted geometrically as sections of bundles on the moduli space of elliptic curves (see [11]). As such, the $\mathcal{H}_1$-action on $\mathcal{Q}(\Gamma)$ demonstrates the amazing versatility of the Hopf algebra $\mathcal{H}_1$ of Connes and Moscovici. Additionally, the use of quasimodular forms helps us to find new operators on the algebra $\mathcal{Q}(\Gamma)$. At the heart of these new operators is the classical Eisenstein series $G_2$ of weight 2 which is not a modular form but only quasimodular (see Section 2 for details). However, we know (see [6, Remark 1]) that $G_2$ plays an important role in defining actions on the modular Hecke algebra . Hence, we feel that working with the quasimodular Hecke algebra allows us to fully involve the Eisenstein series $G_2$ in the theory. The action of these new operators is also expressed in terms of the action of a co-commutative Hopf algebra, which arises as the universal enveloping algebra of a Lie algebra. We also hope that in the future, we can lift the quasimodular Hecke operators to the level of bundles in the same spirit as our work in [2].

We now describe the paper in detail. In Section 2, we briefly recall the notion of modular Hecke algebras of Connes and Moscovici [6, 7]. We let $\mathcal{QM}$ be the "quasimodular tower", i.e., $\mathcal{QM}$ is the colimit over all $N$ of the spaces $\mathcal{QM}(\Gamma(N))$ of quasimodular forms of level $\Gamma(N)$ (see (2.8)). We define a quasimodular Hecke operator of level $\Gamma$ to be a function of finite support from $\Gamma\backslash GL_2^+(\mathbb{Q})$ to the quasimodular tower $\mathcal{QM}$ satisfying a certain covariance condition (see Definition 2.4). We then show that the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ carries an algebra structure $(\mathcal{Q}(\Gamma), *)$ by considering a convolution product over cosets of $\Gamma$ in $GL_2^+(\mathbb{Q})$. Further, the modular Hecke algebra of Connes and Moscovici embeds naturally as a subalgebra of $\mathcal{Q}(\Gamma)$. We also show that the quasimodular Hecke operators of level $\Gamma$ act on quasimodular forms of level $\Gamma$, i.e., $\mathcal{QM}(\Gamma)$ is a left $\mathcal{Q}(\Gamma)$-module. In this section, we will also define a second algebra structure $(\mathcal{Q}(\Gamma), *^r)$ on $\mathcal{Q}(\Gamma)$ by considering the convolution product over cosets of $\Gamma$ in $SL_2(\mathbb{Z})$, a construction that should be compared to the "restricted" modular Hecke algebra from [1, § 4]. When we consider $\mathcal{Q}(\Gamma)$ as an algebra equipped with this latter product $*^r$, it will be denoted by $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

In Section 3, we define Lie algebra and Hopf algebra actions on $\mathcal{Q}(\Gamma)$. Given a quasimodular form $f \in \mathcal{QM}(\Gamma)$ of level $\Gamma$, it is well known that we can write $f$ as a sum

$$f = \sum_{i=0}^{s} a_i(f) \cdot G_2^i, \tag{1.1}$$

where the coefficients $a_i(f)$ are modular forms of level $\Gamma$ and $G_2$ is the classical Eisenstein series of weight 2. Therefore, we can consider two different sets of operators on the quasimodular tower $\mathcal{QM}$: those which act on the powers of $G_2$ appearing in the expression for $f$ and those which act on the modular coefficients $a_i(f)$. The collection of operators acting on the modular coefficients $a_i(f)$ are studied in Section 3.2. These induce on $\mathcal{Q}(\Gamma)$ analogues of operators acting on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici and we show that $\mathcal{Q}(\Gamma)$ carries an action of the same Hopf algebra $\mathcal{H}_1$ of codimension 1 foliations that acts on $\mathcal{A}(\Gamma)$. On the other hand, by considering operators on $\mathcal{QM}$ that act on the powers of $G_2$ appearing in (1.1), we are able to define additional operators $D$, $\{T_k^l\}_{k\geq 1, l\geq 0}$ and $\{\phi^{(m)}\}_{m\geq 1}$ on $\mathcal{Q}(\Gamma)$ (see Section 3.1). Further, we show that these operators satisfy the following commutator relations:

$$\begin{gathered} \left[T_k^l, T_{k'}^{l'}\right] = (k'-k)T_{k+k'-2}^{l+l'} \\ \left[D, \phi^{(m)}\right] = 0, \quad \left[T_k^l, \phi^{(m)}\right] = 0, \quad \left[\phi^{(m)}, \phi^{(m')}\right] = 0 \\ \left[T_k^l, D\right] = \frac{5}{24}(k-1)T_{k-1}^{l+1} - \frac{1}{2}(k-3)T_{k+1}^l. \end{gathered} \tag{1.2}$$

We then consider the Lie algebra $\mathcal{L}$ generated by the symbols $D$, $\{T_k^l\}_{k\geq 1, l\geq 0}$, $\{\phi^{(m)}\}_{m\geq 1}$ satisfying the commutator relations in (1.2). Then, there is a Lie action of $\mathcal{L}$ on $\mathcal{Q}(\Gamma)$. Finally, let $\mathcal{H}$ be the Hopf algebra given by the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of $\mathcal{L}$. Then, we show that $\mathcal{H}$ has a Hopf action with respect to the product $*^r$ on $\mathcal{Q}(\Gamma)$ and this action captures the operators $D$, $\{T_k^l\}_{k\geq 1, l\geq 0}$ and $\{\phi^{(m)}\}_{m\geq 1}$ on $\mathcal{Q}(\Gamma)$. In other words, $\mathcal{H}$ acts on $\mathcal{Q}(\Gamma)$ such that:

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2), \quad \forall\, h \in \mathcal{H},\ F^1, F^2 \in \mathcal{Q}(\Gamma), \tag{1.3}$$

where the coproduct $\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$.

In Section 4, we develop the theory of twisted quasimodular Hecke operators. For any $\sigma \in SL_2(\mathbb{Z})$, we define in Section 4.1 the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, this reduces to the original definition of $\mathcal{Q}(\Gamma)$. In general, $\mathcal{Q}_\sigma(\Gamma)$ is not an algebra but we show that $\mathcal{Q}_\sigma(\Gamma)$ carries a pairing:

$$(\_\_,\_\_)\colon \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma). \tag{1.4}$$

Further, we show that $\mathcal{Q}_\sigma(\Gamma)$ may be equipped with the structure of a right $\mathcal{Q}(\Gamma)$-module. We can also extend the action of the Hopf algebra $\mathcal{H}_1$ of codimension 1 foliations to $\mathcal{Q}_\sigma(\Gamma)$. In fact, we show that $\mathcal{H}_1$ has an action on the right $\mathcal{Q}(\Gamma)$-module $\mathcal{Q}_\sigma(\Gamma)$ and this action is Hopf, i.e.,

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) * h_{(2)}(F^2), \quad \forall\, h \in \mathcal{H}_1,\ F^1 \in \mathcal{Q}_\sigma(\Gamma),\ F^2 \in \mathcal{Q}(\Gamma). \tag{1.5}$$

We recall from [6] that $\mathcal{H}_1$ is equal as an algebra to the universal enveloping algebra of the Lie algebra $\mathcal{L}_1$ with generators $X$, $Y$, $\{\delta_n\}_{n\geq 1}$ satisfying the following relations:

$$[Y,X] = X, \quad [X,\delta_n] = \delta_{n+1}, \quad [Y,\delta_n] = n\delta_n, \quad [\delta_k,\delta_l] = 0, \quad \forall\, k,l,n \geq 1. \tag{1.6}$$

Then, we can consider the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ with two generators $X$, $Y$ satisfying $[Y,X] = X$. If we let $\mathfrak{h}_1$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_1$, we show that the pairing in (1.4) on $\mathcal{Q}_\sigma(\Gamma)$ carries a "Hopf action" of $\mathfrak{h}_1$. In other words, we have:

$$h(F^1,F^2) = \sum \big(h_{(1)}(F^1), h_{(2)}(F^2)\big), \quad \forall\, h \in \mathfrak{h}_1,\ F^1, F^2 \in \mathcal{Q}_\sigma(\Gamma). \tag{1.7}$$

In Section 4.2, we consider operators between the modules $\mathcal{Q}_\sigma(\Gamma)$ as $\sigma$ varies over $SL_2(\mathbb{Z})$. More precisely, for any $\tau, \sigma \in SL_2(\mathbb{Z})$, we define a morphism:

$$X_\tau\colon \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma). \tag{1.8}$$

In particular, this gives us operators acting between the levels of the graded module

$$\mathbb{Q}_\sigma(\Gamma) = \bigoplus_{m\in\mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma), \tag{1.9}$$

where for any $\sigma \in SL_2(\mathbb{Z})$, we set

$$\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma.$$

Further, we generalize the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in (1.4) to a pairing:

$$(\_\_,\_\_)\colon \mathcal{Q}_{\tau_1\sigma}(\Gamma) \otimes \mathcal{Q}_{\tau_2\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma), \tag{1.10}$$

where $\tau_1$, $\tau_2$ are commuting matrices in $SL_2(\mathbb{Z})$. In particular, (1.10) gives us a pairing

$$\mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma), \quad \forall\, m, n \in \mathbb{Z}$$

and hence a pairing on the tower $\mathbb{Q}_\sigma(\Gamma)$. Finally, we consider the Lie algebra $\mathfrak{l}_{\mathbb{Z}} \supseteq \mathfrak{l}_1$ with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the following commutator relations:

$$[Z, X_n] = (n+1)X_n, \quad [X_n, X_{n'}] = 0, \quad \forall\, n, n' \in \mathbb{Z}. \tag{1.11}$$

Then, if we let $\mathfrak{h}_{\mathbb{Z}}$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_{\mathbb{Z}}$, we show that $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$. In other words, for any $F^1$, $F^2 \in \mathbb{Q}_\sigma(\Gamma)$, we have

$$h(F^1, F^2) = \sum \big(h_{(1)}(F^1), h_{(2)}(F^2)\big), \quad \forall\, h \in \mathfrak{h}_{\mathbb{Z}}. \tag{1.12}$$

## 2. The quasimodular Hecke algebra

We begin this section by briefly recalling the notion of quasimodular forms. The notion of quasimodular forms is due to Kaneko and Zagier [10]. The theory has been further developed in Zagier [14]. For an introduction to the basic theory of quasimodular forms, we refer the reader to the exposition of Royer [12].

Throughout, let $\mathbb{H} \subseteq \mathbb{C}$ be the upper half plane. Then, there is a well known action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

$$z \mapsto \frac{az+b}{cz+d}, \quad \forall\, z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{2.1}$$

For any $N \geq 1$, we denote by $\Gamma(N)$ the following principal congruence subgroup of $SL_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, N) \right\}. \tag{2.2}$$

In particular, $\Gamma(1) = SL_2(\mathbb{Z})$. We are now ready to define quasimodular forms.

**Definition 2.1.** Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a holomorphic function and let $N \geq 1, k, s \geq 0$ be integers. Then, the function $f$ is a quasimodular form of level $N$, weight $k$ and depth $s$ if there exist holomorphic functions $f_0, f_1, \dots, f_s : \mathbb{H} \longrightarrow \mathbb{C}$ with $f_s \neq 0$ such that:

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j \tag{2.3}$$

for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. The collection of quasimodular forms of level $N$, weight $k$ and depth $s$ will be denoted by $\mathcal{QM}_k^s(\Gamma(N))$. By convention, we let the zero function $0 \in \mathcal{QM}_k^0(\Gamma(N))$ for every $k \geq 0$, $N \geq 1$.

More generally, for any holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ and any matrix $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$, we define:

$$(f|_k\alpha)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \forall\, k \geq 0. \tag{2.4}$$

Then, we can say that $f$ is quasimodular of level $N$, weight $k$ and depth $s$ if there exist holomorphic functions $f_0, f_1, \ldots, f_s : \mathbb{H} \longrightarrow \mathbb{C}$ with $f_s \neq 0$ such that:

$$(f|_k\alpha)(z) = \sum_{j=0}^{s} f_j(z)\left(\frac{c}{cz+d}\right)^j, \quad \forall\, \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N). \tag{2.5}$$

When the integer $k$ is clear from context, we write $f|_k\alpha$ simply as $f|\alpha$ for any $\alpha \in GL_2^+(\mathbb{Q})$. Also, it is clear that we have a product:

$$\mathcal{QM}_k^s(\Gamma(N)) \otimes \mathcal{QM}_l^t(\Gamma(N)) \longrightarrow \mathcal{QM}_{k+l}^{s+t}(\Gamma(N)) \tag{2.6}$$

on quasi-modular forms. For any $N \geq 1$, we now define:

$$\mathcal{QM}(\Gamma(N)) := \bigoplus_{s=0}^{\infty} \bigoplus_{k=0}^{\infty} \mathcal{QM}_k^s(\Gamma(N)). \tag{2.7}$$

We now consider the direct limit:

$$\mathcal{QM} := \varinjlim_{N \geq 1} \mathcal{QM}(\Gamma(N)), \tag{2.8}$$

which we will refer to as the quasimodular tower. Additionally, for any $k \geq 0$ and $N \geq 1$, we let $\mathcal{M}_k(\Gamma(N))$ denote the collection of usual modular forms of weight $k$ and level $N$. Then, we can define the modular tower $\mathcal{M}$:

$$\mathcal{M} := \varinjlim_{N \geq 1} \mathcal{M}(\Gamma(N)), \quad \mathcal{M}(\Gamma(N)) := \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma(N)). \tag{2.9}$$

We now recall the modular Hecke algebra of Connes and Moscovici [6].

**Definition 2.2** (see [6, § 1]). Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. A modular Hecke operator of level $\Gamma$ is a function of finite support

$$F : \Gamma\backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{M}, \quad \Gamma\alpha \mapsto F_\alpha \tag{2.10}$$

such that for any $\gamma \in \Gamma$, we have:

$$F_{\alpha\gamma} = F_\alpha|\gamma. \tag{2.11}$$

The collection of all modular Hecke operators of level $\Gamma$ will be denoted by $\mathcal{A}(\Gamma)$.

Our first aim is to define a quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ analogous to the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici. For this, we recall the structure theorem for quasimodular forms, proved by Kaneko and Zagier [10].

**Theorem 2.3** (see [10, § 1, Proposition 1]). *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. For any even number $K \geq 2$, let $G_K$ denote the classical Eisenstein series of weight $K$:*

$$G_K(z) := -\frac{B_K}{2K} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d^{K-1}\right)e^{2\pi i n z}, \tag{2.12}$$

*where $B_K$ is the $K$-th Bernoulli number and $z \in \mathbb{H}$. Then, every quasimodular form in $\mathcal{QM}(\Gamma)$ can be written uniquely as a polynomial in $G_2$ with coefficients in $\mathcal{M}(\Gamma)$. More precisely, for any quasimodular form $f \in \mathcal{QM}_k^s(\Gamma)$, there exist functions $a_0(f), a_1(f), \ldots, a_s(f)$ such that:*

$$f = \sum_{i=0}^{s} a_i(f)G_2^i, \tag{2.13}$$

*where $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$ is a modular form of weight $k-2i$ and level $\Gamma$ for each $0 \leq i \leq s$.*

We now consider a quasimodular form $f \in \mathcal{QM}$. For sake of definiteness, we may assume that $f \in \mathcal{QM}_k^s(\Gamma(N))$, i.e. $f$ is a quasimodular form of level $N$, weight $k$ and depth $s$. We now define an operation on $\mathcal{QM}$ by setting:

$$f\|\alpha = \sum_{i=0}^{i}\big(a_i(f)|_{k-2i}\alpha\big)G_2^i, \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}), \tag{2.14}$$

where $\{a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N))\}_{0\leq i\leq s}$ is the collection of modular forms determining $f = \sum_{i=0}^{s} a_i(f)G_2^i$ as in Theorem 2.3. We know that for any $\alpha \in GL_2^+(\mathbb{Q})$, each $(a_i(f)|_{k-2i}\alpha)$ is an element of the modular tower $\mathcal{M}$. This shows that

$$f\|\alpha = \sum_{i=0}^{i}\big(a_i(f)|_{k-2i}\alpha\big)G_2^i \in \mathcal{QM}.$$

However, we note that for arbitrary $\alpha \in GL_2^+(\mathbb{Q})$ and $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N))$, it is not necessary that $(a_i(f)|_{k-2i}\alpha) \in \mathcal{M}_{k-2i}(\Gamma(N))$. In other words, the operation defined in (2.14) on the quasimodular tower $\mathcal{QM}$ does not descend to an endomorphism on each $\mathcal{QM}_k^s(\Gamma(N))$. From the expression in (2.14), it is also clear that:

$$(f\cdot g)\|\alpha = (f\|\alpha)\cdot(g\|\alpha), \quad f\|(\alpha\cdot\beta) = (f\|\alpha)\|\beta, \quad \forall\, f, g \in \mathcal{QM},\ \alpha, \beta \in GL_2^+(\mathbb{Q}). \tag{2.15}$$

We are now ready to define the quasimodular Hecke operators.

**Definition 2.4.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup. A quasimodular Hecke operator of level $\Gamma$ is a function of finite support:

$$F : \Gamma\backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{QM}, \quad \Gamma\alpha \mapsto F_\alpha \tag{2.16}$$

such that for any $\gamma \in \Gamma$, we have:

$$F_{\alpha\gamma} = F_\alpha \| \gamma. \tag{2.17}$$

The collection of all quasimodular Hecke operators of level $\Gamma$ will be denoted by $\mathcal{Q}(\Gamma)$.

We will now introduce the product structure on $\mathcal{Q}(\Gamma)$. In fact, we will introduce two separate product structures $(\mathcal{Q}(\Gamma), *)$ and $(\mathcal{Q}(\Gamma), *^r)$ on $\mathcal{Q}(\Gamma)$.

**Proposition 2.5.** (a) *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the collection of quasimodular Hecke operators of level $\Gamma$. Then, the product defined by:*

$$(F * G)_\alpha := \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta \cdot (G_{\alpha\beta^{-1}} \| \beta), \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}) \tag{2.18}$$

*for all $F, G \in \mathcal{Q}(\Gamma)$ makes $\mathcal{Q}(\Gamma)$ into an associative algebra.*

(b) *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the collection of quasimodular Hecke operators of level $\Gamma$. Then, the product defined by:*

$$(F *^r G)_\alpha := \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} F_\beta \cdot (G_{\alpha\beta^{-1}} \| \beta), \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}) \tag{2.19}$$

*for all $F, G \in \mathcal{Q}(\Gamma)$ makes $\mathcal{Q}(\Gamma)$ into an associative algebra which we denote by $\mathcal{Q}^r(\Gamma)$.*

*Proof.* (a) We need to check that the product in (2.18) is associative. First of all, we note that the expression in (2.18) can be rewritten as:

$$(F * G)_\alpha = \sum_{\alpha_2\alpha_1=\alpha} F_{\alpha_1} \cdot G_{\alpha_2} \| \alpha_1, \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}), \tag{2.20}$$

where the sum in (2.20) is taken over all pairs $(\alpha_1, \alpha_2)$ with $\alpha_2\alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2) \sim (\gamma\alpha_1, \alpha_2\gamma^{-1}), \quad \forall\, \gamma \in \Gamma. \tag{2.21}$$

Hence, for $F, G, H \in \mathcal{Q}(\Gamma)$, we can write:

$$\begin{aligned}(F * (G * H))_\alpha &= \sum_{\alpha_2'\alpha_1=\alpha} F_{\alpha_1} \cdot (G * H)_{\alpha_2'} \| \alpha_1 \\ &= \sum_{\alpha_2'\alpha_1=\alpha} F_{\alpha_1} \cdot \left( \sum_{\alpha_3\alpha_2=\alpha_2'} G_{\alpha_2} \cdot H_{\alpha_3} \| \alpha_2 \right) \| \alpha_1 \\ &= \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F_{\alpha_1} \cdot (G_{\alpha_2} \| \alpha_1) \cdot (H_{\alpha_3} \| \alpha_2\alpha_1),\end{aligned} \tag{2.22}$$

where the sum in (2.22) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3\alpha_2\alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma\alpha_1, \gamma'\alpha_2\gamma^{-1}, \alpha_3\gamma'^{-1}), \quad \forall\, \gamma, \gamma' \in \Gamma \tag{2.23}$$

On the other hand, we have

$$\begin{aligned}((F * G) * H)_\alpha &= \sum_{\alpha_3\alpha_2''=\alpha} (F * G)_{\alpha_2''} \cdot H_{\alpha_3} \| \alpha_2'' \\ &= \sum_{\alpha_3\alpha_2''=\alpha} \left( \sum_{\alpha_2\alpha_1=\alpha_2''} F_{\alpha_1} \cdot G_{\alpha_2} \| \alpha_1 \right) \cdot H_{\alpha_3} \| \alpha_2'' \\ &= \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F_{\alpha_1} \cdot (G_{\alpha_2} \| \alpha_1) \cdot (H_{\alpha_3} \| \alpha_2\alpha_1),\end{aligned} \tag{2.24}$$

where the sum in (2.24) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3\alpha_2\alpha_1 = \alpha$ modulo the equivalence relation in (2.23). From (2.22) and (2.24) the result follows. We can similarly verify (b). □

We know that modular forms are quasimodular forms of depth 0, i.e., for any $k \geq 0$, $N \geq 1$, we have $\mathcal{M}_k(\Gamma(N)) = \mathcal{QM}_k^0(\Gamma(N))$. It follows that the modular tower $\mathcal{M}$ defined in (2.9) embeds into the quasimodular tower $\mathcal{QM}$ defined in (2.8). We are now ready to show that the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici embeds into the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for any congruence subgroup $\Gamma = \Gamma(N)$.

**Proposition 2.6.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Let $\mathcal{A}(\Gamma)$ be the modular Hecke algebra of level $\Gamma$ as defined in Definition 2.2 and let $\mathcal{Q}(\Gamma)$ be the quasimodular Hecke algebra of level $\Gamma$ as defined in Definition 2.4. Then, there is a natural embedding of algebras $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$.*

*Proof.* For any $\alpha \in GL_2^+(\mathbb{Q})$ and any $f \in \mathcal{QM}_k^s(\Gamma)$, we consider the operation $f \mapsto f\|\alpha$ as defined in (2.14):

$$f\|\alpha = \sum_{i=0}^{i} \big(a_i(f)|_{k-2i}\alpha\big) G_2^i \in \mathcal{QM}. \tag{2.25}$$

In particular, if $f \in \mathcal{M}_k(\Gamma) = \mathcal{QM}_k^0(\Gamma)$ is a modular form, it follows from (2.25) that:

$$f\|\alpha = a_0(f)|_k\alpha = f|_k\alpha = f|\alpha \in \mathcal{M}. \tag{2.26}$$

Hence, using the embedding of $\mathcal{M}$ in $\mathcal{QM}$, it follows from (2.11) in the definition of $\mathcal{A}(\Gamma)$ and from (2.17) in the definition of $\mathcal{Q}(\Gamma)$ that we have an embedding $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$ of modules. Further, we recall from [6, § 1] that the product on $\mathcal{A}(\Gamma)$ is given by:

$$(F * G)_\alpha := \sum_{\beta \in \Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta \cdot (G_{\alpha\beta^{-1}}|\beta), \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}),\ F, G \in \mathcal{A}(\Gamma). \tag{2.27}$$

Comparing (2.27) with the product on $\mathcal{Q}(\Gamma)$ described in (2.18) and using (2.26) it follows that $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$ is an embedding of algebras. □

We end this section by describing the action of the algebra $\mathcal{Q}(\Gamma)$ on $\mathcal{QM}(\Gamma)$.

**Proposition 2.7.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$. Then, for any element $f \in \mathcal{QM}(\Gamma)$ the action of $\mathcal{Q}(\Gamma)$ defined by:*

$$F * f := \sum_{\beta \in \Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta \cdot f\|\beta, \quad \forall\, F \in \mathcal{Q}(\Gamma) \tag{2.28}$$

*makes $\mathcal{QM}(\Gamma)$ into a left module over $\mathcal{Q}(\Gamma)$.*

*Proof.* It is easy to check that the right hand side of (2.28) is independent of the choice of coset representatives. Further, since $F \in \mathcal{Q}(\Gamma)$ is a function of finite support, we can choose finitely many coset representatives $\{\beta_1, \beta_2, \ldots, \beta_n\}$ such that

$$F * f = \sum_{j=1}^{n} F_{\beta_j} \cdot f\|\beta_j. \tag{2.29}$$

It suffices to consider the case $f \in \mathcal{QM}_k^s(\Gamma)$ for some weight $k$ and depth $s$. Then, we can express $f$ as a sum:

$$f = \sum_{i=0}^{s} a_i(f) G_2^i, \tag{2.30}$$

where each $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$. Similarly, for any $\beta \in GL_2^+(\mathbb{Q})$, we can express $F_\beta$ as a finite sum:

$$F_\beta = \sum_{r=0}^{t_\beta} a_{\beta r}(F_\beta) \cdot G_2^r \tag{2.31}$$

with each $a_{\beta r}(F_\beta) \in \mathcal{M}$. In particular, we let $t = \max\{t_{\beta_1}, t_{\beta_2}, \dots, t_{\beta_n}\}$ and we can now write:

$$F_{\beta_j} = \sum_{r=0}^{t} a_{\beta_j r}(F_{\beta_j}) \cdot G_2^r \tag{2.32}$$

by adding appropriately many terms with zero coefficients in the expression for each $F_{\beta_j}$. Further, for any $\gamma \in \Gamma$, we know that

$$F_{\beta_j\gamma} = F_{\beta_j} \| \gamma = \sum_{r=0}^{t} \big(a_{\beta_j r}(F_{\beta_j})|\gamma\big) \cdot G_2^r.$$

In other words, we have, for each $j$:

$$F_{\beta_j\gamma} = \sum_{r=0}^{t} a_{\beta_j\gamma r}(F_{\beta_j\gamma}) \cdot G_2^r, \quad a_{\beta_j\gamma r}(F_{\beta_j\gamma}) = \big(a_{\beta_j r}(F_{\beta_j})|\gamma\big). \tag{2.33}$$

The sum in (2.29) can now be expressed as:

$$F * f := \sum_{j=1}^{n} F_{\beta_j} \cdot f \| \beta_j = \sum_{i=0}^{s} \sum_{r=0}^{t} \sum_{j=1}^{n} a_{\beta_j r}(F_{\beta_j}) \cdot \big(a_i(f)|\beta_j\big) \cdot G_2^{r+i}. \tag{2.34}$$

For any $i, r$, we now set:

$$A_{ir}(F, f) := \sum_{j=1}^{n} a_{\beta_j r}(F_{\beta_j}) \cdot \big(a_i(f)|\beta_j\big). \tag{2.35}$$

Again, it is easy to see that the sum $A_{ir}(F, f)$ in (2.35) does not depend on the choice of the coset representatives $\{\beta_1, \beta_2, \dots, \beta_n\}$. Then, for any $\gamma \in \Gamma$, we have:

$$\begin{aligned} A_{ir}(F, f)|\gamma &= \sum_{j=1}^{n} \big(a_{\beta_j r}(F_{\beta_j})|\gamma\big) \cdot \big(a_i(f)|\beta_j\gamma\big) \\ &= \sum_{j=1}^{n} a_{\beta_j\gamma r}(F_{\beta_j\gamma}) \cdot \big(a_i(f)|\beta_j\gamma\big) = A_{ir}(F, f), \end{aligned} \tag{2.36}$$

where the last equality in (2.36) follows from the fact that $\{\beta_1\gamma, \beta_2\gamma, \dots, \beta_n\gamma\}$ is another collection of distinct cosets reprsentatives of $\Gamma$ in $GL_2^+(\mathbb{Q})$. From (2.36), we note that each $A_{ir}(F, f)$ belongs to $\mathcal{M}(\Gamma)$. Then, the sum:

$$F * f = \sum_{i=0}^{s} \sum_{r=0}^{t} A_{ir}(F, f) \cdot G_2^{i+r} \tag{2.37}$$

is an element of $\mathcal{QM}(\Gamma)$. Hence, $\mathcal{QM}(\Gamma)$ is a left module over $\mathcal{Q}(\Gamma)$. $\square$

## 3. The Lie algebra and Hopf algebra actions on $\mathcal{Q}(\Gamma)$

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In this section, we will describe two different sets of operators on the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$. Given a quasimodular form $f \in \mathcal{QM}(\Gamma)$ of level $\Gamma$, we have mentioned in the last section that $f$ can be expressed as a finite sum:

$$f = \sum_{i=0}^{s} a_i(f) \cdot G_2^i, \tag{3.1}$$

where $G_2$ is the classical Eisenstein series of weight 2 and each $a_i(f)$ is a modular form of level $\Gamma$. Then in Section 3.1, we consider operators on the quasimodular tower that act on the powers of $G_2$ appearing in (3.1). These induce operators $D$, $\{T_k^l\}_{k\geq 1, l\geq 0}$ on the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$. In order to understand the action of these operators on products of elements in $\mathcal{Q}(\Gamma)$, we also need to define extra operators $\{\phi^{(m)}\}_{m\geq 1}$. Finally, we show that these operators may all be described in terms of a Hopf algebra $\mathcal{H}$ with a "Hopf action" on $\mathcal{Q}^r(\Gamma)$, i.e.,

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2), \quad \forall\, h \in \mathcal{H},\ F^1, F^2 \in \mathcal{Q}^r(\Gamma), \tag{3.2}$$

where the coproduct $\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$. In Section 3.2, we consider operators on the quasimodular tower $\mathcal{QM}$ that act on the modular coefficients $a_i(f)$ appearing in (3.1). These induce on $\mathcal{Q}(\Gamma)$ analogues of operators acting on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici [6]. Then, we show that $\mathcal{Q}(\Gamma)$ carries a Hopf action of the same Hopf algebra $\mathcal{H}_1$ of codimension 1 foliations that acts on $\mathcal{A}(\Gamma)$.

### 3.1. The operators $D$, $\{T_k^l\}$ and $\{\phi^{(m)}\}$ on $\mathcal{Q}(\Gamma)$.

For any even number $K \geq 2$, let $G_K$ be the classical Eisenstein series of weight $K$ as in (2.12). Since $G_2$ is a quasimodular form, i.e., $G_2 \in \mathcal{QM}$, its derivative $G_2' \in \mathcal{QM}$. Further, it is well known that:

$$G_2' = \frac{5\pi(\sqrt{-1})}{3} G_4 - 4\pi(\sqrt{-1}) G_2^2, \tag{3.3}$$

where $G_4$ is the Eisenstein series of weight 4 (which is a modular form). For our purposes, it will be convenient to write:

$$G_2' = \sum_{j=0}^{2} g_j G_2^j \tag{3.4}$$

with each $g_j$ a modular form. From (3.3), it follows that:

$$g_0 = \frac{5\pi(\sqrt{-1})}{3} G_4, \quad g_1 = 0, \quad g_2 = -4\pi(\sqrt{-1}). \tag{3.5}$$

We are now ready to define the operators $D$ and $\{W_k\}_{k\geq 1}$ on $\mathcal{QM}$. The first operator $D$ differentiates the powers of $G_2$:

$$\begin{aligned} D\colon \mathcal{QM} &\longrightarrow \mathcal{QM} \\ f=\sum_{i=0}^{i} a_i(f)G_2^i &\mapsto -\frac{1}{8\pi(\sqrt{-1})}\left(\sum_{i=0}^{i} i a_i(f)G_2^{i-1}\cdot G_2'\right) \\ &\qquad = -\frac{1}{8\pi(\sqrt{-1})}\sum_{i=0}^{i}\sum_{j=0}^{2} i a_i(f) g_j G_2^{i+j-1}. \end{aligned} \tag{3.6}$$

The operators $\{W_k\}_{k\geq 1}$ are "weight operators" and $W_k$ also steps up the power of $G_2$ by $k-2$. We set:

$$W_k\colon \mathcal{QM}\longrightarrow \mathcal{QM}, \quad f=\sum_{i=0}^{i} a_i(f)G_2^i \mapsto \sum_{i=0}^{i} i a_i(f) G_2^{i+k-2}. \tag{3.7}$$

From the definitions in (3.6) and (3.7), we can easily check that $D$ and $W_k$ are derivations on $\mathcal{QM}$. Finally, for any $\alpha\in GL_2^+(\mathbb{Q})$ and any integer $m\geq 1$, we set

$$\nu_\alpha^{(m)} = -\frac{5}{24}\left(G_4^m|\alpha - G_4^m\right). \tag{3.8}$$

**Lemma 3.1.** (a) *Let $f\in\mathcal{QM}$ be an element of the quasimodular tower and $\alpha\in GL_2^+(\mathbb{Q})$. Then, the operator $D$ satisfies:*

$$D(f)\|\alpha = D(f\|\alpha) + \nu_\alpha^{(1)}\cdot\left(W_1(f)\|\alpha\right), \tag{3.9}$$

*where, using (3.8), we know that $\nu_\alpha^{(1)}$ is given by:*

$$\nu_\alpha^{(1)} := -\frac{1}{8\pi(\sqrt{-1})}\left(g_0|\alpha - g_0\right) = -\frac{5}{24}\left(G_4|\alpha - G_4\right), \qquad \forall\, \alpha\in GL_2^+(\mathbb{Q}). \tag{3.10}$$

(b) *For $f\in\mathcal{QM}$ and $\alpha\in GL_2^+(\mathbb{Q})$, each operator $W_k$, $k\geq 1$ satisfies:*

$$W_k(f)\|\alpha = W_k(f\|\alpha). \tag{3.11}$$

*Proof.* (a) For the sake of definiteness, we assume that $f = \sum_{i=0}^{i} a_i(f)G_2^i$ with each $a_i(f) \in \mathcal{M}$. For $\alpha \in GL_2^+(\mathbb{Q})$, it follows from (3.6) that:

$$\begin{aligned} D(f)\|\alpha &= -\frac{1}{8\pi(\sqrt{-1})}\left(\sum_i \sum_j i a_i(f) g_j G_2^{i+j-1}\right)\|\alpha \\ &= -\frac{1}{8\pi(\sqrt{-1})}\sum_i \sum_j\ i\big(a_i(f)|\alpha\big)(g_j|\alpha)G_2^{i+j-1}, \\ D(f\|\alpha) &= D\left(\sum_i \big(a_i(f)\alpha\big)G_2^i\right) \\ &= -\frac{1}{8\pi(\sqrt{-1})}\sum_i \sum_j\ i\big(a_i(f)|\alpha\big)g_j G_2^{i+j-1}. \end{aligned} \tag{3.12}$$

From (3.12) it follows that:

$$D(f)\|\alpha - D(f\|\alpha) = -\frac{1}{8\pi(\sqrt{-1})}\sum_{i=0}^{s}\sum_{j=0}^{2}\ i\big(a_i(f)|\alpha\big)\big(g_j|\alpha - g_j\big)G_2^{i+j-1}. \tag{3.13}$$

From (3.5), it is clear that $g_j|\alpha - g_j = 0$ for $j = 1$ and $j = 2$. It follows that:

$$\begin{aligned} D(f)\|\alpha - D(f\|\alpha) &= -\frac{1}{8\pi(\sqrt{-1})}\sum_{i=0}^{s}\ i\big(a_i(f)|\alpha\big)\big(g_0|\alpha - g_0\big)G_2^{i-1} \\ &= -\frac{1}{8\pi(\sqrt{-1})}\big(g_0|\alpha - g_0\big)\cdot\left(\sum_{i=0}^{s}\ i\big(a_i(f)|\alpha\big)G_2^{i-1}\right). \end{aligned}$$

This proves the result of (a). The result of part (b) is clear from the definition in (3.7). □

We note here that it follows from (3.8) that for any $\alpha, \beta \in GL_2^+(\mathbb{Q})$, we have:

$$\nu_{\alpha\beta}^{(m)} = \nu_\alpha^{(m)}|\beta + \nu_\beta^{(m)}, \quad \forall\, m \geq 1. \tag{3.14}$$

Additionally, since each $G_4^m$ is a modular form, we know that when $\alpha \in SL_2(\mathbb{Z})$:

$$\nu_\alpha^{(m)} = -\frac{5}{24}\big(G_4^m|\alpha - G_4^m\big) = 0, \quad \forall\, \alpha \in SL_2(\mathbb{Z}), m \geq 1. \tag{3.15}$$

Moreover, from the definitions in (3.6) and (3.7) respectively, it is easily verified that $D$ and $\{W_k\}_{k\geq 1}$ are derivations on the quasimodular tower $\mathcal{QM}$. We now proceed to define operators on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for some principal congruence subgroup $\Gamma = \Gamma(N)$. Choose $F \in \mathcal{Q}(\Gamma)$. We set:

$$\begin{aligned} &D, W_k, \phi^{(m)} : \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma), \quad k \geq 1,\ m \geq 1 \\ &D(F)_\alpha := D(F_\alpha), \quad W_k(F)_\alpha := W_k(F_\alpha), \quad \phi^{(m)}(F)_\alpha := \nu_\alpha^{(m)} \cdot F_\alpha, \\ &\qquad \forall\, \alpha \in GL_2^+(\mathbb{Q}). \end{aligned} \tag{3.16}$$

From Lemma 3.1 and the properties of $\nu_\alpha^{(m)}$ described in (3.14) and (3.15), it may be easily verified that the operators $D$, $W_k$ and $\phi^{(m)}$ in (3.16) are well defined on $\mathcal{Q}(\Gamma)$. We will now compute the commutators of the operators $D$, $\{W_k\}_{k\geq 1}$ and $\{\phi^{(m)}\}_{m\geq 1}$ on $\mathcal{Q}(\Gamma)$. In order to describe these commutators, we need one more operator $E$:

$$E : \mathcal{QM} \longrightarrow \mathcal{QM}, \quad f \mapsto G_4 \cdot f. \tag{3.17}$$

Since $G_4$ is a modular form of level $\Gamma(1) = SL_2(\mathbb{Z})$, i.e., $G_4|\gamma = G_4$ for any $\gamma \in SL_2(\mathbb{Z})$, it is clear that $E$ induces a well defined operator on $\mathcal{Q}(\Gamma)$:

$$E : \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma), \quad E(F)_\alpha := E(F_\alpha) = G_4 \cdot F_\alpha, \\ \forall\, F \in \mathcal{Q}(\Gamma),\ \alpha \in GL_2^+(\mathbb{Q}). \tag{3.18}$$

We will now describe the commutator relations between the operators $D$, $E$, $\{E^l W_k\}_{k\geq 1, l\geq 0}$ and $\{\phi^{(m)}\}_{m\geq 1}$ on $\mathcal{Q}(\Gamma)$.

**Proposition 3.2.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$. The operators $D$, $E$, $\{E^l W_k\}_{k\geq 1, l\geq 0}$ and $\{\phi^{(m)}\}_{m\geq 1}$ on $\mathcal{Q}(\Gamma)$ satisfy the following relations:*

$$\begin{gathered} [E, E^l W_k] = 0, \quad [E, D] = 0, \\ [E, \phi^{(m)}] = 0, \quad [D, \phi^{(m)}] = 0, \quad [W_k, \phi^{(m)}] = 0, \quad [\phi^{(m)}, \phi^{(m')}] = 0, \\ [E^l W_k, D] = \frac{5}{24}(k-1)\big(E^{l+1} W_{k-1}\big) - \frac{1}{2}(k-3) E^l W_{k+1}. \end{gathered} \tag{3.19}$$

*Proof.* For any $F \in \mathcal{Q}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, by definition, we know that $D(F)_\alpha = D(F_\alpha)$, $W_k(F)_\alpha = W_k(F_\alpha)$, and $E(F)_\alpha = E(F_\alpha)$. Hence, in order to prove that $[E, W_k] = 0$ and $[E, D] = 0$, it suffices to show that $[E, W_k](f) = 0$ and $[E, D](f) = 0$, respectively, for any element $f \in \mathcal{QM}$. Both of these are easily verified from the definitions of $D$ and $W_k$ in (3.6) and (3.7) respectively. Further, since $[E, W_k] = 0$, it is clear that $[E, E^l W_k] = 0$.

Similarly, in order to prove the expression for $[E^l W_k, D]$, it suffices to prove that:

$$[E^l W_k, D](f) = \frac{5}{24}(k-1)(E^{l+1} W_{k-1})(f) - \frac{1}{2}(k-3) E^l W_{k+1}(f) \tag{3.20}$$

for any $f \in \mathcal{QM}$. Further, it suffices to consider the case where

$$f = \sum_{i=0}^{s} a_i(f) G_2^i,$$

where the $a_i(f) \in \mathcal{M}$. We now have:

$$
\begin{aligned}
W_kD(f) &= -\frac{1}{8\pi(\sqrt{-1})} W_k\left(\sum_{i=0}^{i}\sum_{j=0}^{2} i a_i(f) g_j G_2^{i+j-1}\right) \\
&= -\frac{1}{8\pi(\sqrt{-1})} \sum_{i=0}^{i}\sum_{j=0}^{2} i(i+j-1) a_i(f) g_j G_2^{i+j+k-3}, \\
DW_k(f) &= D\left(\sum_{i=0}^{i} i a_i(f) G_2^{i+k-2}\right) \\
&= -\frac{1}{8\pi(\sqrt{-1})} \sum_{i=0}^{i}\sum_{j=0}^{2} i(i+k-2) a_i(f) g_j G_2^{i+j+k-3}.
\end{aligned} \tag{3.21}
$$

It follows from (3.21) that:

$$
\begin{aligned}
[W_k, D](f) = &-\frac{1}{8\pi(\sqrt{-1})} \sum_{i=0}^{i}\sum_{j=0}^{2} ij a_i(f) g_j G_2^{i+j+k-3} \\
&+ \frac{1}{8\pi(\sqrt{-1})} \sum_{i=0}^{i}\sum_{j=0}^{2} i(k-1) a_i(f) g_j G_2^{i+j+k-3} \\
= &-\frac{2g_2}{8\pi(\sqrt{-1})} \sum_{i=0}^{i} i a_i(f) G_2^{i+k-1} \\
&+ (k-1)\frac{1}{8\pi(\sqrt{-1})} \sum_{i=0}^{i} i a_i(f) g_0 G_2^{i+k-3} \\
&+ (k-1)\frac{g_2}{8\pi(\sqrt{-1})} \sum_{i=0}^{i} i a_i(f) G_2^{i+k-1},
\end{aligned}
$$

where the second equality uses the fact that $g_1 = 0$. Further, since $g_0 = (5\pi(\sqrt{-1})/3)G_4$ and $g_2 = -4\pi(\sqrt{-1})$, it follows from (3.1) that we have:

$$
\begin{aligned}
[W_k, D](f) &= \frac{5}{24}(k-1)\sum_{i=0}^{i} i G_4 a_i(f) G_2^{i+k-3} - \frac{1}{2}(k-3)\sum_{i=0}^{i} i a_i(f) G_2^{i+k-1} \\
&= \frac{5}{24}(k-1)(EW_{k-1})(f) - \frac{1}{2}(k-3)W_{k+1}(f).
\end{aligned} \tag{3.22}
$$

Finally, since $E$ commutes with $\{W_k\}_{k\geq 1}$ and $D$, it follows from (3.22) that:

$$
\big[E^l W_k, D\big] = \frac{5}{24}(k-1)\big(E^{l+1}W_{k-1}\big) - \frac{1}{2}(k-3)E^l W_{k+1}, \quad \forall\, k \geq 1,\ l \geq 0 \tag{3.23}
$$

as operators on $\mathcal{Q}(\Gamma)$. Finally, it may be easily verified from the definitions that

$$\left[E, \phi^{(m)}\right] = \left[D, \phi^{(m)}\right] = \left[W_k, \phi^{(m)}\right] = 0. \qquad \square$$

The operators $\{E^l W_k\}_{k\geq 1, l\geq 0}$ appearing in Proposition 3.2 above can be described more succintly as:

$$T_k^l\colon \mathcal{QM} \longrightarrow \mathcal{QM}, \quad T_k^l := E^l W_k, \quad \forall\, k \geq 1,\ l \geq 0 \tag{3.24}$$

and

$$T_k^l\colon \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma), \quad T_k^l(F)_\alpha := T_k^l(F_\alpha) = E^l W_k(F_\alpha), \\ \forall\, F \in \mathcal{Q}(\Gamma),\ \alpha \in GL_2^+(\mathbb{Q}). \tag{3.25}$$

We are now ready to describe the Lie algebra action on $\mathcal{Q}(\Gamma)$.

**Proposition 3.3.** *Let $\mathcal{L}$ be the Lie algebra generated by the symbols $D$, $\{T_k^l\}_{k\geq 1, l\geq 0}$, $\{\phi^{(m)}\}_{m\geq 1}$ along with the following relations between the commutators:*

$$\begin{gathered} \left[T_k^l, T_{k'}^{l'}\right] = (k'-k) T_{k+k'-2}^{l+l'}, \\ \left[D, \phi^{(m)}\right] = 0, \quad \left[T_k^l, \phi^{(m)}\right] = 0, \quad \left[\phi^{(m)}, \phi^{(m')}\right] = 0, \\ \left[T_k^l, D\right] = \frac{5}{24}(k-1) T_{k-1}^{l+1} - \frac{1}{2}(k-3) T_{k+1}^l. \end{gathered} \tag{3.26}$$

*Then, for any principal congruence subgroup $\Gamma = \Gamma(N)$, we have a Lie action of $\mathcal{L}$ on the algebra of quasimodular Hecke operators $\mathcal{Q}(\Gamma)$ of level $\Gamma$.*

*Proof.* For any $k \geq 1$ and $l \geq 0$, $T_k^l$ has been defined to be the operator $E^l W_k$ on $\mathcal{Q}(\Gamma)$. We want to verify that:

$$\left[T_k^l, T_{k'}^{l'}\right] = (k-k') T_{k+k'-2}^{l+l'}, \quad \forall\, k, k' \geq 1,\ l, l' \geq 0. \tag{3.27}$$

As in the proof of Proposition 3.2, it suffices to show that the relation in (3.27) holds for any $f \in \mathcal{QM}$. As before, we let $f = \sum_{i=0}^{s} a_i(f) G_2^i$, where each $a_i(f) \in \mathcal{M}$. We now have:

$$\begin{aligned} T_k^l T_{k'}^{l'}(f) &= T_k^l\left(\sum_{i=0}^{i} i a_i(f) G_4^{l'} \cdot G_2^{i+k'-2}\right) \\ &= \sum_{i=0}^{i} i(i+k'-2) a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4}, \\ T_{k'}^{l'} T_k^l(f) &= T_{k'}^{l'}\left(\sum_{i=0}^{i} i a_i(f) G_4^{l} \cdot G_2^{i+k-2}\right) \\ &= \sum_{i=0}^{i} i(i+k-2) a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4}. \end{aligned} \tag{3.28}$$

From (3.28) it follows that:

$$[T_k^l, T_{k'}^{l'}](f) = (k'-k)\sum_{i=0}^{i} i a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4} = (k'-k) T_{k+k'-2}^{l+l'}. \tag{3.29}$$

Hence, the relation (3.27) holds for the operators $T_k^l$, $T_{k'}^{l'}$ acting on $\mathcal{Q}(\Gamma)$. The remaining relations in (3.26) for the Lie action of $\mathcal{L}$ on $\mathcal{Q}(\Gamma)$ follow from (3.19). □

**Lemma 3.4.** *Let $f \in \mathcal{QM}$ be an element of the quasimodular tower and let $\alpha \in GL_2^+(\mathbb{Q})$. Then, for any $k \geq 1$, $l \geq 0$, the operator $T_k^l: \mathcal{QM} \longrightarrow \mathcal{QM}$ satisfies:*

$$T_k^l(f)\|\alpha = T_k^l(f\|\alpha) - \frac{24}{5}\nu_\alpha^{(l)} \cdot \big(T_k^0(f)\|\alpha\big). \tag{3.30}$$

*Proof.* For the sake of definiteness, we assume that $f = \sum_{i=0}^{s} a_i(f) \cdot G_2^i$ with each $a_i(f) \in \mathcal{M}$. We now compute:

$$\begin{aligned}
T_k^l(f)\|\alpha &= (E^l W_k)(f)\|\alpha = \left(\sum_{i=0}^{i} i G_4^l \cdot a_i(f) G_2^{i+k-2}\right)\|\alpha \\
&= \sum_{i=0}^{i} i (G_4^l|\alpha) \cdot \big(a_i(f)|\alpha\big) G_2^{i+k-2}, \\
T_k^l(f\|\alpha) &= (E^l W_k)(f\|\alpha) = (E^l W_k)\left(\sum_{i=0}^{i} \big(a_i(f)|\alpha\big) G_2^i\right) \\
&= \sum_{i=0}^{i} i (G_4^l) \cdot \big(a_i(f)|\alpha\big) G_2^{i+k-2}.
\end{aligned} \tag{3.31}$$

Subtracting, it follows that:

$$\begin{aligned}
T_k^l(f)\|\alpha - T_k^l(f\|\alpha) &= (G_4^l|\alpha - G_4^l) \cdot \left(\sum_{i=0}^{i} i \big(a_i(f)|\alpha\big) G_2^{i+k-2}\right) \\
&= -\frac{24}{5}\nu_\alpha^{(l)} \cdot \big(W_k(f)\|\alpha\big).
\end{aligned} \tag{3.32}$$

Putting $T_k^0 = E^0 W_k = W_k$, we have the result. □

**Proposition 3.5.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$. Then, for any $k \geq 1$, $l \geq 0$, the operator $T_k^l$ satisfies:*

$$T_k^l(F^1 * F^2) = T_k^l(F^1) * F^2 + F^1 * T_k^l(F^2) + \frac{24}{5}\big(\phi^{(l)}(F^1) * T_k^0(F^2)\big)_\alpha, \quad \forall\, F^1, F^2 \in \mathcal{Q}(\Gamma). \tag{3.33}$$

*Further, the operators $\{T_k^l\}_{k\geq 1, l\geq 0}$ are all derivations on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.*

*Proof.* We know that $T_k^l = E^l W_k$ and that $W_k$ is a derivation on $\mathcal{QM}$. We choose quasimodular Hecke operators $F^1$, $F^2 \in \mathcal{Q}(\Gamma)$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we know that:

$$\begin{aligned}
&T_k^l\big(F^1 * F^2\big)_\alpha \\
&\quad = E^l W_k\Bigg(\sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big)\Bigg) \\
&\quad = \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} E^l W_k\big(F_\beta^1 \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big)\big) \\
&\quad = \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} G_4^l \cdot W_k(F_\beta^1) \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big) + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot G_4^l \cdot W_k\big(F^2_{\alpha\beta^{-1}} \| \beta\big) \\
&\quad = \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} G_4^l \cdot W_k(F_\beta^1) \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big) + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot G_4^l \cdot \big(W_k(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\
&\quad = \big(T_k^l(F^1) * F^2\big)_\alpha + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (G_4^l|\beta) \cdot \big(W_k(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\
&\qquad\qquad - \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big(G_4^l|\beta - G_4^l\big) \cdot \big(W_k(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\
&\quad = \big(T_k^l(F^1) * F^2\big)_\alpha + \big(F^1 * T_k^l(F^2)\big)_\alpha \\
&\qquad\qquad + \frac{24}{5} \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \nu_\beta^{(l)} \cdot \big(W_k(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\
&\quad = \big(T_k^l(F^1) * F^2\big)_\alpha + \big(F^1 * T_k^l(F^2)\big)_\alpha + \frac{24}{5}\big(\phi^{(l)}(F^1) * T_k^0(F^2)\big)_\alpha,
\end{aligned}$$

where it is understood that $\phi^{(0)} = 0$. This proves (3.33). Further, since $\nu_\beta^{(l)} = 0$ for any $\beta \in SL_2(\mathbb{Z})$, when we consider the product $*^r$ defined in (2.19) on the algebra $\mathcal{Q}^r(\Gamma)$, the calculation above reduces to

$$T_k^l(F^1 *^r F^2) = T_k^l(F^1) *^r F^2 + F^1 *^r T_k^l(F^2). \tag{3.34}$$

Hence, each $T_k^l$ is a derivation on $\mathcal{Q}^r(\Gamma)$. $\square$

**Proposition 3.6.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$.*

(a) *The operator $D: \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma)$ on the algebra $(\mathcal{Q}(\Gamma), *)$ satisfies:*

$$D(F^1 * F^2) = D(F^1) * F^2 + F^1 * D(F^2) - \phi^{(1)}(F^1) * T_1^0(F^2), \quad \forall\, F^1, F^2 \in \mathcal{Q}(\Gamma). \tag{3.35}$$

*When we consider the product $*^r$, the operator $D$ becomes a derivation on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, i.e.:*

$$D(F^1 *^r F^2) = D(F^1) *^r F^2 + F^1 *^r D(F^2), \quad \forall\, F^1, F^2 \in \mathcal{Q}^r(\Gamma). \tag{3.36}$$

(b) *The operators $\{W_k\}_{k\geq 1}$ and $\{\phi^{(m)}\}_{m\geq 1}$ are derivations on $\mathcal{Q}(\Gamma)$, i.e.,*

$$\begin{aligned} W_k(F^1 * F^2) &= W_k(F^1) * F^2 + F^1 * W_k(F^2), \\ \phi^{(m)}(F^1 * F^2) &= \phi^{(m)}(F^1) * F^2 + F^1 * \phi^{(m)}(F^2) \end{aligned} \tag{3.37}$$

*for any $F^1$, $F^2 \in \mathcal{Q}(\Gamma)$. Additionally, $\{\phi^{(m)}\}_{m\geq 1}$ and $\{W_k\}_{k\geq 1}$ are also derivations on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.*

*Proof.* (a) We choose quasimodular Hecke operators $F^1$, $F^2 \in \mathcal{Q}(\Gamma)$. We have mentioned before that $D$ is a derivation on $\mathcal{QM}$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned} D(F^1 * F^2)_\alpha &= D\Big( \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big)\Big) \\ &= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} D\big(F^1_\beta \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big)\big) \\ &= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} D(F^1_\beta) \cdot \big(F^2_{\alpha\beta^{-1}} \| \beta\big) + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot D\big(F^2_{\alpha\beta^{-1}} \| \beta\big) \\ &= \big(D(F^1) * F^2\big)_\alpha + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \big(D(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\ &\qquad - \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \nu^{(1)}_\beta \cdot \big(W_1(F^2_{\alpha\beta^{-1}}) \| \beta\big) \\ &= \big(D(F^1) * F^2\big)_\alpha + \big(F^1 * D(F^2)\big)_\alpha - \big(\phi^{(1)}(F^1) * T_1^0(F^2)\big)_\alpha. \end{aligned}$$

This proves (3.35). In order to prove (3.36), we note that $\nu^{(1)}_\beta = 0$ for any $\beta \in SL_2(\mathbb{Z})$ (see (3.15)). Hence, when we use the product $*^r$ defined in (2.19), the calculation above reduces to

$$D(F^1 *^r F^2) = D(F^1) *^r F^2 + F^1 *^r D(F^2) \tag{3.38}$$

for any $F^1$, $F^2 \in \mathcal{Q}^r(\Gamma)$.

(b) For any $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and knowing from (3.14) that $\nu_\alpha^{(m)} = \nu_\beta^{(m)} + \nu_{\alpha\beta^{-1}}^{(m)}|\beta$, we have:

$$\begin{aligned}
\phi^{(m)}(F^1 * F^2)_\alpha &= \nu_\alpha^{(m)} \cdot \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big(F^2_{\alpha\beta^{-1}}\|\beta\big) \\
&= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \big(\nu_\beta^{(m)} \cdot F_\beta^1\big) \cdot \big(F^2_{\alpha\beta^{-1}}\|\beta\big) \\
&\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big(\nu^{(m)}_{\alpha\beta^{-1}}|\beta\big) \cdot \big(F^2_{\alpha\beta^{-1}}\|\beta\big) \\
&= \phi^{(m)}(F^1) * F^2 + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big((\nu^{(m)}_{\alpha\beta^{-1}} \cdot F^2_{\alpha\beta^{-1}})\|\beta\big) \\
&= \phi^{(m)}(F^1) * F^2 + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \big(\phi^{(m)}(F^2)_{\alpha\beta^{-1}}\|\beta\big) \\
&= \phi^{(m)}(F^1) * F^2 + F^1 * \phi^{(m)}(F^2).
\end{aligned} \tag{3.39}$$

The fact that each $W_k$ is also a derivation on $\mathcal{Q}(\Gamma)$ now follows from a similar calculation using the fact that $W_k$ is a derivation on the quasimodular tower $\mathcal{QM}$ and that $W_k(f)\|\alpha = W_k(f\|\alpha)$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$ (from (3.11)). Finally, a similar calculation may be used to verify that $\{W_k\}_{k\geq 1}$ and $\{\phi^{(m)}\}_{m\geq 1}$ are all derivations on $\mathcal{Q}^r(\Gamma)$. □

We now introduce the Hopf algebra $\mathcal{H}$ that acts on $\mathcal{Q}^r(\Gamma)$. The Hopf algebra $\mathcal{H}$ is the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of the Lie algebra $\mathcal{L}$ introduced in Proposition 3.3. As such, the coproduct $\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is defined by:

$$\begin{gathered}
\Delta(D) = D \otimes 1 + 1 \otimes D, \quad \Delta(T_k^l) = T_k^l \otimes 1 + 1 \otimes T_k^l, \\
\Delta(\phi^{(m)}) = \phi^{(m)} \otimes 1 + 1 \otimes \phi^{(m)}.
\end{gathered} \tag{3.40}$$

We will now show that $\mathcal{H}$ has a Hopf action on the algebra $\mathcal{Q}^r(\Gamma)$.

**Proposition 3.7.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, there is a Hopf action of $\mathcal{H}$ on the algebra $\mathcal{Q}^r(\Gamma)$, i.e.,*

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2), \quad \forall\, F^1, F^2 \in \mathcal{Q}^r(\Gamma),\ h \in \mathcal{H}, \tag{3.41}$$

*where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$.*

*Proof.* In order to prove (3.41), it suffices to verify the relation for $D$ and each of $\{T_k^l\}_{k\geq 1, l\geq 0}$, $\{\phi^{(m)}\}_{m\geq 1}$. From Proposition 3.5 and Proposition 3.6, we know that for $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$ and any $k \geq 1, l \geq 0, m \geq 1$:

$$\begin{aligned} D(F^1 *^r F^2) &= D(F^1) *^r F^2 + F^1 *^r D(F^2), \\ T_k^l(F^1 *^r F^2) &= T_k^l(F^1) *^r F^2 + F^1 *^r T_k^l(F^2), \\ \phi^{(m)}(F^1 *^r F^2) &= \phi^{(m)}(F^1) *^r F^2 + F^1 *^r \phi^{(m)}(F^2). \end{aligned} \tag{3.42}$$

Comparing with the expressions for the coproduct in (3.40), it is clear that (3.41) holds for each $h \in \mathcal{H}$. □

**3.2. The operators $X$, $Y$, and $\{\delta_n\}$ of Connes and Moscovici.** Let $\Gamma = \Gamma(N)$ be a congruence subgroup. In this subsection, we will show that the algebra $\mathcal{Q}(\Gamma)$ carries an action of the Hopf algebra $\mathcal{H}_1$ of Connes and Moscovici [5]. The Hopf algebra $\mathcal{H}_1$ is part of a larger family $\{\mathcal{H}_n\}_{n\geq 1}$ of Hopf algebras defined in [5] and $\mathcal{H}_1$ is the Hopf algebra corresponding to "codimension 1 foliations". As an algebra, $\mathcal{H}_1$ is identical to the universal enveloping algebra $\mathcal{U}(\mathcal{L}_1)$ of the Lie algebra $\mathcal{L}_1$ generated by $X$, $Y$, $\{\delta_n\}_{n\geq 1}$ satisfying the commutator relations:

$$[Y, X] = X, \quad [X, \delta_n] = \delta_{n+1}, \quad [Y, \delta_n] = n\delta_n, \quad [\delta_k, \delta_l] = 0, \qquad \forall\, k, l, n \geq 1. \tag{3.43}$$

Further, the coproduct $\Delta\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ on $\mathcal{H}_1$ is determined by:

$$\begin{aligned} \Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\ \Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1. \end{aligned} \tag{3.44}$$

Finally, the antipode $S\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is given by:

$$S(X) = -X + \delta_1 Y, \quad S(Y) = -Y, \quad S(\delta_1) = -\delta_1. \tag{3.45}$$

Following Connes and Moscovici [6], we define the operators $X$ and $Y$ on the modular tower: for any congruence subgroup $\Gamma = \Gamma(N)$, we set:

$$Y\colon \mathcal{M}_k(\Gamma) \longrightarrow \mathcal{M}_k(\Gamma), \quad Y(f) := \frac{k}{2} f, \quad \forall\, f \in \mathcal{M}_k(\Gamma). \tag{3.46}$$

Further, the operator $X\colon \mathcal{M}_k(\Gamma) \longrightarrow \mathcal{M}_{k+2}(\Gamma)$ is the Ramanujan differential operator on modular forms:

$$X(f) := \frac{1}{2\pi i}\frac{d}{dz}(f) - \frac{1}{12\pi i}\frac{d}{dz}(\log \Delta) \cdot Y(f), \quad \forall\, f \in \mathcal{M}_k(\Gamma), \tag{3.47}$$

where $\Delta(z)$ is the well known modular form of weight 12 given by:

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}. \tag{3.48}$$

We start by extending these operators to the quasimodular tower $\mathcal{QM}$. Let $f \in \mathcal{QM}_k^s(\Gamma)$ be a quasimodular form. Then, we can express $f = \sum_{i=0}^{i} a_i(f) G_2^i$, where $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$. We set:

$$X(f) = \sum_{i=0}^{s} X\big(a_i(f)\big) \cdot G_2^i, \quad Y(f) = \sum_{i=0}^{s} Y\big(a_i(f)\big) \cdot G_2^i. \tag{3.49}$$

From (3.49), it is clear that $X$ and $Y$ are derivations on $\mathcal{QM}$.

**Lemma 3.8.** *Let $f \in \mathcal{QM}$ be an element of the quasimodular tower. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:*

$$X(f)\|\alpha = X(f\|\alpha) + \big(\mu_{\alpha^{-1}} \cdot Y(f)\big)\|\alpha, \tag{3.50}$$

*where, for any $\delta \in GL_2^+(\mathbb{Q})$, we set:*

$$\mu_\delta := \frac{1}{12\pi i} \frac{d}{dz} \log \frac{\Delta|\delta}{\Delta}. \tag{3.51}$$

*Further, we have $Y(f\|\alpha) = Y(f)\|\alpha$.*

*Proof.* Following [6, Lemma 5], we know that for any $g \in \mathcal{M}$, we have:

$$X(g)|\alpha = X(g|\alpha) + \big(\mu_{\alpha^{-1}} \cdot Y(g)\big)|\alpha, \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}). \tag{3.52}$$

It suffices to consider the case $f \in \mathcal{QM}_k^s(\Gamma)$ for some congruence subgroup $\Gamma$. If we express $f \in \mathcal{QM}_k^s(\Gamma)$ as $f = \sum_{i=0}^{i} a_i(f) G_2^i$ with $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$, it follows that:

$$Xa_i(f)|\alpha = X\big(a_i(f)|\alpha\big) + \big(\mu_{\alpha^{-1}} \cdot Y\big(a_i(f)\big)\big)|\alpha, \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}) \tag{3.53}$$

for each $0 \le i \le s$. Combining (3.53) with the definitions of $X$ and $Y$ on the quasimodular tower in (3.49), we can easily prove (3.50). Finally, it is clear from the definition of $Y$ that $Y(f\|\alpha) = Y(f)\|\alpha$. □

From the definition of $\mu_\delta$ in (3.51), one may verify that (see [6, § 3)]):

$$\mu_{\delta_1\delta_2} = \mu_{\delta_1}|\delta_2 + \mu_{\delta_2}, \quad \forall\, \delta_1, \delta_2 \in GL_2^+(\mathbb{Q}) \tag{3.54}$$

and that $\mu_\delta = 0$ for any $\delta \in SL_2(\mathbb{Z})$. We now define operators $X$, $Y$ and $\{\delta_n\}_{n\ge 1}$ on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for some congruence subgroup $\Gamma = \Gamma(N)$. Let $F \in \mathcal{Q}(\Gamma)$ be a quasimodular Hecke operator of level $\Gamma$; then we define operators:

$$\begin{gathered} X, Y, \delta_n \colon \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma), \\ X(F)_\alpha := X(F_\alpha), \quad Y(F)_\alpha := Y(F_\alpha), \quad \delta_n(F)_\alpha = X^{n-1}(\mu_\alpha) \cdot F_\alpha, \\ \forall\, \alpha \in GL_2^+(\mathbb{Q}). \end{gathered} \tag{3.55}$$

We will now show that the Lie algebra $\mathcal{L}_1$ with generators $X$, $Y$, $\{\delta_n\}_{n\geq 1}$ satisfying the commutator relations in (3.43) acts on the algebra $\mathcal{Q}(\Gamma)$. Additionally, in order to give a Lie action on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, we define at this juncture the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ with generators $X$ and $Y$ satisfying the relation

$$[Y, X] = X. \tag{3.56}$$

Further, we consider the Hopf algebra $\mathfrak{h}_1$ that arises as the universal enveloping algebra $\mathcal{U}(\mathfrak{l}_1)$ of the Lie algebra $\mathfrak{l}_1$. We have used the Hopf algebra $\mathfrak{h}_1$ in a similar manner before to act on a "restricted" version of a modular Hecke algebra in [1, § 4]. We will show that $\mathcal{H}_1$ (resp. $\mathfrak{h}_1$) has a Hopf action on the algebra $\mathcal{Q}(\Gamma)$ (resp. $\mathcal{Q}^r(\Gamma)$). We start by describing the Lie actions.

**Proposition 3.9.** *Let $\mathcal{L}_1$ be the Lie algebra with generators $X$, $Y$ and $\{\delta_n\}_{n\geq 1}$ satisfying the following commutator relations:*

$$[Y, X] = X, \quad [X, \delta_n] = \delta_{n+1}, \quad [Y, \delta_n] = n\delta_n, \quad [\delta_k, \delta_l] = 0, \quad \forall\, k, l, n \geq 1. \tag{3.57}$$

*Then, for any given congruence subgroup $\Gamma = \Gamma(N)$ of $SL_2(\mathbb{Z})$, we have a Lie action of $\mathcal{L}_1$ on the module $\mathcal{Q}(\Gamma)$.*

*Proof.* From [6, § 3], we know that for any element $g \in \mathcal{M}$ of the modular tower, we have $[Y, X](g) = X(g)$. Since the action of $X$ and $Y$ on the quasimodular tower $\mathcal{QM}$ (see (3.49)) is naturally extended from their action on $\mathcal{M}$, it follows that $[Y, X] = X$ on the quasimodular tower $\mathcal{QM}$. In particular, given any quasimodular Hecke operator $F \in \mathcal{Q}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we have $[Y, X](F_\alpha) = X(F_\alpha)$ for the element $F_\alpha \in \mathcal{QM}$. By definition, $X(F)_\alpha = X(F_\alpha)$ and $Y(F_\alpha) = Y(F)_\alpha$ and hence $[Y, X] = X$ holds for the action of $X$ and $Y$ on $\mathcal{Q}(\Gamma)$.

Further, since $X$ is a derivation on $\mathcal{QM}$ and $\delta_n(F)_\alpha = X^{n-1}(\mu_\alpha) \cdot F_\alpha$, we have

$$\begin{aligned} [X, \delta_n](F)_\alpha &= X\big(X^{n-1}(\mu_\alpha) \cdot F_\alpha\big) - X^{n-1}(\mu_\alpha) \cdot X(F_\alpha), \\ &= X\big(X^{n-1}(\mu_\alpha)\big) \cdot F_\alpha = X^n(\mu_\alpha) \cdot F_\alpha = \delta_{n+1}(F)_\alpha. \end{aligned} \tag{3.58}$$

Similarly, since $\mu_\alpha \in \mathcal{M} \subseteq \mathcal{QM}$ is of weight 2 and $Y$ is a derivation on $\mathcal{QM}$, we have:

$$\begin{aligned} [Y, \delta_n](F)_\alpha &= Y\big(X^{n-1}(\mu_\alpha) \cdot F_\alpha\big) - X^{n-1}(\mu_\alpha) \cdot Y(F_\alpha), \\ &= Y\big(X^{n-1}(\mu_\alpha)\big) \cdot F_\alpha = nX^{n-1}(\mu_\alpha) \cdot F_\alpha = n\delta_n(F)_\alpha. \end{aligned} \tag{3.59}$$

Finally, we can verify easily that $[\delta_k, \delta_l] = 0$ for any $k, l \geq 1$. □

From Proposition 3.9, it is also clear that the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ has a Lie action on the module $\mathcal{Q}(\Gamma)$.

**Lemma 3.10.** *Let* $\Gamma = \Gamma(N)$ *be a congruence subgroup of* $SL_2(\mathbb{Z})$ *and let* $\mathcal{Q}(\Gamma)$ *be the algebra of quasimodular Hecke operators of level* $\Gamma$. *Then, the operator* $X\colon \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma)$ *on the algebra* $(\mathcal{Q}(\Gamma), *)$ *satisfies:*

$$X(F^1 * F^2) = X(F^1) * F^2 + F^1 * X(F^2) + \delta_1(F^1) * Y(F^2), \qquad \forall\, F^1, F^2 \in \mathcal{Q}(\Gamma). \tag{3.60}$$

*When we consider the product* $*^r$, *the operator* $X$ *becomes a derivation on the algebra* $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, *i.e.*

$$X(F^1 *^r F^2) = X(F^1) *^r F^2 + F^1 *^r X(F^2), \quad \forall\, F^1, F^2 \in \mathcal{Q}^r(\Gamma). \tag{3.61}$$

*Proof.* We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. Using (3.54), we also note that

$$0 = \mu_1 = \mu_{\beta^{-1}}|\beta + \mu_\beta, \quad \forall\, \beta \in GL_2^+(\mathbb{Q}). \tag{3.62}$$

We have mentioned before that $X$ is a derivation on $\mathcal{QM}$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned}
X(F^1 * F^2)_\alpha &= X\Big(\sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \big(F^2_{\alpha\beta^{-1}}\|\beta\big)\Big)\\
&= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} X\big(F^1_\beta \cdot \big(F^2_{\alpha\beta^{-1}}\|\beta\big)\big)\\
&= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} X(F^1_\beta)\cdot\big(F^2_{\alpha\beta^{-1}}\|\beta\big) + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot X\big(F^2_{\alpha\beta^{-1}}\|\beta\big)\\
&= \big(X(F^1) * F^2\big)_\alpha + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \big(X(F^2_{\alpha\beta^{-1}})\|\beta\big)\\
&\qquad - \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \big((\mu_{\beta^{-1}} \cdot Y(F^2_{\alpha\beta^{-1}}))\|\beta\big)\\
&= \big(X(F^1) * F^2\big)_\alpha + \big(F^1 * X(F^2)\big)_\alpha\\
&\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \big(F^1_\beta \cdot \mu_\beta\big)\cdot\big(Y(F^2_{\alpha\beta^{-1}})\|\beta\big)\\
&= \big(X(F^1) * F^2\big)_\alpha + \big(F^1 * X(F^2)\big)_\alpha + \big(\delta_1(F^1) * Y(F^2)\big)_\alpha.
\end{aligned}$$

This proves (3.60). In order to prove (3.61), we note that $\mu_\beta = 0$ for any $\beta \in SL_2(\mathbb{Z})$. Hence, if we use the product $*^r$, the calculation above reduces to

$$X(F^1 *^r F^2) = X(F^1) *^r F^2 + F^1 *^r X(F^2) \tag{3.63}$$

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$. $\square$

Finally, we describe the Hopf action of $\mathcal{H}_1$ on the algebra $(\mathcal{Q}(\Gamma), *)$ as well as the Hopf action of $\mathfrak{h}_1$ on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

**Proposition 3.11.** *Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$. Then, the Hopf algebra $\mathcal{H}_1$ has a Hopf action on the quasimodular Hecke algebra $(\mathcal{Q}(\Gamma), *)$; in other words, we have:*

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) \otimes h_{(2)}(F^2), \quad \forall\, h \in \mathcal{H}_1,\ F^1, F^2 \in \mathcal{Q}(\Gamma), \tag{3.64}$$

*where the coproduct $\Delta\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}_1$. Similarly, there exists a Hopf action of the Hopf algebra $\mathfrak{h}_1$ on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.*

*Proof.* In order to prove (3.64), it suffices to check the relation for $X$, $Y$ and $\delta_1 \in \mathcal{H}_1$. For the element $X \in \mathcal{H}_1$, this is already the result of Lemma 3.10. Now, for any $F^1$, $F^2 \in \mathcal{Q}(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned}
\delta_1(F^1 * F^2)_\alpha &= \mu_\alpha \cdot \left( \sum_{\beta \in \Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \left(F^2_{\alpha\beta^{-1}} \| \beta\right) \right) \\
&= \sum_{\beta \in \Gamma\backslash GL_2^+(\mathbb{Q})} \left(\mu_\beta \cdot F^1_\beta\right) \cdot \left(F^2_{\alpha\beta^{-1}} \| \beta\right) + \sum_{\beta \in \Gamma\backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot \left((\mu_{\alpha\beta^{-1}} \cdot F^2_{\alpha\beta^{-1}}) \| \beta\right) \\
&= \left(\delta_1(F^1) * F^2\right)_\alpha + \left(F^1 * \delta_1(F^2)\right)_\alpha.
\end{aligned} \tag{3.65}$$

Further, using the fact that $Y$ is a derivation on $\mathcal{QM}$ and $Y(f\|\alpha) = Y(f)\|\alpha$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$, we can easily verify the relation (3.64) for the element $Y \in \mathcal{H}_1$. This proves (3.64) for all $h \in \mathcal{H}_1$.

Finally, in order to demonstrate the Hopf action of $\mathfrak{h}_1$ on $\mathcal{Q}^r(\Gamma)$, we need to check that:

$$\begin{aligned}
X(F^1 *^r F^2) &= X(F^1) *^r F^2 + F^1 *^r X(F^2), \\
Y(F^1 *^r F^2) &= Y(F^1) *^r F^2 + F^1 *^r Y(F^2)
\end{aligned} \tag{3.66}$$

for any $F^1$, $F^2 \in \mathcal{Q}^r(\Gamma)$. The relation for $X$ has already been proved in (3.63). The relation for $Y$ is again an easy consequence of the fact that $Y$ is a derivation on $\mathcal{QM}$ and $Y(f\|\alpha) = Y(f)\|\alpha$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$. □

## 4. Twisted quasimodular Hecke operators

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. For any $\sigma \in SL_2(\mathbb{Z})$, we have developed the theory of $\sigma$-twisted modular Hecke operators in [3]. In this section, we introduce and study the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, $\mathcal{Q}_\sigma(\Gamma)$ coincides with the algebra $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators. In general, we will show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$-module and carries a pairing:

$$(\_\_,\_\_)\colon \mathcal{Q}_\sigma(\Gamma)\otimes\mathcal{Q}_\sigma(\Gamma)\longrightarrow\mathcal{Q}_\sigma(\Gamma). \tag{4.1}$$

We recall from Section 3 the Lie algebra $\mathfrak{l}_1$ with two generators $Y$, $X$ satisfying $[Y,X]=X$. If we let $\mathfrak{h}_1$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_1$, we show in Section 4.1 that the pairing in (4.1) on $\mathcal{Q}_\sigma(\Gamma)$ carries a "Hopf action" of $\mathfrak{h}_1$. In other words, we have:

$$h(F^1,F^2)=\sum\big(h_{(1)}(F^1),h_{(2)}(F^2)\big),\quad \forall\, h\in\mathfrak{h}_1,\ F^1,F^2\in\mathcal{Q}_\sigma(\Gamma), \tag{4.2}$$

where the coproduct $\Delta\colon\mathfrak{h}_1\longrightarrow\mathfrak{h}_1\otimes\mathfrak{h}_1$ is given by $\Delta(h)=\sum h_{(1)}\otimes h_{(2)}$ for any $h\in\mathfrak{h}_1$. In Section 4.2, we consider operators $X_\tau\colon\mathcal{Q}_\sigma(\Gamma)\longrightarrow\mathcal{Q}_{\tau\sigma}(\Gamma)$ for any $\tau$, $\sigma\in SL_2(\mathbb{Z})$. In particular, we consider operators acting between the levels of the graded module:

$$\mathbb{Q}_\sigma(\Gamma)=\bigoplus_{m\in\mathbb{Z}}\mathcal{Q}_{\sigma(m)}(\Gamma), \tag{4.3}$$

where for any $\sigma\in SL_2(\mathbb{Z})$, we set $\sigma(m)=\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\cdot\sigma$. Further, we generalize the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in (4.1) to a pairing:

$$(\_\_,\_\_)\colon \mathcal{Q}_{\sigma(m)}(\Gamma)\otimes\mathcal{Q}_{\sigma(n)}(\Gamma)\longrightarrow\mathcal{Q}_{\sigma(m+n)}(\Gamma),\quad \forall\, m,n\in\mathbb{Z}. \tag{4.4}$$

We show that the pairing in (4.4) is a special case of a more general pairing

$$(\_\_,\_\_)\colon \mathcal{Q}_{\tau_1\sigma}(\Gamma)\otimes\mathcal{Q}_{\tau_2\sigma}(\Gamma)\longrightarrow\mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma), \tag{4.5}$$

where $\tau_1,\tau_2$ are commuting matrices in $SL_2(\mathbb{Z})$. From (4.4), it is clear that we have a graded pairing on $\mathbb{Q}_\sigma(\Gamma)$ that extends the pairing on $\mathcal{Q}_\sigma(\Gamma)$. Finally, we consider the Lie algebra $\mathfrak{l}_\mathbb{Z}$ with generators $\{Z,X_n|n\in\mathbb{Z}\}$ satisfying the commutator relations:

$$[Z,X_n]=(n+1)X_n,\quad [X_n,X_{n'}]=0,\quad \forall\, n,n'\in\mathbb{Z}. \tag{4.6}$$

Then, for $n=0$, we have $[Z,X_0]=X_0$ and hence the Lie algebra $\mathfrak{l}_\mathbb{Z}$ contains the Lie algebra $\mathfrak{l}_1$ acting on $\mathcal{Q}_\sigma(\Gamma)$. Then, if we let $\mathfrak{h}_\mathbb{Z}$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_\mathbb{Z}$, we show that $\mathfrak{h}_\mathbb{Z}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$. In other words, for any $F^1$, $F^2\in\mathbb{Q}_\sigma(\Gamma)$, we have

$$h(F^1,F^2)=\sum\big(h_{(1)}(F^1),h_{(2)}(F^2)\big),\quad \forall\, h\in\mathfrak{h}_\mathbb{Z}, \tag{4.7}$$

where the coproduct $\Delta\colon\mathfrak{h}_\mathbb{Z}\longrightarrow\mathfrak{h}_\mathbb{Z}\otimes\mathfrak{h}_\mathbb{Z}$ is defined by setting $\Delta(h):=\sum h_{(1)}\otimes h_{(2)}$ for each $h\in\mathfrak{h}_\mathbb{Z}$.

**4.1. The pairing on $\mathcal{Q}_\sigma(\Gamma)$ and Hopf action.** Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. We start by defining the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, this reduces to the definition of $\mathcal{Q}(\Gamma)$.

**Definition 4.1.** Choose $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. A $\sigma$-twisted quasimodular Hecke operator $F$ of level $\Gamma$ is a function of finite support:

$$F\colon \Gamma\backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{QM}, \quad \Gamma\alpha \mapsto F_\alpha \in \mathcal{QM} \tag{4.8}$$

such that:

$$F_{\alpha\gamma} = F_\alpha \| \sigma\gamma\sigma^{-1}, \quad \forall\, \gamma \in \Gamma. \tag{4.9}$$

We denote by $\mathcal{Q}_\sigma(\Gamma)$ the collection of $\sigma$-twisted quasimodular Hecke operators of level $\Gamma$.

**Proposition 4.2.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose some $\sigma \in SL_2(\mathbb{Z})$. Then there exists a pairing:*

$$(\_\_,\_\_)\colon \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma) \tag{4.10}$$

*defined as follows:*

$$(F^1, F^2)_\alpha := \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} F^1_{\beta\sigma} \cdot \left(F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \sigma\beta\right), \qquad \forall\, F^1, F^2 \in \mathcal{Q}_\sigma(\Gamma),\ \alpha \in GL_2^+(\mathbb{Q}). \tag{4.11}$$

*Proof.* We choose $\gamma \in \Gamma$. Then, for any $\beta \in SL_2(\mathbb{Z})$, we have:

$$\begin{gathered} F^1_{\gamma\beta\sigma} = F^1_{\beta\sigma}, \\ F^2_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}} \| \sigma\gamma\beta = F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \sigma\gamma^{-1}\sigma^{-1}\sigma\gamma\beta = F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \sigma\beta \end{gathered} \tag{4.12}$$

and hence the sum in (4.11) is well defined, i.e., it does not depend on the choice of coset representatives. We have to show that $(F^1, F^2) \in \mathcal{Q}_\sigma(\Gamma)$. For this, we first note that $F^2_{\gamma\alpha\sigma^{-1}\beta^{-1}} = F^2_{\alpha\sigma^{-1}\beta^{-1}}$ for any $\gamma \in \Gamma$ and hence from the expression in (4.11), it follows that $(F^1, F^2)_{\gamma\alpha} = (F^1, F^2)_\alpha$. On the other hand, for any $\gamma \in \Gamma$, we can write:

$$(F^1, F^2)_{\alpha\gamma} = \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} F^1_{\beta\sigma} \cdot \left(F^2_{\alpha\gamma\sigma^{-1}\beta^{-1}} \| \sigma\beta\right). \tag{4.13}$$

We put $\delta = \beta\sigma\gamma^{-1}\sigma^{-1}$. It is clear that as $\beta$ runs through all the coset representatives of $\Gamma$ in $SL_2(\mathbb{Z})$, so does $\delta$. From (4.9), we know that $F^1_{\delta\sigma\gamma} = F^1_{\delta\sigma}\|\sigma\gamma\sigma^{-1}$. Then, we can rewrite (4.13) as:

$$\begin{aligned}(F^1,F^2)_{\alpha\gamma} &= \sum_{\delta\in\Gamma\backslash SL_2(\mathbb{Z})} F^1_{\delta\sigma\gamma}\cdot\left(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\sigma\delta\sigma\gamma\sigma^{-1}\right)\\ &= \sum_{\delta\in\Gamma\backslash SL_2(\mathbb{Z})} \left(F^1_{\delta\sigma}\|\sigma\gamma\sigma^{-1}\right)\cdot\left(\left(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\sigma\delta\right)\|\sigma\gamma\sigma^{-1}\right)\\ &= \left(\sum_{\delta\in\Gamma\backslash SL_2(\mathbb{Z})} F^1_{\delta\sigma}\cdot\left(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\sigma\delta\right)\right)\|(\sigma\gamma\sigma^{-1})\\ &= (F^1,F^2)_\alpha\|\sigma\gamma\sigma^{-1}.\end{aligned} \tag{4.14}$$

It follows that $(F^1,F^2)\in\mathcal{Q}_\sigma(\Gamma)$ and hence we have a well defined pairing

$$(\_\_,\_\_)\colon \mathcal{Q}_\sigma(\Gamma)\otimes\mathcal{Q}_\sigma(\Gamma)\longrightarrow\mathcal{Q}_\sigma(\Gamma). \qquad \square$$

We now consider the Hopf algebra $\mathfrak{h}_1$ defined in Section 3.2. By definition, $\mathfrak{h}_1$ is the universal enveloping algebra of the Lie algebra $\mathfrak{l}_1$ with two generators $X$ and $Y$ satisfying $[Y,X]=X$. We will now show that $\mathfrak{l}_1$ has a Lie action on $\mathcal{Q}_\sigma(\Gamma)$ and that $\mathfrak{h}_1$ has a "Hopf action" with respect to the pairing on $\mathcal{Q}_\sigma(\Gamma)$.

**Proposition 4.3.** *Let $\sigma\in SL_2(\mathbb{Z})$ and let $\Gamma=\Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$.*

(a) *The Lie algebra $\mathfrak{l}_1$ has a Lie action on $\mathcal{Q}_\sigma(\Gamma)$ defined by:*

$$X(F)_\alpha := X(F_\alpha),\quad Y(F)_\alpha := Y(F_\alpha),\qquad \forall\, F\in\mathcal{Q}_\sigma(\Gamma),\ \alpha\in GL_2^+(\mathbb{Q}). \tag{4.15}$$

(b) *The universal enveloping algebra $\mathfrak{h}_1$ of the Lie algebra $\mathfrak{l}_1$ has a "Hopf action" with respect to the pairing on $\mathcal{Q}_\sigma(\Gamma)$; in other words, we have:*

$$h(F^1,F^2)=\sum\left(h_{(1)}(F^1),h_{(2)}(F^2)\right),\qquad \forall\, F^1,F^2\in\mathcal{Q}_\sigma(\Gamma), h\in\mathfrak{h}_1, \tag{4.16}$$

*where the coproduct $\Delta\colon\mathfrak{h}_1\longrightarrow\mathfrak{h}_1\otimes\mathfrak{h}_1$ is given by $\Delta(h)=\sum h_{(1)}\otimes h_{(2)}$ for any $h\in\mathfrak{h}_1$.*

*Proof.* (a) We need to verify that for any $F\in\mathcal{Q}_\sigma(\Gamma)$ and any $\alpha\in GL_2^+(\mathbb{Q})$, we have $([Y,X](F))_\alpha = X(F)_\alpha$. We know that for any element $g\in\mathcal{QM}$ and hence in particular for the element $F_\alpha\in\mathcal{QM}$, we have $[Y,X](g)=X(g)$. The result now follows from the definition of the action of $X$ and $Y$ in (4.15).

(b) The Lie action of $\mathfrak{l}_1$ on $\mathcal{Q}_\sigma(\Gamma)$ from part (a) induces an action of the universal enveloping algebra $\mathfrak{h}_1$ on $\mathcal{Q}_\sigma(\Gamma)$. In order to prove (4.16), it suffices to prove the result for the generators $X$ and $Y$. We have:

$$\begin{aligned}
\big(X(F^1,F^2)\big)_\alpha &= X\big((F^1,F^2)_\alpha\big) \\
&= X\Bigg(\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} F^1_{\beta\sigma}\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\sigma\beta\big)\Bigg) \\
&= \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} X(F^1_{\beta\sigma})\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\sigma\beta\big) \\
&\qquad\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\sigma\beta\big) \\
&= \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} X(F^1_{\beta\sigma})\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\sigma\beta\big) \\
&\qquad\qquad + \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} F^1_{\beta\sigma}\cdot\big(X(F^2_{\alpha\sigma^{-1}\beta^{-1}})\|\sigma\beta\big) \\
&= \big(X(F^1),F^2)\big)_\alpha + \big(F^1,X(F^2)\big)_\alpha.
\end{aligned} \tag{4.17}$$

In (4.17), we have used the fact that $\sigma\beta\in SL_2(\mathbb{Z})$ and hence

$$X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\sigma\beta\big) = X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)\|\sigma\beta.$$

We can similarly verify the relation (4.16) for $Y\in\mathfrak{h}_1$. This proves the result. □

Our next aim is to show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$-module. Thereafter, we will consider the Hopf algebra $\mathcal{H}_1$ defined in Section 3.2 and show that there is a "Hopf action" of $\mathcal{H}_1$ on the right $\mathcal{Q}(\Gamma)$-module $\mathcal{Q}_\sigma(\Gamma)$.

**Proposition 4.4.** *Let $\sigma\in SL_2(\mathbb{Z})$ and let $\Gamma=\Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, $\mathcal{Q}_\sigma(\Gamma)$ carries a right $\mathcal{Q}(\Gamma)$-module structure defined by:*

$$(F^1 * F^2)_\alpha := \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta\big) \tag{4.18}$$

*for any $F^1\in\mathcal{Q}_\sigma(\Gamma)$ and any $F^2\in\mathcal{Q}(\Gamma)$.*

*Proof.* We take $\gamma\in\Gamma$. Then, since $F^1\in\mathcal{Q}_\sigma(\Gamma)$ and $F^2\in\mathcal{Q}(\Gamma)$, we have:

$$F^1_{\gamma\beta\sigma}=F^1_{\beta\sigma},\quad F^2_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}}|\gamma\beta = F^2_{\alpha\sigma^{-1}\beta^{-1}}|\gamma^{-1}\gamma\beta = F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta. \tag{4.19}$$

It follows that the sum in (4.18) is well defined, i.e., it does not depend on the choice of coset representatives for $\Gamma$ in $GL_2^+(\mathbb{Q})$. Further, it is clear that $(F^1 * F^2)_{\gamma\alpha} = (F^1 * F^2)_\alpha$. In order to show that $F^1 * F^2\in\mathcal{Q}_\sigma(\Gamma)$, it remains to show that

$$(F^1 * F^2)_{\alpha\gamma} = (F^1 * F_2)_\alpha\|\sigma\gamma\sigma^{-1}.$$

By definition, we know that:

$$(F^1 * F^2)_{\alpha\gamma} = \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma} \cdot \left(F^2_{\alpha\gamma\sigma^{-1}\beta^{-1}}|\beta\right) \tag{4.20}$$

We now set $\delta = \beta\sigma\gamma^{-1}\sigma^{-1}$. This allows us to rewrite (4.20) as follows:

$$\begin{aligned}(F^1 * F^2)_{\alpha\gamma} &= \sum_{\delta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\delta\sigma\gamma} \cdot \left(F^2_{\alpha\sigma^{-1}\delta^{-1}}|\delta\sigma\gamma\sigma^{-1}\right) \\ &= \sum_{\delta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \left(F^1_{\delta\sigma}\|\sigma\gamma\sigma^{-1}\right) \cdot \left((F^2_{\alpha\sigma^{-1}\delta^{-1}}|\delta)|\sigma\gamma\sigma^{-1}\right) \\ &= \left(\sum_{\delta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\delta\sigma} \cdot \left(F^2_{\alpha\sigma^{-1}\delta^{-1}}|\delta\right)\right)\|\sigma\gamma\sigma^{-1} \\ &= (F^1 * F^2)_{\alpha}\|\sigma\gamma\sigma^{-1}.\end{aligned} \tag{4.21}$$

Hence, $(F^1 * F^2) \in \mathcal{Q}_\sigma(\Gamma)$. In order to show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$-module, we need to check that $F^1 * (F^2 * F^3) = (F^1 * F^2) * F^3$ for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2, F^3 \in \mathcal{Q}(\Gamma)$. For this, we note that:

$$(F^1 * F^2)_\alpha = \sum_{\alpha_2\alpha_1=\alpha} F^1_{\alpha_1} \cdot \left(F^2_{\alpha_2}|\alpha_1\sigma^{-1}\right), \quad \forall\, \alpha \in GL_2^+(\mathbb{Q}), \tag{4.22}$$

where the sum in (4.22) is taken over all pairs $(\alpha_1, \alpha_2)$ such that $\alpha_2\alpha_1 = \alpha$ modulo the the following equivalence relation:

$$(\alpha_1, \alpha_2) \sim \left(\gamma\alpha_1, \alpha_2\gamma^{-1}\right), \quad \forall\, \gamma \in \Gamma. \tag{4.23}$$

It follows that for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\left((F^1 * F^2) * F^3\right)_\alpha = \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F^1_{\alpha_1} \cdot \left(F^2_{\alpha_2}|\alpha_1\sigma^{-1}\right) \cdot \left(F^3_{\alpha_3}|\alpha_2\alpha_1\sigma^{-1}\right), \tag{4.24}$$

where the sum in (4.24) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_3\alpha_2\alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2, \alpha_3) \sim \left(\gamma\alpha_1, \gamma'\alpha_2\gamma^{-1}, \alpha_3\gamma'^{-1}\right), \quad \forall\, \gamma, \gamma' \in \Gamma. \tag{4.25}$$

On the other hand, we have:

$$\begin{aligned}\left(F^1 * (F^2 * F^3)\right)_\alpha &= \sum_{\alpha_2'\alpha_1=\alpha} F^1_{\alpha_1} \cdot \left((F^2 * F^3)_{\alpha_2'}|\alpha_1\sigma^{-1}\right) \\ &= \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F^1_{\alpha_1} \cdot \left(F^2_{\alpha_2}|\alpha_1\sigma^{-1}\right) \cdot \left(F^3_{\alpha_3}|\alpha_2\alpha_1\sigma^{-1}\right).\end{aligned} \tag{4.26}$$

Again, we see that the sum in (4.26) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_3\alpha_2\alpha_1 = \alpha$ modulo the equivalence relation in (4.25). From (4.24) and (4.26), it follows that $(F^1 * (F^2 * F^3))_\alpha = ((F^1 * F^2) * F^3)_\alpha$. This proves the result. $\square$

We are now ready to describe the action of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{Q}_\sigma(\Gamma)$. From Section 3.2, we know that $\mathcal{H}_1$ is generated by $X$, $Y$, $\{\delta_n\}_{n\geq 1}$ which satisfy the relations (3.43), (3.44), (3.45).

**Proposition 4.5.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose some $\sigma \in SL_2(\mathbb{Z})$.*

(a) *The collection of $\sigma$-twisted quasimodular Hecke operators of level $\Gamma$ can be made into an $\mathcal{H}_1$-module as follows; for any $F \in \mathcal{Q}_\sigma(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$:*

$$X(F)_\alpha := X(F_\alpha), \quad Y(F)_\alpha := Y(F_\alpha), \quad \delta_n(F)_\alpha := X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha, \quad \forall\, n \geq 1. \tag{4.27}$$

(b) *The Hopf algebra $\mathcal{H}_1$ has a "Hopf action" on the right $\mathcal{Q}(\Gamma)$-module $\mathcal{Q}_\sigma(\Gamma)$; in other words, for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2 \in \mathcal{Q}(\Gamma)$, we have:*

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) * h_{(2)}(F^2), \quad \forall\, h \in \mathcal{H}_1, \tag{4.28}$$

*where the coproduct $\Delta\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathcal{H}_1$.*

*Proof.* (a) For any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have already checked in the proof of Proposition 4.3 that $X(F)$, $Y(F) \in \mathcal{Q}_\sigma(\Gamma)$. Further, from (3.54), we know that for any $\alpha \in GL_2^+(\mathbb{Q})$ and $\gamma \in \Gamma$, we have:

$$\begin{aligned} \mu_{\gamma\alpha\sigma^{-1}} &= \mu_\gamma|\alpha\sigma^{-1} + \mu_{\alpha\sigma^{-1}} = \mu_{\alpha\sigma^{-1}}, \\ \mu_{\alpha\gamma\sigma^{-1}} &= \mu_{\alpha\sigma^{-1}}|\sigma\gamma\sigma^{-1} + \mu_{\sigma\gamma\sigma^{-1}} = \mu_{\alpha\sigma^{-1}}|\sigma\gamma\sigma^{-1}. \end{aligned} \tag{4.29}$$

Hence, for any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have:

$$\begin{aligned} \delta_n(F)_{\gamma\alpha} &= X^{n-1}(\mu_{\gamma\alpha\sigma^{-1}}) \cdot F_{\gamma\alpha} = X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha = \delta_n(F)_\alpha, \\ \delta_n(F)_{\alpha\gamma} &= X^{n-1}(\mu_{\alpha\gamma\sigma^{-1}}) \cdot F_{\alpha\gamma} = X^{n-1}(\mu_{\alpha\sigma^{-1}}|\sigma\gamma\sigma^{-1}) \cdot (F_\alpha\|\sigma\gamma\sigma^{-1}) \\ &= \delta_n(F)_\alpha\|\sigma\gamma\sigma^{-1}. \end{aligned} \tag{4.30}$$

Hence, $\delta_n(F) \in \mathcal{Q}_\sigma(\Gamma)$. In order to show that there is an action of the Lie algebra $\mathcal{L}_1$ (and hence of its universal eneveloping algebra $\mathcal{H}_1$) on $\mathcal{Q}_\sigma(\Gamma)$, it remains to check the commutator relations (3.43) between the operators $X$, $Y$ and $\delta_n$ acting on $\mathcal{Q}_\sigma(\Gamma)$. We have already checked that $[Y, X] = X$ in the proof of Proposition 4.3. Since $X$ is a derivation on $\mathcal{QM}$ and $\delta_n(F)_\alpha = X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha$, we have:

$$\begin{aligned} [X, \delta_n](F)_\alpha &= X\big(X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha\big) - X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot X(F_\alpha) \\ &= X\big(X^{n-1}(\mu_{\alpha\sigma^{-1}})\big) \cdot F_\alpha = X^n(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha = \delta_{n+1}(F)_\alpha. \end{aligned} \tag{4.31}$$

Similarly, since $\mu_{\alpha\sigma^{-1}} \in \mathcal{M} \subseteq \mathcal{QM}$ is of weight 2 and $Y$ is a derivation on $\mathcal{QM}$, we have:

$$\begin{aligned}[Y,\delta_n](F)_\alpha &= Y\big(X^{n-1}(\mu_{\alpha\sigma^{-1}})\cdot F_\alpha\big) - X^{n-1}(\mu_{\alpha\sigma^{-1}})\cdot Y(F_\alpha)\\ &= Y\big(X^{n-1}(\mu_{\alpha\sigma^{-1}})\big)\cdot F_\alpha = nX^{n-1}(\mu_{\alpha\sigma^{-1}})\cdot F_\alpha = n\delta_n(F)_\alpha.\end{aligned} \tag{4.32}$$

Finally, we can verify easily that $[\delta_k,\delta_l] = 0$ for any $k, l \geq 1$.

(b) In order to prove (4.28), it is enough to check this equality for the generators $X$, $Y$ and $\delta_1 \in \mathcal{H}_1$. For $F^1 \in \mathcal{Q}_\sigma(\Gamma)$, $F^2 \in \mathcal{Q}(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned}\big(X(F^1*F^2)\big)_\alpha &= X\big((F^1*F^2)_\alpha\big)\\ &= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} X\big(F^1_{\beta\sigma}\cdot(F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta)\big)\\ &= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} X(F^1_{\beta\sigma})\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta\big) + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta\big)\\ &= \big(X(F^1)*F^2\big)_\alpha + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)|\beta\\ &\qquad - \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot\big(\mu_{\beta^{-1}}|\beta\big)\cdot Y\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)|\beta\\ &= \big(X(F^1)*F^2\big)_\alpha + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)|\beta\\ &\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma}\cdot\mu_\beta\cdot Y\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)|\beta\\ &= \big(X(F^1)*F^2\big)_\alpha + \big(F^1*X(F^2)\big)_\alpha\\ &\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \delta_1(F)_{\beta\sigma}\cdot Y(F^2)_{\alpha\sigma^{-1}\beta^{-1}}|\beta\\ &= \big(X(F^1)*F^2\big)_\alpha + \big(F^1*X(F^2)\big)_\alpha + \big(\delta_1(F^1)*Y(F^2)\big)_\alpha.\end{aligned} \tag{4.33}$$

In (4.33) above, we have used the fact that $0 = \mu_{\beta^{-1}\beta} = \mu_{\beta^{-1}}|\beta + \mu_\beta$. For $\alpha$, $\beta \in GL_2^+(\mathbb{Q})$, it follows from (3.54) that

$$\mu_{\alpha\sigma^{-1}} = \mu_{\alpha\sigma^{-1}\beta^{-1}\beta} = \mu_{\alpha\sigma^{-1}\beta^{-1}}|\beta + \mu_\beta. \tag{4.34}$$

Since $F^2 \in \mathcal{Q}(\Gamma)$ we know from (3.55) that $\delta_1(F^2)_{\alpha\sigma^{-1}\beta^{-1}} = \mu_{\alpha\sigma^{-1}\beta^{-1}}\cdot F^2_{\alpha\sigma^{-1}\beta^{-1}}$.

Combining with (4.34), we have:

$$\begin{aligned}
\delta_1\big((F^1 * F^2)\big)_\alpha &= \mu_{\alpha\sigma^{-1}} \cdot (F^1 * F^2)_\alpha \\
&= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \mu_{\alpha\sigma^{-1}} \cdot \big(F^1_{\beta\sigma} \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta)\big) \\
&= \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} \big(\mu_\beta \cdot F^1_{\beta\sigma}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}|\beta\big) \\
&\qquad + \sum_{\beta\in\Gamma\backslash GL_2^+(\mathbb{Q})} F^1_{\beta\sigma} \cdot \big(\mu_{\alpha\sigma^{-1}\beta^{-1}} \cdot F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)|\beta \\
&= \big(\delta_1(F^1) * F^2\big)_\alpha + \big(F^1 * \delta_1(F^2)\big)_\alpha.
\end{aligned} \tag{4.35}$$

Finally, from the definition of $Y$, it is easy to show that

$$\big(Y(F^1 * F^2)\big)_\alpha = \big(Y(F^1) * F^2\big)_\alpha + \big(F^1 * Y(F^2)\big)_\alpha. \qquad \square$$

**4.2. The operators $X_\tau\colon \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma)$ and Hopf action.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and choose some $\sigma \in SL_2(\mathbb{Z})$. In Section 4.1, we have only considered operators $X$, $Y$ and $\{\delta_n\}_{n\geq 1}$ that are endomorphisms of $\mathcal{Q}_\sigma(\Gamma)$. In this section, we will define an operator

$$X_\tau\colon \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma) \tag{4.36}$$

for $\tau \in SL_2(\mathbb{Z})$. In particular, we consider the commuting family $\{\rho_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\}_{n\in\mathbb{Z}}$ of matrices in $SL_2(\mathbb{Z})$ and write $\sigma(n) := \rho_n \cdot \sigma$. Then, we have operators:

$$X_{\rho_n}\colon \mathcal{Q}_{\sigma(m)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma), \quad \forall\, m, n \in \mathbb{Z} \tag{4.37}$$

acting "between the levels" of the graded module $\mathbb{Q}_\sigma(\Gamma) := \bigoplus_{m\in\mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$. We already know that $\mathcal{Q}_\sigma(\Gamma)$ carries an action of the Hopf algebra $\mathfrak{h}_1$. Further, $\mathfrak{h}_1$ has a Hopf action on the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in the sense of Proposition 4.3. We will now show that $\mathfrak{h}_1$ can be naturally embedded into a larger Hopf algebra $\mathfrak{h}_{\mathbb{Z}}$ acting on $\mathbb{Q}_\sigma(\Gamma)$ that incorporates the operators $X_{\rho_n}$ in (4.37). Finally, we will show that the pairing on $\mathcal{Q}_\sigma(\Gamma)$ can be extended to a pairing:

$$(\_\_,\_\_)\colon \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma), \quad \forall\, m, n \in \mathbb{Z}. \tag{4.38}$$

This gives us a pairing on $\mathbb{Q}_\sigma(\Gamma)$ and we prove that this pairing carries a Hopf action of $\mathfrak{h}_{\mathbb{Z}}$. We start by defining the operators $X_\tau$ mentioned in (4.36).

**Proposition 4.6.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$.*

(a) *For each $\tau \in SL_2(\mathbb{Z})$, we have a morphism:*

$$X_\tau\colon \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma), \quad X_\tau(F)_\alpha := X(F_\alpha)\|\tau^{-1}, \qquad \forall\, F \in \mathcal{Q}_\sigma(\Gamma),\ \alpha \in GL_2^+(\mathbb{Q}). \tag{4.39}$$

(b) *Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1\tau_2 = \tau_2\tau_1$. Then, the commutator $[X_{\tau_1}, X_{\tau_2}] = 0$.*

*Proof.* (a) We choose any $F \in \mathcal{Q}_\sigma(\Gamma)$. From (4.39), it is clear that $X_\tau(F)_{\gamma\alpha} = X_\tau(F)_\alpha$ for any $\gamma \in \Gamma$ and $\alpha \in GL_2^+(\mathbb{Q})$. Further, we note that:

$$\begin{aligned} X_\tau(F)_{\alpha\gamma} = X(F_{\alpha\gamma})\|\tau^{-1} &= X\big(F_\alpha\|\sigma\gamma\sigma^{-1}\big)\|\tau^{-1} \\ &= X\big(F_\alpha\|\tau^{-1}\big)\|\tau\sigma\gamma\sigma^{-1}\tau^{-1} \\ &= X_\tau(F_\alpha)\|\big((\tau\sigma)\gamma(\sigma^{-1}\tau^{-1})\big). \end{aligned} \tag{4.40}$$

It follows from (4.40) that $X_\tau(F) \in \mathcal{Q}_{\tau\sigma}(\Gamma)$ for any $F \in \mathcal{Q}_\sigma(\Gamma)$.

(b) Since $\tau_1$ and $\tau_2$ commute, both $X_{\tau_1}X_{\tau_2}$ and $X_{\tau_2}X_{\tau_1}$ are operators from $\mathcal{Q}_\sigma(\Gamma)$ to $\mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) = \mathcal{Q}_{\tau_2\tau_1\sigma}(\Gamma)$. For any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have ($\forall\ \alpha \in GL_2^+(\mathbb{Q})$):

$$\begin{aligned} \big(X_{\tau_1}X_{\tau_2}(F)\big)_\alpha &= X\big(X_{\tau_2}(F)_\alpha\big)\|\tau_1^{-1} \\ &= X^2(F_\alpha)\|\tau_2^{-1}\tau_1^{-1} \\ &= X^2(F_\alpha)\|\tau_1^{-1}\tau_2^{-1} = \big(X_{\tau_2}X_{\tau_1}(F)\big)_\alpha. \end{aligned} \tag{4.41}$$

This proves the result. □

As mentioned before, we now consider the commuting family $\{\rho_n := \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\}_{n\in\mathbb{Z}}$ of matrices in $SL_2(\mathbb{Z})$ and set $\sigma(n) := \rho_n \cdot \sigma$ for any $\sigma \in SL_2(\mathbb{Z})$. We want to define a pairing on the graded module $\mathbb{Q}_\sigma(\Gamma) = \bigoplus_{m\in\mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$ that extends the pairing on $\mathcal{Q}_\sigma(\Gamma)$. In fact, we will prove a more general result.

**Proposition 4.7.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1\tau_2 = \tau_2\tau_1$. Then, there exists a pairing:*

$$(\_\_,\_\_)\colon \mathcal{Q}_{\tau_1\sigma}(\Gamma) \otimes \mathcal{Q}_{\tau_2\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) \tag{4.42}$$

*defined as follows: for any $F^1 \in \mathcal{Q}_{\tau_1\sigma}(\Gamma)$ and any $F^2 \in \mathcal{Q}_{\tau_2\sigma}(\Gamma)$, we set:*

$$(F^1, F^2)_\alpha := \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\big)\cdot\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\big), \qquad \forall\ \alpha \in GL_2^+(\mathbb{Q}). \tag{4.43}$$

*In particular, when $\tau_1 = \tau_2 = 1$, the pairing in (4.43) reduces to the pairing on $\mathcal{Q}_\sigma(\Gamma)$ defined in (4.11).*

*Proof.* We choose some $\gamma \in \Gamma$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, $\beta \in SL_2(\mathbb{Z})$, we have $F^1_{\gamma\beta\sigma} = F^1_{\beta\sigma}$ and:

$$\begin{aligned} \big(F^2_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}}\|\tau_2\sigma\gamma\beta\tau_1^{-1}\tau_2^{-1}\big) &= \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\gamma^{-1}\sigma^{-1}\tau_2^{-1}\tau_2\sigma\gamma\beta\tau_1^{-1}\tau_2^{-1}\big) \\ &= \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\big). \end{aligned}$$

It follows that the sum in (4.43) is well defined, i.e. independent of the choice of coset representatives of $\Gamma$ in $SL_2(\mathbb{Z})$. Additionally, we have:

$$(F^1,F^2)_{\alpha\gamma} := \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\big)\cdot\big(F^2_{\alpha\gamma\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\big). \tag{4.44}$$

We now set $\delta=\beta\sigma\gamma^{-1}\sigma^{-1}$. Since $F^1\in\mathcal{Q}_{\tau_1\sigma}(\Gamma)$, we know that

$$F^1_{\delta\sigma\gamma}=F^1_{\delta\sigma}\|\tau_1\sigma\gamma\sigma^{-1}\tau_1^{-1}.$$

Then, we can rewrite the expression in (4.44) as follows:

$$\begin{aligned}
&(F^1,F^2)_{\alpha\gamma}\\
&\quad=\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}\big(F^1_{\delta\sigma\gamma}\|\tau_2^{-1}\big)\cdot\big(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\tau_2\sigma\delta\sigma\gamma\sigma^{-1}\tau_1^{-1}\tau_2^{-1}\big)\\
&\quad=\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}\big(F^1_{\delta\sigma}\|\tau_1\sigma\gamma\sigma^{-1}\tau_1^{-1}\tau_2^{-1}\big)\cdot\big(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\tau_2\sigma\delta\sigma\gamma\sigma^{-1}\tau_1^{-1}\tau_2^{-1}\big)\\
&\quad=\Bigg(\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}\big(F^1_{\delta\sigma}\|\tau_2^{-1}\big)\cdot\big(F^2_{\alpha\sigma^{-1}\delta^{-1}}\|\tau_2\sigma\delta\tau_1^{-1}\tau_2^{-1}\big)\Bigg)\|\tau_1\tau_2\sigma\gamma\sigma^{-1}\tau_1^{-1}\tau_2^{-1}\\
&\quad=(F^1,F^2)_\alpha\|\tau_1\tau_2\sigma\gamma\sigma^{-1}\tau_1^{-1}\tau_2^{-1}.
\end{aligned}\tag{4.45}$$

From (4.45) it follows that $(F^1,F^2)\in\mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma)$. □

In particular, it follows from the pairing in (4.42) that for any $m,n\in\mathbb{Z}$, we have a pairing

$$(\_\_,\_\_)\colon \mathcal{Q}_{\sigma(m)}(\Gamma)\otimes\mathcal{Q}_{\sigma(n)}(\Gamma)\longrightarrow\mathcal{Q}_{\sigma(m+n)}(\Gamma). \tag{4.46}$$

It is clear that (4.46) induces a pairing on $\mathbb{Q}_\sigma(\Gamma)=\bigoplus_{m\in\mathbb{Z}}\mathcal{Q}_{\sigma(m)}(\Gamma)$ for each $\sigma\in SL_2(\mathbb{Z})$. We will now define operators $\{X_n\}_{n\in\mathbb{Z}}$ and $Z$ on $\mathbb{Q}_\sigma(\Gamma)$. For each $n\in\mathbb{Z}$, the operator $X_n\colon\mathbb{Q}_\sigma(\Gamma)\longrightarrow\mathbb{Q}_\sigma(\Gamma)$ is induced by the collection of operators:

$$X_n^m:=X_{\rho_n}\colon\mathcal{Q}_{\sigma(m)}(\Gamma)\longrightarrow\mathcal{Q}_{\sigma(m+n)}(\Gamma),\quad\forall\, m\in\mathbb{Z}, \tag{4.47}$$

where, as mentioned before, $\rho_n=\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)$. Then, $X_n\colon\mathbb{Q}_\sigma(\Gamma)\longrightarrow\mathbb{Q}_\sigma(\Gamma)$ is an operator of homogeneous degree $n$ on the graded module $\mathbb{Q}_\sigma(\Gamma)$. We also consider:

$$\begin{aligned}
Z\colon\mathcal{Q}_{\sigma(m)}(\Gamma)\longrightarrow\mathcal{Q}_{\sigma(m)}(\Gamma),\quad Z(F)_\alpha:=mF_\alpha+Y(F_\alpha),&\\
\forall\, F\in\mathcal{Q}_{\sigma(m)}(\Gamma),\ \alpha\in GL_2^+(\mathbb{Q}).&
\end{aligned}\tag{4.48}$$

This induces an operator $Z\colon\mathbb{Q}_\sigma(\Gamma)\longrightarrow\mathbb{Q}_\sigma(\Gamma)$ of homogeneous degree 0 on the graded module $\mathbb{Q}_\sigma(\Gamma)$. We will now show that $\mathbb{Q}_\sigma(\Gamma)$ is acted upon by a certain Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ such that the Lie action incorporates the operators $\{X_n\}_{n\in\mathbb{Z}}$ and $Z$

mentioned above. We define $\mathfrak{l}_{\mathbb{Z}}$ to be the Lie algebra with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the following commutator relations:

$$[Z, X_n] = (n+1)X_n, \quad [X_n, X_{n'}] = 0, \quad \forall\, n, n' \in \mathbb{Z}. \tag{4.49}$$

In particular, we note that $[Z, X_0] = X_0$. It follows that the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ contains the Lie algebra $\mathfrak{l}_1$ defined in (3.56). We now describe the action of $\mathfrak{l}_{\mathbb{Z}}$ on $\mathbb{Q}_\sigma(\Gamma)$.

**Proposition 4.8.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ has a Lie action on $\mathbb{Q}_\sigma(\Gamma)$.*

*Proof.* We need to check that $[Z, X_n] = (n+1)X_n$ and $[X_n, X_{n'}] = 0$, $\forall\, n, n' \in \mathbb{Z}$ for the operators $\{Z, X_n | n \in \mathbb{Z}\}$ on $\mathbb{Q}_\sigma(\Gamma)$. From part (b) of Proposition 4.6, we know that $[X_n, X_{n'}] = 0$. From (4.47) and (4.48), it is clear that in order to show that $[Z, X_n] = (n+1)X_n$, we need to check that

$$[Z, X_n^m] = (n+1)X_n^m \colon \mathcal{Q}_{\sigma(m)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma)$$

for any given $m \in \mathbb{Z}$. For any $F \in \mathcal{Q}_{\sigma(m)}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we now check that:

$$\begin{aligned} \big(ZX_n^m(F)\big)_\alpha &= (n+m)X_n^m(F)_\alpha + Y\big(X_n^m(F)_\alpha\big) \\ &= (n+m)X(F_\alpha)\|\rho_n^{-1} + YX(F_\alpha)\|\rho_n^{-1}, \\ \big(X_n^m Z(F)\big)_\alpha &= X\big(Z(F)_\alpha\big)\|\rho_n^{-1} = mX(F_\alpha)\|\rho_n^{-1} + XY(F_\alpha)\|\rho_n^{-1}. \end{aligned} \tag{4.50}$$

Combining (4.50) with the fact that $[Y, X] = X$, it follows that $[Z, X_n^m] = (n+1)X_n^m$ for each $m \in \mathbb{Z}$. Hence, the result follows. □

We now consider the universal enveloping algebra $\mathfrak{h}_{\mathbb{Z}}$ of the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$. Accordingly, the coproduct $\Delta$ on $\mathfrak{h}_{\mathbb{Z}}$ is given by:

$$\Delta(X_n) = X_n \otimes 1 + 1 \otimes X_n, \quad \Delta(Z) = Z \otimes 1 + 1 \otimes Z, \quad \forall\, n \in \mathbb{Z}. \tag{4.51}$$

It is clear that $\mathfrak{h}_{\mathbb{Z}}$ contains the Hopf algebra $\mathfrak{h}_1$, the universal enveloping algebra of $\mathfrak{l}_1$. From Proposition 4.3, we know that $\mathfrak{h}_1$ has a Hopf action on the pairing on $\mathcal{Q}_\sigma(\Gamma)$. We want to show that $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$. For this, we prove the following lemma.

**Lemma 4.9.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1$, $\tau_2$, $\tau_3 \in SL_2(\mathbb{Z})$ be three matrices such that $\tau_i\tau_j = \tau_j\tau_i$, $\forall\, i$, $j \in \{1, 2, 3\}$. Then, for any $F^1 \in \mathcal{Q}_{\tau_1\sigma}(\Gamma)$, $F^2 \in \mathcal{Q}_{\tau_2\sigma}(\Gamma)$, we have:*

$$X_{\tau_3}(F^1, F^2) = \big(X_{\tau_3}(F^1), F^2\big) + \big(F^1, X_{\tau_3}(F^2)\big). \tag{4.52}$$

*Proof.* Consider any $\alpha \in GL_2^+(\mathbb{Q})$. Then, from the definition of $X_{\tau_3}$, it follows that

$$\begin{aligned} X_{\tau_3}(F^1, F^2)_\alpha &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} X\big((F^1_{\beta\sigma}\|\tau_2^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1})\big)\|\tau_3^{-1} \quad (4.53)\\ &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(X(F^1_{\beta\sigma})\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big)\\ &\quad + \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big). \end{aligned}$$

Since $F^1 \in \mathcal{Q}_{\tau_1\sigma}(\Gamma)$, it follows that $X_{\tau_3}(F^1) \in \mathcal{Q}_{\tau_1\tau_3\sigma}(\Gamma)$. Similarly, we see that $X_{\tau_3}(F^2) \in \mathcal{Q}_{\tau_2\tau_3\sigma}(\Gamma)$. It follows that:

$$\begin{aligned} \big(X_{\tau_3}(F^1), F^2\big)_\alpha &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(X_{\tau_3}(F^1)_{\beta\sigma}\|\tau_2^{-1}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big)\\ &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(X(F^1_{\beta\sigma})\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big)\\ (F^1, X_{\tau_3}(F^2))_\alpha &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(X_{\tau_3}(F^2)_{\alpha\sigma^{-1}\beta^{-1}}\|\tau_2\tau_3\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big)\\ &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)\|\tau_3^{-1}\tau_2\tau_3\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big)\\ &= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma}\|\tau_2^{-1}\tau_3^{-1}\big) \cdot \big(X\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)\|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}\big). \end{aligned} \tag{4.54}$$

Comparing (4.53) and (4.54), the result of (4.52) follows. □

As a special case of Lemma 4.9, it follows that for any $F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma)$ and $F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma)$, we have:

$$X_{\rho_n}(F^1, F^2) = X_n(F^1, F^2) = \big(X_n(F^1), F^2\big) + \big(F^1, X_n(F^2)\big), \quad \forall\, n \in \mathbb{Z}. \tag{4.55}$$

We conclude by showing that $\mathfrak{h}_\mathbb{Z}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$.

**Proposition 4.10.** *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Hopf algebra $\mathfrak{h}_\mathbb{Z}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$. In other words, for $F^1$, $F^2 \in \mathbb{Q}_\sigma(\Gamma)$, we have*

$$h(F^1, F^2) = \sum \big(h_{(1)}(F^1), h_{(2)}(F^2)\big), \tag{4.56}$$

*where the coproduct $\Delta\colon \mathfrak{h}_\mathbb{Z} \longrightarrow \mathfrak{h}_\mathbb{Z} \otimes \mathfrak{h}_\mathbb{Z}$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathfrak{h}_\mathbb{Z}$.*

*Proof.* It suffices to prove the result in the case where $F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma)$, $F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma)$ for some $m, m' \in \mathbb{Z}$. Further, it suffices to prove the relation (4.56) for the generators

$\{Z, X_n | n \in \mathbb{Z}\}$ of the Hopf algebra $\mathfrak{h}_{\mathbb{Z}}$. For the generators $X_n$, $n \in \mathbb{Z}$, this is already the result of (4.55) which follows from Lemma 4.9. Since $\Delta(Z) = Z \otimes 1 + 1 \otimes Z$, it remains to show that

$$Z(F^1, F^2) = \big(Z(F^1), F^2\big) + \big(F^1, Z(F^2)\big), \qquad \forall\, F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma),\ F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma). \tag{4.57}$$

By the definition of the pairing on $\mathbb{Q}_\sigma(\Gamma)$, we know that $(F^1, F^2) \in \mathcal{Q}_{\sigma(m+m')}(\Gamma)$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned}
Z(F^1, F^2)_\alpha &= (m+m')(F^1, F^2)_\alpha + Y(F^1, F^2)_\alpha \\
&= (m+m') \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big((F^1_{\beta\sigma} \| \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1})\big) \\
&\qquad + \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} Y\big((F^1_{\beta\sigma} \| \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1})\big) \\
&= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big((mF^1_{\beta\sigma} + Y(F^1_{\beta\sigma}))\| \rho_{m'}^{-1}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}\big) \\
&\quad + \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma} \| \rho_{m'}^{-1}\big) \cdot \big((m'F^2_{\alpha\sigma^{-1}\beta^{-1}} + Y\big(F^2_{\alpha\sigma^{-1}\beta^{-1}}\big)\big) \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}\big) \\
&= \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(Z(F^1)_{\beta\sigma} \| \rho_{m'}^{-1}\big) \cdot \big(F^2_{\alpha\sigma^{-1}\beta^{-1}} \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}\big) \\
&\qquad + \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \big(F^1_{\beta\sigma} \| \rho_{m'}^{-1}\big) \cdot \big(Z(F^2)_{\alpha\sigma^{-1}\beta^{-1}} \| \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}\big) \\
&= \big(Z(F^1), F^2\big)_\alpha + \big(F^1, Z(F^2)\big)_\alpha.
\end{aligned} \tag{4.58}$$

□

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