

The Harish-Chandra isomorphism for quantum GL_2

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Abstract. We construct an explicit Harish-Chandra isomorphism, from the quantum Hamiltonian reduction of the algebra $\mathcal{D}_q(GL_2)$ of quantum differential operators on GL_2 , to the spherical double affine Hecke algebra associated to GL_2 . The isomorphism holds for all deformation parameters $q \in \mathbb{C}^\times$ and $t \neq \pm i$, such that q is not a non-trivial root of unity.

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1. Introduction

A fundamental construction in the geometric representation theory of a reductive algebraic group G is Harish-Chandra's restriction homomorphism: given a conjugation-invariant differential operator on the Lie algebra, \mathfrak{g} , of G , we consider its restriction to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The restricted differential operator is invariant for the Weyl group W , but it may develop poles along hyperplanes stabilized by reflections in W . Conjugating by the discriminant eliminates these poles, and produces a regular, W -equivariant differential operator on \mathfrak{h} . This procedure leads to a homomorphism of algebras,

$$HC: D(\mathfrak{g})^G \rightarrow D(\mathfrak{h})^W,$$

from the algebra $D(\mathfrak{g})^G$ of ad-invariant differential operators on \mathfrak{g} , to the algebra $D(\mathfrak{h})^W$ of W -invariant differential operators on the Cartan subalgebra \mathfrak{h} .

Levasseur and Stafford's theorem [23] states that the restriction homomorphism descends to an isomorphism, called the Harish-Chandra isomorphism,

$$HC: D(\mathfrak{g}) \underset{\ker(\epsilon)}{\parallel} G \xrightarrow{\cong} D(\mathfrak{h})^W.$$

Here, $\epsilon: U\mathfrak{g} \rightarrow \mathbb{C}$ denotes the “co-unit” homomorphism, sending the generating subspace $\mathfrak{g} \subset U\mathfrak{g}$ to zero. More generally, for any ad-equivariant two-sided ideal $\mathcal{I} \subset U\mathfrak{g}$, we denote by

$$D(\mathfrak{g}) //_{\mathcal{I}} G := (D(\mathfrak{g}) / D(\mathfrak{g}) \cdot \widehat{\mu}^\#(\mathcal{I}))^G,$$

the quantum Hamiltonian reduction along the quantum moment map, $\widehat{\mu}^\#: U\mathfrak{g} \rightarrow D(\mathfrak{g})$, corresponding to the derivative of the adjoint action. The word “quantum” here refers to the fact that $D(\mathfrak{g})$ and $U\mathfrak{g}$ quantize the standard symplectic structures on $T^*\mathfrak{g}$ and \mathfrak{g}^* , respectively, and that $\widehat{\mu}^\#$ quantizes a classical moment map $\mu: T^*\mathfrak{g} \rightarrow \mathfrak{g}^*$.

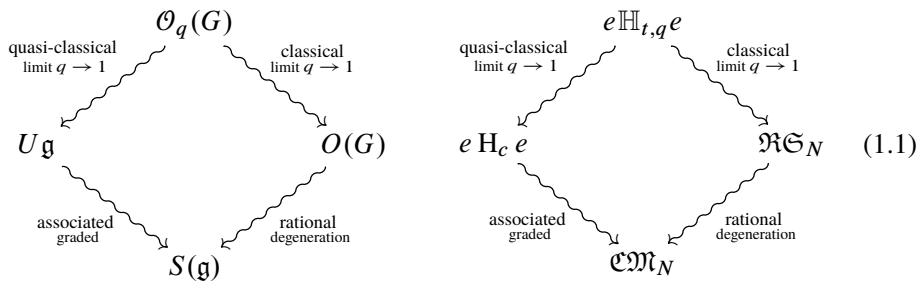
Specializing to the case $G = GL_N$, $\mathfrak{g} = \mathfrak{gl}_N$, $\mathfrak{h} = \mathbb{C}^N$, $W = S_N$, the algebra $D(\mathfrak{h})^W$ admits a canonical deformation to the *spherical rational Cherednik algebra*, denoted $eH_c(G)e$, and depending on a parameter $c \in \mathbb{C}$. In [11], Etingof and Ginzburg extended Levasseur–Stafford’s theorem, by defining a deformed Harish-Chandra isomorphism,

$$HC_c: D(\mathfrak{g}) //_{\mathcal{I}_c} G \xrightarrow{\cong} eH_c(G)e.$$

Here $\mathcal{I}_c \subset U\mathfrak{g}$ is a certain ad-invariant, two-sided ideal quantizing the orbit o_c of traceless matrices $X \in \mathfrak{sl}_N$ such that $X + c \cdot \text{Id}$ has rank at most one. The classical Hamiltonian reduction of $T^*\mathfrak{gl}_N$ along o_c gives rise to the celebrated Calogero–Moser variety \mathcal{CM}_N .

Each of the ingredients of the deformed Harish-Chandra homomorphism HC_c admits a multiplicative deformation:

- The universal enveloping algebra $U\mathfrak{g}$ q -deforms to a subalgebra $\mathcal{O}_q(G)$ of the quantized universal enveloping algebra $U_q\mathfrak{g}$. The algebra $\mathcal{O}_q(G)$ is, moreover, a quantization of G with its Semenov–Tian–Shansky Poisson bracket. On the other hand, the spherical rational Cherednik algebra $eH_c(G)e$ q -deforms to the spherical double affine Hecke algebra (spherical DAHA), denoted $e\mathbb{H}_{t,q}e$, which in turn quantizes the Ruijsenaars–Schneider integrable system \mathfrak{RS}_N . We can encode these basic inputs and outputs in the following “diamonds of degenerations”:



The arrows “classical limit $q \rightarrow 1$ ” and “associated graded” on the right diagram are theorems of Oblomkov [26] and Etingof–Ginzburg [11], respectively.

Each of the elements of Hamiltonian reduction also q -deform: these basic inputs to Hamiltonian reduction are organized into the left hand side of Figure 1. We have:

- The algebra $\mathcal{D}(\mathfrak{g})$ q -deforms to the algebra $\mathcal{D}_q(G)$, of quantum differential operators on G . The algebra $\mathcal{D}_q(G)$ is, moreover, a quantization of the “multiplicative cotangent bundle”, $G \times G$.
- The map $\widehat{\mu}^\#$. This is itself a q -deformation of a multiplicative moment map $\widetilde{\mu}: G \times G \rightarrow G$.
- Each ideal $\mathcal{J}_c \subset U\mathfrak{g}$ q -deforms to an ad-invariant, two-sided ideal $\mathcal{J}_t \subset U_q\mathfrak{g}$, for $t \in \mathbb{C}^\times$; each \mathcal{J}_t , in turn, quantizes an orbit O_t , consisting of special linear matrices $X \in SL_N$ such that $X + t \cdot \text{Id}$ is rank at most one.
- The quantum Hamiltonian reduction $\mathcal{D}_q(G) //_{\mathcal{J}_t} U_q\mathfrak{g}$ is constructed analogously, both to the quantum Hamiltonian reduction of $\mathcal{D}(\mathfrak{g})$, and to the multiplicative Hamiltonian reduction of $G \times G$.

It is thus a natural and important problem to construct the q -analog of the Harish-Chandra isomorphism,

$$HC_{q,t}: \mathcal{D}_q(G) //_{\mathcal{J}_t} U_q\mathfrak{g} \xrightarrow{\cong} e\mathbb{H}_{t,q}e,$$

which would give rise to the degenerations on the right hand-side of Figure 1:

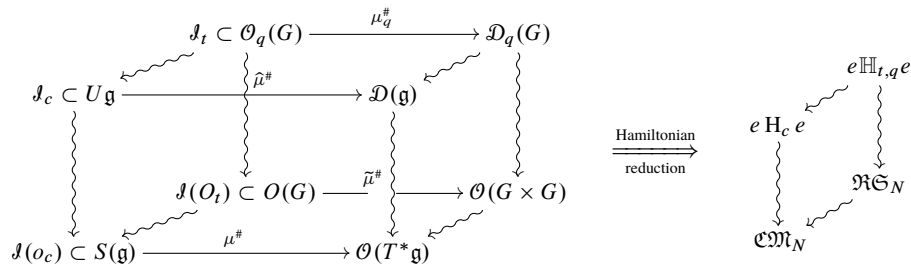


Figure 1. The elements of classical, multiplicative, quantum, and quantum multiplicative Hamiltonian reduction, and their degenerations.

Such quantum Harish-Chandra isomorphisms have been constructed for special parameters, first in [30], when q is a root of unity, via quantum differential operators on $GL_N \times \mathbb{P}^{N-1}$, and subsequently in [18], in the formal neighborhood of $q = 1$, as a special case of a quantum multiplicative quiver variety.

Each of these approaches rely on being “close to” the classical setting: in [30] Azumaya algebras are employed to reduce to a geometric problem on the spectrum of the center, while [18] relies upon formal deformation arguments. Neither method can be applied to arbitrary — or even generic — values of the parameter $q \in \mathbb{C}$.

1.1. Main result. This paper is devoted to constructing a q -deformed Harish-Chandra isomorphism “far from” the classical setting — when q is not a root of unity — in the case $N = 2$. Our main result is Theorem 4.1, in which we construct an explicit isomorphism of algebras:

$$HC_{q,t}: \mathcal{D}_q(GL_2) //_{\mathfrak{d}_t} U_q \mathfrak{gl}_2 \xrightarrow{\cong} e \mathbb{H}_{t,q}(GL_2) e,$$

and which holds for any $q \in \mathbb{C}^\times$ not a root of unity, and for all $t \neq \pm i$.

Our construction of $HC_{q,t}$ consists of giving explicit presentations and PBW bases of both algebras (see Theorems 2.6 and 3.20), and directly defining and computing $HC_{q,t}$ against these PBW bases. To this end, we exploit certain natural \mathbb{Z}^2 -gradings on each algebra; while the graded pieces are infinite-dimensional, we express each algebra as an Ore localization of a certain auxiliary algebra with finite-dimensional graded pieces, and show that $HC_{q,t}$ is a graded isomorphism there, compatibly with the localization.

Remark 1.1. In light of [30], [18], and the present paper, it is natural to conjecture the existence of a q -deformed Harish-Chandra isomorphism for all N . The explicit description of the Harish-Chandra isomorphism which we propose in this paper is unfeasible for large finite N , owing to the increasing complexity in a presentation for the spherical double affine Hecke algebra as the rank N grows. However, Burban, Schiffmann, and Vasserot have defined in [6, 28] a certain “ $N \rightarrow \infty$ limit” of the positive parts of the spherical DAHA, which embeds into the Hall algebra of an elliptic curve, and has many remarkable properties; in particular this limiting algebra can be computed with more directly than its finite N constituents. A certain specialization of Burban and Schiffmann’s algebra appears in [24], where it is related to the skein algebra of the torus. It should be interesting, therefore, to consider the $N \rightarrow \infty$ limit of the Hamiltonian reduction procedure for $\mathcal{D}_q(GL_N)$; the present paper provides an initial step in that direction.

In the paper [19], certain explicit formulas are given for elements c_1, \dots, c_N which q -deform the coefficients of the characteristic polynomial, thus generalizing tr_q and \det_q of Lemma 3.9; these give natural generators for subalgebra of invariants in $\mathcal{O}_q(G)$. The formulas in Theorem 4.1 suggest that such a q -deformed Harish-Chandra homomorphism may be defined simply by sending the collection of generating invariants, c_i , in each $\mathcal{O}_q(G)$ -subalgebra of $\mathcal{D}_q(G)$, to the elementary q -symmetric polynomials P_i and Q_i , respectively.

Remark 1.2 (Roots of unity). When the quantum parameter q is a root of unity, there is an interesting phenomenon, related to the quantum Frobenius Hamiltonian reduction of [14]: the algebra $\mathcal{D}_q(G)$ develops a large center, and many more vectors become invariant for the quantum group action. Meanwhile, the functor of taking invariants is no longer exact, so the simplification of Lemma 3.19 cannot be made. It is interesting to note that, nevertheless, the isomorphism $HC_{q,t}$ is perfectly

well-defined when q is a root of unity, only it identifies the spherical DAHA at a root of unity with an abstract algebra which cannot in turn be identified with the quantum Hamiltonian reduction. This suggests that, when q is a root of unity, the notion of Hamiltonian reduction employed here should be replaced by one involving divided powers quantum groups, so that the isomorphism $HC_{q,t}$ recovers its natural interpretation.

Remark 1.3 (Other types). It is possible to apply the techniques of this paper to other surfaces in place of the marked torus, and to other groups of small rank. For the GL_3 case, see [29]. For the case of SL_2 and the four-punctured sphere, one recovers the (C_1^\vee, C_1) DAHA; this is work in progress of Juliet Cooke.

1.2. Relation to factorization homology. Let us briefly explain the connection of the present work to [1, 5] and to [2], where Theorem 4.1 has already been announced. Since this relation is only motivational in the present paper, we will be informal, referring to those papers for complete details. Let us fix the braided tensor category $\mathcal{A} = \text{Rep}_q G$ of integrable modules for the quantum group, its factorization homology defines an invariant of surfaces valued in “pointed” (a.k.a. E_0 -)categories, i.e. categories endowed with a distinguished object:

$$\begin{aligned} \text{Surfaces} &\longrightarrow \text{Pointed categories,} \\ S &\mapsto \left(\int_S \mathcal{A}, \mathcal{O}_S \right). \end{aligned}$$

The algebra $\mathcal{O}_q(G)$ arises naturally in this context: its representation theory, the category $\mathcal{O}_q(G)\text{-mod}_{\mathcal{A}}$ of $\mathcal{O}_q(G)$ -modules in \mathcal{A} , computes the factorization homology $\int_{\text{Ann}} \mathcal{A}$, of \mathcal{A} on the annulus, and we have $\mathcal{O}_{\text{Ann}} \cong \mathcal{O}_q(G)$ itself.

Given a $U_q\mathfrak{g}$ -equivariant two-sided ideal of $\mathcal{O}_q(G)$, such as \mathcal{I}_t , the category \mathcal{M} of modules for the algebra $\mathcal{O}_q(G)/\mathcal{I}_t$ therefore defines a “braided \mathcal{A} -module category”. Braided \mathcal{A} -module categories serve as the point defects in the associated topological field theory (TFT): the factorization homology of the pair $(U_q\mathfrak{g}, \mathcal{I}_t)$ defines an invariant of surfaces with marked points, again valued in pointed categories:

$$\begin{aligned} \text{Marked surfaces} &\longrightarrow \text{Pointed categories,} \\ (S, x) &\mapsto \left(\int_{(S,x)} (\mathcal{A}, \mathcal{M}), \mathcal{O}_{(S,x)} \right). \end{aligned}$$

A theorem in [2] identifies the endomorphism algebra of the distinguished object for a torus T^2 , marked at a single point with the braided module category determined by \mathcal{I}_t , with the quantum Hamiltonian reduction algebra $\mathcal{D}_q(G) //_{\mathcal{I}_t} U_q\mathfrak{gl}_2$.

Hence the results of this paper identify this endomorphism algebra, in the case $G = GL_2$, with the spherical double affine Hecke algebra $e\mathbb{H}_{t,q}e$. This points to a new construction of the representation theory of the DAHA in topological

terms. The factorization homology studied in [1] and [2] is expected to extend to 3-dimensional TFT: its values on suitably decorated knot complements therefore give rise, functorially, to representations of $e\mathbb{H}_{t,q}e$.

Such an extension, combined with Theorem 4.1, may eventually provide a link between factorization homology and several expected topological invariants valued in the representation theory of the DAHA, such as the quantum A- [13, 17] and DAHA-Jones [9] polynomials associated to knots in S^3 , and the DAHA modules of [16, 27] associated to plane singularities. Our results may be compared to [3], who study appearances of the spherical DAHA as modified skein algebras: in particular, at $t = 1$, both constructions recover [12], and both constructions are aimed at restoring the lost parameter t .

1.3. Outline. An outline of the paper is as follows. In Section 2, we recall the definition of the double affine Hecke algebra $\mathbb{H}_{t,q}$ of GL_2 and its spherical subalgebra $e\mathbb{H}_{t,q}e$, and we give an explicit presentation and a PBW basis for the latter (Theorem 2.6). In Section 3 we recall the definition of $\mathcal{D}_q(GL_2)$ and its Hamiltonian reduction $\mathcal{D}_q(GL_2) //_{\mathfrak{h}_t} U_q\mathfrak{g}$, and we give an explicit presentation and a PBW basis for the latter (Theorem 3.20). Section 4 contains the main result, Theorem 4.1, which constructs an isomorphism between the spherical DAHA and the Hamiltonian reduction of $\mathcal{D}_q(GL_2)$. This result follows directly from the presentations of these two algebras from Theorems 2.6 and 3.20, which constitute the main technical contribution of the paper. The proofs of Theorems 2.6 and 3.20 are delayed until Section 5, and Section 6, respectively.

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2. Double affine Hecke algebra $\mathbb{H}_{t,q}$ of GL_2 and its spherical subalgebra

2.1. Parameters. Let us fix a base ring $R_{t,q} = \mathbb{C}[t^{\pm 1}, q^{\pm 1}][\!(t^2 + 1)^{-1}\!]$, the commutative ring of rational functions in variables t and q whose denominators are products of powers of t , q and $t^2 + 1$.

For any numbers $q', t' \in \mathbb{C}^\times$ with $(t')^2 + 1 \neq 0$, the quotient of $R_{t,q}$ by the corresponding maximal ideal $\mathfrak{m} = \langle (q - q'), (t - t') \rangle$ gives the evaluation map $R_{t,q} \twoheadrightarrow R_{t,q}/\mathfrak{m} \cong \mathbb{C}$. For any algebra \mathbf{A} over $R_{t,q}$, the evaluation map gives its specialization, an algebra \mathbf{A}/\mathfrak{mA} over \mathbb{C} .

For the rest of the paper we will work with algebras depending on parameters q and t ; we will treat the corresponding algebras over $R_{t,q}$ and their specializations simultaneously. Most arguments and results apply in both settings; we will point out the distinction only when necessary. In particular, the condition “ q is not a non-trivial root of unity” means either that we work over $R_{t,q}$ (so the condition is vacuously true), or over \mathbb{C} , but with the stated restriction.

2.2. The double affine Hecke algebra $\mathbb{H}_{t,q}$.

Definition 2.1. The double affine Hecke algebra (DAHA) of type GL_2 is the associative algebra $\mathbb{H}_{t,q}$ with generators

$$T^{\pm 1}, X_1^{\pm 1}, X_2^{\pm 1}, Y_1^{\pm 1}, Y_2^{\pm 1}$$

and relations

$$\begin{aligned} TX_1T &= X_2, & T^{-1}Y_1T^{-1} &= Y_2, & X_1X_2 &= X_2X_1, & Y_1Y_2 &= Y_2Y_1, \\ Y_1X_1X_2 &= q^2X_1X_2Y_1, & X_1^{-1}Y_2 &= Y_2X_1^{-1}T^{-2}, & (T + t^{-1})(T - t) &= 0. \end{aligned} \tag{2.1}$$

Remark 2.2. This definition first appeared in [7]. There are several different conventions in the literature for the presentation of DAHA and the choice of parameters. Definition 2.1 follows the conventions of [15] (see also [7], [8], and [25]) up to replacing the parameters t by t^2 and q by q^2 . This rescaling is not significant, and it is dictated by the need to align the parameters for DAHA with the parameters for $\mathcal{D}_q(GL_2)$ in the next section.

Remark 2.3. Let S_2 be the symmetric group of order 2. The algebra $\mathbb{H}_{1,1}$ is isomorphic to the semidirect product $\mathbb{C}[S_2] \ltimes \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, Y_1^{\pm 1}, Y_2^{\pm 1}]$, and the algebra $\mathbb{H}_{t,q}$ is a flat deformation of $\mathbb{H}_{1,1}$.

The following proposition gives the reordering relations for any two generators X_i, Y_j and the PBW basis. Similar relations can be written for any two generators $X_i^{\pm 1}, Y_j^{\pm 1}$.

Proposition 2.4. (1) *The following relations are also satisfied in $\mathbb{H}_{t,q}$:*

$$\begin{aligned} T^2 &= (t - t^{-1})T + 1, & T^{-1} &= T - (t - t^{-1}), & (2.2) \\ X_1Y_1 &= q^{-2}T^{-2}Y_1X_1, & X_1Y_2 &= Y_2X_1 + (t - t^{-1})T^{-1}Y_1X_1, \\ X_2Y_1 &= Y_1X_2 + (t - t^{-1})q^{-2}T^{-1}Y_1X_1, & X_2Y_2 &= q^{-2}Y_2X_2T^{-2}. \end{aligned} \tag{2.3}$$

(2) *The set*

$$\{T^\epsilon Y_1^{a_1} Y_2^{a_2} X_1^{b_1} X_2^{b_2} \mid \epsilon \in \{0, 1\}, a_i, b_i \in \mathbb{Z}\}$$

forms a basis of $\mathbb{H}_{t,q}$.

Proof. Part (1) is proved by direct computations. Part (2) was proved in [8]. For illustration we prove the relations (2.3).

First, rewriting the relation $X_1^{-1}Y_2 = Y_2X_1^{-1}T^{-2}$ from (2.1) as $Y_2T^2X_1 = X_1Y_2$, we have

$$\begin{aligned} X_1Y_2 &= Y_2T^2X_1 \\ &\stackrel{(2.2)}{=} Y_2X_1 + (t - t^{-1})Y_2TX_1 \\ &\stackrel{(2.1)}{=} Y_2X_1 + (t - t^{-1})T^{-1}Y_1X_1. \end{aligned} \quad (2.4)$$

Conjugating the relation $X_1^{-1}Y_2 = Y_2X_1^{-1}T^{-2}$ by T^{-1} and rewriting, we have

$$\begin{aligned} T^{-1}X_1^{-1}Y_2T &= T^{-1}Y_2X_1^{-1}T^{-1} \\ X_2^{-1}Y_1 &= T^{-2}Y_1X_2^{-1}. \end{aligned} \quad (2.5)$$

Next, we have

$$\begin{aligned} X_1Y_1 &= X_2^{-1}X_1X_2Y_1 \\ &\stackrel{(2.1)}{=} q^{-2}X_2^{-1}Y_1X_1X_2 \\ &\stackrel{(2.5)}{=} q^{-2}T^{-2}Y_1X_1. \end{aligned} \quad (2.6)$$

Conjugating (2.6) by T , we have

$$\begin{aligned} X_2Y_2 &\stackrel{(2.1)}{=} TX_1Y_1T^{-1} \\ &\stackrel{(2.6)}{=} q^{-2}T^{-1}Y_1X_1T^{-1} \\ &\stackrel{(2.1)}{=} q^{-2}Y_2X_2T^{-2}. \end{aligned} \quad (2.7)$$

Finally, combining (2.4) and (2.6), we get

$$\begin{aligned} X_2Y_1 &\stackrel{(2.1)}{=} TX_1T^2Y_2T \\ &\stackrel{(2.2)}{=} TX_1Y_2T + (t - t^{-1})TX_1TY_2T \\ &\stackrel{(2.4)}{=} TY_2X_1T + (t - t^{-1})Y_1X_1T + (t - t^{-1})TX_1TY_2T \\ &\stackrel{(2.1)}{=} Y_1T^{-1}X_1T + (t - t^{-1})Y_1X_1T + (t - t^{-1})TX_1Y_1 \\ &\stackrel{(2.2)}{=} Y_1TX_1T + (t - t^{-1})TX_1Y_1 \\ &\stackrel{(2.6)}{=} Y_1X_2 + (t - t^{-1})q^{-2}T^{-1}Y_1X_1. \end{aligned} \quad (2.8)$$

□

The algebra $\mathbb{H}_{t,q}$ is \mathbb{Z}^2 -graded by

$$\deg X_1 = \deg X_2 = (0, 1), \quad \deg Y_1 = \deg Y_2 = (1, 0), \quad \deg T = (0, 0).$$

This grading is inner: it follows from (2.3) that any $h \in \mathbb{H}_{t,q}$ with $\deg h = (M, N)$ satisfies

$$Y_1 Y_2 h = q^{2N} h Y_1 Y_2, \quad X_1 X_2 h = q^{-2M} h X_1 X_2. \tag{2.9}$$

2.3. The spherical double affine Hecke algebra $e\mathbb{H}_{t,q}e$. The subalgebra of the double affine Hecke algebra generated by T and T^{-1} is the (finite) Hecke algebra of the symmetric group S_2 . For $t^2 + 1 \neq 0$, this algebra is isomorphic to the group algebra of S_2 . Denoting the only nontrivial element of the symmetric group by s , the identification is given by

$$T = \frac{t + t^{-1}}{2}s + \frac{t - t^{-1}}{2}.$$

The trivial idempotent of S_2 is

$$e = \frac{1 + s}{2} = \frac{1 + tT}{1 + t^2} \in \mathbb{H}_{t,q}.$$

It satisfies

$$eT = Te = te, \quad eT^{-1} = T^{-1}e = t^{-1}e. \tag{2.10}$$

This action of this idempotent on any representation is the projection to the trivial representation of S_2 , or equivalently to the representation of the Hecke algebra in which T acts by the eigenvalue t .

Definition 2.5. The spherical double affine Hecke algebra is the subalgebra $e\mathbb{H}_{t,q}e$ of $\mathbb{H}_{t,q}$.

We will give an explicit presentation of the spherical DAHA by generators and relations, and the resulting PBW type basis for it.

Theorem 2.6. (1) *The spherical double affine Hecke algebra $e\mathbb{H}_{t,q}e$ is isomorphic to the algebra with generators $P_1, P_2^{\pm 1}, Q_1, Q_2^{\pm 1}, R$ and relations:*

$$P_2 P_1 = P_1 P_2, \tag{2.11} \quad P_2 R = q^{-2} R P_2, \tag{2.17}$$

$$Q_2 Q_1 = Q_1 Q_2, \tag{2.12} \quad R Q_2 = q^{-2} Q_2 R, \tag{2.18}$$

$$P_2 Q_2 = q^{-4} Q_2 P_2, \tag{2.13} \quad P_1 R = q^{-2} R P_1 + (1 - q^{-2}) Q_1 P_2,$$

$$P_2 Q_1 = q^{-2} Q_1 P_2, \tag{2.14} \tag{2.19}$$

$$P_1 Q_2 = q^{-2} Q_2 P_1, \tag{2.15} \quad R Q_1 = q^{-2} Q_1 R + (1 - q^{-2}) Q_2 P_1, \tag{2.20}$$

$$P_1 Q_1 = Q_1 P_1 + (q^{-2} - 1)R, \tag{2.16} \tag{2.21}$$

$$R^2 = (1 + t^2)(q^{-2} + t^{-2})Q_2 P_2 - q^{-2} Q_2 P_1^2 - q^{-2} Q_1^2 P_2 + q^{-2} Q_1 R P_1.$$

The isomorphism is given by

$$\begin{aligned} \Phi(P_1) &= e(X_1 + X_2)e, & \Phi(P_2) &= e(X_1 X_2)e, & \Phi(R) &= e(t^{-2}Y_1 X_1 + Y_2 X_2)e, \\ \Phi(Q_1) &= e(Y_1 + Y_2)e, & \Phi(Q_2) &= e(Y_1 Y_2)e. \end{aligned} \tag{2.22}$$

The \mathbb{Z}^2 -grading is given on the generators by

$$\deg P_i = (0, i), \quad \deg Q_i = (i, 0), \quad \deg R = (1, 1),$$

and Φ is a graded isomorphism.

(2) After the identification given by the isomorphism Φ , the set

$$\{Q_1^{a_1} Q_2^{a_2} R^\epsilon P_1^{b_1} P_2^{b_2} \mid \epsilon \in \{0, 1\}, a_1, b_1 \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}\}$$

forms a basis of the spherical double affine Hecke algebra $e\mathbb{H}_{t,q}$.

The presentation of the spherical DAHA by generators and relations given in Theorem 2.6 is one of the main technical steps of the paper and is used in the proof of the main result, Theorem 4.1. Section 5 is dedicated to the proof of its proof.

3. The algebra $\mathcal{D}_q(GL_2)$ of quantum differential operators and its Hamiltonian reduction

In this section, we recall the definitions quantum enveloping algebra $U_q\mathfrak{gl}_2$, the quantum coordinate algebra $\mathcal{O}_q(GL_2)$, the algebra $\mathcal{D}_q(GL_2)$ of polynomial quantum differential operators, and the Hamiltonian reduction $\mathcal{D}_q(GL_2) //_{\mathfrak{g}_t} U$ of $\mathcal{D}_q(GL_2)$. We will follow the conventions of [21] and [18].

3.1. The quantum enveloping algebra $U = U_q\mathfrak{gl}_2$. We use the conventions from [21].

Definition 3.1. The quantum enveloping algebra $U = U_q\mathfrak{gl}_2$ is the Hopf algebra with:

- generators

$$E, F, K_1^{\pm 1}, K_2^{\pm 1};$$

- relations

$$\begin{aligned} K_1 E K_1^{-1} &= qE, & K_2 E K_2^{-1} &= q^{-1}E, \\ K_1 F K_1^{-1} &= q^{-1}F, & K_2 F K_2^{-1} &= qF, \\ K_1 K_2 &= K_2 K_1, & EF - FE &= \frac{K_1 K_2^{-1} - K_1^{-1} K_2}{q - q^{-1}}; \end{aligned}$$

- the Hopf structure:

$$\begin{aligned} \Delta(E) &= E \otimes K_1 K_2^{-1} + 1 \otimes E, & S(E) &= -EK_1^{-1}K_2, & \varepsilon(E) &= 0, \\ \Delta(F) &= F \otimes 1 + K_1^{-1}K_2 \otimes F, & S(F) &= -K_1K_2^{-1}F, & \varepsilon(F) &= 0, \\ \Delta(K_i) &= K_i \otimes K_i, & S(K_i) &= K_i^{-1}, & \varepsilon(K_i) &= 1. \end{aligned}$$

The vector representation V of U has ordered basis $\{e_{-1}, e_1\}$, with action:

$$\begin{aligned} \rho_V(K_1) &= \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix}, & \rho_V(K_2) &= \begin{bmatrix} 1 & 0 \\ 0 & q^{-1} \end{bmatrix}, \\ \rho_V(E) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & \rho_V(F) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The Hopf algebra U is quasi-triangular, with the R-matrix

$$\mathcal{R} = q^{H \otimes H} \cdot \sum_{n \geq 0} q^{n(n-1)/2} (q - q^{-1})^n \frac{1}{[n]_q!} E^n \otimes F^n.$$

Here $q^{H \otimes H}$ is the operator which acts by a scalar on the tensor product of any two weight spaces. More precisely, for V_1, V_2 representations of U , and $v_1 \in V_1, v_2 \in V_2$ vectors such that $K_i \triangleright v_j = q^{\alpha_{ij}} v_j$, the action of $q^{H \otimes H}$ on their tensor product is given by $q^{H \otimes H} \triangleright (v_1 \otimes v_2) = q^{\alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22}} (v_1 \otimes v_2)$.

3.2. A presentation of U via L-matrices. We recall an alternative presentation of U which uses the R-matrix \mathcal{R} , its inverse \mathcal{R}^{-1} and the vector representation ρ_V (see [21]). Define \underline{R}, L^+, L^- by

$$\begin{aligned} \underline{R} &= (\rho_V \otimes \rho_V)(\mathcal{R}) \in \text{Mat}_{2 \times 2} \otimes \text{Mat}_{2 \times 2}, \\ L^+ &= (\text{id} \otimes \rho_V)(\mathcal{R}) \in \text{Mat}_{2 \times 2}(U), \\ L^- &= (\rho_V \otimes \text{id})(\mathcal{R}^{-1}) \in \text{Mat}_{2 \times 2}(U), \end{aligned}$$

where $\text{Mat}_{2 \times 2}$ denotes the set of 2×2 matrices with entries in the ground ring ($R_{t,q}$ or \mathbb{C}), and $\text{Mat}_{2 \times 2}(U)$ denotes the set of 2×2 matrices with entries in the algebra U . Let $L_1^\pm = L^\pm \otimes \text{id}, L_2^\pm = \text{id} \otimes L^\pm$.

Proposition 3.2 ([21, Theorem 8.33]). *The Hopf algebra U is isomorphic to the Hopf algebra with:*

- generators given by the entries of the matrices

$$L^+ = \begin{bmatrix} l_1^{+1} & l_2^{+1} \\ 0 & l_2^{+2} \end{bmatrix}, \quad L^- = \begin{bmatrix} l_1^{-1} & 0 \\ l_1^{-2} & l_2^{-2} \end{bmatrix};$$

- relations given by the entries of the matrix equations

$$L_1^\pm L_2^\pm \underline{R} = \underline{R} L_2^\pm L_1^\pm, \quad L_1^- L_2^+ \underline{R} = \underline{R} L_2^+ L_1^-, \quad \text{and} \quad l_i^{-i} = (l_i^i)^{-1};$$

- which may be written explicitly as:

$$\begin{aligned} l_1^{+1} l_1^{-1} &= l_2^{+2} l_2^{-2} = 1, & l_1^{\pm 1} l_2^{\pm 2} &= l_2^{\pm 2} l_1^{\pm 1}, \\ l_1^{+1} l_2^{+1} &= q^{-1} l_2^{+1} l_1^{+1}, & l_2^{+2} l_2^{+1} &= q l_2^{+1} l_2^{+2}, \\ l_1^{+1} l_1^{-2} &= q l_1^{-2} l_1^{+1}, & l_2^{+2} l_1^{-2} &= q^{-1} l_1^{-2} l_2^{+2}, \\ l_1^{-2} l_2^{+1} - l_2^{+1} l_1^{-2} &= (q - q^{-1})(l_2^{+2} l_1^{-1} - l_1^{+1} l_2^{-2}); \end{aligned}$$

- the Hopf structure given by:

$$S(L^\pm) = (L^\pm)^{-1}, \quad \Delta(l_j^{\pm i}) = \sum_k l_k^{\pm i} \otimes l_j^{\pm k}, \quad \varepsilon(l_j^{\pm i}) = \delta_{ij}.$$

The isomorphism is given by:

$$l_i^{\pm i} = K_i^{\mp 1}, \quad l_2^{+1} = (q - q^{-1})K_1^{-1}E, \quad l_1^{-2} = -(q - q^{-1})FK_1.$$

3.3. The adjoint action and the quantum coordinate algebra $\mathcal{O}_q(GL_2)$. The Hopf algebra U acts on itself by the *adjoint action*, defined as

$$x \triangleright y = x_{(1)}yS(x_{(2)}) \quad \text{for} \quad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

The locally finite subspace with respect to this action is the subalgebra generated by the entries of the matrix $L^+S(L^-)$, and by $(l_1^{+1}l_2^{+2})^{-1}$. This subspace has another interpretation, as the quantum algebra of functions on the group GL_2 (see Proposition 3.6:

Definition 3.3. Let $\mathcal{O}_q^+(GL_2)$ be the algebra with

- generators l_j^i , organized in a matrix

$$L = \begin{bmatrix} \ell_1^1 & \ell_2^1 \\ \ell_1^2 & \ell_2^2 \end{bmatrix};$$

- relations given by the entries of the matrix equation

$$R_{21}L_1RL_2 = L_2R_{21}L_1R,$$

- which may be written explicitly as:

$$\begin{aligned} \ell_2^1 \ell_1^1 &= \ell_1^1 \ell_2^1 + (1 - q^{-2})\ell_2^1 \ell_2^2, & \ell_2^2 \ell_1^1 &= \ell_1^1 \ell_2^2, \\ \ell_1^2 \ell_1^1 &= \ell_1^1 \ell_2^1 - (1 - q^{-2})\ell_2^2 \ell_2^1, & \ell_2^2 \ell_2^1 &= q^2 \ell_2^1 \ell_2^2, \\ \ell_2^2 \ell_1^1 &= \ell_2^1 \ell_2^1 + (1 - q^{-2})(\ell_1^1 \ell_2^1 - \ell_2^2 \ell_2^1), & \ell_2^2 \ell_2^1 &= q^{-2} \ell_2^1 \ell_2^2. \end{aligned} \tag{3.1}$$

Proposition 3.4. *The element $\det_q(L) := \ell_1^1 \ell_2^2 - q^2 \ell_2^1 \ell_1^2$ is central in $\mathcal{O}_q^+(GL_2)$.*

Definition 3.5. Let

$$\mathcal{O}_q(GL_2) = \mathcal{O}_q^+(GL_2)[\det_q(L)^{-1}]$$

denote the algebra obtained from $\mathcal{O}_q^+(GL_2)$ localizing at the central element $\det_q(L)$. We call $\mathcal{O}_q(GL_2)$ the quantum coordinate algebra of GL_2 .¹

Proposition 3.6 ([21]). *There is a unique algebra homomorphism, $\phi: \mathcal{O}_q(GL_2) \hookrightarrow U$, defined on generators by $L \mapsto L^+ S(L^-)$, i.e.*

$$\phi: \ell_j^i \mapsto \sum_k l_k^{+i} S(l_j^{-k}).$$

Moreover, this is an algebra embedding, whose image is contained in the locally finite part of U with respect to the adjoint action.

Remark 3.7. It was proved in [20] (see also [21, 6.2.7]) that the analogous map ϕ for \mathfrak{sl}_2 (more generally, for a simply connected semi-simple group) is an isomorphism onto the locally finite part. A straightforward computation gives that $\phi(\det_q(L)) = (l_1^{+1} l_2^{+2})^2$. Hence, while the image of ϕ does not contain the central element,

$$K_1 K_2 = l_1^{+1} l_2^{+2} = \sqrt{\det_q(L)},$$

comparison with Rosso’s isomorphism for \mathfrak{sl}_2 gives that the locally finite part is generated by the image of ϕ and this central element. The entire quantum group is furthermore generated by adjoining K_1 .

Using this embedding, we can pull back the restriction of the adjoint action of U on the locally finite part of U , and get an action of U on $\mathcal{O}_q(GL_2)$. Unpacking these definitions and identifications, we have the following:

Lemma 3.8. *The embedding $\mathcal{O}_q(GL_2) \hookrightarrow U$ of the quantum coordinate algebra into the quantum enveloping algebra and the resulting restriction of the adjoint action are given on generators by the following formulas.*

- *The embedding:*

$$\begin{aligned} \ell_1^1 &= K_1^{-2} + q^{-1}(q - q^{-1})^2 K_1^{-1} K_2^{-1} E F, & \ell_2^1 &= q^{-1}(q - q^{-1}) K_1^{-1} K_2^{-1} E, \\ \ell_1^2 &= (q - q^{-1}) K_2^{-2} F, & \ell_2^2 &= K_2^{-2}. \end{aligned}$$

- *The action:*

$$\begin{aligned} K_m \triangleright \ell_j^i &= q^{\delta_{im} - \delta_{jm}} \ell_j^i, \\ E \triangleright \ell_j^i &= \delta_{j=1} \ell_2^i - \delta_{i=2} q^{2\delta_{j=2}} \ell_j^1, \\ F \triangleright \ell_j^i &= \delta_{j=2} q^{2\delta_{i=2} - 1} a_1^i - \delta_{i=1} q^{-1} \ell_j^2. \end{aligned}$$

¹In fact there are two candidates for this name, and for the notation $\mathcal{O}_q(\mathcal{G})$, the other being the restricted dual to U_q , also known as the Fadeev–Reshetikhin–Takhtajan (FRT) algebra. We will not make use of the FRT algebra in this paper.

Lemma 3.9. *The center of $\mathcal{O}_q(GL_2)$ is equal to the space $\mathcal{O}_q(GL_2)^U$ of U -invariants. It is the subalgebra generated by the q -trace $\text{tr}_q(L)$, the q -determinant $\det_q(L)$, and its inverse, where the q -trace and the q -determinant are defined as:*

$$\begin{aligned} \text{tr}_q(L) &= \ell_1^1 + q^{-2}\ell_2^2, \\ \det_q(L) &= \ell_1^1\ell_2^2 - q^2\ell_2^1\ell_1^2. \end{aligned} \tag{3.2}$$

Proof. Because $\mathcal{O}_q(GL_2)$ is a flat deformation of $\mathcal{O}(G)$, it follows that the subalgebra of invariants is generated by a unique element in degree one, and two. The formulas here follow by direct computation. \square

Remark 3.10. Up to changes in convention, these formulas first appeared in [22]. For $N > 2$, the description of the center of $\mathcal{O}_q(GL_N)$, and hence $U_q\mathfrak{gl}_N$, in terms of q -deformed minors is taken up in [19].

3.4. The algebra $\mathcal{D}_q(GL_2)$ of quantum differential operators. Next, we recall the definition of the algebra of polynomial quantum differential operators on GL_2 .

All the conventions follow [18], and correspond to the quiver $\bullet \overset{v}{\curvearrowright} \overset{e}{\curvearrowright} \bullet$ in their notation. They first appeared in this formulation in [30].

Definition 3.11. We let $\mathcal{D}_q^+(GL_2)$ denote the algebra with:

- generators a_j^i and $\partial_j^i, i, j = 1, 2$, organized in matrices

$$A = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \partial_1^1 & \partial_2^1 \\ \partial_1^2 & \partial_2^2 \end{bmatrix};$$

- relations given by the entries of the following matrix equations

$$R_{21}A_1RA_2 = A_2R_{21}A_1R, \tag{3.3}$$

$$R_{21}D_1RD_2 = D_2R_{21}D_1R, \tag{3.4}$$

$$R_{21}D_1RA_2 = A_2R_{21}D_1R_{21}^{-1}; \tag{3.5}$$

- which may be written out explicitly as follows:

$$\begin{aligned} a_2^1a_1^1 &= a_1^1a_2^1 + (1 - q^{-2})a_2^1a_2^2 & a_2^2a_1^1 &= a_1^1a_2^2 \\ a_2^1a_1^2 &= a_1^2a_2^1 - (1 - q^{-2})a_2^2a_1^1 & a_2^2a_1^2 &= q^2a_2^1a_2^2 \\ a_1^2a_2^1 &= a_2^1a_1^2 + (1 - q^{-2})(a_1^1a_2^2 - a_2^2a_2^2) & a_2^2a_1^2 &= q^{-2}a_1^2a_2^2 \end{aligned} \tag{3.6}$$

$$\begin{aligned} \partial_2^1\partial_1^1 &= \partial_1^1\partial_2^1 + (1 - q^{-2})\partial_2^1\partial_2^2 & \partial_2^2\partial_1^1 &= \partial_1^1\partial_2^2 \\ \partial_2^2\partial_1^1 &= \partial_1^1\partial_2^2 - (1 - q^{-2})\partial_2^2\partial_2^1 & \partial_2^2\partial_1^2 &= q^2\partial_2^1\partial_2^2 \\ \partial_2^2\partial_1^2 &= \partial_2^1\partial_2^2 + (1 - q^{-2})(\partial_1^1\partial_2^2 - \partial_2^2\partial_2^2) & \partial_2^2\partial_1^2 &= q^{-2}\partial_1^2\partial_2^2 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 \partial_1^1 \cdot a_1^1 &= -(1 - q^{-2}) \cdot \partial_2^1 \cdot a_1^2 + q^{-2} \cdot a_1^1 \cdot \partial_1^1 + (q^{-2} - q^{-4}) \cdot a_2^1 \cdot \partial_1^2 \\
 \partial_1^1 \cdot a_2^1 &= (q^{-2} - 1) \cdot \partial_2^1 \cdot a_2^2 + q^{-2} \cdot a_2^1 \cdot \partial_1^1 \\
 \partial_1^1 \cdot a_1^2 &= (1 - q^2) \cdot \partial_1^2 \cdot a_1^1 - (q - q^{-1})^2 \cdot \partial_2^2 \cdot a_1^2 + a_1^2 \cdot \partial_1^1 + (1 - q^{-2}) \cdot a_2^2 \cdot \partial_1^2 \\
 \partial_1^1 \cdot a_2^2 &= (1 - q^2) \cdot \partial_1^2 \cdot a_2^1 - (q - q^{-1})^2 \cdot \partial_2^2 \cdot a_2^2 + a_2^2 \cdot \partial_1^1 \\
 \partial_2^1 \cdot a_1^1 &= a_1^1 \cdot \partial_2^1 + (1 - q^{-2}) \cdot a_2^1 \cdot \partial_2^2 + (q^{-2} - 1) \cdot a_2^1 \cdot \partial_1^1 \\
 \partial_2^1 \cdot a_2^1 &= q^{-2} \cdot a_2^1 \cdot \partial_2^1 \\
 \partial_2^1 \cdot a_1^2 &= (q^{-2} - 1) \cdot \partial_2^2 \cdot a_1^1 + a_2^2 \cdot \partial_2^1 + (q^{-2} - 1) \cdot a_2^2 \cdot \partial_1^1 + (1 - q^{-2}) \cdot a_2^2 \cdot \partial_2^2 \\
 \partial_2^1 \cdot a_2^2 &= (q^{-2} - 1) \cdot \partial_2^2 \cdot a_2^1 + q^{-2} \cdot a_2^2 \cdot \partial_2^1 \\
 \partial_1^2 \cdot a_1^1 &= q^{-2} \cdot a_1^1 \cdot \partial_1^2 - (1 - q^{-2}) \cdot \partial_2^2 \cdot a_1^2 \\
 \partial_1^2 \cdot a_2^1 &= (q^{-2} - 1) \cdot \partial_2^2 \cdot a_2^2 + a_2^1 \cdot \partial_1^2 \\
 \partial_1^2 \cdot a_1^2 &= a_2^2 \cdot \partial_1^2 \\
 \partial_2^2 \cdot a_1^1 &= a_1^1 \cdot \partial_2^2 + (1 - q^2) \cdot a_2^1 \cdot \partial_1^2 \\
 \partial_2^2 \cdot a_2^1 &= a_2^1 \cdot \partial_2^2 \\
 \partial_2^2 \cdot a_1^2 &= q^{-2} \cdot a_1^2 \cdot \partial_2^2 + (q^{-2} - 1) \cdot a_2^2 \cdot \partial_1^2 \\
 \partial_2^2 \cdot a_2^2 &= q^{-2} \cdot a_2^2 \cdot \partial_2^2.
 \end{aligned} \tag{3.8}$$

Proposition 3.12. (1) *The elements*

$$\det_q(A) := a_1^1 a_2^2 - q^2 a_2^1 a_1^2, \quad \det_q(D) := \partial_1^1 \partial_2^2 - q^2 \partial_2^1 \partial_1^2$$

satisfy:

$$\begin{aligned}
 \det_q(A) a_j^i &= a_j^i \det_q(A), & \det_q(D) a_j^i &= q^{-2} a_j^i \det_q(D), \\
 \partial_j^i \det_q(A) &= q^{-2} \det_q(A) \partial_j^i, & \partial_j^i \det_q(D) &= \det_q(D) \partial_j^i.
 \end{aligned}$$

In particular, the multiplicative set $\{\det_q(A)^m \det_q(D)^n \mid m, n \in \mathbb{N}\}$ satisfies the Ore condition.

(2) *The set of elements*

$$a_{j_1}^{i_1} a_{j_2}^{i_2} \dots a_{j_m}^{i_m} \partial_{l_1}^{k_1} \partial_{l_2}^{k_2} \dots a_{l_n}^{k_n}$$

with $n, m \in \mathbb{N}_0, i_p, j_p, k_p, l_p \in \{1, 2\}$, subject to the conditions

$$\begin{aligned}
 \forall p \quad (i_p < i_{p+1}) \quad \text{or} \quad (i_p = i_{p+1} \text{ and } j_p \leq j_{p+1}), \\
 \forall p \quad (k_p < k_{p+1}) \quad \text{or} \quad (k_p = k_{p+1} \text{ and } l_p \leq l_{p+1})
 \end{aligned}$$

forms a basis of $\mathcal{D}_q^+(GL_2)$.

Proof. Claim (1) is a straightforward computation; claim (2) is Theorem 5.3 in [18]. □

Definition 3.13. We let $\mathcal{D}_q(GL_2)$ denote the algebra,

$$\mathcal{D}_q(GL_2) := \mathcal{D}_q^+(GL_2)[\det_q(A)^{-1}\det_q(D)^{-1}],$$

obtained by localizing $\mathcal{D}_q^+(GL_2)$ at the element $\det_q(A)\det_q(D)$. We call this the algebra of polynomial quantum differential operators on GL_2 .

Remark 3.14. The two evident copies of $\mathcal{O}_q(GL_2)$ inside $\mathcal{D}_q(GL_2)$, embedded by $L \mapsto A$ and $L \mapsto D$, carry an action of U as described in Lemma 3.8. This extends to the action of U on the algebra $\mathcal{D}_q(GL_2)$, a consequence of the construction of $\mathcal{D}_q(GL_2)$ via a U -linear map, see Section 3 in [18].

The algebra $\mathcal{D}_q(GL_2)$ is \mathbb{Z}^2 -graded by

$$\deg(a_j^i) = (1, 0), \quad \deg(\partial_j^i) = (0, 1). \tag{3.9}$$

This grading is inner, in the sense that for any $h \in \mathcal{D}_q(GL_2)$ with $\deg(h) = (M, N)$ we have

$$\det_q(A) \cdot h = q^{2N} h \cdot \det_q(A), \quad \det_q(D) \cdot h = q^{-2M} h \cdot \det_q(D). \tag{3.10}$$

3.5. The quantum Hamiltonian reduction $\mathcal{D}_q(GL_2) //_{\mathfrak{g}_t} U$. In addition to the two embeddings of $\mathcal{O}_q(GL_2)$ into $\mathcal{D}_q(GL_2)$ given by $L \mapsto A$ and $L \mapsto D$, we will use a third, more involved embedding, given by the “quantum moment map”. We will make use of the following quantum cofactor matrices \tilde{D} and \tilde{A} , and the quantum inverse matrices, D^{-1} and A^{-1} :

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} a_2^2 & -q^2 a_2^1 \\ -q^2 a_1^2 & q^2 a_2^2 + (1 - q^2) a_2^2 \end{bmatrix}, & \tilde{D} &= \begin{bmatrix} \partial_2^2 & -q^2 \partial_2^1 \\ -q^2 \partial_1^2 & q^2 \partial_2^2 + (1 - q^2) \partial_2^2 \end{bmatrix}, \\ A^{-1} &= (\det_q(A))^{-1} \tilde{A}, & D^{-1} &= (\det_q(D))^{-1} \tilde{D}. \end{aligned} \tag{3.11}$$

The notation A^{-1} and D^{-1} is justified by the fact that $AA^{-1} = A^{-1}A = I$ and $DD^{-1} = D^{-1}D = I$.

Definition/Proposition 3.15 ([18, Proposition 7.20, Definition 7.24]). We have a homomorphism of algebras,

$$\mu_q: \mathcal{O}_q(GL_2) \rightarrow \mathcal{D}_q(GL_2) \quad \text{defined on generators by } \mu_q(L) = DA^{-1}D^{-1}A.$$

The homomorphism μ_q is called the *quantum moment map*.

For $t \in \mathbb{C}^\times$, let $X_t \in GL_2$ denote the matrix,

$$X_t := \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix},$$

and let O_t denote its conjugacy class in GL_2 . The multiplicative Calogero–Moser variety is the set

$$C_t := \{(A, B) \in GL_2 \times GL_2, \text{ such that } AB^{-1}A^{-1}B \in O_t\}.$$

The GL_2 -varieties O_t and C_t admit canonical equivariant q -deformations, constructed as follows.

Definition 3.16. For $t \in \mathbb{C}^\times$, let $\mathfrak{J}_t \subset \mathcal{O}_q(GL_2)$ denote the two-sided ideal in $\mathcal{O}_q(GL_2)$ generated by the central elements:

$$Z_t := \text{tr}_q(L - q^4 \cdot X_t) = \text{tr}_q(L) - q^4 \cdot (t^{-2} + q^{-2}t^2), \quad \text{and } \det_q(L) - q^8.$$

Consider also the left $\mathcal{D}_q(GL_2)$ ideal $\mathcal{D}_q(GL_2) \cdot \mu_q(\mathfrak{J}_t)$.

Remark 3.17. The two-sided ideal \mathfrak{J}_t and the left ideal $\mathcal{D}_q(GL_2) \cdot \mu_q(\mathfrak{J}_t)$ are ad-equivariant q -deformations of the defining ideals of O_t and C_t , respectively.

We note that $\mu_q(\det_q(L) - q^8) = 0$, so that

$$\mathcal{D}_q(GL_2) \cdot \mu_q(\mathfrak{J}_t) = \mathcal{D}_q(GL_2) \cdot \mu_q(Z_t)$$

is in fact a principal left ideal. Since the principal generator

$$\mu_q(Z_t) = \text{tr}_q(\mu_q(L)) - q^4 \cdot (t^{-2} + q^{-2}t^2)$$

is U -invariant by Lemma 3.9, it follows that $\mathcal{D}_q(GL_2) \cdot \mu_q(Z_t)$ is preserved by U , and the action of U descends to the quotient,

$$M := \mathcal{D}_q(GL_2) / \mathcal{D}_q(GL_2) \cdot \mu_q(Z_t).$$

Definition 3.18. The quantum Hamiltonian reduction of $\mathcal{D}_q(GL_2)$ by the moment map μ_q along the orbit O_t is defined to be the subspace

$$\mathcal{D}_q(GL_2) //_{\mathfrak{J}_t} U := (\mathcal{D}_q(GL_2) / \mathcal{D}_q(GL_2) \cdot \mu_q(Z_t))^U$$

of U -invariants in the quotient of $\mathcal{D}_q(GL_2)$ by the left ideal generated by $\mu_q(\mathfrak{J}_t)$.

Any endomorphism ρ of the cyclic $\mathcal{D}_q(GL_2)$ -module M is uniquely determined by $\rho(1)$, which should be invariant for the U -action, and should have the property that $\mathfrak{J}_t \cdot \rho(1) \subset \mathfrak{J}_t$. The quantum moment map condition, $\mu(x) \cdot y = (x_{(1)} \cdot y)\mu(x_{(2)})$ ensures that the former property implies the latter, as it implies that \mathfrak{J}_t commutes with $\rho(1)$. Hence we have an isomorphism,

$$\mathcal{D}_q(GL_2) //_{\mathfrak{J}_t} U \cong \text{End}(\mathcal{D}_q(GL_2) / \mathcal{D}_q(GL_2) \cdot \mu_q(Z_t)).$$

We therefore regard the quantum Hamiltonian reduction as an algebra, with this multiplication. This algebra structure can be understood more directly, as follows:

Lemma 3.19. *When q is not a nontrivial root of unity, we have a natural isomorphism of algebras:*

$$\mathcal{D}_q(GL_2) \Big/ \Big/_{\mathfrak{J}_t} U \cong \mathcal{D}_q(GL_2)^U \Big/ \mathcal{D}_q(GL_2)^U \cdot \mu_q(Z_t).$$

Proof. Locally finite representations of U are semisimple, so taking invariants is exact and we have

$$\begin{aligned} \mathcal{D}_q(GL_2) \Big/ \Big/_{\mathfrak{J}_t} U &= (\mathcal{D}_q(GL_2) \Big/ \mathcal{D}_q(GL_2) \cdot \mu_q(Z_t))^U \\ &\cong \mathcal{D}_q(GL_2)^U \Big/ (\mathcal{D}_q(GL_2) \cdot \mu_q(Z_t))^U. \end{aligned}$$

As the generator $\mu_q(Z_t)$ of the ideal is U -invariant, we also have

$$(\mathcal{D}_q(GL_2) \cdot \mu_q(Z_t))^U = \mathcal{D}_q(GL_2)^U \cdot \mu_q(Z_t). \quad \square$$

The usefulness of this lemma is that it reduces the problem of finding the presentation of $\mathcal{D}_q(GL_2) \Big/ \Big/_{\mathfrak{J}_t} U$ to the problem of finding the presentation of $\mathcal{D}_q(GL_2)^U$. This is employed in Section 6.

The ideal $\mathcal{D}_q(GL_2) \cdot \mu_q(Z_t)$ is only a one-sided ideal in $\mathcal{D}_q(GL_2)$. However, its sub-space of invariants, $(\mathcal{D}_q(GL_2) \cdot \mu_q(Z_t))^U$, is a two-sided ideal in $\mathcal{D}_q(GL_2)^U$. This equips $\mathcal{D}_q(GL_2) \Big/ \Big/_{\mathfrak{J}_t} U$ with an algebra structure, which coincides with the above description of it as the algebra of endomorphisms of the module M . The following theorem gives a presentation of this algebra by generators and relations, and the corresponding PBW basis for it.

Theorem 3.20. *Assume that $q, t \neq 0$, and that q is not a nontrivial root of unity.*

- (1) *The quantum Hamiltonian reduction $\mathcal{D}_q(GL_2) \Big/ \Big/_{\mathfrak{J}_t} U$ is isomorphic to the algebra with generators $c_1, c_2^{\pm 1}, d_1, d_2^{\pm 1}, r$ and relations:*

$$c_2c_1 = c_1c_2, \quad (3.12) \quad d_2r = q^{-2}rd_2, \quad (3.17)$$

$$d_2d_1 = d_1d_2, \quad (3.13) \quad rc_2 = q^{-2}c_2r, \quad (3.18)$$

$$d_2c_2 = q^{-4}c_2d_2, \quad (3.14) \quad d_1r = q^{-2}rd_1 + (1 - q^{-2})q^{-2}c_1d_2, \quad (3.19)$$

$$d_2c_1 = q^{-2}c_1d_2, \quad (3.15) \quad rc_1 = q^{-2}c_1r + (1 - q^{-2})q^{-2}c_2d_1, \quad (3.20)$$

$$d_1c_2 = q^{-2}c_2d_1, \quad (3.16) \quad d_1c_1 = c_1d_1 + (q^{-2} - 1)r, \quad (3.21)$$

$$r^2 = q^{-4}(1 + t^2)(q^{-2} + t^{-2})c_2d_2 - q^{-4}c_2d_1^2 - q^{-4}c_1^2d_2 + q^{-2}c_1rd_1. \quad (3.22)$$

The isomorphism is given by

$$\begin{aligned} \Psi(c_1) &= \text{tr}_q(A), & \Psi(c_2) &= \det_q(A), & \Psi(r) &= q^2 \text{tr}_q(DA), \\ \Psi(d_1) &= \text{tr}_q(D), & \Psi(d_2) &= \det_q(D). \end{aligned} \quad (3.23)$$

The \mathbb{Z}^2 grading is given by

$$\deg c_i = (i, 0), \quad \deg d_i = (0, i), \quad \deg r = (1, 1),$$

and Ψ is a graded isomorphism.

(2) After the identification given by Ψ , the set

$$\{c_1^{a_1} c_2^{a_2} r^\epsilon d_1^{b_1} d_2^{b_2} \mid \epsilon \in \{0, 1\}, a_1, b_1 \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}\}$$

forms a basis of the algebra $\mathcal{D}_q(GL_2) //_{\mathfrak{g}_t} U$.

Section 6 is dedicated to the proof of this theorem.

4. The quantum Harish-Chandra isomorphism

Theorem 4.1. Assume that $q, t \neq 0, t^2 + 1 \neq 0$, and q is not a nontrivial root of 1. There exists a unique graded isomorphism of algebras,

$$HC_{t,q}: \mathcal{D}_q(GL_2) //_{\mathfrak{g}_t} U \xrightarrow{\cong} e\mathbb{H}_{t,q}e$$

such that:

$$\begin{aligned} HC_{t,q}(d_1) &= P_1, & HC_{t,q}(d_2) &= q^2 P_2, & HC_{t,q}(r) &= R, \\ HC_{t,q}(c_1) &= Q_1, & HC_{t,q}(c_2) &= q^2 Q_2. \end{aligned}$$

Proof. This follows directly by comparing the presentation of $e\mathbb{H}_{t,q}e$ given in Theorem 2.6 and the presentation of $\mathcal{D}_q(GL_2) //_{\mathfrak{g}_t} U$ given in Theorem 3.20. \square

Remark 4.2. We note that the isomorphism $HC_{t,q}$ extends, even for q a root of unity, to give an isomorphism:

$$HC_{t,q}: \mathcal{B} / \mathcal{B} \cdot (w - (t^{-2} + q^{-2}t^2)c_2d_2) \xrightarrow{\cong} e\mathbb{H}_{t,q}e,$$

where \mathcal{B} is an explicitly given algebra with generators and relations (see Prop. 6.1), which recovers the Hamiltonian reduction when q is not a root of unity. In the case the q is a non-trivial root of unity, one should take more care with defining the Hamiltonian reduction, so that the isomorphism will still hold. It should be interesting to compare this directly to [30].

5. Proof of Theorem 2.6

This section contains the proof of Theorem 2.6. The aim is to give a presentation of the spherical DAHA by generators and relations (Theorem 2.6 (1)), and at the same time to find a PBW type basis for it (Theorem 2.6 (2)).

Throughout this section, let \mathcal{A} be the algebra with generators $P_1, P_2^{\pm 1}, Q_1, Q_2^{\pm 1}, R$ and relations (2.11)–(2.21). It is \mathbb{Z}^2 -graded by

$$\begin{aligned} \deg(P_1) &= (0, 1), & \deg(P_2) &= (0, 2), \\ \deg(Q_1) &= (1, 0), & \deg(Q_2) &= (2, 0), & \deg(R) &= (1, 1). \end{aligned}$$

We first describe a PBW basis of this algebra.

Lemma 5.1. *The set*

$$\{Q_1^{a_1} Q_2^{a_2} R^\epsilon P_1^{b_1} P_2^{b_2} \mid \epsilon \in \{0, 1\}, a_1, b_1 \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}\}$$

forms a basis of the algebra \mathcal{A} .

Proof. We will use the Diamond Lemma from [4]. In the language of that paper, the defining relations (2.11)–(2.21) of \mathcal{A} are *straightening* relations, prescribing how to replace (straighten) certain monomials by linear combinations of other, simpler (with respect to some ordering), monomials.

For a monomials ABC such that AB can be straightened and BC can be straightened, we say that *the straightening diamond* holds if the two resulting straightenings of ABC can be further straightened to a common value: in other words, the results of straightening ABC should be independent on the choice to first straighten AB or to first straighten BC . As proved in [4], the necessary and sufficient condition for such a straightening algorithm to be well defined, and for the resulting set of straightened monomials to be a basis, is that the straightening diamonds hold for all monomials ABC .

To apply this theorem to the algebra \mathcal{A} , we must establish the straightening diamonds for all of the following monomials:

$$\begin{aligned} &P_2 P_1 R, P_2 P_1 Q_2, P_2 P_1 Q_1, P_2 R Q_2, P_2 R Q_1, \\ &P_2 Q_2 Q_1, P_1 R Q_2, R^2 Q_2, P_1 Q_2 Q_1, R Q_2 Q_1, \\ &P_2 R^2, P_1 R Q_1, P_1 R^2, R^2 Q_1, R^3. \end{aligned}$$

This is done by direct computation. Because P_2 and Q_2 each q -commute with all other generators, the only non-trivial checks involve the final four monomials. For

illustration, we prove that the straightening diamond for $P_1 R Q_1$ holds. Straightening $P_1 R$ first, we get

$$\begin{aligned} (P_1 R)Q_1 &\stackrel{(2.19)}{=} q^{-2} R P_1 Q_1 + (1 - q^{-2}) Q_1 P_2 Q_1 \\ &\stackrel{(2.16),(2.14)}{=} q^{-2} R Q_1 P_1 + q^{-2}(q^{-2} - 1) R^2 + (1 - q^{-2}) q^{-2} Q_1^2 P_2 \\ &\stackrel{(2.20)}{=} q^{-4} Q_1 R P_1 + q^{-2}(1 - q^{-2}) Q_2 P_1^2 + q^{-2}(q^{-2} - 1) R^2 \\ &\quad + q^{-2}(1 - q^{-2}) Q_1^2 P_2. \end{aligned}$$

Straightening $R Q_1$ first instead, we get

$$\begin{aligned} P_1(R Q_1) &\stackrel{(2.20)}{=} q^{-2} P_1 Q_1 R + (1 - q^{-2}) P_1 Q_2 P_1 \\ &\stackrel{(2.16),(2.15)}{=} q^{-2} Q_1 P_1 R + q^{-2}(q^{-2} - 1) R^2 + q^{-2}(1 - q^{-2}) Q_2 P_1^2 \\ &\stackrel{(2.19)}{=} q^{-4} Q_1 R P_1 + q^{-2}(1 - q^{-2}) Q_1^2 P_2 + q^{-2}(q^{-2} - 1) R^2 \\ &\quad + q^{-2}(1 - q^{-2}) Q_2 P_1^2. \end{aligned}$$

By inspection, these two expressions agree.

This proves the corresponding claim for the subalgebra \mathcal{A}^+ generated by P_1, P_2, Q_1, Q_2, R . The claim for \mathcal{A} follows by localization at $P_2 Q_2$. \square

Next, we show that the defining relations of \mathcal{A} are satisfied in the spherical DAHA.

Lemma 5.2. *The map $\Phi: \mathcal{A} \rightarrow e\mathbb{H}_{t,q}e$ given by (2.22) is a graded algebra homomorphism.*

Proof. The images of P_1, P_2, R, Q_1, Q_2 under the proposed map Φ lie in the spherical subalgebra of $\mathbb{H}_{t,q}$, and the images of P_2 and Q_2 are invertible. It remains to show that $\Phi(P_1), \Phi(P_2), \Phi(Q_1), \Phi(Q_2)$, and $\Phi(R)$ satisfy the relations (2.11)–(2.21). This is a straightforward, if lengthy computation, using relations from Definition 2.1 and Proposition 2.4.

First, notice that in $\Phi(P_1), \Phi(P_2), \Phi(Q_1), \Phi(Q_2), \Phi(R)$, it is enough to write e on the left:

$$\begin{aligned} \Phi(P_1) &= e(X_1 + X_2)e = (X_1 + X_2)e = e(X_1 + X_2), \\ \Phi(Q_1) &= e(Y_1 + Y_2)e = (Y_1 + Y_2)e = e(Y_1 + Y_2), \\ \Phi(P_2) &= e(X_1 X_2)e = (X_1 X_2)e = e(X_1 X_2), \\ \Phi(Q_2) &= e(Y_1 Y_2)e = (Y_1 Y_2)e = e(Y_1 Y_2), \\ \Phi(R) &= e(t^{-2} Y_1 X_1 + Y_2 X_2)e = e(t^{-2} Y_1 X_1 + Y_2 X_2). \end{aligned} \tag{5.1}$$

This simplifies some computations by allowing us to calculate in the subalgebra generated by X_1, X_2, Y_1, Y_2 .

For example, using the relation $X_1 X_2 = X_2 X_1$ from Definition 2.1, we have that

$$\begin{aligned} \Phi(P_2 P_1) &= e(X_1 X_2)e \cdot e(X_1 + X_2)e \\ &= e(X_1 X_2)(X_1 + X_2)e \\ &= e(X_1 + X_2)(X_1 X_2)e \\ &= \Phi(P_1 P_2), \end{aligned}$$

proving (2.11). In a similar way, the relation (2.12) follows from $Y_1 Y_2 = Y_2 Y_1$, and relations (2.13), (2.14), (2.15), (2.17), and (2.18) follow from (2.9). For illustration we include the slightly more involved proof of relation (2.16); relations (2.19), (2.20), and (2.21) are proved analogously.

$$\begin{aligned} \Phi(P_1 Q_1) &= e(X_1 + X_2)e \cdot e(Y_1 + Y_2)e \\ &\stackrel{(5.1)}{=} e(X_1 Y_1 + X_1 Y_2 + X_2 Y_1 + X_2 Y_2)e \\ &\stackrel{(2.3)}{=} e(q^{-2} T^{-2} Y_1 X_1 + Y_2 X_1 + (t - t^{-1}) T^{-1} Y_1 X_1 \\ &\quad + Y_1 X_2 + (t - t^{-1}) q^{-2} T^{-1} Y_1 X_1 + q^{-2} Y_2 X_2 T^{-2})e \\ &\stackrel{(2.2)}{=} e(q^{-2} T^{-2} Y_1 X_1 + Y_2 X_1 + (t - t^{-1}) T^{-1} Y_1 X_1 \\ &\quad + Y_1 X_2 + (t - t^{-1}) q^{-2} T^{-1} Y_1 X_1 + q^{-2} Y_2 X_2 \\ &\quad - (t - t^{-1}) q^{-2} T^{-1} Y_1 X_1)e \\ &\stackrel{(2.10)}{=} e((1 - t^{-2} + q^{-2} t^{-2}) Y_1 X_1 + Y_1 X_2 + Y_2 X_1 + q^{-2} Y_2 X_2)e \\ &= e((Y_1 + Y_2)(X_1 + X_2) + (q^{-2} - 1)(t^{-2} Y_1 X_1 + Y_2 X_2))e \\ &= \Phi(Q_1 P_1 + (q^{-2} - 1)R). \quad \square \end{aligned}$$

This proves that $\Phi(P_1), \Phi(P_2), \Phi(Q_1), \Phi(Q_2), \Phi(R)$ really satisfy the relations (2.11)–(2.21) (i.e. that Φ is a homomorphism). It remains to show that they generate the whole spherical double affine Hecke algebra (i.e. that Φ is surjective), and that the stated relations are exhaustive (i.e. that Φ is injective). We tackle injectivity first.

Lemma 5.3. *The set*

$$\{\Phi(Q_1^{a_1} Q_2^{a_2} R^\epsilon P_1^{b_1} P_2^{b_2}) \mid \epsilon \in \{0, 1\}, a_1, b_1 \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}\}$$

is linearly independent in $e\mathbb{H}_{t,q}e$. The homomorphism Φ is injective.

Proof. We will use the PBW theorem for $\mathbb{H}_{t,q}$ (Proposition 2.4 (2)) to prove that the above set is linearly independent in $e\mathbb{H}_{t,q}e \subseteq \mathbb{H}_{t,q}$.

Assume that for some finite indexing set I , some collection of nonzero scalars $\alpha_i, i \in I$, and some $\epsilon_i \in \{0, 1\}, a_{i,1}, b_{i,1} \in \mathbb{N}_0, a_{i,2}, b_{i,2} \in \mathbb{Z}, i \in I$, we have

$$\sum_{i \in I} \alpha_i \Phi(Q_1^{a_{i,1}} Q_2^{a_{i,2}} R^{\epsilon_i} P_1^{b_{i,1}} P_2^{b_{i,2}}) = 0.$$

Using the definition of the map Φ and the observation (5.1), we can write this as

$$\sum_{i \in I} \alpha_i e (Y_1 + Y_2)^{a_{i,1}} (Y_1 Y_2)^{a_{i,2}} (t^{-2} Y_1 X_1 + Y_2 X_2)^{\epsilon_i} (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}} = 0.$$

Using $e = (1 + tT)/(1 + t^2)$ and the PBW theorem for DAHA, it follows that also

$$\sum_{i \in I} \alpha_i (Y_1 + Y_2)^{a_{i,1}} (Y_1 Y_2)^{a_{i,2}} (t^{-2} Y_1 X_1 + Y_2 X_2)^{\epsilon_i} (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}} = 0. \tag{5.2}$$

The subalgebra generated by $Y_1^{\pm 1}, Y_2^{\pm 1}$ is commutative. After multiplying (5.2) on the left by some power of $Y_1 Y_2$ if necessary, we can assume that $a_{i,2} \geq 0$ for all $i \in I$, and that there exists at least one $i \in I$ for which $a_{i,2} = 0$. Similarly, by multiplying (5.2) on the right by some power of $X_1 X_2$, we can assume that $b_{i,2} \geq 0$ for all $i \in I$.

As explained after Definition 2.1, the algebra $\mathbb{H}_{t,q}$ is \mathbb{Z}^2 graded. We may assume that all terms in the expression (5.2) are of the same bigraded degree (M, N) . This means that for all $i \in I$ we have

$$\begin{aligned} a_{i,1} + 2a_{i,2} + \epsilon_i &= M, \\ b_{i,1} + 2b_{i,2} + \epsilon_i &= N. \end{aligned}$$

The PBW theorem for DAHA (Proposition 2.4 (2)) implies that, as a vector space, $\mathbb{H}_{t,q}$ is also \mathbb{Z}^4 graded by $(\deg(Y_1), \deg(Y_2), \deg(X_1), \deg(X_2))$. The highest power of Y_1 in the term

$$\alpha_i (Y_1 + Y_2)^{a_{i,1}} (Y_1 Y_2)^{a_{i,2}} (t^{-2} Y_1 X_1 + Y_2 X_2)^{\epsilon_i} (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}}$$

of the expression (5.2) is equal to

$$a_{i,1} + a_{i,2} + \epsilon_i = M - a_{i,2}.$$

Using the assumption that there exists an $i \in I$ with $a_{i,2} = 0$ and the PBW theorem for $\mathbb{H}_{t,q}$, we get that the coefficient of Y_1^M in (5.2) is

$$\sum_{\substack{i \in I, \\ a_{i,2}=0, \epsilon_i=0}} \alpha_i (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}} + \sum_{\substack{i \in I, \\ a_{i,2}=0, \epsilon_i=1}} \alpha_i t^{-2} X_1 (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}} = 0, \tag{5.3}$$

while at least one of the coefficients α_i in (5.3) is nonzero. Let

$$f = \sum_{\substack{i \in I, \\ a_{i,2}=0, \epsilon_i=0}} \alpha_i (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}}, \quad g = \sum_{\substack{i \in I, \\ a_{i,2}=0, \epsilon_i=1}} \alpha_i t^{-2} (X_1 + X_2)^{b_{i,1}} (X_1 X_2)^{b_{i,2}}. \tag{5.4}$$

Formula (5.3) states that

$$f + X_1g = 0. \tag{5.5}$$

Polynomials f and g are symmetric in X_1, X_2 , so symmetrizing (5.5) we get that

$$f + \frac{X_1 + X_2}{2}g = 0.$$

Subtracting this from (5.5) we get that

$$(X_1 - X_2)g = 0.$$

This and the PBW basis imply that $g = 0$, and so also $f = 0$.

At least one of the coefficients α_i in at least one of the expressions in (5.4) is nonzero. On the other hand, another use of the PBW theorem for DAHA implies that the set $\{(X_1 + X_2)^{b_{i,1}}(X_1 X_2)^{b_{i,2}}\}$ is linearly independent. This is a contradiction. \square

To show that the injective graded homomorphism $\Phi: \mathcal{A} \rightarrow e\mathbb{H}_{t,q}e$ is surjective, we would like to employ a dimension argument. However, these algebras are infinite dimensional, and bigraded with infinite dimensional bigraded pieces. To work around this problem, we introduce the following auxiliary algebras.

Definition 5.4. Let $\mathbb{H}_{t,q}^+$ denote the subalgebra of $\mathbb{H}_{t,q}$ generated by $T^{\pm 1}, X_1, X_2, Y_1, Y_2$. Let \mathcal{A}^+ denote the subalgebra of \mathcal{A} generated by P_1, P_2, Q_1, Q_2 and R .

These subalgebras are also \mathbb{Z}^2 graded, and have finite dimensional bigraded pieces. The homomorphism Φ restricts to a graded homomorphism $\mathcal{A}^+ \rightarrow e\mathbb{H}_{t,q}^+e$. By the next lemma, we can easily recover information about $\mathbb{H}_{t,q}, e\mathbb{H}_{t,q}e$ and \mathcal{A} from the information about $\mathbb{H}_{t,q}^+, e\mathbb{H}_{t,q}^+e$ and \mathcal{A}^+ .

Lemma 5.5. (1) *The multiplicative set $\{(Y_1 Y_2)^a (X_1 X_2)^b\}$ in $\mathbb{H}_{t,q}$ satisfies the Ore condition, and the multiplicative set $\{Q_2^a P_2^b\}$ in \mathcal{A} satisfies the Ore condition.*

(2) *The algebra $\mathbb{H}_{t,q}$ is the localization of $\mathbb{H}_{t,q}^+$ by $(Y_1 Y_2)^{-1} (X_1 X_2)^{-1}$, and the algebra \mathcal{A} is the localization of \mathcal{A}^+ by $Q_2^{-1} P_2^{-1}$.*

We will first prove that the restriction of the homomorphism Φ to $\mathcal{A}^+ \rightarrow e\mathbb{H}_{t,q}^+e$ is an isomorphism, and then use this lemma to deduce the same about $\Phi: \mathcal{A} \rightarrow e\mathbb{H}_{t,q}e$.

Let \mathbf{A} be any algebra over \mathbb{C} or over $R_{t,q}$ with a \mathbb{Z}^2 grading. If working over the ground ring \mathbb{C} , assume that every bigraded piece $\mathbf{A}[M, N]$ is a finite dimensional \mathbb{C} -vector space with dimension $\dim(\mathbf{A}[M, N])$. If working over the ground ring $R_{t,q}$, assume that every bigraded piece $\mathbf{A}[M, N]$ is a free $R_{t,q}$ -modules of finite rank $\dim(\mathbf{A}[M, N])$. In either case, we record the dimensions of the bigraded pieces as a Hilbert series in two variables given by

$$\text{Hilb}_{\mathbf{A}}(u, v) = \sum_{(M,N)} \dim(\mathbf{A}[M, N]) u^M v^N.$$

Let us compute the Hilbert series of the algebras \mathcal{A}^+ and $e\mathbb{H}_{t,q}^+e$.

Lemma 5.6. *The set*

$$\{Q_1^{a_1} Q_2^{a_2} R^\epsilon P_1^{b_1} P_2^{b_2} \mid \epsilon \in \{0, 1\}, a_i, b_i \in \mathbb{N}_0\}$$

is a basis of the algebra \mathcal{A}^+ . The Hilbert series of this bigraded algebra is given by

$$\text{Hilb}_{\mathcal{A}^+}(u, v) = \frac{1}{1-u} \cdot \frac{1}{1-u^2} \cdot \frac{1}{1-v} \cdot \frac{1}{1-v^2} \cdot (1+uv).$$

Proof. The first assertion follows directly from Lemma 5.1. To calculate the Hilbert series, note that $\deg(Q_1^{a_1} Q_2^{a_2} R^\epsilon P_1^{b_1} P_2^{b_2}) = (a_1 + 2a_2 + \epsilon, b_1 + 2b_2 + \epsilon)$, so

$$\begin{aligned} \text{Hilb}_{\mathcal{A}^+}(u, v) &= \sum_{\substack{a_1, a_2 \geq 0, b_1, b_2 \geq 0 \\ \epsilon = 0, 1}} u^{a_1+2a_2+\epsilon} v^{b_1+2b_2+\epsilon} \\ &= \frac{1}{1-u} \cdot \frac{1}{1-u^2} \cdot \frac{1}{1-v} \cdot \frac{1}{1-v^2} \cdot (1+uv). \quad \square \end{aligned}$$

Lemma 5.7. *The Hilbert series of the bigraded algebra $e\mathbb{H}_{t,q}^+e$ is given by*

$$\text{Hilb}_{e\mathbb{H}_{t,q}^+e}(u, v) = \frac{1}{1-u} \cdot \frac{1}{1-u^2} \cdot \frac{1}{1-v} \cdot \frac{1}{1-v^2} \cdot (1+uv).$$

Proof. Consider $e\mathbb{H}_{t,q}^+$, as a vector space over \mathbb{C} or as a free module over $R_{t,q}$. By Proposition 2.4 (2), a basis of $e\mathbb{H}_{t,q}^+$ is the set $\{eY_1^{a_1} Y_2^{a_2} X_1^{b_1} X_2^{b_2} \mid a_i, b_i \in \mathbb{N}_0\}$. The group S_2 acts on $e\mathbb{H}_{t,q}^+$ by right multiplication, and the space of invariants is precisely the spherical subalgebra $e\mathbb{H}_{t,q}^+e$.

This S_2 action depends polynomially on $t^{\pm 1}$, does not depend on q , and it preserves the finite dimensional bigraded pieces $eH_{t,q}^+[N, M]$. The dimension of the space of invariants in every bigraded piece can be calculated using group characters as

$$\dim(e\mathbb{H}_{t,q}^+[N, M]e) = \frac{1}{2} \cdot (\text{Tr}(\text{id}_{e\mathbb{H}_{t,q}^+[N, M]}) + \text{Tr}(s_{e\mathbb{H}_{t,q}^+[N, M]})).$$

This expression takes values in \mathbb{N}_0 , does not depend on q , and the only term which depends on t is $\text{Tr}(s_{e\mathbb{H}_{t,q}^+[N, M]})$, which is in $\mathbb{C}[t^{\pm 1}][t^2 + 1]^{-1}$. Thus, the above formula gives a polynomial function from $\{t \in \mathbb{C}^\times \mid t^2 \neq -1\}$ to \mathbb{Z} . The only such functions are constants. This proves that $\dim(e\mathbb{H}_{t,q}^+[N, M]e)$ does not depend on t and q .

We will calculate the Hilbert series of $e\mathbb{H}_{t,q}^+e$ by calculating it at $t = 1, q = 1$. The graded action of S_2 on $eH_{1,1}^+$ can be identified with the usual permutation action of S_2 on the space $\mathbb{C}[X_1, X_2, Y_1, Y_2]$. Denoting the trivial character of the group S_2 by χ_+ and the sign character by χ_- , we see that the two dimensional permutation

representation spanned by X_1, X_2 has graded character $(\chi_+ + \chi_-)u$. Consequently, the bigraded characters of $\mathbb{C}[X_1, X_2]$ and $\mathbb{C}[X_1, X_2, Y_1, Y_2]$ can be computed as

$$\begin{aligned} \chi_{\mathbb{C}[X_1, X_2]}(u, v) &= \frac{1}{1 - v\chi_+} \cdot \frac{1}{1 - v\chi_-} \cdot \chi_+, \\ \chi_{\mathbb{C}[X_1, X_2, Y_1, Y_2]}(u, v) &= \frac{1}{1 - u\chi_+} \cdot \frac{1}{1 - u\chi_-} \cdot \frac{1}{1 - v\chi_+} \cdot \frac{1}{1 - v\chi_-} \cdot \chi_+ \\ &= \frac{\chi_+}{1 - u} \cdot \frac{\chi_+ + u\chi_-}{1 - u^2} \cdot \frac{\chi_+}{1 - v} \cdot \frac{\chi_+ + v\chi_-}{1 - v^2} \\ &= \frac{(1 + uv)\chi_+ + (u + v)\chi_-}{(1 - u)(1 - u^2)(1 - v)(1 - v^2)}. \end{aligned}$$

Reading off the χ_+ coefficient, we get that the bigraded Hilbert series of the spherical DAHA equals

$$\begin{aligned} \text{Hilb}_{e\mathbb{H}_{t,q}^+e}(u, v) &= \text{Hilb}_{e\mathbb{H}_{1,1}^+e}(u, v) \\ &= \frac{1}{1 - u} \cdot \frac{1}{1 - u^2} \cdot \frac{1}{1 - v} \cdot \frac{1}{1 - v^2} \cdot (1 + uv). \quad \square \end{aligned}$$

We are now ready to prove the main Theorem of section 2, Theorem 2.6, which describes the spherical DAHA by generators and relations and gives a PBW basis for it.

Proof of Theorem 2.6. Let us prove that the map $\Phi: \mathcal{A} \rightarrow e\mathbb{H}_{t,q}e$, defined on generators by (2.22), is an isomorphism.

It is a \mathbb{Z}^2 graded algebra homomorphism by Lemma 5.2, and injective by Lemma 5.3. Its restriction $\Phi|_{\mathcal{A}^+}: \mathcal{A}^+ \rightarrow e\mathbb{H}_{t,q}^+e$ to the algebra \mathcal{A}^+ from Definition 5.4 is thus also a graded injective homomorphism. By Lemma 5.6 and Lemma 5.7, the algebras \mathcal{A}^+ and $e\mathbb{H}_{t,q}^+e$ have the same Hilbert series, so $\Phi|_{\mathcal{A}^+}$ is an isomorphism. By Lemma 5.5, the homomorphism Φ is a localization of the isomorphism $\Phi|_{\mathcal{A}^+}$, and thus it is also an isomorphism.

In particular, the image under Φ of the basis given in Lemma 5.1 is a basis of the spherical DAHA $e\mathbb{H}_{t,q}e$. □

6. Proof of Theorem 3.20

This section contains the proof of Theorem 3.20, which gives the presentation of the quantum Hamiltonian reduction

$$\begin{aligned} \mathcal{D}_q(GL_2) //_{\mathfrak{sl}_t} U &= (\mathcal{D}_q(GL_2) / \mathcal{D}_q(GL_2) \cdot \mu_q(\mathfrak{sl}_t))^U \\ &\cong \mathcal{D}_q(GL_2)^U / \mathcal{D}_q(GL_2)^U \cdot \mu_q(Z_t) \end{aligned}$$

by generators and relations, and a PBW type basis for it. To this end, we first give a presentation of $\mathcal{D}_q(GL_2)^U$, in Proposition 6.1.

For most of this section we allow q, t formal, or $q, t \in \mathbb{C}^\times$ arbitrary. Lemma 6.5 requires the additional assumption that q is not a nontrivial root of 1. The spirit of the proof is very similar to the proof of Theorem 2.6 given in Section 5.

6.1. A presentation of $\mathcal{D}_q(GL_2)^U$. The main result of this subsection, and the only part of it which is used in the rest of the paper, is the following proposition.

Proposition 6.1. *Assume that q is not a non-trivial root of unity.*

(1) *Let \mathcal{B} be the algebra with generators:*

$$c_1, c_2^{\pm 1}, d_1, d_2^{\pm 1}, r, w$$

and relations:

$$c_2c_1 = c_1c_2, \tag{6.1} \quad d_2r = q^{-2}rd_2, \tag{6.7}$$

$$d_2d_1 = d_1d_2, \tag{6.2} \quad rc_2 = q^{-2}c_2r, \tag{6.8}$$

$$d_2c_2 = q^{-4}c_2d_2, \tag{6.3} \quad d_1r = q^{-2}rd_1 + (1 - q^{-2})q^{-2}c_1d_2, \tag{6.9}$$

$$d_2c_1 = q^{-2}c_1d_2, \tag{6.4} \quad rc_1 = q^{-2}c_1r + (1 - q^{-2}q^{-2})c_2d_1, \tag{6.10}$$

$$d_1c_2 = q^{-2}c_2d_1, \tag{6.5} \quad r^2 = q^{-4}w + (q^{-4} + q^{-6})c_2d_2$$

$$d_1c_1 = c_1d_1 + (q^{-2} - 1)r, \tag{6.6} \quad -q^{-4}c_2d_1^2 - q^{-4}c_1^2d_2 + q^{-2}c_1rd_1, \tag{6.11}$$

$$wr = rw, \tag{6.12}$$

$$wc_1 = q^{-2}c_1w, \tag{6.13}$$

$$wc_2 = q^{-4}c_2w, \tag{6.14}$$

$$wd_1 = q^2d_1w, \tag{6.15}$$

$$wd_2 = q^4d_2w. \tag{6.16}$$

There is an isomorphism of algebras

$$\bar{\Psi}: \mathcal{B} \rightarrow (\mathcal{D}_q(GL_2))^U$$

given on the generators by

$$\begin{aligned} \bar{\Psi}(c_1) &= \text{tr}_q(A), & \bar{\Psi}(c_2) &= \det_q(A), \\ \bar{\Psi}(d_1) &= \text{tr}_q(D), & \bar{\Psi}(d_2) &= \det_q(D), \\ \bar{\Psi}(r) &= q^2 \text{tr}_q(DA), & \bar{\Psi}(w) &= \text{tr}_q(D\tilde{A}\tilde{D}A). \end{aligned} \tag{6.12}$$

(2) *Let \mathcal{B}^+ be the subalgebra of \mathcal{B} generated by c_1, c_2, d_1, d_2, r, w . The isomorphism $\bar{\Psi}$ restricts to the isomorphism of algebras*

$$\bar{\Psi}|_{\mathcal{B}^+}: \mathcal{B}^+ \rightarrow (\mathcal{D}_q^+(GL_2))^U.$$

The algebras \mathcal{B} and \mathcal{B}^+ are \mathbb{Z}^2 -graded by

$$\begin{aligned} \deg(c_1) &= (1, 0), & \deg(d_1) &= (0, 1), & \deg(w) &= (2, 2), \\ \deg(c_2) &= (2, 0), & \deg(d_2) &= (0, 2), & \deg(r) &= (1, 1). \end{aligned} \tag{6.13}$$

The multiplicative set generated by c_2d_2 and w is q -central, and therefore Ore in \mathcal{B}^+ , and \mathcal{B} is a noncommutative localization of \mathcal{B}^+ by this set.

Let us first show that the map $\bar{\Psi}$ is indeed a homomorphism.

Lemma 6.2. *The elements*

$$\text{tr}_q(A), \text{det}_q(A), \text{tr}_q(D), \text{det}_q(D), q^2 \text{tr}_q(DA), \text{tr}_q(D\tilde{A}\tilde{D}A),$$

which appear on the right hand sides of formulas (6.12), are U -invariant and satisfy defining relations (6.1)–(6.16) of the algebra \mathcal{B} . Therefore, (6.12) defines a graded algebra homomorphism $\bar{\Psi}: \mathcal{B} \rightarrow (\mathcal{D}_q(GL_2))^U$.

Proof. The maps $L \mapsto A, L \mapsto D, L \mapsto DA$ and $L \mapsto \mu_q(L)$ are all homomorphisms of U -modules, so the U -invariance of these elements follows from Lemma 3.9.

Seeing that these elements satisfy relations (6.1)–(6.16) is also a direct computation, using the explicit form of defining relations of $\mathcal{D}_q(GL_2)$ given by (3.6), (3.7), and (3.8). For instance, (3.10) immediately implies (6.1), (6.2), (6.3), (6.4), (6.5), (6.7), (6.8), (6.14), (6.16).

For illustration, let us also prove (6.6). We claim that

$$\bar{\Psi}(d_1)\bar{\Psi}(c_1) - \bar{\Psi}(c_1)\bar{\Psi}(d_1) = (q^{-2} - 1)\bar{\Psi}(r).$$

The left hand side is equal to

$$\begin{aligned} &\bar{\Psi}(d_1)\bar{\Psi}(c_1) - \bar{\Psi}(c_1)\bar{\Psi}(d_1) \\ &= \text{tr}_q(D) \text{tr}_q(A) - \text{tr}_q(A) \text{tr}_q(D) \\ &= (\partial_1^1 + q^{-2}\partial_2^2)(a_1^1 + q^{-2}a_2^2) - (a_1^1 + q^{-2}a_2^2)(\partial_1^1 + q^{-2}\partial_2^2) \\ &\stackrel{(3.8)}{=} (q^{-2} - 1)(a_1^1\partial_1^1 + (q^{-2} - 1)a_1^1\partial_2^2 + q^2a_2^1\partial_1^2 + a_1^2\partial_2^1 + (q^{-2} - 1)a_2^2\partial_1^1 \\ &\quad + (1 - q^{-2} + q^{-4})a_2^2\partial_2^2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{\Psi}(r) &= q^2 \text{tr}_q(DA) \\ &= q^2(\partial_1^1a_1^1 + \partial_2^1a_1^2) + (\partial_1^2a_2^1 + \partial_2^2a_2^2) \\ &\stackrel{(3.8)}{=} a_1^1\partial_1^1 + (q^{-2} - 1)a_1^1\partial_2^2 + q^2a_2^1\partial_1^2 + q^2a_2^1\partial_1^2 + a_1^2\partial_2^1 \\ &\quad + (q^{-2} - 1)a_2^2\partial_1^1 + (1 - q^{-2} + q^{-4})a_2^2\partial_2^2. \end{aligned} \tag{6.14}$$

Comparing these two expressions proves (6.6). Relations (6.9), (6.10), (6.11), (6.12), (6.13) and (6.15) are proved by similar direct computations. \square

We now describe a PBW basis of the algebra \mathcal{B} and the Hilbert series of \mathcal{B}^+ .

Lemma 6.3. (1) *The set*

$$\{c_1^{a_1} c_2^{a_2} r^\epsilon d_1^{b_1} d_2^{b_2} w^c \mid a_1, b_1, c \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}, \epsilon \in \{0, 1\}\}$$

is a basis of \mathcal{B} .

(2) *The set*

$$\{c_1^{a_1} c_2^{a_2} r^\epsilon d_1^{b_1} d_2^{b_2} w^c \mid a_1, b_1, a_2, b_2, c \in \mathbb{N}_0, \epsilon \in \{0, 1\}\}$$

is a basis of \mathcal{B}^+ .

(3) *The Hilbert series of the algebra \mathcal{B}^+ is given by*

$$\text{Hilb}_{\mathcal{B}^+}(u, v) = \frac{1}{(1-u)(1-v)(1-u^2)(1-v^2)(1-uv)}.$$

Proof. (1) The first claim follows from the second by localization at c_2 and d_2 .

(2) The second claim is an application of the Diamond Lemma from [4], similar to the proof of Lemma 5.1. Relations (6.1)–(6.11) are straightening relations, describing how to reorder monomials. In order to apply the Diamond Lemma, it remains to establish the straightening diamonds for the following 26 monomials:

$$\begin{aligned} & wd_2d_1, wd_2r, wd_2c_2, wd_2c_1, wd_1r, wd_1c_2, wd_1c_1, wrc_2, wrc_1, wc_2c_1, \\ & d_2d_1r, d_2d_1c_2, d_2d_1c_1, d_2rc_2, d_2rc_1, d_2c_2c_1, d_1rc_2, r^2c_2, d_1c_2c_1, rc_2c_1, \\ & wr^2, d_2r^2, d_1r^2, d_1rc_1, r^2c_1, r^3. \end{aligned}$$

This is a direct calculation, very similar to that in the proof of Lemma 5.1. As there, all but the final four monomials follow immediately from the fact that c_2, d_2 and w q -commute with all generators. For illustration, we include a proof that the straightening diamond for r^2c_1 holds. Straightening r^2 first, we find:

$$\begin{aligned} (r^2)c_1 &\stackrel{(6.11)}{=} (q^{-2}c_1rd_1 - q^{-4}c_1^2d_2 - q^{-4}c_2d_1^2 + q^{-4}w + (q^{-4} + q^{-6})c_2d_2)c_1 \\ &\stackrel{(6.6), (6.13), (6.4)}{=} q^{-2}c_1rc_1d_1 + (q^{-4} - q^{-2})c_1r^2 - q^{-6}c_1^3d_2 - q^{-4}c_2d_1c_1d_1 \\ &\quad - (q^{-6} - q^{-4})c_2d_1r + q^{-6}c_1w + (q^{-6} + q^{-8})c_2c_1d_2 \\ &\stackrel{(6.10), (6.11), (6.6), (6.9)}{=} q^{-4}c_1^2rd_1 + (q^{-4} - q^{-6})c_1c_2d_1^2 \\ &\quad + (q^{-6} - q^{-4})c_1^2rd_1 \\ &\quad - (q^{-8} - q^{-6})c_1^3d_2 - (q^{-8} - q^{-6})c_1c_2d_1^2 + (q^{-8} - q^{-6})c_1w \\ &\quad + (q^{-4} - q^{-2})(q^{-4} + q^{-6})c_1c_2d_2 - q^{-6}c_1^3d_2 - q^{-4}c_1c_2d_1^2 \\ &\quad - (q^{-6} - q^{-4})c_2rd_1 - (q^{-8} - q^{-6})c_2rd_1 \\ &\quad - (q^{-6} - q^{-4})(q^{-2} - q^{-4})c_1c_2d_2 + q^{-6}c_1w + (q^{-6} + q^{-8})c_1c_2d_2 \\ &= q^{-6}c_1^2rd_1 - q^{-8}c_1c_2d_1^2 - q^{-8}c_1^3d_2 + q^{-8}c_1w + (q^{-4} - q^{-8})c_2rd_1 \\ &\quad + (2q^{-10} - q^{-8} + q^{-6})c_1c_2d_2. \end{aligned}$$

Straightening rc_1 first, we find:

$$\begin{aligned}
 r(rc_1) &\stackrel{(6.10)}{=} q^{-2}rc_1r + (q^{-2} - q^{-4})rc_2d_1 \\
 &\stackrel{(6.10),(6.8)}{=} q^{-4}c_1r^2 + (q^{-4} - q^{-6})c_2d_1r + (q^{-4} - q^{-6})c_2rd_1 \\
 &\stackrel{(6.11),(6.9)}{=} q^{-6}c_1^2rd_1 - q^{-8}c_1^3d_2 - q^{-8}c_1c_2d_1^2 + q^{-8}c_1w + (q^{-8} + q^{-10})c_1c_2d_2 \\
 &\quad + (q^{-6} - q^{-8})c_2rd_1 + (q^{-4} - q^{-6})(q^{-2} - q^{-4})c_1c_2d_2 \\
 &\quad + (q^{-4} - q^{-6})c_2rd_1 \\
 &\stackrel{(6.11),(6.9)}{=} q^{-6}c_1^2rd_1 - q^{-8}c_1^3d_2 - q^{-8}c_1c_2d_1^2 + q^{-8}c_1w + (q^{-4} - q^{-8})c_2rd_1 \\
 &\quad + (2q^{-10} - q^{-8} + q^{-6})c_1c_2d_2.
 \end{aligned}$$

By inspection, these two expressions agree. Similar computations show that straightening diamonds hold for the remaining expression; hence, by the Diamond Lemma, the given set of straightened monomials is indeed a PBW basis for \mathcal{B}^+ .

(3) Using (2), we calculate the Hilbert series of \mathcal{B}^+ as

$$\begin{aligned}
 \text{Hilb}_{\mathcal{B}^+}(u, v) &= \sum_{\substack{a_1, b_1, a_2, b_2, c \geq 0 \\ \epsilon \in \{0, 1\}}} u^{a_1+2a_2+\epsilon+2c} v^{b_1+2b_2+\epsilon+2c} \\
 &= \frac{1 + uv}{(1 - u)(1 - u^2)(1 - v)(1 - v^2)(1 - u^2v^2)} \\
 &= \frac{1}{(1 - u)(1 - v)(1 - u^2)(1 - v^2)(1 - uv)}. \quad \square
 \end{aligned}$$

Next, we show that $\overline{\Psi}|_{\mathcal{B}^+}$ is injective.

Lemma 6.4. *The image of the set*

$$\{c_1^{a_1} c_2^{a_2} r^\epsilon d_1^{b_1} d_2^{b_2} w^c \mid a_1, b_1, c \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}, \epsilon \in \{0, 1\}\} \tag{6.15}$$

under the map $\overline{\Psi}$ is linearly independent in $(\mathcal{D}_q(GL_2))^U$. The map $\overline{\Psi}: \mathcal{B} \rightarrow (\mathcal{D}_q(GL_2))^U$ is injective.

Proof. Let us consider the set

$$\{c_1^{a_1} c_2^{a_2} r^d d_1^{b_1} d_2^{b_2} \mid a_1, b_1, d \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}\} \tag{6.16}$$

with a partial order on it given by the degree d of r . Looking at the defining relations (6.1)–(6.11) of the algebra \mathcal{B} , we see that

$$c_1^{a_1} c_2^{a_2} r^\epsilon d_1^{b_1} d_2^{b_2} w^c = q^{-2b_1-4b_2+4c} c_1^{a_1} c_2^{a_2} r^{2c+\epsilon} d_1^{b_1} d_2^{b_2} + \text{lower order terms.}$$

In other words, the set (6.16) is also a basis of \mathcal{B} , and the change of basis matrix between (6.15) and (6.16) (in some order) is upper triangular with powers of q on the

diagonal. So, it is enough to prove that the image under $\bar{\Psi}$ of the set (6.16) is linearly independent in $(\mathcal{D}_q(GL_2))^U$.

Suppose, for the sake of contradiction, that there is a nontrivial linear combination of the images under $\bar{\Psi}$ of some elements of (6.16).

$$\sum_{\substack{a_1, b_1, d \in \mathbb{N}_0, \\ a_2, b_2 \in \mathbb{Z}}} \alpha_{a_1, a_2, d, b_1, b_2} \bar{\Psi}(c_1^{a_1} c_2^{a_2} r^d d_1^{b_1} d_2^{b_2}) = 0. \tag{6.17}$$

By this assumption, the set

$$E = \{ \mathbf{e} = (a_1, a_2, d, b_1, b_2) \mid a_1, b_1, d \in \mathbb{N}_0, a_2, b_2 \in \mathbb{Z}, \alpha_{a_1, a_2, d, b_1, b_2} \neq 0 \}$$

is not empty. After multiplying (6.17) on the left by some power of $\bar{\Psi}(c_2)$ and on the right by some power of $\bar{\Psi}(d_2)$, we may assume that

$$E = E \cap \mathbb{N}_0^5,$$

meaning that all exponents a_2, b_2 are nonnegative integers. Let $p_1, \dots, p_5: E \rightarrow \mathbb{N}_0$ be projections from E to the coordinates of E . We have

$$\sum_{\mathbf{e} \in E} \alpha_{\mathbf{e}} \operatorname{tr}_q(A)^{a_1} \det_q(A)^{a_2} (q^2 \operatorname{tr}_q(DA))^d \operatorname{tr}_q(D)^{b_1} \det_q(D)^{b_2} = 0. \tag{6.18}$$

Let

$$N = \max \{ (p_2 + p_3)(\mathbf{e}) \mid \mathbf{e} \in E \}.$$

The set $(p_2 + p_3)^{-1}(N) \subseteq E$ is nonempty. Putting (6.18) in PBW order using (3.2), (6.14), and (3.6), we read off that the leading nonzero term of a_2^1 is equal to

$$\sum_{\mathbf{e} \in (p_2 + p_3)^{-1}(N)} \alpha_{\mathbf{e}} (a_1^1 + q^{-2} a_2^2)^{a_1} (-q^2)^N (a_2^2)^N (a_1^2)^{a_2} (\partial_1^2)^d \operatorname{tr}_q(D)^{b_1} \det_q(D)^{b_2} = 0.$$

This can be rewritten as

$$\sum_{\mathbf{e} \in (p_2 + p_3)^{-1}(N)} \sum_{i=0}^{a_1} \alpha_{\mathbf{e}} \binom{a_1}{i} q^{-2i+2di} (a_1^1)^{a_1-i} (a_2^1)^N (a_1^2)^{a_2} (a_2^2)^i (\partial_1^2)^d \operatorname{tr}_q(D)^{b_1} \det_q(D)^{b_2} = 0. \tag{6.19}$$

Fix any $(a_1, a_2, d, b_1, b_2) \in (p_2 + p_3)^{-1}(N)$. Then the set

$$p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d)$$

is not empty. Using the PBW theorem for $\mathcal{D}_q(GL_2)$, (Proposition 3.12) and equation (6.19), we see that

$$\sum_{\mathbf{e} \in p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d)} \alpha_{\mathbf{e}} \operatorname{tr}_q(D)^{p_4(\mathbf{e})} \det_q(D)^{p_5(\mathbf{e})} = 0. \tag{6.20}$$

Let

$$M = \max \{p_5(\mathbf{e}) \mid \mathbf{e} \in p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d)\}.$$

The set

$$p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d) \cap p_5^{-1}(M)$$

is nonempty. The leading ∂_2^1 term in (6.20) is then

$$\sum_{\mathbf{e} \in p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d) \cap p_5^{-1}(M)} \alpha_{\mathbf{e}} \operatorname{tr}_q(D)^{p_4(\mathbf{e})} (\partial_2^1)^M (\partial_1^2)^M = 0. \tag{6.21}$$

From this it follows that

$$\alpha_{\mathbf{e}} = 0 \quad \text{for all } \mathbf{e} \in p_1^{-1}(a_1) \cap p_2^{-1}(a_2) \cap p_3^{-1}(d) \cap p_5^{-1}(M) \subseteq E,$$

which contradicts the definition of the set E .

So, the assumption that there exists a nontrivial linear combination of the images under $\bar{\Psi}$ of some elements of (6.16) is wrong. This proves the lemma. \square

We have now shown that the map $\bar{\Psi}$ from the statement of Proposition 6.1 is a graded injective homomorphism of algebras. To see that it is an isomorphism, we employ a dimension argument. The algebras \mathcal{B} and $\mathcal{D}_q(GL_2)$ have infinite dimensional bigraded pieces, so we first deal with their subalgebras \mathcal{B}^+ and $\mathcal{D}_q^+(GL_2)$.

Lemma 6.5. *Assume that q is not a nontrivial root of unity. Then the Hilbert series of the algebra $\mathcal{D}_q^+(GL_2)^U$ is*

$$\operatorname{Hilb}_{\mathcal{D}_q^+(GL_2)^U}(u, v) = \frac{1}{(1-u)(1-v)(1-u^2)(1-v^2)(1-uv)}.$$

Proof. As a bigraded U -module, $\mathcal{D}_q^+(GL_2)$ is isomorphic to $\mathcal{O}_q^+(GL_2) \otimes \mathcal{O}_q^+(GL_2)$. In order to find the invariants, we will decompose the representation $\mathcal{O}_q^+(GL_2)$ of U into irreducible direct summands. The multiplicities of irreducible modules in this decomposition is the same for a variable q , for all $q \in \mathbb{C}^\times$ which are not roots of unity, and for $q = 1$. (For $q \neq 1$ such that $q^n = 1$, there are genuinely more invariants.) In particular,

$$\operatorname{Hilb}_{\mathcal{D}_q^+(GL_2)^U}(u, v) = \operatorname{Hilb}_{\mathcal{D}_1^+(GL_2)^{U(\mathfrak{gl}_2)}}(u, v).$$

We will calculate this Hilbert series at $q = 1$.

At $q = 1$, the algebra $U = U(\mathfrak{gl}_2)$ is the universal enveloping algebra of \mathfrak{gl}_2 . Let V_n denote the irreducible representation of $U(\mathfrak{gl}_2)$ which factors through the quotient $U(\mathfrak{gl}_2) \rightarrow U(\mathfrak{sl}_2)$ and whose highest weight is n . We have $\dim V_n = n + 1$. Then

$$\operatorname{span}\{a_j^i \mid i, j \in \{1, 2\}\} \cong V_0 \oplus V_2$$

as $U(\mathfrak{gl}_2)$ modules. For any $U(\mathfrak{gl}_2)$ module V , let $S(V)$ denote the symmetric algebra on V with the obvious grading. There is an isomorphism of graded $U(\mathfrak{gl}_2)$ modules

$$\mathcal{O}_1^+(GL_2) = \mathbb{C}[a_j^i \mid i, j \in \{1, 2\}] \cong S(V_0 \oplus V_2) \cong S(V_0) \otimes S(V_2). \tag{6.22}$$

Let us record the decomposition of graded pieces into irreducible direct summands as a one-variable Grothendieck group expression. In that language,

$$[S(V_0)] = \sum_{k \geq 0} [V_0] u^k. \tag{6.23}$$

Using weights, we decompose the symmetric algebra on V_2 as a direct sum of irreducible representations, and similarly get that

$$[S(V_2)] = \sum_{n \geq 0} [\oplus_{i \geq 0} V_{2n-4i}] u^n = \sum_{n \geq 0} \sum_{i \geq 0} [V_{2n-4i}] u^n. \tag{6.24}$$

Combining (6.22), (6.23) and (6.24), we get

$$[\mathcal{O}_1^+(GL_2)] = \left(\sum_{k \geq 0} [V_0] u^k \right) \left(\sum_{n \geq 0} \sum_{i \geq 0} [V_{2n-4i}] u^n \right) = \sum_{k \geq 0} \sum_{n \geq 0} \sum_{i \geq 0} [V_{2n-4i}] u^{n+k}. \tag{6.25}$$

We are interested in the multiplicity of $[V_0]$ in $\mathcal{O}_1^+(GL_2) \otimes \mathcal{O}_1^+(GL_2)$. For that purpose, recall that

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|m-n|}.$$

The trivial representation V_0 appears as a summand in the decomposition of $V_m \otimes V_n$ if and only if $m = n$, and in that case it appears with multiplicity 1. So,

$$[\mathcal{O}_1^+(GL_2) \otimes \mathcal{O}_1^+(GL_2)] = \sum_{k \geq 0} \sum_{n \geq 0} \sum_{i \geq 0} [V_{2n-4i}] u^{n+k} \cdot \sum_{l \geq 0} \sum_{m \geq 0} \sum_{j \geq 0} [V_{2m-4j}] v^{m+l}$$

and

$$\begin{aligned} & [(\mathcal{O}_1^+(GL_2) \otimes \mathcal{O}_1^+(GL_2))^{U(\mathfrak{gl}_2)}] \\ &= \sum_{\substack{k \geq 0, n \geq 0, \\ 0 \leq i \leq n/2}} \sum_{\substack{l \geq 0, m \geq 0, \\ 0 \leq j \leq m/2}} [(V_{2n-4i} \otimes V_{2m-4j})^{U(\mathfrak{gl}_2)}] u^{n+k} v^{m+l} \\ &= [V_0] \sum_{\substack{k \geq 0, n \geq 0, \\ 0 \leq i \leq n/2}} \sum_{\substack{l \geq 0, m \geq 0, \\ 0 \leq j \leq m/2}} \delta_{2n-4i=2m-4j} u^{n+k} v^{m+l}. \end{aligned}$$

The Hilbert series of the space of invariants is thus

$$\begin{aligned}
 \text{Hilb}_{\mathcal{D}_1^+(GL_2)^{U(\mathfrak{gl}_2)}}(u, v) &= \sum_{\substack{k \geq 0, n \geq 0, \\ 0 \leq i \leq n/2}} \sum_{\substack{l \geq 0, m \geq 0, \\ 0 \leq j \leq m/2}} \delta_{2n-4i=2m-4j} u^{n+k} v^{m+l} \\
 &= \frac{1}{(1-u)(1-v)} \sum_{i \geq 0} \sum_{j \geq 0} \sum_{n \geq 2i} \sum_{m \geq 2j} \delta_{2n-4i=2m-4j} u^n v^m \\
 &= \frac{1}{(1-u)(1-v)} \sum_{i \geq 0} \sum_{j \geq 0} \sum_{n \geq 2i} u^n v^{n-2i+2j} \\
 &= \frac{1}{(1-u)(1-v)} \sum_{i \geq 0} \sum_{j \geq 0} \sum_{n \geq 0} u^{n+2i} v^{n+2j} \\
 &= \frac{1}{(1-u)(1-v)(1-u^2)(1-v^2)(1-uv)}. \quad \square
 \end{aligned}$$

We are now ready to prove the main statement of this subsection.

Proof of Proposition 6.1. Assume that $q, t \neq 0$, and that q is not a nontrivial root of unity.

The map $\bar{\Psi}$, defined on generators of \mathcal{B} in (6.12) extends uniquely to a graded homomorphism of algebras $\bar{\Psi}: \mathcal{B} \rightarrow (\mathcal{D}_q(GL_2))^U$ by Lemma 6.2. By Lemma 6.3 and Lemma 6.4, the image of a basis in \mathcal{B} is linearly independent in $(\mathcal{D}_q(GL_2))^U$, so the map $\bar{\Psi}$ is injective. The restriction of $\bar{\Psi}$ to \mathcal{B}^+ is thus also a graded injective homomorphism of algebras $\bar{\Psi}|_{\mathcal{B}^+}: \mathcal{B}^+ \rightarrow (\mathcal{D}_q^+(GL_2))^U$. The Hilbert series of \mathcal{B}^+ and $(\mathcal{D}_q^+(GL_2))^U$ given in Lemmas 6.3 and 6.5 coincide, so $\bar{\Psi}|_{\mathcal{B}^+} \rightarrow (\mathcal{D}_q^+(GL_2))^U$ is an isomorphism. The algebra \mathcal{B} is a localization of \mathcal{B}^+ by the Ore set generated by c_2d_2 , the algebra $(\mathcal{D}_q(GL_2))^U$ is a localization of $(\mathcal{D}_q^+(GL_2))^U$ by the Ore set generated by $\det_q(A)\det_q(D) = \bar{\Psi}(c_2d_2)$, and the map $\bar{\Psi}$ is a localization of the map $\bar{\Psi}|_{\mathcal{B}^+}$. So, $\bar{\Psi}: \mathcal{B} \rightarrow (\mathcal{D}_q(GL_2))^U$ is an isomorphism of algebras. \square

6.2. A presentation of $\mathcal{D}_q(GL_2) //_{\mathfrak{J}_t} U$. We now combine Lemma 3.19 and Proposition 6.1, to prove Theorem 3.20.

Proof of Theorem 3.20. By Lemma 3.19 and Proposition 6.1, we have

$$\mathcal{D}_q(GL_2) //_{\mathfrak{J}_t} U \cong \mathcal{D}_q(GL_2)^U / \mathcal{D}_q(GL_2)^U \cdot \mu_q(Z_t) \cong \mathcal{B} / \mathcal{B} \cdot \bar{\Psi}^{-1}(\mu_q(Z_t))$$

Let us first calculate the generator $\bar{\Psi}^{-1}(\mu_q(Z_t))$ of this principal \mathcal{B} ideal.

$$\begin{aligned}
 \bar{\Psi}^{-1}(\mu_q(Z_t)) &\stackrel{\text{Def. 3.16}}{=} \bar{\Psi}^{-1}(\mu_q(\text{tr}_q(L - q^4 X_t))) \\
 &\stackrel{\text{Def. 3.15}}{=} \bar{\Psi}^{-1}(\text{tr}_q(DA^{-1}D^{-1}A - q^4 X_t))
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(3.11)}{=} \bar{\Psi}^{-1}(\text{tr}_q(D \det_q(A)^{-1} \tilde{A} \det_q(D)^{-1} \tilde{D} A - q^4 X_t)) \\
 &= \bar{\Psi}^{-1}(\det_q(A)^{-1} \det_q(D)^{-1} \text{tr}_q(D \tilde{A} \tilde{D} A) - q^4 \text{tr}_q(X_t)) \\
 &\stackrel{(6.12)}{=} c_2^{-1} d_2^{-1} w - q^4(t^{-2} + q^{-2}t^2) \\
 &= c_2^{-1} d_2^{-1} (w - q^4(t^{-2} + q^{-2}t^2) d_2 c_2) \\
 &\stackrel{(6.3)}{=} c_2^{-1} d_2^{-1} (w - (t^{-2} + q^{-2}t^2) c_2 d_2).
 \end{aligned}$$

Thus $\mathcal{B} \cdot \bar{\Psi}^{-1}(\mu_q(Z_t))$ is the principal left ideal in \mathcal{B} generated by the element

$$c_2^{-1} d_2^{-1} (w - (t^{-2} + q^{-2}t^2) c_2 d_2),$$

or equivalently, by the element

$$w - (t^{-2} + q^{-2}t^2) c_2 d_2.$$

A presentation of the quotient,

$$\mathcal{B} / \mathcal{B} \cdot (w - (t^{-2} + q^{-2}t^2) c_2 d_2)$$

can be deduced from the presentation of the algebra \mathcal{B} given in Proposition 6.1, as follows:

- Generators are: $c_1, c_2^{\pm 1}, d_1, d_2^{\pm 1}, r$.
- Relations are (6.1)–(6.10), and an extra relation obtained from (6.11) as

$$\begin{aligned}
 r^2 &= q^{-4} w + (q^{-4} + q^{-6}) c_2 d_2 - q^{-4} c_2 d_1^2 - q^{-4} c_1^2 d_2 + q^{-2} c_1 r d_1 \\
 &= q^{-4} (t^{-2} + q^{-2} t^2) c_2 d_2 + (q^{-4} + q^{-6}) c_2 d_2 \\
 &\quad - q^{-4} c_2 d_1^2 - q^{-4} c_1^2 d_2 + q^{-2} c_1 r d_1 \\
 &= q^{-4} (1 + t^2) (q^{-2} + t^{-2}) c_2 d_2 - q^{-4} c_2 d_1^2 - q^{-4} c_1^2 d_2 + q^{-2} c_1 r d_1. \tag{6.26}
 \end{aligned}$$

This proves that $\mathcal{D}_q(GL_2) //_{\mathfrak{J}_t} U$ is isomorphic to the algebra

$$\mathcal{B} / \mathcal{B} \cdot (w - (t^{-2} + q^{-2}t^2) c_2 d_2),$$

whose presentation is stated in Theorem 3.20, with the isomorphism Ψ induced by $\bar{\Psi}$. □

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