

Calabi–Yau deformations and negative cyclic homology

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Abstract. In this paper we relate the deformation theory of Ginzburg Calabi–Yau algebras to negative cyclic homology. We do this by exhibiting a DG-Lie algebra that controls this deformation theory and whose homology is negative cyclic homology. We show that the bracket induced on negative cyclic homology coincides with Menichi’s string topology bracket. We show in addition that the obstructions against deforming Calabi–Yau algebras are annihilated by the map to periodic cyclic homology. In the commutative we show that our DG-Lie algebra is homotopy equivalent to $(T^{\text{poly}}[[u]], -u \text{ div})$.

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1. Introduction

Throughout k is a field of characteristic zero. In this paper we discuss the deformation theory of Calabi–Yau k -algebras in the sense of Ginzburg [13]. Recall that a k -algebra A is d -Calabi–Yau if it is perfect as A^e -module and there is an isomorphism¹ in $D(A^e)$

$$\eta: \text{RHom}_{A^e}(A, A^e) \longrightarrow \Sigma^{-d} A. \quad (1.1)$$

In the rest of this introduction we fix a d -CY algebra A . Here and throughout the paper we take the point of view that η is part of the structure of A .

The definition of a d -Calabi–Yau algebra can be “relativized” without any difficulty. Hence there is an associated deformation theory. Our first result in this paper is the construction of a DG-Lie algebra which controls this deformation theory.

To be more precise: let Nilp be the category of commutative, finite dimensional, local k -algebras (R, m) such that $R/m = k$. For $R \in \text{Nilp}$ let $\text{Def}_{A, \eta}(R)$ be the

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¹This isomorphism is sometimes required to satisfy a certain symmetry condition but this happens to be automatically satisfied. See [29, App. C].

category of R -algebras B which are d -Calabi–Yau (with respect to R) and which are in addition equipped with an isomorphism $B \otimes_R k \cong A$ respecting η . We view $\text{Def}_{A,\eta}$ as a pseudo-functor from Nilp to the category of groupoids Gd .

For a nilpotent DG-Lie algebra \mathfrak{h} let $\text{MC}(\mathfrak{h})$ be the groupoid of solutions to the Maurer–Cartan equation in \mathfrak{h} (see §5). For an arbitrary DG-Lie algebra \mathfrak{g} we have an associated “deformation functor”

$$\mathcal{MC}(\mathfrak{g}): \text{Nilp} \longrightarrow \text{Gd}: (R, m) \mapsto \text{MC}(\mathfrak{g} \otimes_k m).$$

In this paper we introduce a DG-Lie algebra $\mathfrak{D}^\bullet(A, \eta)$ (see §6) which controls the deformation theory of (A, η) .²

Theorem 1.1 (A combination of Prop. 4.2 and Thm. 6.1 below). *There is a morphism of pseudo-functors $\pi: \mathcal{MC}(\mathfrak{D}^\bullet(A, \eta)) \longrightarrow \text{Def}_{A,\eta}$ which when evaluated on an arbitrary $R \in \text{Nilp}$ is essentially surjective on objects and surjective on morphisms.*

We obtain in particular for $R \in \text{Nilp}$ a bijection between $\text{MC}(\mathfrak{D}^\bullet(A) \otimes_k m)/\cong$ and $\text{Def}_{A,\eta}(R)/\cong$. In this sense the deformation theory of A is controlled by the DG-Lie algebra $\mathfrak{D}^\bullet(A, \eta)$.

$\mathfrak{D}^\bullet(A, \eta)$ is constructed as a twisted semi-direct product of the Hochschild cochain complex with the negative cyclic chain complex of A . So the construction is similar in spirit to [26] which treats finite dimensional A_∞ -algebras with a non-degenerate inner product. However our algebras are not finite dimensional and they do not carry an inner product.

The construction of $\mathfrak{D}^\bullet(A, \eta)$ yields a morphism

$$\phi: \mathfrak{D}^\bullet(A, \eta) \longrightarrow \bar{\mathfrak{C}}^\bullet(A),$$

where $\bar{\mathfrak{C}}^\bullet(A)$ is the (shifted) Hochschild cochain complex of A . As is well known, $\bar{\mathfrak{C}}^\bullet(A)$ controls the deformation theory of A as algebra. The morphism ϕ corresponds to “forgetting η ” as is explained in §7.

The next result is the construction of an explicit quasi-isomorphism of complexes

$$\mathfrak{D}^\bullet(A, \eta) \xrightarrow{\cong} \Sigma^{-d+1} \text{CC}_\bullet^-(A) \tag{1.2}$$

between $\mathfrak{D}^\bullet(A, \eta)$ and the shifted negative cyclic complex $\text{CC}_\bullet^-(A)$. As a result we obtain the following information about the deformation theory of Calabi–Yau algebras.

Theorem 1.2. (1) *The tangent space to the deformation space of a d -Calabi–Yau algebra is $\text{HC}_{d-2}^-(A)$.*

²In fact this is slightly imprecise as $\mathfrak{D}^\bullet(A, \eta)$ is only determined up to a non-unique isomorphism. The actual definition of $\mathfrak{D}^\bullet(A, \eta)$ depends on the lift of η to an explicit cycle in a suitable complex but we will ignore this subtlety in the introduction.

(2) *The obstructions against deforming a d -Calabi–Yau algebra are in*

$$\ker \left(\mathrm{HC}_{d-3}^-(A) \rightarrow \mathrm{HC}_{d-3}^{\mathrm{per}}(A) \right).$$

The first statement is a formal consequence of (1.2). The second statement is Theorem 10.1 below. It follows in particular that if $\mathrm{HC}_{d-3}^-(A) \rightarrow \mathrm{HC}_{d-3}^{\mathrm{per}}(A)$ is injective then the deformation theory of A as Calabi–Yau algebra is unobstructed. This happens for example if $d \leq 3$ (see Corollary 10.5 and Lemma 10.6 below).

Our next result is the description of the Lie bracket on $\mathrm{HC}_\bullet(A)$ induced by (1.2):

Theorem 1.3 (Theorem 8.2 below). *The Lie bracket on negative cyclic homology induced by (1.2) is the “string topology” Lie bracket introduced in [20] by Menichi.*

Let us now specialize to the case where A is commutative. Let $T^{\mathrm{poly},\bullet}(A)$ be the Lie algebra of poly-vector fields on A . Then η in (1.1) may be interpreted as a volume form (see §9 below). Let div be the divergence operator on $T^{\mathrm{poly},\bullet}(A)$ associated to η . Using Willwacher’s “formality for cyclic chains” [32] (see also [7, 23, 28]) we show that there is an isomorphism

$$(T^{\mathrm{poly},\bullet}[[u]], -u \mathrm{div}) \xrightarrow{\cong} \mathfrak{D}^\bullet(A, \eta) \quad (|u| = 2)$$

in the homotopy category of DG-Lie algebras which fits in a commutative diagram

$$\begin{array}{ccc} (T^{\mathrm{poly},\bullet}(A)[[u]], -u \mathrm{div}) & \xrightarrow{\cong} & \mathfrak{D}^\bullet(A) \\ \downarrow u \mapsto 0 & & \downarrow \phi \\ T^{\mathrm{poly},\bullet}(A) & \xrightarrow{\cong} & \bar{\mathfrak{C}}^\bullet(A) \end{array} \tag{1.3}$$

The lower arrow is a globalized version of Kontsevich’s formality quasi-isomorphism [16]. This diagram gives a conceptual explanation of Dolgushev’s result [8] that the Kontsevich \ast -product associated to a divergence free Poisson bracket is Calabi–Yau.³

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Notation and conventions. All rings and ring homomorphisms are unital. We mix homological and cohomological indices, using the convention $X_i = X^{-i}$.

2. Preliminaries on the Hochschild and cyclic complexes

In this section remind the reader about the basic operations on the Hochschild and cyclic complexes. The reason for putting this section first is that it also allows us to introduce some notation. Readers vaguely familiar with the material may safely skip to the next section.

³This is a special case Dolgushev’s result. Dolgushev does not restrict to the Calabi–Yau case.

2.1. Notation. Let R be a commutative ring and assume that B is an R -algebra. Let $C_\bullet(B)$ and $C^\bullet(B)$ denote the usual relative Hochschild chain and cochain complexes of B/R . Thus

$$C^\bullet(B) = \bigoplus_n \text{Hom}_R(\Sigma B^{\otimes n}, B)$$

$$C_\bullet(B) = \bigoplus_n B \otimes (\Sigma B)^{\otimes n}$$

where, here and below, all unadorned tensor products are over R . We also use

$$\mathcal{C}^\bullet(B) = \Sigma C^\bullet(B)$$

$$= \bigoplus_n \text{Hom}_R(\Sigma B^{\otimes n}, \Sigma B).$$

We will also consider the *normalized versions* of these objects:

$$\bar{C}^\bullet(B) = \bigoplus_n \text{Hom}_R(\Sigma(B/R)^{\otimes n}, B),$$

$$\bar{C}_\bullet(B) = \bigoplus_n B \otimes \Sigma(B/R)^{\otimes n},$$

and a similar definition for $\bar{\mathcal{C}}^\bullet(B)$. It is well known that the obvious maps

$$\bar{C}^\bullet(B) \longrightarrow C^\bullet(B) \quad \text{and} \quad C_\bullet(B) \longrightarrow \bar{C}_\bullet(B)$$

are quasi-isomorphisms [31, Thm. 8.3.8, Lem. 8.3.7].

If $x \in \mathcal{C}^n(B)$ then we write $|x| = n - 1$. Thus $|x|$ refers to the cohomological degree of x .

2.2. The Hochschild cochain complex. The standard algebraic structures on the Hochschild cochain complex can all be deduced from its structure as a *brace algebra* [11]. Recall that the braces are maps

$$\mathcal{C}^\bullet(B) \otimes \cdots \otimes \mathcal{C}^\bullet(B) \longrightarrow \mathcal{C}^\bullet(B): x \otimes x_1 \otimes \cdots \otimes x_m \mapsto x\{x_1, \dots, x_m\}$$

defined by

$$x\{x_1, \dots, x_m\}(b_1, \dots, b_n)$$

$$= \sum_{0 \leq i_1 \leq \dots \leq i_m \leq n} (-1)^\epsilon x(b_1, \dots, b_{i_1}, x_1(b_{i_1+1}, \dots, b_{i_1+|x_1|+1}), \dots,$$

$$\dots, b_{i_m}, x_m(b_{i_m+1}, \dots, b_{i_m+|x_m|+1}), \dots, b_n),$$

where $\epsilon = \sum_1^m |x_k| i_k$. The corresponding *Gerstenhaber Lie bracket* on $\mathcal{C}^\bullet(B)$ is

$$[x, y] = x\{y\} - (-1)^{|x||y|} y\{x\}.$$

Let $\mu \in \mathfrak{C}^1(B) = \text{Hom}(\Sigma B \otimes \Sigma B, \Sigma B)$ denote the “inverse” multiplication $\mu(b_1, b_2) = -b_1 b_2$. Then $[\mu, \mu] = 0$ and hence

$$dx = [\mu, x] \quad (2.1)$$

defines a differential of degree one on $\mathfrak{C}^\bullet(B)$.

The *cupproduct* on $\mathfrak{C}^\bullet(B)$ is defined by

$$x \cup y = (-1)^{|x|} \mu\{x, y\}.$$

This is an associative product of degree one on $\mathfrak{C}^\bullet(B)$, or equivalently an associative product of degree zero on $C^\bullet(B)$. One has [11]

- (1) $(\mathfrak{C}^\bullet(B), d, [,])$ is a DG-Lie algebra;
- (2) $(C^\bullet(B), d, \cup)$ is a DG-algebra, commutative up to homotopy;
- (3) More generally: $(C^\bullet(B), d, [,], \cup)$ is a so-called DG-“Gerstenhaber algebra” up to homotopy.

2.3. The Hochschild chain complex. When combining the Hochschild cochain complex with the Hochschild chain complex things becomes much more intricate [2, 24]. We will content ourselves by giving formulas for the basic operations and stating some relations between them. We refer to [24] for more details.

The first basic operation is the *contraction*

$$i_x(b_0 \otimes \cdots \otimes b_n) := b_0 x(b_1, \dots, b_d) \otimes b_{d+1} \otimes \cdots \otimes b_n$$

for $x \in \mathfrak{C}^\bullet(B)$ and $b_0 \otimes \cdots \otimes b_n \in C_\bullet(B)$. This is an action of $\mathfrak{C}^\bullet(B)$ on $C_\bullet(B)$ satisfying $|i_x| = |x| + 1$ and

$$i_x i_y = (-1)^{(|x|+1)(|y|+1)} i_{y \cup x}. \quad (2.2)$$

The contraction is often written as a *cappproduct*: $i_x(-) = x \cap -$.

The second basic operation is the *Lie derivative*

$$\begin{aligned} L_x(b_0 \otimes \cdots \otimes b_n) &:= \sum_{i=0}^{n-|x|-1} (-1)^{|x|i} b_0 \otimes \cdots \otimes b_i \otimes x(b_{i+1}, \dots, b_{i+|x|+1}) \otimes \cdots \otimes b_n \\ &+ \sum_{i=n-|x|}^n (-1)^{n(i+1)+|x|} x(b_{i+1}, \dots, b_n, b_0, \dots, b_{|x|-n+i}) \otimes \cdots \otimes b_i. \end{aligned}$$

The Lie derivative defines a graded Lie action of $\mathfrak{C}^\bullet(B)$ on $C_\bullet(B)$. Explicitly: $|L_x| = |x|$ and

$$[L_x, L_y] = L_{[x, y]}. \quad (2.3)$$

The *Hochschild differential* on $\mathfrak{C}_\bullet(B)$ is defined as $\mathfrak{b} = L_\mu$. From (2.1) and (2.3) one obtains

$$[\mathfrak{b}, L_x] = L_{dx}. \tag{2.4}$$

Hence $(\mathfrak{C}_\bullet(B), \mathfrak{b})$ is a DG-Lie module over $\mathfrak{C}^\bullet(B)$. One also has compatibility with the contraction:

$$[\mathfrak{b}, i_x] + i_{dx} = 0. \tag{2.5}$$

The last basic operation we need is the *Connes differential*.

$$\mathfrak{B}: \mathfrak{C}_\bullet(B) \longrightarrow \mathfrak{C}_\bullet(B)$$

with formula

$$\begin{aligned} \mathfrak{B}(b_0 \otimes \cdots \otimes b_n) &= \sum_{i=0}^n (-1)^{ni} 1 \otimes b_i \otimes \cdots \otimes b_n \otimes b_0 \otimes \cdots \otimes b_{i-1} \\ &\quad + \sum_0^n (-1)^{n(i+1)} b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_n \otimes b_0 \otimes \cdots \otimes b_{i-2}. \end{aligned}$$

It is well known that $|\mathfrak{B}| = -1$, $\mathfrak{b}\mathfrak{B} + \mathfrak{B}\mathfrak{b} = 0$, $\mathfrak{B}^2 = 0$.

Some of the following identities hold only for normalized chains/cochains. Note that if $x \in \bar{\mathfrak{C}}^\bullet(B)$ then i_x, L_x are well-defined operations on $\bar{\mathfrak{C}}_\bullet(B)$.

Lemma 2.1. *Assume $x \in \bar{\mathfrak{C}}^\bullet(B)$. Then on $\mathfrak{C}_\bullet(B)$ we have*

$$[\mathfrak{B}, L_x] = 0. \tag{2.6}$$

The formula (2.6) does not hold for unnormalized cochains.

2.4. The negative cyclic complex. Let u be a variable of degree two and put

$$\overline{\mathfrak{CC}}_\bullet^-(B) = \bar{\mathfrak{C}}_\bullet(B)[[u]].$$

Equipped with the *cyclic differential* $\mathfrak{b} + u\mathfrak{B}$, this is the *normalized negative cyclic complex*. In the sequel operations on $\bar{\mathfrak{C}}_\bullet(B)$ will be (tacitly) extended to $\overline{\mathfrak{CC}}_\bullet^-(B)$ by making them u -linear. This applies in particular to i_x and L_x . Combining (2.4) and (2.6) we obtain

$$[\mathfrak{b} + u\mathfrak{B}, L_x] = L_{dx}. \tag{2.7}$$

The compatibility of i_x with the cyclic differential is more subtle. In [24] (see also [12]) Tamarkin and Tsygan define for $x \in \bar{\mathfrak{C}}^\bullet(B)$ a graded endomorphism S_x of $\bar{\mathfrak{C}}_\bullet(B)$ (depending linearly on x) such that $|S_x| = |x| - 1$ and such that the following identity holds

$$[\mathfrak{b} + u\mathfrak{B}, i_x + uS_x] + i_{dx} + uS_{dx} = uL_x \tag{2.8}$$

on $\overline{\mathfrak{CC}}_\bullet^-(B)$. This identity will be important for us in the sequel. Note that it implies (2.7).

The following is a special case of [24, Prop. 3.3.4].

Lemma 2.2. *Let $x, y \in \bar{\mathcal{C}}^\bullet(B)$ be such that $dx = dy = 0$. Then $[L_x, i_y + uS_y]$ is homotopic to $(-1)^{|x|}(i_{[x,y]} + uS_{[x,y]})$.*

2.5. A comment on base change. If A is a k -algebra and R is a commutative k -algebra then for $B = A \otimes_k R$ it is clear that $C_{R,\bullet}(B) \cong C_\bullet(A) \otimes_k R$ (where contrary to our usual conventions we have now made the base ring explicit in the notation). Since the negative cyclic complex involves a product this is not true for $CC_{R,\bullet}^-(B)$. However it is true if R is finite dimensional. Similarly in that case we have $C_R^\bullet(B) \cong C^\bullet(A) \otimes_k R$. In the sequel we will not mention these base change isomorphisms explicitly.

2.6. Some comments on signs. In the previous sections the operations $i_x, L_x, S_x, \mathfrak{b}, \mathfrak{B}$ of degree $|x| + 1, |x|, |x| - 1, 1, -1$ were defined as acting on $\bar{C}_\bullet(B)$. We define corresponding operations on shifts $\Sigma^r \bar{C}_\bullet(B)$ in the usual way:

$$\begin{aligned} i_x(s^r b) &= (-1)^{r(|x|+1)} s^r i_x(b), \\ L_x(s^r b) &= (-1)^{r|x|} s^r L_x(b), \\ S_x(s^r b) &= (-1)^{r(|x|-1)} s^r S_x(b), \\ \mathfrak{b}(s^r b) &= (-1)^r s^r \mathfrak{b}(b), \\ \mathfrak{B}(s^r b) &= (-1)^r s^r \mathfrak{b}(B), \end{aligned}$$

where s is the *degree change operator* $|sb| = |b| - 1$.

The relations between $i_x, L_x, S_x, \mathfrak{b}, \mathfrak{B}$ stated in §2.3, §2.3 carry over to all shifts $\Sigma^r \bar{C}_\bullet(B)$ without any sign changes, since all terms in the identities (necessarily) have the same degree.

3. Preliminaries on Calabi–Yau algebras

In this section we extend Ginzburg’s definition of Calabi–Yau algebras to the relative case.

Let R be a commutative ring. For an R -algebra B we put $B^e = B \otimes_R B^\circ$. We use without further comment the standard equivalences between the categories of left B^e -modules, right B^e -modules and B -bimodules which are R -central.

A B^e -module is called *perfect* if it has a finite resolution by finitely generated projective B^e -modules. If B is R -flat and B is a perfect B^e -module then we say that B is *homologically smooth* over R . The implicit flatness hypothesis ensures that $B^e = B \otimes_R B^\circ$ is the correct definition from a derived point of view. We could have avoided this hypothesis by first replacing B by an R -flat DG-resolution but for simplicity we have chosen not to do this.

Definition 3.1 (Ginzburg [13]). An R -Calabi–Yau algebra of dimension d is a pair (B, η) where

- (1) B is an R -algebra B which is homologically smooth over R ;
- (2) η is an isomorphism $\mathrm{RHom}_{B^e}(B, B^e) \rightarrow \Sigma^{-d} B$ in $D(B^e)$.

Remark 3.2. Note that the amount of freedom for η is quite limited. If $(B, \eta), (B, \eta')$ are Calabi–Yau then there exists $z \in Z(B)$ such that $\eta' = z\eta$ (see [6]).

Recall that if M is a complex of B^e -module then its *Hochschild homology and cohomology* are respectively defined as

$$\begin{aligned} \mathrm{HH}_i(B, M) &= H^{-i}(M \otimes^L B), \\ \mathrm{HH}^i(B, M) &= H^i(\mathrm{RHom}_{B^e}(B, M)). \end{aligned}$$

As usual we write $\mathrm{HH}_i(B) = \mathrm{HH}_i(B, B)$ and similarly $\mathrm{HH}^i(B) = \mathrm{HH}^i(B, B)$. One has

$$\mathrm{HH}_i(B) = H^{-i}(C_\bullet(B)),$$

and if B is a projective R -module then

$$\mathrm{HH}^i(B) = H^i(C^\bullet(B)).$$

The operations $[\ , \], \cup, \cap, L, \mathbf{B}$ introduced in §2 descend to homology and make the pair $(\mathrm{HH}^\bullet(B), \mathrm{HH}_\bullet(B))$ into a so-called *calculus* structure [24]. Up to suitable, and for us irrelevant, signs \cup is the Yoneda products on $\mathrm{HH}^\bullet(B) = \mathrm{Ext}_{B^e}^\bullet(B, B)$ and \cap is the action of $\mathrm{HH}^\bullet(B)$ on $\mathrm{HH}_\bullet(B) = H^{-\bullet}(B \otimes_{B^e}^L B)$ through its action on the second factor (see e.g. [3, Prop. 11.1, 12.1]).

Lemma 3.3. *Let B be a homologically smooth algebra. Then for M a perfect B^e -module there is a canonical isomorphism*

$$\mathrm{HH}_i(B, M) \cong \mathrm{Hom}_{B^e}(\Sigma^i \mathrm{RHom}_{B^e}(M, B^e), B) \tag{3.1}$$

in $D(R)$.

Proof. Since M is perfect we may replace it with a complex of finitely generated projective B^e -modules. In this way we reduce to $M = B^e$ which is an easy verification. □

Definition 3.4. Let B be a homologically smooth algebra R and $\eta \in \mathrm{HH}_d(B)$. We say that η is *nondegenerate* if its image under (3.1) is an isomorphism.

This allows us to redefine a d -Calabi–Yau algebra over R as a couple (B, η) where B is a homologically smooth R -algebra and η is a non-degenerate element of $\mathrm{HH}_d(B)$. Below we will massage this new definition further. Recall the following:

Proposition 3.5 (“Poincare duality”). *Assume that (B, η) is a d -Calabi–Yau R -algebra. Then for each i , the map*

$$\mathrm{HH}^i(B) \longrightarrow \mathrm{HH}_{d-i}(B): \mu \mapsto \mu \cap \eta \tag{3.2}$$

is an isomorphism.

Proof. The existence of the isomorphism was first stated in [30] without the explicit formula (3.2). The formula (3.2) is folklore. For completeness we include a possible proof.

The proof of Lemma 3.3 shows that there is a canonical isomorphism in $D(R)$

$$\mathrm{RHom}_{B^e}(\mathrm{RHom}_{B^e}(B, B^e), \overset{\downarrow}{B}) \cong B \otimes_{B^e}^L \overset{\downarrow}{B}, \tag{3.3}$$

which is compatible with the $\mathrm{RHom}_{B^e}(B, B)$ -actions on the marked copies of B .

By definition $\eta \in H^{-d}(B \otimes_{B^e}^L B)$ corresponds under (3.3) to an isomorphism $\eta^+ : \mathrm{RHom}_{B^e}(B, B^e) \rightarrow \Sigma^{-d} B$. This yields an isomorphism

$$\mathrm{RHom}_{B^e}(\Sigma^{-d} B, \overset{\downarrow}{B}) \longrightarrow \mathrm{RHom}_{B^e}(\mathrm{RHom}_{B^e}(B, B^e), \overset{\downarrow}{B}): \theta \mapsto \theta \circ \eta^+ \tag{3.4}$$

also compatible with the marked $\mathrm{RHom}_{B^e}(B, B)$ -actions. Composing (3.3) and (3.4) gives us an isomorphism

$$\xi : \mathrm{RHom}_{B^e}(\Sigma^{-d} B, \overset{\downarrow}{B}) \longrightarrow B \otimes_{B^e}^L \overset{\downarrow}{B},$$

which sends Id_B to η .

According to the discussion preceding Lemma 3.3, the compatibility with the $\mathrm{RHom}_{B^e}(B, B)$ -actions implies that ξ transforms \cup into \cap on the level of cohomology. More precisely,

$$\xi(\mu \cup \sigma) = \pm \mu \cap \xi(\sigma).$$

The lemma now follows by taking $\sigma = \mathrm{Id}_B$. □

Corollary 3.6. *Assume that (B, η) is a d -Calabi–Yau R -algebra. Then*

$$\begin{aligned} \mathrm{HH}^i(B) &= 0 \quad \text{for } i \notin [0, d], \\ \mathrm{HH}_i(B) &= 0 \quad \text{for } i \notin [0, d]. \end{aligned}$$

As before let $\mathrm{CC}_\bullet^-(B) = (C_\bullet(B)[[u]], b + uB)$ be the negative cyclic complex and denote its corresponding homology by $\mathrm{HC}_\bullet^-(B)$.

Proposition 3.7. *Let (B, η) be a d -Calabi–Yau R -algebra. Then, $\mathrm{HC}_i^-(B) = 0$ for $i > d$ and, furthermore, the map*

$$\pi : \mathrm{CC}_\bullet^-(B) \longrightarrow C_\bullet(B): \sum b_i u^i \mapsto b_0$$

induces an isomorphism $\mathrm{HC}_d^-(B) \cong \mathrm{HH}_d(B)$.

Proof. We use a spectral sequence argument. We view $CC_{\bullet}^{-}(B)$ as a double complex with b pointing vertically upwards and uB pointing horizontally to the right. By Corollary 3.6 we have $HH_i(B) = 0$ for $i > d$. Hence if we filter $CC_{\bullet}^{-}(B)$ by column degree then the E^1 term of the resulting spectral sequence looks like

$$\begin{array}{ccccccc}
 0 & & HH_{d-2}(B) & \xrightarrow{uB} & u HH_{d-1}(B) & \xrightarrow{uB} & u^2 HH_d(B) \\
 & & & & & & \\
 0 & & HH_{d-1}(B) & \xrightarrow{uB} & u HH_d(B) & & 0 \\
 & & & & & & \\
 0 & & HH_d(B) & & 0 & & 0 \\
 & & & & & & \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

From this the result follows. □

Definition 3.8. Let B be a homologically smooth R -algebra. We say that an element $\eta \in HC_d^{-}(B)$ is non-degenerate if $\pi(\eta)$ is non-degenerate.

This leads to the following redefinition of a Ginzburg d -Calabi–Yau R -algebra, which we use below.

Definition 3.9 (Restatement of Definition 3.1). A Calabi–Yau algebra of dimension d over R is a couple (B, η) where B is a homologically smooth R -algebra and η is a non-degenerate element of $HC_d^{-}(B)$.

We have shown that this definition is equivalent to Ginzburg’s original definition. In the more general setting of DG-algebras this is no longer the case. It is generally believed that Definition 3.9 is the “correct” definition for a d -Calabi–Yau algebra in the DG-case. This is the point of view of Kontsevich–Soibelman in [17] and also of Keller [15].

4. Deformations of Calabi–Yau algebras

In this section we fix a d -Calabi–Yau k -algebra (A, η_0) as in Definition 3.9. We will study the deformations of A as a Calabi–Yau algebra.

Let Nilp be the category of commutative, finite dimensional, local k -algebras (R, m) such that $R/m = k$. For $(R, m) \in \text{Nilp}$ we define a groupoid $\text{Def}_{A, \eta_0}(R)$ as follows: the objects in $\text{Def}_{A, \eta_0}(R)$ are triples (B, s, η) such that B is an R -flat R -algebra, $s: B \rightarrow A$ is an R -algebra map inducing an isomorphism $B \otimes_R k \rightarrow A$ and η is an element of $HC_d^{-}(B)$ such that $s(\eta) = \eta_0$.

A morphism $(B_1, s_1, \eta_1) \rightarrow (B_2, s_2, \eta_2)$ is a commutative diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\phi} & B_2 \\ & \searrow s_1 & \swarrow s_2 \\ & & A \end{array}$$

such $\eta_2 = \phi(\eta_1)$. One sees that Def_{A, η_0} becomes a pseudo-functor $\text{Nilp} \rightarrow \text{Gd}$ in the obvious way.

To be able to rightfully claim that Def_{A, η_0} describes the Calabi–Yau deformations of (A, η_0) we need the following elementary lemma.

Lemma 4.1. *Assume that $(B, s, \eta) \in \text{Def}_{A, \eta_0}(R)$. Then (B, η) is d -Calabi–Yau.*

Proof. We have to show that B is a perfect B^e -module and that η induces an isomorphism $\eta^+ : \text{RHom}_{B^e}(B, B^e) \rightarrow \Sigma^{-d} B$.

Since R is finite dimensional every flat R -module is R -projective. This applies in particular to B and B^e . Let

$$0 \rightarrow P_u \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0$$

be a finite resolution of A by finitely generated projective A^e -modules. It is easy to see that this resolution can be lifted step by step to a resolution

$$0 \rightarrow Q_u \rightarrow \dots \rightarrow Q_0 \rightarrow B \rightarrow 0,$$

where the Q_i are finitely generated projective B^e -modules satisfying $Q_i \otimes_R k \cong P_i$. In particular B is perfect.

It also follows that $H = \text{cone } \eta^+$ is perfect. It is easy to see that $\eta^+ \otimes^L k \cong \eta_0^+$, and hence

$$(\text{cone } \eta^+) \otimes^L k \cong \text{cone}(\eta^+ \otimes^L k) \cong \text{cone } \eta_0^+ = 0.$$

It now suffices to note that if H is perfect and $H \otimes^L k = 0$ then $H = 0$. □

We now introduce a variant of the groupoid $\text{Def}_{A, \eta_0}(R)$ which is easier to describe cohomologically. We remind the reader of the base change convention exhibited in §2.5 which we will use throughout. As in §2.2 let $-\mu_0 \in \mathcal{C}^1(A)$ be the multiplication map on A and let $\hat{\eta}_0$ be a lift of η_0 to $\overline{\text{CC}}_d(A)$. We define an associated groupoid $\text{Def}_{A, \hat{\eta}_0}^b(R)$ as follows. The objects are couples (μ, η) where

- (1) $\mu \in \mathcal{C}^1(A) \otimes_k R$ is such that $-\mu$ defines a unital associative multiplication on $A \otimes_k R$;
- (2) $\mu \bmod m = \mu_0$;
- (3) $\eta \in \overline{\text{CC}}_d(A) \otimes_k R$;
- (4) $(L_\mu + uB)(\eta) = 0$;
- (5) $\eta \bmod m = \hat{\eta}_0$.

For (4), recall that $L_\mu + uB$ is the cyclic differential for the algebra $(A \otimes_k R, \mu)$. A morphism $(\mu_1, \eta_1) \rightarrow (\mu_2, \eta_2)$ in $\text{Def}_{A, \mu_0, \hat{\eta}_0}^b(R)$ is a couple (ϕ, ξ) where

- (1) ϕ is a unital map of R -algebras $\phi: (A \otimes_k R, -\mu_1) \rightarrow (A \otimes_k R, -\mu_2)$;
- (2) ϕ is the identity modulo m ;
- (3) ξ is an element of $\overline{\text{CC}}_{d+1}^-(A) \otimes_k m$;
- (4) $(L_{\mu_2} + uB)(\xi) = \phi(\eta_1) - \eta_2$.

The composition of morphisms

$$(\mu_1, \eta_1) \xrightarrow{(\phi', \xi')} (\mu_2, \eta_2) \xrightarrow{(\phi, \xi)} (\mu_2, \eta_2)$$

is defined by

$$(\phi, \xi) \circ (\phi', \xi') = (\phi \circ \phi', \phi(\xi') + \xi). \tag{4.1}$$

Below we will often use the notation $\bar{\eta}$ for the cohomology class of a cocycle η .

Proposition 4.2. *The morphism of groupoids*

$$\begin{aligned} \text{Ob}(\text{Def}_{A, \hat{\eta}_0}^b(R)) &\longrightarrow \text{Ob}(\text{Def}_{A, \eta_0}(R)): (\mu, \eta) \mapsto ((A \otimes_k R, -\mu), \text{“mod } m\text{”}, \bar{\eta}) \\ \text{Mor}(\text{Def}_{A, \hat{\eta}_0}^b(R)) &\longrightarrow \text{Mor}(\text{Def}_{A, \eta_0}(R)): (\phi, \xi) \mapsto \phi \end{aligned}$$

is essentially surjective on objects and surjective on morphisms.

Proof. We first prove essential surjectivity. Let $(B, s, \psi) \in \text{Def}_{A, \eta_0}(R)$. Then, since R is finite dimensional local and B is R -flat we have an isomorphism $B \cong A \otimes_k R$ as R -modules and it is easy to see that this isomorphism may be chosen to make the following diagram commutative

$$\begin{array}{ccc} B & \xrightarrow{\phi} & A \otimes_k R \\ & \searrow s & \swarrow \text{mod } m \\ & & A \end{array}$$

We now transfer the multiplication on B to $A \otimes_k R$ where it becomes an element of $-\mu \in \mathcal{C}^1(A) \otimes_k R$ which modulo m is equal to $-\mu_0$. We do the same with $\psi \in \text{HC}_d^-(B)$ and we choose an element $\eta \in \overline{\text{CC}}_d^-(A) \otimes_k R$ such that $(L_\mu + uB)(\eta) = 0$, $\bar{\eta} = \phi(\psi)$. Thus in $\text{Def}_{A, \eta_0}(R)$ we have

$$(B, s, \psi) \cong ((A \otimes_k R, -\mu), - \text{mod } m, \bar{\eta}).$$

This proves essential surjectivity. Now we prove surjectivity on morphisms. Let $(\mu_1, \eta_1), (\mu_2, \eta_2) \in \text{Ob}(\text{Def}_{A, \hat{\eta}_0}^b(R))$ and let ϕ be a unital algebra morphism

$$(A \otimes_k R, -\mu_1) \rightarrow (A \otimes_k R, -\mu_2)$$

inducing the identity modulo m and satisfying $\phi(\bar{\eta}_1) = \bar{\eta}_2$.

It follows that $\phi(\eta_1) - \eta_2$ is a boundary in the negative cyclic complex of $(A \otimes_k R, -\mu_2)$. In other words there exists $\xi \in \overline{\text{CC}}_{d+1}^-(A) \otimes_k R$ such that

$$\phi(\eta_1) - \eta_2 = (L_{\mu_2} + uB)(\xi).$$

We have to show that we may choose $\xi \in \overline{\text{CC}}_{d+1}^-(A) \otimes_k m$

Since ϕ is the identity modulo m we have

$$\begin{aligned} \phi(\eta_1) - \eta_2 \text{ mod } m &= \eta_1 - \eta_2 \text{ mod } m \\ &= \hat{\eta}_0 - \hat{\eta}_0 \\ &= 0. \end{aligned}$$

It follows that $\phi(\eta_1) - \eta_2 \in \overline{\text{CC}}_d^-(A) \otimes_k m$ and hence $d\xi \text{ mod } m = 0$. Since $\text{HC}_{d+1}^-(A) = 0$ by Proposition 3.7 we see that there exists $\gamma \in \overline{\text{CC}}_{d+2}^-(A) \otimes_k R$ such that $(L_{\mu_2} + uB)(\gamma) \cong \xi \text{ mod } m$. In other words

$$\xi' = \xi - (L_{\mu_2} + uB)(\gamma) \in \overline{\text{CC}}_d^-(A) \otimes_k m.$$

Then the couple (ϕ, ξ') is a pre-image for ϕ . □

For completeness we state the following.

Proposition 4.3. *Let $\hat{\eta}'_0 \in \overline{\text{CC}}_0^-(A)$ be a different lift of η_0 . Then $\text{Def}_{A, \hat{\eta}'_0}^b(R)$ and $\text{Def}_{A, \hat{\eta}_0}^b(R)$ are isomorphic.*

We could easily prove this here directly, however we will postpone the proof until §6 where we reinterpret $\text{Def}_{A, \hat{\eta}_0}^b(R)$ in terms of the Maurer–Cartan equation.

5. The Maurer–Cartan formalism

In this section we briefly recall the construction of the deformation functor associated to a DG-Lie algebra.

Let \mathfrak{h}^\bullet be a DG-Lie algebra over k . The set

$$\text{MC}(\mathfrak{h}^\bullet) \stackrel{\text{def}}{=} \left\{ y \in \mathfrak{h}^1 \mid dy + \frac{1}{2}[y, y] = 0 \right\}$$

is the set of solutions to the *Maurer–Cartan equation* in \mathfrak{h}^\bullet . It has a natural structure of a groupoid which we now describe.

Assume that \mathfrak{n} is a nilpotent Lie algebra and let $\widehat{U}(\mathfrak{n})$ be the enveloping algebra of \mathfrak{n} , completed at the augmentation ideal. Then the group $\exp(\mathfrak{n})$ is by definition the set of group like elements in $\widehat{U}(\mathfrak{n})$. It is well known and easy to see that there is a bijection

$$\exp: \mathfrak{n} \longrightarrow \exp(\mathfrak{n}): n \mapsto e^n$$

between the primitive and the group like elements in $\widehat{U}(\mathfrak{n})$.

Now assume that \mathfrak{h}^\bullet is nilpotent. Then $\widehat{U}(\mathfrak{h}^0)$ acts on the graded Lie algebra \mathfrak{h}^\bullet using the adjoint action and hence so does the *gauge group* $G(\mathfrak{h}^\bullet) \stackrel{\text{def}}{=} \exp(\mathfrak{h}^0)$. This action does not commute with the differential and in particular it does not preserve $\text{MC}(\mathfrak{h}^\bullet)$. However the following modified *gauge action* does:

$$\begin{aligned} \exp(x) * y &\stackrel{\text{def}}{=} e^{\text{ad } x}(y) - \frac{e^{\text{ad } x} - 1}{\text{ad } x}(dx) \\ &= e^{\text{ad } x}(y) - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\text{ad } x)^n(dx), \end{aligned} \tag{5.1}$$

where $x \in \mathfrak{h}^0$, $y \in \mathfrak{h}^1$ and $(\text{ad } x)(u) = [x, u]$.

An elegant derivation of this action is given by Manetti [19, §V.4]. One first formally adjoins an element δ of degree one to \mathfrak{h}^\bullet such that $dx = [\delta, x]$, $d\delta = 0$ and $[\delta, \delta] = 0$. Then (5.1) can be rewritten as:

$$\exp(x) * y = e^{\text{ad } x}(y + \delta) - \delta. \tag{5.2}$$

This action preserves $\text{MC}(\mathfrak{h}^\bullet)$ since for $y \in \mathfrak{h}^1$:

$$y \in \text{MC}(\mathfrak{h}^\bullet) \iff [y + \delta, y + \delta] = 0.$$

In the sequel we view $\text{MC}(\mathfrak{h}^\bullet)$ as a groupoid through the $G(\mathfrak{h}^\bullet)$ -action.

If $y \in \text{MC}(\mathfrak{h}^\bullet)$ then by definition \mathfrak{h}_y^\bullet is the DG-Lie algebra which is \mathfrak{h}^\bullet as graded Lie algebra but which has the deformed differential $d_y = d + [y, -]$. Using (5.2) one easily shows that for $x \in \mathfrak{h}^0$

$$e^{\text{ad } x}: \mathfrak{h}_y^\bullet \longrightarrow \mathfrak{h}_{\exp(x)*y}^\bullet \tag{5.3}$$

is an isomorphism of DG-Lie algebras

Assume $(R, m) \in \text{Nilp}$. Given an arbitrary DG-Lie algebra \mathfrak{g}^\bullet over k , the vector space $\mathfrak{g}^\bullet \otimes_k m$ becomes a nilpotent DG-Lie algebra. We define $\mathcal{MC}(R)$ as $\text{MC}(\mathfrak{g}^\bullet \otimes_k m)$ equipped with the groupoid structure introduced above. In this way we obtain a pseudo-functor $\mathcal{MC}: \text{Nilp} \longrightarrow \text{Gd}$. This is the “deformation functor” associated to \mathfrak{g}^\bullet .

6. The DG-Lie algebra $\mathfrak{D}^\bullet(A, \eta)$

Below $(A, \bar{\eta}_0)$ is a d -Calabi–Yau k -algebra where $\eta_0 \in \overline{\text{CC}}_d(A)$ satisfies

$$(L_{\mu_0} + uB)(\eta_0) = 0,$$

with $-\mu_0 \in \mathcal{C}^1(A)$ being the multiplication on A . In this section we associate a DG-Lie algebra $\mathfrak{D}^\bullet(A, \eta_0)$ to A and prove that its deformation functor (see §5) is isomorphic to the functor Def_{A, η_0}^b introduced in §4.

If \mathfrak{g}^\bullet is a DG-Lie algebra and M^\bullet a \mathfrak{g}^\bullet -module then the direct sum complex $\mathfrak{g}^\bullet \oplus M^\bullet$ becomes a DG-Lie algebra when endowed with the following bracket:

$$[(g, m), (g', m')] := ([g, g'], gm' - (-1)^{|g'| |m|} g'm)$$

The resulting DG-Lie algebra is called the *semi-direct product* of \mathfrak{g}^\bullet and M^\bullet and is denoted by $\mathfrak{g}^\bullet \ltimes M^\bullet$

By (2.3) and (2.7) (see also §2.6) we have a DG-Lie action

$$\bar{\mathfrak{C}}^\bullet(A) \times \Sigma^{-d-1} \overline{\mathbb{C}\mathbb{C}_\bullet}^-(A) \longrightarrow \Sigma^{-d-1} \overline{\mathbb{C}\mathbb{C}_\bullet}^-(A): (x, \eta) \mapsto L_x \eta$$

and we can form the corresponding semi-direct product

$$\mathfrak{D}^\bullet(A)^\# = \bar{\mathfrak{C}}^\bullet(A) \ltimes \Sigma^{-d-1} \overline{\mathbb{C}\mathbb{C}_\bullet}^-(A).$$

The element $x = (0, s^{-d-1} \eta_0) \in \mathfrak{D}^\bullet(A)^\#$ satisfies $dx = 0$ and $[x, x] = 0$. So it satisfies the Maurer–Cartan equation. Put $\mathfrak{D}^\bullet(A, \eta_0) = \mathfrak{D}^\bullet(A)_x^\#$, with notation as in §5.

Theorem 6.1. *Let $(R, m) \in \text{Nilp}$. There is an isomorphism of groupoids*

$$\Phi(R): \text{MC}(\mathfrak{D}^\bullet(A, \eta_0) \otimes_k m) \longrightarrow \text{Def}_{A, \eta_0}^b(R),$$

which on objects is given by

$$(\mu, s^{-d-1} \eta) \mapsto (\mu_0 + \mu, \eta_0 + \eta). \tag{6.1}$$

Corollary 6.2. *There is a natural transformation of pseudo-functors*

$$\Phi: \mathcal{MC}(\mathfrak{D}^\bullet(A, \eta_0)) \longrightarrow \text{Def}_{A, \eta_0}^b,$$

which, when evaluated on $R \in \text{Nilp}$, is an isomorphism of groupoids.

We shall prove Theorem 6.1 by combining some lemmas. Throughout we fix $(R, m) \in \text{Nilp}$. The following lemma says that $\Phi(R)$ behaves correctly on objects.

Lemma 6.3. *Let $\mu \in \bar{\mathfrak{C}}_1^\bullet(A) \otimes_k m$ and $\eta \in \overline{\mathbb{C}\mathbb{C}_d}^-(A) \otimes_k m$. The following are equivalent:*

- (1) $(\mu, s^{-d-1} \eta) \in \text{MC}(\mathfrak{D}^\bullet(A, \eta_0) \otimes_k m)$;
- (2) $(\mu_0 + \mu, \eta_0 + \eta) \in \text{Def}_{A, \eta_0}^b(R)$.

Proof. We will work out what it means for $(\mu, s^{-d-1}\eta) \in \mathfrak{D}^1(A, \eta_0) \otimes_k m$ to satisfy the Maurer–Cartan equation. To simplify the notations we write $\eta'_0 = s^{-d-1}\eta_0$, $\eta' = s^{-d-1}\eta$. We compute

$$\begin{aligned} & \frac{1}{2}[(\mu, \eta'), (\mu, \eta')] + d_{\mathfrak{D}}(\mu, \eta') \\ &= \frac{1}{2}([\mu, \mu], 2L_{\mu}(\eta')) + ([\mu_0, \mu], (L_{\mu_0} + u\mathbf{B})(\eta')) + [(0, \eta'_0), (\mu, \eta')] \\ &= \frac{1}{2}([\mu, \mu], 2L_{\mu}(\eta')) + ([\mu_0, \mu], (L_{\mu_0} + u\mathbf{B})(\eta')) + [0, L_{\mu}(\eta'_0)] \\ &= \left(\frac{1}{2}[\mu, \mu] + [\mu_0, \mu], L_{\mu}(\eta') + (L_{\mu_0} + u\mathbf{B})(\eta') + L_{\mu}(\eta'_0)\right) \\ &= ([\mu_0 + \mu, \mu_0 + \mu], (L_{\mu+\mu_0} + u\mathbf{B})(\eta' + \eta'_0)), \end{aligned}$$

where in the last line we have used $[\mu_0, \mu_0] = 0$, $(L_{\mu_0} + u\mathbf{B})(\eta'_0) = 0$. Thus if $(\mu_0 + \mu, \eta_0 + \eta) \in \text{Def}_{A, \eta_0}^{\text{fb}}(R)$ then $(\mu, s^{-d-1}\eta) \in \text{MC}(\mathfrak{D}^{\bullet}(A, \eta_0) \otimes_k m)$. To prove the converse the only thing we still need to check is that $-(\mu_0 + \mu)$ defines a unital multiplication on $A \otimes_k R$. This follows immediately from the fact that $-\mu_0$ is unital and μ is normalized. \square

The next two lemmas will help us describing the gauge group action of $G(\mathfrak{D}^{\bullet}(A, \eta))$.

Lemma 6.4. *Let \mathfrak{n} be a nilpotent Lie algebra over k and let M a representation of \mathfrak{n} . Then there is an isomorphism of groups*

$$\exp(\mathfrak{n}) \times M \longrightarrow \exp(\mathfrak{n} \ltimes M): (\exp(n), m) \mapsto \exp(n, 0) \exp(0, m).$$

Proof. This is a straightforward verification from the definition of “exp” in §5 using the fact that

$$U(\mathfrak{h} \ltimes M) \cong U(\mathfrak{h}) \times \text{Sym}(M). \quad \square$$

Lemma 6.5. *Let \mathfrak{g}^{\bullet} be a nilpotent DG-Lie algebra over k and M^{\bullet} a nilpotent DG-module. Consider the DG-algebra \mathfrak{h}^{\bullet} which is $\mathfrak{g}^{\bullet} \ltimes M^{\bullet}$ as graded Lie algebras and which is equipped with a deformed differential $(d_{\mathfrak{g}}, d_M) + d_0$ where $d_0: \mathfrak{g}^{\bullet} \rightarrow M$ is of the form $g \mapsto (-1)^{|g|} g m_0$ for suitable $m_0 \in M^1$. Then for $g \in \mathfrak{g}^0$, $m \in M^0$, and $(g_1, m_1) \in \mathfrak{h}^1$ we have*

$$\begin{aligned} \exp(g, 0) * (g_1, m_1) &= (\exp(g) * g_1, e^g(m_1 - m_0) + m_0), \\ \exp(0, m) * (g_1, m_1) &= (g_1, m_1 - (g_1 + d_M)m). \end{aligned}$$

Proof. We compute

$$\begin{aligned} \exp(g, 0) * (g_1, m_1) &= e^{\text{ad}(g,0)}(g_1, m_1) - \sum_n \frac{1}{(n+1)!} \text{ad}^n(g, 0)(d_{\mathfrak{h}}(g, 0)) \\ &= (e^{\text{ad}g} g_1, e^g m_1) - \sum_n \frac{1}{(n+1)!} \text{ad}^n(g, 0)(d_{\mathfrak{g}}g, d_0g) \\ &= (e^{\text{ad}g} g_1, e^g m_1) - \sum_n \frac{1}{(n+1)!} (\text{ad}^n(g)(d_{\mathfrak{g}}g), g^{n+1}m_0) \\ &= (e^g * g_1, e^g(m_1 - m_0) + m_0). \end{aligned}$$

Similarly:

$$\begin{aligned} \exp(0, m) * (g_1, m_1) &= e^{\text{ad}(0,m)}(g_1, m_1) - \sum_n \frac{1}{(n+1)!} \text{ad}^n(0, m)(d_{\mathfrak{h}}(0, m)) \\ &= (g_1, m_1) - (0, g_1m) - (0, d_Mm) \\ &= (g_1, m_1 - (g_1 + d_M)m). \quad \square \end{aligned}$$

We will also use the following variant of (5.2)

Lemma 6.6. *Let \mathfrak{h}^\bullet be a nilpotent DG-Lie algebra with inner differential $d = [\mu_0, -]$. Then for $x \in \mathfrak{h}^0$, $y \in \mathfrak{h}^1$ one has*

$$\exp(x) * y = e^{\text{ad}x}(y + \mu_0) - \mu_0.$$

Proof. Direct evaluation of the righthand side yields the formula (5.1) for $\exp(x) * y$. □

Proof of Theorem 6.1. We start by verifying that (6.1) yields indeed a map of groupoids. To this end we have to define $\Phi(R)$ on maps. Note that by Lemma 6.4 each element of $\exp(\mathfrak{D}^0(A, \mu_0, \eta_0) \otimes m)$ can be uniquely written as $\exp(0, s^{-d-1}\xi) \exp(f, 0)$ for

$$f \in \overline{\mathfrak{C}}^0(A) \otimes_k m = \text{Hom}(A/k, A) \otimes_k m \subset \text{Hom}(A, A) \otimes_k m$$

and $\xi \in \overline{\text{CC}}_{d+1}^-(A) \otimes_k m.$

We put $\phi = e^f$. Then $\phi \in \text{Hom}(A, A) \otimes_k R$ is such that $\phi \text{ mod } m = \text{Id}_A$.

Assume that

$$\exp(0, s^{-d-1}\xi) * \exp(f, 0) * (\mu_1, s^{-d-1}\eta_1) = (\mu_2, s^{-d-1}\eta_2). \tag{6.2}$$

We define $\Phi(R)$ on maps as follows

$$\Phi(R)(\exp(0, s^{-d-1}\xi) \exp(f, 0)) = (e^f, (-1)^d \xi). \tag{6.3}$$

For this to be well defined we should have a morphism

$$(\phi, (-1)^d \xi): (\mu_0 + \mu_1, \eta_0 + \eta_1) \longrightarrow (\mu_0 + \mu_2, \eta_0 + \eta_2)$$

in $\text{Def}_{A, \eta_0}^b(R)$. In other words:

- (a) $\phi: (A \otimes_k R, -(\mu_0 + \mu_1)) \longrightarrow (A \otimes_k R, -(\mu_0 + \mu_2))$ is an R -algebra morphism;
- (b) $\phi(\eta_0 + \eta_1) = \eta_0 + \eta_2 + (-1)^d (L_{\mu_0 + \mu_2} + u\mathbf{B})(\xi)$.

Put $\eta'_i = s^{-d-1} \eta_i$ for $i = 0, 1, 2$, $\xi' = s^{-d-1} \xi$. We invoke Lemma 6.5 with $m_0 = -\eta'_0$. Then (6.2) yields

$$(\mu_2, \eta'_2) = (\exp(f) * \mu_1, e^f(\eta'_0 + \eta'_1) - \eta'_0 - L_{\exp(f)*\mu_1}(\xi') - (L_{\mu_0} + u\mathbf{B})(\xi')). \tag{6.4}$$

We may compute $\exp(f) * \mu_1$ inside unnormalized cochains $C^\bullet(A)$ and then we may invoke Lemma 6.6. We find

$$\exp(f) * \mu_1 = e^{\text{ad } f}(\mu_0 + \mu_1) - \mu_0.$$

Furthermore a direct computation shows that

$$\begin{aligned} e^{\text{ad } f}(\mu_0 + \mu_1) &= e^f \circ (\mu_0 + \mu_1) \circ (e^{-f}, e^{-f}) \\ &= \phi \circ (\mu_0 + \mu_1) \circ (\phi^{-1}, \phi^{-1}). \end{aligned}$$

Hence (6.4) translates into

$$\begin{aligned} \mu_0 + \mu_2 &= \phi \circ (\mu_0 + \mu_1) \circ (\phi^{-1}, \phi^{-1}) \\ \eta'_0 + \eta'_2 &= \phi(\eta'_0 + \eta'_1) - (L_{\mu_0 + \mu_2} + u\mathbf{B})(\xi'). \end{aligned}$$

The first of these equations yields (a). The second yields (b) taking into account that $L_{\mu_0 + \mu_2} + u\mathbf{B}$ has degree one, which induces a sign change.

It remains to show that our assignment respects compositions. By Lemma 6.4 we have for $f, g, h \in \overline{\mathcal{C}}^0(A) \otimes_k m$ such that $\exp(h) = \exp(g)\exp(f)$, $v, \xi \in \overline{\mathcal{C}}_{-d-1}^-(A) \otimes_k m$:

$$\begin{aligned} &\Phi(R)(\exp(0, s^{-d-1}v)\exp(g, 0) \circ \exp(0, s^{-d-1}\xi)\exp(f, 0)) \\ &= \Phi(R)(\exp(0, s^{-d-1}v)\exp(0, s^{-d-1}e^g\xi)\exp(g, 0)\exp(f, 0)) \\ &= \Phi(R)(\exp(0, s^{-d-1}(v + e^g\xi))\exp(h, 0)) \\ &= (e^h, (-1)^d(v + e^g\xi)) \\ &= (e^g e^f, (-1)^d(v + e^g\xi)) \end{aligned}$$

and

$$\begin{aligned} &\Phi(R)(\exp(0, s^{-d-1}v)\exp(g, 0)) \circ \Phi(R)(\exp(0, s^{-d-1}\xi)\exp(f, 0)) \\ &= (e^g, (-1)^d v)(e^f, (-1)^d \xi) \\ &= (e^g e^f, (-1)^d(v + e^g\xi)) \end{aligned}$$

by (4.1). We conclude that $\Phi(R)$ is indeed a map of groupoids. By Lemma 6.3 it is bijective on objects, and running the above computation backwards, starting from (6.3), we see that it also bijective on maps. Thus $\Phi(R)$ is an isomorphism of groupoids. \square

The following result implies Proposition 4.3.

Proposition 6.7. *Assume that $\eta_0, \eta'_0 \in \overline{\text{CC}}_d(A)$ induce the same element in $\text{HC}_d^-(A)$. Then $\mathcal{D}^\bullet(A, \eta_0) \cong \mathcal{D}^\bullet(A, \eta'_0)$.*

Proof. From (5.3) one sees that it is sufficient to show that $(0, s^{-s-1}\eta_0), (0, s^{-d-1}\eta'_0)$ are in the same $G(\mathcal{D}^\bullet(A)^\#)$ orbit. Pick $\xi \in \overline{\text{CC}}_{d+1}(A)$ such that

$$\eta'_0 = \eta_0 + (-1)^d(L_{\mu_0} + uB)\xi.$$

We compute using (5.1)

$$\begin{aligned} \exp(0, s^{-d-1}\xi) * (0, s^{-d-1}\eta_0) &= (0, s^{-d-1}\eta_0) - (0, (L_{\mu_0} + uB)(s^{-d-1}\xi)) \\ &= (0, s^{-d-1}\eta'_0). \end{aligned} \quad \square$$

7. Relation with Hochschild cohomology

Let $(A, \bar{\eta}_0)$ be a d -Calabi–Yau k -algebra and let $-\mu_0$ be the multiplication of A . Let $(R, m) \in \text{Nilp}$. We may define pseudo-functors $\text{Def}_A, \text{Def}_A^b: \text{Nilp} \rightarrow \text{Gd}$ in the same way as $\text{Def}_{A, \eta_0}, \text{Def}_{A, \eta_0}^b$, ignoring η_0 . The induced morphism

$$\text{Def}_A^b(R) \rightarrow \text{Def}_A(R)$$

is essentially surjective on objects and surjective on morphisms. Furthermore there is an isomorphism of groupoids

$$\Phi(R): \text{MC}(\bar{\mathcal{C}}^\bullet(A) \otimes_k m) \rightarrow \text{Def}_A^b(R): \mu \mapsto \mu_0 + \mu.$$

The obvious morphism of DG-Lie algebras

$$\phi: \mathcal{D}^\bullet(A, \eta_0) \rightarrow \bar{\mathcal{C}}^\bullet(A): (\mu, \eta) \mapsto \mu$$

makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{MC}(\mathcal{D}^\bullet(A, \eta_0)) & \xrightarrow{\phi} & \mathcal{MC}(\bar{\mathcal{C}}^\bullet(A)) \\ \Phi \downarrow & & \downarrow \Phi \\ \text{Def}_{A, \eta} & \xrightarrow{\text{forget } \eta} & \text{Def}_A \end{array}$$

8. Homology of $\mathfrak{D}^\bullet(A, \eta)$

Let $(A, \bar{\eta}_0)$ be a d -Calabi–Yau algebra as before with multiplication $-\mu_0$. In this section we prove that the homology of $\mathfrak{D}^\bullet(A, \eta_0)$ is isomorphic to $\mathrm{HC}_{-+d-1}^-(A)$. Furthermore we show that the induced Lie bracket on $\mathrm{HC}_{-+d-1}^-(A)$ is given Menichi’s string topology bracket [20].

In our statements and computations we will use the following conventions:

- Taking homology classes is indicated by overlining.
- Depending on context \cong will mean either “up to homotopy” (when discussing maps) or “up to addition of a coboundary” (when discussing elements).

Theorem 8.1. *The map*

$$\begin{aligned} \Psi: \mathfrak{D}^\bullet(A, \eta_0) &\longrightarrow \Sigma^{-d+1}\overline{\mathrm{CC}}_\bullet^-(A): (\mu, s^{-d-1}\eta) \\ &\mapsto (-1)^{|\mu|-1}(i_\mu + uS_\mu)(s^{-d+1}\eta_0) + us^{-d+1}\eta \end{aligned}$$

is a quasi-isomorphism of complexes.

Proof. To simplify the notation we put

$$I_\mu = i_\mu + uS_\mu.$$

We first check that Ψ does indeed commute with differentials. Write $\eta'_0 = s^{-d-1}\eta_0$, $\eta' = s^{-d-1}\eta$. Then

$$\Psi(\mu, \eta') = s^2((-1)^{|\mu|-1}I_\mu\eta'_0 + u\eta'), \tag{8.1}$$

and hence

$$\begin{aligned} (d \circ \Psi)(\mu, \eta') &= (\mathbf{b} + u\mathbf{B})s^2((-1)^{|\mu|-1}I_\mu\eta'_0 + u\eta') \\ &= s^2((-1)^{|\mu|-1}(\mathbf{b} + u\mathbf{B})I_\mu\eta'_0 + u(\mathbf{b} + u\mathbf{B})\eta') \\ &= s^2((-1)^{|\mu|-1}[\mathbf{b} + u\mathbf{B}, I_\mu](\eta'_0) + u(\mathbf{b} + u\mathbf{B})\eta') && \text{(since } (\mathbf{b} + u\mathbf{B})\eta'_0 = 0) \\ &= s^2((-1)^{|\mu|-1}(uL_\mu - Id_\mu)\eta'_0 + u(\mathbf{b} + u\mathbf{B})\eta') && \text{(by (2.8))} \\ &= s^2((-1)^{|\mu|}Id_\mu\eta'_0 + u((\mathbf{b} + u\mathbf{B})\eta' - (-1)^{|\mu|}L_\mu\eta'_0)) \\ &= \Psi(d\mu, (\mathbf{b} + u\mathbf{B})\eta' - (-1)^{|\mu|}L_\mu\eta'_0) && \text{(by (8.1))} \\ &= (\Psi \circ d)(\mu, \eta'). \end{aligned}$$

To see that Ψ is indeed a quasi-isomorphism, consider the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma^{-d-1}\overline{\text{CC}}_{\bullet}^{-}(A) & \longrightarrow & \mathfrak{D}^{\bullet}(A, \eta_0) & \longrightarrow & \overline{\mathfrak{C}}^{\bullet}(A) \longrightarrow 0 \\
 & & \Psi \downarrow & & \Psi \downarrow & & \downarrow \overline{\Psi} \\
 0 & \longrightarrow & u\Sigma^{-d+1}\overline{\text{CC}}_{\bullet}^{-}(A) & \longrightarrow & \Sigma^{-d+1}\overline{\text{CC}}_{\bullet}^{-}(A) & \longrightarrow & \Sigma^{-d+1}\overline{\mathfrak{C}}_{\bullet}(A) \longrightarrow 0
 \end{array}$$

The map on the left is multiplication by u which is an isomorphism. The map $\overline{\Psi}$ is given on cohomology by

$$\overline{\mu} \mapsto \pm \overline{I_{\mu} \eta_0} \text{ mod } u = \pm \overline{i_{\mu} \pi(\eta_0)},$$

where π is as in Proposition 3.7. Hence $\overline{\Psi}$ is an isomorphism by Proposition 3.5. From the five lemma we conclude that the middle arrow is an isomorphism on cohomology as well. \square

We now describe the Lie bracket on $\text{HC}_{\bullet}^{-}(A)$ induced by the quasi-isomorphism Ψ . As already used in the above proof the map

$$- \cap \pi(\overline{\eta}_0): \text{HH}^i(A) \longrightarrow \text{HH}_{d-i}(A)$$

is invertible by Proposition 3.5. Let us denote its inverse by j . Using j , one can transport the cup product on $\text{HH}^{\bullet}(A)$ to a product on $\text{HH}_{\bullet}(A)$

$$\cdot : \text{HH}_i(A) \times \text{HH}_j(A) \longrightarrow \text{HH}_{i+j-d}(A)$$

with explicit formula

$$a \cdot b = (j(a) \cup j(b)) \cap \pi(\overline{\eta}_0)$$

or in a form more suitable for us below

$$i_{\mu_1} \pi(\overline{\eta}_0) \cdot i_{\mu_2} \pi(\overline{\eta}_0) = i_{\mu_1 \cup \mu_2} \pi(\overline{\eta}_0). \tag{8.2}$$

Theorem 8.2. *The Lie bracket induced on*

$$H^{\bullet}(\Sigma^{-d+1}\overline{\text{CC}}_{\bullet}^{-}(A)) = \text{HC}_{\bullet+d-1}(A)$$

by the quasi-isomorphism Ψ is given by

$$\begin{aligned}
 [-, -]: \text{HC}_n^{-}(A) \times \text{HC}_m^{-}(A) &\longrightarrow \text{HC}_{n+m-d+1}^{-}(A): (\eta_1, \eta_2) \\
 &\mapsto (-1)^{|\eta_1|+d} \mathbf{B}(\pi(\eta_1) \cdot \pi(\eta_2)),
 \end{aligned}$$

where \mathbf{B} is given by

$$\mathbf{B}: \text{HH}_q(A) \longrightarrow \text{HC}_{q+1}^{-}(A): \bar{v} \mapsto \overline{\mathbf{B}v}.$$

We first need the following technical lemma.

Lemma 8.3. *Let $\mu \in \overline{\mathcal{C}}^\bullet(A)$ and $\eta \in \overline{\mathcal{CC}}_\bullet(A)$ be cocycles. Then $L_\mu\eta$ and $\text{Bi}_\mu\pi(\eta)$ are both cocycles in $\overline{\mathcal{CC}}_\bullet(A)$ and $\overline{\text{Bi}_\mu\pi(\eta)} = \overline{L_\mu\eta}$ in $\text{HC}_\bullet(A)$.*

Proof. $L_\mu\eta$ is a cocycle by (2.7). $\text{Bi}_\mu\pi(\eta)$ is a cocycle since $\pi(\eta)$ is a cocycle in $\overline{\mathcal{C}}_\bullet(A)$ and

$$(\mathbf{b} + u\mathbf{B})(\text{Bi}_\mu\pi(\eta)) = \mathbf{b}\text{Bi}_\mu\pi(\eta) = -\mathbf{B}\text{bi}_\mu\pi(\eta) = 0,$$

where the last equality follows from (2.5).

For the second claim, we first multiply by u :

$$\begin{aligned} u(L_\mu\eta - \text{Bi}_\mu\pi(\eta)) &= [\mathbf{b} + u\mathbf{B}, I_\mu]\eta - u\text{Bi}_\mu\pi(\eta) && \text{(by (2.8))} \\ &= (\mathbf{b} + u\mathbf{B})I_\mu\eta - u\text{Bi}_\mu\pi(\eta) && \text{(since } (\mathbf{b} + u\mathbf{B})\eta = 0) \\ &= (\mathbf{b} + u\mathbf{B})(I_\mu\eta - i_\mu\pi(\eta)) && \text{(since } \text{bi}_\mu\pi(\eta) = 0). \end{aligned}$$

Now, $\pi(I_\mu\eta - i_\mu\pi(\eta)) = i_\mu\pi(\eta) - i_\mu\pi(\eta) = 0$, which means that $I_\mu\eta - i_\mu\pi(\eta)$ is divisible by u . Thus it follows that

$$L_\mu\eta - \text{Bi}_\mu\pi(\eta) = (\mathbf{b} + u\mathbf{B})(u^{-1}(I_\mu\eta - i_\mu\pi(\eta))),$$

hence the claim. □

Proof of Theorem 8.2. Let $(\mu_1, s^{-d-1}\eta_1)$ and $(\mu_2, s^{-d-1}\eta_2)$ be two cocycles in $\mathcal{D}^\bullet(A, \eta_0)$. We must prove for $\eta'_i = s^{-d-1}\eta_i$

$$\overline{s^{d-1}\Psi([\mu_1, \eta'_1], [\mu_2, \eta'_2])} = [\overline{s^{d-1}\Psi(\mu_1, \eta'_1)}, \overline{s^{d-1}\Psi(\mu_2, \eta'_2)}]. \tag{8.3}$$

We will first compute the lefthand side of (8.3). Writing out the differential in $\mathcal{D}^\bullet(A, \eta'_0)$ explicitly, the fact that $(\mu_1, \eta'_1), (\mu_2, \eta'_2)$ are cocycles implies

$$\begin{aligned} d\mu_1 &= d\mu_2 = 0, \\ (\mathbf{b} + u\mathbf{B})\eta'_1 - (-1)^{|\mu_1|}L_{\mu_1}\eta'_0 &= (\mathbf{b} + u\mathbf{B})\eta'_2 - (-1)^{|\mu_2|}L_{\mu_2}\eta'_0 = 0, \end{aligned} \tag{8.4}$$

where $\eta'_0 = s^{-d-1}\eta'_0$. We compute

$$\begin{aligned} x &\stackrel{\text{def}}{=} s^{d-1}\Psi([\mu_1, \eta'_1], [\mu_2, \eta'_2]) \\ &= s^{d-1}\Psi([\mu_1, \mu_2], L_{\mu_1}\eta'_2 - (-1)^{|\mu_1||\eta'_2|}L_{\mu_2}\eta'_1) \\ &= s^{d+1}((-1)^{|\mu_1|+|\mu_2|-1}I_{[\mu_1, \mu_2]}\eta'_0 + u(L_{\mu_1}\eta'_2 - (-1)^{|\mu_1||\mu_2|}L_{\mu_2}\eta'_1)), \end{aligned} \tag{8.5}$$

where we have used (8.1) and the fact that $|\eta'_2| = |\mu_2|$.

We now consider the boundary element $(\mathbf{b} + u\mathbf{B})I_{\mu_1}\eta'_2$. By (2.8), we have

$$(\mathbf{b} + u\mathbf{B})I_{\mu_1}\eta'_2 - (-1)^{|\mu_1|+1}I_{\mu_1}(\mathbf{b} + u\mathbf{B})\eta'_2 + I_{d\mu_1}\eta'_2 = uL_{\mu_1}\eta'_2.$$

Taking into account (8.4) this becomes

$$\begin{aligned} (\mathbf{b} + u\mathbf{B})I_{\mu_1}\eta'_2 &= (-1)^{|\mu_1|-1}I_{\mu_1}(\mathbf{b} + u\mathbf{B})\eta'_2 + uL_{\mu_1}\eta'_2 \\ &= (-1)^{|\mu_1|-1+|\mu_2|}I_{\mu_1}L_{\mu_2}\eta'_0 + uL_{\mu_1}\eta'_2 \end{aligned}$$

and similarly

$$(\mathbf{b} + u\mathbf{B})I_{\mu_2}\eta'_1 = (-1)^{|\mu_2|-1+|\mu_1|}I_{\mu_2}L_{\mu_1}\eta'_0 + uL_{\mu_2}\eta'_1.$$

We now subtract both boundaries with appropriate sign from (8.5) to obtain the following homologous cocycle

$$\begin{aligned} x &\cong (-1)^{|\mu_1|+|\mu_2|-1}s^{d+1}(I_{[\mu_1,\mu_2]}\eta'_0 - I_{\mu_1}L_{\mu_2}\eta'_0 + (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1}\eta'_0) \\ &= (-1)^{|\mu_1|+|\mu_2|-1}s^{d+1}(I_{[\mu_1,\mu_2]} - I_{\mu_1}L_{\mu_2} + (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1})\eta'_0. \end{aligned} \tag{8.6}$$

By Lemma 2.2 and (8.4):

$$[L_{\mu_1}, I_{\mu_2}] - (-1)^{|\mu_1|}I_{[\mu_1,\mu_2]} \cong 0.$$

Thus

$$\begin{aligned} I_{[\mu_1,\mu_2]} &\cong (-1)^{|\mu_1|}(L_{\mu_1}I_{\mu_2} - (-1)^{L_{\mu_1}||\mu_2|}I_{\mu_2}L_{\mu_1}) \\ &= (-1)^{|\mu_1|}(L_{\mu_1}I_{\mu_2} - (-1)^{|\mu_1|(|\mu_2|+1)}I_{\mu_2}L_{\mu_1}) \\ &= (-1)^{|\mu_1|}L_{\mu_1}I_{\mu_2} - (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1}. \end{aligned}$$

Substituting this in (8.6) we obtain

$$x \cong (-1)^{|\mu_1|+|\mu_2|-1}s^{d+1}((-1)^{|\mu_1|}L_{\mu_1}I_{\mu_2}\eta'_0 - I_{\mu_1}L_{\mu_2}\eta'_0). \tag{8.7}$$

Next we observe, using (2.8)

$$\begin{aligned} &[\mathbf{b} + u\mathbf{B}, I_{\mu_1}I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}I_{\mu_2\cup\mu_1}] \\ &= [\mathbf{b} + u\mathbf{B}, I_{\mu_1}]I_{\mu_2} + (-1)^{|\mu_1|+1}I_{\mu_1}[\mathbf{b} + u\mathbf{B}, I_{\mu_2}] \\ &\quad - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}[\mathbf{b} + u\mathbf{B}, I_{\mu_2\cup\mu_1}] \\ &= u(L_{\mu_1}I_{\mu_2} + (-1)^{|\mu_1|+1}I_{\mu_1}L_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}L_{\mu_2\cup\mu_1}), \end{aligned} \tag{8.8}$$

and also using (2.2):

$$\begin{aligned} I_{\mu_1}I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}I_{\mu_2\cup\mu_1} \bmod u &= i_{\mu_1}i_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}i_{\mu_2\cup\mu_1} \\ &= 0. \end{aligned}$$

In other words $I_{\mu_1}I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)}I_{\mu_2\cup\mu_1}$ is divisible by u and we obtain from (8.8)

$$L_{\mu_1}I_{\mu_2} + (-1)^{|\mu_1|+1}I_{\mu_1}L_{\mu_2} \cong (-1)^{(|\mu_1|+1)(|\mu_2|+1)}L_{\mu_2\cup\mu_1}.$$

Substituting this back in (8.7) we find

$$\begin{aligned} x &\cong (-1)^{|\mu_1|(|\mu_2|+1)} s^{d+1} L_{\mu_2 \cup \mu_1} \eta'_0 \\ &= (-1)^{|\mu_1|(|\mu_2|+1)} s^{d+1} \mathbf{B}i_{\mu_2 \cup \mu_1} \pi(\eta'_0) && \text{(by Lemma 8.3)} \\ &\cong (-1)^{|\mu_2|+1} s^{d+1} \mathbf{B}i_{\mu_1 \cup \mu_2} \pi(\eta'_0) && \text{(by §2.2)} \\ &\cong (-1)^{|\mu_2|+1} (-1)^{(|\mu_1|+|\mu_2|+1)(d+1)} \mathbf{B}i_{\mu_1 \cup \mu_2} \pi(\eta_0), \end{aligned}$$

and hence by (8.2)

$$\bar{x} = (-1)^{|\mu_2|+1} (-1)^{(|\mu_1|+|\mu_2|+1)(d+1)} \mathbf{B}(i_{\mu_1} \pi(\bar{\eta}_0) \cdot i_{\mu_2} \pi(\bar{\eta}_0)).$$

To compute the righthand side of (8.3) we note

$$\begin{aligned} \pi(s^{d-1} \Psi(\mu_i, \eta'_i)) &= \pi(s^{d+1} ((-1)^{|\mu_i|-1} I_{\mu_i} \eta'_0 + u \eta'_i)) && \text{(by (8.1))} \\ &= (-1)^{|\mu_i|-1} (-1)^{(|\mu_i|+1)(d+1)} i_{\mu_i} \pi(\eta_0), \end{aligned}$$

so that

$$\begin{aligned} &\overline{[s^{d-1} \Psi(\mu_1, \eta'_1), s^{d-1} \Psi(\mu_2, \eta'_2)]} \\ &= (-1)^{|\mu_1|+d} \mathbf{B}(\overline{s^{d-1} \Psi(\mu_1, \eta'_1)} \cdot \overline{s^{d-1} \Psi(\mu_1, \eta'_1)}) \\ &= (-1)^{|\mu_1|+d+|\mu_1|+|\mu_2|} (-1)^{(|\mu_1|+|\mu_2|)(d+1)} \mathbf{B}(\overline{i_{\mu_1} \pi(\eta_0)} \cdot \overline{i_{\mu_2} \pi(\eta_0)}) \\ &= (-1)^{|\mu_2|+1} (-1)^{(|\mu_1|+|\mu_2|+1)(d+1)} \mathbf{B}(i_{\mu_1} \pi(\bar{\eta}_0) \cdot i_{\mu_2} \pi(\bar{\eta}_0)), \end{aligned}$$

finishing the proof. □

9. The commutative case

In this section we will use formality results from [7, 16, 23, 28, 32] so we will assume that the ground field k contains \mathbb{R} . It is likely that this condition can be removed by using the methods from [9, 10] but we have not checked it.

Let $A = \mathcal{O}(X)$ where X is a smooth affine d -dimensional Calabi–Yau variety over k . Let $T^{\text{poly}, \bullet}(A)$ be the Lie algebra of poly-vector fields on X . We assume that $T^{\text{poly}, \bullet}(A)$ is shifted in such a way that the Lie bracket has degree zero. Similarly let $\Omega^\bullet(A)$ be the differential forms on X (not shifted).

We fix a volume form $\eta \in \Omega^d(A)$. The Hochschild–Kostant–Rosenberg map furnishes an isomorphism $\text{HH}_d(A) \cong \Omega^d(A)$. So we may consider η as an element in $\text{HH}_d(A)$ and hence by Proposition 3.7 as a cycle (still denoted by η) in $\text{CC}_d^-(A)$. It is well known and easy to see that (A, η) is a Calabi–Yau algebra in the sense of Ginzburg. Let

$$\text{div}: T^{\text{poly}, \bullet}(A) \longrightarrow T^{\text{poly}, \bullet-1}(A)$$

be the divergence operator corresponding to η (see §9.4 below). The divergence is a differential which acts as a derivation with respect to the Lie bracket on $T^{\text{poly}, \bullet}(A)$.

In this section we will prove the following result.

Theorem 9.1. *There exists an isomorphism*

$$(T^{\text{poly}, \bullet}(A)[[u]], -u \text{ div}) \cong \mathcal{D}^\bullet(A, \eta)$$

in the homotopy category of DG-Lie algebras which fits in a diagram like (1.3).

Recall that the homotopy category of DG-Lie algebras is the category of DG-Lie algebras with quasi-isomorphisms formally inverted.

9.1. Semi-direct products for L_∞ -algebras. We remind the reader of a few basic definition regarding L_∞ -algebras and modules. Let \mathfrak{h}^\bullet be a graded k -vector space. Recall that an L_∞ -structure on \mathfrak{h}^\bullet is a square zero, degree one coderivation Q on the symmetric coalgebra $S^c(\Sigma\mathfrak{h}^\bullet)$. Such an L_∞ -structure is determined by its Taylor coefficients $\partial^n Q$ which are maps $S^n(\Sigma\mathfrak{h}^\bullet) \rightarrow \Sigma\mathfrak{h}^\bullet$. Here and in related situations below we always assume that zeroth order Taylor coefficient are zero.

A DG-Lie algebra can be made into an L_∞ -algebra by putting

$$\partial^1 Q(sg) = -sdg, \quad \partial^2 Q(sg, sh) = (-1)^{|g|}s[g, h], \quad \partial^n Q = 0 \quad \text{for } n \geq 3.$$

A morphism of L_∞ -algebras $\psi : (\mathfrak{g}^\bullet, Q) \rightarrow (\mathfrak{h}^\bullet, Q)$ is a coalgebra morphism $\psi : S^c(\Sigma\mathfrak{g}^\bullet) \rightarrow S^c(\Sigma\mathfrak{h}^\bullet)$ commuting with Q . It is also determined by its Taylor coefficients $\partial^n \psi : S^n(\Sigma\mathfrak{g}^\bullet) \rightarrow \Sigma\mathfrak{h}^\bullet$.

If V^\bullet is a graded k -vector space then an L_∞ - \mathfrak{h}^\bullet -module structure on V^\bullet is a square zero, degree one differential $R : S^c(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet \rightarrow S^c(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet$ satisfying

$$(Q \otimes \text{Id}_{S^c\mathfrak{h}} \otimes \text{Id}_V + \text{Id}_{S^c\mathfrak{h}} \otimes R) \circ (\Delta \otimes \text{Id}_V) = (\Delta \otimes \text{Id}_V) \circ R$$

as maps from $S^c(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet$ to $S^c(\Sigma\mathfrak{h}^\bullet) \otimes S^c(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet$. An L_∞ - \mathfrak{h}^\bullet -module structure R on V^\bullet is entirely determined by the maps $\partial^{n+1} R : S^n(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet \rightarrow V^\bullet$. If \mathfrak{h}^\bullet is a DG-Lie algebra and V^\bullet is a DG-module over it then V^\bullet can be made into an L_∞ -module over \mathfrak{h} by putting

$$\partial^1 R(v) = dv, \quad \partial^2 R(sg, v) = g \cdot v, \quad \partial^n R = 0 \quad \text{for } n \geq 3.$$

If V^\bullet is an L_∞ - \mathfrak{h}^\bullet -module then so are $\Sigma^m V^\bullet$ for all m using the obvious sign convention

$$\partial^{n+1} R(sg_1, \dots, sg_n, s^m v) = (-1)^{m(n+|g_1|+\dots+|g_n|)} \partial^{n+1} R(sg_1, \dots, sg_n, v).$$

We may combine the L_∞ -structures on \mathfrak{h}^\bullet and ΣV^\bullet to make the direct sum $\mathfrak{h}^\bullet \oplus V^\bullet$ into an L_∞ -algebra. We will denote the resulting L_∞ -algebra by $\mathfrak{h}^\bullet \ltimes V^\bullet$ and call it the *semi-direct product* of \mathfrak{h}^\bullet . This is an obvious generalization of the semi-direct product of a DG-Lie algebra with a DG-module which was used in §6.

Assume that $(V^\bullet, R), (W^\bullet, R)$ are L_∞ - \mathfrak{h}^\bullet -modules. An L_∞ morphism $\mu: V^\bullet \rightarrow W^\bullet$ is a comodule map $\mu: S^c(\Sigma\mathfrak{g}) \otimes V^\bullet \rightarrow S^c(\Sigma\mathfrak{g}) \otimes W^\bullet$, commuting with R . It is determined by its Taylor coefficients $\partial^n \mu: S^n(\Sigma\mathfrak{h}^\bullet) \otimes V^\bullet \rightarrow W^\bullet$.

Given in addition an L_∞ -morphism $\psi: \mathfrak{g}^\bullet \rightarrow \mathfrak{h}^\bullet$ the pullback V_ψ^\bullet of V^\bullet is defined as follows:

$$\begin{aligned} & \partial^{n+1} R_\psi(sg_1, \dots, sg_n, v) \\ &= \sum_{t, 1 \leq m_1 < \dots < m_{t-1} < n} \pm \partial^{t+1} R(\partial^{m_1} \psi(sg_{i_1}, \dots, sg_{i_{m_1}}), \partial^{m_2 - m_1} \psi(sg_{i_{m_1+1}}, \dots, sg_{i_{m_2}}), \dots, \\ & \dots, \partial^{n - m_{t-1} - 1} \psi(sg_{i_{m_{t-1}+1}}, \dots, sg_n), v), \end{aligned}$$

where for all $j: i_{m_{j+1}} < \dots < i_{m_{j+1}}$ and the sign is the Koszul sign of the corresponding shuffle of the $(sg_i)_i$. By construction we have a canonical L_∞ -morphism

$$\psi_V: \mathfrak{g}^\bullet \times V_\psi^\bullet \rightarrow \mathfrak{h}^\bullet \times V^\bullet,$$

which restricted to $S^n(\Sigma\mathfrak{g})$ coincides with $\partial^n \psi$.

9.2. Twisting. Assume that $\psi: \mathfrak{g}^\bullet \rightarrow \mathfrak{h}^\bullet$ is a L_∞ -morphism between L_∞ -algebras equipped with some type of topology and let $\omega \in \mathfrak{g}^1$ be a Maurer–Cartan element in \mathfrak{g}^1 , i.e. a solution of the L_∞ -Maurer–Cartan equation

$$\sum_{i \geq 1} \frac{1}{i!} (\partial^i Q)(\underbrace{\omega \cdots \omega}_i) = 0.$$

One has to assume that one is in a situation where all occurring series are convergent and standard series manipulations are allowed. In our application below the series are in fact finite.

Define Q_ω, ψ_ω and ω' by [33]

$$(\partial^i Q_\omega)(\gamma) = \sum_{j \geq 0} \frac{1}{j!} (\partial^{i+j} Q)(\underbrace{\omega \cdots \omega}_j \gamma) \quad (\text{for } i > 0) \tag{9.1}$$

$$(\partial^i \psi_\omega)(\gamma) = \sum_{j \geq 0} \frac{1}{j!} (\partial^{i+j} \psi)(\underbrace{\omega \cdots \omega}_j \gamma) \quad (\text{for } i > 0) \tag{9.2}$$

$$\omega' = \sum_{j \geq 1} \frac{1}{j!} (\partial^j \psi)(\underbrace{\omega \cdots \omega}_j) \tag{9.3}$$

for $\gamma \in S^i(\Sigma\mathfrak{g}^\bullet)$. Then e.g. by [33] ω' is a solution of the Maurer–Cartan equation on \mathfrak{h}^\bullet and furthermore $\mathfrak{g}^\bullet, \mathfrak{h}^\bullet$, when equipped with $Q_\omega, Q_{\omega'}$ are again L_∞ -algebras. Let us denote these by $\mathfrak{g}_\omega^\bullet$ and $\mathfrak{h}_{\omega'}^\bullet$. Finally ψ_ω is an L_∞ map $\mathfrak{g}_\omega^\bullet \rightarrow \mathfrak{h}_{\omega'}^\bullet$.

9.3. Applying formality to $\mathfrak{D}(A, \eta)$. By [4, 16, 33] there is an L_∞ -quasi-isomorphism

$$\mathfrak{U}: T^{\text{poly}, \bullet}(A) \rightarrow \overline{\mathfrak{C}}^\bullet(A)$$

such that $\partial^1 \mathfrak{U}$ is the standard Hochschild–Kostant–Rosenberg quasi-isomorphism.

View $(\overline{\text{CC}}_\bullet^-(A), \mathfrak{b} + u\mathfrak{B})$ as an L_∞ -module over $T^{\text{poly}, \bullet}(A)$ via \mathfrak{U} as in §9.1. We also view $(\Omega^\bullet(A)[[u]], ud)$ as a DG-Lie module over $T^{\text{poly}, \bullet}(A)$ via the Lie derivative. Then by [7, 23, 28, 32] there is an L_∞ -quasi-morphism of L_∞ -modules over $T^{\text{poly}, \bullet}(A)$

$$\mathfrak{S}: (\overline{\text{CC}}_\bullet^-(A), \mathfrak{b} + u\mathfrak{B}) \longrightarrow (\Omega^\bullet(A)[[u]], ud),$$

where $\partial^1 \mathfrak{S}$ is again the HKR quasi-isomorphism. Thus we get a roof of L_∞ -quasi-morphisms of graded DG-Lie algebras

$$\begin{array}{ccc} & T^{\text{poly}, \bullet}(A) \times \Sigma^{-d-1} \overline{\text{CC}}_\bullet^-(A) & \\ \mathfrak{S} \swarrow & & \searrow \mathfrak{U} \\ T^{\text{poly}, \bullet}(A) \times \Sigma^{-d-1} \Omega^\bullet(A)[[u]] & & \overline{\mathfrak{C}}^\bullet(A) \times \Sigma^{-d-1} \overline{\text{CC}}_\bullet^-(A) \end{array} \tag{9.4}$$

We obtain a new roof by twisting with $(0, \eta')$ where $\eta' = s^{-d-1} \eta$.

$$\begin{array}{ccc} & (T^{\text{poly}, \bullet}(A) \times \Sigma^{-d-1} \overline{\text{CC}}_\bullet^-(A))_{(0, \eta')} & \\ \mathfrak{S}_{(0, \eta')} \swarrow & & \searrow \mathfrak{U}_{(0, \eta')} \\ \mathfrak{T}^\bullet(A, \eta) & & \mathfrak{D}^\bullet(A, \eta) \end{array} \tag{9.5}$$

where

$$\mathfrak{T}^\bullet(A, \eta) = (T^{\text{poly}, \bullet}(A) \times \Sigma^{-d-1} \Omega^\bullet(A)[[u]])_{(0, \eta')}.$$

The complexes here are 2-step filtered. The arrows are quasi-isomorphisms since if we take the associated graded complexes for the 2-step filtrations we find the same arrows as in (9.4).

9.4. Divergence etc. . . . The divergence operator is defined by

$$\text{div}: T^{\bullet, \text{poly}}(A) \longrightarrow T^{\bullet-1, \text{poly}}(A)$$

via the following identity

$$d(\gamma \cap \eta) = \text{div } \gamma \cap \eta.$$

We conclude immediately

$$\text{div}^2 = 0$$

and furthermore the following is true [21]:

$$(-1)^{|\gamma_1|} [\gamma_1, \gamma_2] = \text{div}(\gamma_1 \gamma_2) - \text{div}(\gamma_1) \gamma_2 - (-1)^{|\gamma_1|+1} \gamma_1 \text{div } \gamma_2.$$

So $(T^{\text{poly}, \bullet}(A), -\text{div}, \cup)$ is a BV-algebra (see App. A).

Proposition 9.2. *There is an L_∞ -isomorphism of DG-Lie algebras*

$$\delta: (T^{\text{poly}, \bullet}(A)[[u]], -u \operatorname{div}) \longrightarrow \mathfrak{T}^\bullet(A).$$

Proof. According to Proposition A.4 there exists an L_∞ -isomorphism

$$(T^{\text{poly}, \bullet}(A)[[u]], -u \operatorname{div}) \longrightarrow (T^{\text{poly}, \bullet}(A) \ltimes \mathfrak{a}, -u \operatorname{div}),$$

where \mathfrak{a} is the abelian graded Lie algebra on the vector space $uT^{\text{poly}, \bullet}(A)[[u]]$. The action of $T^{\text{poly}, \bullet}(A)$ on \mathfrak{a} is given by

$$\gamma \star a = [\gamma, a] + (-1)^{|\gamma|} \operatorname{div} \gamma \cup a.$$

To finish the proof it is sufficient to show that the following map

$$\begin{aligned} \delta': (T^{\text{poly}, \bullet}(A) \ltimes \mathfrak{a}, -u \operatorname{div}) &\longrightarrow \mathfrak{T}^\bullet(A, \eta) \\ &= (T^{\text{poly}, \bullet}(A) \ltimes \Sigma^{-d-1} \Omega^\bullet(A)[[u]])_{(0, \eta')}: (\gamma, a) \mapsto (\gamma, (-1)^{|a|} u^{-1} a \cap \eta') \end{aligned}$$

is an isomorphism of DG-Lie algebras. First we show that

$$\delta': \mathfrak{a} \longrightarrow \Sigma^{-d-1} \Omega^\bullet(A)[[u]]: a \mapsto (-1)^{|a|} u^{-1} (a \cap \eta')$$

is compatible with the action of $T^{\text{poly}, \bullet}(A)$. We compute for $\gamma \in T^{\text{poly}, \bullet}(A)$ and $a \in \mathfrak{a}$.

$$\begin{aligned} \delta'(\gamma \star a) &= \delta'([\gamma, a] + (-1)^{|\gamma|} \operatorname{div} \gamma \cup a) \\ &= (-1)^{|\gamma|+|a|} u^{-1} ([\gamma, a] + (-1)^{|\gamma|} \operatorname{div} \gamma \cup a) \cap \eta' \\ &= (-1)^{|\gamma|+|a|} u^{-1} ((-1)^{|\gamma|} \operatorname{div}(\gamma \cup a) - (-1)^{|\gamma|} \operatorname{div}(\gamma) \cup a \\ &\quad + \gamma \cup \operatorname{div}(a) + (-1)^{|\gamma|} \operatorname{div}(\gamma) \cup a) \cap \eta' \\ &= (-1)^{|\gamma|+|a|} u^{-1} ((-1)^{|\gamma|} \operatorname{div}(\gamma \cup a) + \gamma \cup \operatorname{div}(a)) \cap \eta' \\ &= (-1)^{|\gamma|+|a|} u^{-1} ((-1)^{|\gamma|} d(\gamma \cap (a \cap \eta')) + \gamma \cap d(a \cap \eta')) \\ &= L_\gamma(\delta'(a)). \end{aligned}$$

Now we check compatibility with the differential of δ' on an element $a \in \mathfrak{a}$.

$$\begin{aligned} \delta'(-u \operatorname{div} a) &= -(-1)^{|a|+1} \operatorname{div} a \cap \eta' \\ &= (-1)^{|a|} d(a \cap \eta') \\ &= d(\delta'(a)). \end{aligned}$$

Finally we check compatibility with the differential of δ' on $\gamma \in T^{\text{poly}, \bullet}(A)$.

$$\begin{aligned} \delta'(-u \operatorname{div} \gamma) &= -(-1)^{|\gamma|+1} \operatorname{div} \gamma \cap \eta' \\ &= (-1)^{|\gamma|} d(\gamma \cap \eta') \\ &= (-1)^{|\gamma|} L_a \eta' \\ &= [(0, \eta'), (\gamma, 0)]. \end{aligned}$$

□

Proof of Theorem 9.1. It suffices to combine diagram (9.5) with Proposition 9.2, taking into account that an L_∞ -quasi-isomorphism yields an isomorphism in the homotopy category of DG-Lie algebras via the bar cobar construction. \square

10. Obstructions

Let \mathfrak{g}^\bullet be a DG-Lie algebra and let $(S, n) \rightarrow (R, m)$ be a surjective morphism in Nilp with one-dimensional kernel $ks \subset n$. Let $x \in \mathfrak{g}^1 \otimes m$ be a solution to the Maurer–Cartan equation. Lift x to an arbitrary element \hat{x} of $\mathfrak{g}^1 \otimes n$ and let $p(\hat{x}) \in \mathfrak{g}^2$ be such that $p(\hat{x})s = d\hat{x} + \frac{1}{2}[\hat{x}, \hat{x}]$. Then clearly $dp(\hat{x}) = 0$ and furthermore the cohomology class

$$o(x) \stackrel{\text{def}}{=} \overline{p(\hat{x})} \in H^1(\mathfrak{g}^\bullet)$$

does not depend on the chosen lift \hat{x} of x . It is easy to see that $o(x) = 0$ if and only if x can be lifted to an element of $\text{MC}(\mathfrak{g}^\bullet \otimes n)$. Consequently $o(x)$ is called the *obstruction class* of x .

The *obstruction space* $O(\mathfrak{g}^\bullet)$ is the linear span in \mathfrak{g}^2 of all $o(x)$ for all morphisms $(S, n) \rightarrow (R, m)$ with one-dimensional kernel and all $x \in \text{MC}(\mathfrak{g}^\bullet \otimes m)$ as above.

Clearly $o(x)$ and hence $O(\mathfrak{g}^\bullet)$ is functorial under DG-Lie algebra morphisms. It is well known and easy to see that this functoriality extends to L_∞ -morphisms.

Recall that the *periodic cyclic complex* $\text{CC}_\bullet^{\text{per}}(A)$ of a k -algebra A is obtained by inverting u in $\text{CC}_\bullet^-(A)$. Its homology will be denoted by $\text{HC}_\bullet^{\text{per}}(A)$. The following is the main result of this section.

Theorem 10.1. *Let $(A, \bar{\eta})$ be a d -Calabi–Yau algebra. Then the composition*

$$O(\mathfrak{D}^\bullet(A, \eta)) \hookrightarrow H^2(\mathfrak{D}^\bullet(A, \eta)) \stackrel{\text{Thm 8.1}}{\cong} \text{HC}_{d-3}^-(A) \rightarrow \text{HC}_{d-3}^{\text{per}}(A)$$

is zero.

The proof depends on the following beautiful result by Tsygan and Daletskii [27, Thm. 1] (see also [5]).

Theorem 10.2. *The Lie action of $\mathfrak{C}^\bullet(A)$ on $\text{CC}_\bullet^-(A)$ can be extended to a u -linear L_∞ -action of the DG-Lie algebra $(\mathfrak{C}^\bullet(A)[u, \epsilon], d + u\partial/\partial\epsilon)$, with $|\epsilon| = 1, \epsilon^2 = 0$ and such that*

$$\begin{aligned} \partial^1 R(\gamma) &= d\gamma \\ \partial^2 R(s\sigma, \gamma) &= L_\sigma \gamma \\ \partial^2 R(s(\epsilon\sigma), \gamma) &= I_\sigma \gamma \end{aligned}$$

for $\sigma \in \mathfrak{C}^\bullet(A), \gamma \in \text{CC}_\bullet^-(A)$.

The statement about $\partial^2 R(s(\epsilon\sigma), \gamma)$ does not occur in [28, Thm. 1] but it follows easily from the proof.

In the rest of this section $(A, \bar{\eta})$ is a d -Calabi–Yau algebra.

Lemma 10.3. *There is a commutative diagram of complexes*

$$\begin{array}{ccc}
 (\mathfrak{C}^\bullet(A) \times \Sigma^{-d-1} \text{CC}_\bullet^-(A))_{(0,\eta')} & \xrightarrow{\hspace{10em}} & (\mathfrak{C}^\bullet(A)[u, \epsilon] \times \Sigma^{-d-1} \text{CC}_\bullet^-(A))_{(0,\eta')} \\
 \searrow \Psi & & \swarrow \Psi' \\
 & \Sigma^{-d+1} \text{CC}_\bullet^-(A) &
 \end{array}
 \tag{10.1}$$

where

- Ψ was introduced in Theorem 8.1;
- $\eta = s^{-d-1} \eta'$;
- the horizontal map is a twist (see §9.2) of the map obtained from the obvious inclusion of DG-Lie algebras

$$(\mathfrak{C}^\bullet(A), d) \hookrightarrow (\mathfrak{C}^\bullet(A)[u, \epsilon], d + \partial/\partial\epsilon);$$

- Ψ' restricted to $\mathfrak{C}^\bullet(A)[u, \epsilon]$ is u -linear and satisfies

$$\begin{aligned}
 \Psi'(\sigma) &= (-1)^{|\sigma|+1} I_\sigma \eta' \\
 \Psi'(\epsilon\sigma) &= 0
 \end{aligned}
 \tag{10.2}$$

for $\sigma \in \mathfrak{C}^\bullet(A)$;

- Ψ' restricted to $\Sigma^{-d-1} \text{CC}_\bullet^-(A)$ is multiplication by u .

Proof. The commutativity of the diagram is clear. We only have to show that Ψ' commutes with the differential. For Ψ' restricted to $\Sigma^{-d-1} \text{CC}_\bullet^-(A)$ this is obvious. As far as the restriction of Ψ' to $\mathfrak{C}^\bullet(A)[u, \epsilon]$ is concerned: the only non-trivial case (given that Ψ already commutes with the differential) is the evaluation on an element of $\epsilon\mathfrak{g}$.

Using (9.1) we find for $\sigma \in \mathfrak{C}^\bullet(A)$

$$d_{(0,\eta')}(\epsilon\sigma) = (d(\epsilon\sigma), (-1)^{|\sigma|} I_\sigma \eta').$$

Given (10.2) we have to show

$$\Psi'(d_{(0,\eta')}(\epsilon\sigma)) = 0.$$

We compute

$$\begin{aligned}
 \Psi'(d_{(0,\eta')}(\epsilon\sigma)) &= \Psi'(d(\epsilon\sigma), (-1)^{|\sigma|} I_\sigma \eta') \\
 &= \Psi'(-\epsilon d\sigma + u\sigma, (-1)^{|\sigma|} I_\sigma \eta') \\
 &= (-1)^{|\sigma|+1} u I_\sigma \eta' + (-1)^{|\sigma|} u I_\sigma \eta' \\
 &= 0.
 \end{aligned}
 \tag{□}$$

Lemma 10.4. *Consider $\Sigma^{-d+1} \text{CC}_\bullet^{\text{per}}(A)$ as an abelian DG-Lie algebra. Then there exists an L_∞ -morphism*

$$\Delta: \mathfrak{D}^\bullet(A, \eta) \rightarrow \Sigma^{-d+1} \text{CC}_\bullet^{\text{per}}(A)$$

such that the following diagram is commutative

$$\begin{array}{ccc} H^\bullet(\mathfrak{D}^\bullet(A, \eta)) & \xrightarrow{H^\bullet(\Psi)} & H^\bullet(\Sigma^{-d+1} \text{CC}_\bullet^-(A)) \\ & \searrow H^\bullet(\Delta) & \downarrow \text{canonical} \\ & & H^\bullet(\Sigma^{-d+1} \text{CC}_\bullet^{\text{per}}(A)) \end{array}$$

Proof. To simplify the notations put

$$\mathfrak{g}^\bullet = \mathfrak{C}^\bullet(A), \quad V^- = \Sigma^{-d-1} \text{CC}_\bullet^-(A), \quad V^{\text{per}} = \Sigma^{-d-1} \text{CC}_\bullet^{\text{per}}(A).$$

Thus we get L_∞ -morphisms (see §§9.1, 9.2)

$$\begin{aligned} (\mathfrak{g}^\bullet \ltimes V^-)_{(0, \eta')} &\rightarrow (\mathfrak{g}^\bullet[u, \epsilon] \ltimes V^-)_{(0, \eta')} \rightarrow (\mathfrak{g}^\bullet[u, u^{-1}, \epsilon] \ltimes V^{\text{per}})_{(0, \eta')} \\ &\xleftarrow{c} (0 \ltimes V^{\text{per}})_{(0, \eta')} \cong V^{\text{per}} \xrightarrow{\times u} \Sigma^2 V^{\text{per}}. \end{aligned} \quad (10.3)$$

Here c goes in the wrong direction but it is easy to see that $(\mathfrak{g}^\bullet[u, u^{-1}, \epsilon], d + u\partial/\partial\epsilon)$ is acyclic. Hence c is an quasi-isomorphism. This means that there is an L_∞ -quasi-isomorphism c' which goes in the opposite direction and which inverts c on the level of cohomology. Taking the composition of everything we obtain an L_∞ -morphism

$$(\mathfrak{g}^\bullet \ltimes V^-)_{(0, \eta')} \longrightarrow \Sigma^2 V^{\text{per}},$$

which is the desired Δ .

It remains to show that Δ and Ψ are compatible on the level of cohomology. This follows from the following commutative diagram whose upper row is a compressed version of (10.3) and whose lower row we obtain from (10.1).

$$\begin{array}{ccccccc} & & \Delta & & & & \\ & \curvearrowright & & \curvearrowleft & & & \\ (\mathfrak{g}^\bullet \ltimes V^-)_{(0, \eta')} & \longrightarrow & (\mathfrak{g}^\bullet[u, u^{-1}, \epsilon] \ltimes V^{\text{per}})_{(0, \eta')} & \xrightarrow{c'} & V^{\text{per}} & \xrightarrow{\times u} & \Sigma^2 V^{\text{per}} \\ & & \cong & \xleftarrow{c} & \parallel & & \parallel \\ \Psi \downarrow & & \Psi' \downarrow & & V^{\text{per}} & \xrightarrow{\times u} & \Sigma^2 V^{\text{per}} \\ \Sigma^2 V^- & \longrightarrow & \Sigma^2 V^{\text{per}} & \xleftarrow{\times u} & & & \\ & \curvearrowright & & \curvearrowleft & & & \\ & & \text{canonical} & & & & \square \end{array}$$

Proof of Theorem 10.1. The theorem follows from Lemma 10.4 together with the functoriality of obstruction spaces under L_∞ -morphisms and the fact that the obstruction space of an abelian Lie algebra is trivial. \square

Corollary 10.5. *If the map $\mathrm{HC}_{d-3}^-(A) \longrightarrow \mathrm{HC}_{d-3}^{\mathrm{per}}(A)$ is injective then the deformation theory of A is unobstructed.*

This corollary applies for example in the case $d \leq 3$ by the following well-known lemma.

Lemma 10.6. $\mathrm{HC}_n^-(A) \longrightarrow \mathrm{HC}_n^{\mathrm{per}}(A)$ is an isomorphism for $n \leq 0$.

Proof. There is an exact sequence

$$\mathrm{HC}_{n-1}(A) \longrightarrow \mathrm{HC}_n^-(A) \longrightarrow \mathrm{HC}_n^{\mathrm{per}}(A) \longrightarrow \mathrm{HC}_{n-2}(A)$$

(e.g. [18, Prop. 5.1.5]) where $\mathrm{HC}_\bullet(A)$ denotes ordinary cyclic homology. The complex computing ordinary cyclic homology is concentrated in homological degrees ≥ 0 . Hence $\mathrm{HC}_n(A) = 0$ for $n < 0$. This finishes the proof. \square

Remark 10.7. Many 3-dimensional Calabi–Yau algebras are obtained from superpotentials (see [1, 29]). For those it is not very surprising that the deformation theory is unobstructed (the deformations come from deforming the superpotential). However there are examples of 3-dimensional Calabi–Yau algebras which are not obtained from superpotentials. See e.g. [6]. Simple examples are given by 3-dimensional smooth commutative Calabi–Yau algebras with no exact volume form.

A. A technical result on BV-algebras

Recall that a DG-BV-algebra is a quadruple $(\mathfrak{g}^\bullet, d, \Delta, \cup)$ where $(\mathfrak{g}^\bullet, d)$ is a complex, \cup is a commutative, associative product of degree⁴ 1 on \mathfrak{g}^\bullet compatible with d , Δ is a differential of degree -1 , $(\mathfrak{g}^\bullet, d, [-, -])$ is a DG-Lie algebra with $[-, -]$ defined by:

$$[g, h] = (-1)^{|g|+1}(\Delta(g \cup h) - \Delta g \cup h - (-1)^{|g|+1}g \cup \Delta h)$$

and $\cup, [-, -]$ are related by the Leibniz rule:

$$[g, h_1 \cup h_2] = [g, h_1] \cup h_2 + (-1)^{|g|(|h_1|+1)}h_1 \cup [g, h_2]$$

It is shown in [14, 25] that if \mathfrak{h}^\bullet is a DG-BV-algebra then $(\mathfrak{h}^\bullet((u)), d + u\Delta)$ is homotopy abelian. The same proof works for $u\mathfrak{h}^\bullet[[u]], d + u\Delta)$ but not for $(\mathfrak{h}^\bullet[[u]], d + u\Delta)$. Our aim in this section is to make $(\mathfrak{h}^\bullet[[u]], d + u\Delta)$ as

⁴As always our grading conventions are such that Lie brackets have degree zero.

“commutative as possible” (see Proposition A.4 below) by making at least its sub-DG-Lie algebra $(u\mathfrak{h}^\bullet[[u]], d + u\Delta)$ abelian. This is not completely straightforward since in order to do this we have to twist the action of \mathfrak{h}^\bullet on $u\mathfrak{h}^\bullet[[u]]$.

The fact that $(\mathfrak{h}^\bullet((u)), d + u\Delta)$ and $(u\mathfrak{h}^\bullet[[u]], d + u\Delta)$ are homotopy abelian is in fact a special case of a general result in [22]. For the benefit of the reader we repeat the proof of this result. Afterwards we will reuse the proof to treat $(\mathfrak{h}^\bullet[[u]], d + u\Delta)$.

It is convenient to use the following adhoc definition.

Definition A.1. A BV_- algebra is a DG-Lie algebra \mathfrak{g}^\bullet equipped with a commutative, associative product \cup of degree -1 , compatible with d , such that

$$[g, h] = (-1)^{|g|+1}(d(g \cup h) - dg \cup h - (-1)^{|g|+1}g \cup dh) \tag{A.1}$$

and

$$[g, h_1 \cup h_2] = [g, h_1] \cup h_2 + (-1)^{|g|(|h_1|+1)}h_1 \cup [g, h_2]. \tag{A.2}$$

Lemma A.2 ([22]). *Let \mathfrak{g}^\bullet be a BV_- -algebra and let \mathfrak{a}^\bullet be the same as \mathfrak{g}^\bullet but with the Lie bracket set to zero. Then there is a L_∞ -morphism $\psi: \mathfrak{g}^\bullet \rightarrow \mathfrak{a}^\bullet$ such $\partial^1\psi$ is the identity. In other words \mathfrak{g}^\bullet is homotopy abelian.*

Example A.3. Let $(\mathfrak{h}^\bullet, d, \Delta, \cup)$ be a DG-BV-algebra. Then

$$(u\mathfrak{h}^\bullet[[u]], d + u\Delta, [-, -], u^{-1}(- \cup -))$$

is a BV_- -algebra and hence by the previous lemma $(u\mathfrak{h}^\bullet[[u]], d + u\Delta)$ is homotopy abelian. The same reasoning applies to $(\mathfrak{h}^\bullet((u)), d + u\Delta)$.

Proof of Lemma A.2. Put $V^\bullet = \Sigma\mathfrak{g}^\bullet$. The coderivation Q on S^cV^\bullet corresponding to the DG-Lie structure is given by

$$\begin{aligned} \partial^1 Q: V^\bullet &\rightarrow V^\bullet: sg \mapsto -s dg \\ \partial^2 Q: S^2V^\bullet &\rightarrow V^\bullet: (sg, sh) \mapsto (-1)^{|g|}s[g, h] \end{aligned}$$

and all other $\partial^n Q$ are zero.

For simplicity of notation we put

$$sg_1 \cdot sg_2 \cdots sg_n = s(g_1 \cup \cdots \cup g_n).$$

From (A.1) and (A.2) we obtain:

$$\begin{aligned} \partial^1 Q(v_1 \cdot v_2 \cdots v_n) &= \sum_i \epsilon_i \partial^1 Q(v_i) v_1 \cdots \hat{v}_i \cdots v_n \\ &\quad + \sum_{i < j} \epsilon_{i,j} \partial^2 Q(v_i, v_j) v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n, \tag{A.3} \end{aligned}$$

where the signs are determined by

$$\begin{aligned} v_1 \cdot v_2 \cdots v_n &= \epsilon_i v_i \cdot v_1 \cdots \hat{v}_i \cdots v_n \\ &= \epsilon_{i,j} v_i \cdot v_j \cdot v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n. \end{aligned}$$

Consider $\partial^1 Q$ as a coderivation of $S^c V^\bullet$ and let $\psi: S^c V^\bullet \rightarrow S^c V^\bullet$ be the coalgebra automorphism determined by

$$\partial^n \psi(v_1, \dots, v_n) = v_1 \cdot v_2 \cdots v_n.$$

Then (A.3) becomes

$$\partial^1 Q \circ \psi = \psi \circ Q,$$

which finishes the proof. □

Proposition A.4. *Let $(\mathfrak{h}^\bullet, d, \Delta, \cup)$ be a DG-BV-algebra. Let \mathfrak{a}^\bullet be the graded vector space $u\mathfrak{h}^\bullet[[u]]$. The following operation*

$$h \star a = [h, a] + (-1)^{|h|+1} \Delta(h) \cup a \tag{A.4}$$

for $h \in \mathfrak{h}^\bullet, a \in \mathfrak{a}^\bullet$ makes \mathfrak{a}^\bullet into a graded \mathfrak{h}^\bullet -representation. Furthermore $d + u\Delta$ defines a derivation on the Lie algebra $\mathfrak{h}^\bullet \ltimes \mathfrak{a}^\bullet$ and finally there is an L_∞ -isomorphism

$$\phi: \mathfrak{h}^\bullet[[u]] \rightarrow (\mathfrak{h}^\bullet \ltimes \mathfrak{a}^\bullet, d + u\Delta)$$

such that $\partial^1 \phi$ is the identity.

Proof. In the proof below we identify the underlying vector spaces of $\mathfrak{h}^\bullet[[u]]$ and $\mathfrak{h}^\bullet \ltimes \mathfrak{a}^\bullet$ in the obvious way. The fact that (A.4) defines indeed a representation as well as compatibility with differentials is an easy direct verification: Now put $V^\bullet = \Sigma \mathfrak{a}^\bullet, W^\bullet = \Sigma \mathfrak{h}^\bullet$. Let Q be the coderivation on $S^c(W^\bullet \oplus V^\bullet)$ corresponding to $\mathfrak{h}^\bullet[[u]]$. We observe that $\partial^1 Q|_{W^\bullet} = \partial^1 Q_1 + \partial^2 Q_2$ where $\partial^1 Q_1 = -d$ and $\partial^1 Q_2 = -u\Delta$. Let Q' be the coderivation on $S^c(W^\bullet \oplus V^\bullet)$ corresponding to $(\mathfrak{h}^\bullet \ltimes \mathfrak{a}^\bullet, d + u\Delta)$. We have $\partial^1 Q' = \partial^1 Q$. Furthermore

$$\begin{aligned} \partial^2 Q'(w_1, w_2) &= \partial^2 Q(w_1, w_2) \quad \text{for } w_1, w_2 \in W^\bullet, \\ \partial^2 Q'(v_1, v_2) &= 0 \quad \text{for } v_1, v_2 \in V^\bullet, \end{aligned}$$

and for $h \in \mathfrak{h}^\bullet, a \in \mathfrak{a}^\bullet$

$$\begin{aligned} \partial^2 Q'(sh, sa) &= (-1)^{|h|} s(h \star a) \\ &= (-1)^{|h|} s[h, a] - s(\Delta h \cup a) \\ &= \partial^2 Q(sh, sa) + \partial^1 Q_2(sh) \cdot sa, \end{aligned}$$

where as above $x \cdot y = u^{-1}(x \cup y)$. In other words

$$\partial^2 Q'(w, v) = \partial^2 Q(w, v) + \partial^1 Q_2(w) \cdot v \quad \text{for } w \in W^\bullet, v \in V^\bullet. \tag{A.5}$$

We now construct the desired L_∞ -morphism. By definition $\partial^n \psi = \text{Id}$ for $n = 1$. For $n > 1, i \geq 1, w_1, \dots, w_i \in W^\bullet, v_1, \dots, v_j \in V^\bullet$ we put

$$\partial^n \psi(w_1, \dots, w_i, v_1, \dots, v_j) = 0$$

and

$$\partial^n \psi(v_1, \dots, v_j) = v_1 \cdot v_2 \cdots v_n.$$

We now verify

$$\psi \circ Q = Q' \circ \psi$$

We must evaluate both sides on $S^i W^\bullet \otimes S^j V^\bullet$. If $i = 0$ then the desired equality follows from the proof of Lemma A.2. If $i > 2$ then both sides are zero so this case is trivial as well. If $i = 2$ then both sides are zero unless $j = 0$ in which case we reduce to $\partial^2 Q|_{S^2 W^\bullet} = \partial^2 Q'|_{S^2 W^\bullet}$.

We concentrate on the case $i = 1$. We find

$$(Q' \circ \psi)(w_1, v_1, \dots, v_j) = \partial^2 Q'(w_1, v_1 \cdot v_2 \cdots v_n)$$

and

$$\begin{aligned} (\psi \circ Q)(w_1, v_1, \dots, v_j) &= \partial^1 Q_2(w_1) \cdot v_1 \cdots v_j + \sum_l \pm \partial^2 Q(w_1, v_l) \cdot v_1 \cdots \hat{v}_l \cdots v_j \\ &= \partial^1 Q_2(w_1) \cdot v_1 \cdots v_j + \partial^2 Q(w_1, v_1 \cdot v_2 \cdots v_j). \end{aligned}$$

We conclude by (A.5). □

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