

On invariants of C^* -algebras with the ideal property

Kun Wang

Dedicated to the 60th birthday of Professor Chunlan Jiang

Abstract. In this paper, we study the relation between the extended Elliott invariant and the Stevens invariant of C^* -algebras. We show that in general the Stevens invariant can be derived from the extended Elliott invariant in a functorial manner. We also show that these two invariants are isomorphic for C^* -algebras satisfying the ideal property. A C^* -algebra is said to have the ideal property if each of its closed two-sided ideals is generated by projections inside the ideal. Both simple, unital C^* -algebras and real rank zero C^* -algebras have the ideal property. As a consequence, many classes of non-simple C^* -algebras can be classified by their extended Elliott invariants, which is a generalization of Elliott's conjecture.

Mathematics Subject Classification (2010). 46L35; 46L05.

Keywords. Classification of C^* -algebras, ideal property, extended Elliott invariant, Stevens invariant.

1. Introduction

George Elliott initiated the classification program of nuclear C^* -algebras since his classification of approximately finite-dimensional (AF) algebras via their scaled, ordered K_0 -groups ([6]). Successful classification results have been obtained for AH algebras (the inductive limits of matrix algebras over metric spaces) with slow dimension growth for cases of real rank zero (see [2–5, 9, 10, 12, 13, 16, 17]) and simple AH algebras (see [7, 8, 11, 18, 25]) by using the so called Elliott invariant, which consists of the ordered K_0 -group, the K_1 -group, the simplex of tracial state space and the natural pairing between the tracial state space and the K_0 -group.

A C^* -algebra is said to have the ideal property if each of its closed two-sided ideals is generated (as a closed two-sided ideal) by projections inside the ideal. It is obvious that both simple, unital C^* -algebras and real rank zero C^* -algebras have the ideal property. There are many other examples of C^* -algebras arising from dynamical systems which have the ideal property (see [20, 28–32], etc.). In 1995, K. Stevens classified all unital approximately divisible AI algebras with the ideal

property ([39]). Conel Pasnicu studied C^* -algebras with the ideal property and obtained a characterization theorem for AH algebras with the ideal property which are quite useful for classification theory (see [27]). In 2011, K. Ji and C. Jiang improved Stevens' result by dropping the conditions unital and approximate divisible (see [21]). Subsequently, Jiang and the present author completely classified all inductive limits of splitting interval algebras (ASI) with the ideal property ([22]). AI algebra is a special case of ASI algebra. The invariant we used to classify ASI algebras in our paper was first proposed by Stevens. We call it Stevens invariant. Stevens invariant of a C^* -algebra A consists of the K_0 -group of A , the K_1 -group of A and the tracial state spaces of all hereditary C^* -subalgebras of the form eAe with certain compatibility conditions, where e is any projection in A . Stevens invariant is also used to classify AH-algebras with the ideal property (see [19]).

We know that for simple C^* -algebras, traces are assumed to be bounded in the unital cases, and lower semicontinuous and densely defined in the non-unital case. But these two kinds of traces will not suffice for the classification of non-simple C^* -algebras. That is, for non-simple C^* -algebras, in many cases, all finite traces or densely defined lower semi-continuous traces are identically zero on a proper ideal. Therefore, neither finite traces nor densely defined traces can give information of the ideals. In this paper, we propose to include the extended valued traces (the value could be infinity) in the traditional Elliott invariant — called the *Extended Elliott Invariant*. Another sign for considering the extended valued traces is that all lower semicontinuous traces on a C^* -algebra constitute a non-cancellative cone that in particular determines the lattice of closed two-sided ideals, an important invariant in its own right.

It is natural to ask what is the connection between the extended Elliott invariant and the Stevens invariant? Does the extended Elliott invariant still work for classifying C^* -algebras with the ideal property? In this paper, we partially answered these two questions. The following theorems are our main results in this paper:

Theorem 1.1. *Let A be a C^* -algebra with the ideal property. Then the Stevens invariant of A is equivalent to the extended Elliott invariant of A .*

Theorem 1.2. *Let A, B be two C^* -algebras with the ideal property. If A and B have isomorphic extended Elliott Invariant, then A and B have isomorphic Stevens Invariant — and vice versa.*

The paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, we define two categories \mathcal{S} and \mathcal{E} corresponding to Stevens invariant and extended Elliott invariant respectively. We show that there are canonical non-trivial maps between the object set of \mathcal{E} and the object set of \mathcal{S} . Moreover, the Stevens invariant of a C^* -algebra can always be derived from its extended Elliott invariant. The converse is true when the C^* -algebra has the ideal property. In Section 4, we extend the maps defined in Section 3 to be functors between two sub-categories of \mathcal{S} and \mathcal{E} and prove Theorem 1.2. Finally, we show that there is a class

of C^* -algebras without the ideal property whose extended Elliott invariants cannot be derived from their Stevens invariants.

Acknowledgements. The result of this paper is part of my thesis. I benefit a lot from discussions with my advisor, Professor Guihua Gong in writing this paper. I would like to express my gratitude to him, for his support, patience, and encouragement throughout my graduate studies.

2. Preliminaries

For convenience of the reader, we recall some definitions and lemmas (see [33] for more details).

Definition 2.1. Let A be a C^* -algebra. A weight on A is a function $\phi: A_+ \rightarrow [0, +\infty]$ such that:

- (i) $\phi(\alpha x) = \alpha\phi(x)$, if $x \in A_+$ and $\alpha \in \mathbb{R}_+$;
- (ii) $\phi(x + y) = \phi(x) + \phi(y)$, if x and y belong to A_+ .

Moreover, ϕ is lower semi-continuous if for each $\alpha \in \mathbb{R}_+$ the set

$$\{x \in A_+ \mid \phi(x) \leq \alpha\}$$

is closed.

Definition 2.2. Let A be a C^* -algebra. A trace on A is a weight ϕ such that $\phi(u^*xu) = \phi(x)$ for all $x \in A_+$ and all unitary $u \in \tilde{A}$, where \tilde{A} is the unitization of A .

Remark 2.3. In this paper, we denote by $T(A)$ the collection of all lower semi-continuous traces on A . This set is a non-cancellative cone endowed with operations of pointwise addition and pointwise scalar multiplication by strictly positive real numbers (see [14] for details). Let $T_F(A)$ denote the set of all finite traces on A .

The following two propositions are properties of traces quoted from [33].

Let A be a C^* -algebra and ϕ be a trace on A . Let A_+^ϕ be a subset of A_+ defined by

$$A_+^\phi := \{x \in A_+ \mid \phi(x) < \infty\}.$$

Proposition 2.4 (see [33, 5.1.2]). *For each trace ϕ on a C^* -algebra A the linear span A^ϕ of A_+^ϕ is a not necessary closed ideal of A with $(A^\phi)_+ = A_+^\phi$, and there is a unique extension of ϕ to a positive linear functional on A^ϕ . Moreover, the set*

$$A_2^\phi = \{x \in A \mid x^*x \in A_+^\phi\}$$

*is an ideal of A such that $y^*x \in A^\phi$ for any $x, y \in A_2^\phi$.*

Proposition 2.5 (see [33, 5.2.2]). *If ϕ is a trace on a C^* -algebra A , then*

$$\phi(yx) = \phi(xy)$$

for each x in A^ϕ and y in \tilde{A} . Moreover, if ϕ is lower semi-continuous, then

$$\phi(x^*x) = \phi(xx^*)$$

for all x in A and $\phi(xy) = \phi(yx)$ for all x and y in A_2^ϕ .

Next we want to discuss how to extend traces.

Definition 2.6. We define an equivalence relation in A_+ by setting $x \approx y$ if there is a finite set $\{z_n\}$ in A such that $x = \sum z_n^* z_n$ and $y = \sum z_n z_n^*$. And we use the notation $y \preceq x$ to mean $y \approx x_1$, $x_1 \leq x$.

Theorem 2.7 (see [33, 5.2.7]). *Let B be a hereditary C^* -subalgebra of a C^* -algebra A , and let ϕ be a lower semi-continuous weight on B . For each x in A_+ define*

$$\tilde{\phi}(x) = \sup \{ \phi(y) \mid y \in B_+, y \preceq x \}.$$

Then $\tilde{\phi}$ is a lower semi-continuous trace on A and $\tilde{\phi}|_{B_+}$ is the smallest trace dominating ϕ .

The following two definitions are some usual notations.

Definition 2.8. Let A be a C^* -algebra. Let $\mathcal{P}(A)$ be the set of all projections in A . Let $K_0(A)$ be the K_0 -group of A and $K_0(A)^+ \subseteq K_0(A)$ be the semigroup of $K_0(A)$ generated by $[p] \in K_0(A)$, where $p \in \mathcal{P}_\infty(A)$. Define

$$\Sigma A = \{ [p] \in K_0(A)^+ : p \text{ is a projection in } A \}.$$

Then $(K_0(A), K_0(A)^+, \Sigma A)$ is a scaled ordered group.

Definition 2.9. Let X be any convex set.

- (1) Let $\text{Aff}(X)^+$ be the collection of all affine maps from X to $[0, \infty]$.
- (2) Let $\text{Aff}_b(X)$ be the collection of all affine maps from X to \mathbb{R} .
- (3) Let $\text{Aff}_b(X)^+$ be the subset of $\text{Aff}_b(X)$ consisting of all nonnegative affine functions.

Any affine map $\xi: X \rightarrow Y$ induces a linear map $\xi^*: \text{Aff}(Y) \rightarrow \text{Aff}(X)$ by

$$\xi^*(f)(\tau) = f(\xi(\tau)),$$

for all $f \in \text{Aff}(Y)$ and $\tau \in X$.

3. Two invariants and their relevant categories

In this section, we construct two categories \mathcal{E} and \mathcal{F} in which the extended Elliott invariant and Stevens invariant sit, respectively. We show that there is a canonical non-trivial map from the object set of \mathcal{E} to the object set of \mathcal{F} , which induces a map from the extended Elliott invariant of a stably finite C^* -algebra to its Stevens invariant. We also construct a canonical map from the object set of \mathcal{F} to the object set of \mathcal{E} , which induces a map from the Stevens invariant to the extended Elliott invariant for a stably finite C^* -algebra with the ideal property.

Let \mathcal{E} denote the category whose objects are four-tuples

$$((G_0, G_0^+), \Sigma G, G_1, X),$$

where (G_0, G_0^+) is a partially ordered abelian group; G_1 is a countable abelian group; ΣG is a subset of G_0^+ ; X is a cone closed under addition and positive scalar multiplication such that there exists a positive linear map s^G from G_0^+ to $\text{Aff}(X)^+$. And X is also a complete lattice cone when endowed with the order structure induced by its addition operation (i.e., $\tau_1 \leq \tau_2$ if there exists $\tau_3 \in X$ such that $\tau_1 + \tau_3 = \tau_2$).

A morphism

$$\Theta: ((G_0, G_0^+), \Sigma G, G_1, X) \rightarrow ((H_0, H_0^+), \Sigma H, H_1, Y)$$

in \mathcal{E} is a three-tuple

$$\Theta = (\theta_0, \theta_1, \zeta),$$

where $\theta_0: (G_0, G_0^+, \Sigma G) \rightarrow (H_0, H_0^+, \Sigma H)$ is an order-preserving homomorphism satisfying $\theta_0(\Sigma G) \subseteq \Sigma H$; $\theta_1: G_1 \rightarrow H_1$ is any homomorphism and $\zeta: Y \rightarrow X$ is a continuous affine map that makes the diagram below commutative:

$$\begin{array}{ccc} G_0^+ & \xrightarrow{\theta_0} & H_0^+ \\ \downarrow s^G & & \downarrow s^H \\ \text{Aff}(X) & \xrightarrow{\zeta^*} & \text{Aff}(Y). \end{array}$$

Definition 3.1. For a C^* -algebra A , the extended Elliott invariant of A is

$$((K_0(A), K_0(A)^+), \Sigma A, K_1(A), T(A)),$$

with the natural pairing between $K_0(A)^+$ and $T(A)$, i.e., let

$$s^A: K_0(A)^+ \rightarrow \text{Aff}(T(A))$$

be defined by evaluating a given trace at a K_0 -class.

Obviously, if A is a stably finite C^* -algebra, then the extended Elliott invariant of A is an object in the category \mathcal{E} (see [14] for more details). Given a class of stably finite C^* -algebras, say \mathcal{A} , let $\mathcal{E}_{\mathcal{A}}$ denote the subcategory of \mathcal{E} whose objects can be realised as the extended Elliott invariant of a member of \mathcal{A} .

Definition 3.2. A preordered cone (F, \leq) is said to have the Riesz property if for $f, g, h \in F$ with $f \leq g + h$ there are always $\hat{g}, \hat{h} \in F$ with $\hat{g} \leq g, \hat{h} \leq h$ such that $f = \hat{g} + \hat{h}$.

Proposition 3.3 (see [15, Theorem 2.6.8]). *Let (F, \leq) be a lattice cone. Then its positive dual cone F_+^* has the Riesz property and it is a complete lattice cone.*

Let \mathcal{S} denote the category whose objects are four-tuples

$$((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+}),$$

where (G_0, G_0^+) is a partially ordered abelian group; G_1 is a countable abelian group; ΣG is a subset of G_0^+ ; for each $p \in G_0^+$, there is a positive cone Δ_p^G with a base of a simplex, and a positive linear map

$$s_p^G: G_0^p \longrightarrow \text{Aff}_b(\Delta_p^G),$$

where G_0^p is the subgroup of G_0 generated by the set

$$\{e \in G_0 : 0 \leq e \leq np \text{ for some } n \in \mathbb{Z}\}.$$

For any $p' \in G_0^+$ with $p' \leq p$, there is an affine map $\lambda_{p,p'}^G: \Delta_p^G \rightarrow \Delta_{p'}^G$ satisfying the following conditions:

- (1) If $p'' \leq p' \leq p$, then $\lambda_{p,p''}^G = \lambda_{p',p''}^G \circ \lambda_{p,p'}^G$.
- (2) $s_p^G(e)(\tau) = s_{p'}^G(e)(\lambda_{p,p'}(\tau))$ for all $\tau \in \Delta_p^G, e \in G_0^{p'}$.
- (3) The map $(\lambda_{p,p'}^G)^*: \text{Aff}_b(\Delta_{p'}^G) \rightarrow \text{Aff}_b(\Delta_p^G)$ induced by $\lambda_{p,p'}^G$ is hereditary, i.e., if $f \in \text{Aff}_b(\Delta_{p'}^G)$ and $g \in \text{Aff}_b(\Delta_p^G)$ satisfying $(\lambda_{p,p'}^G)^*(f) \geq g$, then there exists $h \in \text{Aff}_b(\Delta_{p'}^G)$ such that

$$g = (\lambda_{p,p'}^G)^*(h).$$

- (4) For each $f \in \text{Aff}_b(\Delta_{p+q})$, there exist $f_1 \in \text{Aff}_b(\Delta_p), f_2 \in \text{Aff}_b(\Delta_q)$ such that

$$f = \lambda_{p+q,p}^*(f_1) + \lambda_{p+q,q}^*(f_2),$$

where $p, q \in G_0^+$.

Remark 3.4. By the definition of Δ_p^G in the Stevens invariant, we know that Δ_p^G is a lattice cone for each p .

A morphism

$$\Theta: ((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+}) \rightarrow ((H_0, H_0^+), \Sigma H, H_1, \{\Delta_e^H\}_{e \in H_0^+})$$

in \mathcal{S} is a three-tuple

$$\Theta = (\theta_0, \theta_1, \{\xi^p\}_{p \in G_0^+}),$$

where $\theta_0: (G_0, G_0^+, \Sigma G) \rightarrow (H_0, H_0^+, \Sigma H)$ is an order-preserving homomorphism satisfying $\theta_0(\Sigma G) \subseteq \Sigma H$; $\theta_1: G_1 \rightarrow H_1$ is any homomorphism; for each $p \in G_0^+$, there is a continuous affine map $\xi^p: \Delta_{\theta_0(p)}^H \rightarrow \Delta_p^G$ that makes the diagram below commutative:

$$\begin{array}{ccc} \Delta_q^H & \xrightarrow{\xi^p} & \Delta_p^G \\ \downarrow \lambda_{q,q'}^H & & \downarrow \lambda_{p,p'}^G \\ \Delta_{q'}^H & \xrightarrow{\xi^{p'}} & \Delta_{p'}^G \end{array}$$

where $q = \theta_0(p)$, $q' = \theta_0(p')$, $p, p' \in G_0^+$ satisfying $p' \leq p$.

Definition 3.5. For a C^* -algebra A , the Stevens invariant is

$$((K_0(A), K_0(A)^+), \Sigma A, K_1(A), \{T_F(\overline{pAp})\}_{p \in K_0(A)^+}),$$

with a natural pairing s_p^A between $K_0(A)^p$ and $T_F(\overline{pAp})$ given by evaluating a given trace at a K_0 -class and $\lambda_{p,q}^A: T_F(\overline{pAp}) \rightarrow T_F(\overline{qAq})$ defined by restriction. That is,

$$\lambda_{p,q}^A(\tau) = \tau|_{\overline{qAq}}.$$

It is easy to see that the Stevens invariant of A is an object in the category \mathcal{S} when A is stably finite. Given a class \mathcal{A} of stably finite C^* -algebras, let $\mathcal{S}_{\mathcal{A}}$ denote the subcategory of \mathcal{S} whose objects can be realised as the Stevens invariant of a member of \mathcal{A} .

Lemma 3.6 (see [35, Lemma 10.4]). *If P is a lattice and P_1 is a hereditary subcone of P , then P_1 is a lattice.*

Lemma 3.7. *Let $((G_0, G_0^+), \Sigma G, G_1, X)$ be an object in \mathcal{E} . For any $p \in G_0^+$, let*

$$\begin{aligned} \Delta'_p &= \{\tau \in X : 0 \leq s^G(p)(\tau) < \infty\}, \\ \Delta_p^G &= \Delta'_p / \sim, \end{aligned}$$

where $\tau_1 \sim \tau_2$ if and only if $f(\tau_1) = f(\tau_2)$ for all $f \in \text{Aff}(X)$ satisfying f is bounded on Δ'_p . Then Δ_p^G is a positive cone with a base of a simplex.

Proof. For $\tau \in \Delta'_p$, let $[\tau]$ represent the equivalent class of τ in Δ_p^G . It's easy to see that for all $\alpha \in \mathbb{R}_+$ and $\tau \in \Delta'_p$,

$$\alpha[\tau] = [\alpha\tau] \in \Delta_p^G.$$

If $[\tau_1], [\tau_2] \in \Delta_p^G$ and $t \in [0, 1]$, then

$$t[\tau_1] + (1 - t)[\tau_2] = [t\tau_1 + (1 - t)\tau_2] \in \Delta_p^G.$$

Therefore, Δ_p^G is a positive cone. Similarly, Δ'_p is a positive subcone of X .

Claim. Δ'_p is a hereditary subcone of X . If $\phi_1 \in \Delta'_p$, $\phi_2 \in X$ satisfying $\phi_2 \leq \phi_1$, then by the definition of the order on X there exists $\phi_3 \in X$ such that

$$\phi_1 = \phi_2 + \phi_3.$$

Thus,

$$\begin{aligned} s^G(p)\left(\frac{1}{2}\phi_1\right) &= s^G(p)\left(\frac{1}{2}\phi_2 + \frac{1}{2}\phi_3\right) \\ &= \frac{1}{2}s^G(p)(\phi_2) + \frac{1}{2}s^G(p)(\phi_3). \end{aligned}$$

Since $s^G(p)(\phi_1) < \infty$, we have $s^G(p)(\phi_2) < \infty$. Thus, $\phi_2 \in \Delta'_p$ and the claim is true. By Lemma 3.6, Δ'_p is a lattice.

Define $\|\tau\| = s^G(p)(\tau)$ for all $\tau \in \Delta'_p$. It is easy to see that $\|\cdot\|$ is a norm on Δ'_p . Therefore, Δ'_p can be embedded into a norm space. Let

$$T'_p = \{\tau \in \Delta'_p : s^G(p)(\tau) = 1\}.$$

Then T'_p is a convex base of the cone Δ'_p with Δ'_p a lattice. Therefore, T'_p is a simplex (see the second paragraph of page 52 in [35]).

Let $T_p^G = T'_p / \sim$, which is a simplex base of Δ_p^G . □

Theorem 3.8. *There is a natural nontrivial transformation \mathcal{G} that maps the objects in \mathcal{E} to the objects in \mathcal{S} .*

Proof. Let $((G_0, G_0^+), \Sigma G, G_1, X)$ be any object in \mathcal{E} . For any $p \in G_0^+$, define Δ'_p and Δ_p^G as in the Lemma 3.7. Then Δ_p^G is a positive cone with a base of a simplex.

Let

$$G_0^{p+} = G_0^p \cap G_0^+$$

and $s_p^G: G_0^{p+} \rightarrow \text{Aff}_b(\Delta_p^G)^+$ be defined by

$$s_p^G(q)([\tau]) = s^G(q)(\tau)$$

for $q \in G_0^{p+}$ and $\tau \in \Delta'_p$. Since $s_p^G(q)(\tau) < \infty$ for all $q \in G_0^{p+}$, $\tau \in \Delta'_p$, we can extend s_p^G to be a map (still denote it by s_p^G) from G_0^p to $\text{Aff}_b(\Delta_p^G)$ by

$$s_p^G(q_1 - q_2)([\tau]) = s_p^G(q_1)(\tau) - s_p^G(q_2)(\tau)$$

for $q_1, q_2 \in G_0^{p+}$ and $\tau \in \Delta'_p$. It is easy to see that s_p^G is a well-defined linear map.

For $q \in G_0^+$ with $q \leq p$, let $\lambda_{p,q}^G: \Delta_p^G \rightarrow \Delta_q^G$ be the map induced by the inclusion map from Δ_p' to Δ_q' . It is easy to see that $\lambda_{p,q}^G$ is well-defined and moreover,

- (1) If $p'' \leq p' \leq p$, then $\lambda_{p,p''}^G = \lambda_{p',p''}^G \circ \lambda_{p,p'}^G$.
- (2) $s_p^G(e)([\tau]) = s_{p'}^G(e)(\lambda_{p,p'}([\tau]))$ for all $\tau \in \Delta_p'$, $e \in G_0^{p'}$.
- (3) If $f \in \text{Aff}_b(\Delta_q^G)$ and $g \in \text{Aff}_b(\Delta_p^G)$ satisfying $(\lambda_{p,q}^G)^*(f) \geq g$, let T_q' be a simplex base of Δ_q' and let E_q' be the set of extreme points of T_q' . Define $h: E_q' \rightarrow \mathbb{R}$ by the following

$$h(\tau) = \begin{cases} g(\tau), & \text{if } \tau \in \Delta_p' \cap E_q', \\ f(\tau), & \text{otherwise.} \end{cases}$$

Then h can be extended to an affine map from Δ_q' to \mathbb{R} (still denoted by h). Since any element in Δ_p' can not be written as a linear combination of elements in $\Delta_q' \setminus \Delta_p'$, h induces a map from Δ_p^G to \mathbb{R} satisfying

$$g = \lambda_{p,q}^*(h) \quad \text{and} \quad h \leq f.$$

Thus, the map $(\lambda_{p,p'}^G)^*: \text{Aff}(\Delta_q^G) \rightarrow \text{Aff}(\Delta_p^G)$ is hereditary.

- (4) Let p, q be any two elements in G_0^+ . Since Δ_{p+q}' is a lattice cone, by Proposition 3.3, $\text{Aff}_b(\Delta_{p+q}')$ has the Riesz property. For each $f \in \text{Aff}_b(\Delta_{p+q})$, there exists a constant M such that

$$f \leq M(s^G(p) + s^G(q)).$$

By the Riesz property, there exist $f_1 \in \text{Aff}_b(\Delta_p)$, $f_2 \in \text{Aff}_b(\Delta_q)$ such that

$$f = \lambda_{p+q,p}^*(f_1) + \lambda_{p+q,q}^*(f_2).$$

Therefore, $((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+})$ is an object in \mathcal{S} . Finally let

$$\mathcal{E}: \{\text{Objects in } \mathcal{E}\} \rightarrow \{\text{Objects in } \mathcal{S}\}$$

be the map defined by sending

$$((G_0, G_0^+), \Sigma G, G_1, X) \mapsto ((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+}),$$

which completes the proof. □

Corollary 3.9. *Let \mathcal{A} be the class of stably finite C^* -algebras. Then*

$$\mathcal{E}(\mathcal{E}(A)) \cong \mathcal{S}(A)$$

for all $A \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}$. We know that

$$\mathcal{E}(A) = ((K_0(A), K_0(A)^+), \Sigma A, K_1(A), T(A)).$$

Let $\mathcal{G}(\mathcal{E}(A)) = ((K_0(A), K_0(A)^+), \Sigma A, K_1(A), \{\Delta_p^G\}_{p \in K_0(A)^+})$. By the definition of \mathcal{G} , for each $p \in K_0(A)^+$,

$$\begin{aligned} \Delta'_p &= \{\tau \in T(A) : 0 \leq \tau(p) < \infty\}, \\ \Delta_p^A &= \Delta'_p / \sim, \end{aligned}$$

where $\tau_1 \sim \tau_2$ if and only if $f(\tau_1) = f(\tau_2)$ for all $f \in \text{Aff}(T(A))$ satisfying f is bounded on Δ'_p . It is enough to show that

$$\Delta_p^A \cong T_F(\overline{pAp})$$

for each $p \in K_0(A)^+$.

For any $\tau \in \Delta'_p$, $\tau|_{\overline{pAp}}$ is a finite trace on \overline{pAp} . Define

$$\gamma: \Delta_p^A \rightarrow T_F(\overline{pAp})$$

by

$$\gamma([\tau]) = \tau|_{\overline{pAp}}.$$

Thus γ is a positive affine map and it is easy to see that γ is well-defined.

Notice that any element in $\text{Aff}T(\overline{pAp})$ can be realized by self-adjoint element in \overline{pAp} . Thus, if $[\tau_1] \neq [\tau_2]$ in Δ_p^A , then there exists an element $f \in \text{Aff}T(\overline{pAp})$ and a self-adjoint element $a \in \overline{pAp}$ such that

$$\tau_1(a) = f(\tau_1) \neq f(\tau_2) = \tau_2(a).$$

Therefore, $\gamma([\tau_1]) \neq \gamma([\tau_2])$. γ is injective.

What's left is to show that γ is a surjection. Let ϕ be any finite trace on \overline{pAp} . By Theorem 2.7, there is a lower semi-continuous trace $\tilde{\phi}$ on A such that $\tilde{\phi}|_{\overline{pAp}} = \phi$. Thus,

$$\gamma(\tilde{\phi}) = \phi.$$

Therefore, γ is a one-to-one and onto affine map. Thus, we have

$$\mathcal{G}(\mathcal{E}(A)) \cong \mathcal{S}(A). \quad \square$$

Theorem 3.10. *There is a natural nontrivial transformation \mathcal{F} that maps the objects in \mathcal{S} to the objects in \mathcal{E} .*

Proof. Let $((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+})$ be any object in \mathfrak{S} . We say that Λ is an ideal of G_0^+ if it is a hereditary sub-semigroup of G_0^+ . Let X be the collection of all sets of the form $\{\tau_p\}_{p \in \Lambda}$ satisfying

- (1) Λ is an ideal of G_0^+ ;
- (2) $\tau_p \in \Delta_p$ for each $p \in \Lambda$ and $\lambda_{p,q}(\tau_p) = \tau_q$ whenever $q < p$.

The multiplication in X is the usual multiplication and the addition operation in X is defined as follows:

$$\{\tau_p\}_{p \in \Lambda_1} + \{\phi_p\}_{p \in \Lambda_2} = \{\tau_p + \phi_p\}_{p \in \Lambda_1 \cap \Lambda_2}.$$

We define an order relation \leq on X by the following:

$$\{\tau_p\}_{p \in \Lambda_1} \leq \{\phi_p\}_{p \in \Lambda_2} \text{ if and only if } \Lambda_1 \supseteq \Lambda_2 \text{ and } \tau_p \leq \phi_p \text{ for } p \in \Lambda_2.$$

Then X naturally has a lattice structure induced by the lattice structures of Δ_p 's. That is,

$$\begin{aligned} \{\tau_p\}_{p \in \Lambda_1} \vee \{\phi_p\}_{p \in \Lambda_2} &= \{\tau_p \vee \phi_p\}_{p \in \Lambda_1 \cap \Lambda_2}, \\ \{\tau_p\}_{p \in \Lambda_1} \wedge \{\phi_p\}_{p \in \Lambda_2} &= \{\psi_p\}_{p \in \mathfrak{S}\{\Lambda_1, \Lambda_2\}}, \end{aligned}$$

where $\mathfrak{S}\{\Lambda_1, \Lambda_2\}$ is the ideal of G_0^+ generated by Λ_1 and Λ_2 , and ψ_p will be defined later.

Let $\Lambda = \Lambda_1 \cap \Lambda_2$. Now let's define a map

$$\hat{\alpha}: \{\text{Aff}_b(\Delta_p)^+\}_{p \in S(\Lambda_1, \Lambda_2)} \rightarrow \mathbb{R}^+$$

by the following:

- If $f \in \text{Aff}_b(\Delta_p)^+$ for some $p \in \Lambda_1$, let

$$\hat{\alpha}(f) = \inf \{f_1(\tau_{p_1}) + f_2(\tau_{p_2} \wedge \phi_{p_2}) : f = \lambda_{p,p_1}^*(f_1) + \lambda_{p,p_2}^*(f_2), f_i \in \text{Aff}(\Delta_{p_i})^+, p_2 \in \Lambda\}.$$

- If $f \in \text{Aff}_b(\Delta_p)^+$ for some $p \in \Lambda_2$, let

$$\hat{\alpha}(f) = \inf \{f_1(\phi_{p_1}) + f_2(\tau_{p_2} \wedge \phi_{p_2}) : f = \lambda_{p,p_1}^*(f_1) + \lambda_{p,p_2}^*(f_2), f_i \in \text{Aff}(\Delta_{p_i})^+, p_2 \in \Lambda\}.$$

- If $f \in \text{Aff}(\Delta_p)^+$ for some $p \in S(\Lambda_1, \Lambda_2)$, let

$$\hat{\alpha}(f) = \inf \{\hat{\alpha}(f_1) + \hat{\alpha}(f_2) : f = \lambda_{p,p_1}^*(f_1) + \lambda_{p,p_2}^*(f_2), f_i \in \text{Aff}(\Delta_{p_i})^+, p_i \in \Lambda_i, i = 1, 2\}.$$

Claim. $\hat{\alpha}(\lambda_{p,q}^*(g)) = \hat{\alpha}(g)$, for all $q \leq p \in S(\Lambda_1, \Lambda_2)$ and $g \in \text{Aff}_b(\Delta_q)$.

It is enough to prove the above equation for $p \in \Lambda_1$. We know that:

$$\hat{\alpha}(g) = \inf \{f_1(\tau_{q'}) + f_2(\tau_e \wedge \phi_e) : g = \lambda_{q,q'}^*(f_1) + \lambda_{q,e}^*(f_2), \\ f_2 \in \text{Aff}(\Delta_e)^+, e \in \Lambda, f_1 \in \text{Aff}(\Delta_{q'})^+, q' \leq q\}$$

and

$$\hat{\alpha}(\lambda_{p,q}^*(g)) = \inf \{g_1(\tau_{p'}) + g_2(\tau_{e'} \wedge \phi_{e'}) : \lambda_{p,q}^*(g) = \lambda_{p,p'}^*(g_1) + \lambda_{p,e'}^*(g_2), \\ e' \in \Lambda, g_1 \in \text{Aff}(\Delta_{p'})^+, g_2 \in \text{Aff}(\Delta_{e'})^+, p' \leq p\}.$$

For any $g = \lambda_{q,q'}^*(f_1) + \lambda_{q,e}^*(f_2)$, we have $\lambda_{p,q}^*(g) = \lambda_{p,q'}^*(f_1) + \lambda_{p,e}^*(f_2)$. Thus,

$$\hat{\alpha}(\lambda_{p,q}^*(g)) \leq \hat{\alpha}(g).$$

On the other hand, if $\lambda_{p,q}^*(g) = \lambda_{p,p'}^*(g_1) + \lambda_{p,e'}^*(g_2)$, then

$$\lambda_{p,q}^*(g) \geq \lambda_{p,p'}^*(g_1) \quad \text{and} \quad \lambda_{p,q}^*(g) \geq \lambda_{p,e'}^*(g_2).$$

By the hereditary property, there exist $h_1, h_2 \in \text{Aff}_b(\Delta_q)$ such that

$$\lambda_{p,p'}^*(g_1) = \lambda_{p,q}^*(h_1) \quad \text{and} \quad \lambda_{p,e'}^*(g_2) = \lambda_{p,q}^*(h_2).$$

That is,

$$\lambda_{p,q}^*(g) = \lambda_{p,q}^*(h_1) + \lambda_{p,q}^*(h_2).$$

Therefore, $g - h_1 - h_2 \in \ker(\lambda_{p,q}^*)$. Since any element in $\ker(\lambda_{p,q}^*)$ can be written as a difference of two elements in $\ker(\lambda_{p,q}^*) \cap \text{Aff}(\Delta_q)^+$, there are $j_1, j_2 \in \ker(\lambda_{p,q}^*) \cap \text{Aff}(\Delta_q)^+$ such that

$$g + j_2 = h_1 + h_2 + j_1.$$

By redefining h_1 as $h_1 + j_1$, we have $g \leq h_1 + h_2$. Therefore, $\hat{\alpha}(\lambda_{p,q}^*(g)) \geq \hat{\alpha}(g)$ and the claim is true.

Then $\hat{\alpha}$ is well-defined. It is well-known that there is a one-to-one correspondence between the elements in Δ_p and the affine maps from $\text{Aff}_b(\Delta_p)^+$ to \mathbb{R}^+ . If the image of $\hat{\alpha}$ is finite when restricted on $\text{Aff}_b(\Delta_p)^+$ for some p , then there exists an element in Δ_p (we denote it by α_p) such that $\hat{\alpha}(f) = f(\alpha_p)$ for all $f \in \text{Aff}(\Delta_p)^+$.

It is routine to check

$$\hat{\alpha}(f) = t\hat{\alpha}(g_1) + (1-t)\hat{\alpha}(g_2),$$

for all $t \in [0, 1]$, $f, g_i \in \text{Aff}_b(\Delta_p)^+$ and $p \in G_0^+$ satisfying $f = tg_1 + (1-t)g_2$.

Finally, for $p \in S(\Lambda_1, \Lambda_2)$, let $\psi_p = \alpha_p$. If $q \leq p$, then

$$g(\psi_q) = g(\alpha_q) = \lambda_{p,q}^*(g)(\alpha_p) = \lambda_{p,q}^*(g)(\psi_p) = g(\lambda_{p,q}(\psi_p))$$

for all $g \in \text{Aff}(\Delta_q)^+$. That is, ψ_q and $\lambda_{p,q}(\psi_p)$ correspond to the same map from $\text{Aff}(\Delta_q)$ to \mathbb{R}^+ . Therefore, $\psi_q = \lambda_{p,q}(\psi_p)$. Thus, $\{\psi_p\}_{p \in S(\Lambda_1, \Lambda_2)}$ is an element in X .

Define $s^G: G_0^+ \rightarrow \text{Aff}(X)^+$ as follows:

$$s^G(q)(\{\tau_p\}_{p \in \Lambda}) = \begin{cases} s_q^G(q)(\tau_q), & \text{if } q \in \Lambda, \\ \infty, & \text{otherwise.} \end{cases}$$

for all $q \in G_0^+$.

It is easy to see that $((G_0, G_0^+), \Sigma G, G_1, X)$ is an object in \mathcal{E} . Let

$$\mathcal{F}: \{\text{Objects in } \mathcal{S}\} \rightarrow \{\text{Objects in } \mathcal{E}\}$$

be the map defined by sending

$$((G_0, G_0^+), \Sigma G, G_1, \{\Delta_p^G\}_{p \in G_0^+}) \mapsto ((G_0, G_0^+), \Sigma G, G_1, X),$$

which completes the proof. □

Lemma 3.11. *Let $a, b \in A_+$ be such that $\|a - b\| < \varepsilon$. Then $(a - \varepsilon)_+ \preceq b$.*

Proof. By Lemma 2.2 of [24], there is $d \in A$ with $\|d\| \leq 1$ and $(a - \varepsilon)_+ = dbd^*$. Hence, $(a - \varepsilon)_+ \sim b^{1/2}d^*db^{1/2} \leq b$. □

Lemma 3.12. *Let A be a C*-algebra and τ be a lower semicontinuous trace on A . Let A^τ be the ideal of A defined in Proposition 2.4. If $p \in \overline{A^\tau}$ (the closure of A^τ) is a projection, then $\tau(p) < \infty$.*

Proof. If $p \in A^\tau$, then $\tau(p) < \infty$ by definition. If $p \in \overline{A^\tau} \setminus A^\tau$, then there exists a sequence $\{x_i\}_{i=1}^\infty \subseteq A_+^\tau$ with $\lim_{i \rightarrow \infty} x_i = p$. For any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that:

$$\|x_k - p\| < \varepsilon.$$

By Lemma 3.11 $(p - \varepsilon)_+ \preceq x_k$. Since $(p - \varepsilon)_+ = (1 - \varepsilon)p$,

$$\tau((1 - \varepsilon)p) = \tau((p - \varepsilon)_+) \leq \tau(x_k) < \infty.$$

Therefore, $\tau(p) < \infty$. □

Lemma 3.13. *Let ϕ and ϕ' be two lower semi-continuous traces on a C*-algebra A and e be a projection in A . Suppose that both ϕ and ϕ' are finite when restricted on $(\overline{eAe})_+$ and*

$$\phi(x) = \phi'(x) \text{ for all } x \in (\overline{eAe})_+.$$

Then

$$\phi(x) = \phi'(x) \text{ for all } x \in (\overline{AeA})_+,$$

where \overline{AeA} stands for the closed two-sided ideal generated by e .

Proof. For $x \in A_+$, let

$$\psi(x) = \sup \{ \phi(y) : y \preceq x, y \in (\overline{eAe})_+ \}.$$

By Theorem 2.7, ψ is a lower semi-continuous trace on A and $\phi(x) = \psi(x)$ for all $x \in (\overline{eAe})_+$. We only need to show that $\phi(x) = \psi(x)$ for all $x \in (\overline{AeA})_+$.

For any $x \in (\overline{AeA})_+$, $y \in (\overline{eAe})_+$ satisfying $y \preceq x$, we have

$$\psi(y) = \phi(y) \leq \phi(x).$$

Taking supremum on both sides for all such y we get

$$\psi(x) \leq \phi(x) \text{ for all } x \in (\overline{AeA})_+. \tag{3.1}$$

Let

$$\Omega = \{ \sum_{k=1}^n a_k e b_k : a_k, b_k \in A, n \in \mathbb{Z} \}$$

be a subset of \overline{AeA} .

Claim. $\phi(x) = \psi(x)$ for all $x \in \Omega_+$.

In fact, for $x = aeb \in A_+$ with $a, b \in A$, we have

$$\phi((ae)^*(ae)) = \phi(ea^*ae) < \infty$$

and

$$\phi((eb)^*(eb)) = \phi((eb)(eb)^*) = \phi(ebb^*e) < \infty.$$

Similarly,

$$\psi((ae)^*(ae)) = \psi(ea^*ae) < \infty,$$

$$\psi((eb)^*(eb)) = \psi((eb)(eb)^*) = \psi(ebb^*e) < \infty.$$

Therefore, by Proposition 2.5,

$$\phi(x) = \phi(aeeb) = \phi(ebae) = \psi(ebae) = \psi(aeb) = \psi(x).$$

Thus, the claim is true.

Let x be a positive element in \overline{AeA} and $\{y_n\}$ be an increasing sequence in Ω_+ with $\lim_{n \rightarrow \infty} y_n = x$. Then by the above claim

$$\phi(x) \leq \lim \phi(y_n) = \lim \psi(y_n) \leq \psi(x),$$

where the first inequality is due to ϕ being lower semi-continuous.

Therefore, combined with the inequality (3.1), we get $\phi(x) = \psi(x)$ for all $x \in (\overline{AeA})_+$. That is,

$$\phi(x) = \sup \{ \phi(y) : y \preceq x, y \in (\overline{eAe})_+ \} \text{ for all } x \in (\overline{AeA})_+.$$

Similarly,

$$\phi'(x) = \sup \{ \phi'(y) : y \preceq x, y \in (\overline{eAe})_+ \} \text{ for all } x \in (\overline{AeA})_+.$$

Therefore,

$$\phi'(x) = \phi(x) \text{ for all } x \in (\overline{AeA})_+. \quad \square$$

Lemma 3.14. *Let A be a C^* -algebra with the ideal property. Suppose that for any projection $e \in A$, there is a lower semi-continuous trace ϕ_e on \overline{AeA} satisfying*

$$\phi_{e_1}(x) = \phi_{e_2}(x) \text{ for all } x \in (\overline{Ae_1A})_+ \cap (\overline{Ae_2A})_+,$$

where e_1, e_2 are two projections in A . Then there is a lower semi-continuous trace ϕ on A such that

$$\phi(x) = \phi_e(x)$$

for any $x \in (\overline{AeA})_+$ and any projection $e \in A$.

Proof. If A is generated by a single projection, say e , then let $\phi = \phi_e$. In this case, if e' is any projection in A , then

$$\phi(x) = \phi_e(x) = \phi_{e'}(x) \text{ for all } x \in \overline{Ae'A}.$$

If A is not generated by a single projection, let J be an ideal of A and ϕ_J be a lower semi-continuous trace on J satisfying desired properties, i.e.

$$\phi_J(x) = \phi_e(x) \text{ for all } x \in (\overline{AeA})_+,$$

where e is any projection in J . Let p be a projection outside J and J' be the closed ideal generated by J and p . Let $\phi_{J'}$ be an extension of ϕ_J on J' defined as follows.

Since $(J')_+ = J_+ + (\overline{ApA})_+$ for any $x \in (J')_+$, there exist $x_1 \in J_+$ and $x_2 \in (\overline{ApA})_+$ such that $x = x_1 + x_2$. Define

$$\phi_{J'}(x) = \phi_J(x_1) + \phi_p(x_2).$$

If there exist $y_1 \in J_+$ and $y_2 \in (\overline{ApA})_+$ such that $x_1 + x_2 = x = y_1 + y_2$, then

$$x_1 - y_1 = y_2 - x_2 \in J_+ \cap (\overline{ApA})_+.$$

For any projection $e \in J \cap (\overline{ApA})$, we have

$$\phi_J(z) = \phi_e(z) = \phi_p(z)$$

for all $z \in \overline{AeA}$. Since A has the ideal property, $J \cap (\overline{ApA})$ is generated by projections. Therefore, $\phi_J(x_1 - y_1) = \phi_p(y_2 - x_2)$. That is

$$\phi_J(x_1) + \phi_p(x_2) = \phi_J(y_1) + \phi_p(y_2).$$

Thus, $\phi_{J'}$ is well-defined. It is obvious that $\phi_{J'}$ is lower semi-continuous trace on J' satisfying desired properties.

To complete the proof, we apply Zorn’s lemma. Let X be the set of all pairs

$$(J, \phi_J),$$

where J is a sub-ideal of I and ϕ_J is a lower semi-continuous trace satisfying the desired properties. Define the relation $<$ on X by

$$(J_1, \phi_{J_1}) < (J_2, \phi_{J_2}) \text{ if and only if } J_1 \subseteq J_2 \text{ and } \phi_{J_2}|_{J_1} = \phi_{J_1}.$$

This is a partial ordering with the property that any totally ordered subset has a maximal element. Zorn’s lemma says that X has a maximal element, say $(\hat{J}, \phi_{\hat{J}})$. If \hat{J} is a proper subspace of A , i.e., $\hat{J} \neq A$, then the argument given before produces an extension of $\phi_{\hat{J}}$ to a larger ideal, contradicting the maximality of $(\hat{J}, \phi_{\hat{J}})$. \square

Corollary 3.15. *Let \mathcal{J} be the class of all stably finite C^* -algebras satisfying the ideal property. Then*

$$\mathcal{E}(A) \cong \mathcal{F}(\mathcal{S}(A))$$

for all $A \in \mathcal{J}$.

Proof. Let A be any stably finite C^* -algebra satisfying the ideal property. We have

$$\mathcal{S}(A) = ((K_0(A), K_0(A)^+), \Sigma A, K_1(A), \{\text{TF}(\overline{pAp})\}_{p \in K_0(A)^+}).$$

Let

$$\mathcal{F}(\mathcal{S}(A)) = ((K_0(A), K_0(A)^+), \Sigma A, K_1(A), X),$$

where X is the collection of all sets of the form $\{\tau_p\}_{p \in \Lambda}$ defined as in the proof of Theorem 3.10. We will show that TA is isomorphic to X .

For any $\tau \in \text{T}(A)$, let

$$A_+^\tau = \{x \in A : \tau(x) < \infty\}$$

and A^τ be the linear span of A_+^τ , which is an ideal of A by Lemma 2.4. Let

$$\mathcal{P}_\tau := \{p \in \overline{A_+^\tau} : p \text{ is a projection}\},$$

By Lemma 3.12, $\mathcal{P}_\tau \subseteq \overline{A_+^\tau}$. Since A is a C^* -algebra with the ideal property, $\overline{A_+^\tau}$ is generated by \mathcal{P}_τ . Thus, \mathcal{P}_τ is not empty. For any $f \in \mathcal{P}_\tau$, we have

$$\tau|_{\overline{fAf}} \triangleq \tau_f$$

is a finite trace on \overline{fAf} . Let Λ_τ be the sub-semigroup of $K_0(A)^+$ generated by \mathcal{P}_τ . For each $p \in K_0(A)^+$, we can define τ_p by extending of τ_f ’s for $f \in \mathcal{P}_\tau$ to matrix algebra. Therefore, $\{\tau_f\}_{f \in K_0(A)^+}$ is an element in X .

Define $\beta: \text{T}(A) \rightarrow X$ by sending τ to $\{\tau_f\}_{f \in K_0(A)^+}$. It is obvious to see that β is injective since A has the ideal property.

Let $\{\tau_p\}_{p \in \Lambda}$ be any element in X . By the definition of X , τ_p is a finite trace on \overline{pAp} for each p . By Theorem 2.7, for each p , we can extend τ_p to be a lower semi-continuous trace on the ideal \overline{pAp} , still denoted by τ_p .

Claim. $\tau_{p_1}(x) = \tau_{p_2}(x)$ for all $x \in (\overline{Ap_1A}) \cap (\overline{Ap_2A})$, where p_1, p_2 are in Λ .

In fact, since A has the ideal property, it is enough to show

$$\tau_{p_1}(x) = \tau_{p_2}(x) \text{ for all } x \in \overline{AqA},$$

where q is any projection in $(\overline{Ap_1A}) \cap (\overline{Ap_2A})$. By Lemma 4.4, there are integers m, n such that $q \leq np_1$ and $q \leq mp_2$. Thus,

$$\tau_{p_1}(x) = \tau_{np_1}(x) = \tau_q(x) = \tau_{mp_2}(x) = \tau_{p_2}(x).$$

Therefore, the claim is true.

Applying Lemma 3.14, we know that there is a lower semi-continuous trace $\hat{\tau}$ on the ideal of A generated by $\Lambda \cap A$. Let

$$\tau(x) = \begin{cases} \hat{\tau}(x), & \text{if } x \text{ is in the ideal generated by } \Lambda, \\ \infty, & \text{otherwise.} \end{cases}$$

Then τ is a lower semi-continuous trace on A satisfying

$$\beta(\tau) = \{\tau_p\}_{p \in \Lambda}.$$

Therefore, β is also surjective. Thus, TA is isomorphic to X . □

Remark 3.16. Theorem 1.1 follows from Corollaries 3.9 and 3.15 immediately.

4. Main theorem

In this section, we extend the maps defined in Section 3 to be functors between two sub-categories of \mathfrak{S} and \mathfrak{E} and prove Theorem 1.2. Finally, we show that there are C^* -algebras without the ideal property whose Elliott invariant cannot be derived from the Stevens invariant.

Definition 4.1. Let A, B be two C^* -algebras. Let α from $K_0(A)$ to $K_0(B)$ be a homomorphism, and $\xi: TB \rightarrow TA$ be an affine map. We say that α and ξ are compatible if

$$\tau(\alpha(x)) = (\xi(\tau))(x)$$

for all $x \in K_0(A)_+$ and $\tau \in TB$.

Proposition 4.2. *Let A, B be two C^* -algebras. If there is a morphism*

$$\Theta = (\theta_0, \theta_1, \{\xi^p\}_{p \in K_0(A)_+}): \mathfrak{S}(A) \rightarrow \mathfrak{S}(B),$$

then θ_0 and ξ^e are compatible for all $e \in K_0(A)^+$.

Proof. See 1.11 in [21]. □

Lemma 4.3. *Let A be a C^* -algebra with the ideal property and B be a C^* -algebra satisfying:*

- (1) *There exists a scaled ordered isomorphism $\alpha: K_0A \rightarrow K_0B$;*
- (2) *There is an isomorphism $\xi: TB \rightarrow TA$ which is compatible with α .*

Let $\tau \in TB$ and I be the closed ideal of A generated by the set

$$\{e \in \mathcal{P}(A) : \tau(\alpha(e)) < \infty\}.$$

Then

$$\xi(\tau)(x) = +\infty \text{ for all } x \in A_+ \setminus I_+.$$

Proof. Let $\phi = \xi(\tau)$. For any projection $p \in \overline{A^\phi}$, by Lemma 3.12, $\phi(p) < +\infty$. Since ξ is compatible with α ,

$$\tau(\alpha(p)) = \xi(\tau)(p) = \phi(p) < +\infty.$$

That is $\alpha(p) \in B_+^\tau$ and $p \in I$. Therefore $\mathcal{P}(\overline{A^\phi}) \subseteq \mathcal{P}(I)$. Since A is a C^* -algebra with the ideal property, $\overline{A^\phi} \subseteq I$. So

$$A \setminus I \subseteq A \setminus \overline{A^\phi}.$$

Hence $\phi(x) = +\infty$ for all $x \in A_+ \setminus I_+$. □

The following lemma is well known (see [26, Lemma 3.3.6]).

Lemma 4.4. *If p is a projection in A , $b \in A_+$ and p is in the ideal generated by b , then there are $x_1, x_2, \dots, x_k \in A$ such that $p = \sum_{i=1}^k x_i b x_i^*$.*

The following theorem is the main theorem of this section.

Theorem 4.5. *Let $\mathcal{E}_\mathcal{J}$ and $\mathcal{S}_\mathcal{J}$ be two sub-categories of \mathcal{E} and \mathcal{S} , respectively, where \mathcal{J} is the class of C^* -algebras with the ideal property. Then $\mathcal{E}_\mathcal{J}$ and $\mathcal{S}_\mathcal{J}$ are isomorphic. That is, there are canonical functors*

$$\mathcal{F}: \mathcal{S}_\mathcal{J} \rightarrow \mathcal{E}_\mathcal{J}, \quad \mathcal{G}: \mathcal{E}_\mathcal{J} \rightarrow \mathcal{S}_\mathcal{J}$$

such that

$$\mathcal{G} \circ \mathcal{F} = \text{Id}_{\mathcal{S}_\mathcal{J}}, \quad \mathcal{F} \circ \mathcal{G} = \text{Id}_{\mathcal{E}_\mathcal{J}}.$$

Proof. We divide the proof into three steps.

Step I. Construction of the functor $\mathcal{F}: \mathcal{S}_\mathcal{J} \rightarrow \mathcal{E}_\mathcal{J}$.

Let \mathcal{F} map the objects in $\mathcal{S}_\mathcal{J}$ to the objects in $\mathcal{E}_\mathcal{J}$ be defined by

$$\mathcal{F}(\mathcal{S}(\bullet)) = \mathcal{E}(\bullet).$$

By Corollary 3.15, this map is well-defined. Let $A, B \in \mathcal{J}$ such that there is an arrow

$$\Phi = (\theta_0, \theta_1, \{\xi^e\}_{e \in K_0(A)_+})$$

from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$. We need to construct an affine map ζ from TB to TA which is compatible with θ_0 .

For $\tau \in TB$, let

$$B_+^\tau = \{x \in B : \tau(x) < \infty\}$$

and B^τ be the linear span of B_+^τ , which is an ideal of B by Lemma 2.4. Let

$$\begin{aligned} \mathcal{P}_\tau &:= \{p \in \overline{B_+^\tau} : p \text{ is a projection}\}, \\ \Lambda_\tau &= \{f \in \mathcal{P}_\tau : f \text{ is in the image of } \theta_0\}. \end{aligned}$$

Then $\Lambda_\tau \subseteq \mathcal{P}_\tau \subseteq B_+^\tau$, where the last inclusion is by Lemma 3.12.

For any $f \in \Lambda_\tau$, we have $\tau(f) < \infty$ and

$$\tau|_{\overline{fBf}} \triangleq \tau_f$$

is a finite trace on \overline{fBf} . Let $e \in A$ be a pre-image of f under θ_0 . Define $\phi_e = \xi^e(\tau_f)$. Since ξ^e and $\theta_0|_{\overline{eAe}}$ are compatible, ϕ_e is a finite trace on \overline{eAe} . We can extend ϕ_e to a lower semi-continuous trace (still denoted by ϕ_e) on \overline{AeA} by

$$\phi_e(x) := \sup \{\phi_e(y) : y \preceq x, y \in (\overline{eAe})_+\}, \text{ for all } x \in (\overline{AeA})_+.$$

Claim. If $x \in J_0 := (\overline{Ae_1A}) \cap (\overline{Ae_2A})$ is a positive element, where e_1, e_2 are two projections in A satisfying $\theta_0(e_i) = f_i$ for some $f_i \in \Lambda_\tau, i = 1, 2$, then

$$\phi_{e_1}(x) = \phi_{e_2}(x).$$

In fact, since A is a C^* -algebra with the ideal property, J_0 is generated by projections inside it. Let I_0 be a closed ideal of B generated by the set

$$\{q : q = \theta_0(p) \text{ for some } p \text{ in } J_0\}.$$

Then we have

$$I_0 = (\overline{Bf_1B}) \cap (\overline{Bf_2B}) \quad \text{and} \quad \tau_{f_1}|_{I_0} = \tau_{f_2}|_{I_0}.$$

Let p be any projection in J_0 and let $q = \theta_0(p)$. By Lemma 4.4, there exist natural numbers n_1, n_2 such that $p \leq n_i e_i$ for $i = 1, 2$. Therefore, by the compatible condition, the following diagrams are commutative:

$$\begin{array}{ccc} T_F(\overline{\tilde{f}_1 B \tilde{f}_1}) & \xrightarrow{\xi^{\tilde{e}_1}} & T_F(\overline{\tilde{e}_1 A \tilde{e}_1}) \\ \downarrow i & & \downarrow i \\ T_F(\overline{qBq}) & \xrightarrow{\xi^p} & T_F(\overline{pAp}), \end{array}$$

$$\begin{array}{ccc} T_F(\overline{\tilde{f}_2 B \tilde{f}_2}) & \xrightarrow{\xi^{\tilde{e}_2}} & T_F(\overline{\tilde{e}_2 A \tilde{e}_2}) \\ \downarrow i & & \downarrow i \\ T_F(\overline{q B q}) & \xrightarrow{\xi^p} & T_F(\overline{p A p}), \end{array}$$

where $\tilde{f}_i = n_i f$ and $\tilde{e}_i = n_i e_i$ for $i = 1, 2$. Since

$$\begin{aligned} T_F(\overline{\tilde{f}_i B \tilde{f}_i}) &= T_F(M_{n_i}(\overline{f_i B f_i})) = T_F(\overline{f_i B f_i}), \\ T_F(\overline{\tilde{e}_i A \tilde{e}_i}) &= T_F(M_{n_i}(\overline{e_i A e_i})) = T_F(\overline{e_i A e_i}), \end{aligned}$$

we can get the following commutative diagrams:

$$\begin{array}{ccc} T_F(\overline{f_1 B f_1}) & \xrightarrow{\xi^{e_1}} & T_F(\overline{e_1 A e_1}) \\ \downarrow i & & \downarrow i \\ T_F(\overline{q B q}) & \xrightarrow{\xi^p} & T_F(\overline{p A p}), \end{array}$$

$$\begin{array}{ccc} T_F(\overline{f_2 B f_2}) & \xrightarrow{\xi^{e_2}} & T_F(\overline{e_2 A e_2}) \\ \downarrow i & & \downarrow i \\ T_F(\overline{q B q}) & \xrightarrow{\xi^p} & T_F(\overline{p A p}). \end{array}$$

Therefore,

$$i \circ \xi^{e_1}(\tau_{f_1}) = \xi^p \circ i(\tau_{f_1}), \quad i \circ \xi^{e_2}(\tau_{f_2}) = \xi^p \circ i(\tau_{f_2}).$$

That is

$$\phi_{e_1}|_{\overline{p A p}} = \xi^p(\tau_{f_1}|_{\overline{q B q}}), \quad \phi_{e_2}|_{\overline{p A p}} = \xi^p(\tau_{f_2}|_{\overline{q B q}}).$$

Since

$$\tau_{f_1}|_{\overline{q B q}} = \tau_{f_2}|_{\overline{q B q}}, \quad \phi_{e_1}|_{\overline{p A p}} = \phi_{e_2}|_{\overline{p A p}}.$$

By Theorem 3.13, we have

$$\phi_{e_1}|_{\overline{A p A}} = \phi_{e_2}|_{\overline{A p A}}.$$

Therefore, $\phi_{e_1}|_{J_0} = \phi_{e_2}|_{J_0}$ since J_0 is generated by projections inside it. Thus, we have proved the claim.

Let I be the ideal generated by $\theta_0^{-1}(\Lambda_\tau)$ (the preimage of Λ_τ). Applying Lemma 3.14, there is a lower semi-continuous trace ϕ' on I satisfying certain properties. Extend ϕ' to a lower semi-continuous trace on A (denoted by ϕ) as follows:

$$\phi(x) = \begin{cases} \phi'(x), & \text{if } x \in I, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $\zeta(\tau) = \phi$. Then it remains to check that ζ is an affine map compatible with θ_0 .

For any $\tau \in TB, e \in K_0(A)_+,$ let $f = \theta_0(e)$ and $\zeta(\tau) = \phi$. If $\tau(f) < \infty,$ then

$$\zeta(\tau)(e) = \phi(e) = \phi_e(e) = \xi^*(\tau_f)(e) = \tau_f(\theta_0(e)) = \tau(\theta_0(e)).$$

If $\tau(f) = \infty,$ then e is not in the preimage of Λ_τ under θ_0 . Thus,

$$\zeta(\tau)(e) = \phi(e) = \infty = \tau(f).$$

Therefore, $\zeta(\tau)(e) = \tau(\theta_0(e))$ for all $\tau \in TB$ and $e \in K_0(A)_+.$ That is, ζ is compatible with θ_0 .

Let $\rho', \rho'' \in TB$ and $\rho = t\rho' + (1-t)\rho''$ for $t \in (0, 1).$ Let $\mu = \zeta(\rho), \mu' = \zeta(\rho'), \mu'' = \zeta(\rho'').$ If x is a positive element in the ideal generated by the preimage of Λ_ρ under $\theta_0,$ (without loss of generality, we can assume $x \in \overline{AeA}$ and $\theta_0(e) = f$), then by the above construction, we have

$$\zeta(\rho)(x) = \mu(x) = \mu|_{\overline{AeA}}(x) := \mu_e(x) = \xi^e(\rho_f)(x).$$

Since ξ^e is an affine map, we have

$$\begin{aligned} \xi^e(\rho_f)(x) &= \xi^e(t\rho'_f + (1-t)\rho''_f)(x) \\ &= t\xi^e(\rho'_f)(x) + (1-t)\xi^e(\rho''_f)(x) \\ &= t\mu'_e(x) + (1-t)\mu''_e(x) \\ &= [t\zeta(\rho') + (1-t)\zeta(\rho'')](x). \end{aligned}$$

If $x \in A_+$ is not in the ideal generated by the preimage of $\Lambda_\rho,$ by Lemma 4.3,

$$\zeta(\rho)(x) = \mu(x) = \infty.$$

Since $\Lambda_\rho = \Lambda_{\rho'} \cap \Lambda_{\rho''},$ x is not in the ideal generated by $\theta_0^{-1}(\Lambda_{\rho'}) \cap \theta_0^{-1}(\Lambda_{\rho''}),$ where $\theta_0^{-1}(\cdot)$ means the preimage set under $\theta_0.$ Thus,

$$\zeta(\rho')(x) = \infty \quad \text{or} \quad \zeta(\rho'')(x) = \infty.$$

Therefore,

$$\zeta(\rho)(x) = t\zeta(\rho')(x) + (1-t)\zeta(\rho'')(x)$$

for all $x \in A_+$ and all $t \in (0, 1).$ As a consequence, $\Psi = (\theta_0, \theta_1, \zeta)$ is a morphism from $\mathcal{E}(A)$ to $\mathcal{E}(B)$ in $\mathcal{E}.$ Define

$$\mathcal{F}(\Phi) = \Psi.$$

(1) It is obvious that \mathcal{F} maps objects to objects and morphisms to morphisms.

(2) Let A be a C^* -algebra with the ideal property. For any $\mu \in TA$, let J be the ideal generated by projections in Λ_μ . Let μ' be a lower semi-continuous trace on A defined by

$$\mu'(x) = \begin{cases} \mu(x), & \text{if } x \in J, \\ \infty, & \text{otherwise.} \end{cases}$$

By Lemma 4.3, $\mu'(x) = \mu(x)$ for all x in A . Therefore,

$$\mathcal{F}(\text{id}_{\mathcal{S}(A)}) = \text{id}_{\mathcal{E}(A)} = \text{id}_{\mathcal{F}(\mathcal{S}(A))}.$$

(3) Let A_1, A_2, A_3 be C^* -algebras with the ideal property. Suppose

$$\Psi_1: \mathcal{S}(A_1) \rightarrow \mathcal{S}(A_2), \quad \Psi_2: \mathcal{S}(A_2) \rightarrow \mathcal{S}(A_3)$$

are two morphisms. By the functoriality of K_0 and T , we can get

$$\mathcal{F}(\Psi_2 \circ \Psi_1) = \mathcal{F}(\Psi_2) \circ \mathcal{F}(\Psi_1).$$

Therefore, \mathcal{F} is a functor from the \mathcal{S}_J to \mathcal{E}_J .

Step II. Construction of the map $\mathcal{G}: \mathcal{E}_J \rightarrow \mathcal{S}_J$.

Let \mathcal{G} map the objects in \mathcal{E}_J to the objects in \mathcal{S}_J be defined by

$$\mathcal{G}(\mathcal{E}(\bullet)) = \mathcal{S}(\bullet).$$

By Corollary 3.9, this map is well-defined. Let $A, B \in \mathcal{I}$ such that there is an arrow $\Psi = (\theta_0, \theta_1, \zeta)$ from $\mathcal{E}(A)$ to $\mathcal{E}(B)$. We need to construct an arrow from $\mathcal{S}(A)$ to $\mathcal{S}(B)$. Let e be any projection in A . First we want to construct an affine map ξ^e from $T_F(\overline{\theta_0(e)B\theta_0(e)})$ to $T_F(\overline{eAe})$.

Let $f = \theta_0(e)$. For $\tau_f \in T_F(\overline{fBf})$, define a trace τ'_f on the closed ideal \overline{BfB} by

$$\tau'_f(x) = \sup \{ \tau_f(y) : y \preceq x, y \in (\overline{fBf})_+ \}, \text{ for all } x \in \overline{BfB}.$$

Then by Theorem 2.7, τ'_f is lower semi-continuous. Let τ be a trace on B defined by

$$\tau(x) = \begin{cases} \tau'_f(x), & \text{if } x \in \overline{BfB}, \\ \infty, & \text{otherwise.} \end{cases}$$

Then τ is a lower semi-continuous trace on B . Let $\phi = \zeta(\tau)$. By Lemma 4.3,

$$\phi(x) = \infty \text{ for any } x \in A \setminus (\overline{AeA}).$$

Let $\phi'_e = \phi|_{\overline{AeA}}$ and $\phi_e = \phi|_{\overline{eAe}}$. Then

$$\phi_e(e) = \phi(e) = \zeta(\tau)(e) = \tau(\theta_0(e)) = \tau(f) < \infty.$$

Thus ϕ_e is a finite trace on \overline{eAe} . Define $\xi^e(\tau_f) = \phi_e$. Then it is routine to check that ξ^e is an affine map.

Let $e' \in \mathcal{P}(A)$ and $f' \in \mathcal{P}(B)$ be such that $e' \leq e, f' \leq f$ and $\theta_0(e') = f'$. Let $\tau_{f'} \in T_F(\overline{f'Bf'})$ be any finite trace. We need to show the following diagram commutes:

$$\begin{array}{ccc} T_F(\overline{fBf}) & \xrightarrow{\xi^e} & T_F(\overline{eAe}) \\ \downarrow i & & \downarrow i \\ T_F(\overline{f'Bf'}) & \xrightarrow{\xi^{e'}} & T_F(\overline{e'Ae'}). \end{array}$$

That is, we need to show

$$i \circ \xi^e(\tau_f) = \xi^{e'} \circ i(\tau_{f'}).$$

Let $i(\tau_f) = \tau_{f'}$ and let $\tau, \tau' \in TB$ be extensions of τ_f and $\tau_{f'}$ respectively defined as above. Let $\phi = \zeta(\tau)$ and $\phi' = \zeta(\tau')$. By the definition of $\tau_{f'}$, we know that $\tau_{f'} = \tau_f|_{\overline{f'Bf'}}$. By Theorem 3.13, $\tau' = \tau$ on $\overline{Bf'B}$. Thus,

$$t\tau(x) + (1-t)\tau'(x) = \tau'(x), \text{ for all } x \in \overline{Bf'B} \text{ and } 0 \leq t \leq 1.$$

If $x \notin \overline{Bf'B}$, then $\tau'(x) = \infty$. Thus,

$$t\tau(x) + (1-t)\tau'(x) = \tau'(x), \text{ for all } x \notin \overline{Bf'B} \text{ and } 0 \leq t < 1.$$

Therefore, we have the following equality

$$t\tau + (1-t)\tau' = \tau', \text{ for any } 0 \leq t < 1.$$

Taking $t = 1/2$ and since ζ is an affine map, we get

$$\frac{1}{2}\zeta(\tau) + \frac{1}{2}\zeta(\tau') = \zeta(\tau').$$

So

$$\frac{1}{2}\phi|_{\overline{e'Ae'}} + \frac{1}{2}\phi'|_{\overline{e'Ae'}} = \phi'|_{\overline{e'Ae'}}.$$

Since both $\phi|_{\overline{e'Ae'}}$ and $\phi'|_{\overline{e'Ae'}}$ are finite,

$$i \circ \xi^e(\tau_f) = \phi|_{\overline{e'Ae'}} = \phi'|_{\overline{e'Ae'}} = \xi^{e'} \circ i(\tau_{f'}).$$

Thus ξ^e and $\xi^{e'}$ are compatible. Therefore, $\Phi = (\theta_0, \theta_1, \{\xi^e\}_{e \in \Sigma(A)})$ is a morphism from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$ in \mathfrak{S}_f . Define

$$\mathcal{G}(\Psi) = \Phi.$$

The following properties of \mathcal{G} are obvious:

(1) \mathcal{G} maps objects to objects and morphisms to morphisms by the above construction.

(2) For every C^* -algebra A with the ideal property

$$\mathcal{G}(\text{id}_{\mathcal{E}(A)}) = \text{id}_{\mathcal{S}(A)}.$$

(3) Let A_1, A_2, A_3 be C^* -algebras. Suppose that

$$\Phi_1: \mathcal{E}(A_1) \rightarrow \mathcal{E}(A_2), \quad \Phi_2: \mathcal{E}(A_2) \rightarrow \mathcal{E}(A_3)$$

are two morphisms. Then

$$\mathcal{G}(\Phi_2 \circ \Phi_1) = \mathcal{G}(\Phi_2) \circ \mathcal{G}(\Phi_1).$$

Therefore, \mathcal{G} is a functor from the \mathcal{E}_I to \mathcal{S}_I .

Step III. Check the identity of the theorem.

(1) Let A, B be C^* -algebras with the ideal property. Suppose we have the following maps

$$\begin{array}{ccccc} \mathcal{S}(A) & \xrightarrow{\mathcal{F}} & \mathcal{E}(A) & \xrightarrow{\mathcal{G}} & \mathcal{S}(A) \\ \downarrow \Phi=(\theta_0, \theta_1, \xi) & & \downarrow \Psi=(\theta_0, \theta_1, \zeta) & & \downarrow \hat{\Phi}=(\theta_0, \theta_1, \hat{\xi}) \\ \mathcal{S}(B) & \xrightarrow{\mathcal{F}} & \mathcal{E}(B) & \xrightarrow{\mathcal{G}} & \mathcal{S}(B), \end{array}$$

where $\xi = \{\xi^p\}_{p \in \Sigma(A)}$, $\hat{\xi} = \{\hat{\xi}^p\}_{p \in \Sigma(A)}$, $\mathcal{F}(\Phi) = \Psi$ and $\mathcal{G}(\Psi) = \hat{\Phi}$. We need to show $\hat{\Phi} = \Phi$.

Let $\tau_f \in T_F(\overline{fBf})$ be any finite trace, where f is a projection in B with $\theta_0(e) = f$. By the construction of Steps II and I, we have

$$\hat{\xi}^e(\tau_f) = \zeta(\tau)|_{\overline{eAe}} = \xi^e(\tau_f).$$

Therefore, $\xi = \hat{\xi}$ and $\Phi = \hat{\Phi}$. That is $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathcal{S}(A)}$.

(2) To prove $\mathcal{F} \circ \mathcal{G} = \text{id}_{\mathcal{E}(A)}$, assume we have the following maps

$$\begin{array}{ccccc} \mathcal{E}(A) & \xrightarrow{\mathcal{G}} & \mathcal{S}(A) & \xrightarrow{\mathcal{F}} & \mathcal{E}(A) \\ \downarrow \Psi=(\theta_0, \theta_1, \zeta) & & \downarrow \Phi=(\theta_0, \theta_1, \xi) & & \downarrow \hat{\Psi}=(\theta_0, \theta_1, \hat{\zeta}) \\ \mathcal{E}(B) & \xrightarrow{\mathcal{G}} & \mathcal{S}(B) & \xrightarrow{\mathcal{F}} & \mathcal{E}(B), \end{array}$$

where $\xi = \{\xi^p\}_{p \in \Sigma(A)}$, $\mathcal{G}(\Psi) = \Phi$ and $\mathcal{F}(\Phi) = \hat{\Psi}$. We need to show that $\hat{\Psi} = \Psi$.

Let $\tau \in TB$, $\zeta(\tau) = \phi$ and $\hat{\zeta}(\tau) = \hat{\Phi}$. Then

$$\hat{\Phi}|_{\overline{eAe}} = \hat{\zeta}(\tau)|_{\overline{eAe}} = \xi^e(\tau_f) = \zeta(\tau)|_{\overline{eAe}} = \phi|_{\overline{eAe}}$$

for all pairs $e \in \mathcal{P}(A)$, $f \in \mathcal{P}(B)$ with $\theta_0(e) = f$ and $f \in B^\tau$. Therefore, $\widehat{\zeta}(x) = \zeta(x)$ for all x in the ideal I which is generated by projections in the set of $\theta_0^{-1}(B^\tau)$. By Lemma 4.3, $\phi(x) = +\infty$ for all $x \in A \setminus I$. By our construction of $\widehat{\Phi}$, we know $\widehat{\Phi}(x) = +\infty$ for all $x \in A \setminus I$. Therefore, $\widehat{\Phi} = \phi$, which completes the proof. \square

Remark 4.6. Theorem 1.2 follows from the above theorem.

Example 4.7 (Counter-Example). In [36], Shaloub Razak classified a class of C^* -algebras which are inductive limits of certain specified building blocks by using their Elliott invariants. Those C^* -algebras Razak classified are simple, stably projectionless and with trivial K -theory. Their Elliott invariants cannot be derived from their Stevens invariants since their Stevens invariants are all trivial.

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Received 03 January, 2017; revised 20 May, 2017

K. Wang, Department of Mathematics, Texas A & M University,
College Station, TX 77843, USA; and
Department of Mathematics, University of Puerto Rico,
Rio Piedras, PR 00936, USA
E-mail: kun.wang@upr.edu