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Simple nuclear C*-algebras not equivariantly isomorphic to their opposites

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Abstract. We exhibit examples of simple separable nuclear C*-algebras, along with actions of the circle group and outer actions of the integers, which are not equivariantly isomorphic to their opposite algebras. In fact, the fixed point subalgebras are not isomorphic to their opposites. The C*-algebras we exhibit are well behaved from the perspective of structure and classification of nuclear C*-algebras: they are unital C*-algebras in the UCT class, with finite nuclear dimension. One is an AH-algebra with unique tracial state and absorbs the CAR algebra tensorially. The other is a Kirchberg algebra.

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1. Introduction

Let A be a C*-algebra. We denote by A^{op} the opposite algebra: the same Banach space with the same involution, but with reversed multiplication. The question of constructing operator algebras not isomorphic to their opposites goes back to [3], which constructs factors not isomorphic to their opposites. Separable simple C*-algebras not isomorphic to their opposites were constructed in [22, 23], but those examples are not nuclear. The proofs that they are not isomorphic to their opposites depend on embedding them as weak operator dense subalgebras of von Neumann algebras not isomorphic to their opposites, so that the method can't be used to produce nuclear examples. In the nuclear setting, there are nonsimple C*-algebras not isomorphic to their opposites ([21]; see also [26] for a related discussion). These examples are in fact continuous trace C*-algebras, and the proofs depend on the Dixmier–Douady invariant. It is also consistent with ZFC that there are nonseparable simple AF algebras and nonseparable purely infinite simple C*-algebras not isomorphic to their opposites ([8]). The proofs depend strongly

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on nonseparability, because all separable C*-algebras of the types considered in that paper are already known to be isomorphic to their opposites. The question of whether there are separable simple nuclear C*-algebras not isomorphic to their opposites is an important and difficult open question, particularly due to its connection with the Elliott classification program. The Elliott invariant, as well as the Cuntz semigroup, cannot distinguish a C*-algebra from its opposite. Thus, existence of a simple separable nuclear C*-algebra not isomorphic to its opposite would reveal an entirely new phenomenon.

In this paper, we address the equivariant situation. We exhibit examples of simple separable unital nuclear C*-algebras A along with outer actions of $\mathbb{Z}$ and with actions of the circle group $\mathbb{T}$ which are not *equivariantly* isomorphic to their opposites. The C*-algebras A are well behaved from the perspective of structure and classification of C*-algebras. In one set of examples, A is AH with no dimension growth, has a unique tracial state, and tensorially absorbs the CAR algebra. In the other, A is a Kirchberg algebra satisfying the Universal Coefficient Theorem.

In fact, in our examples the fixed point algebras are not isomorphic to their opposite algebras. In particular, we give outer actions of $\mathbb{Z}$ on a simple separable unital nuclear C*-algebra with tracial rank zero and on a unital Kirchberg algebra, both satisfying the Universal Coefficient Theorem, such that the fixed point algebras are not isomorphic to their opposites.

These examples illustrate some of the difficulties one would encounter if one wished to extend the current classification results to the equivariant setting, for actions of both $\mathbb{Z}$ and $\mathbb{T}$.

We outline our construction. We find a suitable compact manifold L (which is $S^n \times M$ in the proofs of Theorem 3.1 and Theorem 3.2, with $n \geq 5$ and odd, and with M as used in Example 2.5 of [21]) and a torsion class $\eta_0 \in H^3(L; \mathbb{Z})$ for which we can prove that there is no homeomorphism of L which sends η_0 to $-\eta_0$. Then there is a locally trivial continuous field E over L whose fiber is a finite dimensional matrix algebra and whose Dixmier–Douady invariant is η_0. The condition on η_0 ensures that the section algebra $\Gamma(E)$ is not stably isomorphic to its opposite. (See Chapters 4 and 5 of [25] for the general theory of the Dixmier–Douady invariant. Some of it is summarized at the beginning of Section 1 of [21].) Next, use an existence result of Fathi and Herman (and the presence of the factor S^n in the construction of L) to find a uniquely ergodic minimal diffeomorphism h of L which is homotopic to the identity. Then $h^*(E) \cong E$, allowing one to construct an automorphism α of $\Gamma(E)$ which induces h on $\mathrm{Prim}(\Gamma(E))$. The basic version of our example is $A_0 = \Gamma(E) \rtimes_\alpha \mathbb{Z}$, with the action of $\mathbb{T}$ being the dual action. The fixed point algebra of this dynamical system is isomorphic to $\Gamma(E)$ and the fixed point algebra of the opposite system is isomorphic to $\Gamma(E)^{\mathrm{op}}$, so the system is not isomorphic to its opposite. The algebra A_0 is simple and separable, has finite nuclear dimension, and satisfies the Universal Coefficient Theorem.

Stronger conditions on the algebra in the system are obtained by tensoring the system with the trivial action on M_{2^∞} or on $\mathcal{O}_\infty$. This change tensors the fibers of E

with M_{2^∞} or $\mathcal{O}_\infty$. To prove that the new section algebras are not isomorphic to their opposites, we need a substitute for the Dixmier–Douady invariant which applies to continuous fields whose fibers are stably isomorphic to M_{2^∞} or $\mathcal{O}_\infty$. Fortunately, a suitable theory is already available in [4].

In the rest of the introduction, we recall a few general facts about opposite algebras. Section 2 contains some preparatory lemmas, and Section 3 contains the construction of our examples. In Section 4 we collect several remarks on our construction, outline a shorter construction which gives examples with some of the properties of our main examples, and state some open questions.

If A is a C*-algebra, we denote by $A^\#$ its conjugate algebra. As a real C*-algebra it is the same as A, but it has the reverse complex structure. That is, if we denote its scalar multiplication by $(\lambda, a) \mapsto \lambda \bullet_\# a$, then $\lambda \bullet_\# a = \overline{\lambda} a$ for any $a \in A$ and any $\lambda \in \mathbb{C}$. We recall the following easy fact.

Lemma 1.1. *Let A be a C*-algebra. Then the map $a \mapsto a^*$ is an isomorphism $A^{\mathrm{op}} \to A^\#$.*

In this paper, we often find it more convenient to use $A^\#$.

Let G be a locally compact Hausdorff group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a point-norm continuous action. For $g \in G$, the same map α_g, viewed as a map from $A^\#$ to itself, is also an automorphism. To see that, we note that it is clearly a real C*-algebra automorphism, and for each $\lambda \in \mathbb{C}$ and $a \in A$, we have

$$\alpha_g(\lambda \bullet_\# a) = \alpha_g\big(\overline{\lambda} a\big) = \overline{\lambda}\alpha_g(a) = \lambda \bullet_\# \alpha_g(a).$$

Thus, the same map gives us an action $\alpha^\#$ of G on $A^\#$, which we call the *conjugate action*. The definition of the crossed product shows that $(A \rtimes_\alpha G)^\# \cong A^\# \rtimes_{\alpha^\#} G$: they are identical as real C*-algebras, and the complex structure on $A \rtimes_\alpha G$ comes from the complex structure on A. The same map also gives an action α^{op} of G on A^{op}, which we call the *opposite action*. The map from Lemma 1.1 intertwines α^{op} with $\alpha^\#$ and hence $(G, A^{\mathrm{op}}, \alpha^{\mathrm{op}}) \cong (G, A^\#, \alpha^\#)$. The identification of the crossed product, however, is less direct.

If (G, A, α) and (G, B, β) are G-C*-algebras and are G-equivariantly isomorphic, then $A \rtimes_\alpha G \cong B \rtimes_\beta G$. Thus, an equivariant version of the problem of whether C*-algebras are isomorphic to their opposites is whether (G, A, α) is isomorphic to $(G, A^\#, \alpha^\#)$.

We denote the algebra of compact operators on a separable infinite dimensional Hilbert space by K.

2. Preparatory lemmas

Our construction requires two lemmas from cohomology, some properties of a particular finite group, a result on quasidiagonality of crossed products of integer actions on section algebras of continuous fields, a result on invariant traces on section algebras, and a lemma concerning tracial states on crossed products by an automorphism with finite Rokhlin dimension.

The following lemma generalizes Lemma 3.6 of [21], which is part of an example of a topological space with specific properties originally suggested by Greg Kuperberg.

Lemma 2.1. *Let $n \in \mathbb{Z}_{>0}$, let M be a connected compact orientable manifold of dimension $4n$ and with no boundary, and let $h\colon M \to M$ be a continuous function such that $h_*\colon H_*(M;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ is an isomorphism. Suppose that the signature of M is nonzero. Then h is orientation preserving.*

Proof. We recall the definition of the signature, starting with the bilinear form $\omega\colon H^{2n}(M;\mathbb{Z}) \times H^{2n}(M;\mathbb{Z}) \to \mathbb{Z}$ defined as follows. Let $e_0 \in H_0(M;\mathbb{Z})$ be the standard generator. Thus there is an isomorphism $\nu\colon H_0(M;\mathbb{Z}) \to \mathbb{Z}$ such that $\nu(ke_0) = k$ for all $k \in \mathbb{Z}$. Further let $c \in H_{4n}(M;\mathbb{Z})$ be the generator corresponding to the orientation of M (the fundamental class). Also recall the cup product

$$(\alpha, \beta) \mapsto \alpha \smile \beta$$

from $H^k(M;\mathbb{Z}) \times H^l(M;\mathbb{Z})$ to $H^{k+l}(M;\mathbb{Z})$, and the cap product

$$(\alpha, \beta) \mapsto \alpha \frown \beta$$

from $H^k(M;\mathbb{Z}) \times H_l(M;\mathbb{Z})$ to $H_{l-k}(M;\mathbb{Z})$. Then ω is given by

$$\omega(\alpha, \beta) = \nu\big([\alpha \smile \beta] \frown c\big)$$

for $\alpha, \beta \in H^{2n}(M;\mathbb{Z})$. The signature of the form gotten by tensoring with $\mathbb{R}$ is, by definition, the signature of M.

The Universal Coefficient Theorem (see [12, Theorem 3.2 and p. 198]) and the Five Lemma imply that

$$h^*\colon H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$$

is an isomorphism. In particular,

$$h^*\colon H^{2n}(M;\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})$$

is an isomorphism. Therefore the bilinear form ρ on $H^{2n}(M;\mathbb{Z})$, given by

$$\rho(\alpha, \beta) = \omega\big(h^*(\alpha),\, h^*(\beta)\big) = \nu\big([h^*(\alpha) \smile h^*(\beta)] \frown c\big)$$

for $\alpha, \beta \in H^{2n}(M;\mathbb{Z})$, is equivalent to ω. In particular, ρ has the same signature as ω.

Now define a bilinear form ω_0 on $H^{2n}(M;\mathbb{Z})$ by

$$\omega_0(\alpha,\beta) = \nu\big([h^*(\alpha) \smile h^*(\beta)] \frown (h_*)^{-1}(c)\big)$$

for $\alpha, \beta \in H^{2n}(M;\mathbb{Z})$. The formula for ω_0 differs from the formula for ρ only in that c has been replaced by $(h_*)^{-1}(c)$. The maps $\nu \circ h_*$ and ν agree on $H_0(M;\mathbb{Z})$. (This is true for any continuous map $h: M \to M$.) Naturality of the cup and cap products therefore implies that $\omega = \omega_0$. If $(h_*)^{-1}(c) = -c$, then $\omega_0 = -\rho$, so ω_0 and ρ have opposite signatures. Since $\omega_0 = \omega$ and ρ have the same signature by the previous paragraph, we find that the signature of ω is zero. This contradiction shows that $(h_*)^{-1}(c) \neq -c$.

Since h_* is an isomorphism and $H_{4n}(M;\mathbb{Z}) \cong \mathbb{Z}$, it follows that $h_*(c) = \pm c$. The previous paragraph rules out $h_*(c) = -c$, so $h_*(c) = c$. □

Lemma 2.2. *Let $m \in \mathbb{Z}_{>0}$ and let M be a connected compact orientable manifold of dimension m. Let $n \in \mathbb{Z}_{>0}$ satisfy $n > m$, and let $h: S^n \times M \to S^n \times M$ be a continuous function. Let $y_0 \in S^n$, let $i: M \to S^n \times M$ be $i(x) = (y_0, x)$ for $x \in M$, and let $p: S^n \times M \to M$ be the projection on the second factor. Then:*

(1) *If*

$$h_*: H_*(S^n \times M;\, \mathbb{Z}) \to H_*(S^n \times M;\, \mathbb{Z})$$

is an isomorphism then

$$(p \circ h \circ i)_*: H_*(M;\mathbb{Z}) \to H_*(M;\mathbb{Z})$$

is an isomorphism.

(2) *If*

$$h_*: \pi_1(S^n \times M) \to \pi_1(S^n \times M)$$

is an isomorphism then

$$(p \circ h \circ i)_*: \pi_1(M) \to \pi_1(M)$$

is an isomorphism. (We omit the choice of basepoints in π_1, since the spaces in question are path connected.)

Proof. We prove (1). Let $k \in \mathbb{Z}_{\geq 0}$; we show that

$$(p \circ h \circ i)_*: H_k(M;\mathbb{Z}) \to H_k(M;\mathbb{Z})$$

is an isomorphism. For $k > m$, $H_k(M;\mathbb{Z}) = 0$, so this is immediate. Accordingly, we may assume that $0 \leq k \leq m$.

Let e_0 be the usual generator of $H_0(S^n;\mathbb{Z}) \cong \mathbb{Z}$ and let e_n be a generator of $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Since $H_*(S^n;\mathbb{Z})$ is free, the Künneth formula [12, Theorem 3B.6] implies that the standard pairing $(\eta, \mu) \mapsto \eta \times \mu$ yields a graded isomorphism

$$\omega: H_*(S^n;\mathbb{Z}) \otimes H_*(M;\mathbb{Z}) \to H_*(S^n \times M;\, \mathbb{Z}).$$

Since $H_l(M;\mathbb{Z}) = 0$ for $l \geq n$, it follows that $\mu \mapsto \omega(e_0 \otimes \mu)$ defines an isomorphism

$$\beta\colon H_k(M;\mathbb{Z}) \to H_k(S^n \times M;\, \mathbb{Z})$$

(and, similarly, $\mu \mapsto e_n \times \mu$ defines an isomorphism $H_k(M;\mathbb{Z}) \to H_{n+k}(S^n \times M;\, \mathbb{Z})$). Moreover, $\beta = i_*$.

Let $p_0\colon S^n \to \{y_0\}$ be the unique map from S^n to $\{y_0\}$. Since $(p_0)_*(e_0)$ is a generator of $H_0(\{y_0\};\mathbb{Z}) \cong \mathbb{Z}$, naturality in the Künneth formula implies that $p_*(e_0 \times \mu) = \mu$ for $\mu \in H_k(M;\mathbb{Z})$. (By contrast, $p_*(e_n \times \mu) = 0$.) Thus

$$p_*\colon H_k(S^n \times M;\, \mathbb{Z}) \to H_k(M;\mathbb{Z})$$

is an isomorphism. (In fact, $p_* = \beta^{-1}$.)

We factor $(p \circ h \circ i)_*\colon H_k(M;\mathbb{Z}) \to H_k(M;\mathbb{Z})$ as

$$H_k(M;\mathbb{Z}) \xrightarrow{i_*} H_k(S^n \times M;\, \mathbb{Z}) \xrightarrow{h_*} H_k(S^n \times M;\, \mathbb{Z}) \xrightarrow{p_*} H_k(M;\, \mathbb{Z}).$$

We have just shown that the first and last maps are isomorphisms, and the middle map is an isomorphism by hypothesis. So $(p \circ h \circ i)_*$ is an isomorphism.

Part (2) follows immediately as soon as we know that p_* and i_* are isomorphisms. This fact follows from [12, Proposition 1.12]. □

We will start our constructions with the manifold M used in [21, Example 3.5], with

$$\pi_1(M) \cong \langle a, b \mid a^3 = b^7 = 1,\ aba^{-1} = b^2 \rangle.$$

There is a gap in the proof for [21, Example 3.5]; we need to know that there is no automorphism of $\pi_1(M)$ which sends the image of a in the abelianization to the image of a^2. We prove that here; for convenience of the reader and to establish notation in the proof, we prove all the properties of this group from scratch.

Lemma 2.3. *Let G be the group with presentation in terms of generators and relations given by*

$$G = \langle a, b \mid a^3 = b^7 = 1,\ aba^{-1} = b^2 \rangle.$$

Then G is a finite group with 21 elements, its abelianization is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and is generated by the image of a, and every automorphism of G induces the identity automorphism on its abelianization.

Proof. Rewrite the last relation as $ab = b^2 a$. It follows that for all $r, s \in \mathbb{Z}_{\geq 0}$ there is $t \in \mathbb{Z}_{\geq 0}$ such that $a^r b^s = b^t a^r$. Since a and b have finite order, we therefore have

$$G = \left\{ b^t a^r : r, t \in \mathbb{Z}_{\geq 0} \right\}. \tag{2.1}$$

Since $a^3 = b^7 = 1$, it follows that G has at most 21 elements.

Write $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ and $\mathbb{Z}/7\mathbb{Z} = \{0, 1, \ldots, 6\}$. One checks that there is an automorphism γ of $\mathbb{Z}/7\mathbb{Z}$ such that $\gamma(1) = 2$, and that $\gamma^3 = \mathrm{id}_{\mathbb{Z}/7\mathbb{Z}}$. Thus, there is a semidirect product group $S = \mathbb{Z}/7\mathbb{Z} \rtimes_\gamma \mathbb{Z}/3\mathbb{Z}$. Moreover, the elements $(0,1)$ and $(1,0)$ satisfy the relations defining G. Therefore there is a surjective homomorphism $\psi\colon G \to S$ such that $\psi(a) = (1,0)$ and $\psi(b) = (0,1)$. So G has exactly 21 elements and the subgroup $\langle b \rangle \subset G$ is normal and has order 7.

Let H be the abelianization of G and let $\pi\colon G \to H$ be the associated map. The relations for G show that there is a surjective homomorphism $\kappa\colon G \to \mathbb{Z}/3\mathbb{Z}$ such that $\kappa(a) = 1$ and $\kappa(b) = 0$. Therefore $\mathbb{Z}/3\mathbb{Z}$ is a quotient of H. Since $\mathrm{card}(G)/\mathrm{card}(H)$ is prime and G is not abelian, we get $H \cong \mathbb{Z}/3\mathbb{Z}$, generated by $\pi(a)$. Moreover, $\pi(b)$ is the identity element of H.

Now let $\varphi\colon G \to G$ be an automorphism, and let $\overline{\varphi}\colon H \to H$ be the induced automorphism of H. To show that $\overline{\varphi} = \mathrm{id}_H$, we must rule out $\overline{\varphi}(\pi(a)) = \pi(a^2)$. So assume $\overline{\varphi}(\pi(a)) = \pi(a^2)$. Use (2.1), $a^3 = b^7 = 1$, and $\pi(b) = 1$ to find $r \in \{0, 1, \ldots, 6\}$ such that $\varphi(a) = b^r a^2$. Since $\langle b \rangle$ is a normal Sylow 7-subgroup, all elements of G of order 7 are contained in $\langle b \rangle$, so there is $s \in \{1, 2, \ldots, 6\}$ such that $\varphi(b) = b^s$. Apply φ to the relation $aba^{-1} = b^2$ to get

$$b^r a^2 b^s a^{-2} b^{-r} = b^{2s}. \tag{2.2}$$

The relation $aba^{-1} = b^2$ also implies that $ab^s a^{-1} = b^{2s}$, so $a^2 b^s a^{-2} = b^{4s}$. Substituting in (2.2) gives $b^{4s} = b^{2s}$. Thus $(b^s)^2 = 1$. Since $\langle b \rangle$ is cyclic of odd order, we get $b^s = 1$, so $\varphi(b) = 1$, a contradiction. □

The proof of the following lemma is motivated by ideas from the proof of Theorem 9 in [24].

Lemma 2.4. *Let A be a separable continuous trace C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Set $X = \mathrm{Prim}(A)$, let $h\colon X \to X$ be homeomorphism induced by α (so that if $P \subset A$ is a primitive ideal, then $h(P) = \alpha(P)$), and assume that X is an infinite compact metrizable space and that h is minimal. Then $A \rtimes_\alpha \mathbb{Z}$ is simple and quasidiagonal.*

Proof. Simplicity of $A \rtimes_\alpha \mathbb{Z}$ follows from the corollary to Theorem 1 in [1].

We claim that there is a nonzero homomorphism

$$\varphi\colon A \rtimes_\alpha \mathbb{Z} \to C_{\mathrm{b}}(\mathbb{Z}_{>0}, K)/C_0(\mathbb{Z}_{>0}, K).$$

Since $A \rtimes_\alpha \mathbb{Z}$ is simple, it will then follow that φ is injective. Since A is nuclear, so is $A \rtimes_\alpha \mathbb{Z}$. Therefore we can lift φ to a completely positive contraction

$$T\colon A \rtimes_\alpha \mathbb{Z} \to C_{\mathrm{b}}(\mathbb{Z}_{>0},\ K).$$

We thus get a sequence $(T_n)_{n \in \mathbb{Z}_{>0}}$ of completely positive contractions

$$T_n\colon A \rtimes_\alpha \mathbb{Z} \to K$$

such that

$$\lim_{n\to\infty} \|T_n(ab) - T_n(a)T_n(b)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|T_n(a)\| = \|a\| \tag{2.3}$$

for all $a, b \in A \rtimes_\alpha \mathbb{Z}$. We can pick a sequence $(p_n)_{n \in \mathbb{Z}_{>0}}$ in K consisting of finite rank projections such that, if for all $n \in \mathbb{Z}_{>0}$ we replace T_n by $a \mapsto p_n T_n(x) p_n$, the resulting sequence of maps still satisfies (2.3). Thus, we may assume that there is a sequence $(l(n))_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$, we actually have a completely positive contraction

$$T_n \colon A \rtimes_\alpha \mathbb{Z} \to M_{l(n)};$$

moreover, the sequence $(T_n)_{n \in \mathbb{Z}_{>0}}$ satisfies (2.3). Thus $A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal.

It remains to prove the claim. It suffices to prove the claim for $A \otimes K$ and $\alpha \otimes \mathrm{id}_K$ in place of A and α. By Proposition 5.59 in [25], we may therefore assume that A is the section algebra of a locally trivial continuous field E over X with fiber K.

Fix a point $x_0 \in X$. Choose a closed neighborhood S of x_0 such that $E|_S$ is trivial, and let $\kappa \colon A \to C(S, K)$ be the composition of the quotient map $A \to \Gamma(E|_S)$ and a trivialization $\Gamma(E|_S) \to C(S, K)$. For $x \in S$, let $\mathrm{ev}_x \colon C(S, K) \to K$ be evaluation at x. For $n \in \mathbb{Z}$ define

$$\sigma_n = \mathrm{ev}_{x_0} \circ \kappa \circ \alpha^n \colon A \to K,$$

a surjective homomorphism with kernel $h^{-n}(x_0) \in X = \mathrm{Prim}(A)$.

Since h is minimal, there is a sequence $(k(n))_{n \in \mathbb{Z}_{>0}}$ in $\mathbb{Z}_{>0}$ such that

$$\lim_{n \to \infty} k(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} h^{-k(n)}(x_0) = x_0.$$

Without loss of generality $h^{-k(n)}(x_0) \in S$ for all $n \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{>0}$, the homomorphisms $\sigma_{k(n)}$ and $\mathrm{ev}_{h^{-k(n)}(x_0)} \circ \kappa$ are surjective homomorphisms from A to K with the same kernel (namely $h^{-k(n)}(x_0) \in X = \mathrm{Prim}(A)$), so they are unitarily equivalent irreducible representations. That is, there is a unitary $v_n \in M(K) = L(l^2)$ such that

$$v_n \sigma_{k(n)}(a) v_n^{-1} = \big(\mathrm{ev}_{h^{-k(n)}(x_0)} \circ \kappa\big)(a)$$

for all $a \in A$. Choose $c_n \in M(K)_{\mathrm{sa}}$ with $\|c_n\| \leq \pi$ such that $v_n = \exp(i c_n)$, and set $w_n = \exp(i k(n)^{-1} c_n)$. Then

$$\|w_n - 1\| \leq \frac{\pi}{k(n)} \quad \text{and} \quad w_n^{k(n)} = v_n.$$

Define $\rho_n \colon A \to M_{k(n)}(K)$ by

$$\rho_n(a) = \mathrm{diag}\big(\sigma_0(a),\, w_n \sigma_1(a) w_n^{-1},\, w_n^2 \sigma_2(a) w_n^{-2}, \ldots \\ \ldots,\, w_n^{k(n)-1} \sigma_{k(n)-1}(a) w_n^{-(k(n)-1)}\big)$$

for $a \in A$. Further define the permutation unitary $u_n \in M(M_{k(n)}(K))$ by

$$u_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

We want to show that for all $a \in A$ we have

$$\lim_{n \to \infty} \|\rho_n(\alpha(a)) - u_n \rho_n(a) u_n^*\| = 0.$$

To do this, let $a \in A$. Then

$$\begin{aligned}
&\rho_n(\alpha(a)) \\
&= \operatorname{diag}\big(\sigma_1(a),\ w_n \sigma_2(a) w_n^{-1},\ w_n^2 \sigma_3(a) w_n^{-2}, \ldots,\ w_n^{k(n)-1} \sigma_{k(n)}(a) w_n^{-(k(n)-1)}\big) \\
&= u_n \operatorname{diag}\big(w_n^{k(n)-1} \sigma_{k(n)}(a) w_n^{-(k(n)-1)},\ \sigma_1(a),\ w_n \sigma_2(a) w_n^{-1}, \ldots \\
&\qquad \ldots,\ w_n^{k(n)-2} \sigma_{k(n)-1}(a) w_n^{-(k(n)-2)}\big) u_n^*.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\rho_n(\alpha(a)) - u_n \rho_n(a) u_n^*\| \\
&= \|\rho_n(a) - u_n^* \rho_n(\alpha(a)) u_n\| \\
&= \max\Big(\big\|\sigma_0(a) - w_n^{k(n)-1} \sigma_{k(n)}(a) w_n^{-(k(n)-1)}\big\|, \\
&\qquad \big\|w_n \sigma_1(a) w_n^{-1} - \sigma_1(a)\big\|,\ \big\|w_n^2 \sigma_2(a) w_n^{-2} - w_n \sigma_2(a) w_n^{-1}\big\|, \ldots \\
&\qquad \ldots,\ \big\|w_n^{k(n)-1} \sigma_{k(n)-1}(a) w_n^{-(k(n)-1)} - w^{k(n)-2} \sigma_{k(n)-1}(a) w_n^{-(k(n)-2)}\big\| \Big).
\end{aligned}$$

Every term except the first on the right hand side of this estimate is dominated by

$$2\|w_n - 1\| \|a\| \le \frac{2\pi \|a\|}{k(n)}.$$

The first term is estimated as follows:

$$\begin{aligned}
&\big\|\sigma_0(a) - w_n^{k(n)-1} \sigma_{k(n)}(a) w_n^{-(k(n)-1)}\big\| \\
&\qquad \le \|\sigma_0(a) - v_n \sigma_{k(n)}(a) v_n^*\| \\
&\qquad\quad + \big\|w_n^{k(n)} \sigma_{k(n)}(a) w_n^{-k(n)} - w_n^{k(n)-1} \sigma_{k(n)}(a) w_n^{-(k(n)-1)}\big\| \\
&\qquad \le \big\|(\mathrm{ev}_{x_0} \circ \kappa)(a) - \big(\mathrm{ev}_{h^{-k(n)}(x_0)} \circ \kappa\big)(a)\big\| + 2\|w_n - 1\| \|a\| \\
&\qquad \le \big\|(\mathrm{ev}_{x_0} \circ \kappa)(a) - \big(\mathrm{ev}_{h^{-k(n)}(x_0)} \circ \kappa\big)(a)\big\| + \frac{2\pi \|a\|}{k(n)}.
\end{aligned}$$

Now let $\varepsilon > 0$. Since $\kappa(a) \in C(S, K)$ is continuous and $\lim_{n\to\infty} h^{-k(n)}(x_0) = x_0$, there is $N_1 \in \mathbb{Z}_{>0}$ such that for all $n \geq N_1$ we have

$$\big\| (\mathrm{ev}_{x_0} \circ \kappa)(a) - \big(\mathrm{ev}_{h^{-k(n)}(x_0)} \circ \kappa\big)(a) \big\| < \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} k(n) = \infty$, there is $N_2 \in \mathbb{Z}_{>0}$ such that for all $n \geq N_2$ we have

$$\frac{2\pi \|a\|}{k(n)} < \frac{\varepsilon}{2}.$$

For $n \geq \max(N_1, N_2)$, we then have $\|\rho_n(\alpha(a)) - u_n \rho_n(a) u_n^*\| < \varepsilon$, as desired.

For $n \in \mathbb{Z}_{>0}$ choose an isomorphism $\psi_n \colon M_{k(n)}(K) \to K$, and use the same symbol for the induced isomorphism $M_{k(n)}(M(K)) \to M(K)$. Let

$$u \in M\big(C_{\mathrm{b}}(\mathbb{Z}_{>0}, K)/C_0(\mathbb{Z}_{>0}, K)\big)$$

be the image there of

$$\big(\psi_1(u_1),\ \psi_2(u_2),\ \ldots\big) \in C_{\mathrm{b}}(\mathbb{Z}_{>0}, M(K)),$$

and for $a \in A$ let $\psi(a) \in C_{\mathrm{b}}(\mathbb{Z}_{>0}, K)/C_0(\mathbb{Z}_{>0}, K)$ be the image there of

$$\big((\psi_1 \circ \rho_1)(a),\ (\psi_2 \circ \rho_2)(a),\ \ldots\big) \in C_{\mathrm{b}}(\mathbb{Z}_{>0}, K).$$

Then $u\psi(a)u^* = \psi(\alpha(a))$ for all $a \in A$, so u and ψ together define a homomorphism

$$\varphi \colon A \rtimes_\alpha \mathbb{Z} \to C_{\mathrm{b}}(\mathbb{Z}_{>0}, K)/C_0(\mathbb{Z}_{>0}, K).$$

This homomorphism is nonzero because if we choose $c \in K \setminus \{0\}$ then there is $a \in A$ such that $\kappa(a)$ is the constant function with value c, and $\|\psi(a)\|$ is easily checked to be $\|c\|$. This completes the proof of the claim, and thus of the lemma. □

To show that the crossed product is quasidiagonal, it isn't actually necessary that h be minimal. It suffices to assume that every point of X is chain recurrent. The basic idea is the same, but the notation gets messier.

Lemma 2.5. *Let G be a group, and let X be a compact metrizable G-space. Let A be a unital C*-algebra with a unique tracial state σ. Let E be a locally trivial continuous field over X with fiber A, and let $\alpha \colon G \to \Gamma(E)$ be a continuous action of G on the section algebra $\Gamma(E)$ such that $\alpha_g(\mathrm{Ker}(\mathrm{ev}_x)) = \mathrm{Ker}(\mathrm{ev}_{gx})$ for all $g \in G$ and $x \in X$. Then there is a faithful G-equivariant conditional expectation $P \colon \Gamma(E) \to C(X)$ such that for every $s \in \Gamma(E)$, $x \in X$, and isomorphism $\psi \colon E_x \to A$, we have*

$$P(s)(x) = \sigma\big(\psi(s(x))\big). \tag{2.4}$$

Moreover, the formula $\tau_\mu(s) = \int_X P(s)\, d\mu$ defines an affine homeomorphism from the G-invariant Borel probability measures μ on X to the G-invariant tracial states on $\Gamma(E)$.

Proof. For each $x \in X$ choose some isomorphism $\varphi_x \colon E_x \to A$. Now for $s \in \Gamma(E)$ define a function $P(s) \colon X \to \mathbb{C}$ by $P(s)(x) = \sigma(\varphi_x(s(x)))$ for $x \in X$.

We claim that (2.4) holds for any choice of isomorphism $\psi \colon E_x \to A$. To prove the claim, simply observe that E_x has a unique tracial state, so that $\sigma \circ \psi = \sigma \circ \varphi_x$.

Let $s \in \Gamma(E)$. We claim that $P(s)$ is continuous. Let $x_0 \in X$; it is enough to find an open set $U \subset X$ with $x_0 \in U$ such that $P(s)|_U$ is continuous. Choose U such that there is a trivialization of $E|_U$, that is, isomorphisms $\psi_x \colon E_x \to A$ for $x \in U$ such that the map $(a, x) \mapsto \psi_x^{-1}(a)$ is a homeomorphism from $A \times U$ to $E|_U$. Then the map $x \mapsto \psi_x(s(x))$ from U to A is continuous, so $x \mapsto \sigma(\psi_x(s(x)))$ is continuous. The previous claim implies that $\sigma(\varphi_x(s(x))) = \sigma(\psi_x(s(x)))$ for all $x \in U$, so the claim follows.

We now know that $P \colon \Gamma(E) \to C(X)$ is well defined. This map is clearly linear, positive, bounded (by 1), and equivariant. The conditional expectation property is immediate from the fact that σ is tracial. We prove that P is faithful. Suppose $s \in \Gamma(E)$ and $P(s^*s) = 0$. Let $x \in X$. Since σ is faithful and $\sigma(\varphi_x(s^*)\varphi_x(s)) = P(s^*s)(x) = 0$, we have $\varphi_x(s) = 0$. Since this is true for all $x \in X$, we have $s = 0$, as desired.

It remains to prove the last sentence. For any C*-algebra B, we denote by $\mathrm{T}(B)$ the tracial state space of B. We further identify the space of Borel probability measures on X with $\mathrm{T}(C(X))$ in the standard way, sending a measure μ to the tracial state $f \mapsto \int_X f \, d\mu$. Then the formula for τ_μ in the statement of the lemma becomes $\tau_\mu = \mu \circ P$. We regard this as a map from $\mathrm{T}(C(X))$ to $\mathrm{T}(\Gamma(E))$. This map is obviously affine and weak* to weak* continuous. It is equivariant for the obvious actions of G on these spaces. Since $\mathrm{T}(C(X))$ and $\mathrm{T}(\Gamma(E))$ are compact, it therefore suffices to prove that $\mu \mapsto \mu \circ P$ is bijective (ignoring the G-invariance restriction).

The fact that A is unital gives a unital homomorphism $\iota \colon C(X) \to \Gamma(E)$. Clearly $P \circ \iota = \mathrm{id}_{C(X)}$. Therefore $\tau_\mu \circ \iota = \mu$ for all $\mu \in \mathrm{T}(C(X))$. So $\mu \mapsto \mu \circ P$ is injective.

We claim that every extreme point of $\mathrm{T}(\Gamma(E))$ is of the form $\sigma \circ \varphi_x$ for some $x \in X$. This claim implies that all extreme points of $\mathrm{T}(\Gamma(E))$ are in the range of $\mu \mapsto \mu \circ P$. So the Krein–Milman Theorem implies that this map is surjective, finishing the proof.

So let $\tau \in \mathrm{T}(\Gamma(E))$. Set

$$J = \bigl\{ b \in \Gamma(E) \colon \tau(b^*b) = 0 \bigr\},$$

which is a closed ideal in $\Gamma(E)$. Using local triviality of E and simplicity of A, it is not hard to show that there is a compact set $L \subset X$ such that

$$J = \bigl\{ b \in \Gamma(E) \colon \varphi_x(b) = 0 \text{ for all } x \in L \bigr\}.$$

If $L = \varnothing$ then $J = \Gamma(E)$, which is clearly impossible. If there is $x \in X$ such that $L = \{x\}$, then $J = \mathrm{Ker}(\varphi_x)$. It follows that τ factors through $\Gamma(E)/\mathrm{Ker}(\varphi_x) \cong A$. By uniqueness of the tracial state on A, we must have $\tau = \sigma \circ \varphi_x$.

Finally, assume that there exist distinct $x, y \in L$. Choose disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$. Choose $f, g \in C(X)$ such that

$$0 \leq g \leq f \leq 1, \quad fg = g, \quad g(x) = 1, \quad \text{and} \quad \operatorname{supp}(f) \subset U.$$

Let $\iota\colon C(X) \to \Gamma(E)$ be as above. Then $\iota(g^{1/2}) \notin J$ because $(\varphi_x \circ \iota)(g) = 1$ and $\iota((1-f)^{1/2}) \notin J$ because $(\varphi_y \circ \iota)(1-f) = 1$. So $\tau(\iota(g)) > 0$ and $\tau(\iota(1-f)) > 0$. Also $\tau(\iota(f)) > 0$ because $g \leq f$. Therefore there are tracial states τ_1, τ_2 on $\Gamma(E)$ such that

$$\tau_1(b) = \frac{\tau(\iota(f)b)}{\tau(\iota(f))} \quad \text{and} \quad \tau_2(b) = \frac{\tau(\iota(1-f)b)}{\tau(\iota(1-f))}$$

for all $b \in \Gamma(E)$. We have $\tau_1(g) \neq 0$ since $fg = g$ and $\tau(\iota(g)) \neq 0$, but $\tau_2(\iota(g)) = 0$ since $(1-f)g = 0$. Therefore $\tau_1 \neq \tau_2$. Since

$$\tau(\iota(f))\tau_1 + \tau(\iota(1-f))\tau_2 = \tau$$

and the coefficients are strictly positive, we have shown that τ is not an extreme point in $\mathrm{T}(\Gamma(E))$. □

Corollary 2.6. *In the situation of Lemma 2.5, if G is amenable, the action of G on X is mimimal, and A is simple, then there exists a faithful G-invariant tracial state on $\Gamma(E)$.*

Proof. Since G is amenable and X is compact, there is a G-invariant Borel probability measure μ on X. Since the action is minimal, μ must have full support, so that, if $f \in C(X)_+$ and $\int_X f\, d\mu = 0$, then $f = 0$. Now let $P\colon \Gamma(E) \to C(X)$ be as in Lemma 2.5, and let τ_μ be the G-invariant tracial state on $\Gamma(E)$ obtained from μ, given by $\tau_\mu(s) = \int_X P(s)\, d\mu$ for $s \in \Gamma(E)$. We show that τ_μ is faithful. Let $s \in \Gamma(E)$, and suppose that $\tau_\mu(s^*s) = 0$. Then $P(s^*s) = 0$ since μ has full support. So $s = 0$ since P is faithful. □

The next lemma shows that for crossed products by automorphisms with finite Rokhlin dimension, any tracial state on the crossed product arises from an invariant tracial state on the original algebra. We refer to [13] for a discussion of finite Rokhlin dimension in the nonunital setting. (See Definition 1.21 of [13].)

Lemma 2.7. *Let A be a separable C*-algebra, and let $\alpha \in \operatorname{Aut}(A)$ be an automorphism with finite Rokhlin dimension. Let $P\colon A \rtimes_\alpha \mathbb{Z} \to A$ be the canonical conditional expectation. Then for any tracial state τ on $A \rtimes_\alpha \mathbb{Z}$ there is an α-invariant tracial state ρ on A such that $\tau = \rho \circ P$.*

Proof. Let $d \in \mathbb{Z}_{\geq 0}$ be the Rokhlin dimension of α. Apply the proof of [14, Proposition 2.8] and [14, Remark 2.9] to Definition 1.21 of [13], to get the following single tower version of Rokhlin dimension, in which d is replaced by $2d+1$. For any finite set $F \subset A$, any $p > 0$, and any $\varepsilon > 0$, there are positive contractions

$$f_0^{(l)},\ f_1^{(l)},\ \ldots,\ f_{p-1}^{(l)} \in A$$

for $l = 0, 1, \ldots, 2d+1$, such that:

(1) $\| f_j^{(l)} f_k^{(l)} b \| < \varepsilon$ for $l = 0, 1, \ldots, 2d+1$, $j, k = 0, 1, \ldots, p-1$ with $j \neq k$, and all $b \in F$.

(2) $\left\| \left(\sum_{l=0}^{2d+1} \sum_{j=0}^{p-1} f_j^{(l)} \right) b - b \right\| < \varepsilon$ for all $b \in F$.

(3) $\| [f_j^{(l)}, b] \| < \varepsilon$ for $l = 0, 1, \ldots, 2d+1$, $j = 0, 1, \ldots, p-1$, and all $b \in F$.

(4) $\| (\alpha (f_j^{(l)}) - f_{j+1}^{(l)}) b \| < \varepsilon$ for $l = 0, 1, \ldots, 2d+1$, $j = 0, 1, \ldots, p-2$, and all $b \in F$.

(5) $\| (\alpha (f_{p-1}^{(l)}) - f_0^{(l)}) b \| < \varepsilon$ for $l = 0, 1, \ldots, 2d+1$ and all $b \in F$.

By projectivity of cones (as in the argument of Remark 1.18 of [13]), we can replace (1) by the stronger condition:

(6) $f_k^{(l)} f_j^{(l)} = 0$ for $l = 0, 1, \ldots, 2d+1$ and $j, k = 0, 1, \ldots, p-1$ with $j \neq k$.

Let u be the canonical unitary in $M(A \rtimes_\alpha \mathbb{Z})$. Since A contains an approximate identity for $A \rtimes_\alpha \mathbb{Z}$, the restriction $\tau|_A$ has norm 1. Therefore $\tau|_A$ is an α-invariant tracial state. Thus, it suffices to show that $\tau(a u^n) = 0$ for all $a \in A$ and for all $n \in \mathbb{Z} \setminus \{0\}$. We may assume that $\|a\| \leq 1$. Since $a u^{-n} = (u^n a^*)^* = [\alpha^n(a^*) u^n]^*$, it suffices to treat the case $n > 0$. Fix $\varepsilon > 0$; we prove that $|\tau(a u^n)| < \varepsilon$.

Define

$$\varepsilon_0 = \frac{\varepsilon}{(2d+1)(n+1)+1}.$$

Then $\varepsilon_0 > 0$. An argument using polynomial approximations to the function $\lambda \mapsto \lambda^{1/2}$ on $[0, \infty)$ provides $\delta > 0$ such that $\delta \leq \varepsilon_0$ and whenever C is a C*-algebra and $b, c, x \in C$ satisfy

$$\|b\| \leq 1, \quad \|c\| \leq 1, \quad \|x\| \leq 1, \quad b \geq 0, \quad c \geq 0, \quad \text{and} \quad \|bx - xc\| < \delta,$$

then $\|b^{1/2} x - x c^{1/2}\| < \varepsilon_0$. Set $\delta_0 = \delta/(n+1)$.

Apply the single tower property above with (6) in place of (1), with $p = n+1$, with

$$F = \{ a,\ a^*,\ \alpha^{-1}(a^*),\ \alpha^{-2}(a^*), \ldots,\ \alpha^{-(n-1)}(a^*) \},$$

and with δ_0 in place of ε, getting positive contractions $f_0^{(l)}, f_1^{(l)}, \ldots, f_n^{(l)} \in A$ as above. In particular, whenever $j \neq k$ we have $f_j^{(l)} f_k^{(l)} = 0$, so $(f_j^{(l)})^{1/2} (f_k^{(l)})^{1/2} = 0$.

In the following estimates, we interpret all subscripts in expressions $f_k^{(l)}$ as elements of $\{0, 1, \ldots, n\}$ by reduction modulo $n+1$. For $l = 0, 1, \ldots, 2d+1$, for $k = 0, 1, \ldots, p-1$, and for $b \in A$, we have

$$\left\| \left(\alpha^n \left(f_{k-n}^{(l)} \right) - f_k^{(l)} \right) b \right\| \leq \sum_{m=1}^{n} \left\| \alpha^{n-m} \left(\left(\alpha \left(f_{k-n+m-1}^{(l)} \right) - f_{k-n+m}^{(l)} \right) \alpha^{m-n}(b) \right) \right\|.$$

Putting $b = a^*$, and using (4), (5), and the definition of F, it follows that

$$\left\| \left(\alpha^n \left(f_{k-n}^{(l)} \right) - f_k^{(l)} \right) a^* \right\| < n\delta_0.$$

Using $u^n b u^{-n} = \alpha^n(b)$ for $b \in A$ and taking adjoints, we get

$$\left\| a \left(f_k^{(l)} - u^n f_{k-n}^{(l)} u^{-n} \right) \right\| < n\delta_0.$$

Therefore, also using (3),

$$\begin{aligned} &\left\| f_k^{(l)} a u^n - a u^n f_{k-n}^{(l)} \right\| \\ &\qquad \leq \left\| f_k^{(l)} a - a f_k^{(l)} \right\| \|u^n\| + \left\| a \left(f_k^{(l)} - u^n f_{k-n}^{(l)} u^{-n} \right) \right\| \|u^n\| \\ &\qquad < \delta_0 + n\delta_0 = \delta. \end{aligned}$$

Using the choice of δ and $\left\| f_k^{(l)} \right\| \leq 1$ at the second step, we now get

$$\begin{aligned} &\left\| f_k^{(l)} a u^n - \left(f_k^{(l)} \right)^{1/2} a u^n \left(f_k^{(l)} \right)^{1/2} \right\| \\ &\qquad \leq \left\| \left(f_k^{(l)} \right)^{1/2} \right\| \cdot \left\| \left(f_k^{(l)} \right)^{1/2} a u^n - a u^n \left(f_{k-n}^{(l)} \right)^{1/2} \right\| < \varepsilon_0. \end{aligned}$$

Using the trace property at the second step and, at the third step, the fact that k and $k-n$ are not equal modulo $n+1$, we now get

$$\begin{aligned} \left| \tau \left(f_k^{(l)} a u^n \right) \right| &< \left| \tau \left(\left(f_k^{(l)} \right)^{1/2} a u^n \left(f_{k-n}^{(l)} \right)^{1/2} \right) \right| + \varepsilon_0 \\ &= \left| \tau \left(\left(f_{k-n}^{(l)} \right)^{1/2} \left(f_k^{(l)} \right)^{1/2} a u^n \right) \right| + \varepsilon_0 = \varepsilon_0. \end{aligned}$$

Using (2) and $\delta_0 \leq \varepsilon_0$, we therefore get

$$|\tau(a u^n)| < \left| \tau \left(\sum_{l=0}^{2d+1} \sum_{j=0}^{n} f_j^{(l)} a u^n \right) \right| + \delta_0 < (2d+1)(n+1)\varepsilon_0 + \varepsilon_0 = \varepsilon.$$

This completes the proof. □

3. Constructing the examples

In this section, we construct our examples.

Theorem 3.1. *There exist a simple unital separable AH-algebra A with a unique tracial state and satisfying $A \cong A \otimes M_{2^\infty}$, and a continuous action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$, with the following properties:*

(1) *The fixed point subalgebra A^γ is not isomorphic to its opposite.*

(2) *The crossed product $A \rtimes_\gamma \mathbb{T}$ is not isomorphic to its opposite.*

(3) *The C*-algebra A is not $\mathbb{T}$-equivariantly isomorphic to its opposite.*

Theorem 3.2. *There exist a unital Kirchberg algebra B satisfying the Universal Coefficient Theorem, and a continuous action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B)$, with the following properties:*

(1) *The fixed point subalgebra B^γ is not isomorphic to its opposite.*

(2) *The crossed product $B \rtimes_\gamma \mathbb{T}$ is not isomorphic to its opposite.*

(3) *The C*-algebra B is not $\mathbb{T}$-equivariantly isomorphic to its opposite.*

The proofs of Theorem 3.1 and Theorem 3.2 are the same until nearly the end, so we prove them together.

Proofs of Theorem 3.1 and Theorem 3.2. We start with the compact connected four dimensional manifold M used in [21, Example 3.5], whose fundamental group can be identified with the group G of Lemma 2.3 and whose signature is nonzero. It follows from [12, Theorem 2A.1] that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$, generated by the image of a under the map $\pi_1(M) \to H_1(M;\mathbb{Z})$, so Poincaré duality ([12, Theorem 3.30]) gives $H^3(M;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$.

Let $\eta \in H^3(M;\mathbb{Z})$ be a generator. We claim that if $h\colon M \to M$ is a continuous map such that the induced maps $h_*\colon H_*(M;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ and $h_*\colon \pi_1(M) \to \pi_1(M)$ are isomorphisms, then $h^*(\eta) = \eta$.

To prove the claim, note first that the Universal Coefficient Theorem (as in the proof of Lemma 2.1) and the Five Lemma imply that $h^*\colon H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is an isomorphism as well. By Lemma 2.1, h is orientation preserving. Since $h_*\colon \pi_1(M) \to \pi_1(M)$ is an automorphism, and since, by Lemma 2.3, there is no automorphism of $\pi_1(M)$ which sends a to a^{-1}, it follows by naturality of the Hurewicz map that $h_*\colon H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is the identity. Since h_* fixes the orientation class, it follows by naturality in Poincaré duality that h^* is also the identity on $H^3(M;\mathbb{Z})$, as required.

To proceed, we would have liked to have a minimal homeomorphism of M. In Remark 4.1 below, we explain why no such homeomorphism exists. We remedy this situation by giving ourselves more space, as follows.

Choose an odd integer $n \geq 5$. Let $\eta_0 \in H^3(S^n \times M;\, \mathbb{Z})$ be the product of the standard generator of $H^0(S^n;\mathbb{Z})$ and η. Let $h\colon S^n \times M \to S^n \times M$ be a continuous function such that the induced maps

$$h_*\colon H_*(S^n \times M;\, \mathbb{Z}) \to H_*(S^n \times M;\, \mathbb{Z})$$

and

$$h_*\colon \pi_1(S^n \times M) \to \pi_1(S^n \times M)$$

are isomorphisms. Applying Lemma 2.2 and following the notation there, the induced maps

$$(p \circ h \circ i)_*\colon H_*(M;\mathbb{Z}) \to H_*(M;\mathbb{Z}) \quad \text{and} \quad (p \circ h \circ i)_*\colon \pi_1(M) \to \pi_1(M)$$

are isomorphisms. By the claim above, $(p \circ h \circ i)^*(\eta) = \eta$. It follows from the definitions of the maps and of η_0 that $h^*(\eta_0) = \eta_0$.

By [11, Corollary 1.7], there exist $N \in \mathbb{Z}_{>0}$ and a locally trivial continuous field E over $S^n \times M$ with fiber M_N whose section algebra $\Gamma(E)$ has Dixmier–Douady invariant η_0. We identify $\operatorname{Prim}(\Gamma(E))$ with $S^n \times M$ in the obvious way. Recall that, by [25, Theorem 6.3], $\Gamma(E)^{\mathrm{op}}$ has Dixmier–Douady invariant $-\eta_0$. Thus, an isomorphism from $\Gamma(E) \otimes K$ to $\Gamma(E)^{\mathrm{op}} \otimes K$ would induce a homeomorphism from $\operatorname{Prim}(\Gamma(E))$ to itself whose induced action on $H^3(S^n \times M)$ sends η_0 to $-\eta_0$. Since no such homeomorphism exists, it follows that $\Gamma(E)$ is not stably isomorphic to its opposite algebra.

Since S^n admits a free action of $\mathbb{T}$, so does $S^n \times M$. By [9, Theorem 1 and Theorem 4], there exists a uniquely ergodic minimal diffeomorphism $h\colon S^n \times M \to S^n \times M$ which is homotopic to the identity. Thus $h^*(E) \cong E$. Therefore there exists an automorphism $\alpha\colon \Gamma(E) \to \Gamma(E)$ which induces h on $\operatorname{Prim}(\Gamma(E))$. Set $A_0 = \Gamma(E) \rtimes_\alpha \mathbb{Z}$. Then A_0 is a separable unital nuclear C*-algebra satisfying the Universal Coefficient Theorem. The algebra A_0 is simple and quasidiagonal by Lemma 2.4.

We claim that A_0 has finite nuclear dimension and a unique tracial state. To prove the claim, use [17, Corollary 3.10] to see that the decomposition rank of $\Gamma(E)$ is $\dim(S^n \times M)$, hence finite. Therefore $\Gamma(E)$ has finite nuclear dimension. Since the center of $\Gamma(E)$ is isomorphic to $C(S^n \times M)$, it follows that $\alpha|_{Z(\Gamma(E))}$ is an automorphism of $C(S^n \times M)$ arising from a minimal homeomorphism. Therefore, by [14, Theorem 6.1] or [28, Corollary 2.6], $\alpha|_{Z(\Gamma(E))}$ has finite Rokhlin dimension. It follows immediately from the definition of finite Rokhlin dimension that if α is an automorphism of a C*-algebra C and the restriction of α to the center of C has finite Rokhlin dimension, then α has finite Rokhlin dimension as well (with commuting towers). Therefore A_0 has finite nuclear dimension by [14, Theorem 4.1]. Since h is uniquely ergodic, $\Gamma(E)$ admits a unique invariant tracial state by Lemma 2.5. By Lemma 2.7, A_0 has a unique tracial state.

The fixed point subalgebra of the dual action γ of $\mathbb{T}$ on A_0 is isomorphic to $\Gamma(E)$. Therefore it is not isomorphic to its opposite algebra. By Takai duality, the crossed product of A_0 by the dual action is stably isomorphic to $\Gamma(E)$, and therefore also not isomorphic to its opposite. In general, if two C*-algebras with a G-action are equivariantly isomorphic, it follows immediately that their fixed point subalgebras are isomorphic. Thus, $(\mathbb{T}, A_0, \gamma)$ is not equivariantly isomorphic to its opposite $(\mathbb{T}, A_0^{\mathrm{op}}, \gamma^{\mathrm{op}})$.

The remainder of the proof consists of showing that the same properties remain after we tensor everything with $\mathcal{O}_\infty$ (for Theorem 3.2) or with M_{2^∞} (for Theorem 3.1).

We use the following property of locally trivial fields whose fibers are isomorphic to a fixed C*-algebra D. We claim that two such fields E and F are isomorphic as continuous fields over X if and only if their section C*-algebras $\Gamma(E)$ and $\Gamma(F)$ are isomorphic via a $C(X)$-linear isomorphism. The proof of this claim is an elementary exercise in bundle theory. The case when D is isomorphic to the compact operators is contained in the proof of Proposition 5.59 of [25]. The same arguments apply for an arbitrary C*-algebra D. Indeed, suppose that $(U_i)_{i\in I}$ is a finite cover of X by closed sets such that E and F are obtained by gluing trivial fields $U_i \times D$ over nonempty intersections $U_i \cap U_j$ using cocycles $\gamma_{i,j}^E, \gamma_{i,j}^F : U_i \cap U_j \to \mathrm{Aut}(D)$. Then any isomorphism $\gamma : \Gamma(E) \to \Gamma(F)$ induces $C(U_i)$-linear isomorphisms $\eta_i : C(U_i, D) \to C(U_i, D)$ which we may identify with continuous maps $\eta_i : U_i \to \mathrm{Aut}(D)$. Being induced by a global map γ, the family $(\eta_i)_{i\in I}$ must satisfy the compatibility conditions $\eta_i(x)\gamma_{i,j}^E(x) = \gamma_{i,j}^F(x)\eta_j(x)$ for all $i, j \in I$ and $x \in U_i \cap U_j$. It follows that the family of maps $U_i \times D \to U_i \times D$, given by $(x, d) \mapsto (x, \eta_i(x)d)$ for $x \in U_i$ and $d \in D$, defines an isomorphism of continuous fields from E to F. This completes the proof of the claim.

For any continuous field F over X and any nuclear C*-algebra D, denote by $F \otimes D$ the continuous field whose fiber over $x \in X$ is $F_x \otimes D$. (This is in fact a continuous field by [16, Theorem 4.5].) Suppose that F_1 and F_2 are two continuous fields over X with fibers M_N, and that $\Gamma(F_1 \otimes \mathcal{O}_\infty \otimes K) \cong \Gamma(F_2 \otimes \mathcal{O}_\infty \otimes K)$. Since the fibers of these fields are simple, it follows that there is a homeomorphism $g : X \to X$ such that $g^*(F_2 \otimes \mathcal{O}_\infty \otimes K) \cong F_1 \otimes \mathcal{O}_\infty \otimes K$. Apply [4, Corollary 4.9], noting that $\mathbb{C}$ is included among the strongly selfabsorbing C*-algebras there. (See the beginning of [4, Section 2.1].) We conclude that $g^*(F_2) \otimes K \cong F_1 \otimes K$, so $\Gamma(F_2) \otimes K \cong \Gamma(F_1) \otimes K$. Taking $F_1 = E$ and $F_2 = E^\#$ (the fiberwise conjugate field, with fibers $(E^\#)_x = (E_x)^\#$), the fact that $\Gamma(E)$ is not stably isomorphic to its opposite algebra now gives the second step of the following calculation, while $(\mathcal{O}_\infty \otimes K)^\# \cong \mathcal{O}_\infty \otimes K$ gives the third step:

$$\begin{aligned}\Gamma(E) \otimes \mathcal{O}_\infty \otimes K &\cong \Gamma(E \otimes \mathcal{O}_\infty \otimes K)\\ &\ncong \Gamma(E^\# \otimes \mathcal{O}_\infty \otimes K)\\ &\cong \Gamma\big(E^\# \otimes (\mathcal{O}_\infty \otimes K)^\#\big) \cong \big(\Gamma(E) \otimes \mathcal{O}_\infty \otimes K\big)^\#.\end{aligned} \tag{3.1}$$

Set $B = A_0 \otimes \mathcal{O}_\infty$ and let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B)$ be the tensor product of the dual action on A_0 and the trivial action on $\mathcal{O}_\infty$. Then

$$B \rtimes_\gamma \mathbb{T} \cong \Gamma(E) \otimes \mathcal{O}_\infty \otimes K, \quad B^\# \rtimes_{\gamma^\#} \mathbb{T} \cong \big(\Gamma(E) \otimes \mathcal{O}_\infty \otimes K\big)^\#,$$
$$B^\gamma \cong \Gamma(E) \otimes \mathcal{O}_\infty, \quad \text{and} \quad (B^\#)^{\gamma^\#} \cong \big(\Gamma(E) \otimes \mathcal{O}_\infty\big)^\#.$$

So parts (1) and (2) of Theorem 3.2 follow from (3.1) and Lemma 1.1. Part (3) is now immediate. Since B is a unital Kirchberg algebra which satisfies the Universal Coefficient Theorem, we have proved Theorem 3.2.

Next we are going to show that if we tensor E fiberwise with the CAR algebra M_{2^∞}, the section algebra will still fail to be stably isomorphic to its opposite algebra. Set $D = M_{2^\infty}$ and let $\bar{E}^*_D(X)$ be the (reduced) cohomology theory which arises as in [4, Corollary 3.9] from the infinite loop structure of the classifying space of $\mathrm{Aut}_0(D \otimes K)$, the component of the identity of the automorphism group of $D \otimes K$. Thus, by general theory of fiber bundles, locally trivial bundles with fiber $D \otimes K$ and structure group $\mathrm{Aut}_0(D \otimes K)$ over a finite connected CW complex X are classified by the set $[X, B\mathrm{Aut}_0(D \otimes K)]$ of homotopy classes and hence by the group $\bar{E}^1_D(X)$.

The unital map $\mathbb{C} \to D$ induces a morphism $\mathrm{Aut}_0(K) \to \mathrm{Aut}_0(D \otimes K)$ and a natural transformation of cohomology theories $T\colon \bar{E}^*_{\mathbb{C}}(X) \to \bar{E}^*_D(X)$. Let

$$t\colon H^3(X;\mathbb{Z}) \to H^3\big(X;\, \mathbb{Z}[1/2]\big)$$

be the coefficient map induced by

$$\mathbb{Z} \xrightarrow{\cong} \pi_2\big(\mathrm{Aut}_0(K)\big) \longrightarrow \pi_2\big(\mathrm{Aut}_0(D \otimes K)\big) \xrightarrow{\cong} K_0(D) \xrightarrow{\cong} \mathbb{Z}[1/2].$$

Using naturality of the Atiyah–Hirzebruch spectral sequence (see after Corollary 4.3 of [4] for the description of the E_2-pages of the spectral sequences for $\bar{E}^*_{\mathbb{C}}(X)$ and $\bar{E}^*_D(X)$) one obtains a commutative diagram

$$\begin{array}{ccc} \bar{E}^1_{\mathbb{C}}(X) & \xrightarrow{\ T\ } & \bar{E}^1_D(X) \\ \Big\downarrow{\scriptstyle \bar{\delta}_1} & & \Big\downarrow{\scriptstyle \bar{\delta}_1} \\ H^3(X;\mathbb{Z}) & \xrightarrow{\ t\ } & H^3\big(X;\, \mathbb{Z}[1/2]\big) \end{array}$$

in which the vertical maps are the edge homomorphisms. All the rows of the E_2-page of the spectral sequence for $E^*_{\mathbb{C}}(X)$ are null with the exception of the (-2)-row whose entries are $H^p(X;\mathbb{Z})$ for $p \geq 0$. Thus the first vertical map is an isomorphism and it can be identified with the Dixmier–Douady map since $\bar{E}^1_{\mathbb{C}}(X) \cong H^3(X;\mathbb{Z})$. In particular, T is injective whenever t is injective.

In the case of $X = S^n \times M$ with M and n as above, $H^3(X;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$. Since X is a compact manifold, its integral cohomology is finitely generated. It therefore follows from the cohomology Universal Coefficient Theorem given for chain complexes in Theorem 10 in Section 5 of Chapter 5 of [27] that t is bijective. Hence the map $T\colon \overline{E}^1_{\mathbb{C}}(X) \to \overline{E}^1_D(X)$ is injective.

Now suppose that F_1 and F_2 are two continuous fields over X with fibers M_N, and that $\Gamma(F_1 \otimes D \otimes K) \cong \Gamma(F_2 \otimes D \otimes K)$. As in the argument above for the case $D = \mathcal{O}_\infty$, there is a homeomorphism $h\colon X \to X$ such that

$$h^*(F_2 \otimes D \otimes K) \cong F_1 \otimes D \otimes K.$$

Since $\overline{E}^1_{\mathbb{C}}(X) \to \overline{E}^1_D(X)$ is injective, it follows that

$$h^*(F_2 \otimes K) \cong F_1 \otimes K.$$

Taking $F_1 = E$ and $F_2 = E^{\#}$ as before, we deduce as before that

$$\Gamma(E) \otimes D \otimes K \not\cong \big(\Gamma(E) \otimes D \otimes K\big)^{\#}.$$

Now define $A = A_0 \otimes D$ and let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ be the tensor product of the dual action on A_0 and the trivial action on D. Proceed as before to deduce parts (1), (2), and (3) of Theorem 3.1.

Since $A = A \otimes M_{2^\infty}$, and since A is quasidiagonal by Lemma 2.4, it follows from [20, Theorem 6.1] that A is tracially AF and from [20, Corollary 6.2] that A isomorphic to an AH-algebra with real rank zero and no dimension growth. (Corollary 6.2 of [20] provides conditions which allow the use of classification results from [18] and [30].) This concludes the proof of Theorem 3.1. □

Corollary 3.3. *There exist a simple unital separable AH-algebra A with a unique tracial state and satisfying $A \cong A \otimes M_{2^\infty}$, and an automorphism $\alpha \in \mathrm{Aut}(A)$ such that α^n is outer for all $n \neq 0$, with the following properties:*

(1) *The fixed point subalgebra A^α is not isomorphic to its opposite.*

(2) *The C*-algebra A is not $\mathbb{Z}$-equivariantly isomorphic to its opposite.*

Corollary 3.4. *There exist a unital Kirchberg algebra B satisfying the Universal Coefficient Theorem and an automorphism $\alpha \in \mathrm{Aut}(B)$ such that α^n is outer for all $n \neq 0$, with the following properties:*

(1) *The fixed point subalgebra B^α is not isomorphic to its opposite.*

(2) *The C*-algebra B is not $\mathbb{Z}$-equivariantly isomorphic to its opposite.*

Proofs of Corollary 3.3 and Corollary 3.4. The proofs of both corollaries are the same. Let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ or $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B)$ be the circle action from Theorem 3.1 or Theorem 3.2 as appropriate. Let $\zeta \in \mathbb{T}$ be an irrational angle, so that $\mathbb{Z} \cdot \zeta$ is dense in $\mathbb{T}$. Set $\alpha = \gamma_\zeta$. Then $A^\gamma = A^\alpha$ or $B^\gamma = B^\alpha$. If α is chosen suitably, then α^n will be outer for all $n \neq 0$. Such choices exist by Lemma 3.5 below. □

In these corollaries, we do not claim that the crossed products are not isomorphic. In particular, for the actions used in the proof of Corollary 3.4, we will show that the crossed products actually are at least sometimes isomorphic; probably this is true in general. We need a lemma and a corollary, which we state in greater generality than we need.

Lemma 3.5. *Let A be a separable unital C*-algebra. Let $\alpha \in \mathrm{Aut}(A)$. Suppose A has a faithful invariant tracial state τ. Let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_\alpha \mathbb{Z})$ be the dual action on the crossed product. Then for all but countably many $\lambda \in \mathbb{T}$, the automorphism γ_λ is outer.*

Proof. Let $\pi\colon A \to L(H)$ be the Gelfand-Naimark-Segal representation associated with τ, and let $\xi \in H$ be the associated cyclic vector. Using the α-invariance of τ, we find that

$$\langle \pi(\alpha(a))\xi,\, \pi(\alpha(b))\xi \rangle = \langle \pi(a)\xi,\, \pi(b)\xi \rangle$$

for all $a, b \in A$, from which it follows that there is a unique isometry $s \in L(H)$ such that $s\pi(a)\xi = \pi(\alpha(a))\xi$ for all $a \in A$. Applying the same argument with α^{-1} in place of α, we find that s is unitary.

Let $\lambda \in \mathbb{T}$. We claim that if γ_λ is inner, then $\overline{\lambda}$ is an eigenvalue of s. Since H is separable, s has at most countably many eigenvalues, and the lemma will follow.

To prove the claim, suppose there is $v \in A \rtimes_\alpha \mathbb{Z}$ such that $\gamma_\lambda(b) = vbv^*$ for all $b \in A \rtimes_\alpha \mathbb{Z}$. Let $Q\colon A \rtimes_\alpha \mathbb{Z} \to A$ be the standard conditional expectation. Let $u \in A \rtimes_\alpha \mathbb{Z}$ be the canonical unitary of the crossed product, so that $uau^* = \alpha(a)$ for all $a \in A$. Then

$$u^*vu = u^*(vuv^*)v = u^*(\lambda u)v = \lambda v. \tag{3.2}$$

For $n \in \mathbb{Z}$ let $a_n = Q(vu^{-n}) \in A$ be the nth coefficient of v in the crossed product, so that (see [6, Theorem 8.2.2]) v is given by the limit of the Cesàro means:

$$v = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) a_n u^n.$$

Applying (3.2) and $u^*a_nu = \alpha^{-1}(a_n)$, we get

$$\lambda v = u^*vu = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \alpha^{-1}(a_n) u^n.$$

It follows that for all $n \in \mathbb{Z}$ we have

$$\lambda a_n = Q(\lambda v u^{-n}) = \alpha^{-1}(a_n).$$

Choose $n \in \mathbb{Z}$ such that $a_n \neq 0$. We have $\langle \pi(a_n)\xi, \pi(a_n)\xi \rangle = \tau(a_n^* a_n)$, which is nonzero because τ is faithful. Therefore $\pi(a_n)\xi \in H$ is nonzero and satisfies

$$s^*\pi(a_n)\xi = \pi(\alpha^{-1}(a_n))\xi = \lambda\pi(a_n)\xi. \qquad \square$$

Corollary 3.6. *Under the hypotheses of Lemma 3.5, for all but countably many $\lambda \in \mathbb{T}$, the automorphism γ_λ has the property that γ_λ^n is outer for all $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. This is immediate from Lemma 3.5. □

Corollary 3.6 applies to our setting as follows. Let X be a compact metric space, let $n \in \mathbb{Z}_{>0}$, let E be a locally trivial bundle over X with fiber M_n, let $h: X \to X$ be a minimal homeomorphism, and let $\alpha \in \mathrm{Aut}(\Gamma(E))$ be an automorphism which induces the map h on $\mathrm{Prim}(A)$. Set $A = \Gamma(E)$ and $B = A \rtimes_\alpha \mathbb{Z}$. Corollary 2.6 provides a faithful α-invariant tracial state τ on A. It follows from Lemma 3.5 that for all but countably many choices of $\zeta \in \mathbb{T}$ in the proof of Corollary 3.4, the automorphism γ_ζ used there is outer.

We use [29, Theorem 1] to see that the tensor product of any automorphism with an outer automorphism is outer. If B is a unital Kirchberg algebra satisfying the Universal Coefficient Theorem, and α^n is outer for all $n \in \mathbb{Z} \setminus \{0\}$, then the crossed product $B \rtimes_\alpha \mathbb{Z}$ is also a Kirchberg algebra. (Pure infiniteness follows from [15, Corollary 4.4].) By the Five Lemma, $B \rtimes_\alpha \mathbb{Z}$ and $B^{\mathrm{op}} \rtimes_\alpha \mathbb{Z}$ have the same K-theory, so they are isomorphic.

It seems very likely that suitable generalizations of Theorem 12 in Section V of [7] and Theorem 11 in Section VI of [7] will show that, in the proof of Corollary 3.4, the automorphism γ_ζ is outer for all $\zeta \notin \exp(2\pi i \mathbb{Q})$. The results of [7] are stated for automorphisms of $C(X)$ for connected compact spaces X, and one would need to generalize them to automorphisms of section algebras of locally trivial M_n-bundles over such spaces.

4. Remarks and questions

We collect here several remarks: we show that the manifold M used in the proofs above does not itself admit any minimal homeomorphisms, and we describe a shorter construction of examples, with the disadvantages that it does not give unital algebras and that we don't have proofs of some of the extra properties of the examples. We finish with several open questions.

Remark 4.1. We explain here why the manifold M we started with in the proofs of Theorem 3.1 and Theorem 3.2 does not admit minimal homeomorphisms. The Euler characteristic of M is at least 2, because $H_1(M; \mathbb{Z})$ and $H_3(M; \mathbb{Z})$ are torsion groups, so this fact follows from Theorem 3 of [10]. We provide here a different argument, which might be of independent interest.

We first make the following purely algebraic claim: if $a \in M_n(\mathbb{R})$, then there exists $k \in \{1, 2, \ldots, n+1\}$ such that $\mathrm{Tr}(a^k) \geq 0$. We are indebted to Ilya Tyomkin for providing us with the argument. Assume for contradiction that $\mathrm{Tr}(a^k) < 0$ for $k = 1, 2, \ldots, n+1$. Define polynomials $e_m(t_1, t_2, \ldots, t_n)$ and $p_m(t_1, t_2, \ldots, t_n)$

of n variables $t_1, t_2, \ldots, t_n$ as follows. For $m = 0, 1, \ldots, n$, take e_m be the mth elementary symmetric function ([19, p. 19]; the formulas in [19] are actually written in terms of formal infinite linear combinations of monomials in infinitely many variables, and we use the result of setting $t_{n+1} = t_{n+2} = \cdots = 0$). For $m = 1, 2, \ldots, n$, set

$$p_m(t_1, t_2, \ldots, t_n) = \sum_{k=1}^{n} t_k^m$$

([19, p. 23]). Newton's formula (equation (2.11′) on page 23 of [19]) states that

$$me_m(t_1, t_2, \ldots, t_n) = \sum_{r=1}^{m} (-1)^{r-1} p_r(t_1, t_2, \ldots, t_n) e_{m-r}(t_1, t_2, \ldots, t_n) \tag{4.1}$$

for $m = 1, 2, \ldots, n$.

Now let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a, counting multiplicity. Then $\mathrm{Tr}(a^k) = p_k(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the characteristic polynomial of a is

$$q(x) = \sum_{k=0}^{n} (-1)^k e_k(\lambda_1, \lambda_2, \ldots, \lambda_n) x^{n-k}.$$

Our assumption implies that $p_k(\lambda_1, \lambda_2, \ldots, \lambda_n) < 0$ for $m = 1, 2, \ldots, n$. An induction argument using (4.1) shows that $(-1)^k e_k(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0$ for $k = 0, 1, \ldots, n$. Therefore

$$\mathrm{Tr}(aq(a)) = \sum_{k=0}^{n} (-1)^k e_k(\lambda_1, \lambda_2, \ldots, \lambda_n) \mathrm{Tr}(a^{n-k+1}) < 0.$$

But the Cayley–Hamilton theorem implies that $q(a) = 0$, so $\mathrm{Tr}(aq(a)) = 0$, a contradiction. This proves the claim.

Now let $h \colon M \to M$ be a homeomorphism. We claim that h has a periodic point, and therefore cannot be minimal. The groups $H_1(M; \mathbb{Q})$ and $H_3(M; \mathbb{Q})$ are trivial. Since M has nonzero signature, Lemma 2.1 implies that h is orientation preserving. So h acts as the identity on $H_0(M; \mathbb{Q})$ and $H_4(M; \mathbb{Q})$. Now $H_2(M; \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$. Therefore, by the claim above, there exists some $k > 0$ such that the map

$$h_* \colon H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$$

satisfies $\mathrm{Tr}((h_*)^k) \geq 0$. It now follows from the Lefschetz fixed point theorem that h^k has a fixed point, as claimed.

Remark 4.2. We describe a different method to construct an example as in Theorem 3.1. The argument is shorter and does not rely on the existence theorem of [9] to produce a minimal homeomorphism, but has the disadvantage that the resulting algebra is not unital. In particular, we do not get the detailed properties given in Theorem 3.1, because the results needed to get them are not known in the nonunital case.

Fix $n \in \mathbb{Z}_{>0}$ with $n \geq 15$. Set $X = \mathbb{T}^n$. Choose a uniquely ergodic minimal homeomorphism $h\colon X \to X$ which is homotopic to id_X. (For example, choose $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$ such that $1, \theta_1, \theta_2, \ldots, \theta_n$ are linearly independent over $\mathbb{Q}$, and define

$$h(\zeta_1, \zeta_2, \ldots, \zeta_n) = \left(e^{2\pi i\theta_1}\zeta_1,\ e^{2\pi i\theta_2}\zeta_2,\ \ldots,\ e^{2\pi i\theta_n}\zeta_n\right)$$

for $\zeta_1, \zeta_2, \ldots, \zeta_n \in \mathbb{T}$.)

Let $D = M_{\mathbb{Q}}$ be the universal UHF algebra. As in the proof of Theorem 3.1, let $\overline{E}_D^*(-)$ be the (reduced) cohomology theory which arises as in [4, Corollary 3.9] from the infinite loop structure of the classifying space of $\mathrm{Aut}_0(D \otimes K)$. By statement (ii) at the beginning of the proof of [4, Corollary 4.5],

$$\overline{E}_D^1(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X; \mathbb{Q}).$$

Let F be a locally trivial continuous field of C*-algebras over X with fibers isomorphic to $M_{\mathbb{Q}} \otimes K$ and structure group $\mathrm{Aut}_0(M_{\mathbb{Q}} \otimes K)$. As in [4, Corollary 3.9], F is determined up to isomorphism of bundles by its class in $[F] \in \overline{E}_D^1(X)$:

$$[F] = (\delta_1(F), \delta_2(F), \delta_3(F), \ldots) \in H^3(X;\mathbb{Q}) \oplus H^5(X;\mathbb{Q}) \oplus H^7(X;\mathbb{Q}) \oplus \cdots .$$

By [5, Theorem 3.4], the opposite bundle F^{op} satisfies $\delta_k(F^{\mathrm{op}}) = (-1)^k \delta_k(F)$ for $k \in \mathbb{Z}_{>0}$. Therefore the class of F^{op} is given by

$$[F^{\mathrm{op}}] = \left(-\delta_1(F),\ \delta_2(F),\ -\delta_3(F),\ \ldots\right).$$

Let $\xi \in H^1(\mathbb{T};\mathbb{Q})$ be the standard generator. For $k = 1, 2, \ldots, n$ let $p_k\colon X \to \mathbb{T}$ be the projection on the kth coordinate, and define $\xi_k = p_k^*(\xi) \in H^1(X;\mathbb{Q})$. It is known that $H^*(X;\mathbb{Q}) \cong \bigwedge^*(\mathbb{Q}^n)$ as graded rings, with $\xi_1, \xi_2, \ldots, \xi_n$ forming a basis of $H^1(X;\mathbb{Q})$. Define

$$\eta_1 \in H^3(X;\mathbb{Q}), \quad \eta_2 \in H^5(X;\mathbb{Q}), \quad \text{and} \quad \eta_3 \in H^7(X;\mathbb{Q})$$

to be the cup products

$$\eta_1 = \xi_1 \smile \xi_2 \smile \xi_3, \quad \eta_2 = \xi_4 \smile \xi_5 \smile \cdots \smile \xi_8,$$

and

$$\eta_3 = \xi_9 \smile \xi_{10} \smile \cdots \smile \xi_{15}.$$

Then $\eta_3 \smile \eta_5 \smile \eta_7 \neq 0$. Using the correspondence above, choose a locally trivial continuous field E over X with fiber $M_{\mathbb{Q}} \otimes K$ such that

$$\delta_1(E) = \eta_1, \quad \delta_2(E) = \eta_2, \quad \delta_3(E) = \eta_3, \quad \delta_7(E) = \eta_1 \smile \eta_2 \smile \eta_3,$$

and $\delta_k(E) = 0$ for all other values of k. Then

$$\delta_1(E^{\mathrm{op}}) = -\eta_1, \quad \delta_2(E^{\mathrm{op}}) = \eta_2, \quad \delta_3(E^{\mathrm{op}}) = -\eta_3,$$
$$\delta_7(E^{\mathrm{op}}) = -\eta_1 \smile \eta_2 \smile \eta_3,$$

and $\delta_k(E^{\mathrm{op}}) = 0$ for all other values of k.

Suppose $\Gamma(E^{\mathrm{op}}) \cong \Gamma(E)$. Then, by reasoning analogous to that in the proofs of Theorem 3.1 and Theorem 3.2, there must be a homeomorphism $g\colon X \to X$ such that

$$g^*(\delta_k(E^{\mathrm{op}})) = \delta_k(E)$$

for all $k \in \mathbb{Z}_{>0}$. But g^* is a morphism of graded rings

$$g^*\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}).$$

Thus, if

$$g^*(-\eta_1) = \eta_1, \quad g^*(\eta_2) = \eta_1, \quad \text{and} \quad g^*(-\eta_3) = \eta_3,$$

then

$$g^*(-\eta_1 \smile \eta_2 \smile \eta_3) = -\eta_1 \smile \eta_2 \smile \eta_3 \neq \eta_1 \smile \eta_2 \smile \eta_3.$$

So $\Gamma(E^{\mathrm{op}}) \not\cong \Gamma(E)$.

Presumably $\Gamma(E)$ has no tracial states. If we want to use $\Gamma(E)$ in place of A_0 in the proof of Theorem 3.1, we need nonunital analogs of the theorems cited in that proof, many of which are not known.

One may also use the space $X = S^3 \times S^5 \times S^7$, taking

$$\eta_1 \in H^3(X;\mathbb{Q}), \quad \eta_2 \in H^5(X;\mathbb{Q}), \quad \text{and} \quad \eta_3 \in H^7(X;\mathbb{Q})$$

to be the classes coming from generators of $H^3(S^3;\mathbb{Q})$, $H^5(S^5;\mathbb{Q})$, and $H^7(S^7;\mathbb{Q})$, except that for the existence of minimal homeomorphisms one appeals to [9] as in the proof of Theorem 3.1 and Theorem 3.2.

One might hope to distinguish an action of $\mathbb{T}$ from its opposite (replacing the algebra with its opposite but keeping the same formula for the action) using the Bentmann–Meyer invariant ([2]). This appeared as a question in an earlier draft of this paper. However, Rasmus Bentmann has informed us that he has proved that this invariant never distinguishes an action and its opposite.

We explain how his result implies that our two actions are $KK^{\mathbb{T}}$-equivalent to their opposites. First, we apply Proposition 3.1 of [2], according to which, in particular, if a crossed product $B \rtimes_\gamma \mathbb{T}$ is in the ordinary bootstrap class then the system $(\mathbb{T}, B, \gamma)$ is in the bootstrap class for actions of $\mathbb{T}$. In our case, γ is the tensor product of a dual action with a trivial action. In the notation of the proofs of Theorem 3.1 and Theorem 3.2, it follows that $B \rtimes_\gamma \mathbb{T}$ is stably isomorphic to $\Gamma(E) \otimes \mathcal{O}_\infty$ or to $\Gamma(E) \otimes M_{2^\infty}$, both of which are clearly in the ordinary bootstrap class. An analogous statement is true for the opposite actions. Accordingly, the discussion after Proposition 3.1 of [2] applies, and in particular Theorem 2.7 of [2] applies in our situation. Thus, since the invariants of our circle actions and their opposites agree, it follows that they are $KK^{\mathbb{T}}$-equivalent to their opposites.

Similar reasoning is likely to apply to any examples constructed by related methods. However, the following question remains open.

Question 4.3. Is there any circle action on an algebra as in Theorem 3.1 or Theorem 3.2 which is not $KK^{\mathbb{T}}$-equivalent to its opposite action?

We conclude with two other questions, which we have not seriously investigated.

Question 4.4. What happens when we restrict the actions of Theorem 3.1 and Theorem 3.2 to finite subgroups of $\mathbb{T}$? What happens if we consider these actions as actions of $\mathbb{T}$ but with the discrete topology?

Question 4.5. In Theorem 3.1, is it possible to take A to be a UHF algebra?

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