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A regulator for smooth manifolds and an index theorem

Ulrich Bunke

Abstract. For a smooth manifold X and an integer $d > \dim(X)$ we construct and investigate a natural map

$$\sigma_d \colon K_d(C^\infty(X)) \to \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X)\ .$$

Here $K_d(C^\infty(X))$ is the algebraic K-theory group of the algebra of complex valued smooth functions on X, and $\mathbf{ku}\,\mathbb{C}/\mathbb{Z}^*$ is the generalized cohomology theory called connective complex K-theory with coefficients in $\mathbb{C}/\mathbb{Z}$.

If the manifold X is closed of odd dimension $d-1$ and equipped with a Dirac operator $\not{D}$, then we state and partially prove the conjecture stating that the following two maps

$$K_d(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}$$

coincide:

1. Pair the result of σ_d with the K-homology class of $\not{D}$.
2. Compose the Connes–Karoubi multiplicative character with the classifying map of the d-summable Fredholm module of $\not{D}$.

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1. Introduction

The torsion subgroup of the algebraic K-theory of the field $\mathbb{C}$ of complex numbers has been calculated by Suslin [36]. An important tool for this calculation was a collection of homomorphisms

$$r_{2n+1} \colon K_{2n+1}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z} \tag{1}$$

for $n \in \mathbb{N}$ which turned out to induce isomorphisms of torsion subgroups

$$K_{2n+1}(\mathbb{C})_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

One may interpret $\mathbb{C}$ as the algebra of complex-valued smooth functions $C^\infty(X)$ on the one-point manifold $X = *$. The first goal of the present paper is to generalize the construction of the homomorphism (1) to higher-dimensional smooth manifolds X. In order to state the result we need the following notation:

1. We write $\mathbf{ku} := \mathbf{K}^{top}(\mathbb{C})$ for the connective topological complex K-theory spectrum. Its homotopy groups are given by

$$\pi_n(\mathbf{ku}) \cong \begin{cases} 0, & n < 0, \\ \mathbb{Z}, & n \geq 0 \text{ even}, \\ 0, & n \geq 0 \text{ odd}. \end{cases} \tag{2}$$

2. In general, we write $\mathbf{E}^*(X)$ for the cohomology groups of the manifold X with coefficients in the spectrum $\mathbf{E}$.
3. If A is an abelian group, then we let $\mathbf{M}A$ denote the Moore spectrum of A and write

$$\mathbf{E}A := \mathbf{E} \wedge \mathbf{M}A\ . \tag{3}$$

For example, we can form the spectrum $\mathbf{ku}\,\mathbb{C}/\mathbb{Z}$. In view of (2) and [5, Eq. (2.1)] its homotopy groups are given by

$$\pi_n(\mathbf{ku}\,\mathbb{C}/\mathbb{Z}) \cong \begin{cases} 0, & n < 0, \\ \mathbb{C}/\mathbb{Z}, & n \geq 0 \text{ even}, \\ 0, & n \geq 0 \text{ odd}. \end{cases} \tag{4}$$

Theorem 1.1. *Let X be a smooth manifold and $d \in \mathbb{N}$. If $d > \dim(X)$, then we have a construction of a homomorphism*

$$\sigma_d\colon K_d(C^\infty(X)) \to \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X)$$

which is natural in X and induces the map (1) in homotopy groups for $X = *$

Remark 1.2. The main point of the theorem is the assertion that there is some interesting generalization of the homomorphism (1) to higher-dimensional manifolds. In this paper we just give a construction of such a natural homomorphism. We do not address the problem of characterizing it by a collection of natural properties. Example 1.4 below shows that the map σ_d contains certain "higher information". As opposed to the case $X = *$, in the higher-dimensional case we do not understand its kernel or cokernel.

Remark 1.3. The classical construction of the homomorphism (1) relies on the observation by Quillen that the natural map $K_*(\mathbb{C}) \to K_*^{top}(\mathbb{C})$ from the algebraic to the topological K-theory of $\mathbb{C}$ vanishes rationally in positive degrees. In order to employ this fact for the construction of r_{2n+1} we work in the stable ∞-category $\mathbf{Sp}$ of spectra. We consider the diagram

$$\begin{tikzcd} \mathbf{K}(\mathbb{C})[1..\infty] \arrow[r, dotted] \arrow[d] \arrow[ddr, dashed] & \Sigma^{-1}\,\mathbf{ku}\,\mathbb{C}/\mathbb{Z} \arrow[d] \\ \mathbf{K}(\mathbb{C}) \arrow[r] & \mathbf{ku} \arrow[d] \\ & \mathbf{ku}\,\mathbb{C}\,. \end{tikzcd} \tag{5}$$

Here $\mathbf{K}(\mathbb{C})[1..\infty] \to \mathbf{K}(\mathbb{C})$ denotes the connected covering of the connective algebraic K-theory spectrum $\mathbf{K}(\mathbb{C})$. The right column is a segment of a Bockstein fibre sequence for the topological K-theory spectrum $\mathbf{ku}$. Finally, the middle horizontal map is the canonical map from algebraic to topological K-theory. Now, by Quillen's observation, the dashed arrow induces the trivial map in homotopy groups. Since its target is rational the dashed arrow actually vanishes as a map of spectra. A choice of a zero homotopy of this arrow induces the dotted arrow which in turn induces the map r_{2n+1} after applying $\pi_{2n+1}(-)$ and identifying $\pi_{2n+1}(\Sigma^{-1}\,\mathbf{ku}\,\mathbb{C}/\mathbb{Z})$ with $\mathbb{C}/\mathbb{Z}$ using (4). Note that the restriction of r_{2n+1} to the torsion subgroup does not depend on the choice of the zero homotopy. A reference for this construction is [26, 7.19ff] where instead of $\Sigma^{-1}\,\mathbf{ku}\,\mathbb{C}/\mathbb{Z}^*$ the periodic version $\Sigma^{-1}\,\mathbf{KU}\,\mathbb{C}/\mathbb{Z}$ (called multiplicative K-theory) is used. Another reference for the construction of (1) formulated in a slightly different language is [39]. In the present paper we will generalize the alternative construction [12, Ex. 6.9]. $\square$

Example 1.4. For a unital algebra A let

$$\iota\colon A^\times \to K_1(A) \tag{6}$$

be the natural homomorphism from the group $A^\times$ of units of A to the first algebraic K-theory group of A.

A complex-valued smooth function $f \in C^\infty(S^1)$ gives rise to an invertible function $\exp(f) \in C^\infty(S^1)$. If $u \in C^\infty(S^1)$ is a second invertible function, then we have two algebraic K-theory classes $\iota(\exp(f))$, $\iota(u) \in K_1(C^\infty(S^1))$. Using the multiplicative structure of the algebraic K-theory for commutative algebras we define the class

$$\iota(\exp(f)) \cup \iota(u) \in K_2(C^\infty(S^1))\,.$$

We use the identification

$$\mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-3}(S^1) \cong \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-4}(*) \cong \mathbb{C}/\mathbb{Z}$$

given by suspension and (4). With this identification we have

$$\sigma_2(\iota(\exp(f)) \cup \iota(u)) = \left[\frac{1}{(2\pi i)^2}\int_{S^1} f\,\frac{du}{u}\right]_{\mathbb{C}/\mathbb{Z}}\,. \tag{7}$$

The formula (7) is a special case of (57). $\square$

In [12, Example 6.9] we explained how one can construct the map (1) using techniques of differential cohomology. The differential cohomology approach in particular provides a canonical choice for the dotted arrow in (5). The main idea for the construction of σ_d is to apply the framework of differential cohomology to the algebraic K-theory of the Fréchet algebra $C^\infty(X)$. The details will be worked out in Section 2. The final construction of σ_d will be given in Definition 2.36.

We now come to the second theme of this paper. Let us assume that X is a closed manifold of odd dimension d which carries a generalized Dirac operator $\not{D}$. This Dirac operator gives rise to a K-homology class $[\not{D}] \in \mathbf{KU}_d(X)$ and a $d+1$-summable Fredholm module which we will describe in Subsection 3.2 in greater detail. This Fredholm module is classified by a homomorphism

$$b_{\not{D}} : C^\infty(X) \to \mathcal{M}_d \ ,$$

which is unique up to unitary equivalence, and where $\mathcal{M}_d$ denotes the classifying algebra for $d+1$-summable Fredholm modules introduced in [19], see Remark 3.10 for an explicit description. In [19] Connes and Karoubi further introduced the multiplicative character

$$\delta : K_{d+1}(\mathcal{M}_d) \to \mathbb{C}/\mathbb{Z} \ .$$

Since $\mathbf{KU}\,\mathbb{C}/\mathbb{Z}$ is a $\mathbf{KU}$-module spectrum we can define a map

$$r_{\not{D}} : \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \to \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \xrightarrow{\langle -,[\not{D}]\rangle} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-2d-2}(*) \cong \mathbb{C}/\mathbb{Z}$$

induced by the pairing with the K-homology class $[\not{D}]$. An explicit construction of this map using elements of local index theory will be given in (80). We now make the following conjecture:

Conjecture 1.5. *Assume that X is a closed odd-dimensional manifold of dimension d with a Dirac operator $\not{D}$. Then the following diagram commutes:*

$$\begin{tikzcd} & \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \arrow[dr, "r_{\not{D}}"] & \\ K_{d+1}(C^\infty(X)) \arrow[ur, "\sigma_{d+1}"] \arrow[dr, "b_{\not{D}}"'] & & \mathbb{C}/\mathbb{Z} \\ & K_{d+1}(\mathcal{M}_d). \arrow[ur, "\delta"'] & \end{tikzcd}$$

This conjecture is supported by our second main result which asserts that it holds true if one replaces $K_d(C^\infty(X))$ by its subgroup of classes which are topologically trivial. Note that we do not know any example of a topologically non-trivial class in $K_n(C^\infty(X))$ for $n > \dim(X)$, see Remark 1.10. In order to explain what topologically trivial means we consider the homotopification fibre sequence in spectra (see (34) for details)

$$\mathbf{K}^{rel}(C^\infty(X)) \xrightarrow{\partial} \mathbf{K}(C^\infty(X)) \to \mathbf{K}^{top}(C^\infty(X)) \to \Sigma\mathbf{K}^{rel}(C^\infty(X))$$

relating the algebraic K-theory spectrum of $C^\infty(X)$ with its topological and relative K-theory spectra. A class in $K_{d+1}(C^\infty(X))$ is called topologically trivial if its image in $K^{top}_{d+1}(C^\infty(X))$ vanishes, or equivalently, if this class belongs to the image of $\partial\colon K^{rel}_{d+1}(C^\infty(X)) \to K_{d+1}(C^\infty(X))$. We have the following theorem:

Theorem 1.6. *Assume that X is a closed odd-dimensional manifold of dimension d with a Dirac operator $\slashed{D}$. Then the following diagram commutes:*

$$\begin{tikzcd}
 & \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \arrow[dr, "r_{\slashed{D}}"] & \\
K^{rel}_{d+1}(C^\infty(X)) \arrow[ur, "\sigma_{d+1}\circ\partial"] \arrow[dr, "b_{\slashed{D}}\circ\partial"'] & & \mathbb{C}/\mathbb{Z} \\
 & K_{d+1}(\mathcal{M}_d). \arrow[ur, "\delta"'] &
\end{tikzcd}$$

In the remainder of this introduction we describe how our constructions are related with other results in the literature relating index and spectral theory of operators with algebraic K-theory of smooth functions.

Since $C^\infty(X)$ is a commutative algebra, the algebraic K-theory $K_*(C^\infty(X))$ is a graded commutative ring. As in Example 1.4 we can use the map

$$\iota\colon C^\infty(X)^\times \xrightarrow{(6)} K_1(C^\infty(X))$$

and the $\cup$-product in algebraic K-theory in order to construct higher algebraic K-theory classes.

Example 1.7. Assume that X is a closed odd-dimensional manifold with a Dirac operator $\slashed{D}$. If $f \in C^\infty(X)$, then we can form the unit $e^f \in C^\infty(X)^\times$, and we can consider the class $\iota(e^f) \in K_1(C^\infty(X))$. Note that this element is topologically trivial. Indeed, we can consider $\iota(e^{tf}) \in K_1(C^\infty(I \times X))$, where t is the coordinate of the interval. This class restricts to zero at $t = 0$ and to $\iota(e^f)$ at $t = 1$.

Given a collection $f_1, \dots, f_d$ of such smooth functions we can form the topologically trivial algebraic K-theory class

$$\{e^{f_1}, \dots, e^{f_d}\} \in \iota(e^{f_1}) \cup \dots \cup \iota(e^{f_d}) \in K_d(C^\infty(X)) .$$

The main result of [22] is an explicit formula [22, (1.2)] for the number

$$(\delta \circ b_{\slashed{D}})\big(\{e^{f_1}, \dots, e^{f_d}\}\big) \in \mathbb{C}/\mathbb{Z} .$$

It involves the traces of algebraic expressions build from the f_i and the positive spectral projection of $\not{D}$. Kaad's formula can be considered as the analytical side of an index formula. One can interpret our Conjecture 1.5 as providing the topological counterpart. Indeed, since $\{e^{f_1},\ldots,e^{f_d}\}$ is topologically trivial, by Theorem 1.6 we have the equality

$$(\delta \circ b_{\not{D}})\big(\{e^{f_1},\ldots,e^{f_d}\}\big) = (\rho_{\not{D}} \circ \sigma_d)\big(\{e^{f_1},\ldots,e^{f_d}\}\big),$$

where $d-1=\dim(X)$.

Example 1.8. We consider the case $X = S^1$ with the Dirac operator $\not{D} := i\partial_t$ acting as an unbounded essentially selfadjoint operator with domain $C^\infty(S^1)$ on the Hilbert space $L^2(S^1)$. Let $u_1, u_2 \in C^\infty(S^1)^\times$ be two invertible complex-valued functions. Then we form the algebraic K-theory class

$$\{u_1,u_2\} \in K_2(C^\infty(S^1)) \ .$$

Let $P \in B(L^2(S^1))$ be the projection onto the subspace of positive Fourier modes, i.e. the positive spectral projection of $\not{D}$. For $f \in C^\infty(S^1)$ we consider the Toeplitz operator

$$T_f := PfP \in B(L^2(S^1)) \ ,$$

where f acts as multiplication operator. For two functions $f_1, f_2 \in C^\infty(S^1)$ the difference $T_{f_1}T_{f_2} - T_{f_1 f_2}$ is a trace class operator.

We let $\mathcal{A} \subset B(L^2(S^1))$ be the algebra generated by all Toeplitz operators T_f for $f \in C^\infty(S^1)$ and the algebra of trace class operators $\mathcal{L}^1 := \mathcal{L}^1(L^2(S^1))$. We then get the Toeplitz extension

$$0 \to \mathcal{L}^1 \to \mathcal{A} \to C^\infty(S^1) \to 0 \ . \tag{8}$$

Associated to an extension of the trace class operators one has the determinant invariant (see e.g. [6])

$$d := \det \circ \partial \colon K_2(C^\infty(S^1)) \to \mathbb{C}^* \ ,$$

where $\partial\colon K_2(C^\infty(S^1)) \to K_1(\mathcal{A},\mathcal{L}^1)$ is the boundary operator in algebraic K-theory associated to the sequence (8) and $\det\colon K_1(\mathcal{A},\mathcal{L}^1) \to \mathbb{C}^*$ is induced by the Fredholm determinant. The diagram [23, (3)] states that

$$d\big(\{u_1,u_2\}\big) = \exp\big(2\pi i\, \delta\big(b_{\not{D}}\big(\{u_1,u_2\}\big)\big)\big) \ . \tag{9}$$

The determinant invariant was identified by Carey–Pincus [16] with the joint torsion $\tau(A,B) \in \mathbb{C}^*$ which is defined for the pair A, B of Fredholm operators which commute up to trace class operators. In the special case of the pair T_{u_1}, T_{u_2} on $\mathtt{im}(P)$ we thus have

$$d\big(\{u_1,u_2\}\big) = \tau(T_{u_1},T_{u_2}) \ .$$

We refer to [33] and [24] for a gentle introduction to joint torsion.

The joint torsion $\tau(T_{u_1}, T_{u_2})$ in turn has been calculated explicitly. In the special case where $u_1 = e^{f_1}$ we have

$$\tau(T_{u_1}, T_{u_2}) = \exp\left(\frac{1}{2\pi i}\int f_1 \, d\log u_2\right) .$$

In order to state the result of the calculation in the general case in a comprehensive way we will use the cup product in Deligne cohomology. Using the isomorphism (11) (to be explained below) the invertible functions u_i can be interpreted as classes in Deligne cohomology $H^1_{Del}(S^1, \mathbb{Z})$. Their cup product is the class

$$u_1 \cup u_2 \in H^2_{Del}(S^1, \mathbb{Z}) .$$

We have an isomorphism

$$\langle -, [S^1]\rangle \colon H^2_{Del}(S^1, \mathbb{Z}) \xrightarrow{\cong} \mathbb{C}/\mathbb{Z}$$

given by evaluation. In [16, (1.2), (1.3)] Carey–Pincus calculate the determinant invariant and joint torsion:

$$d\big(\{u_1, u_2\}\big) = \tau(T_{u_1}, T_{u_2}) = \exp\big(2\pi i \langle u_1 \cup u_2, [S^1]\rangle\big) .$$

Combining this equality with (9) we get the equality

$$\delta\big(b_{\not{D}}\big(\{u_1, u_2\}\big)\big) = \langle u_1 \cup u_2, [S^1]\rangle . \tag{10}$$

Using the multiplicative features of the differential regulator map $\hat{\mathtt{reg}}_X$ (see Remark 2.30) one can also calculate $r_{\not{D}}(\sigma_2(\{u_1, u_2\}))$ explicitly. The result is again the right-hand side of (10) as expected by Conjecture 1.5. We will not give the details of the multiplicative theory since it requires a set-up which is similar to [15] but differs from the one used in the present paper.

Note that the class $2\{u_1, u_2\} \in K_2(C^\infty(S^1))$ is topologically trivial. In order to see this let $n_i \in \mathbb{Z}$ denote the mapping degree of $\frac{u_i}{|u_i|} \colon S^1 \to S^1$ for $i = 1, 2$. Then u_i can be deformed through invertible smooth functions to the invertible function $z^{n_i} \colon S^1 \to \mathbb{C}^*$. So the difference $\iota(u_i) - \iota(z^{n_i})$ is topologically trivial. We now have $\iota(z^n) = n\, \iota(z)$ and hence

$$\iota(z^{n_1}) \cup \iota(z^{n_2}) = n_1 n_2\, \iota(z) \cup \iota(z) .$$

Since the cup product on $K_*(C^\infty(S^1))$ is graded commutative the square of a class in degree one is two-torsion. This implies the assertion. Moreover, if one of the degrees n_i is even, then $\{u_1, u_2\}$ itself is topologically trivial. Thus Theorem 1.6 gives a proof of the equality

$$2\delta\big(b_{\not{D}}\big(\{u_1, u_2\}\big)\big) = 2r_{\not{D}}\big(\sigma_2\big(\{u_1, u_2\}\big)\big) .$$

independently of the calculations of Kaad and Carey–Pincus and a multiplicative version of the theory. If the degree of one of the maps u_i is even then we can even remove the factor 2 on both sides.

Remark 1.9. In this remark we recall the basic features of Deligne cohomology used above. For $p \in \mathbb{N}$ the Deligne cohomology group $H^p_{Del}(X,\mathbb{Z})$ is defined as the pth hypercohomology of the complex of sheaves

$$0 \to \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0 \ .$$

We refer to [17] for a first definition and to [7], [8, Sec. 3], or [12, 4.3]) for introductions to Deligne cohomology. In the original paper [17] Deligne cohomology classes are called differential characters and a different grading convention was used. We have a cup product

$$\cup\colon H^p_{Del}(X,\mathbb{Z}) \otimes H^q_{Del}(X,\mathbb{Z}) \to H^{p+q}_{Del}(X,\mathbb{Z}),$$

which turns Deligne cohomology into a graded commutative ring. Moreover, we have a natural isomorphism of groups

$$H^1_{Del}(X,\mathbb{Z}) \cong C^\infty(X)^\times \ . \tag{11}$$

Note that we get invertible complex-valued functions since Ω^* is the de Rham complex of complex-valued forms. Finally, for a closed connected and oriented manifold M of dimension $n-1$ we have an evaluation isomorphism

$$\langle -, [M]\rangle\colon H^n_{Del}(M,\mathbb{Z}) \stackrel{\cong}{\to} \mathbb{C}/\mathbb{Z} \ . \qquad \square$$

Remark 1.10. We refer to [26, Appendix 4] for some information about the algebraic K-theory of the algebra of smooth functions on a manifold.

Let X be a compact manifold and $n \in \mathbb{N}$ be odd. By [26, Thm. A.4.6] the rank of

$$\mathtt{im}\left(K_n(C^\infty(X)) \to K_n^{top}(C^\infty(X))\right)$$

is at least $\dim H^n(X;\mathbb{R})$. By [26, Thm. A.4.6] this is true also if X is oriented and $n = \dim(X)$ (not necessarily odd).

We have a decomposition

$$K_*(C^\infty(X)) \cong K_*(\mathbb{C}) \oplus \tilde{K}_*(C^\infty(X)) \ ,$$

where the first summand is induced by the inclusion $\mathbb{C} \to C^\infty(X)$ as constant functions, and the second summand is the kernel of the restriction to some point in X. There are non-trivial classes in $K_d(C^\infty(X))$ for arbitrary large odd $d \in \mathbb{N}$. Note that classes coming from the summand $K_d(\mathbb{C})$ are topologically trivial.

Theorem [26, Thm. A.4.3] shows that the group $\tilde{K}_n(C^\infty(X))$ itself is huge for $1 \le n \le \dim(X)$.

We do not know whether there exists topologically non-trivial classes in degrees strictly larger than $\dim(X)$. $\square$

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2. The construction of σ_d

2.1. Elements of sheaf theory on manifolds. Our main idea is to analyse the algebraic K-theory spectrum $\mathbf{K}(C^\infty(X))$ of the algebra of complex-valued smooth functions on a manifold X using the techniques of differential cohomology theory as developed in [12]. In the following we recall some of the basic notions.

Let **Mf** denote the site of smooth manifolds with corners with the open covering topology.

Remark 2.1. An n-dimensional manifold with corners is locally modeled by open subsets of the subspace $[0,\infty)^n \subset \mathbb{R}^n$. The category of manifolds with corners contains the unit interval $[0,1]$, manifolds with boundary, simplices Δ^n. Furthermore, the category of manifolds with corners is closed under taking products. □

For a presentable ∞-category $\mathcal{C}$ (see [30, Ch. 5]) we will consider the ∞-category of presheaves $\mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})$ and its full subcategory of sheaves $\mathbf{Sh}_{\mathcal{C}}(\mathbf{Mf})$ with values in $\mathcal{C}$ on the site **Mf**.

Definition 2.2. A presheaf $G \in \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})$ is a sheaf if for every manifold M and every open covering $U \to M$ the natural map

$$G(M) \to \lim_{\Delta} G(U^\bullet) \tag{12}$$

is an equivalence.

In this definition the simplicial manifold $U^\bullet \in \mathbf{Mf}^{\Delta^{op}}$ is the Čech nerve of the open covering and the map (12) is induced from the natural map $U^\bullet \to M$, were M is considered as a constant simplicial manifold. By an application of the general theory [30, 6.2.2.7] we get that $\mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})$ and $\mathbf{Sh}_{\mathcal{C}}(\mathbf{Mf})$ are again presentable ∞-categories and that there is an adjunction

$$L\colon \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf}) \leftrightarrows \mathbf{Sh}_{\mathcal{C}}(\mathbf{Mf})\colon \mathrm{incl}$$

between the inclusion of sheaves into presheaves and the sheafification functor L.

We use the unit interval $I := [0,1]$ in order to define the notion of homotopy invariance.

Definition 2.3. A sheaf or presheaf G on **Mf** is called homotopy invariant, if the map

$$G(M) \to G(I \times M)$$

induced by the projection $I \times M \to M$ is an equivalence for every smooth manifold M.

As in [12, Sec. 2] one argues that the full subcategories of homotopy invariant sheaves $\mathbf{Sh}^h_{\mathcal{C}}(\mathbf{Mf})$ or homotopy invariant presheaves $\mathbf{PSh}^h_{\mathcal{C}}(\mathbf{Mf})$ are presentable. Their inclusions into all sheaves or presheaves fit into adjunctions

$$\mathcal{H}^{pre}\colon \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf}) \leftrightarrows \mathbf{PSh}^h_{\mathcal{C}}(\mathbf{Mf})\colon \mathrm{incl}\ , \quad \mathcal{H}\colon \mathbf{Sh}_{\mathcal{C}}(\mathbf{Mf}) \leftrightarrows \mathbf{Sh}^h_{\mathcal{C}}(\mathbf{Mf})\colon \mathrm{incl}\ . \quad (13)$$

The left adjoints are called homotopification functors. The homotopification functors for sheaves and presheaves are related by the equivalence

$$\mathcal{H} \simeq L \circ \mathcal{H}^{pre} \circ \mathrm{incl}\ , \quad (14)$$

see [12, Prop. 2.6].

Let X be a smooth manifold. By $i_X\colon \mathbf{Mf} \to \mathbf{Mf}$ we denote the map of sites given by $M \mapsto X \times M$. It induces a pull-back

$$i_X^*\colon \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf}) \to \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})\ . \quad (15)$$

The functor i_X^* has the following properties:

Lemma 2.4. 1. *The functor i_X^* preserves sheaves.*

2. *The map i_X^* preserves homotopy invariant presheaves and sheaves.*
3. *For presheaves i_X^* commutes with homotopification, i.e. the natural transformation $\mathcal{H}^{pre} \circ i_X^* \to i_X^* \circ \mathcal{H}^{pre}$ is an equivalence.*
4. *If X is compact, then the analogous statement holds true for sheaves, i.e. the natural map $\mathcal{H} \circ i_X^* \to i_X^* \circ \mathcal{H}$ is an equivalence.*

Proof. If $U \to M$ is an open covering covering of M, then $i_X(U) \to i_X(M)$ is an open covering of $X \times M$. If G is a presheaf, then the descent map of $i_X^* G$ with respect to $U \to M$ is the same as the descent map of G with respect to $i_X(U) \to i_X(M)$. This implies the first assertion.

A sheaf or presheaf G is homotopy invariant by definition if the natural transformation $G \to i_I^* G$ (induced by the map $I \to *$) is an equivalence. We have equivalences of functors $i_I^* i_X^* \simeq i_{I\times X}^* \simeq i_X^* i_I^*$. This implies that i_X^* preserves homotopy invariant sheaves or presheaves.

In order to see that i_X^* commutes with homotopification of presheaves we use the explicit formula for the homotopification given in [12, Section 7]. We define the functor

$$\mathbf{s}\colon \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf}) \to \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})\ , \quad \mathbf{s}(G) := \mathtt{colim}_{\Delta^{op}}\, i_{\Delta^\bullet}^* G\ , \quad (16)$$

where $\Delta^\bullet$ is the cosimplicial manifold of standard simplices. Then the homotopification on presheaves $\mathcal{H}^{pre}$ is given by

$$\mathcal{H}^{pre} \simeq \mathbf{s}\ . \quad (17)$$

Since the colimit for presheaves is taken object wise and $i^*_{\Delta^\bullet} i^*_X \simeq i^*_X i^*_{\Delta^\bullet}$ we see that the homotopification for presheaves commutes with i^*_X.

We now assume that X is compact and G is a sheaf. For $n \in \mathbb{N}$ we let $S^n \in \mathbf{Mf}$ denote the n-dimensional sphere. For every $n \in \mathbb{N}$, using [12, Prop. 7.6] at the marked places, we get the following chain of equivalences:

$$\begin{aligned}(i^*_X \mathcal{H} G)(S^n) &\simeq (\mathcal{H} G)(X \times S^n)\\ &\overset{!}{\simeq} (\mathcal{H}^{pre} G)(X \times S^n)\\ &\simeq (i^*_X \mathcal{H}^{pre} G)(S^n) \simeq (\mathcal{H}^{pre} i^*_X G)(S^n) \overset{!}{\simeq} (\mathcal{H} i^*_X G)(S^n)\,.\end{aligned}$$

Assertion 4. now follows from [12, Lemma 7.3] which states that an equivalence between objects of $\mathbf{Sh}_{\mathcal{C}}(\mathbf{Mf})$ can be detected on the collection of spheres S^n, $n \in \mathbb{N}$. $\square$

Remark 2.5. Since i^*_X preserves sheaves we have a natural transformation

$$L \circ i^*_X \to i^*_X \circ L\,. \tag{18}$$

In general it is not an equivalence. For example, let $\mathcal{C} := \mathbf{Ab}$ and $\underline{\mathbb{Z}}^{pre}$ be the constant presheaf with value $\mathbb{Z}$ and X consist of two points. Then we have

$$i^*_X(L(\underline{\mathbb{Z}}^{pre}))(*) \cong \mathbb{Z} \oplus \mathbb{Z},$$

but $L(i^*_X \underline{\mathbb{Z}}^{pre})(*) \cong \mathbb{Z}$ and the map (18) is the diagonal inclusion.

In the present paper we will use the language of diffeological algebras.

Definition 2.6. A diffeological structure on an algebra A over $\mathbb{C}$ is a subsheaf of algebras A^∞ of the sheaf of algebras $M \mapsto \mathtt{Hom}_{\mathbf{Set}}(M, A)$ such that $A^\infty(*) = A$. A diffeological algebra is an algebra equipped with a diffeological structure.

Remark 2.7. A sheaf F of sets on $\mathbf{Mf}$ which is a subsheaf of the sheaf of set-valued functions to $F(*)$ is also called a concrete sheaf. We refer to [35] for a discussion of various variants of the definition of a diffeology. Our version is most similar to the notion of a Chen space, but not equal. A Chen space is a concrete sheaf of sets on the site of convex subsets with non-empty interior of euclidean spaces. In contrast, our sheaves are defined on all manifolds with corners. $\square$

Example 2.8. In the following we list some examples of diffeological algebras.

1. The constant sheaf $\underline{A}$ generated by A is the minimal diffeological structure, while the sheaf $M \mapsto \mathtt{Hom}_{\mathbf{Set}}(M, A)$ is the maximal diffeological structure on A.

2. If A is a diffeological algebra and X is a smooth manifold, then we define the algebra $C^\infty(X, A) := A^\infty(X)$. It has again a diffeological structure given by the sheaf $i^*_X A^\infty$.

3. The algebra $\mathbb{C}$ has a diffeological structure such that $\mathbb{C}^\infty$ is the sheaf $M \mapsto C^\infty(M)$ of smooth $\mathbb{C}$-valued functions on **Mf**.

4. For a manifold X we equip $C^\infty(X)$ with the diffeological structure defined in 2.

5. If A is a locally convex algebra, then we have a natural notion of a smooth function $X \to A$. The diffeological structure is given by $A^\infty(X) := C^\infty(X, A)$, where $C^\infty(X, A)$ denotes the algebra of smooth functions on X with values in A. See Remark 2.9 for more details. □

Remark 2.9. In this remark we fix our conventions about smooth functions on manifolds with values in a locally convex algebra. A locally convex vector space is a complex vector space whose topology is defined by a collection of seminorms. A locally convex algebra is a locally convex vector space such that the product induces a continuous bilinear map $A \times A \to A$.

A locally convex vector space has a natural uniform structure. Therefore the notions of completeness and completion are defined.

We now consider smooth functions with values in a locally convex vector space A (see e.g. [37, Sec. 40]). Let $U \subseteq \mathbb{R}^n$ open and consider a continuous function $f : U \to A$.

Definition 2.10. The function f continuously differentiable if there exists a continuous function $f' : U \to \mathtt{Hom}(\mathbb{R}^n, A)$ such that for every seminorm p on A and every compact subset $K \subset U$ we have

$$\lim_{D \to 0} \sup_{x \in K} p\left(\frac{f(x+D) - f(x) - f'(x)(D)}{\|D\|}\right) = 0 .$$

We call $\partial_i f := f'(-)(e^i)$ the partial derivative of f in the ith direction. We call f smooth if it has all iterated continuous partial derivatives.

We denote the iterated partial derivatives by $f^{(k)}_{i_1,\dots,i_k}$. We equip the complex vector space $C^\infty(U, A)$ with the locally convex structure determined by the seminorms

$$f \mapsto \sup_{x \in K} p(f^{(k)}_{i_1,\dots,i_k}(x)) .$$

The set of seminorms which generates the topology of $C^\infty(U, A)$ is thus indexed by compact subsets $K \subset U$, tuples $(i_1, \dots, i_k)$ of elements of $\{1, \dots, n\}$, and seminorms p of A. If A is complete, then so is $C^\infty(U, A)$.

This definition of smooth A-valued functions extends to manifolds in a straightforward manner.

Let X be a smooth manifold. Then the algebra $C^\infty(X, A)$ has two diffeological structures:

1. The first comes from the construction 2. in Example 2.8 above.

2. The second is induced from its locally convex structure.

These two structures coincide in view of the exponential law:

$$C^\infty(X, C^\infty(Y, A)) \cong C^\infty(X \times Y, A) .$$

Since $C^\infty(M, A)$ is a subset of the set of all functions from M to A it is clear that these spaces of smooth functions for varying M define a concrete sheaf, i.e. a diffeological structure on A.

In the non-commutative geometry literature instead of $C^\infty(X, A)$ one often uses the projective tensor product $C^\infty(X) \otimes_\pi A$. If A is complete, then this gives an equivalent structure as we will explain below. We have a natural map

$$C^\infty(X) \otimes A \to C^\infty(X, A),$$

which is continuous with respect to projective topology on the algebraic tensor product.

In general, for locally convex vector spaces V, W we let $V \otimes_\pi W$ denote the completion of the algebraic tensor product $V \otimes W$ with respect to the projective topology. If A is a complete locally convex vector space, then we get an isomorphism

$$C^\infty(X) \otimes_\pi A \xrightarrow{\cong} C^\infty(X, A) .$$

Here is a reference for this classical fact:

1. It follows from [37, Thm. 44.1] that for a complete A we have an isomorphism
$$C^\infty(X) \otimes_\epsilon A \xrightarrow{\cong} C^\infty(X, A),$$
where $\otimes_\epsilon$ denotes the completion of the algebraic tensor product in the ϵ-topology.
2. It follows from [37, Thm. 50.1] that the locally convex vector space $C^\infty(X)$ is nuclear.
3. If one of the tensor factors is nuclear, then the natural map from the π- to the ϵ-tensor product is an isomorphism by [37, Thm. 50.1]. □

Remark 2.11. In this remark we explain the relationship between the notions of homotopy invariance according to Definition 2.3 and diffeotopy invariance of functors defined on locally convex algebras as considered e.g. in [21, Sec. 4.1].

Consider the category $\mathcal{L}oc\mathcal{A}lg_1$ of unital complete locally convex algebras. We have a functor

$$\mathcal{L}oc\mathcal{A}lg_1 \to \mathbf{Sh}_{\mathcal{L}oc\mathcal{A}lg_1}(\mathbf{Mf}) , \quad A \mapsto A^\infty := (M \mapsto C^\infty(M) \otimes_\pi A) .$$

Let $\mathcal{C}$ be a presentable ∞-category.

Lemma 2.12. *A functor $F : \mathcal{L}oc\mathcal{A}lg_1 \to \mathcal{C}$ is a diffeotopy invariant functor in the sense of* [21, Sec. 4.1] *if and only if the presheaf $F(A^\infty) \in \mathbf{PSh}_{\mathcal{C}}(\mathbf{Mf})$ is homotopy invariant in the sense of Definition 2.3.*

Proof. This is immediate from the definitions if one uses the associativity of $\otimes_\pi$ and

$$C^\infty(I \times X) \cong C^\infty(I) \otimes_\pi C^\infty(X) . \qquad \square$$

2.2. Algebraic K-theory and cyclic homology of smooth functions.

2.2.1. Chain complexes and spectra. The purpose of this paragraph is to fix our conventions concerning chain complexes and spectra. We fix some notation and introduce some basic constructions.

Let **Ch** be the category of (in general unbounded) chain complexes of abelian groups and chain morphisms. If R is a ring, then we use $\mathbf{Ch}_R$ in order to denote the category of chain complexes of R-modules.

We identify chain complexes with cochain complexes such that the chain complex

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

corresponds to the cochain complex

$$\cdots \to C^{-n-1} \to C^{-n} \to C^{-n+1} \to \cdots .$$

For an integer $p \in \mathbb{N}$ and a chain complex $(C, d) \in \mathbf{Ch}$ we define its shift by p by $C[p]^n := C^{n+p}$. The differential of the shifted complex is given by $(-1)^p d$.

For $n \in \mathbb{N}$ we let $H^n\colon \mathbf{Ch} \to \mathbf{Ab}$ denote the nth cohomology functor. A chain map is a quasi-isomorphism if it induces an isomorphism in cohomology. If we invert the quasi-isomorphisms in **Ch**, then we get a stable ∞-category $\mathbf{Ch}[W^{-1}]$. There are various ways to construct a model of this ∞-category e.g. using model categories or dg-enhancements. Since the constructions in the present paper are model independent we will not discuss the details.

We have a natural functor $\iota\colon \mathbf{Ch} \to \mathbf{Ch}[W^{-1}]$. Our usual notation convention is that the italic letter C denotes an object of **Ch**, and the corresponding roman letter C denotes the object $\iota(C) \in \mathbf{Ch}[W^{-1}]$. By the universal property of $\mathbf{Ch}[W^{-1}]$ the cohomology functors descent to functors $H^n\colon \mathbf{Ch}[W^{-1}] \to \mathbf{Ab}$.

For an integer $p \in \mathbb{Z}$ and a chain complex $C \in \mathbf{Ch}$

$$\cdots \to C^{p-1} \to C^p \to C^{p+1} \to \cdots$$

we define its naive truncations $\sigma^{\geq p}C$ and $\sigma^{<p}C$ at p by

$$\cdots \to 0 \to C^p \to C^{p+1} \to \cdots , \quad \cdots \to C^{p-2} \to C^{p-1} \to 0 \to \cdots . \tag{19}$$

We have natural inclusion and projection morphisms

$$\sigma^{\geq p}C \to C , \quad C \to \sigma^{<p}C . \tag{20}$$

Remark 2.13. Note that $\iota(\sigma^{\geq p}C)$ is well-defined, but $\sigma^{\geq p}\iota(C)$ does not make sense. □

By $\Omega \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf})$ we denote the sheaf of de Rham complexes on **Mf** of complex-valued differential forms. By [12, Lemma 7.12], for every $p \in \mathbb{Z}$ its truncation $\iota(\sigma^{\geq p}\Omega)$ is a sheaf, i.e.

$$\iota(\sigma^{\geq p}\Omega) \in \mathbf{Sh}_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf}) . \tag{21}$$

Note that $H^p(\iota(\sigma^{\geq p}\Omega)) \cong \Omega^p_{cl} \in \mathbf{Sh}_{\mathbf{Ab}}(\mathbf{Mf})$ is the sheaf of closed p-forms.

Let **Sp** denote the stable ∞-category of spectra. Again we will not discuss explicit models. For every $n \in \mathbb{Z}$ we have a functor $\pi_n : \mathbf{Sp} \to \mathbf{Ab}$ which maps a spectrum to its nth homotopy group. The collection of these functors for all $n \in \mathbb{Z}$ detects equivalences in **Sp**.

We will frequently use the Eilenberg–MacLane correspondence $H : \mathbf{Ch}[W^{-1}] \to \mathbf{Sp}$ (see [31, 8.1.2.13]) which maps a chain complex to its associated Eilenberg–MacLane spectrum. For $\mathrm{C} \in \mathbf{Ch}[W^{-1}]$ we have the relations

$$\pi_n(H(\mathrm{C})) \cong H^{-n}(\mathrm{C}) , \quad H(\mathrm{C}[p]) \simeq \Sigma^p H(\mathrm{C})$$

between the homotopy groups of $H(\mathrm{C})$ and the cohomology groups of C on the one hand, and the shifts by $p \in \mathbb{Z}$ in spectra and chain complexes, on the other.

The Eilenberg–MacLane equivalence preserves limits. Hence it induces a map

$$H : \mathbf{Sh}_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf}) \to \mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf})$$

by objectwise application. For example, by (21) we have the sheaf

$$H(\iota(\sigma^{\geq p}\Omega)) \in \mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf}) . \tag{22}$$

We have $\pi_{-p}(H(\iota(\sigma^{\geq p}\Omega))) \cong \Omega^p_{cl}$.

2.2.2. Algebraic K-theory. In this paragraph we call some basic facts from algebraic K-theory. We let **Alg** denote the category of associative unital algebras. We have a functor

$$\mathbf{K} : \mathbf{Alg} \to \mathbf{Sp}$$

which maps an associative unital algebra to its connective algebraic K-theory spectrum $\mathbf{K}(A)$.

Remark 2.14. One way to construct this functor is as the following composition:

$$\mathbf{K}(A) := \mathrm{sp}(\mathrm{GrCompl}(\mathtt{N}(\mathrm{Iso}(\mathtt{Proj}(A))))) .$$

Here $\mathtt{Proj}(A)$ is the symmetric monoidal category of finitely generated projective A-modules with respect to the direct sum and Iso takes the underlying groupoid. The functor $\mathtt{N}$ maps a symmetric monoidal category to its nerve which is a commutative monoid in spaces. The group completion functor GrCompl turns this monoid into a commutative group (i.e. an E_∞-space) or equivalently into an infinite loop space. Finally, the functor sp maps the infinite loop space to the corresponding spectrum. We refer to [12, Sec. 6] and [11, Sec. 6] for more details. In the present paper will not need any explicit construction of the algebraic K-theory functor. □

Let $\mathbf{Alg}_{\mathbb{C}}$ denote the category of unital algebras over $\mathbb{C}$. We have a functor

$$CC^{-} : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Ch},$$

which maps an associative unital algebra A over $\mathbb{C}$ to its negative cyclic homology complex $CC^-(A)$. We define the negative cyclic homology of A by

$$HC_*^-(A) := H_*(CC^-(A)) . \tag{23}$$

Remark 2.15. For concreteness we will choose for $CC^-(A)$ the standard negative cyclic homology complex denoted by ToT $\mathcal{B}C^-$ in [28, 5.1.7]. Some constructions in the present paper will use this model explicitly. $\square$

We define the functor $\mathbf{CC}^-\colon \mathbf{Alg}_\mathbb{C} \to \mathbf{Sp}$ as the composition

$$\mathbf{Alg}_\mathbb{C} \stackrel{CC^-}{\to} \mathbf{Ch} \stackrel{\iota}{\to} \mathbf{Ch}[W^{-1}] \stackrel{H}{\to} \mathbf{Sp} \ .$$

We have a natural transformation of functors

$$\mathbf{ch}^{gj}\colon \mathbf{K} \to \mathbf{CC}^- \ , \tag{24}$$

given by the Goodwillie–Jones Chern character. For the construction of the Goodwillie –Jones Chern character we refer to [32] and [29, Sec. 5].

2.2.3. Algebraic K-theory sheaves. The goal of this paragraph is to introduce some basic notation which we will use throughout the rest of the paper. We consider a diffeological algebra A (see Definition 2.6) and form the presheaf of spectra

$$\check{\mathbf{K}}_A \in \mathbf{PSh_{Sp}}(\mathbf{Mf}) \ , \quad M \mapsto \mathbf{K}(A^\infty(M)) \ . \tag{25}$$

Its sheafification is a sheaf of spectra and will be denoted by

$$\mathbf{K}_A := L(\check{\mathbf{K}}_A) \in \mathbf{Sh_{Sp}}(\mathbf{Mf}) \ . \tag{26}$$

We apply these constructions to the diffeological algebra $\mathbb{C}$. We then have the equivalences

$$\check{\mathbf{K}}_{C^\infty(X)} \simeq i_X^* \check{\mathbf{K}}_\mathbb{C} \ , \quad \mathbf{K}_{C^\infty(X)} \simeq L i_X^* \check{\mathbf{K}}_\mathbb{C} \ . \tag{27}$$

The first follows from the definition of the diffeological structure on $C^\infty(X)$, and the second is then a reformulation of the definition above.

Applying the negative cyclic homology complex to the sheaf $\mathbb{C}^\infty$ we obtain the presheaf of chain complexes

$$CC^-(\mathbb{C}^\infty) \in \mathbf{PSh_{Ch}}(\mathbf{Mf}) \ , \quad M \mapsto CC^-(C^\infty(M)) \ .$$

Remark 2.16. Note that we do not complete or sheafify the tensor products involved in the definition of the negative cyclic homology complex, but see Remark 2.18. $\square$

In the following we define a differential geometric analog DD^- of $CC^-(\mathbb{C}^\infty)$ and a comparison map

$$\pi^-\colon CC^-(\mathbb{C}^\infty) \to DD^- \ .$$

Definition 2.17. We define the sheaf of chain complexes $DD^- \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf})$

$$DD^- := \prod_{p\in\mathbb{Z}} DD^-(p)\ , \quad DD^-(p) := (\sigma^{\geq p}\Omega)[2p]\ .$$

We further define a map of presheaves of chain complexes

$$\pi^-\colon CC^-(\mathbb{C}^\infty) \to DD^- \tag{28}$$

by

$$CC^-(\mathbb{C}^\infty(X))_{q,p} \ni f_0 \otimes \cdots \otimes f_{p-q} \mapsto \frac{1}{(p-q)!} f_0 df_1 \wedge \cdots \wedge df_{p-q}$$
$$\in F^{p-q}\Omega^{p-q}(X) \subset DD^-(p)(X)^{-p-q}\ .$$

Here the index $(\dots)_{q,p}$ refers to the component of the bicomplex $\mathcal{B}C^-$ in [28, 5.1.7]. Using the formulas [28, Sec. 2.3.2] we conclude that π^- is compatible with the differentials.

Remark 2.18. For a manifold X let $CC^{cont,-}(C^\infty(X))$ be the analog of $CC^-(\mathbb{C}^\infty(X))$ defined using completed (but not sheafified) tensor products. Then we have a factorization of π^-:

$$CC^-(\mathbb{C}^\infty(X)) \to CC^{cont,-}(C^\infty(X)) \stackrel{\pi^{cont,-}}{\to} DD^-(X)\ . \tag{29}$$

The second map is quasi-isomorphism by the well-known calculation of the continuous negative cyclic homology of the algebra of smooth functions on a smooth manifold. We will use the continuous version of cyclic homology and $\pi^{cont,-}$ in Subsection 3.5 below. □

We further define the presheaves

$$\mathbf{CC}^-_{\mathbb{C}} := H \circ \iota \circ CC^-(\mathbb{C}^\infty)\ , \quad \mathrm{DD}^- := \iota(DD^-)\ , \quad \mathtt{DD}^- := H \circ \mathrm{DD}^-\ . \tag{30}$$

The $\mathbf{Ch}[W^{-1}]$-valued presheaf DD^- is a sheaf by (22). As remarked above, the Eilenberg–MacLane functor H preserves sheaves. Therefore $\mathtt{DD}^-$ is a $\mathbf{Sp}$-valued sheaf.

By its naturality the Goodwillie–Jones Chern character (24) provides a map

$$\mathbf{ch}^{gj}\colon \check{\mathbf{K}}_{\mathbb{C}} \to \mathbf{CC}^-_{\mathbb{C}}\ .$$

between presheaves of spectra.

Definition 2.19. We define the regulator morphism $\mathtt{r\check{e}g}$ of presheaves of spectra as the composition

$$\mathtt{r\check{e}g}\colon \check{\mathbf{K}}_{\mathbb{C}} \stackrel{\mathbf{ch}^{gj}}{\to} \mathbf{CC}^-_{\mathbb{C}} \stackrel{\pi^-}{\to} \mathtt{DD}^-\ . \tag{31}$$

Furthermore, for a smooth manifold X, we define the morphism of sheaves of spectra

$$\mathtt{reg}_X\colon \mathbf{K}_{C^\infty(X)} \stackrel{(27)}{\simeq} Li^*_X \check{\mathbf{K}}_{\mathbb{C}} \stackrel{Li^*_X \check{\mathtt{reg}}}{\to} Li^*_X\, \mathtt{DD}^- \simeq i^*_X\, \mathtt{DD}^- \ . \tag{32}$$

The last equivalence in (32) follows from the fact that $\mathtt{DD}^-$ and therefore $i^*_X\,\mathtt{DD}^-$ are sheaves. □

Remark 2.20. In this paper we usually call transformations from K-theory to cyclic homology Chern characters, and transformations from K-theory to differential forms regulators. There is one exception, namely the usual Chern character from topological K-theory to cohomology with complex coefficients calculated by the de Rham cohomology.

2.3. Homotopification and regulator maps. For a sheaf G with values in a stable ∞-category (e.g. $\mathbf{Ch}[W^{-1}]$ or $\mathbf{Sp}$) we have a functorial homotopification fibre sequence of sheaves

$$\mathcal{A}(G) \to G \to \mathcal{H}(G) \to \Sigma\mathcal{A}(G)\ , \tag{33}$$

see [12, Def. 3.1]. The map $G \to \mathcal{H}(G)$ is the unit of the homotopification functor $\mathcal{H}$ introduced in (13), and $\mathcal{A}$ by definition takes the fibre of this unit. The sheaf G is homotopy invariant if and only if $\mathcal{A}(G) \simeq 0$.

Let A be a diffeological algebra (Definition 2.6) and $\mathbf{K}_A$ be as in (26).

Definition 2.21. We define the sheaves of spectra

$$\mathbf{K}^{top}_A := \mathcal{H}(\mathbf{K}_A)\ , \quad \mathbf{K}^{rel}_A := \mathcal{A}(\mathbf{K}_A)\ .$$

We call the evaluations $\mathbf{K}^{top}(A) := \mathbf{K}^{top}_A(*)$ and $\mathbf{K}^{rel}(A) := \mathbf{K}^{rel}_A(*)$ the topological and relative K-theory spectra of A.

Note that the topological and relative K-theory spectra depend on the diffeological structure on A. They fit into the fibre sequence of spectra

$$\mathbf{K}^{rel}(A) \to \mathbf{K}(A) \to \mathbf{K}^{top}(A) \to \Sigma\mathbf{K}^{rel}(A) \tag{34}$$

derived from (33) by evaluation at $*$.

Remark 2.22. A Fréchet algebra has a natural diffeological structure such that $A^\infty(M)$ is the algebra of smooth maps $M \to A$. In this case our definition of $\mathbf{K}^{top}(A)$ coincides with that given in [19, Sec. 3.1]. Indeed, in this reference the authors apply Quillen's $+$-construction to the classifying space of the simplicial group $GL(C^\infty(\Delta^\bullet, A))$. Using (16) we can identify the resulting space with $\Omega^\infty(\mathbf{s}(\mathbf{K}_A)(*))$. We now use the equivalence $\mathbf{s}(\mathbf{K}_A)(*) \simeq \mathbf{K}^{top}_A(*)$ which follows from the combination of (17) and (14). As a consequence, the relative K-theory of a Fréchet algebra defined in [19, Sec. 3.1] is isomorphic to our version.

More generally, if A is a complete locally convex algebra, then in [21, Def. 4.1.3] the notion of the diffeotopy K-theory spectrum was defined. This definition is just $\mathbf{s}(\mathbf{K}_A)(*)$ written down in different symbols. Therefore our topological or relative K-theory of a complete locally convex algebra also coincides with the diffeotopy or relative K-theory of [21]. $\square$

The homotopification of the sheaf $\mathrm{DD}^- \in \mathbf{Sh}_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf})$ defined in (30) can be calculated explicitly again in terms of differential forms. To this end we introduce the two-periodic de Rham complex.

$$DD^{per} := \prod_{p\in\mathbb{Z}} \Omega[2p] \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf}) \,. \tag{35}$$

Its cohomology

$$HP^*(X) := H^*(DD^{per}(X)) \tag{36}$$

is called the periodic cohomology of the manifold X. The periodicity is implemented by the shift isomorphism, which for $k \in \mathbb{Z}$ is given by

$$\iota_{2k}\colon HP^*(X) \xrightarrow{\cong} HP^{*+2k}(X)\,, \quad \iota_{2k}\big([\omega(p)]\big)_{p\in\mathbb{Z}} = \big([\omega(p-k)]\big)_{p\in\mathbb{Z}}\,. \tag{37}$$

We further define

$$\mathrm{DD}^{per} := \iota(DD^{per}) \in \mathbf{Sh}^h_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf})\,. \tag{38}$$

A priori we have $\mathrm{DD}^{per} \in \mathbf{PSh}_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf})$. In order to see that DD^{per} is a sheaf we use [12, Lemma 7.12]. Moreover, since Ω resolves the constant sheaf $\underline{\mathbb{C}}$, the sheaf $\iota(\Omega)$ is homotopy invariant. Consequently, the sheaf DD^{per} is homotopy invariant, too.

In view of Definition 2.17 of DD^- and (20) we have a natural inclusion of sheaves of chain complexes $DD^- \to DD^{per}$.

Lemma 2.23. *The induced map* $\mathrm{DD}^- \to \mathrm{DD}^{per}$ *is equivalent to the homopification morphisms of* DD^-. *In particular we have an equivalence* $\mathcal{H}(\mathrm{DD}^-) \simeq \mathrm{DD}^{per}$.

Proof. By [12, Lemma 7.15] we know that the inclusion $\iota(\sigma^{\geq p}\Omega) \to \iota(\Omega)$ is equivalent to the homotopification map. This implies that the natural inclusion $\mathrm{DD}^- \to \mathrm{DD}^{per}$ is equivalent to the homotopification map $\mathrm{DD}^- \to \mathcal{H}(\mathrm{DD}^-)$. $\square$

We now provide chain complex model for $\mathcal{A}(\mathrm{DD}^-)$. We define the sheaf of chain complexes

$$DD := \prod_{p\in\mathbb{Z}} DD(p) \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf})\,, \quad DD(p) := (\sigma^{\leq p}\Omega)[2p]\,. \tag{39}$$

It fits into exact sequence of sheaves of chain complexes

$$0 \to DD^{-} \to DD^{per} \to DD[2] \to 0\,. \tag{40}$$

The second map in this sequence is induced by the family of maps of chain complexes

$$DD^{per}(p) \cong \Omega[2p] \stackrel{(20)}{\to} (\sigma^{<p}\Omega)[2p] \cong \big((\sigma^{\leq p-1}\Omega)[2(p-1)]\big)[2] \cong DD(p-1)[2]\,. \tag{41}$$

By [12, Lemma 7.12] the object $\mathrm{DD} := \iota(DD)$ is a sheaf with values in $\mathbf{Ch}[W^{-1}]$. From (40) we get a fibre sequence

$$\cdots \to \mathrm{DD}[1] \to \mathrm{DD}^{-} \to \mathrm{DD}^{per} \to \mathrm{DD}[2] \to \cdots\,. \tag{42}$$

Lemma 2.23 implies that this sequence is equivalent to the homotopification sequence (33) applied to DD^{-}.

Corollary 2.24. *We have an equivalence* $\mathcal{A}(\mathrm{DD}^{-}) \simeq \mathrm{DD}[1]$.

We define the sheaves of spectra

$$\mathtt{DD}^{per} := H(\mathrm{DD}^{per})\,, \quad \mathtt{DD} := H(\mathrm{DD})\,.$$

If we apply H to the sequence (40), then we get the fibre sequence of spectra

$$\Sigma\,\mathtt{DD} \to \mathtt{DD}^{-} \to \mathtt{DD}^{per} \to \Sigma^2\,\mathtt{DD}\,. \tag{43}$$

Corollary 2.25. *The fibre sequence of spectra* (43) *is equivalent to the homotopification sequence* (33) *applied to* $\mathtt{DD}^{-}$.

Let now X be a manifold. If we apply the homotopification sequence (33) to the morphism $\mathtt{reg}_X\colon \mathbf{K}_{C^\infty} \to i_X^*\,\mathtt{DD}^{-}$, then we get the first two lines of the following diagram:

$$\begin{tikzcd}
\mathbf{K}^{rel}_{C^\infty(X)} \arrow[r, "\partial"] \arrow[d] \arrow[dd, bend left, dotted, "\mathtt{reg}^{rel}_X"] & \mathbf{K}_{C^\infty(X)} \arrow[r] \arrow[d, "\mathtt{reg}_X"] & \mathbf{K}^{top}_{C^\infty(X)} \arrow[r] \arrow[d] \arrow[dd, bend left, dotted, "\mathtt{reg}^{top}_X"] & \Sigma\mathbf{K}^{rel}_{C^\infty(X)} \arrow[d] \arrow[dd, bend left, dotted, "\mathtt{reg}^{rel}_X"] \\
\mathcal{A}(i_X^*\,\mathtt{DD}^{-}) \arrow[r] \arrow[d, dashed] & i_X^*\,\mathtt{DD}^{-} \arrow[r] \arrow[d, equal] & \mathcal{H}(i_X^*\,\mathtt{DD}^{-}) \arrow[r] \arrow[d, dashed, "1"] & \Sigma\mathcal{A}(i_X^*\,\mathtt{DD}^{-}) \arrow[d, dashed] \\
i_X^*\,\Sigma\,\mathtt{DD} \arrow[r] & i_X^*\,\mathtt{DD}^{-} \arrow[r] & i_X^*\,\mathtt{DD}^{per} \arrow[r] & i_X^*\,\Sigma^2\,\mathtt{DD}\,.
\end{tikzcd} \tag{44}$$

Since $i_X^*\,\mathtt{DD}^{per}$ is homotopy invariant we obtain the dashed morphism denoted by 1 and the lower middle square from the universal property of the homotopification. The lower part of the diagram is now defined by extension of the lower middle square to a morphism of fibre sequences. We use the diagram in order to define the relative and and topological regulator maps $\mathtt{reg}^{rel}_X$ and $\mathtt{reg}^{top}_X$ as indicated.

Remark 2.26. If X is a compact manifold, then by Lemma 2.4.4. we have equivalences

$$i_X^* \circ \mathcal{A} \simeq \mathcal{A} \circ i_X^* , \quad i_X^* \circ \mathcal{H} \simeq \mathcal{H} \circ i_X^* . \tag{45}$$

If we apply i_X^* to (43), then the resulting sequence

$$i_X^* \Sigma \mathtt{DD} \to i_X^* \mathtt{DD}^- \to i_X^* \mathtt{DD}^{per} \to i_X^* \Sigma^2 \mathtt{DD} \tag{46}$$

is equivalent to the homotopification sequence of $i_X^* \mathtt{DD}^-$. In particular, for a compact manifold X the lower two lines in (44) are equivalent, and we have the equivalences $\mathtt{reg}_X^{rel} \simeq \mathcal{A}(\mathtt{reg}_X)$ and $\mathtt{reg}_X^{top} \simeq \mathcal{H}(\mathtt{reg}_X)$. □

By its definition (32) we have the following factorization of the regulator $\mathtt{reg}_X$:

$$\mathbf{K}_{C^\infty(X)} \stackrel{\text{def}}{=} L i_X^* \check{\mathbf{K}}_{\mathbb{C}} \stackrel{(18)}{\to} i_X^* L \check{\mathbf{K}}_{\mathbb{C}} \stackrel{\text{def}}{=} i_X^* \mathbf{K}_{\mathbb{C}} \to i_X^* \mathtt{DD}^- \ . \tag{47}$$

Using the universal property of the homotopification morphism $\mathbf{K}_{C^\infty(X)} \to \mathbf{K}^{top}_{C^\infty(X)}$ we get the factorization of $\mathtt{reg}_X^{top}$:

$$\begin{tikzcd} \mathbf{K}^{top}_{C^\infty(X)} \arrow[r, "!!!"] \arrow[rr, bend left, dotted, "\mathtt{reg}_X^{top}"] & i_X^* \mathbf{K}^{top}_{\mathbb{C}} \arrow[r, "i_X^* \mathtt{reg}^{top}"] \arrow[d, "\simeq"] & i_X^* \mathtt{DD}^{per} \\ & i_X^* \underline{\mathbf{ku}} . \arrow[ur, "i_X^* \mathbf{ch}^{gj}"'] & \end{tikzcd} \tag{48}$$

The identification of $\mathbf{K}^{top}_{\mathbb{C}} \simeq \underline{\mathbf{ku}}$ follows from [12, Lemma 6.3], where $\underline{\mathbf{ku}}$ is the constant sheaf of spectra generated by the connective topological K-theory spectrum $\mathbf{ku} \simeq \mathbf{K}^{top}_{\mathbb{C}}(*)$ of $\mathbb{C}$. We define $\mathbf{ch}^{gj} \colon \underline{\mathbf{ku}} \to \mathtt{DD}^{per}$ so that the diagram commutes.

If we evaluate $i_X^* \mathtt{reg}^{top}$ at a point, identify $(i_X^* \mathbf{K}^{top}_{\mathbb{C}})(*) \simeq \underline{\mathbf{ku}}(X)$, and take homotopy groups, then we get a homomorphism (natural in X)

$$\mathbf{ch}^{gj} \colon \mathbf{ku}^*(X) \to \pi_{-*}(\mathtt{DD}^{per}(X)) \stackrel{(36)}{\cong} HP^*(X) \ .$$

The origin of this map is the Goodwillie–Jones Chern character $\mathbf{ch}^{gj}$, see (24). In this situation we also have the usual Chern character defined by Chern–Weil theory

$$\mathbf{ch}^{cw} \colon \mathbf{ku}^*(X) \to HP^*(X) \ .$$

We refer to [12, Sec. 6.1] for a construction of the Chern character $\mathbf{ch}^{cw}$ using differential cohomology methods. The following lemma is probably well known. For completeness we sketch an argument.

Lemma 2.27. *We have the equality of Chern character maps*

$$\mathbf{ch}^{gj} = \mathbf{ch}^{cw} \colon \mathbf{ku}^*(X) \to HP^*(X).$$

Proof. Since both Chern characters arise from maps between homotopy invariant sheaves of spectra they are characterized by their evaluation at the point. Since the target is rational they are determined by their actions on homotopy groups. Hence it suffices to check that

$$\mathbf{ch}^{gj} = \mathbf{ch}^{cw} \colon \mathbf{ku}^{\ell}(*) \to HP^{\ell}(*)$$

for all $\ell \in \mathbb{Z}$. One can now use the fact that both Chern characters are multiplicative (see [32] for the multiplicitivity of $\mathbf{ch}^{gj}$) in order to reduce to the case $\ell = 2$. For $\ell = 2$ we argue as follows. We know by an explicit calculation that

$$\mathbf{ch}^{gj} \colon \mathbf{K}_1(C^\infty(S^1)) \to H_1(CC^{\,cont,-}(C^\infty(S^1))) \cong \Omega^1(S^1)$$

maps the class $\iota(u) \in \mathbf{K}_1(C^\infty(S^1))$ of a unit $u \in C^\infty(S^1)^\times$ to the form

$$\frac{1}{2\pi i} d \log u \in \Omega^1(S^1)\,.$$

We now consider specifically the embedding $u \colon S^1 \to \mathbb{C}^*$. The class $\iota(u)$ is then mapped to the generator $x \in \mathbf{ku}^{-1}(S^1) \cong \mathbb{Z}$ under the composition

$$\mathbf{K}_1(C^\infty(S^1)) \to \mathbf{K}_1^{top}(C^\infty(S^1)) \xrightarrow{(48),!!!} \pi_1(\mathbf{K}_{\mathbb{C}}^{top}(S^1)) \cong \mathbf{ku}^{-1}(S^1)\,.$$

Therefore $\mathbf{ch}^{gj}(x) \in HP^{-1}(S^1)$ is given by the class $(c(p))_{p\in\mathbb{Z}} \in H^{-1}(DD^{per}(S^1))$ with

$$c(p) := \begin{cases} [\mathrm{vol}_{S^1}] \in H^{-1}(DD^{per}(p)(S^1)), & p = 1, \\ 0, & \text{else.} \end{cases}$$

where $\mathrm{vol}_{S^1} = \frac{1}{2\pi i} d \log u$ is the normalized volume form of S^1. It follows by suspension that $\mathbf{ch}^{gj} \colon \mathbf{ku}^{-2}(*) \to HP^{-2}(*)$ maps the generator of $\mathbf{ku}^{-2}(*) \cong \mathbb{Z}$ to the class $(b(p))_{p\in\mathbb{Z}} \in H^{-2}(DD^{per}(*))$ given by

$$b(p) := \begin{cases} [1] \in H^{-2}(DD^{per}(p)(*)), & p = 1, \\ 0, & \text{else.} \end{cases}$$

The same holds true for $\mathbf{ch}^{cw}$. □

2.4. The differential regulator map. In this subsection we introduce a version $\widehat{\mathbf{ku}}$ of the Hopkins–Singer differential cohomology associated to $\mathbf{ku}$ and a differential regulator map

$$\widehat{\mathtt{reg}}_X \colon \mathbf{K}(C^\infty(X)) \to \widehat{\mathbf{ku}}(X)\,.$$

Definition 2.28. We define the Hopkins–Singer differential connective complex K-theory $\widehat{\mathbf{ku}} \in \mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf})$ by

$$\begin{tikzcd} \widehat{\mathbf{ku}} \arrow[r, "R"] \arrow[d, "I"] & \mathtt{DD}^{-} \arrow[d] \\ \underline{\mathbf{ku}} \arrow[r, "\mathbf{ch}^{cw}"] & \mathtt{DD}^{per} . \end{tikzcd} \tag{49}$$

We further define the differential connective complex K-theory groups of a manifold X by

$$\widehat{ku}^{-*}(X) := \pi_*(\widehat{\mathbf{ku}}(X)) .$$

For a general discussion of Hopkins–Singer differential cohomology theories we refer to [12, Sec. 4.4]. The map I takes the underlying **ku**-class, and the map R is called the curvature morphism.

We fix a manifold X and recall the definition (32) of $\mathtt{reg}_X$ and the (Chern–Weil version) of the Chern character $\mathbf{ch}^{cw}$ which is equivalent to $\mathbf{ch}^{gj}$ by Lemma 2.27.

The middle square of the diagram (44) together with (47) and Lemma 2.27 gives a commutative square

$$\begin{tikzcd} \mathbf{K}_{C^\infty(X)} \arrow[r, "\mathtt{reg}_X"] \arrow[d] & i_X^* \mathtt{DD}^{-} \arrow[d] \\ i_X^* \underline{\mathbf{ku}} \arrow[r, "i_X^* \mathbf{ch}^{cw}"] & i_X^* \mathtt{DD}^{per} . \end{tikzcd} \tag{50}$$

Definition 2.29. For a smooth manifold X we define the differential regulator map

$$\widehat{\mathtt{reg}}_X \colon \mathbf{K}_{C^\infty(X)} \to i_X^* \widehat{\mathbf{ku}}$$

as the map between sheaves of spectra naturally induced by the square (50) and the universal property of the pull-back square (49). The evaluation of $\widehat{\mathtt{reg}}_X$ at a point gives a map of spectra

$$\widehat{\mathtt{reg}}_X \colon \mathbf{K}(C^\infty(X)) \to \widehat{\mathbf{ku}}(X) .$$

Remark 2.30. For commutative algebras it is possible to refine the Goodwillie–Jones Chern character $\mathbf{ch}^{gj}$ to a morphism between commutative ring spectra. The spectra occuring on the right corners of the diagrams (49) and (50) are obtained by an application of $H \circ \iota$ to sheaves of commutative differential graded algebras and therefore are commutative ring spectra, as well. Since the morphisms $\mathtt{reg}_X$ and $\mathbf{ch}^{cw}$ and the diagram (50) have refinements to morphisms between sheaves of commutative ring spectra, the differential regulator naturally becomes a morphism between commutative ring spectra, too.

The multiplicative structure is helpful if one wants to calculate σ_d of products like

$$\iota(u_1) \cup \cdots \cup \iota(u_d) \in K_d(C^\infty(X))$$

for a collection of invertible functions $(u_i)_{i=1,\dots,d}$ in $C^\infty(X)$. Since our main results do not use the multiplicative structure, in the present paper we will not discuss its details further. □

Remark 2.31. In general, for an algebraic K-theory class $x \in K_d(C^\infty(X))$ the differential K-theory class $\widehat{\mathtt{reg}}_X(x) \in \widehat{\mathbf{ku}}^{-d}(X)$ contains strictly more information than just the underlying topological class

$$x^{top} := I(\widehat{\mathtt{reg}}_X(x)) \in \mathbf{ku}^{-d}(X)$$

and the regulator

$$\mathtt{reg}_X(x) = R(\widehat{\mathtt{reg}}_X(x)) \in H^{-d}(DD^-(X)).$$

It is essentially this additional secondary information which we detect by the map σ_d in Theorem 1.1.

2.5. The construction of σ_d.

We consider an integer $d \in \mathbb{N}$ and a smooth manifold X such that its dimension satisfies $\dim(X) \le d-1$. The goal of the present subsection is to construct the homomorphism

$$\sigma_d \colon K_d(C^\infty(X)) \to \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X)$$

asserted in Theorem 1.1. The idea is to obtain this map by specializing the differential regulator map $\widehat{\mathtt{reg}}_X$.

For the moment let X be an arbitrary smooth manifold. We fix an integer $d \in \mathbb{Z}$. In the following definition the symbol R denotes the curvature map the differential cohomology theory $\widehat{\mathbf{ku}}$, see (49).

Definition 2.32. We define the subgroup of flat classes

$$\widehat{ku}^{-d}_{flat}(X) := \ker\left(R \colon \widehat{ku}^{-d}(X) \to \pi_d(\mathtt{DD}^-(X))\right).$$

The flat subgroup of a Hopkins–Singer differential cohomology theory can be calculated explicitly.

Lemma 2.33. *If* $\dim(X) \le d$, *then we have a natural isomorphism*

$$\widehat{ku}^{-d}_{flat}(X) \cong \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X). \tag{51}$$

Proof. The complex of sheaves of abelian groups Ω is a resolution of the constant sheaf $\underline{\mathbb{C}}$. Consequently, DD^{per} resolves the constant sheaf $\underline{\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]}$. We thus obtain an equivalence

$$\mathtt{DD}^{per} \simeq \underline{H\left(\iota\left(\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]\right)\right)}. \tag{52}$$

We have a fibre sequence of pull-back squares

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^{-1}\mathtt{DD}^- \arrow[r, equal] \arrow[d] & \Sigma^{-1}\mathtt{DD}^- \arrow[r] \arrow[d] & \underline{\mathbf{F}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \widehat{\mathbf{ku}} \arrow[r] \arrow[d] & \mathtt{DD}^- \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & 0 & \underline{\mathbf{ku}} \arrow[r] & \mathtt{DD}^{per} & \underline{\mathbf{ku}} \arrow[r] & \mathtt{DD}^{per} &
\end{tikzcd}$$

We calculate the fibre $\mathbf{F}$ of

$$\mathbf{ch}^{cw}\colon \mathbf{ku} \to H\big(\iota\big(\textstyle\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]\big)\big)\ . \tag{53}$$

Using the decomposition

$$\prod_{p\in\mathbb{Z}} \mathbb{C}[2p] \cong \prod_{p\in\mathbb{N}} \mathbb{C}[2p] \oplus \prod_{p=1}^{\infty} \mathbb{C}[-2p]$$

and the fact that $\mathbf{ku}$ is connective we get the decomposition of $\mathbf{F}$ into the sum of the fibres of the morphisms of spectra

$$\mathbf{ku} \to H\big(\iota\big(\textstyle\prod_{p\in\mathbb{N}} \mathbb{C}[2p]\big)\big)\ , \quad 0 \to H\big(\iota\big(\textstyle\prod_{p=1}^{\infty} \mathbb{C}[-2p]\big)\big)\ .$$

The fibre of the first morphism is equivalent to $\Sigma^{-1}\,\mathbf{ku}\,\mathbb{C}/\mathbb{Z}$, and the fibre of the second is $\Sigma^{-1} H(\iota(\prod_{p=1}^{\infty} \mathbb{C}[-2p]))$. Consequently

$$\mathbf{F} \simeq \Sigma^{-1}\,\mathbf{ku}\,\mathbb{C}/\mathbb{Z} \oplus \Sigma^{-1} H\big(\iota\big(\textstyle\prod_{p=1}^{\infty} \mathbb{C}[-2p]\big)\big)\ .$$

We have a long exact sequence

$$\cdots \to \pi_{d+1}(\mathtt{DD}^-(X)) \to \mathbf{F}^{-d}(X) \to \widehat{ku}^{-d}(X) \xrightarrow{R} \pi_d(\mathtt{DD}^-(X)) \to \cdots\ .$$

Note that

$$H^{-k}(DD^-(X)) \cong \prod_{p\in\mathbb{Z}} H^{-k}\big(\sigma^{\geq p}\Omega(X)[2p]\big) \cong \prod_{p\in\mathbb{Z}} H^{2p-k}(\sigma^{\geq p}\Omega(X))\ .$$

If this group is non-zero, then for some $p \in \mathbb{Z}$ we have the relations

$$2p - k \leq \dim(X)\ , \quad 2p - k \geq p\ .$$

This implies $\dim(X) \geq k$. Hence for $\dim(X) \leq d$ we have

$$\pi_{d+1}(\mathtt{DD}^-(X)) = H^{-d-1}(DD^-(X)) = 0\ .$$

Therefore $\widehat{ku}_{flat}^{-d}(X) \cong \mathbf{F}^{-d}(X)$. Since also

$$\Sigma^{-1} H\big(\iota\big(\textstyle\prod_{p=1}^{\infty} \mathbb{C}[-2p]\big)\big)^{-d}(X) = \textstyle\prod_{p=1}^{\infty} H^{-2p-d-1}(X;\mathbb{C}) = 0$$

we have the isomorphism

$$\mathbf{F}^{-d}(X) \cong \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X)\ .$$

□

We decompose the curvature morphism R in (49) into a product of components $R(p)$, $p \in \mathbb{Z}$, according to the product decomposition of DD^-, see (2.17). Similary we decompose $\mathtt{reg}_X$ into a product of components $\mathtt{reg}_X(p)$ for $p \in \mathbb{Z}$. Note that $\pi_d(\mathtt{DD}^-(d)(X)) \cong \Omega^d_{cl}(X)$.

Lemma 2.34. *If* $\dim(X) \le d$ *and* $x \in K_d(C^\infty(X))$, *then we have*

$$R(p)(\hat{\mathtt{reg}}_X(x)) = \begin{cases} 0, & p \neq d \\ \mathtt{reg}_X(d)(x) \in \Omega^d_{cl}(X), & p = d \end{cases} .$$

Proof. Let $x \in K_d(C^\infty(X)))$. For $p \in \mathbb{Z}$ the component $R(p)(x)$ of the curvature of x is represented by the class $\mathtt{reg}_X(p)(x) \in H^{-d}(DD^-(p)(X))$. By the calculations in the proof of Lemma 2.33 it can only be non-trivial if $p = d = \dim(X)$. Note that $H^{-d}(DD^-(d)(X)) \cong \Omega^d_{cl}(X)$. $\square$

Corollary 2.35. *If* $\dim(X) \le d - 1$, *then* $\hat{\mathtt{reg}}_X$ *maps to the subgroup of flat classes.*

In view of Lemma 2.33 we can make the following definition.

Definition 2.36. If $\dim(X) \le d-1$, then we define σ_d as the composition

$$\sigma_d \colon K_d(C^\infty(X)) \stackrel{\hat{\mathtt{reg}}_X}{\to} \widehat{ku}^{-d}_{flat}(X) \cong \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-1}(X)\,. \tag{54}$$

This finishes the proof of Theorem 1.1. $\square$

Remark 2.37. At the moment we have no example which shows that the map σ_d can be non-trivial if $\dim(X) < d-1$. We refer to Example 2.41, 7. and 8. for some vanishing results in this direction.

2.6. Restriction to relative K-theory and explicit calculations.

In this subsection we derive an explicit formula for the restriction of σ_d defined in 2.36 to topologically trivial classes. The result will be formulated as Corollary 2.39. It will be used in the proof of Theorem 1.6. The idea for the calculation of $\sigma_d(x)$ for a topologically trivial class $x \in K_d(X^\infty(X))$ is to use a deformation $\tilde{x}$ of x to zero. Using the homotopy formula for $\widehat{\mathbf{ku}}$ the class $\sigma_d(x)$ can be expressed in terms of a transgression of the curvature of $\hat{\mathtt{reg}}_X(\tilde{x})$.

Let X be a manifold. From the definition of $\widehat{\mathbf{ku}}$ and the differential regulator $\hat{\mathtt{reg}}_X$ we get the diagram

$$\begin{tikzcd}
\mathbf{K}^{rel}_{C^\infty(X)} \arrow[r, dotted, "\mathtt{reg}^{rel}_X"] \arrow[d, "\partial"] & i^*_X \Sigma\, \mathtt{DD} \arrow[r, equal] \arrow[d, "a"] & i^*_X \Sigma\, \mathtt{DD} \arrow[d] \\
\mathbf{K}_{C^\infty(X)} \arrow[r, "\hat{\mathtt{reg}}_X"] \arrow[d] & i^*_X \widehat{\mathbf{ku}} \arrow[r, "R"] \arrow[d, "I"] & i^*_X \mathtt{DD}^- \arrow[d] \\
\mathbf{K}^{top}_{C^\infty(X)} \arrow[r] & i^*_X \underline{\mathbf{ku}} \arrow[r, "\mathbf{ch}^{cw}"] & i^*_X \mathtt{DD}^{per}\,.
\end{tikzcd} \tag{55}$$

The upper right square follows from the fibre sequence (43) and the fact that the lower right square is a pull-back, namely the definition (49) of $\widehat{\mathbf{ku}}$. The columns are fibre sequences.

Remark 2.38. If X is a compact manifold, then the columns in (55) are instances of the homotopification sequence (33). Indeed, the whole diagram can then be obtained by applying the homotopification sequence to the middle raw. See Example 2.26. □

Corollary 2.39. *We have the equality*

$$a \circ \mathtt{reg}_X^{rel} = \hat{\mathtt{reg}}_X \circ \partial \colon K_d(C^\infty(X)) \to \widehat{ku}^{-d}(X)\,.$$

We assume that X is a manifold of dimension $\dim(X) = d-1$. We use the homotopy formula for $\widehat{\mathbf{ku}}$ in order to provide a formula for $\sigma_d(x)$ for certain topologically trivial classes $x \in K_d(C^\infty(X))$. We consider a class $\tilde{x} \in K_d(C^\infty(I \times X))$ which has the property that $\tilde{x}_{|\{0\}\times X} = 0$. We define

$$x := \tilde{x}_{|\{1\}\times X} \in K_d(C^\infty(X))\,.$$

By construction the class x is topologically trivial, i.e. it belongs to the kernel of $K_d(C^\infty(X)) \to K_d^{top}(C^\infty(X))$.

Since the degree of $\tilde{x}$ and the dimension of $I \times X$ match, by Lemma 2.34 the only non-trivial component of the regulator of $\tilde{x}$ is

$$\mathtt{reg}_{I\times X}(d)(\tilde{x}) \in \Omega^d_{cl}(I \times X)\,.$$

Furthermore we have a map

$$i_d \colon \Omega^{d-1}_{cl}(X) \to H^{d-1}\big(\sigma^{<d}\Omega(X)\big) \to H^{-d-1}\big((\sigma^{<d}\Omega(X))[2d]\big) \to \pi_d(\Sigma\,\mathtt{DD}(X))\,.$$

Proposition 2.40. *We have*

$$\hat{\mathtt{reg}}_X(x) = a\left(i_d \int_I \mathtt{reg}_{I\times X}(d)(\tilde{x})\right).$$

Proof. By the homotopy formula [12, (27)] we have

$$\hat{\mathtt{reg}}_X(x) = a\left(\int_I R\big(\hat{\mathtt{reg}}_{I\times X}(\tilde{x})\big)\right).$$

By Lemma 2.34 we have

$$R(d)\big(\hat{\mathtt{reg}}_{I\times X}(\hat{x})\big) = \mathtt{reg}_{I\times X}(d)(\tilde{x})\,,$$

and all other factors of the curvature vanish. So our final formula is

$$\hat{\mathtt{reg}}_X(x) = a\left(i_d \int_I \mathtt{reg}_{I\times X}(d)(\tilde{x})\right). \qquad \square$$

Example 2.41. We now perform some explicit calculations:

1. We consider $X = S^1$. Let $u: S^1 \to \mathbb{C}$ be the inclusion. We consider u as a unit in $C^\infty(S^1)$ and $\iota(u) \in K_1(C^\infty(S^1))$. Note that

$$\pi_1(\mathtt{DD}^-(S^1)) \cong \pi_1(\mathtt{DD}^-(1)(S^1)) \cong \Omega^1(S^1) .$$

Under this identification have

$$\mathtt{reg}_{S^1}(1)(\iota(u)) = \frac{1}{2\pi i}\frac{du}{u} .$$

2. Next we consider the inclusion $t: \mathbb{R}_+ \to \mathbb{C}$ as a unit in $C^\infty(\mathbb{R}_+)$. We obtain

$$\mathtt{reg}_{\mathbb{R}_+}(1)(\iota(t)) = \frac{1}{2\pi i}\frac{dt}{t} .$$

3. We now consider the inclusion $z: \mathbb{C}^* \to \mathbb{C}$. We write $z = u(z)t(z)$ with $u := z/|z|^{-1}$ and $t := |z|$. Then we get by naturality and additivity of the regulator

$$\mathtt{reg}_{\mathbb{C}^*}(1)(\iota(z)) = \frac{1}{2\pi i}\left(\frac{d(z/|z|^{-1})}{z/|z|^{-1}} + \frac{d|z|}{|z|}\right) = \frac{1}{2\pi i}\frac{dz}{z} .$$

4. We consider the unit $\exp(z) = \exp^* z \in C^\infty(\mathbb{C})$. We get

$$\mathtt{reg}_{\mathbb{C}}(1)(\iota(\exp(z)) = \exp^*(\mathtt{reg}_{\mathbb{C}^*}(1)(\iota(z))) = \frac{1}{2\pi i}dz .$$

5. Let now X be a smooth manifold and $f \in C^\infty(X)$. Then we have

$$\mathtt{reg}_X(1)(\iota(\exp(f)) = f^*(\mathtt{reg}_{\mathbb{C}}(1)(\iota(\exp(z))) = \frac{1}{2\pi i}df . \tag{56}$$

6. The regulator $\mathtt{reg}_X: K_*(C^\infty(X)) \to \pi_*(\mathtt{DD}^-(X))$ is given by the composition of the Goodwillie–Jones Chern character and the map (28). Since $C^\infty(X)$ is commutative both maps are in fact multiplicative, where the ring structure on $\mathtt{DD}^-$ is induced by the obvious bigraded differential algebra structure on $\prod_{p\in\mathbb{Z}}(\sigma^{\geq p}\Omega)[2p]$.

Now assume that $\dim(X) = d-1$. We consider a collection $u_2, \ldots, u_d$ of units in $C^\infty(X)$. Let $t \in I$ be the coordinate. On $I \times X$ we consider the collection of d units

$$\exp(tf), u_2, \ldots, u_d$$

and define

$$\tilde{x} := \iota(\exp(tf)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d) \in K_d(C^\infty(I \times X)) .$$

By multiplicativity of the regulator and (56) we have

$$\mathtt{reg}_{I\times X}(d)\big(\iota(\exp(tf))\cup\iota(u_2)\cup\cdots\cup\iota(u_d)\big) = \frac{1}{(2\pi i)^d}(fdt+tdf)\wedge\frac{du_2}{u_2}\wedge\cdots\wedge\frac{du_d}{u_d}$$

(and this is the only non-trivial component). From Proposition 2.40 we conclude that

$$\widehat{\mathtt{reg}}_X\big(\iota(\exp(f)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d)\big) = a\left(i_d\left(\frac{1}{(2\pi i)^d} f \frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_d}{u_d}\right)\right). \tag{57}$$

7. If we consider in 6. more than d factors, then we get a trivial result. Assume that $\dim(X) = d-1$ and $u_2, \ldots, u_{d+1}$ are invertible smooth functions, and f is a smooth function. Then we have

$$\widehat{\mathtt{reg}}_X\big(\iota(\exp(f)) \cup \iota(u_2) \cup \cdots \cup \iota(u_{d+1})\big) = 0\,.$$

To this end, using multiplicativity of $\widehat{\mathtt{reg}}_X$, we write the left-hand side as a product in $\widehat{\mathbf{ku}}$-theory

$$\widehat{\mathtt{reg}}_X\big(\iota(\exp(f)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d)\big) \cup \widehat{\mathtt{reg}}_X(\iota(u_{d+1}))\,.$$

Using (57) and the rule $a(\omega) \cup x = a(\omega \wedge R(x))$ for the product in $\widehat{\mathbf{ku}}$ we rewrite this as

$$\begin{aligned} a\left(i_d\left(\frac{1}{(2\pi i)^d} f \frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_d}{u_d}\right)\right) &\cup \widehat{\mathtt{reg}}_X(\iota(u_{d+1})) \\ &= a\left(i_{d+1}\left(\frac{1}{(2\pi i)^{d+1}} f \frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_{d+1}}{u_{d+1}}\right)\right). \end{aligned}$$

The argument of a on the right-hand side is a d-form on a manifold of dimension $d-1$ and therefore vanishes.

8. Let again $\dim(X) \le d-1$, $x \in K_d(C^\infty(X))$ and $y \in K_{d'}(C^\infty(X))$ for some $d' \in \mathbb{N}$. Then

$$\widehat{\mathtt{reg}}_X(x \cup y) = \widehat{\mathtt{reg}}_X(x) \cup \widehat{\mathtt{reg}}_X(y) = \sigma_d(x) \cup y^{\mathit{top}}\,,$$

where $y^{\mathit{top}} := I(\widehat{\mathtt{reg}}_X(y)) \in \mathbf{ku}^{-d'}(X)$, using the $\mathbf{ku}$-module structure of $\mathbf{ku}\,\mathbb{C}/\mathbb{Z}$ on the right-hand side.

Note that $\mathbf{ku}(X)$ has a multiplicative Atiyah–Hirzebruch filtration $(F^p\,\mathbf{ku}(X))_{p\in\mathbb{Z}}$, where $F^p\,\mathbf{ku}(X)$ consists classes whose pull-backs to $p-1$-dimensional CW-complexes vanish. For an invertible function u we have $\iota(u)^{\mathit{top}} \in F^1\,\mathbf{ku}^{-1}(X)$. If $y = \iota(u_1) \cup \cdots \cup \iota(u_d)$, then $y^{\mathit{top}} = 0$. We conclude that

$$\widehat{\mathtt{reg}}_X\big(x \cup \iota(u_1) \cup \cdots \cup \iota(u_d)\big) = 0\,. \qquad \square$$

3. An index theorem

3.1. The index pairing. In this subsection we introduce the pairing between Dirac operators and the Hopkins–Singer version of differential periodic complex K-theory.

This pairing refines the pairing between periodic complex K-theory and the K-homology. If the Dirac operator comes from a Spin^c-structure, then our pairing is a special case of the integration in differential cohomology. In any case, the pairing has a simple description in terms of standard constructions of local index theory.

First we introduce the Hopkins–Singer version $\widehat{KU}^*(-)$ of differential periodic complex K-theory $\mathbf{KU}$. By

$$\mathbf{ch}^{cw}_{per} \colon \mathbf{KU} \to H\big(\iota\big(\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]\big)\big)$$

we denote the usual Chern character. We will use the same symbol also in order to denote the composition of morphisms of sheaves of spectra

$$\mathbf{ch}^{cw}_{per} \colon \underline{\mathbf{KU}} \stackrel{\mathbf{ch}^{cw}_{per}}{\to} \underline{H\big(\iota\big(\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]\big)\big)} \stackrel{(52)}{\simeq} \mathtt{DD}^{per} \ .$$

Remark 3.1. The Chern character $\mathbf{ch}^{cw}$ discussed in Lemma 2.27 is equivalent to the composition

$$\mathbf{ku} \to \mathbf{KU} \stackrel{\mathbf{ch}^{cw}_{per}}{\to} H\big(\iota\big(\prod_{p\in\mathbb{Z}} \mathbb{C}[2p]\big)\big) \ ,$$

where the first map

$$\mathbf{ku} \to \mathbf{KU} \tag{58}$$

is the connective covering map of $\mathbf{KU}$. □

We set

$$^{\geq 0}\mathtt{DD}^{per} := H\big(\iota\big(\sigma^{\geq 0} DD^{per}\big)\big) \ .$$

The inclusion of sheaves of chain complexes $\sigma^{\geq 0} DD^{per} \to DD^{per}$ induces a morphism of sheaves of spectra $^{\geq 0}\mathtt{DD}^{per} \to \mathtt{DD}^{per}$.

Definition 3.2. We define the Hopkins–Singer differential cohomology theory associated to $\mathbf{KU}$ as the sheaf of spectra $\widehat{\mathbf{KU}} \in \mathbf{Sh_{Sp}}(\mathbf{Mf})$ given by the following pull-back

$$\begin{tikzcd} \widehat{\mathbf{KU}} \arrow[r, "R"] \arrow[d, "I"] & {}^{\geq 0}\mathtt{DD}^{per} \arrow[d] \\ \underline{\mathbf{KU}} \arrow[r, "\mathbf{ch}^{cw}_{per}"] & \mathtt{DD}^{per} \ . \end{tikzcd}$$

We further define the differential periodic complex K-theory groups of a manifold X by

$$\widehat{KU}^*(X) := \pi_{-*}\big(\widehat{\mathbf{KU}}(X)\big)$$

(compare with Definition 2.28).

The differential periodic complex K-theory in degree zero fits into the exact sequence

$$\mathbf{KU}^{-1}(X) \stackrel{\mathbf{ch}^{cw}_{per}}{\to} DD^{per}(X)^{-1}/\mathtt{im}(d) \stackrel{a_{\widehat{\mathbf{KU}}}}{\to} \widehat{KU}^{0}(X) \stackrel{I}{\to} \mathbf{KU}^{0}(X) \to 0\,. \tag{59}$$

This exact sequence is one of the basic features of differential periodic complex K-theory, see e.g. [13, Prop. 2.20].

By [14] for a compact manifold X the group $\widehat{KU}^{0}(X)$ is canonically isomorphic to the differential K-theory groups defined using geometric models [13,34]. In these models geometric vector bundles are cycles for differential K-theory classes. Recall that a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\mathbf{V} := (V, h, \nabla)$ is a triple consisting of a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle $V \to X$, a hermitean metric h on V such that the even and odd summands are orthogonal, and a connection ∇ which preserves h and the grading. In the geometric models for $\widehat{KU}^{0}(X)$ a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\mathbf{V} := (V, h, \nabla)$ tautologically represents a class

$$[\mathbf{V}] \in \widehat{KU}^{0}(X)\,.$$

We refer to [12, Sec. 6.1] for an alternative construction of this class in terms of a cycle map.

We now assume that X is a closed Riemannian manifold of odd dimension d. We further assume that we are given a generalized Dirac operator $\not{D}$ on X. By definition, $\not{D}$ is the Dirac operator associated to a Dirac bundle, see e.g. [9, Sec. 3.1].

Remark 3.3. A generalized Dirac operator provides a K-homology class which can be paired with K-theory classes on X. The basic idea of the following Lemma is that the Dirac operator as a geometric object provides a sort of differential refinement of its K-homology class which can be paired with differential K-theory classes. The map $\rho_{\not{D}}$ defined in Proposition 3.4 below only captures the secondary information contained in this pairing. Its value on a differential K-theory class can be considered as the reduced η-invariant of the Dirac operator twisted with this class. A very similar construction has been used in order to define the intrinsic universal η invariant in [10]. □

Proposition 3.4. *We have a canonical evaluation map*

$$\rho_{\not{D}} \colon \widehat{KU}^{0}(X) \to \mathbb{C}/\mathbb{Z}\,.$$

Proof. Let $x \in \widehat{KU}^{0}(X)$. In view of the sequence (59) we can choose a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\mathbf{V} := (V, h, \nabla)$ and a form $\gamma \in DD^{per}(X)^{-1}/\mathtt{im}(d)$ such that the following identity holds true in $\widehat{KU}^{0}(X)$:

$$x = [\mathbf{V}] + a_{\widehat{\mathbf{KU}}}(\gamma)\,.$$

We need the following standard constructions from local index theory:

1. We form the twist $\not{D} \otimes \mathbf{V}$ of the Dirac operator by $\mathbf{V}$ (see [9, Sec. 3.1] for details if necessary).

2. We let $\xi(\not{D} \otimes \mathbf{V}) \in \mathbb{R}/\mathbb{Z}$ denote the reduced η-invariant of $\not{D} \otimes \mathbf{V}$ given by

$$\xi(\not{D} \otimes \mathbf{V}) := \left[\frac{\eta(\not{D} \otimes \mathbf{V}) + \dim(\ker(\not{D} \otimes \mathbf{V}))}{2}\right], \tag{60}$$

where $\eta(\not{D} \otimes \mathbf{V})$ is the Atiyah–Patodi–Singer η-invariant introduced in [1].

3. We let $\hat{\mathbf{A}}(\not{D}) \in \prod_{p \in \mathbb{Z}} (\Omega(X, \Lambda)[2p])^0_{cl}$ denote the local index form associated to $\not{D}$, where Λ denotes the orientation twist of X.

Remark 3.5. The local index form has the following explicit description. Locally on X we can write $\not{D} = \not{D}_{spin} \otimes \mathbf{E}$ for the spin Dirac operator $\not{D}_{spin}$ and a geometric $\mathbb{Z}/2\mathbb{Z}$-graded twisting bundle $\mathbf{E} = (E, h^E, \nabla^E)$. If we can write $\not{D}$ in this way, then

$$\hat{\mathbf{A}}(\not{D}) = \big([\hat{\mathbf{A}}(\nabla^{LC}) \wedge \mathbf{ch}(\nabla^E)]_{2p}\big)_{p \in \mathbb{Z}} \in \prod_{p \in \mathbb{Z}} \big(\Omega(X, \Lambda)[2p]\big)^0_{cl} ,$$

where ∇^{LC} is the Levi-Civita connection of X, $[\omega]_{2p}$ denotes the degree-$2p$-component of the inhomogeneous even form ω, and $\hat{\mathbf{A}}(\nabla)$ and $\mathbf{ch}(\nabla)$ are the usual characteristic forms defined in terms of the curvature of the connections (including the $2\pi i$-factors), see [9, Sec. 4.3] for explicit formulas.

The following observation will make it unnecessary to use the explicit formula for the index density. This fact will be particularly helpful in the proof of Lemma 3.19 below. We define the integral

$$\int_X : \prod_{p \in \mathbb{Z}} \Omega(X, \Lambda)[2p] \to \mathbb{C} , \quad \int_X (\omega(p))_{p \in \mathbb{Z}} := \sum_{p \in \mathbb{Z}} \int_X [\omega(p)]_{\dim(X)} . \tag{61}$$

It induces an evaluation of cohomology classes which we will denote by the same symbol. By the Atiyah–Singer index theorem we can calculate for every class $u \in \mathbf{KU}^{-1}(X)$ the index pairing by

$$\langle [u], [\not{D}] \rangle = \int_X [\hat{\mathbf{A}}(\not{D})] \cup \iota_{d+1} \mathbf{ch}^{cw}_{per}(u) \in \mathbb{Z} , \tag{62}$$

where ι_{d+1} is the shift isomorphism defined in (37). □

Using the integral (61) we now define

$$\rho_{\not{D}}(x) := \xi(\not{D} \otimes \mathbf{V}) + \left[\int_X \hat{\mathbf{A}}(\not{D}) \wedge \iota_{d+1} \gamma\right] \in \mathbb{C}/\mathbb{Z} . \tag{63}$$

We must show that $\rho_{\not{D}}$ is a well-defined homomorphism.

1. First observe that the right-hand side of (63) does not depend on the choice of γ. Indeed, by (59) two choices differ by a closed form representing an element in the image of $\mathbf{ch}^{cw}_{per}\colon \mathbf{KU}^{-1}(X) \to DD^{per}(X)^{-1}/\,\mathtt{im}(d)$, and the integral of the product of those elements with $\hat{\mathbf{A}}(\not{D})$ is an integer by the Atiyah–Singer index theorem, see Remark 3.5.

2. We observe that the right-hand side of (63) is invariant under stabilization by bundles which admit an odd $\mathbb{Z}/2\mathbb{Z}$ symmetry.

3. Next we observe, using the variation formula for the classes $[\mathbf{V}]$ (the homotopy formula for $\widehat{KU}^0$) and $\xi(\not{D} \otimes \mathbf{V})$, that the right-hand side of (63) does not depend on the choice of the geometry of V.

4. If we choose a different bundle $\mathbf{V}'$ and form γ', then after stabilization we can assume that $V \cong V'$ as graded bundles. Therefore the right-hand side of (63) does not depend on the choice of $\mathbf{V}$.

5. Finally we observe that $\rho_{\not{D}}$ is a homomorphism. □

Remark 3.6. For any integer $k \in \mathbb{Z}$ we can define a version $\widehat{KU}^{k,*}$ of differential K-theory by replacing the cut-off ${}^{\geq 0}$ in Definition 3.2 with ${}^{\geq k}$. Assume that X is a Riemannian spin manifold of odd dimension d. Then the map $p\colon X \to *$ is differentially K-oriented and we have an integration

$$\hat{p}_!\colon \widehat{KU}^{0,0}(X) \to \widehat{KU}^{-d,-d}(*) \cong \mathbb{C}/\mathbb{Z}\ .$$

We refer to [13, Sec. 3] and [8, Sec. 4.10 and 4.11] for details on the integration in differential cohomology.

Let us now assume that $\not{D} = \not{D}_{spin} \otimes \mathbf{E}$ for some twist $\mathbf{E} = (E, h^E, \nabla^E)$. From [13, Cor. 5.5] we conclude that $\rho_{\not{D}}$ can be expressed in terms of the integration $\hat{p}_!$ in differential K-theory as follows

$$\rho_{\not{D}}(x) = \hat{p}_!\big([\mathbf{E}] \cup x\big)\ , \quad x \in \widehat{KU}^0(X)\ .$$

The spin structure on X provides the underlying topological K-orientation of X given by the fundamental class $[\not{D}_{spin}] \in \mathbf{KU}_d(X)$. The restriction of $\rho_{\not{D}}$ to the flat subgroup corresponds under the identification

$$\widehat{KU}^0_{flat}(X) \cong \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X) \tag{64}$$

(this is the analog of Lemma 2.33) to the evaluation pairing

$$\begin{aligned}\langle - \cup [E], [\not{D}_{spin}]\rangle\colon \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X) &\xrightarrow{-\cup[E]} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X)\\ &\xrightarrow{\langle -,[\not{D}_{spin}]\rangle} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-d-1}(*) \cong \mathbb{C}/\mathbb{Z}\ ,\end{aligned}$$

where the first map uses the $\mathbf{KU}$-module structure of $\mathbf{KU}\,\mathbb{C}/\mathbb{Z}$. □

Remark 3.7. In this remark we explain the relation between the evaluation map $\rho_{\not{D}}$ and the index theorem for flat vector bundles by Atiyah–Patodi–Singer [2, Thm. 5.3]. Let $\mathbf{V}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded flat geometric bundle of virtual dimension zero. It represents a class $[\mathbf{V}] \in \widehat{KU}^{0}_{flat}(X) \cong \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X)$. In this case

$$\rho_{\not{D}}([\mathbf{V}]) = \xi(\not{D} \otimes \mathbf{V})$$

is exactly the analytic index of the flat bundle introduced by Atiyah–Patodi–Singer. Their index theorem for flat bundles states that this analytic index is equal to the pairing of the class $[\mathbf{V}]$ with the K-homology class of $\not{D}$. This also follows from the last assertion in Remark 3.6. □

Example 3.8. We consider S^1 as a spin manifold with the standard metric of length 1 and with the non-bounding spin structure. The spinor bundle is one-dimensional and can be trivialized such that $\not{D}_{spin} = i\partial_t$. Assume now that L is a geometric line bundle with holonomy $v \in U(1)$. Then we can trivialize L such that its connection is given by $\nabla^L = d - \log(v)dt$. We get

$$\not{D}_{spin} \otimes \mathrm{L} = i(\partial_t - \log(v))\ .$$

Its spectrum is $\{2\pi n - \log v \mid n \in \mathbb{Z}\}$ with multiplicity 1. For $v \neq 1$ we get by an explicit calculation

$$\eta(\not{D}_{spin} \otimes \mathrm{L}) = 1 - \frac{\log(v)}{\pi i}\ ,$$

where the branch of the logarithm is chosen such that $\frac{\log(v)}{\pi i} \in (0,2)$. Using (60) and (63) we get

$$\rho_{\not{D}_{spin}}([\mathrm{L}]) = \left[\frac{1}{2} - \frac{\log(v)}{2\pi i}\right]_{\mathbb{C}/\mathbb{Z}}$$

because in this case we can take $\gamma = 0$. This formula holds true also for $v = 1$.

If L is trivial, then $[\mathrm{L}] = 1$ and we have $\rho_{\not{D}_{spin}}(1) = [\frac{1}{2}]$. We have an isomorphism

$$\widehat{\mathbf{KU}}^{0,0}_{flat}(S^1) \cong \mathbb{C}/\mathbb{Z}\,,$$

which maps $[\mathrm{L}] - 1$ to $\frac{\log v}{2\pi i}$. With this identification the restriction of the evaluation map to the flat subgroup is given by the homomorphism

$$\rho_{\not{D}_{spin}}: \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}\,, \quad [z] \mapsto [-z]\,.$$ □

3.2. Fredholm modules. We consider a closed Riemannian manifold X of odd dimension d with a generalized Dirac operator $\not{D}$ associated to a Dirac bundle $\mathbf{E}$. The Dirac operator $\not{D}$ gives rise to a $d+1$-summable Fredholm module (H, P) over $C^\infty(X)$ as follows (see [18]):

1. The Hilbert space of the Fredholm module is $H := L^2(X, E)$. The algebra $C^\infty(X)$ acts on H in the usual way by multiplication operators.
2. The operator $P \in B(H)$ is the orthogonal projection P^+ onto the positive eigenspace of $\not{D}$.
3. The condition that (H, P) is $d+1$-summable means that

$$[P, f] \in \mathcal{L}^{d+1}(H) ,$$

for all $f \in C^\infty(X)$, where $\mathcal{L}^{d+1}(H)$ denotes the $d+1$th Schatten class.

Remark 3.9. In some references odd Fredholm modules are denoted by (H, F), where $F \in B(H)$ is a selfadjoint involution such that $[F, f] \in \mathcal{L}^{d+1}(H)$. The relation with our notation is given by the equation $F = P - (1 - P)$.

We let $\mathcal{M}^d$ be the universal algebra for $d+1$-summable Fredholm modules introduced by Connes–Karoubi [19]. Then we get a homomorphism

$$b_{\not{D}} : C^\infty(X) \to \mathcal{M}^d \tag{65}$$

classifying the Fredholm module (H, P). Note that $b_{\not{D}}$ is uniquely determined up to unitary equivalence.

Remark 3.10. In this remark we give an explicit description of $b_{\not{D}}$. The algebra $\mathcal{M}^d$ for odd $d \in \mathbb{N}$ is a subalgebra of the algebra of 2×2-matrices of bounded operators on the standard separable Hilbert space ℓ^2 consisting of the matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \quad a_{12}, a_{21} \in \mathcal{L}^{d+1}(\ell^2) , \quad a_{11}, a_{22} \in B(\ell^2) . \tag{66}$$

Let P^+, P^- be the positive and non-positive spectral projections of $\not{D}$. Then we choose identifications $\ell^2 \cong \mathtt{im}(P^+) \cong \mathtt{im}(P^-)$. The homomorphism $b_{\not{D}} : C^\infty(X) \to \mathcal{M}^d$ is then given by

$$f \mapsto \begin{pmatrix} P^+ f P^+ & P^+ f P^- \\ P^- f P^+ & P^- f P^- \end{pmatrix} . \tag{67}$$

$\square$

3.3. The multiplicative character. In [19, 4.10] Connes and Karoubi constructed the "multiplicative" character

$$\delta : K_{d+1}(\mathcal{M}^d) \to \mathbb{C}/\mathbb{Z} . \tag{68}$$

In this subsection we explain how the construction of the multiplicative character δ fits into the framework of differential cohomology. The details of the construction will be needed later in Subsection 3.6.

We consider a unital locally convex algebra A. It has a natural diffeological structure A^∞, see Example 2.8, 5. From the sheaf of algebras A^∞ we derive the sheaves of spectra

$$\mathbf{CC}^-_A := L(H(\iota(CC^-(A^\infty)))) , \quad \mathbf{K}_A \overset{(26)}{=} L(\mathbf{K}(A^\infty)) .$$

The Goodwillie–Jones Chern character (24) gives a morphism

$$\mathbf{ch}^{gj} : \mathbf{K}_A \to \mathbf{CC}^-_A \ . \tag{69}$$

Remark 3.11. In principle we want to apply the homotopification sequence (33) to $\mathbf{ch}^{gj}$. This leads to the problem of understanding the homotopification of $\mathbf{CC}^-_A$. The known facts are contained e.g. in [21, Cor. 4.1.2]. In particular the homotopification of $\mathbf{CC}^-_A$ is equivalent to the homotopification of its periodic version $\mathbf{CC}^{per}_A$, and the homotopification of the cyclic homology $\mathbf{CC}_A$ vanishes. The problem is that $\mathbf{CC}^{per}_A$ is not known to be homotopy invariant. We will get a better theory if we use the continuous versions of cyclic homology. The main advantage is that the continuous periodic cyclic homology for complete locally convex algebras is known to be diffeotopy invariant, see Theorem 3.12. □

If we define the cyclic bicomplex $\mathcal{B}C^{cont}(A)$ of a locally convex algebra A similarly as in [28, 5.1.7] but using projective tensor products, then we get the continuous versions of cyclic, negative cyclic and and periodic cyclic homology complexes

$$CC^{cont}(A) , \quad CC^{cont,-}(A) , \quad CC^{cont,per}(A) . \tag{70}$$

In the natural extension of the notation of [28, 5.1.7] to the continuous case these complexes would have been denoted by Tot $\mathcal{B}C^{cont}$, ToT $\mathcal{B}C^{cont,-}$, and ToT $\mathcal{B}C^{cont,per}$. We have an exact sequence of chain complexes

$$0 \to CC^{cont,-}(A) \to CC^{cont,per}(A) \xrightarrow{q} CC^{cont}(A)[2] \to 0 . \tag{71}$$

Note that A^∞ is a presheaf of locally convex algebras by Remark 2.9. In (71) we can thus replace A by A^∞ in order to get an exact sequence of presheaves with values in $\mathbf{Ch}$. Then we apply $L \circ H \circ \iota$ in order to get the fibre sequence of sheaves of spectra

$$\Sigma\, \mathbf{CC}^{cont}_A \to \mathbf{CC}^{cont,-}_A \to \mathbf{CC}^{cont,per}_A \to \Sigma^2\, \mathbf{CC}^{cont}_A , \tag{72}$$

which is very similar to (43).

We now use the well-known fact that the continuous periodic cyclic homology is diffeotopy invariant [27, Theoreme 2.7] (for Fréchet algebras) and [38] (for complete locally convex algebras):

Theorem 3.12. *Assume that A is a complete locally convex algebra. Then the projection $I \times M \to M$ induces a quasi-isomorphism*

$$CC^{cont,per}(C^\infty(M, A)) \to CC^{cont,per}(C^\infty(I \times M, A)) .$$

As a consequence, the sheaf $\mathbf{CC}_A^{cont,per}$ is homotopy invariant in the sense of Definition 2.3.

From now on we assume that A is complete. We apply the homotopification sequence (33) to the morphism (69). Using the Definition 2.21 we obtain the upper two columns of the following diagram:

$$\begin{tikzcd}
\mathbf{K}_A^{rel} \arrow[r] \arrow[d, "\mathcal{A}(\mathbf{ch}^{gj})"] & \mathbf{K}_A \arrow[r] \arrow[d, "\mathbf{ch}^{gj}"] & \mathbf{K}_A^{top} \arrow[r] \arrow[d, "\mathcal{H}(\mathbf{ch}^{gj})"] & \Sigma\mathbf{K}_A^{rel} \arrow[d] \\
\mathcal{A}(\mathbf{CC}_A^{-}) \arrow[r] \arrow[d, dashed, "ii"] & \mathbf{CC}_A^{-} \arrow[r] \arrow[d, "t"] & \mathcal{H}(\mathbf{CC}_A^{-}) \arrow[r] \arrow[d, dotted, "i"] & \mathcal{A}(\mathbf{CC}_A^{-}) \arrow[d, dashed, "ii"] \\
\Sigma\,\mathbf{CC}_A^{cont} \arrow[r] & \mathbf{CC}_A^{cont,-} \arrow[r, "p"] & \mathbf{CC}_A^{cont,per} \arrow[r] & \Sigma^2\,\mathbf{CC}_A^{cont}\,.
\end{tikzcd} \tag{73}$$

The lower sequence is (72). The map $t\colon \mathbf{CC}_A^{-} \to \mathbf{CC}_A^{cont,-}$ is induced by the canonical map from algebraic to projectively completed tensor products. The composition $p \circ t$ maps to a homotopy invariant target. The dotted arrow marked by i and the filler of the lower middle square are obtained from the universal property of the homotopification as a left adjoint in (13). The dashed arrows marked by ii and the corresponding fillers are now induced naturally.

We now drop out the middle row and evaluate the diagram at $*$. We then get the map of fibre sequences of spectra

$$\begin{tikzcd}
\mathbf{K}^{rel}(A) \arrow[r] \arrow[d, "\mathbf{ch}^{cont}_{rel}"] & \mathbf{K}(A) \arrow[r] \arrow[d, "\mathbf{ch}^{cont}"] & \mathbf{K}^{top}(A) \arrow[r] \arrow[d, "\mathbf{ch}^{cont}_{per}"] & \Sigma\mathbf{K}^{rel}(A) \arrow[d] \\
\Sigma\,\mathbf{CC}^{cont}(A) \arrow[r] & \mathbf{CC}^{cont,-}(A) \arrow[r, "p"] & \mathbf{CC}^{cont,per}(A) \arrow[r, "q"] & \Sigma^2\,\mathbf{CC}^{cont}(A),
\end{tikzcd} \tag{74}$$

which defines various versions of the continuous Chern character.

Remark 3.13. By [21, Lemma 4.2.2] the lower sequence in (74) can be identified with the homotopification sequence of $\mathbf{CC}^{cont,-}(A)$. The whole diagram is thus the result of applying the homotopification sequence to the map $t\circ\mathbf{ch}^{gj}\colon \mathbf{K}(A) \to \mathbf{CC}^{cont,-}(A)$. The construction of the various versions of the continuous Chern characters above is thus completely parallel to what is done in [21, Sec. 4.2]. The diagram (74) is exactly the last diagram in [21, Sec. 4.2]. $\square$

We finally define the Chern character $\mathbf{ch}^{cont}_{top}$ by the following diagram which involves the factorization of q over the S-operator:

$$\begin{tikzcd}
\mathbf{K}^{top}(A) \arrow[rr] \arrow[d, "\mathbf{ch}^{cont}_{per}"'] \arrow[dr, "\mathbf{ch}^{cont}_{top}"] & & \Sigma\mathbf{K}^{rel}(A) \arrow[d, "\mathbf{ch}^{cont}_{rel}"] \\
\mathbf{CC}^{cont,per}(A) \arrow[r] \arrow[rr, bend right, dotted, "q"'] & \mathbf{CC}^{cont}(A) \arrow[r, "S"] & \Sigma^2\,\mathbf{CC}^{cont}(A).
\end{tikzcd} \tag{75}$$

For $\sharp \in \{\emptyset, -, per\}$ we let $HC^{cont,\sharp}_*(A) := H_*(CC^{cont,\sharp}(A))$ denote the respective versions of continuous cyclic homology groups of A.

We can now explain the construction of Connes–Karoubi character [19], see also [21, Sec. 7.3]. The algebra $\mathcal{M}^d$ has a natural Fréchet structure so that the notions of topological and relative K-theory used in [19] or [21] coincide with our versions, see Remark 2.22. We start with the diagram derived from the right part of (75) and the upper sequence in (73) (see [19, 4.10])

$$\begin{tikzcd}
K^{top}_{d+2}(\mathcal{M}^d) \arrow[r] \arrow[d, "\mathbf{ch}^{cont}_{top}"] \arrow[dd, bend right, dotted] & K^{rel}_{d+1}(\mathcal{M}^d) \arrow[r, "\alpha"] \arrow[d, "\mathbf{ch}^{cont}_{rel}"] & K_{d+1}(\mathcal{M}^d) \arrow[r, "0"] \arrow[dd, dotted, "\delta"] & K^{top}_{d+1}(\mathcal{M}^d) \\
HC^{cont}_{d+2}(\mathcal{M}^d) \arrow[r, "S"] & HC^{cont}_{d}(\mathcal{M}^d) \arrow[d, "\phi_d"] & & \\
\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}/\mathbb{Z}. &
\end{tikzcd} \tag{76}$$

It is a theorem of Karoubi [25] that the right upper map (marked by 0) vanishes. The map ϕ_d is given by the pairing with an explicit continuous cocycle $\phi_d \in HC^d_{cont}(\mathcal{M}^d)$ which we will describe in (77) below. It has been verified in [19] that elements coming from $\mathbb{Z} \cong K^{top}_{d+2}(\mathcal{M}^d)$ are mapped to integers under the obvious composition indicated by the left dotted arrow. The right dotted arrow is the multiplicative character. It is defined by the obvious diagram chase.

Remark 3.14. In this remark we describe the cocycle ϕ_d explicitly. The formula will be used in the Remarks 3.15 and 3.20 below. Our description of ϕ_d employs the chain complex $C^{\lambda,cont}_*(A)$ given in [28, 2.1.4] in order to calculate $HC^{cont}_*(A)$ for a unital locally convex algebra over $\mathbb{C}$. In particular $C^{\lambda,cont}_n(A)$ is the space of coinvariants for the action of the cyclic permutation group on $A^{\otimes_\pi n+1}$. We use the notation $[a^0 \otimes \cdots \otimes a^n]$ in order to denote elements in $C^{\lambda,cont}_n(A)$.

Furthermore, for a locally convex algebra A we calculate the cyclic cohomology $HC^d_{cont}(A)$ using the complex $C^*_{\lambda,cont}(A)$, where $C^n_{\lambda,cont}(A)$ is the $\mathbb{C}$-vector space of continuous multilinear and cyclically invariant maps $A^{\times n+1} \to \mathbb{C}$. We have a natural pairing

$$C^n_{\lambda,cont}(A) \times C_n^{\lambda,cont}(A) \to \mathbb{C}$$

given by

$$\big(\phi, [a^0 \otimes \cdots \otimes a^n]\big) \to \phi(a^0, \ldots, a^n)\ .$$

Using these conventions the map $\phi_d \colon HC_d^{cont}(\mathcal{M}^d) \to \mathbb{C}$ in (76) is given for odd d by the pairing with the cocycle (using the notation introduced in Remark 3.10)

$$\phi_d(a^0, \ldots, a^d) := (-1)^{\frac{d-1}{2}} \frac{d!}{(2\pi i)^{\frac{d-1}{2}}(\frac{d-1}{2})!} \mathtt{Tr}\left[z \begin{pmatrix} 0 & a^0_{12} \\ a^0_{21} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & a^d_{12} \\ a^d_{21} & 0 \end{pmatrix}\right], \tag{77}$$

where

$$z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

Remark 3.15. In this remark we approach an explicit formula for the composition

$$\delta \circ K(b_{\not{D}}) \circ \partial \colon K^{rel}_{d+1}(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}\ .$$

Here for a homomorphism b between algebras we denote by $K(b)$ or $HC(b)$ the induced maps in K-theory or cyclic homology. In view of (76) and the naturality of $\mathbf{ch}^{cont}_{rel}$ we have the equality

$$\delta \circ K(b_{\not{D}}) \circ \partial = [-]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ \mathbf{ch}^{cont}_{rel} \circ K(b_{\not{D}}) = [-]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ HC(b_{\not{D}}) \circ \mathbf{ch}^{cont}_{rel}\ .$$

It is clearly complicated to write down an explicit formula for the relative Chern character $\mathbf{ch}^{cont}_{rel}$. But we can give an explicit formula for the composition

$$\phi_d \circ HC(b_{\not{D}}) : HC_d(C^\infty(X)) \to HC_d(\mathcal{M}^d) \to \mathbb{C}\ .$$

We continue with the notation introduced in Remark 3.10. For $f \in C^\infty(X)$ we have

$$\big[(P^+ - P^-), f\big] = 2(P^+ f P^- - P^- f P^+) = 2\begin{pmatrix} 0 & P^+ f P^- \\ -P^- f P^+ & 0 \end{pmatrix}.$$

Combining (77) with (67) we see that $\phi_d \circ HC(b_{\not{D}})$ is represented by the cochain

$$\begin{aligned} &(f_0, \ldots, f_d) \\ &\mapsto \frac{-2^{d+1} d!}{(2\pi i)^{\frac{d-1}{2}}(\frac{d-1}{2})!} \mathtt{Tr}\Big((P^+ - P^-)\big[(P^+ - P^-), f_0\big] \cdots \big[(P^+ - P^-), f_d\big]\Big) \end{aligned} \tag{78}$$

This formula is a first step in the direction of the main result of [22]. On the other hand it is still a complicated non-local formula. By standard methods of local index theory using e.g. the heat kernel and Getzler rescaling one can produce local cocyles representing the same cohomology class, see e.g. [4, 20]. In Lemma 3.19 we avoid complicated analysis and the struggle with normalizations by using the Atiyah–Singer index theorem. The explicit local formula will be stated in Remark 3.20. □

3.4. The conjecture. The connective covering morphism of spectra (58) induces a morphism of spectra

$$\mathbf{ku}\,\mathbb{C}/\mathbb{Z} \to \mathbf{KU}\,\mathbb{C}/\mathbb{Z}\,. \tag{79}$$

Let X be a closed Riemannian manifold of odd dimension d equipped with a generalized Dirac operator $\not{D}$. Then we define the map $r_{\not{D}}$ as the following composition:

$$\begin{aligned} r_{\not{D}}\colon \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) &\stackrel{(79)}{\to} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \\ &\stackrel{\iota_{d+1}}{\cong} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X) \cong \widehat{KU}^{0}_{\mathit{flat}}(X) \stackrel{\rho_{\not{D}}}{\to} \mathbb{C}/\mathbb{Z}\,, \end{aligned} \tag{80}$$

where $\iota_{2k}\colon \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{p}(X) \stackrel{\cong}{\to} \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{p+2k}(X)$ is again a periodicity operator. We now consider the diagram:

$$\begin{tikzcd} & \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \arrow[dr, "r_{\not{D}}"] & \\ K_{d+1}(C^{\infty}(X)) \arrow[ur, "\sigma_{d+1}"] \arrow[dr, "b_{\not{D}}"'] & & \mathbb{C}/\mathbb{Z} \\ & K_{d+1}(\mathcal{M}^{d}), \arrow[ur, "\delta"'] & \end{tikzcd} \tag{81}$$

where $b_{\not{D}}$ is defined in (65) and classifies the Fredholm module of $\not{D}$, δ is the multiplicative character of Connes–Karoubi (68), and σ_{d+1} is defined in Definition 2.36.

Conjecture 3.16. *Let X be a closed Riemannian manifold of odd dimension d equipped with a generalized Dirac operator $\not{D}$. Then the diagram (81) commutes.*

In the present paper we show this conjecture for topologically trivial classes in $K_{d+1}(C^{\infty}(X))$. The precise formulation of this result is Theorem 1.6.

3.5. Comparison of certain cocycles. In this subsection we prepare the proof of Theorem 1.6 by providing a differential geometric formula for the composition $\delta \circ HC(b_{\not{D}})$. The main result of this subsection is Lemma 3.19.

In the following we define the cyclic homology $HC_*(A)$ of an associative unital algebra A over $\mathbb{C}$ as the homology of the standard cyclic complex $CC_*(A)$. For details we refer to [28, 2.1.9] where this complex is denoted by $\mathtt{Tot}\ \mathcal{B}(A)$. Explicitly, we define

$$CC_n(A) := \begin{cases} \bigoplus_{k=0}^{n/2} A^{\otimes 2k+1}, & n \text{ even}, \\ \bigoplus_{k=0}^{(n-1)/2} A^{\otimes 2k}, & n \text{ odd}. \end{cases}$$

As in Subsection 3.3, for a unital locally convex algebra we define the continuous cyclic homology complex $CC^{cont}(A)$ and its homology $HC_*^{cont}(A)$ similarly but using projective tensor products $\otimes_\pi$ instead of algebraic ones, see Remark (2.9).

Remark 3.17. As shown in [28, 2.1.4] there is a natural quasi-isomorphism

$$CC(A) \to C^\lambda(A), \tag{82}$$

which we use in order to compare the present definition of cyclic homology with the one used in Remark 3.14. The quasi-isomorphism (82) is induced by a chain-complex level projection map, which in degree n is the homomorphism $CC_n(A) \to C_n^\lambda(A)$ given by (we write the formula for odd n)

$$\oplus_{k=0}^{(n-1)/2} \; a_0^k \otimes \cdots \otimes a_{2k+1}^k \mapsto \left[a_0^{(n-1)/2} \otimes \cdots \otimes a_n^{(n-1)/2}\right]. \tag{83}$$

There is a similar quasi-isomorphism in the continuous case. □

We define a morphism of graded groups (see (39) for the definition of $DD(X)$)

$$\pi\colon CC^{cont}(C^\infty(X)) \to DD(X) \tag{84}$$

by the following prescription:

1. If n is odd, then we define $CC_n^{cont}(C^\infty(X)) \to \prod_{p\in\mathbb{Z}} (\sigma^{\le p}\Omega)[2p]^{-n}(X)$ by

$$\bigoplus_{k=0}^{(n-1)/2} f_0^k \otimes \cdots \otimes f_{2k+1}^k \mapsto \sum_{k=0}^{(n-1)/2} \frac{b^{\frac{n+1}{2}+k}}{(2k+1)!} f_0^k df_1^k \wedge \cdots \wedge df_{2k+1}^k \,. \tag{85}$$

2. If n is even, then we define $CC_n^{cont}(C^\infty(X)) \to \prod_{p\in\mathbb{Z}} (\sigma^{\le p}\Omega)[2p]^{-n}(X)$ by

$$\bigoplus_{k=0}^{n/2} f_0^k \otimes \cdots \otimes f_{2k}^k \mapsto \sum_{k=0}^{n/2} \frac{b^{\frac{n}{2}+k}}{(2k)!} f_0^k df_1^k \wedge \cdots \wedge df_{2k}^k \,.$$

In these formulas we use the variable b of degree -2 and the identification

$$\prod_{p\in\mathbb{Z}} \Omega[2p](X) \cong \Omega[b, b^{-1}](X)\,.$$

Under this identification the series $\sum_{p\in\mathbb{Z}} b^p\omega(p) \in \Omega[b, b^{-1}](X)$ corresponds to the family $(\omega(p))_{p\in\mathbb{Z}} \in \prod_{p\in\mathbb{Z}} \Omega[2p](X)$. By [28, 2.3.6] the map π is a morphism of chain complexes. By the calculation of the continuous cyclic homology of $C^\infty(X)$ by Connes π is actually a quasi-isomorphism.

We have a projection

$$\psi\colon DD^{per}(X) \to DD(X) \tag{86}$$

induced by the projections in the components

$$DD^{per}(p) \to DD(p)\ , \quad \Omega[2p] \to (\sigma^{\le p}\Omega)[2p]$$

for all $p \in \mathbb{Z}$.

Remark 3.18. This projection (86) must not be confused with the projection (41). The latter is given by

$$DD^{per}(X) \stackrel{(86)}{\to} DD(X) \stackrel{S}{\to} DD(X)[2]\ ,$$

where $S((\omega(p))_{p\in\mathbb{Z}}) = (\omega(p+1))_{p\in\mathbb{Z}}$. □

We now use that $d \in \mathbb{N}$ is odd and that $\dim(X) = d$. Under these assumptions the map ψ in (86) induces an isomorphism

$$\psi_d\colon HP^{-d}(X) \stackrel{\text{def}}{=} H^{-d}(DD^{per}(X)) \stackrel{\psi}{\to} H^{-d}(DD(X))\ . \tag{87}$$

We consider the isomorphism π_d defined as the following composition of isomorphisms

$$\pi_d\colon HC_d^{cont}(C^\infty(X)) \xrightarrow{\pi} H^{-d}(DD(X)) \stackrel{\psi_d^{-1}}{\to} HP^{-d}(X) \xrightarrow{\iota_{d-1}} HP^{-1}(X)\ . \tag{88}$$

Let $\not{D}$ be a generalized Dirac operator on X. Using the local index density $\hat{\mathbf{A}}(\not{D})$ (see Remark 3.5) we define the map

$$\tilde{\rho}_{\not{D}}\colon HP^{-1}(X) \to \mathbb{C}\ , \quad \tilde{\rho}_{\not{D}}([\gamma]) := \int_X \hat{\mathbf{A}}(\not{D}) \wedge \iota_{d+1}\gamma\ . \tag{89}$$

Lemma 3.19. *Let X be a closed manifold of odd dimension d and $\not{D}$ be a generalized Dirac operator on X. Then the square*

$$\begin{tikzcd} HC_d^{cont}(C^\infty(X)) \arrow[r, "HC(b_{\not{D}})"] \arrow[d, "\pi_d"] & HC_d^{cont}(\mathcal{M}^d) \arrow[d, "\phi_d"] \\ HP^{-1}(X) \arrow[r, "\tilde{\rho}_{\not{D}}"] & \mathbb{C} \end{tikzcd}$$

commutes.

Proof. Our task is to compare the composition of the quasi-isomorphism (82) with the map (78) on the one hand, and the map $\tilde{\rho}_{\not{D}}$ defined in (89) on the other. It seems to be difficult to do this by an explicit calculation. Therefore we give an indirect argument based on the Atiyah–Singer index theorem. Our argument will not use explicit formulas. The convention for fixing normalizations described in Remark 3.5 automatically takes care of the correct normalizations of $\hat{\mathbf{A}}(\not{D})$ and ϕ_d in Remark 3.14.

We consider the composition of the map marked by !!! in (48) with the Chern character given by the third column in (73) in the case $A = \mathbb{C}$:

$$\mathbf{K}^{top}_{C^\infty(X)} \stackrel{!!!}{\to} i_X^* \mathbf{K}^{top}_{\mathbb{C}} \stackrel{\mathbf{ch}^{cont}_{per}}{\to} i_X^* \, \mathbf{CC}^{cont,per}_{\mathbb{C}} \ .$$

By evaluation at $*$ and taking the $(d+2)$th homotopy group we obtain the left triangle in the following diagram:

$$\begin{tikzcd}
K^{top}_{d+2}(C^\infty(X)) \arrow[r] \arrow[dr] & \pi_{d+2}(\mathbf{K}^{top}_{\mathbb{C}}(X)) \arrow[r, "!"] \arrow[d] \arrow[dr, dotted, "\iota_{d+1}\circ \mathbf{ch}^{cw}_{per}"] & HC^{cont}_d(C^\infty(X)) \arrow[r, "HC(b_{\not{D}})"] \arrow[d, "\pi_d"] & HC^{cont}_d(\mathcal{M}^d) \arrow[d, "\phi_d"] \\
& HC^{cont,per}_{d+2}(C^\infty(X)) \arrow[r] \arrow[ur, "q"] & HP^{-1}(X) \arrow[r, "\tilde{\rho}_{\not{D}}"] & \mathbb{C}
\end{tikzcd} \quad . \tag{90}$$

The map marked by q is induced by the map marked by this symbol in (71). By construction the three solid triangles on the left of (90) commute.

Let $K^{C^*}(-)$ denote the usual K-theory for C^*-algebras [3]. Since $-d-2<0$ the connective covering (58) induces an isomorphism marked by $*$ in the following chain of isomorphisms:

$$\pi_{d+2}(\mathbf{K}^{top}_{\mathbb{C}}(X)) \cong \mathbf{ku}^{-d-2}(X) \stackrel{*}{\cong} \mathbf{KU}^{-d-2}(X) \cong K^{C^*}_{d+2}(C(X)) \ . \tag{91}$$

Under this identification the map

$$\mathbf{KU}^{-d-2}(X) \to HP^{-1}(X)$$

induced by the dotted arrow in (90) is the composition $\iota_{d+1} \circ \mathbf{ch}^{cw}_{per}$ of the usual Chern character and the shift, use Lemma 2.27. In particular, its image is a lattice of full rank in $HP^{-1}(X)$. Since π_d is an isomorphism, in order to show that the right square in (90) commutes is suffices to verify that the right hexagon (omit the left upper corner) commutes.

We consider the map

$$K^{C^*}_{d+2}(C(X)) \overset{(91)}{\cong} \pi_{d+2}(\mathbf{K}^{top}_{\mathbb{C}}(X)) \xrightarrow{!} HC^{cont}_d(C^\infty(X)) \tag{92}$$

in the upper line of (90). The cocycle ϕ_d is normalized exactly such that the composition of (92) with $\phi_d \circ HC(b_{\not{D}})$ is the integer-valued function obtained by the pairing of $K^{C^*}_{d+2}(C(X))$ with the Fredholm module of $\not{D}$.

The down-right-composition in the right hexagon of (90) maps the K-theory class $x \in K^{C^*}_{d+2}(C(X))$ to

$$\int_X \left[\hat{\mathbf{A}}(\not{D})\right] \cup \iota_{d+1}\, \mathbf{ch}^{cw}_{per}(x)\ , \tag{93}$$

which is apriori a complex number. The Atiyah–Singer index theorem encoded in equation (62) shows that (93) is an integer and equal to the index pairing. So the down-right-composition coincides with the right-down-composition. □

Remark 3.20. This is a continuation of Remark 3.15. The following two cocycles (a) and (b) on $CC^{cont}_d(C^\infty(X))$ represent the same map $HC^{cont}_d(C^\infty(X)) \to \mathbb{C}$. We describe result of the application of the two cocyles to the chain

$$\bigoplus_{k=0}^{(d-1)/2} f_0^k \otimes \cdots \otimes f_{2k+1}^k \in CC^{cont}_d(C^\infty(X)) :$$

(a) $$\frac{-2^{d+1}d!}{(2\pi i)^{\frac{d-1}{2}}(\frac{d-1}{2})!}\mathtt{Tr}\Big((P^+ - P^-)\big[(P^+ - P^-), f_0^{(d-1)/2}\big]\cdots \cdots\big[(P^+ - P^-), f_d^{(d-1)/2}\big]\Big)\ ,$$

(b) $$\sum_{k=0}^{(d-1)/2} \frac{1}{(2k+1)!}\int_X [\hat{\mathbf{A}}(\not{D})]_{d-2k-1} \wedge f_0^k\, df_1^k \wedge \cdots \wedge df_{2k+1}^k\ .$$

The formula (a) is one for $\phi_d \circ CC(b_{\not{D}})$ obtained by combining (83) and (78). The formula (b) gives $\tilde{\rho}_{\not{D}} \circ \pi_d$ and is derived from a combination of (89) and (85). The equality of the cohomology classes of (a) and (b) is the assertion of Lemma 3.19. □

3.6. Proof of the conjecture for topologically trivial classes. In this subsection we prove Theorem 1.6. We must show the equality of homomorphisms

$$r_{\not{D}} \circ \sigma_{d+1} \circ \partial = \delta \circ K(b_{\not{D}}) \circ \partial \colon K^{rel}_{d+1}(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}\ .$$

This goal is achieved by the following chain of equalities:

$$
\begin{aligned}
r_{\slashed{D}} \circ \sigma_{d+1} \circ \partial &= r_{\slashed{D}} \circ a \circ \mathtt{reg}^{rel}_X \qquad &(94)\\
&= r_{\slashed{D}} \circ a \circ \psi_d \circ \psi_d^{-1} \circ \mathtt{reg}^{rel}_X &(95)\\
&= r_{\slashed{D}} \circ a \circ \psi_d \circ \iota_{-d+1} \circ \iota_{d-1} \circ \psi_d^{-1} \circ \mathtt{reg}^{rel}_X &(96)\\
&= \rho_{\slashed{D}} \circ a_{\widehat{\mathbf{KU}}} \circ \iota_{d-1} \circ \psi_d^{-1} \circ \mathtt{reg}^{rel}_X &(97)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \widetilde{\rho}_{\slashed{D}} \circ \iota_{d-1} \circ \psi_d^{-1} \circ \mathtt{reg}^{rel}_X &(98)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \widetilde{\rho}_{\slashed{D}} \circ \iota_{d-1} \circ \psi^{-1} \circ \pi \circ \pi^{-1} \circ \mathtt{reg}^{rel}_X &(99)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \widetilde{\rho}_{\slashed{D}} \circ \pi_d \circ \pi^{-1} \circ \mathtt{reg}^{rel}_X &(100)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ HC(b_{\slashed{D}}) \circ \pi^{-1} \circ \mathtt{reg}^{rel}_X &(101)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ HC(b_{\slashed{D}}) \circ \pi^{-1} \circ \pi \circ \mathbf{ch}^{cont}_{rel} &(102)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ HC(b_{\slashed{D}}) \circ \mathbf{ch}^{cont}_{rel} &(103)\\
&= [\cdots]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ \mathbf{ch}^{cont}_{rel} \circ K^{rel}(b_{\slashed{D}}) &(104)\\
&= \delta \circ K(b_{\slashed{D}}) \circ \partial\,. &(105)
\end{aligned}
$$

In the following we explain the steps in detail:

1. For (94) we use Corollary 2.39 and the Definition 2.36 of σ_{d+1} in terms of $\widehat{\mathtt{reg}}_X$.
2. At (95) we insert $\mathtt{id} = \psi_d^{-1} \circ \psi_d$, where ψ_d is defined in (87).
3. At (96) we insert $\iota_{-d+1} \circ \iota_{d-1} = \mathtt{id}$, where ι_k is the periodicity operator introduced in (37) for every $k \in 2\mathbb{Z}$. Note that $d-1$ is even.
4. At (97) we use the commutative diagram

$$
\begin{tikzcd}
HP^{-1}(X) \arrow[r, "a_{\widehat{\mathbf{KU}}}"] \arrow[d, "\iota_{-d+1}"] & \widehat{KU}^0_{flat}(X) \arrow[r, "\cong"] \arrow[rrr, dotted, bend left, "\rho_{\slashed{D}}"] & \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-1}(X) \arrow[r, "\cong", "\iota_{-d-1}"'] & \mathbf{KU}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) & \mathbb{C}/\mathbb{Z}\,. \\
HP^{-d}(X) \arrow[r, "\psi_d"] & \pi_{d+1}(\Sigma\,\mathtt{DD}(X)) \arrow[r, "a"] & \widehat{ku}^{-d-1}_{flat}(X) \arrow[r, "\cong"] & \mathbf{ku}\,\mathbb{C}/\mathbb{Z}^{-d-2}(X) \arrow[u, "(79)"'] \arrow[ur, "r_{\slashed{D}}"] &
\end{tikzcd}
$$

5. At (98) we use the definition (89) of $\widetilde{\rho}_{\slashed{D}}$ and the observation based on formula (63) that its composition with $[\cdots]_{\mathbb{C}/\mathbb{Z}}$ coincides with the composition $\rho_{\slashed{D}} \circ a_{\widehat{\mathbf{KU}}}$.
6. At (99) we insert $\pi \circ \pi^{-1} = \mathtt{id}$, where the isomorphism π is defined in (84).
7. At (100) we insert the definition (88) of π_d.
8. At (101) we use Lemma 3.19.
9. At (102) we use the equality $\mathtt{reg}^{rel}_X = \pi \circ \mathbf{ch}^{cont}_{rel}$.
10. At (103) we delete $\pi^{-1} \circ \pi$.

11. At (104) we use the naturality of $\mathbf{ch}^{cont}_{rel}$ expressed by the commutative diagram:

$$\begin{tikzcd} K^{rel}_{d+1}(C^\infty(X)) \arrow[r, "\mathbf{ch}^{cont}_{rel}"] \arrow[d, "K^{rel}(b_{\not{D}})"] & HC^{cont}_d(C^\infty(X)) \arrow[d, "HC(b_{\not{D}})"] \\ K^{rel}_{d+1}(\mathcal{M}^d) \arrow[r, "\mathbf{ch}^{cont}_{rel}"] & HC^{cont}_d(\mathcal{M}^d). \end{tikzcd}$$

12. Finally, at (105) we use the definition of δ in terms of the commutative diagram (76). □

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U. Bunke, Mathematische Fakultät, Universität Regensburg,
93040 Regensburg, Germany

E-mail: ulrich.bunke@mathematik.uni-regensburg.de