

Pullbacks and nontriviality of associated noncommutative vector bundles

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Abstract. Our main theorem is that the pullback of an associated noncommutative vector bundle induced by an equivariant map of quantum principal bundles is a noncommutative vector bundle associated via the same finite-dimensional representation of the structural quantum group. At the level of K_0 -groups, we realize the induced map by the pullback of explicit matrix idempotents. We also show how to extend our result to the case when the quantum-group representation is infinite dimensional, and then apply it to the Ehresmann–Schauenburg quantum groupoid. Finally, using noncommutative Milnor’s join construction, we define quantum quaternionic projective spaces together with noncommutative tautological quaternionic line bundles and their duals. As a key application of the main theorem, we show that these bundles are stably non-trivial as noncommutative complex vector bundles.

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1. Introduction

Our result is motivated by the search of K_0 -invariants. The main idea is to use equivariant homomorphisms to facilitate computations of such invariants by moving them from more complicated algebras to simpler ones. This strategy was recently successfully applied in [19] to distinguish the K_0 -classes of noncommutative line bundles over two different types of quantum complex projective spaces. Herein we generalize from associated noncommutative line bundles to associated noncommutative *vector* bundles. Then we apply our general theorem to noncommutative vector bundles associated with a finitely iterated equivariant noncommutative join [13] of $SU_q(2)$ with itself.

Recall that the n -times iterated join $SU(2) * \cdots * SU(2)$ yields the odd sphere S^{4n+3} , and the diagonal $SU(2)$ -action on the join defines a fibration producing the n -th quaternionic projective space: $S^{4n+3}/SU(2) = \mathbb{H}P^n$. Therefore, we denote the n -iterated equivariant noncommutative join of $C(SU_q(2))$ with itself

by $C(S_q^{4n+3})$, and consider the fixed-point subalgebra $C(S_q^{4n+3})^{SU_q(2)}$ as the defining C^* -algebra $C(\mathbb{H}\mathbb{P}_q^n)$ of a *quantum quaternionic projective space*. Then we define the noncommutative tautological quaternionic line bundle and its dual as noncommutative complex vector bundles associated through the contragredient representation of the fundamental representation of $SU_q(2)$ and the fundamental representation itself, respectively. We observe that, as in the classical case, they are isomorphic as noncommutative (complex) vector bundles.

A classical argument that proves the non-triviality of a vector bundle associated with a principal bundle by restricting the vector bundle to an appropriate subspace uses the fact that the thus restricted vector bundle is associated with the restricted principal bundle. The latter restriction is encoded by an equivariant map of total spaces of principal bundles, which induces a natural transformation of Chern characters of the vector bundles in question.

More precisely, let X and X' be compact Hausdorff right G -spaces. Assume that the G -action on X is free. If $f: X' \rightarrow X$ is a continuous G -equivariant map, the G -action on X' is automatically free as well, and therefore we have a G -equivariant homeomorphism of compact principal bundles over X'/G :

$$X' \ni x' \mapsto (x'G, f(x')) \in X'/G \times_{X/G} X. \tag{1.1}$$

Its inverse, given by means of the *translation map* $\tau: X \times_{X/G} X \rightarrow G$, $\tau(x, xg) = g$, is as follows:

$$X'/G \times_{X/G} X \ni (x'G, x) \mapsto x'\tau(f(x'), x) \in X'. \tag{1.2}$$

Here the fiber product over X/G is given by the canonical quotient map $X \rightarrow X/G$, $x \mapsto xG$, and by the f -induced map $f/G: X'/G \rightarrow X/G$, $xG \mapsto f(x)G$.

Therefore, G -equivariant continuous maps between total spaces of compact principal G -bundles are equivalent to continuous maps between their base spaces. In particular, the isomorphism class of a compact principal G -bundle is uniquely determined by the homotopy class of a map from its base space to a compact approximation of the classifying space BG . As shown by Milnor [20], the quotient map $G * \dots * G \rightarrow (G * \dots * G)/G$ is a principal G -bundle which is a compact approximation of a universal principal G -bundle $EG \rightarrow BG$.

To decide whether a given compact principal G -bundle is nontrivial, it is sufficient to prove that at least one of its associated vector bundles is nontrivial. Furthermore, every associated vector bundle is a pullback of a universal vector bundle, both corresponding to the same representation $G \rightarrow GL(V)$. This is a consequence of the compatibility of associating and pulling back

$$X' \times^G V = (X'/G \times_{X/G} X) \times^G V = X'/G \times_{X/G} (X \times^G V) = (f/G)^*(X \times^G V), \tag{1.3}$$

which is afforded by the G -equivariant homeomorphism (1.1).

In particular, for $G = SU(2)$ (when $BG = \mathbb{H}\mathbb{P}^\infty$), the restriction of the tautological quaternionic line bundle $\tau_{\mathbb{H}\mathbb{P}^n}$ from $\mathbb{H}\mathbb{P}^n$ to $\mathbb{H}\mathbb{P}^1$ is the tautological quaternionic line bundle $\tau_{\mathbb{H}\mathbb{P}^1}$ over $\mathbb{H}\mathbb{P}^1$, so that the Chern character computation proving the non-triviality of $\tau_{\mathbb{H}\mathbb{P}^1}$ proves also the nontriviality of $\tau_{\mathbb{H}\mathbb{P}^n}$.

The present paper generalizes this reasoning to the noncommutative setting as follows. First, we use the Gelfand–Naimark theorem to encode compact Hausdorff spaces as commutative unital C^* -algebras. Next, we employ the Peter–Weyl theory to describe a compact group as a Hopf algebra of representative functions. Then, we take advantage of the Peter–Weyl theory extended from compact groups to compact principal bundles [3] to express a compact principal bundle as a comodule algebra over the Hopf algebra of representative functions. Finally, we use the Serre–Swan theorem to encode a vector bundle as a finitely generated projective module. In particular, we describe an associated vector bundle as an associated finitely generated projective module [4] using the Milnor–Moore cotensor product [21]. Having all these basic structures given in terms of commutative algebras, we generalize by dropping the assumption of commutativity.

In precise technical terms, we proceed as follows. For any finite-dimensional corepresentation V of a coalgebra \mathcal{C} coacting principally on an algebra \mathcal{A} , we can use the cotensor product $\square^{\mathcal{C}}$ to form an associated finitely generated projective module $\mathcal{A} \square^{\mathcal{C}} V$ over the coaction-invariant subalgebra \mathcal{B} . The module $\mathcal{A} \square^{\mathcal{C}} V$ is the section module of the associated noncommutative vector bundle. If \mathcal{A}' is an algebra with a principal coaction of \mathcal{C} , and \mathcal{B}' is its coaction-invariant subalgebra, then any equivariant (colinear) algebra homomorphism $\mathcal{A} \rightarrow \mathcal{A}'$ restricts and corestricts to an algebra homomorphism $\mathcal{B} \rightarrow \mathcal{B}'$ making \mathcal{B}' a $(\mathcal{B}' - \mathcal{B})$ -bimodule. The theorem of the paper is:

Theorem 1.1. *The finitely generated left \mathcal{B}' -modules $\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} V)$ and $\mathcal{A}' \square^{\mathcal{C}} V$ are isomorphic. In particular, for any equivariant $*$ -homomorphism $f: \mathcal{A} \rightarrow \mathcal{A}'$ between unital C^* -algebras equipped with a free action of a compact quantum group, the induced K -theory map $f_*: K_0(\mathcal{B}) \rightarrow K_0(\mathcal{B}')$, where \mathcal{B} and \mathcal{B}' are the respective fixed-point subalgebras, satisfies $f_*([\mathcal{A} \square^{\mathcal{C}} V]) = [\mathcal{A}' \square^{\mathcal{C}} V]$.*

We begin by stating our main result in the standard and easily accessible Hopf-algebraic setting (Theorem 2.1). Then we state and prove two slightly different coalgebraic generalizations of the result: Theorem 2.3 based on faithful flatness and coflatness (cosemisimple coalgebras but with corepresentations of any dimension), and Theorem 2.8 based on Chern–Galois theory [8] (arbitrary coalgebras but with corepresentations of finite dimensions).

An advantage of the faithful-flatness-and-coflatness approach is a possibility to apply it in the case of infinite-dimensional fibers. In particular, we prove (Theorem 2.5) that the pullback of the Ehresmann–Schauenburg quantum groupoid of a quantum principal bundle is the Ehresmann–Schauenburg quantum groupoid of the pullback of the quantum principal bundle.

An advantage of the Chern–Galois approach is a possibility to compute explicitly a K -theoretic invariant whose non-vanishing for a given principal extension proves that the extension is not cleft [22, §8.2]. The aforementioned coalgebraic generality is necessary already in the case of celebrated non-standard Podleś spheres, which are fundamental examples of noncommutative geometry going beyond the reach of Hopf–Galois theory [5–7, 24, 28].

Finally, we use the Peter–Weyl functor to make the result applicable to free actions of compact quantum groups on unital C^* -algebras [2]. In the C^* -algebraic setting we consider our example and main application: the stable non-triviality of the noncommutative tautological quaternionic line bundles and their duals.

2. Pushing forward modules associated with Galois-type coactions

Let \mathcal{C} be a coalgebra, $\delta_M: M \rightarrow M \otimes \mathcal{C}$ a right coaction, and ${}_N\delta: N \rightarrow \mathcal{C} \otimes N$ a left coaction. The *cotensor* product of M with N is $M \square^{\mathcal{C}} N := \ker(\delta_M \otimes \text{id} - \text{id} \otimes {}_N\delta)$. In what follows, we will also use the Heyneman–Sweedler notation (with the summation sign suppressed) for comultiplications and right coactions:

$$\Delta(c) =: c_{(1)} \otimes c_{(2)}, \quad \delta_M(m) =: m_{(0)} \otimes m_{(1)}. \quad (2.1)$$

Next, let \mathcal{H} be a Hopf algebra with bijective antipode S , comultiplication Δ , and counit ε . Also, let $\delta_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{H}$ be a coaction rendering \mathcal{A} a right \mathcal{H} -comodule algebra. The subalgebra of coaction invariants $\{b \in \mathcal{A} \mid \delta_{\mathcal{A}}(b) = b \otimes 1\}$ is called the coaction-invariant (or fixed-point) subalgebra. We say that \mathcal{A} is a *principal* comodule algebra iff there exists a *strong connection* [8, 12, 17], i.e., a unital linear map $\ell: \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{A}$ satisfying:

- (1) $(\text{id} \otimes \delta_{\mathcal{A}}) \circ \ell = (\ell \otimes \text{id}) \circ \Delta$, $((S^{-1} \otimes \text{id}) \circ \text{flip} \circ \delta_{\mathcal{A}}) \otimes \text{id} \circ \ell = (\text{id} \otimes \ell) \circ \Delta$;
- (2) $m \circ \ell = \varepsilon$, where $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication map.

Let \mathcal{H} be a Hopf algebra with bijective antipode. In [18], the principality of an \mathcal{H} -comodule algebra was defined by requiring the bijectivity of the canonical map (see Definition 2.2 (1)) and equivariant projectivity (see Definition 2.6 (2)). One can prove (see [9] and references therein) that an \mathcal{H} -comodule algebra is principal in this sense if and only if it admits a strong connection. Therefore, we will treat the existence of a strong connection as a condition defining the principality of a comodule algebra and avoid the original definition of a principal comodule algebra. The latter is important when going beyond coactions that are algebra homomorphisms — then the existence of a strong connection is implied by principality but we do not have the reverse implication [8].

Theorem 2.1. *Let \mathcal{A} and \mathcal{A}' be right \mathcal{H} -comodule algebras for a Hopf algebra \mathcal{H} with bijective antipode, and V be a finite-dimensional left \mathcal{H} -comodule. Denote by \mathcal{B} and \mathcal{B}' the respective coaction-invariant subalgebras. Assume that \mathcal{A} is principal and that there exists an \mathcal{H} -equivariant algebra homomorphism $f: \mathcal{A} \rightarrow \mathcal{A}'$. The restriction-corestriction of f to $\mathcal{B} \rightarrow \mathcal{B}'$ makes \mathcal{B}' a $(\mathcal{B}' - \mathcal{B})$ -bimodule such that the associated left \mathcal{B}' -modules $\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{H}} V)$ and $\mathcal{A}' \square^{\mathcal{H}} V$ are isomorphic. In particular, the induced map*

$$f_*: K_0(\mathcal{B}) \longrightarrow K_0(\mathcal{B}') \quad \text{satisfies } f_*([\mathcal{A} \square^{\mathcal{H}} V]) = [\mathcal{A}' \square^{\mathcal{H}} V].$$

As will be explained later on, the above Theorem 2.1 specializes Theorem 2.8, and Theorem 3.1 is a common denominator of Theorem 2.3 and Theorem 2.1.

2.1. Faithfully flat coalgebra-Galois extensions.

Definition 2.2 ([9]). Let \mathcal{C} be a coalgebra coaugmented by a group-like element $e \in \mathcal{C}$, and \mathcal{A} an algebra and a right \mathcal{C} -comodule via $\delta_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{C}$ such that $\delta_{\mathcal{A}}(1) = 1 \otimes e$. Put

$$\mathcal{B} := \{b \in \mathcal{A} \mid \forall a \in \mathcal{A}: \delta_{\mathcal{A}}(ba) = b\delta_{\mathcal{A}}(a)\}$$

(coaction-invariant subalgebra). We say that the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is an *e-coaugmented \mathcal{C} -Galois extension* iff

- (1) the canonical map $can: \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, $a \otimes a' \mapsto a\delta_{\mathcal{A}}(a')$ is bijective,
- (2) $\delta_{\mathcal{A}}(1) = 1 \otimes e$.

With any \mathcal{C} -Galois extension one can associate the *canonical entwining* [7]:

$$\psi: \mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{C}, \quad c \otimes a \longmapsto can (can^{-1}(1 \otimes c)a). \tag{2.2}$$

When the \mathcal{C} -Galois extension is *e-coaugmented*, then combining [10, Proposition 2.2] with [7, Proposition 4.2] yields the following presentations of the coaction-invariant subalgebra:

$$\mathcal{B} = \{b \in \mathcal{A} \mid \delta_{\mathcal{A}}(b) = b \otimes e\} = \{b \in \mathcal{A} \mid \psi(e \otimes b) = b \otimes e\}. \tag{2.3}$$

Theorem 2.3. *Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{A}'$ be e-coaugmented \mathcal{C} -Galois extensions, let V be a left \mathcal{C} -comodule. Assume that \mathcal{A}' is faithfully flat as a right \mathcal{B}' -module and that coalgebra \mathcal{C} is cosemisimple. Then every \mathcal{C} -equivariant algebra map $f: \mathcal{A} \rightarrow \mathcal{A}'$ restricts and corestricts to an algebra homomorphism $\mathcal{B} \rightarrow \mathcal{B}'$, and induces an isomorphism*

$$\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} V) \cong \mathcal{A}' \square^{\mathcal{C}} V$$

of left \mathcal{B}' -modules that is natural in V .

Proof. Since \mathcal{A}' is a faithfully flat right \mathcal{B}' -module, the map of left \mathcal{B}' -modules right \mathcal{C} -comodules

$$\tilde{f} := m_{\mathcal{A}'} \circ (\text{id}_{\mathcal{B}'} \otimes_{\mathcal{B}} f): \mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}', \quad m_{\mathcal{A}'}(b' \otimes a') = b'a', \quad (2.4)$$

is an isomorphism if and only if the map of left \mathcal{A}' -modules right \mathcal{C} -comodules

$$\text{id}_{\mathcal{A}'} \otimes_{\mathcal{B}'} \tilde{f}: \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}' \quad (2.5)$$

is an isomorphism. Replacing the left-hand-side \mathcal{B}' by \mathcal{A} , the latter is an isomorphism if and only if

$$\text{id}_{\mathcal{A}'} \otimes_{\mathcal{A}} (f \otimes_{\mathcal{B}'} f): \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}' \quad (2.6)$$

is an isomorphism. Thus, from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}'} \otimes_{\mathcal{A}} (f \otimes_{\mathcal{B}'} f)} & \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}' & (2.7) \\ \text{id}_{\mathcal{A}'} \otimes_{\mathcal{A}} \text{can} \downarrow & & \downarrow \text{can}' & \\ \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes \mathcal{C} & \xrightarrow{\cong} & \mathcal{A}' \otimes \mathcal{C} & \end{array}$$

and the bijectivity of the canonical maps, we infer that \tilde{f} is an isomorphism.

Furthermore, as \tilde{f} is a homomorphism of left \mathcal{B}' -modules right \mathcal{C} -comodules, we conclude that

$$\tilde{f} \square^{\mathcal{C}} \text{id}_V: (\mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A}) \square^{\mathcal{C}} V \longrightarrow \mathcal{A}' \square^{\mathcal{C}} V \quad (2.8)$$

is an isomorphism of left \mathcal{B}' -modules. Finally, since \mathcal{C} is cosemisimple, and any comodule over a cosemisimple coalgebra is injective [11, Theorem 3.1.5 (iii)], whence coflat [11, Theorem 2.4.17 (i)–(iii)], the balanced tensor product $\mathcal{B}' \otimes_{\mathcal{B}} (-)$ and the cotensor product $(-) \square^{\mathcal{C}} V$ commute. Therefore, there is a natural-in- V isomorphism of left \mathcal{B}' -modules

$$\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} V) \cong (\mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A}) \square^{\mathcal{C}} V \cong \mathcal{A}' \square^{\mathcal{C}} V, \quad (2.9)$$

as claimed. □

Note that the algebra \mathcal{A} in the above theorem is given as a right \mathcal{C} -comodule. However, it also enjoys a natural left \mathcal{C} -comodule structure provided that the canonical entwining is bijective. Indeed, one can then define a left coaction as follows:

$${}_{\mathcal{A}}\delta: \mathcal{A} \longrightarrow \mathcal{C} \otimes \mathcal{A}, \quad {}_{\mathcal{A}}\delta(a) = \psi^{-1}(a \otimes e). \quad (2.10)$$

Another consequence of invertibility of the canonical entwining ψ and (2.3) is the equality $\mathcal{B} = \{b \in \mathcal{A} \mid {}_{\mathcal{A}}\delta(b) = e \otimes b\}$ and the fact that the left comultiplication ${}_{\mathcal{A}}\delta$ is right \mathcal{B} -linear.

The left and right \mathcal{C} -comodule structures on \mathcal{A} , together with the left \mathcal{B} -linearity of $\delta_{\mathcal{A}}$ and the right \mathcal{B} -linearity of ${}_{\mathcal{A}}\delta$, allow us to construct a \mathcal{B} -bimodule $\mathcal{A} \square^{\mathcal{C}} \mathcal{A}$. In the Hopf–Galois case, it is the Ehresmann–Schauenburg quantum groupoid [27], which is a noncommutative generalization of the Ehresmann groupoid of a principal bundle [26].

Now, we want to apply Theorem 2.3 to $\mathcal{A} \square^{\mathcal{C}} \mathcal{A}$ with the right \mathcal{A} viewed as a left \mathcal{C} -comodule. To this end, we need the following:

Lemma 2.4. *Let \mathcal{A} and \mathcal{A}' be e -coaugmented \mathcal{C} -Galois extensions with invertible canonical entwining. Then, if an algebra map $f: \mathcal{A} \rightarrow \mathcal{A}'$ is right \mathcal{C} -colinear, it is also left \mathcal{C} -colinear.*

Proof. If $f: \mathcal{A} \rightarrow \mathcal{A}'$ is a \mathcal{C} -colinear algebra homomorphism, then it intertwines the canonical maps can and can' of \mathcal{A} and \mathcal{A}' respectively in the following way:

$$(f \otimes id_{\mathcal{E}}) \circ can = can' \circ (f \otimes_{\mathcal{B}} f). \tag{2.11}$$

Since both can and can' are invertible, this implies that

$$(can')^{-1} \circ (f \otimes id_{\mathcal{E}}) = (f \otimes_{\mathcal{B}} f) \circ can^{-1}. \tag{2.12}$$

Therefore, canonical entwining ψ and ψ' are related as follows:

$$\begin{aligned} ((f \otimes id_{\mathcal{E}}) \circ \psi)(c \otimes a) &= (f \otimes id_{\mathcal{E}})(can((can^{-1}(1 \otimes c)a))) \\ &= (can' \circ (f \otimes_{\mathcal{B}} f))(can^{-1}(1 \otimes c)a) \\ &= can'((f \otimes_{\mathcal{B}} f)(can^{-1}(1 \otimes c)a)) \\ &= can'((f \otimes_{\mathcal{B}} f)(can^{-1}(1 \otimes c))f(a)) \\ &= can'(((can')^{-1}(f(1) \otimes c))f(a)) \\ &= can'(((can')^{-1}(1 \otimes c))f(a)) \\ &= \psi'(c \otimes f(a)) \\ &= (\psi' \circ (id_{\mathcal{E}} \otimes f))(c \otimes a). \end{aligned} \tag{2.13}$$

As c and a are arbitrary, we conclude that

$$(f \otimes id_{\mathcal{E}}) \circ \psi = \psi' \circ (id_{\mathcal{E}} \otimes f). \tag{2.14}$$

Now it follows from the invertibility of the entwining that

$$(\psi')^{-1} \circ (f \otimes id_{\mathcal{E}}) = (id_{\mathcal{E}} \otimes f) \circ \psi^{-1}. \tag{2.15}$$

Finally, evaluating the above equation on $a \otimes e$, we get

$$((\psi')^{-1} \circ (f \otimes id_{\mathcal{E}}))(a \otimes e) = ((id_{\mathcal{E}} \otimes f) \circ \psi^{-1})(a \otimes e), \tag{2.16}$$

which reads

$$({}_{\mathcal{A}'}\delta \circ f)(a) = ((id_{\mathcal{E}} \otimes f) \circ {}_{\mathcal{A}}\delta)(a). \tag{2.17}$$

Since a is arbitrary, we infer the desired left \mathcal{C} -colinearity of f . □

Note that in the Hopf-algebraic setting of comodule algebras, the invertibility of the canonical entwining ψ is equivalent to the bijectivity of the antipode S . Then the left-coaction formula (2.10) reads ${}_{\mathcal{A}}\delta = (S^{-1} \otimes \text{id}) \circ \text{flip} \circ \delta_{\mathcal{A}}$, and the above lemma is trivially true.

The following theorem generalizes the fact that the pullback of the Ehresmann groupoid of a principal bundle is the Ehresmann groupoid of the pullback principal bundle.

Theorem 2.5. *Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{A}'$ be e -coaugmented \mathcal{C} -Galois extensions with bijective canonical entwining. Assume that \mathcal{A}' is faithfully flat as a left and right \mathcal{B}' -module, and that the coalgebra \mathcal{C} is cosemisimple. Then every \mathcal{C} -equivariant algebra map $f: \mathcal{A} \rightarrow \mathcal{A}'$ restricts and corestricts to an algebra homomorphism $\mathcal{B} \rightarrow \mathcal{B}'$, and induces an isomorphism of \mathcal{B}' -bimodules*

$$\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} \mathcal{A}) \otimes_{\mathcal{B}} \mathcal{B}' \cong \mathcal{A}' \square^{\mathcal{C}} \mathcal{A}'.$$

Proof. Observe first that, thanks to Lemma 2.4, Theorem 2.3 admits the left \mathcal{C} -colinear right \mathcal{B} -linear (reversed) version. Now, using the associativity of the tensor product of \mathcal{B}' -bimodules, Theorem 2.3 applied to $V = \mathcal{A}$, and the reversed version of Theorem 2.3 applied to $V = \mathcal{A}'$, we compute:

$$\begin{aligned} \mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} \mathcal{A}) \otimes_{\mathcal{B}} \mathcal{B}' &\cong (\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} \mathcal{A})) \otimes_{\mathcal{B}} \mathcal{B}' & (2.18) \\ &\cong (\mathcal{A}' \square^{\mathcal{C}} \mathcal{A}) \otimes_{\mathcal{B}} \mathcal{B}' \cong \mathcal{A}' \square^{\mathcal{C}} \mathcal{A}'. & \square \end{aligned}$$

2.2. Principal coactions.

Definition 2.6 ([8]). Let $\mathcal{B} \subseteq \mathcal{A}$ be an e -coaugmented \mathcal{C} -Galois extension. We call such an extension a *principal \mathcal{C} -extension* iff

- (1) $\psi: \mathcal{C} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{C}$, $c \otimes a \mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a)$ is bijective (invertibility of the canonical entwining),
- (2) there exists a left \mathcal{B} -linear right \mathcal{C} -colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A}$ (*equivariant projectivity*).

Next, let us consider $\mathcal{A} \otimes \mathcal{A}$ as a \mathcal{C} -bicomodule via the right coaction $\text{id} \otimes \delta_{\mathcal{A}}$ and the left coaction ${}_{\mathcal{A}}\delta \otimes \text{id}$, and \mathcal{C} as a \mathcal{C} -bicomodule via its comultiplication.

Definition 2.7. A *strong connection* is a \mathcal{C} -bilinear map $\ell: \mathcal{C} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that $\ell(e) = 1 \otimes 1$ and $m \circ \ell = \varepsilon$, where m and ε stand for the multiplication in \mathcal{A} and the counit of \mathcal{C} , respectively.

It is clear that the above definition of strong connection coincides with its Hopf-algebraic counterpart by choosing $e = 1$ (see the beginning of Section 2).

Theorem 2.8. *Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{A}'$ be principal \mathcal{C} -extensions, and V a finite-dimensional left \mathcal{C} -comodule. Then every \mathcal{C} -equivariant algebra map $f: \mathcal{A} \rightarrow \mathcal{A}'$*

restricts and corestricts to an algebra homomorphism $\mathcal{B} \rightarrow \mathcal{B}'$, and induces an isomorphism of finitely generated projective left \mathcal{B}' -modules

$$\mathcal{B}' \otimes_{\mathcal{B}} (\mathcal{A} \square^{\mathcal{C}} V) \cong \mathcal{A}' \square^{\mathcal{C}} V$$

that is natural in V . In particular, the induced map $f_*: K_0(\mathcal{B}) \rightarrow K_0(\mathcal{B}')$ satisfies

$$f_*([\mathcal{A} \square^{\mathcal{C}} V]) = [\mathcal{A}' \square^{\mathcal{C}} V].$$

Proof. Note first that combining [8, Lemma 2.2] with [8, Lemma 2.3] implies that a principal \mathcal{C} -extension always admits a strong connection:

$$\ell: \mathcal{C} \longrightarrow \mathcal{A} \otimes \mathcal{A}, \quad \sum_{\mu} a_{\mu} \otimes r_{\mu}(c) := \ell(c) =: \ell(c)^{(1)} \otimes \ell(c)^{(2)} \quad (2.19)$$

(summation suppressed), where $\{a_{\mu}\}_{\mu}$ is a basis of \mathcal{A} . Given a unital linear functional $\varphi: \mathcal{A} \rightarrow \mathbb{C}$, one can construct [18] a left \mathcal{B} -linear map $\sigma: \mathcal{A} \rightarrow \mathcal{B}$,

$$\sigma(a) := a_{(0)} \ell(a_{(1)})^{(1)} \varphi(\ell(a_{(1)})^{(2)}), \quad (2.20)$$

such that $\sigma(b) = b$ for all $b \in \mathcal{B}$. For a finite-dimensional left \mathcal{C} -comodule V with a basis $\{v_i\}_i$, we define the coefficient matrix of the coaction $\varrho: V \rightarrow \mathcal{C} \otimes V$ with respect to $\{v_i\}_i$ by $\varrho(v_i) =: \sum_j c_{ij} \otimes v_j$. By [8, Theorem 3.1], we can now apply (2.19) and (2.20) to the c_{ij} to obtain a finite-size (say N) idempotent matrix e with entries

$$e_{(\mu,i)(\nu,j)} := \sigma(r_{\mu}(c_{ij})a_{\nu}) \in \mathcal{B} \quad (2.21)$$

such that $\mathcal{A} \square^{\mathcal{C}} V \cong \mathcal{B}^N e$ as left \mathcal{B} -modules. Consequently, $\mathcal{A} \square^{\mathcal{C}} V$ is finitely generated projective, and its class in $K_0(\mathcal{B})$ can be represented by e .

Since $f: \mathcal{A} \rightarrow \mathcal{A}'$ satisfies the assumptions of Lemma 2.4,

$$\ell' := (f \otimes f) \circ \ell: \mathcal{C} \longrightarrow \mathcal{A}' \otimes \mathcal{A}' \quad (2.22)$$

is a strong connection on \mathcal{A}' . Next, we choose bases $\{a_{\mu} \mid \mu \in J\}$ and $\{a'_{\mu} \mid \mu \in J'\}$ of \mathcal{A} and \mathcal{A}' respectively in such a way that

$$\{a'_{\mu} = f(a_{\mu}) \mid \mu \in I\} \text{ is a basis of } f(\mathcal{A}) \quad \text{and} \quad \{a_{\mu} \mid \mu \notin I\} \text{ is a basis of } \ker f.$$

Under the above choices, using (2.19) and (2.22), we compute

$$\sum_{\mu} a'_{\mu} \otimes r'_{\mu}(c) := \ell'(c) = \sum_{\mu} f(a_{\mu}) \otimes f(r_{\mu}(c)) = \sum_{\mu} a'_{\mu} \otimes f(r_{\mu}(c)). \quad (2.23)$$

Thus we obtain $r'_{\mu}(h) = f(r_{\mu}(h))$ for all $\mu \in I$.

Now we choose a unital functional φ' on \mathcal{A}' , and take $\varphi := \varphi' \circ f$. For σ' produced from φ' and ℓ' as in (2.20), we check that the diagram

$$\begin{array}{ccc}
 & \mathbb{C} & \\
 \varphi \nearrow & & \nwarrow \varphi' \\
 \mathcal{A} & \xrightarrow{f} & \mathcal{A}' \\
 \cup \downarrow & & \cup \downarrow \\
 B & \xrightarrow{f|_B} & B'
 \end{array}
 \quad (2.24)$$

commutes by the following calculation. First, we compute

$$\begin{aligned}
 \sigma'(a') &= a'_{(0)} \ell'(a'_{(1)})^{(1)} \varphi'(\ell'(a'_{(1)})^{(2)}) \\
 &= a'_{(0)} f(\ell(a'_{(1)})^{(1)}) \varphi'(f(\ell(a'_{(1)})^{(2)})) \\
 &= a'_{(0)} f(\ell(a'_{(1)})^{(1)}) \varphi(\ell(a'_{(1)})^{(2)}).
 \end{aligned}
 \quad (2.25)$$

Next, we plug in $a' = f(a)$ to get

$$\begin{aligned}
 \sigma'(f(a)) &= f(a)_{(0)} f(\ell(f(a)_{(1)})^{(1)}) \varphi(\ell(f(a)_{(1)})^{(2)}) \\
 &= f(a)_{(0)} f(\ell(a_{(1)})^{(1)}) \varphi(\ell(a_{(1)})^{(2)}) \\
 &= f(a)_{(0)} \ell(a_{(1)})^{(1)} \varphi(\ell(a_{(1)})^{(2)}) \\
 &= f(\sigma(a)).
 \end{aligned}
 \quad (2.26)$$

Hence

$$\begin{aligned}
 f(e_{(\mu,i)(v,j)}) &= f(\sigma(r_\mu(c_{ij})a_v)) = \sigma'(f(r_\mu(c_{ij})a_v)) \\
 &= \sigma'(f(r_\mu(c_{ij}))f(a_v)) = \sigma'(r'_\mu(c_{ij})a'_v).
 \end{aligned}
 \quad (2.27)$$

Note that $f(e_{(\mu,i)(v,j)})$ is zero for $v \notin I$ because then $f(a_v) = 0$.

Furthermore, applying [8, Theorem 3.1] to the strong connection ℓ' , the basis $\{a'_\mu\}_\mu$, and the matrix coefficients c_{ij} , for all $\mu, v \in I, i, j \in \{1, \dots, \dim V\}$, we obtain

$$\sigma'(r'_\mu(c_{ij})a'_v) =: e'_{(\mu,i)(v,j)}, \quad (2.28)$$

where the $e'_{(\mu,i)(v,j)}$ are the entries of an idempotent matrix e' such that $\mathcal{B}^{N'} e' \cong \mathcal{A}' \square^{\mathcal{C}} V$ as left \mathcal{B}' -modules. Thus, in the block matrix notation, we arrive at the following crucial equality

$$f(e) = \begin{pmatrix} e' & 0 \\ d & 0 \end{pmatrix}, \quad (2.29)$$

where d is unspecified and f is applied entrywise. Now, taking into account that $f(e)$ is an idempotent matrix, we derive the equality $de' = d$, which allows us to verify that

$$\begin{pmatrix} e' & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}^{-1}. \quad (2.30)$$

Hence, the corresponding finitely generated projective left \mathcal{B}' -modules are isomorphic:

$$\mathcal{B}' \otimes_{\mathcal{B}} (A \square^{\mathcal{C}} V) \cong \mathcal{B}' \otimes_{\mathcal{B}} (B^N e) \cong \mathcal{B}'^N f(e) \cong \mathcal{B}'^{N'} e' \cong \mathcal{A}' \square^{\mathcal{C}} V. \quad (2.31)$$

In particular, $(f|_{\mathcal{B}})_*[e] := [f(e)] = [e'] \in K_0(\mathcal{B}')$. □

2.3. The Hopf-algebraic case revisited. We end this section by arguing that Theorem 2.8 specializes to Theorem 2.1 in the Hopf-algebraic setting. First, observe that the lacking assumption of the principality of \mathcal{A}' in Theorem 2.1 is redundant. Indeed, if ℓ is a strong connection on \mathcal{A} and $f: \mathcal{A} \rightarrow \mathcal{A}'$ is an \mathcal{H} -equivariant algebra homomorphism, then $(f \otimes f) \circ \ell$ is immediately a strong connection on \mathcal{A}' . Furthermore, the bijectivity of the antipode S is equivalent to the invertibility of the canonical entwining, and the coaugmentation is readily provided by $1 \in \mathcal{H}$. Finally, as is explained at the beginning of Section 2, the existence of a strong connection implies both the bijectivity of the canonical map and equivariant projectivity.

3. Pulling back noncommutative vector bundles associated with free actions of compact quantum groups

Let (H, Δ) be a compact quantum group [32]. Let A be a unital C^* -algebra and $\delta_A: A \rightarrow A \otimes_{\min} H$ an injective unital $*$ -homomorphism, where \otimes_{\min} denotes the minimal tensor product of C^* -algebras. We call δ_A a *right coaction* of H on A (or a *right action of the compact quantum group on a compact quantum space*) [25] iff

- (1) $(\delta_A \otimes \text{id}_H) \circ \delta_A = (\text{id}_H \otimes \Delta) \circ \delta_A$ (coassociativity),
- (2) $\{\delta_A(a)(1 \otimes h) \mid a \in A, h \in H\}^{\text{cls}} = A \otimes_{\min} H$ (counitality).

Here “cls” stands for “closed linear span”. Furthermore, a coaction δ_A is called *free* [16] iff

$$\{(a \otimes 1)\delta_A(\tilde{a}) \mid a, \tilde{a} \in A\}^{\text{cls}} = A \otimes_{\min} H. \quad (3.1)$$

Given a compact quantum group (H, Δ) , we denote by $\mathcal{O}(H)$ its dense cosemisimple Hopf $*$ -subalgebra spanned by the matrix coefficients of irreducible (or finite-dimensional) unitary corepresentations [32]. In the same spirit, we define the *Peter–Weyl subalgebra* of A (see [2], cf. [25, 29]) as

$$\mathcal{P}_H(A) := \{a \in A \mid \delta_A(a) \in A \otimes \mathcal{O}(H)\}. \quad (3.2)$$

It follows from Woronowicz’s definition of a compact quantum group that the left and right coactions of H on itself by the comultiplication are free. Also, it is easy to check that $\mathcal{P}_H(H) = \mathcal{O}(H)$, and $\mathcal{P}_H(H)$ is a right $\mathcal{O}(H)$ -comodule via the

restriction-corestriction of δ_A [2]. Its coaction-invariant subalgebra coincides with the fixed-point subalgebra

$$B := \{b \in A \mid \delta_A(b) = b \otimes 1\}. \quad (3.3)$$

A fundamental result concerning Peter–Weyl comodule algebras is that the freeness of an action of a compact quantum group (H, Δ) on a unital C^* -algebra A is *equivalent* to the principality of the Peter–Weyl $\mathcal{O}(H)$ -comodule algebra $\mathcal{P}_H(A)$ [2]. The result bridges algebra and analysis allowing us to conclude from Theorem 2.1 the following crucial claim.

Theorem 3.1. *Let (H, Δ) be a compact quantum group, let A and A' be (H, Δ) - C^* -algebras, B and B' the corresponding fixed-point subalgebras, and $f: A \rightarrow A'$ an equivariant $*$ -homomorphism. Then, if the coaction of (H, Δ) on A is free and V is a representation of (H, Δ) , the following left B' -modules are isomorphic*

$$B' \otimes_B (\mathcal{P}_H(A) \square^{\mathcal{O}(H)} V) \cong \mathcal{P}_H(A') \square^{\mathcal{O}(H)} V.$$

In particular, if V is finite dimensional, then the induced map $f_: K_0(B) \rightarrow K_0(B')$ satisfies*

$$f_*([\mathcal{P}_H(A) \square^{\mathcal{O}(H)} V]) = [\mathcal{P}_H(A') \square^{\mathcal{O}(H)} V].$$

Proof. Note that, since $\mathcal{O}(H)$ is cosemisimple, any comodule is a direct sum of finite-dimensional comodules, so that it suffices to prove the theorem for finite-dimensional representations of (H, Δ) . By [2], the freeness of the (H, Δ) -action is equivalent to principality of the Peter–Weyl comodule algebra $\mathcal{P}_H(A)$, which (as explained at the beginning of Section 2) is tantamount to the existence of a strong connection: $\ell: \mathcal{O}(H) \rightarrow \mathcal{P}_H(A) \otimes \mathcal{P}_H(A)$. Now the claim follows from Theorem 2.1 applied to the case $\mathcal{A} = \mathcal{P}_H(A)$, $\mathcal{H} = \mathcal{O}(H)$, and $\mathcal{B} = B$. \square

Recall that the existence of a strong connection implies equivariant projectivity, which (by [28]) is equivalent to faithful flatness. Combining this with the cosemisimplicity of $\mathcal{O}(H)$, we can view the above Theorem 3.1 as a special case of Theorem 2.3.

3.1. Iterated equivariant noncommutative join construction. Let G be a topological group. Recall that the join of two G -spaces is again a G -space for the diagonal action of G . It is this action that is natural for topological constructions. A straightforward generalization of the diagonal action to the realm of compact quantum groups (H, Δ) acting on C^* -algebras would require that there exists a $*$ -homomorphism $H \otimes_{\min} H \rightarrow H$ extending the algebraic multiplication map, which is typically not the case. However, when taking the join $X * G$ of a G -space X with G , the diagonal action of G on $X * G$ can be gauged to the action on the G -component alone. Thus we obtain an equivalent classical construction that is

amenable to noncommutative deformations. (See [13] for details, cf. [23] for an alternative approach.)

Definition 3.2 ([2, 13]). For any compact quantum group (H, Δ) acting on a unital C^* -algebra A via $\delta_A: A \rightarrow A \otimes_{\min} H$, we define its *equivariant join with H* to be the unital C^* -algebra

$$A \otimes^{\delta_A} H := \left\{ f \in C([0, 1], A) \otimes_{\min} H = C([0, 1], A \otimes_{\min} H) \mid f(0) \in \mathbb{C} \otimes H, f(1) \in \delta_A(A) \right\}.$$

Theorem 3.3 ([2]). Let (H, Δ) be a compact quantum group acting on a unital C^* -algebra A . Then the $*$ -homomorphism

$$\text{id} \otimes \Delta: C([0, 1], A) \otimes_{\min} H \longrightarrow C([0, 1], A) \otimes_{\min} H \otimes_{\min} H$$

restricts and corestricts to a $*$ -homomorphism

$$\delta_\Delta: A \otimes^{\delta_A} H \longrightarrow (A \otimes^{\delta_A} H) \otimes_{\min} H$$

defining an action of (H, Δ) on $A \otimes^{\delta_A} H$. If the action of (H, Δ) on A is free, then so is the above action of (H, Δ) on $A \otimes^{\delta_A} H$.

Note that any equivariant $*$ -homomorphism $F: A \rightarrow A'$ of (H, Δ) - C^* -algebras induces an (H, Δ) -equivariant $*$ -homomorphism $A \otimes^{\delta_A} H \rightarrow A' \otimes^{\delta_{A'}} H$. Indeed, since the $*$ -homomorphism F is equivariant, the $*$ -homomorphism

$$\text{id} \otimes F \otimes \text{id}: C([0, 1]) \otimes_{\min} A \otimes_{\min} H \longrightarrow C([0, 1]) \otimes_{\min} A' \otimes_{\min} H \tag{3.4}$$

restricts and corestricts to an equivariant $*$ -homomorphism $f: A \otimes^{\delta_A} H \rightarrow A' \otimes^{\delta_{A'}} H$. It is clear that the composition of equivariant maps induces the composition of the induced equivariant maps. We refer to this fact as the naturality of the noncommutative equivariant join with H .

Starting from $A = H$, we can iterate Definition 3.2 finitely many times. According to Theorem 3.3, the thus n -times iterated equivariant join A_n comes equipped with a free (H, Δ) -action. The following lemma gives a construction of an equivariant map $A_n \rightarrow A_1$ allowing us later on to apply Theorem 3.1.

Lemma 3.4. Let (H, Δ) be a compact quantum group such that the C^* -algebra H admits a character. Then, for any $n \in \mathbb{N} \setminus \{0\}$, there exists an equivariant $*$ -homomorphism from the n -iterated equivariant join A_n to $A_1 := H \otimes^\Delta H$.

Proof. If $\chi: H \rightarrow \mathbb{C}$ is a character, then

$$f_\chi := \text{ev}_{1/2} \otimes \chi \otimes \text{id}: H \otimes^\Delta H \longrightarrow H \tag{3.5}$$

is an equivariant $*$ -homomorphism. More generally, applying f_χ to the leftmost factor in the n -times iterated equivariant join A_n , we obtain an equivariant map to the $(n - 1)$ -times iterated equivariant join A_{n-1} :

$$\text{id}_{C([0,1])}^{\otimes(n-1)} \otimes f_\chi \otimes \text{id}_H^{\otimes(n-1)}: A_n \longrightarrow A_{n-1}. \tag{3.6}$$

Composing all these maps, we obtain an equivariant map $A_n \rightarrow A_1$, as desired. \square

3.2. Noncommutative tautological quaternionic line bundles and their duals.

To fix notation, let us begin by recalling the definition of $SU_q(2)$ [31]. We take $H = C(SU_q(2))$ to be the universal unital C^* -algebra generated by α and γ subject to the relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, & \alpha\gamma^* &= q\gamma^*\alpha, & \gamma\gamma^* &= \gamma^*\gamma, \\ \alpha^*\alpha + \gamma^*\gamma &= 1, & \alpha\alpha^* + q^2\gamma\gamma^* &= 1, \end{aligned} \tag{3.7}$$

where $0 < q \leq 1$. The coproduct $\Delta: C(SU_q(2)) \rightarrow C(SU_q(2)) \otimes_{\min} C(SU_q(2))$ is given by the formula

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha \otimes 1 & -q\gamma^* \otimes 1 \\ \gamma \otimes 1 & \alpha^* \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes \alpha & 1 \otimes -q\gamma^* \\ 1 \otimes \gamma & 1 \otimes \alpha^* \end{pmatrix}. \tag{3.8}$$

We call the matrix

$$u := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \tag{3.9}$$

the *fundamental representation* matrix of $SU_q(2)$. The counit ε and the antipode S of the Hopf algebra $\mathcal{O}(SU_q(2))$ are respectively given by the formulas

$$\varepsilon \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}. \tag{3.10}$$

Combining the antipode S with the matrix transposition, we obtain the matrix

$$u^\vee := S(u^T) = \begin{pmatrix} \alpha^* & -q\gamma \\ \gamma^* & \alpha \end{pmatrix} \tag{3.11}$$

of the representation *contragredient* to the fundamental representation. Note, however, that

$$\begin{pmatrix} \alpha^* & -q\gamma \\ \gamma^* & \alpha \end{pmatrix} = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}^{-1}, \tag{3.12}$$

which means that u^\vee and u are equivalent as complex representations for arbitrary q .

Motivated by the classical situation, we denote the n -iterated equivariant join of $C(SU_q(2))$ by $C(S_q^{4n+3})$. Note that, except for $n = 0$, our quantum spheres are different from Vaksman–Soibelman quantum spheres [30]. Indeed, it follows immediately from the defining relations of the C^* -algebra of any Vaksman–Soibelman quantum sphere that its space of characters is always a circle. In contrast,

the space of characters of $C(S_q^{4n+3})$ always contains the torus \mathbb{T}^{n+1} appearing as the Cartesian product of the circles of characters of copies of $C(SU_q(2))$ in the iterated equivariant join.

The quaternionic projective space $\mathbb{H}\mathbb{P}^n$ is defined as the quotient space $(\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^\times$ with respect to the right multiplication by non-zero quaternions of non-zero quaternionic column vectors in \mathbb{H}^{n+1} . Observe that the above quotienting restricts to quotienting the space of unit vectors $S^{4n+3} \subset \mathbb{H}^{n+1}$ by the right multiplication by unit quaternions. The latter can be identified with the group $SU(2)$. Since, for $q = 1$, the action of $(C(SU_q(2)), \Delta)$ on $C(S_q^{4n+3})$ is equivalent to the above standard right $SU(2)$ -action on S^{4n+3} with $\mathbb{H}\mathbb{P}^n$ as the space of orbits, we propose the following definition.

Definition 3.5. We define the C^* -algebra of the n -th *quantum quaternionic projective space* as the fixed-point subalgebra

$$C(\mathbb{H}\mathbb{P}_q^n) := C(S_q^{4n+3}/SU_q(2)) := C(S_q^{4n+3})^{SU_q(2)} \\ := \{b \in C(S_q^{4n+3}) \mid \delta_{C(S_q^{4n+3})}(b) = b \otimes 1\}.$$

For the sake of brevity, let us denote the Hopf algebra $\mathcal{O}(C(SU_q(2)))$ by $\mathcal{O}(SU_q(2))$, and the $\mathcal{O}(SU_q(2))$ -comodule algebra $\mathcal{P}_{C(SU_q(2))}(C(S_q^{4n+3}))$ by $\mathcal{P}_{SU_q(2)}(S_q^{4n+3})$. Since the $(C(SU_q(2)), \Delta)$ -action on $C(S_q^{4n+3})$ is free, for any finite-dimensional corepresentation V of $\mathcal{O}(SU_q(2))$, the associated left $C(\mathbb{H}\mathbb{P}_q^n)$ -module $\mathcal{P}_{SU_q(2)}(S_q^{4n+3}) \square^{\mathcal{O}(SU_q(2))} V$ is finitely generated projective by [15, Theorem 1.2].

Next, let V and V^\vee be left corepresentations of $\mathcal{O}(SU_q(2))$ on \mathbb{C}^2 given respectively by the fundamental and contragredient representation matrices u and u^\vee . Now, consider the corresponding associated $C(\mathbb{H}\mathbb{P}_q^n)$ -modules

$$\tau_{\mathbb{H}\mathbb{P}_q^n} := \mathcal{P}_{SU_q(2)}(S_q^{4n+3}) \square^{\mathcal{O}(SU_q(2))} V^\vee, \tag{3.13}$$

$$\tau_{\mathbb{H}\mathbb{P}_q^n}^* := \mathcal{P}_{SU_q(2)}(S_q^{4n+3}) \square^{\mathcal{O}(SU_q(2))} V. \tag{3.14}$$

For $q = 1$, under the standard embedding $\mathbb{C} \subset \mathbb{H}$, the first module is the section module of the tautological quaternionic line bundle, and the second module is the section module of its complex dual. Hence, we refer to $\tau_{\mathbb{H}\mathbb{P}_q^n}$ as the section module of the *noncommutative tautological quaternionic line bundle*, and to $\tau_{\mathbb{H}\mathbb{P}_q^n}^*$ as the section module of its complex dual. It follows from the functoriality of the cotensor product and the equivalence (3.12) that the associated modules $\tau_{\mathbb{H}\mathbb{P}_q^n}$ and $\tau_{\mathbb{H}\mathbb{P}_q^n}^*$ are isomorphic.

Note that restricting coefficients from \mathbb{H} to \mathbb{C} is compatible with dualization. This means that, for every right \mathbb{H} -module (vector space) W , the map from the complex

vector space $\text{Hom}_{\mathbb{C}}(W, \mathbb{C})$ to the left \mathbb{H} -module (vector space) $\text{Hom}_{\mathbb{H}}(W, \mathbb{H})$ defined by

$$\alpha \mapsto (w \mapsto \alpha(w) - \alpha(wj)j) \quad (3.15)$$

is an isomorphism of complex vector spaces. Here j is the quaternionic imaginary unit anticommuting with the complex imaginary unit, and the vector-space structures on Hom-spaces are given by the left multiplication on values. In this way, we identify (as complex vector bundles) the complex dual of a right quaternionic vector bundle with its quaternionic dual.

The following result is our main application of Theorem 3.1.

Theorem 3.6. *For any $n \in \mathbb{N} \setminus \{0\}$ and $0 < q \leq 1$, the finitely generated projective left $C(\mathbb{H}\mathbb{P}_q^n)$ -modules $\tau_{\mathbb{H}\mathbb{P}_q^n}$ and $\tau_{\mathbb{H}\mathbb{P}_q^n}^*$ are not stably free, i.e., the noncommutative tautological quaternionic line bundle and its dual are not stably trivial as noncommutative complex vector bundles.*

Proof. It is explained in [1, Section 3.2] how the non-vanishing of an index pairing computed in [14] for the fundamental representation matrix u implies that $\tau_{\mathbb{H}\mathbb{P}_q^1}^*$ is not stably free.

Furthermore, for $H = C(SU_q(2))$ we have a circle of characters, so that Lemma 3.4 yields an equivariant $*$ -homomorphism $C(S_q^{4n+3}) \rightarrow C(S_q^7)$. This allows us to apply Theorem 3.1 to conclude that

$$[\tau_{\mathbb{H}\mathbb{P}_q^n}^*] = 2[1] \in K_0(C(\mathbb{H}\mathbb{P}_q^n)) \Rightarrow [\tau_{\mathbb{H}\mathbb{P}_q^1}^*] = 2[1] \in K_0(C(\mathbb{H}\mathbb{P}_q^1)). \quad (3.16)$$

As this contradicts the stable non-triviality of $\tau_{\mathbb{H}\mathbb{P}_q^1}^*$, we infer that the left $C(\mathbb{H}\mathbb{P}_q^n)$ -module $\tau_{\mathbb{H}\mathbb{P}_q^n}^*$ is not stably free. Finally, the fact that (3.12) induces an isomorphism between $\tau_{\mathbb{H}\mathbb{P}_q^n}^*$ and $\tau_{\mathbb{H}\mathbb{P}_q^n}$ completes the proof. \square

Now it follows from [1, Proposition 3.2] that the noncommutative Borsuk–Ulam type 2 conjecture [1] holds for $C(S_q^{4n+3})$:

Corollary 3.7. *For any $n \in \mathbb{N} \setminus \{0\}$, there does not exist a $(C(SU_q(2)), \Delta)$ -equivariant $*$ -homomorphism*

$$C(SU_q(2)) \longrightarrow C(S_q^{4n+3}) \otimes_{C(S_q^{4n+3})}^{\delta} C(SU_q(2)).$$

In other words, there does not exist an $SU_q(2)$ -equivariant continuous map from S_q^{4n+7} to S_q^3 .

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