

Open-closed modular operads, the Cardy condition and string field theory

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*Dedicated to the memory of Martin Doubek who died in a traffic accident
on 29th of August 2016, in the middle of work on the present paper.*

Abstract. We prove that the modular operad of diffeomorphism classes of Riemann surfaces with both “open” and “closed” boundary components, in the sense of string field theory, is the modular completion of its genus 0 part quotiented by the Cardy condition. We also provide a finitary presentation of a version of this modular two-colored operad and characterize its algebras via morphisms of Frobenius algebras, recovering some previously known results of Kaufmann, Penner and others. As an important auxiliary tool we characterize inclusions of cyclic operads that induce inclusions of their modular completions.

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Contents

1	Participants in the game	1363
2	Modular completion and the Cardy condition	1380
3	Modular completion of a suboperad	1394
4	Modular completions of the stable and Kaufmann–Penner parts	1399
5	Finitary presentations	1411
	References	1423

Introduction

History of the subject. Barton Zwiebach constructed in [30] “string products” on the Hilbert space of closed string field theory satisfying the “master equation” which

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reflected the structure of the set \mathcal{QC} of diffeomorphism classes of Riemann surfaces of arbitrary genera with labelled holes. As we proved in [22], the master equation expresses that the string products form an algebra over the Feynman transform of \mathcal{QC} . We moreover proved that \mathcal{QC} is the modular completion¹ of its cyclic suboperad $\mathcal{Com} \subset \mathcal{QC}$ consisting of Riemann surfaces of genus 0, that is

$$\mathcal{QC} \cong \text{Mod}(\mathcal{Com}). \quad (1a)$$

Later in [6] we proved a similar statement for open strings. Namely, we identified the modular operad \mathcal{QO} of diffeomorphism classes of Riemann surfaces with marked “open” boundaries with the modular completion of its genus zero part \mathcal{Ass} , i.e. we established an isomorphism

$$\mathcal{QO} \cong \text{Mod}(\mathcal{Ass}).$$

As a follow-up to [6] we argued in [24] that \mathcal{QO} is the symmetrization of a more elementary object $\mathcal{Q}\underline{\mathcal{O}}$ bearing the structure of a non- Σ modular operad. The previous isomorphism then follows from a more elementary

$$\mathcal{Q}\underline{\mathcal{O}} \cong \underline{\text{Mod}}(\underline{\mathcal{Ass}}), \quad (1b)$$

where $\underline{\mathcal{Ass}}$ is the non- Σ version of the associative cyclic operad and $\underline{\text{Mod}}(-)$ the non- Σ modular completion functor.

Aims. We complete the story and establish analogs of the results mentioned above for the combined theory of open and closed strings. The central object will be the set $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ of diffeomorphism classes of Riemann surfaces with both open and closed inputs. It behaves as a non- Σ -modular operad in the open and as an ordinary modular operad in the closed inputs; we call these structures *modular hybrids*. Contrary to expectations, it turns out that $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ is not the modular completion of its genus 0 part $\underline{\mathcal{O}}\mathcal{C}$, but the quotient of this completion by the *Cardy conditions* known to physicists [4, 19] that is, symbolically,

$$\mathcal{Q}\underline{\mathcal{O}}\mathcal{C} \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})/\text{Cardy}. \quad (1c)$$

The unusual feature of the Cardy conditions is that they involve both the open and closed interactions. The above isomorphism restricted to closed resp. open parts gives (1a) resp. (1b), so it is indeed the culmination of the development described in the previous paragraph. As a bonus, we obtain a purely combinatorial proof of a result of [15] characterizing algebras over a version of $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ in terms of morphisms of Frobenius algebras. Our results concerning $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$, namely its Kaufmann–Penner variant recalled in Subsection 4.2, are explicit or implicit in [15].

¹Often called a modular *envelope* in recent literature. We take the liberty to keep our original terminology.

Why is the paper so unbearably long? It is so because we establish three different versions of (1c): the “ordinary,” stable and the Kaufmann–Penner version, each having its own merit — “ordinary” version involves everything that makes sense, stability prevents the combinatorial explosion of the Feynman transform [25, II.5.4], while the Kaufmann–Penner version² admits a nice finitary presentation so its algebras can be described easily. In more detail, the cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}$ contains the stable and KP subhybrids $\underline{\mathcal{O}}\mathcal{C}_{\text{st}}$ and $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ such that

$$\underline{\mathcal{O}}\mathcal{C} \supset \underline{\mathcal{O}}\mathcal{C}_{\text{st}} \supset \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}.$$

Since, as demonstrated in Example 43, the modular completion functor need not preserve inclusions, it is not a priori clear whether

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}) \supset \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}}) \supset \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}).$$

A substantial part of this paper is devoted to the proof that it is indeed the case, therefore the stable and KP cases can be treated as the restricted versions of the ordinary one.

Our approach. There are two approaches to the structures of (topological) string field theory. The classical one of [1] interprets surfaces as cobordisms, with the corresponding combinatorial structure being that of a PROP. The second one does not discriminate between “outputs” and “inputs,” and the relevant combinatorial structure is a modular operad. The difference on the algebra level is that, while in the first approach the bilinear form on the underlying space forms a part of the structure, in the second approach, adopted e.g. in [15] and this article, the bilinear form is absorbed in the definition of the modular endomorphism operad.

Main results. The central technical result is *Proposition 34* which, together with *Propositions 49* and *52*, provides a combinatorial description of the modular completion of the cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}$ and its versions. Our description enables one to interpret, in Remark 35, elements of this completion as diffeomorphism classes of certain Riemann surfaces with embedded loops.

Using the above propositions we obtain the main results of this paper — three versions of the isomorphism (1c): the “ordinary” one in *Theorem 38*, stable one in *Theorem 51* and the Kaufmann–Penner in *Theorem 54*. An interesting byproduct is *Theorem 57* describing the KP version $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ of $\underline{\mathcal{O}}\mathcal{C}$ in terms of generators and relations, together with a characterization of algebras over $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ as couples of Frobenius algebras connected by a morphism satisfying the Cardy and centrality conditions given in *Theorem 59*. Variants of these results are known, see e.g. [15] whose approach is geometric and uses transitivity moves on generalized pair of pants decompositions, or [19] working with 2-cobordisms and the standard forms of these.

²Abbreviated “KP” at some places in the sequel.

Additional physics arguments can be found in [20]. Our approach however provides purely combinatorial proofs.

It turns out that $\underline{\mathcal{O}}\mathcal{C}$ and its versions are more than just cyclic hybrids as they admit a partial modular hybrid structure - they are closed under contractions of open inputs belonging to the same boundary component since this operation does not change the geometric genus. We call structures of this type *premodular hybrids* and denote $\underline{\mathcal{O}}\mathcal{C}$ with this extended structure by $\underline{\mathcal{O}}\mathcal{C}^{\text{pre}}$. In *Theorem 41* we prove that $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ can be alternatively described as the modular completion of this premodular hybrid. Since the Cardy condition already lives in $\underline{\mathcal{O}}\mathcal{C}^{\text{pre}}$, no quotienting is necessary. Finally, *Proposition 44* characterizing inclusions of cyclic operads inducing inclusions of their modular completions is interesting in its own right.

Plan of the paper. *Section 1* begins with recalling the necessary facts about cyclic and modular operads, and their non- Σ versions. We then introduce various versions of hybrids as structures that combine ordinary and non- Σ operads. This section also contains definitions of concrete operads featuring in this article. We believe that *Table 1* helps to navigate through them.

Section 2 is devoted to modular completions of cyclic hybrids and to their quotients by the Cardy condition. It contains the main technical results of this paper, *Proposition 34* and *Theorems 38* and *41*.

Section 3 has an auxiliary character. Its *Proposition 44* describes inclusions $\mathcal{B} \hookrightarrow \mathcal{C}$ of cyclic operads that induce inclusions $\text{Mod}(\mathcal{B}) \hookrightarrow \text{Mod}(\mathcal{C})$ of their modular completions.

In *Section 4* we use the results of *Section 3* to derive the stable and Kaufmann–Penner versions of the theorems in *Section 2*.

Theorem 57 of *Section 5* provides a finitary presentation of the KP modular hybrid $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$. As its application we obtain a result of Kaufmann and Penner describing its algebras in terms of morphisms of Frobenius algebras.

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Conventions. We will assume working knowledge of operads and their versions. Suitable references are monographs [21, 25] complemented with [23] and the original sources [8–10]. Modular completions were introduced in [22] and non- Σ modular operads in [24]. Sundry facts about operads relevant for string field theory can be found e.g. in [2, 3, 5, 7, 11–15, 26, 29, 30]. Operads in this article will, with few exceptions, live in the symmetric monoidal category Set of sets. We will denote by $\mathbb{1}_X$ or simply by $\mathbb{1}$ when X is understood, the identity automorphism of an object X (set, operad, vector space, etc.).

We will denote by \mathbb{N}_+ the set $\{1, 2, \dots\}$ of positive integers, by \mathbb{N} the abelian semigroup $\{0, 1, 2, \dots\}$ of non-negative integers, and by $\frac{1}{2}\mathbb{N}$ the semigroup $\{n/2 \mid n \in \mathbb{N}\}$ of half-integers. By \mathbf{Set} we denote the category of sets, by \mathbf{Fin} the category of finite sets; $|S| \in \mathbb{N}$ will denote the cardinality of $S \in \mathbf{Fin}$.

Operads considered in this article may have “inputs” of two types — open and closed. We will tend to use O as the default notation for open inputs, and C for the closed ones. The operations ${}_u \circ_v$ in cyclic operads will be termed “ \circ -operations,” while the operations \circ_{uv} ³ in modular operads will be called “contractions.”

We will use underlining to distinguish between ordinary (“symmetric”) and non- Σ (“nonsymmetric”) versions of operad-like objects. For instance, the ordinary cyclic operad for associative algebras will be denoted by $\mathcal{A}\mathcal{A}\mathcal{A}$ while its non- Σ version by $\mathcal{A}\mathcal{A}\mathcal{A}$. Likewise, the underlined “ \mathcal{O} ” in “ $\mathcal{Q}\mathcal{O}\mathcal{C}$ ” indicates that this object is a non- Σ modular operad in its open sector, see Table 1 for more examples. We will apply the same convention also to functors. Thus the underlined “ M ” in $\mathbf{Mod}(\mathcal{O}\mathcal{C})$ means that $\mathbf{Mod}(-)$ acts as the non- Σ modular completion on the open part.

1. Participants in the game

Most of the material recalled in this section already appeared in the literature or is an harmless modification of the existing notions. The only novel concept is that of premodular operads and premodular hybrids introduced in Definition 15.

1.1. Standard versions. We start with the following innocuous generalization of cyclic operads.

Definition 1. A *genus-graded* cyclic operad is a cyclic operad with an additional grading by the “operadic genus” (or simply the genus) belonging to an abelian unital semigroup \mathbb{S} .

In other words, genus-graded cyclic operads are cyclic operads in the cartesian monoidal category of \mathbb{S} -graded sets. The components $\mathcal{C}(S)$, $S \in \mathbf{Fin}$, of a genus-graded cyclic operad \mathcal{C} are thus disjoint unions

$$\mathcal{C}(S) = \bigsqcup_{G \in \mathbb{S}} \mathcal{C}(S; G)$$

such that the structure maps ${}_a \circ_b: \mathcal{C}(S_1 \sqcup \{a\}) \times \mathcal{C}(S_2 \sqcup \{b\}) \rightarrow \mathcal{C}(S_1 \sqcup S_2)$ restrict to

$${}_a \circ_b: \mathcal{C}(S_1 \sqcup \{a\}; G_1) \times \mathcal{C}(S_2 \sqcup \{b\}; G_2) \longrightarrow \mathcal{C}(S_1 \sqcup S_2; G_1 + G_2)$$

for arbitrary $S_1, S_2 \in \mathbf{Fin}$ and $G_1, G_2 \in \mathbb{S}$. In this article, \mathbb{S} will either be \mathbb{N} or $\frac{1}{2}\mathbb{N}$.

³In ancient times denoted ξ_{uv} . Notation due to R. Kaufmann.

Symbol	Name	Type	Found in:
Ass	associative operad	cyclic operad	Ex. 4
Ass_{st}	stable associative operad	cyclic operad	Ex. 24
QO	quantum open operad	modular operad	Ex. 5
QO_{st}	stable quantum open operad	modular operad	Ex. 25
Com	commutative operad	genus-graded cyclic operad	Ex. 6
Com_{st}	stable commutative operad	genus-graded cyclic operad	Ex. 26
QC	quantum closed operad	modular operad	Ex. 7
QC_{st}	stable quantum closed operad	modular operad	Ex. 27
QOC	quantum open-closed operad	modular operad	Ex. 8
QOC_{st}	stable quantum open-closed operad	modular operad	Ex. 28
OC	open-closed operad	genus-graded cyclic operad	Ex. 10
OC_{st}	stable open-closed operad	genus-graded cyclic operad	Ex. 29
\underline{Ass}	non- Σ associative operad	non- Σ cyclic operad	Ex. 11
\underline{Ass}_{st}	stable non- Σ associative operad	non- Σ cyclic operad	Ex. 31
\underline{QO}	non- Σ quantum open operad	non- Σ modular operad	Ex. 14
\underline{QO}_{st}	stable non- Σ quantum open operad	non- Σ modular operad	Ex. 32
\underline{O}	multiple boundary operad	premodular operad	Sec. 1.3
\underline{OC}	open-closed hybrid	cyclic hybrid	Ex. 20
\underline{QOC}	quantum open-closed hybrid	modular hybrid	Ex. 19
\underline{OC}^{pre}	open-closed hybrid	premodular hybrid	Ex. 21
\underline{OC}_{st}	stable open-closed hybrid	cyclic hybrid	Ex. 29
\underline{OC}_{KP}	K.-P. open-closed hybrid	cyclic hybrid	Ex. 30
\underline{QOC}_{KP}	K.-P. open-closed hybrid	modular hybrid	Def. 53
\underline{QOC}_{st}	stable quantum open-closed hybrid	modular hybrid	Ex. 28

Table 1. Operads and operad-like structures featuring in this article; see also diagrams (12) and (18).

A morphism of genus-graded cyclic operads is a morphism of the underlying cyclic operads preserving the genus. We let $CycOp$ to denote the category of ordinary cyclic operads (no genus grading) and $CycOp_{gg}$ the category of genus-graded ones. Taking the genus 0 part and ignoring the remaining ones leads to the forgetful functor

$$CycOp_{gg} \longrightarrow CycOp.$$

On the other hand, every cyclic operad can be viewed as a genus-graded cyclic one concentrated in genus $0 \in \mathbb{S}$ with empty components in higher genera. This gives an inclusion of categories

$$\iota: \text{CycOp} \hookrightarrow \text{CycOp}_{\text{gg}}$$

which is the left adjoint of the forgetful functor above.

We will modify the standard definition of *modular operads* as well, by allowing the operadic genus G to belong to a semigroup \mathbb{S} containing \mathbb{N} . This is necessary since we want our theory to accommodate the quantum open-closed operad \mathcal{QOC} recalled later in this section, cf. Example 8 and Remark 9. Let ModOp denote the category of these \mathbb{S} -graded modular operads. The concrete \mathbb{S} will always be clear from the context.

Forgetting the operadic contractions

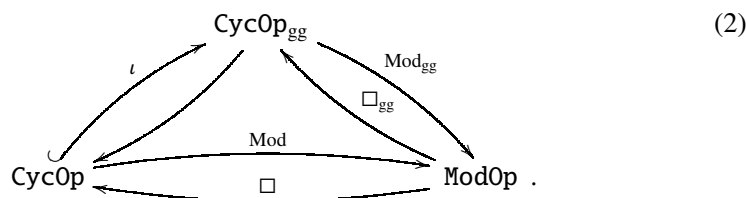
$$\circ_{uv}: \mathcal{M}(S \sqcup \{u, v\}; G) \rightarrow \mathcal{M}(S, G + 1), \quad S \in \text{Fin}, G \in \mathbb{S},$$

every modular operad \mathcal{M} becomes a genus-graded cyclic operad. This gives rise to the forgetful functor

$$\square_{\text{gg}}: \text{ModOp} \rightarrow \text{CycOp}_{\text{gg}}.$$

Its left adjoint will be denoted $\text{Mod}_{\text{gg}}: \text{CycOp}_{\text{gg}} \rightarrow \text{ModOp}$. One also has the “standard” forgetful functor $\square: \text{ModOp} \rightarrow \text{CycOp}$, which replaces everything outside genus 0 by the empty set \emptyset . Its left adjoint $\text{Mod}: \text{CycOp} \rightarrow \text{ModOp}$ is the standard modular completion functor introduced in [22, p. 382].

The above functors fit into the following diagram of adjunctions in which the top arrows are the left adjoints to the bottom ones:



Orders. At this point we need to recall some notions of [24] used later in this article in the definition of non- Σ modular and quantum open-closed operads.

Definition 2. A *cycle* on a finite set $O = \{o_1, \dots, o_n\} \in \text{Fin}$ is an equivalence class of total orders on O modulo the equivalence generated by

$$(o_1, o_2, \dots, o_n) \equiv (o_2, \dots, o_n, o_1).$$

A cycle represented by (o_1, \dots, o_n) will be denoted by $((o_1, \dots, o_n))$. The empty set has a unique cycle on it denoted $(())$. As in [24] we will sometimes call cycles the *pancakes*, imagining them placed in the plane and oriented anticlockwise.

To save the space, we will sometimes leave out the commas, i.e. write e.g. $((o_1o_2o_3))$ instead of $((o_1, o_2, o_3))$. The same simplification will be used also for finite sets, i.e. we will write e.g. $\{abc\}$ instead of $\{a, b, c\}$.

Definition 3. A *multicycle* O on a finite set $O \in \text{Fin}$ consists of

- (i) a disjoint unordered decomposition $O = o_1 \sqcup \dots \sqcup o_b$ of the underlying set O into $b \geq 0$ possibly empty sets, and
- (ii) a cycle o_i on each o_i , $1 \leq i \leq b$.

In the above situation we write $O = o_1 \cdots o_b$.

V. Novák in [27] introduced (partial or total) cyclic orders on a set. It has the property that the disjoint union of cyclically ordered sets bears an induced cyclic order. In Novák’s terminology, a cycle on a finite set O is the same as a total cyclic order of O while a multicycle on O determines a partial cyclic order on O induced from the total cyclic orders of its components. Since we allowed the sets o_i in (ii) to be empty, his cyclic order on O does not determine the multicycle uniquely as it cannot detect the trivial o_i ’s which are part of the structure. Notice that $b = 0$ in Definition 3 is possible only for $O = \emptyset$. We denote the corresponding trivial multicycle by \emptyset .

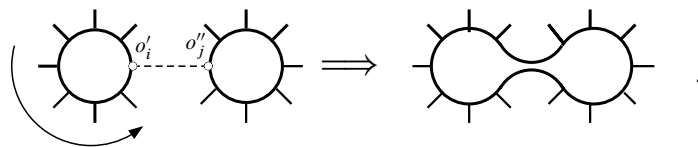
In [24] we introduced two operations on cycles. The *merging* of pancakes $((o'_1, \dots, o'_m))$ and $((o''_1, \dots, o''_n))$ is defined as

$$((o'_1, \dots, o'_m))_{o'_m \circ o''_1} ((o''_1, \dots, o''_n)) := ((o'_1, \dots, o'_{m-1}, o''_1, \dots, o''_n)). \quad (3)$$

Invoking the invariance of cycles under cyclic permutations we see that (3) in fact determines the merging

$$((o'_1, \dots, o'_m))_{o'_i \circ o''_j} ((o''_1, \dots, o''_n))$$

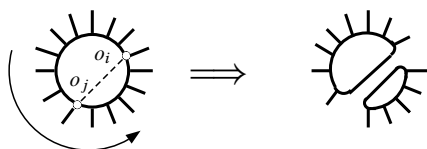
for arbitrary $1 \leq i \leq m, 1 \leq j \leq n$. The following picture explains the terminology:



The second operation on cycles is the *pancake cutting*, defined by the formula

$$\circ_{o_1 o_i} ((o_1, \dots, o_i, \dots, o_n)) := ((o_2, \dots, o_{i-1}))((o_{i+1}, \dots, o_n)), \quad 1 < i \leq n,$$

whose result is a multicycle with two cycles. The invariance under the cyclic group action determines $\circ_{o_i o_j} ((o_1, \dots, o_i, \dots, o_j, \dots, o_n))$ for each $1 \leq i \leq j \leq n$. The intuition behind this operation is explained by the picture below.



The pancake merging of cycles can be extended to multicycles as:

$$\begin{aligned}
 (\mathfrak{o}'_1 \cdots \mathfrak{o}'_{i'} \cdots \mathfrak{o}'_{b'}) \circ_a \circ_b (\mathfrak{o}''_1 \cdots \mathfrak{o}''_{i''} \cdots \mathfrak{o}''_{b''}) \\
 := \mathfrak{o}'_1 \cdots \widehat{\mathfrak{o}'_{i'}} \cdots \mathfrak{o}'_{b'} \circ_a \mathfrak{o}''_1 \cdots \widehat{\mathfrak{o}''_{i''}} \cdots \mathfrak{o}''_{b''} (\mathfrak{o}'_{i'} \circ_a \mathfrak{o}''_{i''}), \quad (4a)
 \end{aligned}$$

where a and b belong to the underlying sets of $\mathfrak{o}'_{i'}$, resp. $\mathfrak{o}''_{i''}$ for some $1 \leq i' \leq b'$, $1 \leq i'' \leq b''$, and the hat indicates the omission as usual. To extend the pancake cutting, i.e. to define $\circ_{uv} (\mathfrak{o}_1 \cdots \mathfrak{o}_b)$, we need to distinguish two cases. It might happen that u and v belong to the same pancake, i.e. $u, v \in \mathfrak{o}_i$ for some $1 \leq i \leq b$. Then we put

$$\circ_{uv} (\mathfrak{o}_1 \cdots \mathfrak{o}_i \cdots \mathfrak{o}_b) := \mathfrak{o}_1 \cdots (\circ_{uv} \mathfrak{o}_i) \cdots \mathfrak{o}_b. \quad (4b)$$

The second possibility is that $u \in \mathfrak{o}_{i'}$ and $v \in \mathfrak{o}_{i''}$ for $1 \leq i' \neq i'' \leq b$. Then

$$\circ_{uv} (\mathfrak{o}_1 \cdots \mathfrak{o}_{i'} \cdots \mathfrak{o}_{i''} \cdots \mathfrak{o}_b) := \mathfrak{o}_1 \cdots \widehat{\mathfrak{o}_{i'}} \cdots \widehat{\mathfrak{o}_{i''}} \cdots \mathfrak{o}_b (\mathfrak{o}_{i'} \circ_u \mathfrak{o}_{i''}). \quad (4c)$$

We have therefore defined the multicycles

$$\mathcal{O}' \circ_a \circ_b \mathcal{O}'' \text{ resp. } \circ_{uv} \mathcal{O}$$

for arbitrary multicycles $\mathcal{O}', \mathcal{O}''$ resp. \mathcal{O} on finite sets $\mathcal{O}', \mathcal{O}''$ resp. \mathcal{O} , with elements $a \in \mathcal{O}', b \in \mathcal{O}''$ resp. $u, v \in \mathcal{O}$. Pancake merging offers an effective definition of the cyclic associative operad \mathcal{A}_{ss} :

Example 4. The component $\mathcal{A}_{\text{ss}}(\mathcal{O})$ of the cyclic associative operad \mathcal{A}_{ss} is, for $\mathcal{O} \in \text{Fin}$, the set cycles on \mathcal{O} , that is

$$\mathcal{A}_{\text{ss}}(\mathcal{O}) := \{ \mathfrak{o} \mid \mathfrak{o} \text{ is a cycle on } \mathcal{O} \}.$$

Clearly $|\mathcal{A}_{\text{ss}}(\mathcal{O})| = (|\mathcal{O}| - 1)!$. The structure operations are given by the pancake merging. An automorphism $\rho \in \text{Aut}(\mathcal{O})$ acts on the set $\mathcal{A}_{\text{ss}}(\mathcal{O})$ by

$$\rho((\mathfrak{o}_1, \dots, \mathfrak{o}_n)) := ((\rho(\mathfrak{o}_1), \dots, \rho(\mathfrak{o}_n))).$$

We will denote by \mathcal{A}_{ss} both the cyclic operad \mathcal{A}_{ss} and its genus-graded version $\iota(\mathcal{A}_{\text{ss}})$. The meaning will always be clear from the context.

Example 5. The $(\mathcal{O}; G)$ -component of the quantum open modular operad \mathcal{QO} is, for $\mathcal{O} \in \text{Fin}$ and $G \in \mathbb{N}$ defined as

$$\mathcal{QO}(\mathcal{O}; G) := \left\{ \left[\begin{smallmatrix} \mathfrak{o}_1 \cdots \mathfrak{o}_b \\ g \end{smallmatrix} \right] \mid b \in \mathbb{N}_+, g \in \mathbb{N}, \right. \\
 \left. G = 2g + b - 1, \mathfrak{o}_1 \cdots \mathfrak{o}_b \text{ is a multicycle on } \mathcal{O} \right\} \quad (5)$$

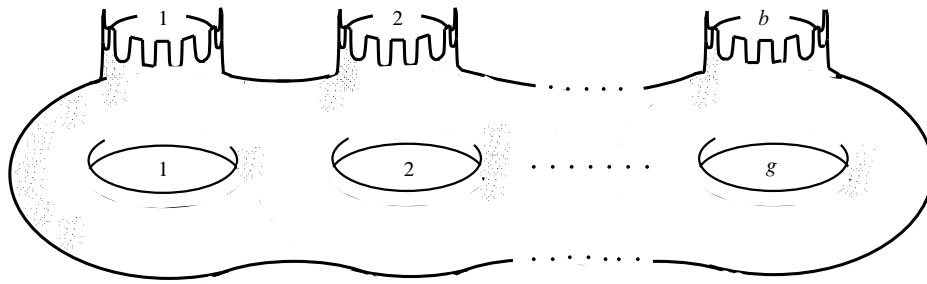


Figure 1. An oriented surface Σ with b toothed boundaries and genus g .

while the other components are empty. In (5), $[o_1 \cdots o_b]$ is a formal symbol depending on the multicycle $o_1 \cdots o_b$ and on a non-negative integer g determined by the operadic genus by the formula $G = 2g + b - 1$. Less formally, $[o_1 \cdots o_b]$ specifies the diffeomorphism class of a two-dimensional oriented genus g surface Σ with b “open” boundaries with teeth labelled by elements of O on boundaries portrayed in Figure 1 taken from [24]. For this reason we call the number g the *geometric genus* of $[o_1 \cdots o_b] \in \mathcal{QO}$. The operadic structure is given by connecting the teeth with ribbons so that the orientability is not violated. Notice that we assume that $b \geq 1$, so Σ has at least one open boundary. The operadic genus G is related to the Euler characteristics χ of Σ by $G = 1 - \chi$.

Notice that not all combinations of G and b are allowed, for instance $G = b = 1$ would imply $g = \frac{1}{2}$. The assumption that $g \in \mathbb{N}$ is precisely the *geometricity* of [24].⁴

The $a \circ_b$ -operations are given by the pancake merging as

$$[o'_1 \cdots o'_{b'}] a \circ_b [o''_1 \cdots o''_{b''}] := [(o'_1 \cdots o'_{b'}) a \circ_b (o''_1 \cdots o''_{b''})]_{g'+g''}, \tag{6a}$$

where $g', g'' \in \mathbb{N}$ and the meaning of the remaining symbols is the same as in (4a). The contractions are given by the pancake cutting, i.e. in the notation of (4b) resp. (4c),

$$\circ_{uv} [o_1 \cdots o_b] := \begin{cases} [o_{uv}(o_1 \cdots o_b)]_g & \text{if } u \text{ and } v \text{ belong to the same pancake, and} \\ [o_{uv}(o_1 \cdots o_b)]_{g+1} & \text{if they belong to different pancakes.} \end{cases} \tag{6b}$$

Automorphisms $\rho \in \text{Aut}(O)$ act according to the formula

$$\rho \left[\left((o_1^1 \cdots o_{n_1}^1) \right) \cdots \left((o_1^b \cdots o_{n_b}^b) \right) \right]_g := \left[\left((\rho(o_1^1) \cdots \rho(o_{n_1}^1)) \right) \cdots \left((\rho(o_1^b) \cdots \rho(o_{n_b}^b)) \right) \right]_g.$$

⁴Notice however that in [24] the symbols g and G are interchanged.

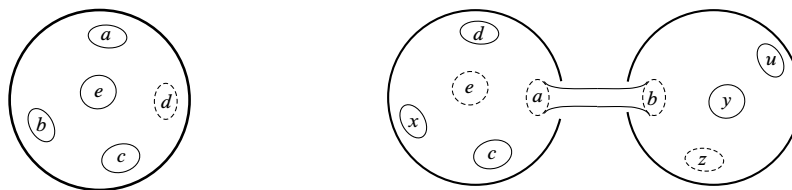


Figure 2. Left: a surface representing the element of $\mathcal{C}om(\{a, b, c, d, e\}; 3)$. Right: the gluing $a \circ b$.

The only solution of $G = 2g + b - 1$ with $G = 0$ for $b \in \mathbb{N}_+$ and $g \in \mathbb{N}$ is $b = 1$, $g = 0$. Therefore the genus 0 component of \mathcal{QO} equals the associative operad $\mathcal{A}ss$, so the injection

$$\mathcal{A}ss \hookrightarrow \mathcal{QO}, \quad \circ \mapsto \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right]$$

defines an isomorphism $\mathcal{A}ss \cong \square(\mathcal{QO})$.

We proved in [6] that $\text{Mod}(\mathcal{A}ss) \cong \mathcal{QO}$. By (2), $\text{Mod}(\mathcal{A}ss) \cong \text{Mod}_{\text{gg}}(\iota(\mathcal{A}ss))$, therefore, under the identification $\iota(\mathcal{A}ss) = \mathcal{A}ss$, one has

$$\text{Mod}_{\text{gg}}(\mathcal{A}ss) \cong \mathcal{QO}.$$

In the following example we introduce a genus-graded version of the cyclic operad describing commutative Frobenius algebras. Its standard definition is modified in such a way that it forms a genus-graded cyclic suboperad of the quantum open-closed operad \mathcal{QOC} recalled in Example 8 below.

Example 6. The component $\mathcal{C}om(C; G)$ of the cyclic genus-graded operad $\mathcal{C}om$ is, for a finite nonempty set $C \in \text{Fin}$ and a non-negative half-integer $G \in \frac{1}{2}\mathbb{N}$ satisfying

$$G = -1 + |C|/2^5 \tag{7}$$

defined as

$$\mathcal{C}om(C; G) := \{C\},$$

while $\mathcal{C}om(C; G)$ is empty for other pairs (C, G) . So all non-empty components of $\mathcal{C}om$ are one-point sets indexed by $C \in \text{Fin}$. The operadic composition is the only possible one and automorphisms from $\text{Aut}(C)$ act trivially.

It is useful in some situations to represent the unique element of $\mathcal{C}om(C; G)$ as the diffeomorphism class of genus-0 compact oriented surfaces with holes indexed by C . In this visualization, the operadic composition is given by connecting these holes by tubes, as indicated in Figure 2 taken from [24].

⁵Notice that $G \in \frac{1}{2}\mathbb{N}$ implies $|C| \geq 2$.

Example 7. The $(C; G)$ -component of the quantum closed modular operad \mathcal{QC} is, for $C \in \mathbf{Fin}$ and $G \in \frac{1}{2}\mathbb{N}$ given by

$$\mathcal{QC}(C; G) := \left\{ \left[\begin{smallmatrix} g \\ C \end{smallmatrix} \right] \mid g \in \mathbb{N}, G = 2g - 1 + |C|/2 \right\},$$

while the other components are empty. Since all non-empty components of $\mathcal{QC}(C; G)$ are one-point sets, the modular operad structure is the unique one with $\text{Aut}(C)$ acting trivially. The symbol $\left[\begin{smallmatrix} g \\ C \end{smallmatrix} \right]$ represents the diffeomorphism class of closed oriented surfaces of genus g with holes indexed by the elements of C . The modular operadic structure is in this representation given by connecting the holes by tubes as in Figure 2.

There is an obvious injection

$$i_C: \mathcal{Com} \hookrightarrow \mathcal{QC}, \quad C \mapsto \left[\begin{smallmatrix} 0 \\ C \end{smallmatrix} \right] \tag{8}$$

which identifies \mathcal{Com} with the cyclic genus-graded suboperad of \mathcal{QC} consisting of elements with $g = 0$. It is easy to verify directly that

$$\text{Mod}_{\text{gg}}(\mathcal{Com}) \cong \mathcal{QC}.$$

A non-genus-graded version of this result appeared in [22, p. 382].

Finally, we recall a two-colored modular operad \mathcal{QOC} containing \mathcal{QO} in the ‘‘open’’ color and \mathcal{QC} in the ‘‘closed’’ one.

Example 8. The $(O, C; G)$ -component $\mathcal{QOC}(O, C; G)$ of the quantum open-closed modular operad is, for $O, C \in \mathbf{Fin}$ and $G \in \frac{1}{2}\mathbb{N}$, defined as the set

$$\left\{ \left[\begin{smallmatrix} o_1 \cdots o_b \\ g \\ C \end{smallmatrix} \right] \mid b \in \mathbb{N}, g \in \mathbb{N}, \right. \\ \left. G = 2g + b - 1 + |C|/2, o_1 \cdots o_b \text{ is a multicycle on } O \right\}. \tag{9}$$

Other components of \mathcal{QOC} are empty. The operadic compositions are

$$\left[\begin{smallmatrix} o'_1 \cdots o'_{b'} \\ g' \\ C' \end{smallmatrix} \right] a \circ b \left[\begin{smallmatrix} o''_1 \cdots o''_{b''} \\ g'' \\ C'' \end{smallmatrix} \right] := \left[\begin{smallmatrix} (o'_1 \cdots o'_{b'}) a \circ b (o''_1 \cdots o''_{b''}) \\ g' + g'' \\ C' \sqcup C'' \end{smallmatrix} \right],$$

if a, b are open inputs, and

$$\left[\begin{smallmatrix} o'_1 \cdots o'_b \\ g' \\ C' \end{smallmatrix} \right] a \circ b \left[\begin{smallmatrix} o''_1 \cdots o''_{b''} \\ g'' \\ C'' \end{smallmatrix} \right] := \left[\begin{smallmatrix} o'_1 \cdots o'_{b'} \ o''_1 \cdots o''_{b''} \\ g' + g'' \\ C' \sqcup C'' \setminus \{a, b\} \end{smallmatrix} \right]$$

if $a \in C', b \in C''$ are closed inputs. We believe that the meaning of the notation, analogous to the one used in previous examples, is clear. The contractions are, for open inputs u and v , given by

$$\circ_{uv} \left[\begin{smallmatrix} o_1 \cdots o_b \\ g \\ C \end{smallmatrix} \right] := \begin{cases} \left[\begin{smallmatrix} \circ_{uv}(o_1 \cdots o_b) \\ g \\ C \end{smallmatrix} \right] & \text{if } u \text{ and } v \text{ belong to the same pancake, and} \\ \left[\begin{smallmatrix} \circ_{uv}(o_1 \cdots o_b) \\ g+1 \\ C \end{smallmatrix} \right] & \text{if they belong to different pancakes.} \end{cases}$$

The contractions for closed inputs $u, v \in C$ are defined by

$$\circ_{uv} \begin{bmatrix} o_1 \dots o_b \\ g \\ C \end{bmatrix} := \begin{bmatrix} o_1 \dots o_b \\ g+1 \\ C \setminus \{u, v\} \end{bmatrix}.$$

There is an obvious action of the group $\text{Aut}(O) \times \text{Aut}(C)$ on the set $\mathcal{QO}\mathcal{C}(O, C; G)$ extending the action of $\text{Aut}(O)$ described in Example 5 and the trivial action of $\text{Aut}(C)$. The symbol

$$\begin{bmatrix} o_1 \dots o_b \\ g \\ C \end{bmatrix}$$

in (9) represents the diffeomorphism class of closed oriented surfaces of genus g with b “open” holes with teeth labelled by elements of O as portrayed in Figure 1, and “closed” holes labelled by elements of C . The operad structure in the open color is given by connecting the teeth via ribbons, the structure in the closed color by connecting the holes via tubes.

There are natural injections $\mathcal{QO} \hookrightarrow \mathcal{QO}\mathcal{C}$ and $\mathcal{QC} \hookrightarrow \mathcal{QO}\mathcal{C}$ given by

$$\begin{bmatrix} o_1 \dots o_b \\ g \\ \emptyset \end{bmatrix} \mapsto \begin{bmatrix} o_1 \dots o_b \\ g \\ \emptyset \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g \\ C \end{bmatrix} \mapsto \begin{bmatrix} \emptyset \\ g \\ C \end{bmatrix}$$

which identify \mathcal{QO} with the genus-graded suboperad of $\mathcal{QO}\mathcal{C}$ consisting of elements with no closed inputs, and \mathcal{QC} with the genus-graded suboperad of $\mathcal{QO}\mathcal{C}$ of elements with $b = 0$ open boundaries.

The genus-graded cyclic operad \mathcal{Com} from Example 6 clearly coincides with the suboperad of $\mathcal{QO}\mathcal{C}$ consisting of elements with $g = b = 0$, i.e. elements of the form

$$\begin{bmatrix} \emptyset \\ 0 \\ C \end{bmatrix}, \quad C \in \text{Fin}.$$

Such an element lives in the operadic genus $G = |C|/2 - 1$. This shall explain the necessity of introducing genus-graded cyclic operads in this article. Notice that the stability assumption $|C| \geq 3$ implies that $G \geq \frac{1}{2}$.

Remark 9. One easily verifies that the Cardy condition

$$\circ_{uv} \left(\begin{bmatrix} ((uqa)) \\ 0 \\ \emptyset \end{bmatrix} \circ_b \begin{bmatrix} ((bvr)) \\ 0 \\ \emptyset \end{bmatrix} \right) = \begin{bmatrix} ((q)) \\ 0 \\ \{c\} \end{bmatrix} \circ_d \begin{bmatrix} ((r)) \\ 0 \\ \{d\} \end{bmatrix} \tag{10}$$

visualized in Figure 3 holds in $\mathcal{QO}\mathcal{C}$. Notice that the Cardy condition involves both “open,” in the left hand side, and “closed,” in the right hand side, structure operations. Let us explain how it forces the operadic genus of $\mathcal{QO}\mathcal{C}$ to be half-integral. The terms

$$\begin{bmatrix} ((uqa)) \\ 0 \\ \emptyset \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} ((bvr)) \\ 0 \\ \emptyset \end{bmatrix}$$

in the left hand side are related by a bijection, hence they have the same operadic genus, say G' . The terms

$$\begin{bmatrix} ((q)) \\ 0 \\ \{c\} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} ((r)) \\ 0 \\ \{d\} \end{bmatrix}$$

in the right hand side also have the same operadic genus, say G'' , by the same reason.

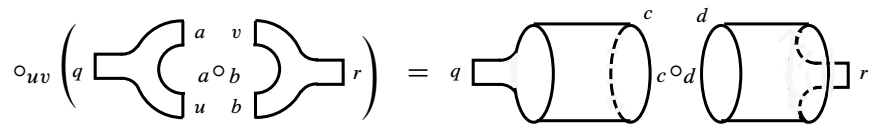


Figure 3. A pictorial form of the Cardy condition. A better picture can be found in [19, (3.44)].

Since \circ_{uv} is required to raise the operadic genus by 1, we have

$$2G' + 1 = 2G''$$

which shows that G' and G'' cannot be simultaneously integral.

The authors of [15] view in Appendix A.3 objects like $\mathcal{QO}\mathcal{C}$ as *c/o-structures*, i.e. objects bigraded by couples (g, χ) , behaving as modular operads with respect to the grading by g in the closed, and as modular operads with respect to $1 - \chi$ in the open structure operations. In our particular case, g would be the geometric genus and χ the Euler characteristic of the surface representing an object of $\mathcal{QO}\mathcal{C}$. The above discussion shows that if one wants to replace *c/o-structures* by a single graded ones, the combined grading has to be given by half-integers.

The modular operad $\mathcal{QO}\mathcal{C}$ contains the following important genus-graded cyclic suboperad.

Example 10. The open-closed cyclic operad \mathcal{OC} is the genus-graded cyclic suboperad of $\mathcal{QO}\mathcal{C}$ consisting of diffeomorphism classes of all surfaces of geometric genus 0, i.e.

$$\mathcal{OC}(O, C; G) := \left\{ \left[\begin{array}{c} o_1 \cdots o_b \\ g \\ c \end{array} \right] \in \mathcal{QO}\mathcal{C}(O, C; G) \mid g = 0 \right\}. \tag{11}$$

The structures encountered so far and their inclusions are organized in the following diagram:

$$\begin{array}{ccc}
 \mathcal{Com} & \hookrightarrow & \mathcal{OC} \\
 \downarrow & & \searrow \\
 \mathcal{OC} & \longrightarrow & \mathcal{QO}\mathcal{C} \\
 \uparrow & & \nearrow \\
 \mathcal{Ass} & \hookrightarrow & \mathcal{QO}
 \end{array} \tag{12}$$

1.2. Non- Σ versions. It turns out that the operad $\mathcal{QO}\mathcal{C}$ from Example 8 is generated, in the sense specified below, by a simpler object $\mathcal{QO}\mathcal{C}$ that behaves in the open color as a non- Σ (non-symmetric) modular operad. Let us briefly recall what these objects are.

Non- Σ cyclic operads were introduced in [25, p.257]. The genus-graded extension of their definition is so obvious that we can safely leave the details to

the reader. We denote their category CycOp resp. CycOp_{gg} in the genus-graded case. The left adjoint

$$\text{Sym}: \text{CycOp}_{\text{gg}} \rightarrow \text{CycOp}_{\text{gg}}$$

to the forgetful functor $\text{Des}: \text{CycOp}_{\text{gg}} \rightarrow \text{CycOp}_{\text{gg}}$ is called the *symmetrization*.

Example 11. The genus-graded cyclic operad $\mathcal{A}_{\mathcal{M}}$ recalled in Example 4 is the symmetrization of the non- Σ cyclic genus-graded operad $\underline{\mathcal{A}}_{\mathcal{M}}$ whose (\mathfrak{o}, G) -component, for a cycle \mathfrak{o} on a finite set O and $G \in \mathbb{S}$, is defined by

$$\underline{\mathcal{A}}_{\mathcal{M}}(\mathfrak{o}; G) := \begin{cases} \{\mathfrak{o}\}, & \text{if } G = 0, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $\underline{\mathcal{A}}_{\mathcal{M}}(\mathfrak{o}; G)$ is either empty or a one-point set, the operadic composition is defined in the only possible manner, and the subgroup $\text{Aut}(\mathfrak{o}) \subset \text{Aut}(O)$ of automorphisms preserving the cyclic order acts trivially. The isomorphism $\mathcal{A}_{\mathcal{M}} \cong \text{Sym}(\underline{\mathcal{A}}_{\mathcal{M}})$ is easy to check.

Since non- Σ modular operads were introduced only very recently [24], we recall their basic features in more detail. While the components of non- Σ cyclic genus-graded operads are indexed by cycles and genera, components of modular non- Σ operads are indexed by *multicycles* and genera. We start by recalling a suitable groupoid of multicycles.

Definition 12. A isomorphism $\mathcal{O}' = \mathfrak{o}'_1 \cdots \mathfrak{o}'_{b'} \longrightarrow \mathcal{O}'' = \mathfrak{o}''_1 \cdots \mathfrak{o}''_{b''}$ of multicycles consists of a bijection $u: \{1, \dots, b'\} \rightarrow \{1, \dots, b''\}$ and of a cyclic order-preserving bijection $\sigma_i: \mathfrak{o}'_i \rightarrow \mathfrak{o}''_{u(i)}$ specified for each $1 \leq i \leq b'$. The groupoid of all multicycles and their isomorphisms will be denoted by MultCyc .

Definition 13. A non- Σ modular operad is a functor

$$\underline{\mathcal{M}}: \text{MultCyc} \times \frac{1}{2}\mathbb{N} \longrightarrow \text{Set},$$

where $\frac{1}{2}\mathbb{N}$ ⁶ is viewed as a discrete category, together with operadic compositions

$$u \circ v: \underline{\mathcal{M}}(\mathcal{O}'; G') \otimes \underline{\mathcal{M}}(\mathcal{O}''; G'') \longrightarrow \underline{\mathcal{M}}(\mathcal{O}' \circ_u \mathcal{O}''; G' + G'') \quad (13)$$

defined for arbitrary disjoint multicycles \mathcal{O}' and \mathcal{O}'' with elements $u \in \mathcal{O}'$, $v \in \mathcal{O}''$ of their underlying sets, and contractions

$$\circ_{uv} = \circ_{vu}: \underline{\mathcal{M}}(\mathcal{O}; G) \longrightarrow \underline{\mathcal{M}}(\circ_{uv}\mathcal{O}; G + 1) \quad (14)$$

given for any multicycle \mathcal{O} and distinct elements $u, v \in \mathcal{O}$ of its underlying set. These data are required to satisfy the expected associativity and equivariance axioms for which we refer the reader to [24].

⁶In [24], $\underline{\mathcal{M}}$ was a functor $\text{MultCyc} \times \mathbb{N} \longrightarrow \text{Set}$, but the difference is irrelevant.

We will denote the category of non- Σ modular operads by $\underline{\text{ModOp}}$. As for cyclic operads, we have the forgetful functor $\text{Des}: \text{ModOp} \rightarrow \underline{\text{ModOp}}$ and its left adjoint $\text{Sym}: \underline{\text{ModOp}} \rightarrow \text{ModOp}$.

Example 14. The quantum open modular operad $\underline{\mathcal{QO}}$ recalled in Example 5 is the symmetrization of the non- Σ modular operad $\underline{\mathcal{QO}} \in \underline{\text{ModOp}}$ whose $(\mathcal{O}; G)$ -component is, for $\mathcal{O} = \circ_1 \cdots \circ_b \in \text{MultCyc}$ and $G \in \mathbb{N}$ defined as

$$\underline{\mathcal{QO}}(\mathcal{O}; G) := \left\{ \begin{bmatrix} \mathcal{O} \\ g \end{bmatrix} \mid g \in \mathbb{N}, G = 2g + b - 1 \right\}$$

while the other components are empty. The structure operations are given by formulas (6a) and (6b). There is a natural inclusion $i_{\mathcal{O}}: \underline{\mathcal{A}} \hookrightarrow \underline{\mathcal{QO}}$ given by

$$i_{\mathcal{O}}(\circ) := \begin{bmatrix} \circ \\ 0 \end{bmatrix}. \tag{15}$$

The symbol $\begin{bmatrix} \mathcal{O} \\ g \end{bmatrix}$ represents, for $\mathcal{O} = \circ_1 \cdots \circ_b$, the diffeomorphism class of surfaces Σ as in Figure 1 with b boundary components and teeth indexed by the underlying set of \mathcal{O} such that each \circ_i , $1 \leq i \leq b$, corresponds to a specific boundary component of Σ , with its total cyclic order induced by the orientation of Σ .

The forgetful functor $\underline{\text{ModOp}} \rightarrow \underline{\text{CycOp}}_{\text{gg}}$ has the left adjoint

$$\underline{\text{Mod}}: \underline{\text{CycOp}}_{\text{gg}} \rightarrow \underline{\text{ModOp}}.$$

The second author proved in [24] that

$$\underline{\text{Mod}}(\underline{\mathcal{A}}) \cong \underline{\mathcal{QO}}.$$

1.3. Premodular operads. Consider the subcollection $\underline{\mathcal{O}}$ of the non- Σ version of the quantum open operad $\underline{\mathcal{QO}}$ of Example 14 consisting of symbols $\begin{bmatrix} \mathcal{O} \\ 0 \end{bmatrix}$ representing surfaces with geometric genus $g = 0$. Inspecting formulas (6a) and (6b) defining the operadic structure of $\underline{\mathcal{QO}}$ we see that, while $\underline{\mathcal{O}}$ is closed under the compositions \circ_b , i.e. it is a genus-graded non- Σ cyclic suboperad of $\underline{\mathcal{QO}}$, $\underline{\mathcal{O}}(\mathcal{O})$ is closed under the contraction \circ_{uv} only if u and v belong to the same cycle of the multicycle $\mathcal{O} = \circ_1 \cdots \circ_b$. The collection $\underline{\mathcal{O}}$ is an example of the following structure.

Definition 15. A *premodular operad* \mathcal{P} is a functor

$$\mathcal{P}: \text{MultCyc} \times \frac{1}{2}\mathbb{N} \longrightarrow \text{Set}$$

together with operadic compositions

$$u \circ_v: \mathcal{P}(\mathcal{O}'; G') \otimes \mathcal{P}(\mathcal{O}''; G'') \longrightarrow \mathcal{P}(\mathcal{O}' \circ_v \mathcal{O}''; G' + G'')$$

defined for arbitrary disjoint multicycles \mathcal{O}' and \mathcal{O}'' and elements $u \in \mathcal{O}'$, $v \in \mathcal{O}''$ of their underlying sets, and contractions

$$\circ_{uv} = \circ_{vu}: \mathcal{P}(\mathcal{O}; G) \longrightarrow \mathcal{P}(\circ_{uv}\mathcal{O}; G + 1)$$

defined only for distinct elements u, v belonging to the *same* underlying set of a cycle of the multicycle O . These data are required to satisfy the axioms analogous to those for non- Σ modular operads. Premodular operads and their morphisms form the category preModOp .

Premodular operads are thus specific partial non- Σ modular operads. Notice that there is no analogue of premodular operads for the standard (Σ -) modular operads.

1.4. Hybrids. We will also need operad-like structures with two colors, “open” and “closed,” which behave differently in each of its colors. Namely, we consider structures that are

- (i) genus-graded non- Σ cyclic operads in the open color and ordinary genus-graded cyclic operads in the closed one,
- (ii) non- Σ modular operads in the open color and ordinary modular operads in the closed one, and
- (iii) premodular operads in the open color and ordinary modular operads in the closed one.

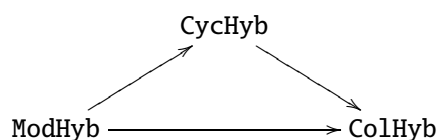
Definition 16. We call a structure of type (i), (ii) resp. (iii) a *cyclic, modular* resp. *premodular hybrid*. Their categories will be denoted CycHyb , ModHyb and PreHyb , respectively.

Modular hybrids should be compared with another formalization of the combinatorial structure of surfaces with open and closed boundaries — *c/o*-structures of [15, Appendix A], cf. also the last paragraph of Remark 9.

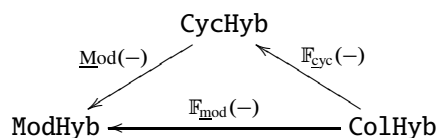
Example 17. Let ColHyb denote the category of hybrid collections which are, by definition, functors

$$E: \text{MultCyc} \times \text{Fin} \times \frac{1}{2}\mathbb{N} \longrightarrow \text{Set}$$

where $\frac{1}{2}\mathbb{N}$ is viewed as a discrete category. Informally, objects of ColHyb are what remains from cyclic (or modular) hybrid when one forgets all \circ -operations (and contractions). We therefore have a commuting diagram



of forgetful functors and the associated commutative diagram



of their left adjoints. The functors $\mathbb{F}_{\text{cyc}}(-)$, $\mathbb{F}_{\text{mod}}(-)$ and $\text{Mod}(-)$ are the free cyclic, free modular and modular hybrid completion functors, respectively.

Example 18. Let \mathbb{k} be a field, not necessarily of characteristic zero. Each (graded) \mathbb{k} -vector space A equipped with a non-degenerate symmetric bilinear form β_A has its modular endomorphism operad $\mathcal{E}nd_A$, see e.g. [25, Example II.5.43], [23, Example 52] or the original source [9, (2.25)]. Given another vector space B with a non-degenerate symmetric bilinear form β_B , one can, in the obvious manner, extend the construction of $\mathcal{E}nd_A$ and create a modular endomorphism hybrid $\mathcal{E}nd_{A,B}$ with components

$$\mathcal{E}nd_{A,B}(\mathcal{O}, C; G) := \text{Lin} \left(\bigotimes_{o \in \mathcal{O}} A_o \otimes \bigotimes_{c \in C} B_c, \mathbb{k} \right). \quad (16)$$

In the above display, \mathcal{O} is the underlying set of the multicyle \mathcal{O} , A_o resp. B_c are the identical copies of the space A resp. B , and \bigotimes is the unordered product in the symmetric monoidal category of graded vector spaces [25, Definition II.1.58]. Due to the presence of non-degenerate bilinear forms, both A and B are finite-dimensional, canonically isomorphic to their duals via raising and lowering indexes. This allows for several formally different but equivalent definitions of the endomorphism modular hybrid. For instance, (16) can be replaced by

$$\mathcal{E}nd_{A,B}(\mathcal{O}, C; G) := \bigotimes_{o \in \mathcal{O}} A_o \otimes \bigotimes_{c \in C} B_c$$

which is more in the spirit of [9].

Having endomorphism hybrids, one can speak about algebras; an *algebra* for a modular hybrid is, by definition, a morphism $\alpha: \mathcal{H} \rightarrow \mathcal{E}nd_{A,B}$. Since $\mathcal{E}nd_{A,B}$ is at the same time also a cyclic hybrid, we define in the same way algebras for cyclic hybrids.

Example 19. The operad $\mathcal{Q}\mathcal{O}\mathcal{C}$ of Example 8 is the symmetrization, in the open color, of the modular hybrid $\mathcal{Q}\mathcal{O}\mathcal{C}$ whose $(\mathcal{O}, C; G)$ -component is, for $\mathcal{O} = o_1 \cdots o_b \in \text{MultCyc}$, $C \in \text{Fin}$ and $G \in \frac{1}{2}\mathbb{N}$ defined as the set of symbols

$$\mathcal{Q}\mathcal{O}\mathcal{C}(\mathcal{O}, C; G) := \left\{ \begin{bmatrix} \mathcal{O} \\ g \\ C \end{bmatrix} \mid g \in \mathbb{N}, G = 2g + b - 1 + |C|/2 \right\}. \quad (17)$$

Example 20. There two-colored genus-graded cyclic operad $\mathcal{O}\mathcal{C}$ from Example 10 is the symmetrization, in the open color, of the cyclic hybrid $\mathcal{O}\mathcal{C}$ defined as the subcollection of $\mathcal{Q}\mathcal{O}\mathcal{C}$ consisting of symbols (17) with $g = 0$. The hybrid $\mathcal{O}\mathcal{C}$ clearly contains both $\mathcal{C}om$ and $\mathcal{A}ss$ as graded cyclic (resp. non- Σ cyclic) suboperads.

Example 21. The cyclic hybrid $\mathcal{O}\mathcal{C}$ from Example 20 is obviously stable under the contractions \circ_{uv} for u and v belonging to the same pancake. It therefore forms a premodular hybrid which we denote by $\mathcal{O}\mathcal{C}^{\text{pre}}$.

One clearly has the following non- Σ version of the diagram in (12) composed of ordinary and non- Σ operads, and cyclic and modular hybrids:

$$\begin{array}{ccc}
 \mathcal{C}om & \longrightarrow & \mathcal{QC} \\
 \downarrow & & \searrow \\
 \underline{\mathcal{QC}} & \longrightarrow & \underline{\mathcal{QC}} \\
 \uparrow & & \nearrow \\
 \underline{\mathcal{Ass}} & \longrightarrow & \underline{\mathcal{QO}}
 \end{array} \tag{18}$$

1.5. Stable versions. Let us slightly generalize the stability condition for modular operads introduced in [9].

Definition 22. The *stable part* of a cyclic or modular operad \mathcal{P} is the collection defined as

$$\mathcal{P}_{st}(S; G) := \mathcal{P}(S; G)$$

if the *stability*

$$2(G - 1) + |S| > 0 \tag{19}$$

is satisfied, while $\mathcal{P}_{st}(S; G) := \emptyset$ for the remaining $(S; G)$. The operad \mathcal{P} is *stable* if $\mathcal{P} = \mathcal{P}_{st}$.

Inequality (19) is equivalent to the absence of continuous families of automorphisms of a Riemann surface of genus G with $|C|$ distinct marked points; whence its name. Notice, that for genus-graded cyclic operads concentrated in genus 0, (19) says that $|S| \geq 3$.

Definition 22 is easily modified to the non- Σ cases, while for hybrids we replace (19) by

$$2(G - 1) + |O| + |C| > 0. \tag{20}$$

The statements in the following lemma can be verified directly.

Lemma 23. *Inequalities (19) and (20) are preserved by the \circ -operations and contractions. If a contraction of x belongs to the stable part of a (non- Σ) modular operad or of a modular hybrid, then x belongs to the stable part as well.*

Thus the stable part of a (non- Σ) cyclic, (non- Σ) modular, or premodular operad, or of a hybrid, is the structure of the same type, with the operations given by the restrictions of the original ones.

Example 24. The stable version \mathcal{Ass}_{st} of the associative cyclic operad \mathcal{Ass} of Example 4 is obtained by requiring that $\mathcal{Ass}_{st}(O) = \emptyset$ if $|O| \leq 2$, i.e.

$$\mathcal{Ass}_{st}(O) := \begin{cases} \mathcal{Ass}(O), & \text{if } |O| \geq 3, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

The operad $\mathcal{Ass}_{\text{st}}$ governs associative algebras with a non-degenerate invariant bilinear form.⁷

Example 25. The stable version of the quantum open modular operad \mathcal{QO} from Example 5 is defined by

$$\mathcal{QO}_{\text{st}}(O; G) := \begin{cases} \mathcal{QO}(O; G), & \text{if } 2(G-1) + |O| > 0, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

The operadic structure is defined by the same formulas as for \mathcal{QO} .

Example 26. The stable version of the genus-graded cyclic commutative operad \mathcal{Com} from Example 6 is defined by

$$\mathcal{Com}_{\text{st}}(C; G) := \begin{cases} \mathcal{Com}(C; G), & \text{if } |C| \geq 3, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to check that, since G is determined by $|C|$ via (7), the condition $|C| \geq 3$ is equivalent to the stability $2(G-1) + |C| > 0$ as expected. Algebras over $\mathcal{Com}_{\text{st}}$ are commutative Frobenius algebras.

Example 27. The stable variant \mathcal{QC}_{st} of the quantum closed operad \mathcal{QC} recalled in Example 7 is defined by

$$\mathcal{QC}_{\text{st}}(C; G) := \begin{cases} \mathcal{QC}(C; G), & \text{if } 2(G-1) + |C| > 0, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Example 28. The stable version $\mathcal{QOC}_{\text{st}}$ of the quantum open-closed operad \mathcal{QOC} from Example 8 is defined by

$$\mathcal{QOC}_{\text{st}}(O, C; G) := \begin{cases} \mathcal{QOC}(O, C; G), & \text{if } 2(G-1) + |O| + |C| > 0, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

The stability condition for the symbol in (9) in $\mathcal{QOC}(O, C; G)$ expressed in terms of its geometric genus and number of boundaries reads

$$4g + 2b + 2|C| + |O| > 4.$$

The stable subhybrid $\mathcal{QOC}_{\text{st}}$ of the modular hybrid \mathcal{QOC} from Example 19 is defined similarly.

Example 29. The stable version \mathcal{OC}_{st} of the open-closed cyclic operad \mathcal{OC} from Example 10 is the genus-graded cyclic suboperad of $\mathcal{QOC}_{\text{st}}$ consisting of all symbols as in (11) satisfying

$$2b + 2|C| + |O| > 4.$$

⁷I.e. non-commutative Frobenius algebras.

element	b	G	$ O $	$ C $	element	b	G	$ O $	$ C $
$\boxed{1}$	1	0	0	0	$\boxed{2}$	1	0	1	0
$\boxed{3}$	1	0	2	0	$\boxed{4}$	2	1	0	0
$\boxed{5}$	1	$\frac{1}{2}$	0	1	$\boxed{6}$	0	0	0	2

Table 2. Unstable elements in $\underline{\mathcal{O}}\mathcal{C}$.

Likewise, the stable cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}_{st}$ consists of all symbols (17) with $g = 0$ satisfying the same inequality. There are six unstable elements of $\underline{\mathcal{O}}\mathcal{C}$, i.e. elements of $\underline{\mathcal{O}}\mathcal{C} \setminus \underline{\mathcal{O}}\mathcal{C}_{st}$, namely

$$\begin{aligned}
 \boxed{1} &:= \begin{bmatrix} \langle \rangle \\ 0 \\ \emptyset \end{bmatrix}, & \boxed{2} &:= \begin{bmatrix} \langle p \rangle \\ 0 \\ \emptyset \end{bmatrix}, & \boxed{3} &:= \begin{bmatrix} \langle pq \rangle \\ 0 \\ \emptyset \end{bmatrix}, \\
 \boxed{4} &:= \begin{bmatrix} \langle \rangle \langle \rangle \\ 0 \\ \{d\} \end{bmatrix}, & \boxed{5} &:= \begin{bmatrix} \langle \rangle \\ 0 \\ \{d\} \end{bmatrix}, & \boxed{6} &:= \begin{bmatrix} \emptyset \\ 0 \\ \{d, e\} \end{bmatrix}.
 \end{aligned}
 \tag{21}$$

Their operadic genera, number of boundaries, and the cardinalities of O and C are listed in Table 2. The symbol $\begin{bmatrix} \emptyset \\ 0 \\ \{d\} \end{bmatrix}$ is excluded since its operadic genus equals $G = -1/2$.

The stable version of the diagram in (12) can be obtained by decorating everything by the subscript “st.”

Example 30. We will consider also the Kaufmann–Penner cyclic subhybrid $\underline{\mathcal{O}}\mathcal{C}_{KP}$ of the stable cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}_{st}$ obtained by discarding the following types of elements of $\underline{\mathcal{O}}\mathcal{C}_{st}$:

$$\begin{aligned}
 \text{type (i): } & \begin{bmatrix} \langle \rangle \dots \langle \rangle \\ 0 \\ \emptyset \end{bmatrix}, \quad b \geq 3; & \text{type (ii): } & \begin{bmatrix} \langle \rangle \dots \langle \rangle \circ \\ 0 \\ \emptyset \end{bmatrix}, \quad b \geq 2, |sfo| \geq 1; \\
 \text{type (iii): } & \begin{bmatrix} \langle \rangle \dots \langle \rangle \\ 0 \\ \{d\} \end{bmatrix}, \quad b \geq 2.
 \end{aligned}$$

The check that $\underline{\mathcal{O}}\mathcal{C}_{KP}$ is closed under \circ -operations is routine. In the proof of Theorem 55 we will need an explicit list of elements

$$\begin{bmatrix} \circ_1 \dots \circ_b \\ 0 \\ C \end{bmatrix} \in \underline{\mathcal{O}}\mathcal{C}
 \tag{22}$$

that do belong to $\underline{\mathcal{O}}\mathcal{C}_{KP}$. We distinguish three cases depending on the number of boundaries.

- If $b \geq 2$, then (22) belongs to $\underline{\mathcal{O}}\mathcal{C}_{KP}$ if and only if:
 - $|C| \geq 2$, or
 - $|C| = 1$ and at least one of \circ_1, \dots, \circ_b is not empty, or
 - $C = \emptyset$ and at least two of \circ_1, \dots, \circ_b are not empty,

- if $b = 1$, then (22) belongs to $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ if and only if $|O| + 2|C| > 2$, and
- if $b = 0$, then (22) belongs to $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ if and only if $|C| \geq 3$.

Example 31. The stable version $\underline{\mathcal{A}}\mathcal{A}_{\text{st}}$ of the non- Σ associative cyclic operad $\underline{\mathcal{A}}\mathcal{A}$ from Example 11 is defined by

$$\underline{\mathcal{A}}\mathcal{A}_{\text{st}}(\mathfrak{o}; G) := \begin{cases} \underline{\mathcal{A}}\mathcal{A}(\mathfrak{o}; G), & \text{if the cardinality of the underlying set of } \mathfrak{o} \text{ is } \geq 3, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Example 32. The stable version $\underline{\mathcal{Q}}\mathcal{O}_{\text{st}}$ of the non- Σ associative modular operad $\underline{\mathcal{Q}}\mathcal{O}$ from Example 14 is defined by

$$\underline{\mathcal{Q}}\mathcal{O}_{\text{st}}(\mathfrak{O}; G) := \begin{cases} \underline{\mathcal{Q}}\mathcal{O}(\mathfrak{O}; G), & \text{if } 2(G - 1) + |O| > 0, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 33. Most of the objects and constructions discussed in this sections can be interpreted in the language of *Feynman categories*, see e.g. [17]. In this setup, operads appear as strong symmetric monoidal functors $\mathcal{F} \rightarrow \mathcal{C}$ from a Feynman category \mathcal{F} into a symmetric monoidal category \mathcal{C} , which is in our case the category of sets. Various versions of operad-like objects can be obtained from the basic Feynman category \mathfrak{G} of graphs [16, Example 1.8] via decorations [16, Sections 2 and 5] or by restriction to a subcategory. Modular envelopes and diagrams that involve them then appear as particular cases of the push-forward construction and its functoriality [16, Section 3].

2. Modular completion and the Cardy condition

This section forms the core of this article. Proposition 34 below explicitly describes the modular completion of the cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}$ and identifies it with the set of diffeomorphism classes of suitable Riemann surfaces with embedded circles. Its proof occupies nearly seven pages. Theorem 38 is the central result of this paper. It describes $\underline{\mathcal{Q}}\mathcal{O}\mathcal{C}$ as the quotient of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$ by the Cardy condition. Theorem 41 in the last subsection characterizes $\underline{\mathcal{Q}}\mathcal{O}\mathcal{C}$ as the modular completion of the premodular hybrid $\underline{\mathcal{O}}\mathcal{C}^{\text{pre}}$.

2.1. Modular completion of cyclic hybrids. In Example 17 we introduced the modular hybrid completion functor $\underline{\text{Mod}}: \text{CycHyb} \rightarrow \text{ModHyb}$ as the left adjoint to the forgetful functor $\text{ModHyb} \rightarrow \text{CycHyb}$. It is clearly a combination of the non- Σ -modular completion functor $\underline{\text{Mod}}$ [24, Section 5] in the open color and the ordinary modular completion Mod [22, p. 382] in the closed one, as indicated by underlying only the first letter of “Mod.” The aim of this subsection is to describe its value on

the open-closed cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}$ from Example 20. This auxiliary technical result is the main step in proving Theorem 38.

We will need the following terminology. Let $O = o_1 \cdots o_b, b \geq 1$, be a nontrivial multicyle. A *decomposition* of O is a disjoint decomposition

$$\{1, \dots, b\} = B_1 \cup \dots \cup B_a$$

of the set indexing the cycles of O into nonempty subsets. When necessary, we will identify it with a choice O_1, \dots, O_a of multicycles $O_i := \{o_j \mid j \in B_i\}, 1 \leq i \leq a$. In this situation we denote $b_i := |B_i|$; clearly $b = b_1 + \dots + b_a$.

Proposition 34. *The component $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})(O, C; G)$ of the modular hybrid completion of $\underline{\mathcal{O}}\mathcal{C}$ is, for $O \in \text{MultCyc}, C \in \text{Fin}$ and $G \in \frac{1}{2}\mathbb{N}$, the set of all symbols*

$$\left[\begin{array}{c} V(O_1; G_1) \cdots V(O_a; G_a) \\ g \\ C \end{array} \right], \quad \text{abbreviated as } \left[\begin{array}{c} V_1 \cdots V_a \\ g \\ C \end{array} \right], \quad (23)$$

where

- (i) $g \in \mathbb{N}$,
- (ii) O_1, \dots, O_a is a decomposition of O ,
- (iii) $G_i \in \mathbb{N}, 1 \leq i \leq a$, are such that $\underline{\mathcal{O}}\mathcal{C}(O_i; G_i)$ is non-empty, and
- (iv) $V(O_i; G_i)$ is the unique nontrivial element of $\underline{\mathcal{O}}\mathcal{C}(O_i; G_i), 1 \leq i \leq a$.

We moreover assume that

$$G = \sum_{i=1}^a G_i + 2g + a - 1 + |C|/2.$$

For $g \in \mathbb{N}$ satisfying $G = 2g - 1 + |C|/2$ we complete the definition by

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})(\emptyset, C; G) := \left\{ \left[\begin{array}{c} \emptyset \\ g \\ C \end{array} \right] \right\},$$

while $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})(O, C; G)$ is empty in all remaining cases.

The modular operad compositions are defined as follows. If u is an open input of $V'_i, 1 \leq i \leq a'$, and v is an open input of $V''_j, 1 \leq j \leq a''$, then

$$\left[\begin{array}{c} V'_1 \cdots V'_{a'} \\ g' \\ C' \end{array} \right] u \circ v \left[\begin{array}{c} V''_1 \cdots V''_{a''} \\ g'' \\ C'' \end{array} \right] := \left[\begin{array}{c} (V'_i u \circ v V''_j) V'_1 \cdots \widehat{V}'_i \cdots V'_{a'} V''_1 \cdots \widehat{V}''_j \cdots V''_{a''} \\ g' + g'' \\ C' \sqcup C'' \end{array} \right].$$

If $u \in C'$ and $v \in C''$ are closed inputs, then

$$\left[\begin{array}{c} V'_1 \cdots V'_{a'} \\ g' \\ C' \end{array} \right] u \circ v \left[\begin{array}{c} V''_1 \cdots V''_{a''} \\ g'' \\ C'' \end{array} \right] := \left[\begin{array}{c} V'_1 \cdots V'_{a'} V''_1 \cdots V''_{a''} \\ g' + g'' \\ C' \sqcup C'' \setminus \{u, v\} \end{array} \right].$$

If u is an open input of V_i and v an open input of V_j , $1 \leq i, j \leq a, i \neq j$, then

$$\circ_{uv} \begin{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} := \begin{bmatrix} (V_i u \circ_v V_j) V_1 \dots \widehat{V}_i \dots \widehat{V}_j \dots V_a \\ g+1 \\ C \end{bmatrix}.$$

If both u, v are open inputs of the same V_i , $1 \leq i \leq a$, then

$$\circ_{uv} \begin{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} = \begin{bmatrix} (\circ_{uv} V_i) V_1 \dots \widehat{V}_i \dots V_a \\ g \\ C \end{bmatrix}.$$

Finally, if $u, v \in C$ are closed inputs, then

$$\circ_{uv} \begin{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} := \begin{bmatrix} V_1 \dots V_a \\ g+1 \\ C \setminus \{u, v\} \end{bmatrix}.$$

The unit $e: \underline{\mathcal{QC}} \rightarrow \underline{\text{Mod}}(\underline{\mathcal{QC}})$ of the adjunction $\text{ModHyb} \overset{\text{CycHyb}}{\rightleftarrows}$ is given by

$$e \left(\begin{bmatrix} \circ_1 \dots \circ_a \\ 0 \\ C \end{bmatrix} \right) := \begin{bmatrix} \begin{bmatrix} \circ_1 \\ 0 \end{bmatrix} \dots \begin{bmatrix} \circ_a \\ 0 \end{bmatrix} \\ 0 \\ C \end{bmatrix} \text{ for } a \geq 1, \quad \text{and} \quad e \left(\begin{bmatrix} \emptyset \\ 0 \\ C \end{bmatrix} \right) := \begin{bmatrix} \emptyset \\ 0 \\ C \end{bmatrix}. \quad (24)$$

We will use the inclusion e of (24) to view $\underline{\mathcal{QC}}$ as a cyclic subhybrid of $\underline{\text{Mod}}(\underline{\mathcal{QC}})$. A combinatorial characterization of pairs (O_i, G_i) for which the set $\underline{\mathcal{QC}}(O_i; G_i)$ in (iii) is non-empty was given in Example 14. Namely, there must exist $g_i \in \mathbb{N}$ such that $G_i = 2g_i + b_i - 1$, V_i is then the symbol $\begin{bmatrix} O_i \\ g_i \end{bmatrix}$, $1 \leq i \leq a$, and the element in (23) takes the form

$$\begin{bmatrix} \begin{bmatrix} O_1 \\ g_1 \end{bmatrix} \dots \begin{bmatrix} O_a \\ g_a \end{bmatrix} \\ g \\ C \end{bmatrix}.$$

The graphical form of the expression above suggests to call $V_i = \begin{bmatrix} O_i \\ g_i \end{bmatrix} \in \underline{\mathcal{QC}}$ in (23) a *nest*.

Remark 35. Symbols (23) can be represented by oriented surfaces Σ with holes indexed by C , b teathed boundaries with teeth indexed by the multicycle O , and an extra data consisting of a embedded non-intersecting circles dividing Σ into $a + 1$ regions, say R_1, \dots, R_a, R_{a+1} , such that R_i contains teathed boundaries indexed by O_i , $1 \leq i \leq a$, and R_{a+1} all holes indexed by C .

Remark 36. We are going to offer another geometric counterpart of Theorem 34. We are indebted to it to an anonymous referee whose suggestions we follow closely.

The open/closed duality presented in [15, Section 4] implies that any surface of geometric genus g with b boundary components indexed by $i = 1, \dots, b$, each having n_i marked points (teeth), and s punctures interpreted as open boundaries without marked points, can be decomposed into more elementary pieces follows.

(a) One glues to a surface of genus g with b closed boundary components cylinders C_1, \dots, C_b with one closed and one open boundary component with n_i teeth, where one cylinder, say C_1 , has moreover s punctures.

(b1) One then further decomposes the cylinders C_2, \dots, C_b into a cylinder C with one closed and one open boundary component, with the open boundary having only one tooth to which a disk (“cogwheel”) with $n_i + 1$ teeth is glued.

(b2) The exceptional cylinder C_1 decomposes into a cylinder C as in (b1) to which a disk D with $n_1 + 1$ teeth and s open boundaries in the interior is glued.

(c) Finally, the punctured disk D appears as a disk with $n_1 + 1 + 2s$ teeth, where the $2s$ teeth come in neighboring pairs. The interior punctures of D are obtained by glueing the neighboring pairs.

Our new realizations are that (b1) and (b2) only need a non- Σ structure in the open part, while (c) uses the premodular structure.

Proof of Proposition 34. We need to verify the universal property saying that for each modular hybrid \mathcal{H} and morphism of cyclic hybrids $F: \underline{\mathcal{O}\mathcal{C}} \rightarrow \mathcal{H}$ there is a unique morphism $\tilde{F}: \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}}) \rightarrow \mathcal{H}$ of modular hybrids such that the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathcal{O}\mathcal{C}} & \xrightarrow{e} & \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}}) \\ & \searrow F & \downarrow \tilde{F} \\ & & \mathcal{H}. \end{array}$$

Uniqueness. Assume that \tilde{F} exists and prove its uniqueness. We have the diagram:

$$\begin{array}{ccccc} & & \mathcal{C}om & \hookrightarrow & \underline{\mathcal{O}\mathcal{C}} & & (25) \\ & & \swarrow i_C & & \searrow \iota_C & & \\ & \underline{\mathcal{O}\mathcal{C}} & \xrightarrow{\quad} & \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}}) & \xrightarrow{F_{i_C}} & \mathcal{H} & \\ \swarrow i_O & \searrow \iota_O & & \downarrow \tilde{F} & & & \\ \underline{\mathcal{A}ss} & \xrightarrow{\quad} & \underline{\mathcal{Q}\mathcal{O}} & \xrightarrow{F} & \mathcal{H} & & \\ & \searrow F_{i_O} & \downarrow F' & & & & \end{array}$$

In this diagram, the inclusion $\iota_O: \underline{\mathcal{Q}\mathcal{O}} \hookrightarrow \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}})$ is the dotted arrow in

$$\begin{array}{ccc} \underline{\mathcal{A}ss} & \xrightarrow{i_O} & \underline{\mathcal{Q}\mathcal{O}} \\ \downarrow & & \downarrow \iota_O \\ \underline{\mathcal{O}\mathcal{C}} & \xrightarrow{e} & \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}}), \end{array}$$

where e is the unit of the adjunction (24) and i_O the inclusion (15). The dotted arrow exists as $\underline{\text{Mod}}(\underline{\mathcal{A}ss}) \cong \underline{\mathcal{Q}\mathcal{O}}$ by [24]. Likewise, the inclusion $\iota_C: \underline{\mathcal{O}\mathcal{C}} \hookrightarrow \underline{\text{Mod}}(\underline{\mathcal{O}\mathcal{C}})$

is the dotted arrow in

$$\begin{array}{ccc} \mathcal{C}om & \xrightarrow{i_C} & \mathcal{Q}\mathcal{C} \\ \downarrow & & \downarrow \iota_C \\ \underline{\mathcal{Q}}\mathcal{C} & \xrightarrow{e} & \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}), \end{array}$$

where i_C is the inclusion (8). It exists since, by [22, p. 382], $\mathcal{Q}\mathcal{C} \cong \text{Mod}(\mathcal{C}om)$. It is easy to show that

$$\iota_O(V) = \begin{bmatrix} V \\ 0 \\ \emptyset \end{bmatrix} \text{ resp. } \iota_C \begin{bmatrix} g \\ C \end{bmatrix} = \begin{bmatrix} \emptyset \\ g \\ C \end{bmatrix}$$

so the maps ι_O and ι_C are indeed inclusions. To simplify the notation, we use these injections to interpret $\mathcal{Q}\mathcal{C}$ and $\underline{\mathcal{Q}}\mathcal{C}$ as suboperads of $\underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C})$.

Further, $F': \underline{\mathcal{Q}}\mathcal{C} \rightarrow \mathcal{H}$ in (25) is the dotted arrow in

$$\begin{array}{ccc} \underline{\text{Ass}} & \xrightarrow{i_O} & \underline{\mathcal{Q}}\mathcal{C} \\ \downarrow & & \downarrow F' \\ \underline{\mathcal{Q}}\mathcal{C} & \xrightarrow{F} & \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}) \end{array}$$

and $F'': \underline{\mathcal{Q}}\mathcal{C} \rightarrow \mathcal{H}$ the dotted arrow in

$$\begin{array}{ccc} \mathcal{C}om & \xrightarrow{i_C} & \mathcal{Q}\mathcal{C} \\ \downarrow & & \downarrow F'' \\ \underline{\mathcal{Q}}\mathcal{C} & \xrightarrow{F} & \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}). \end{array}$$

By the uniqueness of F' resp. F'' ,

$$\tilde{F} \circ \iota_O = \tilde{F}|_{\underline{\mathcal{Q}}\mathcal{C}} = F' \quad \text{and} \quad \tilde{F} \circ \iota_C = \tilde{F}|_{\mathcal{Q}\mathcal{C}} = F''. \tag{26}$$

Before we proceed, we need to introduce the following terminology. Let \circ be a cycle with underlying set O and p an independent symbol. We will call, only for the purposes of this proof, by an *extension* of \circ a cycle $p\circ$ whose underlying set is $\{p\} \sqcup O$ such that the induced cyclic order on O coincides with \circ . It is clear that extensions exist; if $\circ = ((o_1, o_2, \dots, o_n))$, then $((p, o_1, o_2, \dots, o_n))$ is an extension. On the other hand, extensions are not unique. Although $((o_1, o_2, \dots, o_n)) = ((o_2, \dots, o_n, o_1))$,

$$((p, o_1, o_2, \dots, o_n)) \neq ((p, o_2, \dots, o_n, o_1))$$

if $n \geq 2$. Extensions can be generalized to multicycles. If \mathcal{O} is a multicycle, then an extension of \mathcal{O} by p is a multicycle $p\mathcal{O}$ some of whose cycles has been extended by p .

Using the definition of the hybrid modular structure of $\underline{\text{Mod}}(\underline{\mathcal{O}}\underline{\mathcal{C}})$ we get the following expression for its general element:

$$\begin{bmatrix} [O_1] \dots [O_a] \\ g \\ C \end{bmatrix} = \begin{bmatrix} [p'_1 O_1] \\ g_1 \\ \emptyset \end{bmatrix} p'_1 \circ p''_1 \dots \begin{bmatrix} [p'_a O_a] \\ g_a \\ \emptyset \end{bmatrix} p'_a \circ p''_a \begin{bmatrix} [(p''_1)] \dots [(p''_a)] \\ 0 \\ g \\ C \end{bmatrix} \quad (27)$$

where we may further express

$$\begin{bmatrix} [(p''_1)] \dots [(p''_a)] \\ 0 \\ g \\ C \end{bmatrix} = \circ_{u'_1 u''_1} \dots \circ_{u'_g u''_g} \begin{bmatrix} [(p''_1)] \dots [(p''_a)] \\ 0 \\ C \sqcup \{u'_1, u''_1, \dots, u'_g, u''_g\} \end{bmatrix} \quad (28)$$

with some independent variables $u'_1, u''_1, \dots, u'_g, u''_g$. Notice that the elements

$$\begin{bmatrix} [p'_1 O_1] \\ g_1 \\ \emptyset \end{bmatrix}, \dots, \begin{bmatrix} [p'_a O_a] \\ g_a \\ \emptyset \end{bmatrix}$$

in the right hand side of (27) belong to the image of $\iota_{\mathcal{O}}$ and therefore are identified with

$$\begin{bmatrix} p'_1 O_1 \\ g_1 \end{bmatrix}, \dots, \begin{bmatrix} p'_a O_a \\ g_a \end{bmatrix} \in \underline{\mathcal{O}}\underline{\mathcal{C}},$$

while the term

$$\begin{bmatrix} [(p''_1)] \dots [(p''_a)] \\ 0 \\ C \sqcup \{u'_1, u''_1, \dots, u'_g, u''_g\} \end{bmatrix}$$

in the right hand side of (28) belongs to the image of $e: \underline{\mathcal{O}}\underline{\mathcal{C}} \rightarrow \underline{\text{Mod}}(\underline{\mathcal{O}}\underline{\mathcal{C}})$ and therefore is identified with

$$\begin{bmatrix} (p''_1) \dots (p''_a) \\ 0 \\ C \sqcup \{u'_1, u''_1, \dots, u'_g, u''_g\} \end{bmatrix} \in \underline{\mathcal{O}}\underline{\mathcal{C}}.$$

Combining these observations we see that

$$\begin{aligned} \tilde{F} \begin{bmatrix} [O_1] \dots [O_a] \\ g \\ C \end{bmatrix} &= F' \begin{bmatrix} p'_1 O_1 \\ g_1 \end{bmatrix} p'_1 \circ p''_1 \dots F' \begin{bmatrix} p'_a O_a \\ g_a \end{bmatrix} p'_a \circ p''_a \circ_{u'_1 u''_1} \dots \\ &\dots \circ_{u'_g u''_g} F \begin{bmatrix} (p''_1) \dots (p''_a) \\ 0 \\ C \sqcup \{u'_1, u''_1, \dots, u'_g, u''_g\} \end{bmatrix}. \end{aligned} \quad (29)$$

Since F' is unique, (29) determines \tilde{F} uniquely on elements as the one in the left hand side of (27). The proof of the uniqueness is finished by observing that

$$\tilde{F} \begin{bmatrix} \emptyset \\ g \\ C \end{bmatrix} = F'' \begin{bmatrix} g \\ C \end{bmatrix}. \quad (30)$$

Independence on the choices. The aim of this part is to show that the value of the right hand side of (29) does not depend on the choices of the extensions $p'_1\mathcal{O}_1, \dots, p'_a\mathcal{O}_a$. It will be convenient to rewrite it as

$$\begin{aligned} \tilde{F} \left[\begin{array}{c} [\mathcal{O}_1] \dots [\mathcal{O}_a] \\ g \\ C \end{array} \right] &= \circ_{u'_1 u''_1} \dots \\ &\dots \circ_{u'_g u''_g} \left(F' \left[\begin{array}{c} p'_1 \mathcal{O}_1 \\ g_1 \end{array} \right] p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \right) \end{aligned}$$

with $S := C \sqcup \{u'_1, u''_1, \dots, u'_g, u''_g\}$. To prove the independence of the right hand side of (29) on the choices, it clearly suffices to show the independence of the expression

$$F' \left[\begin{array}{c} p'_1 \mathcal{O}_1 \\ g_1 \end{array} \right] p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right]. \tag{31}$$

Since (31) does not depend on the order of $\mathcal{O}_1, \dots, \mathcal{O}_a$, it suffices to prove that it does not depend on the choice of the extension $p'_1\mathcal{O}_1$.

Assume that $\mathcal{O}_1 = \mathcal{o}_1\mathcal{o}_2 \dots \mathcal{o}_b$, $p'_1\mathcal{O}_1 = p'_1\mathcal{o}_1 \mathcal{o}_2 \dots \mathcal{o}_b$, and prove that (31) does not depend on the choice of the extension $p'_1\mathcal{o}_1$ of the cycle \mathcal{o}_1 . One has

$$\begin{aligned} &F' \left[\begin{array}{c} p'_1 \mathcal{o}_1 \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \\ &= F' \left(\left[\begin{array}{c} ((r')) \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] r' \circ r'' \left[\begin{array}{c} r'' p'_1 \mathcal{o}_1 \\ 0 \end{array} \right] \right) p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_1 \mathcal{o}_1 \\ 0 \end{array} \right] p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_1 \mathcal{o}_1 \\ 0 \\ \emptyset \end{array} \right] p'_1 \circ p''_1 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} p'_2 \mathcal{O}_2 \\ g_2 \end{array} \right] p'_2 \circ p''_2 \dots \\ &\quad \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F' \left[\begin{array}{c} r'' p'_1 \mathcal{o}_1 \\ 0 \\ \emptyset \end{array} \right] p'_1 \circ p''_1 F \left[\begin{array}{c} ((p'_1)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \dots \mathcal{o}_b \\ g_1 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} p'_2 \mathcal{O}_2 \\ g_a \end{array} \right] p'_2 \circ p''_2 \dots F' \left[\begin{array}{c} p'_a \mathcal{O}_a \\ g_a \end{array} \right] p'_a \circ p''_a F \left[\begin{array}{c} r'' \mathcal{o}_1 ((p'_2)) \dots ((p'_a)) \\ 0 \\ S \end{array} \right]. \end{aligned}$$

The expression in the last line is clearly independent of the position at which p'_1 was inserted into the cycle \mathcal{o}_1 .

It remains to show that (31) is independent of the order of $\mathcal{o}_1, \dots, \mathcal{o}_b$, i.e. that, choosing $p'_1\mathcal{O}_1 = p'_1\mathcal{o}_i \dots \mathcal{o}_1 \dots \widehat{\mathcal{o}_i} \dots \mathcal{o}_b$, the value of (31) does not depend on i , $1 \leq i \leq b$. Before we do so, we warn the reader that while

$$F' \left[\begin{array}{c} 0 \\ 0 \end{array} \right] = F \left[\begin{array}{c} 0 \\ \emptyset \end{array} \right]$$

for any cycle \circ , it is not necessarily true that

$$F' \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right] = F \left[\begin{smallmatrix} \circ \\ \emptyset \end{smallmatrix} \right]$$

for a *multicycle* $\mathcal{O} = \circ_1 \circ_2 \cdots$.⁸ The reason is that in general

$$\iota_{\mathcal{O}} \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right] \neq e \left[\begin{smallmatrix} \circ \\ \emptyset \end{smallmatrix} \right].$$

One can however still express $F' \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right]$ very explicitly as follows.

For a totally ordered finite set A denote, as in Definition 2, by $((A))$ the induced cycle. Each cycle \circ is of this form for some (non-unique) totally ordered A . So, let $\circ_1 = ((A_1))$ and $\circ_2 = ((A_2))$ be cycles, x', x'' independent symbols and $\circ := ((A_1 x' A_2 x''))$. Then, in $\mathcal{Q}\mathcal{Q}$, one has the identity $\left[\begin{smallmatrix} \circ_1 \circ_2 \\ \circ \end{smallmatrix} \right] = \circ_{x'x''} \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right]$, therefore, since $F': \mathcal{Q}\mathcal{Q} \rightarrow \mathcal{H}$ is a morphism,

$$F' \left[\begin{smallmatrix} \circ_1 \circ_2 \\ \circ \end{smallmatrix} \right] = \circ_{x'x''} F' \left[\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right] \quad \text{in } \mathcal{H}. \tag{32}$$

It is easy to extend (32) to an arbitrary number of cycles, i.e. to an arbitrary multicycle.

With (32) at hand, we are ready to prove that (31) is independent of the order of \circ_1, \dots, \circ_b . To keep the size of formulas within reasonable limits, we assume that $a = 2$, the general case is analogous. One has

$$\begin{aligned} & F' \left[\begin{smallmatrix} p'_1 \circ_1 \circ_2 \circ_3 \cdots \circ_b \\ g_1 \end{smallmatrix} \right] p'_1 \circ p''_1 F' \left[\begin{smallmatrix} p'_2 \circ_2 \\ g_2 \end{smallmatrix} \right] p'_2 \circ p''_2 F' \left[\begin{smallmatrix} ((p'_1))((p'_2)) \\ \circ \\ S \end{smallmatrix} \right] \\ &= F' \left[\begin{smallmatrix} ((s'))((r')) \circ_3 \cdots \circ_b \\ g_1 \end{smallmatrix} \right] s' \circ s'' r' \circ r'' F' \left[\begin{smallmatrix} s'' p'_1 \circ_1 r'' \circ_2 \\ \circ \end{smallmatrix} \right] \\ & \quad p'_1 \circ p''_1 F' \left[\begin{smallmatrix} p'_2 \circ_2 \\ g_2 \end{smallmatrix} \right] p'_2 \circ p''_2 F' \left[\begin{smallmatrix} ((p'_1))((p'_2)) \\ \circ \\ S \end{smallmatrix} \right] \\ &= F' \left[\begin{smallmatrix} ((s'))((r')) \circ_3 \cdots \circ_b \\ g_1 \end{smallmatrix} \right] s' \circ s'' r' \circ r'' \circ_{x'x''} F' \left[\begin{smallmatrix} s'' p'_1 r'' \circ \\ \circ \end{smallmatrix} \right] \\ & \quad p'_1 \circ p''_1 F' \left[\begin{smallmatrix} p'_2 \circ_2 \\ g_2 \end{smallmatrix} \right] p'_2 \circ p''_2 F' \left[\begin{smallmatrix} ((p'_1))((p'_2)) \\ \circ \\ S \end{smallmatrix} \right] \\ &= \circ_{x'x''} F' \left[\begin{smallmatrix} ((s'))((r')) \circ_3 \cdots \circ_b \\ g_1 \end{smallmatrix} \right] s' \circ s'' r' \circ r'' F' \left[\begin{smallmatrix} s'' p'_1 r'' \circ \\ \circ \\ \emptyset \end{smallmatrix} \right] \\ & \quad p'_1 \circ p''_1 F' \left[\begin{smallmatrix} p'_2 \circ_2 \\ g_2 \end{smallmatrix} \right] p'_2 \circ p''_2 F' \left[\begin{smallmatrix} ((p'_1))((p'_2)) \\ \circ \\ S \end{smallmatrix} \right] \\ &= \circ_{x'x''} F' \left[\begin{smallmatrix} ((s'))((r')) \circ_3 \cdots \circ_b \\ g_1 \end{smallmatrix} \right] s' \circ s'' r' \circ r'' F' \left[\begin{smallmatrix} p'_2 \circ_2 \\ g_2 \end{smallmatrix} \right] p'_2 \circ p''_2 F' \left[\begin{smallmatrix} s'' r'' \circ ((p'_2)) \\ \circ \\ S \end{smallmatrix} \right], \end{aligned}$$

where the relation between \circ_1, \circ_2 in the second and \circ in the third line is as in (32). The term in the last line clearly does not see whether p'_1 was inserted into \circ_1 or \circ_2 . This shows that (31) is invariant under the transposition $\circ_1 \leftrightarrow \circ_2$. The transpositions $\circ_1 \leftrightarrow \circ_i$ for arbitrary $1 < i \leq b$ can be discussed similarly.

⁸This become true under some additional conditions discussed in Proposition 38 below.

Morphism property. Let us define $\tilde{F}: \underline{\text{Mod}}(\mathcal{OC}) \rightarrow \mathcal{H}$ by formulas (29) and (30). It is simple to show that such an \tilde{F} extends F , i.e. that $\tilde{F} \circ e = F$; we leave this as an exercise. It is also clear that \tilde{F} defined in this way is equivariant with respect of the automorphisms of the indexing (multicyclic) sets, and is genus-preserving. To finish the proof of Proposition 34 we need to show that this \tilde{F} commutes with the structure operations of modular hybrids. The commutation with the modular operad structure in the “closed” color is simple and we leave it as an exercise.

Let us show that \tilde{F} commutes with the $u \circ v$ -operations in the “open” color. To save the space, we prove it in the simplest nontrivial case. It will be clear that the general case can be attended analogously. We are therefore going to prove that

$$\tilde{F} \left[\begin{array}{c} \text{O}_1 \\ g_1 \\ g' \\ C_1 \end{array} \right] u \circ v \tilde{F} \left[\begin{array}{c} \text{O}_2 \\ g_1 \\ g'' \\ C_2 \end{array} \right] = \tilde{F} \left(\left(\begin{array}{c} \text{O}_1 \\ g_1 \\ g' \\ C_1 \end{array} \right) u \circ v \left(\begin{array}{c} \text{O}_2 \\ g_1 \\ g'' \\ C_2 \end{array} \right) \right) = \tilde{F} \left[\begin{array}{c} \text{O}_1 u \circ v \text{O}_2 \\ g_1 + g_2 \\ g' + g'' \\ C_1 \sqcup C_2 \end{array} \right]. \quad (33)$$

From the definition of \tilde{F} we get

$$\begin{aligned} & \left(F' \left[\begin{array}{c} p'_1 \text{O}_1 \\ g_1 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] \right) u \circ v \left(F' \left[\begin{array}{c} p'_2 \text{O}_2 \\ g_2 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \right) \\ &= F' \left[\begin{array}{c} p'_1 \text{O}_1 \\ g_1 \end{array} \right] u \circ v F' \left[\begin{array}{c} p'_2 \text{O}_2 \\ g_2 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} p'_1 p'_2 (\text{O}_1 u \circ v \text{O}_2) \\ g_1 + g_2 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right]. \end{aligned}$$

Assume that $\text{O}_1 u \circ v \text{O}_2 = \text{o}_1 \text{o}_2 \cdots \text{o}_b$ and $p'_1 p'_2 (\text{O}_1 u \circ v \text{O}_2) = p'_1 p'_2 \text{o}_1 \text{o}_2 \cdots \text{o}_b$. Then

$$\begin{aligned} & F' \left[\begin{array}{c} p'_1 p'_2 (\text{O}_1 u \circ v \text{O}_2) \\ g_1 + g_2 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} p'_1 p'_2 \text{o}_1 \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_1 p'_2 \text{o}_1 \\ 0 \end{array} \right] p'_1 \circ p'_1 F \left[\begin{array}{c} ((p'_1)) \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_2 \text{o}_1 \\ 0 \\ S_1 \end{array} \right] p'_2 \circ p'_2 F \left[\begin{array}{c} ((p'_2)) \\ 0 \\ S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' \text{o}_1 \\ 0 \\ S_1 \sqcup S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p' \text{o}_1 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' F' \left[\begin{array}{c} ((p'')) \\ 0 \\ S_1 \sqcup S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p' \text{o}_1 \\ 0 \end{array} \right] p' \circ p'' F' \left[\begin{array}{c} ((p'')) \\ 0 \\ S_1 \sqcup S_2 \end{array} \right] \\ &= F' \left[\begin{array}{c} p' \text{o}_1 \text{o}_2 \cdots \text{o}_b \\ g_1 + g_2 \end{array} \right] p' \circ p'' F' \left[\begin{array}{c} ((p'')) \\ 0 \\ S_1 \sqcup S_2 \end{array} \right] = F' \left[\begin{array}{c} p' (\text{O}_1 u \circ v \text{O}_2) \\ g_1 + g_2 \end{array} \right] p' \circ p'' F' \left[\begin{array}{c} ((p'')) \\ 0 \\ S_1 \sqcup S_2 \end{array} \right]. \end{aligned}$$

The last term of the above display equals the right hand side of (33) evaluated using (29).

Let us prove that \tilde{F} commutes with the “open” contractions \circ_{uv} . Again we discuss only the simplest nontrivial case, the general one can be treated similarly. We will verify that

$$\circ_{uv} \tilde{F} \left[\begin{array}{c} [O_1] [O_2] \\ [g_1] [g_2] \\ g \\ C \end{array} \right] = \tilde{F} \left(\circ_{uv} \left[\begin{array}{c} [O_1] [O_2] \\ [g_1] [g_2] \\ g \\ C \end{array} \right] \right). \tag{34}$$

Assume that u, v belongs to the same multicycle, say to O_1 . Then (34) boils to

$$\circ_{uv} \tilde{F} \left[\begin{array}{c} [O_1] [O_2] \\ [g_1] [g_2] \\ g \\ C \end{array} \right] = \tilde{F} \left[\begin{array}{c} \circ_{uv} [O_1] [O_2] \\ g \\ C \end{array} \right] \tag{35}$$

in which, by definition,

$$\circ_{uv} [O_1] = \left[\begin{array}{c} \circ_{uv} O_1 \\ g'_1 \end{array} \right],$$

where g'_1 equals g_1 or $g_1 + 1$ depending on whether u and v belong to the same cycle or the different cycles of O_1 . We therefore rewrite (35) as

$$\circ_{uv} \tilde{F} \left[\begin{array}{c} [O_1] [O_2] \\ [g_1] [g_2] \\ g \\ C \end{array} \right] = \tilde{F} \left[\begin{array}{c} \circ_{uv} O_1 [O_2] \\ [g'_1] [g_2] \\ g \\ C \end{array} \right]. \tag{36}$$

The left hand side of (36) equals

$$\begin{aligned} \circ_{uv} \left(F' \left[\begin{array}{c} p'_1 O_1 \\ g_1 \end{array} \right] p'_1 \circ_{p'_1} F' \left[\begin{array}{c} p'_2 O_2 \\ g_2 \end{array} \right] p'_2 \circ_{p'_2} F \left[\begin{array}{c} ((p'_1))((p'_2)) \\ 0 \\ S \end{array} \right] \right) \\ = \circ_{uv} F' \left[\begin{array}{c} p'_1 O_1 \\ g_1 \end{array} \right] p'_1 \circ_{p'_1} F' \left[\begin{array}{c} p'_2 O_2 \\ g_2 \end{array} \right] p'_2 \circ_{p'_2} F \left[\begin{array}{c} ((p'_1))((p'_2)) \\ 0 \\ S \end{array} \right] \\ = F' \left[\begin{array}{c} p'_1 \circ_{uv} O_1 \\ g'_1 \end{array} \right] p'_1 \circ_{p'_1} F' \left[\begin{array}{c} p'_2 O_2 \\ g_2 \end{array} \right] p'_2 \circ_{p'_2} F \left[\begin{array}{c} ((p'_1))((p'_2)) \\ 0 \\ S \end{array} \right], \end{aligned}$$

which is the right hand side of (36) expressed via (29). Before we go further, we need to prove an auxiliary

Sublemma 37. *Let o_1 and o_2 be cycles, S a finite set and u, v, p' and p'' independent symbols. Then*

$$\circ_{uv} F \left[\begin{array}{c} o_1 \ o_2 \\ 0 \\ S \sqcup \{u, v\} \end{array} \right] = \circ_{p' p''} F \left[\begin{array}{c} p' o_1 \ p'' o_2 \\ 0 \\ S \end{array} \right]. \tag{37}$$

Proof of the sublemma. It follows from the axioms of modular operads that

$$\circ_{uv} \left(F \left[\begin{array}{c} o_1 \\ 0 \\ S \sqcup \{u\} \end{array} \right] p' \circ_{p'} F \left[\begin{array}{c} o_2 \\ 0 \\ \{v\} \end{array} \right] \right) = \circ_{p' p''} \left(F \left[\begin{array}{c} o_1 \\ 0 \\ S \sqcup \{u\} \end{array} \right] u \circ_v F \left[\begin{array}{c} o_2 \\ 0 \\ \{v\} \end{array} \right] \right).$$

Equation (37) is then a consequence of the fact that F is a morphism of cyclic hybrids and of the definition of the structure operations in \mathcal{OC} . \square

If $u \in \mathcal{O}_1$ and $v \in \mathcal{O}_2$, (34) boils to

$$\circ_{uv} \tilde{F} \left[\begin{array}{c} [\mathcal{O}_1][\mathcal{O}_2] \\ [g_1][g_2] \\ g \\ C \end{array} \right] = \tilde{F} \left[\begin{array}{c} [\mathcal{O}_1 u \circ v \mathcal{O}_2] \\ [g_1+g_2] \\ g+1 \\ C \end{array} \right]. \tag{38}$$

The left hand side of the above display equals

$$\begin{aligned} & \circ_{uv} \left(F' \left[\begin{array}{c} p'_1 \mathcal{O}_1 \\ g_1 \end{array} \right] p'_1 \circ p''_1 F' \left[\begin{array}{c} p'_2 \mathcal{O}_2 \\ g_2 \end{array} \right] p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right] \right) \\ &= \circ_{uv} \left(F' \left[\begin{array}{c} p'_1 \mathcal{O}_1 \\ g_1 \end{array} \right] p'_1 \circ p''_1 F' \left[\begin{array}{c} p'_2 \mathcal{O}_2 \\ g_2 \end{array} \right] p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right] \right) \\ &= F' \left[\begin{array}{c} p'_1 p'_2 (\mathcal{O}_1 u \circ v \mathcal{O}_2) \\ g_1+g_2 \end{array} \right] p'_1 \circ p''_1 p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right]. \end{aligned}$$

Assume that $\mathcal{O}_1 u \circ v \mathcal{O}_2 = \mathcal{o}_1 \mathcal{o}_2 \cdots \mathcal{o}_b$ and $p'_1 p'_2 (\mathcal{O}_1 u \circ v \mathcal{O}_2) = p'_1 p'_2 \mathcal{o}_1 \mathcal{o}_2 \cdots \mathcal{o}_b$. Then

$$\begin{aligned} & F' \left[\begin{array}{c} p'_1 p'_2 (\mathcal{O}_1 u \circ v \mathcal{O}_2) \\ g_1+g_2 \end{array} \right] p'_1 \circ p''_1 p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} p'_1 p'_2 \mathcal{o}_1 \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] p'_1 \circ p''_1 p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_1 p'_2 \mathcal{o}_1 \\ 0 \end{array} \right] p'_1 \circ p''_1 p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_1))((p''_2)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] r' \circ r'' \circ p'_2 p''_2 F \left[\begin{array}{c} r'' p'_2 \mathcal{o}_1 ((p''_2)) \\ 0 \\ S \end{array} \right] \\ &= F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] r' \circ r'' \circ_{x',x''} F \left[\begin{array}{c} r'' \mathcal{o}_1 \\ 0 \\ S \sqcup \{x',x''\} \end{array} \right] \\ &= \circ_{x',x''} F' \left[\begin{array}{c} ((r')) \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] r' \circ r'' F' \left[\begin{array}{c} r'' p'_2 \mathcal{o}_1 \\ 0 \end{array} \right] p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_2)) \\ 0 \\ S \sqcup \{x',x''\} \end{array} \right] \\ &= \circ_{x',x''} F' \left[\begin{array}{c} p'_2 \mathcal{o}_1 \mathcal{o}_2 \cdots \mathcal{o}_b \\ g_1+g_2 \end{array} \right] p'_2 \circ p''_2 F \left[\begin{array}{c} ((p''_2)) \\ 0 \\ S \sqcup \{x',x''\} \end{array} \right] \\ &= \circ_{x',x''} F' \left[\begin{array}{c} p' (\mathcal{O}_1 u \circ v \mathcal{O}_2) \\ g_1+g_2 \end{array} \right] p' \circ p'' F \left[\begin{array}{c} ((p'')) \\ 0 \\ S \sqcup \{x',x''\} \end{array} \right], \end{aligned}$$

where in the 4th line we used Sublemma 37. It is clear that the last term equals the right hand side of (38) evaluated via (29). This finishes the proof of Proposition 34. \square

2.2. Modular completion modulo Cardy conditions. In this subsection we identify $\mathcal{Q}\mathcal{O}\mathcal{C}$ with the quotient of $\underline{\text{Mod}}(\mathcal{O}\mathcal{C})$ by the Cardy conditions.

Theorem 38. *Let us consider the ideal \mathcal{I}^9 in the modular hybrid $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$ generated by the single relation*

$$\left[\begin{array}{c} \left[\begin{array}{c} ((q)) \\ 0 \end{array} \right] \left[\begin{array}{c} ((r)) \\ 0 \end{array} \right] \\ \emptyset \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} ((q))((r)) \\ 0 \\ \emptyset \end{array} \right] \end{array} \right].$$

Then

$$\mathcal{Q}\underline{\mathcal{O}}\mathcal{C} \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}) / \mathcal{I}. \tag{39}$$

Consequently, for any modular hybrid \mathcal{H} and any morphism $F: \underline{\mathcal{O}}\mathcal{C} \rightarrow \mathcal{H}$ of cyclic hybrids satisfying the relation

$$\circ_{uv} F \left[\begin{array}{c} ((uqvr)) \\ 0 \\ \emptyset \end{array} \right] = F \left[\begin{array}{c} ((q))((r)) \\ 0 \\ \emptyset \end{array} \right], \tag{40}$$

there is a unique morphism $\hat{F}: \mathcal{Q}\underline{\mathcal{O}}\mathcal{C} \rightarrow \mathcal{H}$ of modular hybrids for which the diagram

$$\begin{array}{ccc} \underline{\mathcal{O}}\mathcal{C} & \hookrightarrow & \mathcal{Q}\underline{\mathcal{O}}\mathcal{C} \\ & \searrow F & \downarrow \hat{F} \\ & & \mathcal{H} \end{array}$$

commutes.

Remark 39. Equation (40) is equivalent to

$$\circ_{uv} \left(F \left[\begin{array}{c} ((uqa)) \\ 0 \\ \emptyset \end{array} \right] \circ_b F \left[\begin{array}{c} ((bvr)) \\ 0 \\ \emptyset \end{array} \right] \right) = F \left[\begin{array}{c} ((q)) \\ 0 \\ \{c\} \end{array} \right] \circ_d F \left[\begin{array}{c} ((r)) \\ 0 \\ \{d\} \end{array} \right]$$

which says that F preserves the Cardy condition (10). To see it, use that F , as a morphism of cyclic hybrids, commutes with \circ_b and \circ_d , and then invoke the definition of the \circ -operations in $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$. The necessity to quotient by the Cardy condition follows, in the light of Remark 36, from [15, Theorem 4.1] and the transitivity of the necessary moves described there.

Remark 40. Isomorphism (39) together with the description of $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ given in Example 19 indicates that $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ might be the terminal object in the conjectural category of modular hybrids satisfying the Cardy condition. The rôle of terminality for objects featuring in string theory was observed in [24], cf. the Principle in the introduction of that article. The terminality was then in [16, Definition 4.5] related to *minimal extensions* of Feynman categories. To prove the terminality of $\mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ and interpret it in terms of Feynman categories seems to be an intriguing problem.

Proof of Theorem 38. Let us consider a map $\alpha: \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}) / \mathcal{I} \rightarrow \mathcal{Q}\underline{\mathcal{O}}\mathcal{C}$ given by¹⁰

$$\left[\begin{array}{c} \left[\begin{array}{c} \mathcal{O}_1 \\ g_1 \end{array} \right] \dots \left[\begin{array}{c} \mathcal{O}_a \\ g_a \end{array} \right] \\ g \\ C \end{array} \right] \mapsto \left[\begin{array}{c} \mathcal{O}_1 \dots \mathcal{O}_a \\ g + \sum_{i=1}^a g_i \\ C \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c} \emptyset \\ g \\ C \end{array} \right] \mapsto \left[\begin{array}{c} \emptyset \\ g \\ C \end{array} \right],$$

⁹The term *congruence* instead of “ideal” might be more appropriate in the context of sets, but we take the liberty to stick to the terminology we are used to.

¹⁰We use the same notation for an element of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$ and its equivalence class. The meaning will always be clear from the context.

and a map $\beta: \mathcal{QC} \rightarrow \underline{\text{Mod}}(\mathcal{QC})/\mathcal{I}$ given by

$$\begin{bmatrix} o_1 \cdots o_b \\ g \\ C \end{bmatrix} \mapsto \begin{bmatrix} [o_1 \cdots o_b] \\ g \\ 0 \\ C \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \emptyset \\ g \\ C \end{bmatrix} \mapsto \begin{bmatrix} \emptyset \\ g \\ C \end{bmatrix}.$$

It is easy to check that α and β are well-defined morphisms of modular hybrids and that $\alpha\beta = \mathbb{1}$. To verify $\beta\alpha = \mathbb{1}$, we have to check that

$$\begin{bmatrix} [O_1] \cdots [O_a] \\ [g_1] \cdots [g_a] \\ g \\ C \end{bmatrix} = \begin{bmatrix} [O_1 \cdots O_a] \\ [g + \sum_{i=1}^a g_i] \\ 0 \\ C \end{bmatrix} \quad \text{in } \underline{\text{Mod}}(\mathcal{QC})/\mathcal{I} \tag{41}$$

for $a \geq 1$; for $a = 0$ is the claim trivial. We start by showing that

$$\begin{bmatrix} [(p)] [(qr)] \\ 0 \\ C \end{bmatrix} = \begin{bmatrix} [(p)(qr)] \\ 0 \\ C \end{bmatrix}. \tag{42}$$

To this end, we rewrite the left hand side as

$$\begin{bmatrix} [(p)] [(s')] \\ 0 \\ \emptyset \end{bmatrix} s' \circ s'' \begin{bmatrix} [(qrs'')] \\ 0 \\ C \end{bmatrix} \tag{43a}$$

and apply the generating relation of \mathcal{I} on the first term. We obtain

$$\begin{bmatrix} [(p)(s')] \\ 0 \\ \emptyset \end{bmatrix} s' \circ s'' \begin{bmatrix} [(qrs'')] \\ 0 \\ C \end{bmatrix} \tag{43b}$$

which equals the right hand side of (42).

As the second step of the proof we verify that

$$\begin{bmatrix} [O_1] [O_2] \cdots [O_a] \\ [g_1] [g_2] \cdots [g_a] \\ g \\ C \end{bmatrix} = \begin{bmatrix} [O_1 O_2 \cdots O_a] \\ [g_1 + g_2 \cdots g_a] \\ g \\ C \end{bmatrix}. \tag{44}$$

Assume that $O_1 = o'_1 o'_2 \cdots o'_{b'}$, $O_2 = o''_1 o''_2 \cdots o''_{b''}$ and rewrite the left hand side as

$$\begin{bmatrix} [p' o'_1 o'_2 \cdots o'_{b'}] \\ [g_1] \\ 0 \\ \emptyset \end{bmatrix} p' \circ p'' \begin{bmatrix} [q' o'_1 o'_2 \cdots o'_{b'}] \\ [g_2] \\ 0 \\ \emptyset \end{bmatrix} q' \circ q'' \begin{bmatrix} [(p'')] [(q''r'')] \\ 0 \\ \emptyset \end{bmatrix} r' \circ r'' \begin{bmatrix} [(r'')] [O_3] \cdots [O_a] \\ [g_3] \cdots [g_a] \\ g \\ C \end{bmatrix}. \tag{45}$$

Applying (42) to the third term, we get

$$\begin{bmatrix} [p' o'_1 o'_2 \cdots o'_{b'}] \\ [g_1] \\ 0 \\ \emptyset \end{bmatrix} p' \circ p'' \begin{bmatrix} [q' o'_1 o'_2 \cdots o'_{b'}] \\ [g_2] \\ 0 \\ \emptyset \end{bmatrix} q' \circ q'' \begin{bmatrix} [(p'')(q''r'')] \\ 0 \\ \emptyset \end{bmatrix} r' \circ r'' \begin{bmatrix} [(r'')] [O_3] \cdots [O_a] \\ [g_3] \cdots [g_a] \\ g \\ C \end{bmatrix},$$

which is easily seen to be the right hand side of (44). Using (44) inductively we conclude that the left hand side of (41) equals

$$\left[\begin{array}{c} [O_1 \cdots O_a] \\ [\sum_{i=1}^a g_i] \\ g \\ C \end{array} \right]. \tag{46}$$

The last step we need to prove that $\beta\alpha = \mathbb{1}$ is the equality

$$\left[\begin{array}{c} [o_1 \cdots o_b] \\ [g_1] \\ g \\ C \end{array} \right] = \left[\begin{array}{c} [o_1 \cdots o_b] \\ [g_1 + 1] \\ g^{-1} \\ C \end{array} \right]. \tag{47}$$

By the definition of the contractions in $\underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C})$ its left hand side equals

$$\circ_{p'p''} \left[\begin{array}{c} [((p'))] [p' o_1 \ o_2 \cdots o_b] \\ [0] [g_1] \\ g^{-1} \\ C \end{array} \right], \tag{48}$$

which, by (44), is the same as

$$\circ_{pp'} \left[\begin{array}{c} [((p)) p' o_1 \ o_2 \cdots o_b] \\ [g_1] \\ g^{-1} \\ C \end{array} \right],$$

which is the right hand side of (47). Applying (47) inductively, we see that (46) equals the right hand side of (41). This finishes the proof of $\beta\alpha = \mathbb{1}$ and establishes (39).

Let us prove the second part of the theorem. Denote by $\pi: \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}) \rightarrow \underline{\mathcal{Q}}\underline{\mathcal{Q}}\mathcal{C}$ the natural projection and by $\tilde{F}: \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}) \rightarrow \mathcal{H}$ the unique extension of F guaranteed by the universal property of the modular completion. Such an \tilde{F} descends to \hat{F} in the diagram

$$\begin{array}{ccccc} \underline{\mathcal{Q}}\mathcal{C} & \hookrightarrow & \underline{\text{Mod}}(\underline{\mathcal{Q}}\mathcal{C}) & \xrightarrow{\pi} & \underline{\mathcal{Q}}\underline{\mathcal{Q}}\mathcal{C} \\ & \searrow F & \downarrow \tilde{F} & \swarrow \hat{F} & \\ & & \mathcal{H} & & \end{array}$$

if and only if \tilde{F} preserves the generating relation of \mathcal{J} . But this is indeed so, since

$$\begin{aligned} \tilde{F} \left[\begin{array}{c} [((q))] [((r))] \\ [0] [0] \\ 0 \\ \emptyset \end{array} \right] &= F \left[\begin{array}{c} ((q))((r)) \\ 0 \\ \emptyset \end{array} \right] = \circ_{uv} F \left[\begin{array}{c} ((uqvr)) \\ 0 \\ \emptyset \end{array} \right] \\ &= \circ_{uv} \tilde{F} \left[\begin{array}{c} ((uqvr)) \\ 0 \\ \emptyset \end{array} \right] = \tilde{F} \left[\begin{array}{c} ((q))((r)) \\ 0 \\ \emptyset \end{array} \right], \end{aligned}$$

where the second equality used (40). The uniqueness of \hat{F} follows from the uniqueness of \tilde{F} and the surjectivity of π . This finishes the proof of Theorem 38. \square

2.3. Modular completion of premodular hybrids. One has the functor

$$\square_{\text{pre}}: \text{ModHyb} \rightarrow \text{PreHyb} \tag{49}$$

from the category of modular hybrids to the category of premodular hybrids which forgets all contractions in the “closed” color and contractions \circ_{uv} in the “open” color for which u and v belong to *different* cycles. In this situation there is another version of the modular completion functor, namely the left adjoint

$$\underline{\text{Mod}}_{\text{pre}}: \text{PreHyb} \rightarrow \text{ModHyb}$$

to (49). We have

Theorem 41. *For the premodular hybrid $\underline{\mathcal{Q}}\mathcal{C}^{\text{pre}}$ from Example 21 one has the isomorphism*

$$\underline{\text{Mod}}_{\text{pre}}(\underline{\mathcal{Q}}\mathcal{C}^{\text{pre}}) \cong \underline{\mathcal{Q}}\mathcal{C} \tag{50}$$

of modular hybrids.

Remark 42. Notice that the Cardy condition (10) is already built in $\underline{\mathcal{Q}}\mathcal{C}^{\text{pre}}$, so we do not need to take in (50) the quotient by it.

Proof of Theorem 41. We need to verify that for a arbitrary modular hybrid \mathcal{H} and for any morphism $F: \underline{\mathcal{Q}}\mathcal{C}^{\text{pre}} \rightarrow \square_{\text{pre}}(\mathcal{H})$ of premodular hybrids, there is a unique morphism $\tilde{F}: \underline{\mathcal{Q}}\mathcal{C} \rightarrow \mathcal{H}$ of modular hybrids such that the diagram

$$\begin{array}{ccc} \underline{\mathcal{Q}}\mathcal{C}^{\text{pre}} & \hookrightarrow & \underline{\mathcal{Q}}\mathcal{C} \\ & \searrow F & \downarrow \hat{F} \\ & & \mathcal{H} \end{array}$$

commutes. Since F is a morphism of premodular hybrids, it automatically satisfies relation (40), because

$$\circ_{uv} F \left[\begin{pmatrix} (uqvr) \\ 0 \\ \emptyset \end{pmatrix} \right] = F \left(\circ_{pp'} \left[\begin{pmatrix} (uqvr) \\ 0 \\ \emptyset \end{pmatrix} \right] \right) = F \left[\begin{pmatrix} (q)(r) \\ 0 \\ \emptyset \end{pmatrix} \right].$$

If we forget the partially defined contractions in $\underline{\mathcal{Q}}\mathcal{C}^{\text{pre}}$, F becomes a morphism of cyclic hybrids so it extends, by Theorem 38, into a unique morphism \tilde{F} of modular hybrids that makes the above diagram commutative. \square

3. Modular completion of a suboperad

The central technical result of this article, Proposition 34 of the previous section, describes the modular completion $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$ of the modular hybrid $\underline{\mathcal{O}}\mathcal{C}$. We need a similar result also for the KP and stable subhybrids

$$\underline{\mathcal{O}}\mathcal{C}_{\text{KP}} \hookrightarrow \underline{\mathcal{O}}\mathcal{C}_{\text{st}} \hookrightarrow \underline{\mathcal{O}}\mathcal{C}, \tag{51}$$

but we do not want to repeat the long technical proof of Proposition 34 for them. We prove instead that the morphisms

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \rightarrow \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}}) \rightarrow \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$$

of modular hybrids induced by the inclusions (51) are injective and describe explicitly the modular completions $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$ and $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$ as subhybrids of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$.

The content of this section will therefore be some results about the induced maps between modular completions. To save the reader from unnecessary technicalities, we formulate and prove them only for the “classical” cyclic operads and the “classical” modular completion functor $\text{Mod}: \text{CycOp} \rightarrow \text{ModOp}$ of [22, p. 382]. It will be clear that obvious analogs of these results hold also for non- Σ cyclic operads and cyclic hybrids.

Let thus \mathcal{C} be a cyclic operad and $\mathcal{B} \subset \mathcal{C}$ its cyclic suboperad. We are going to investigate the induced map $\varpi: \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{C})$. The following example shows that, in some situations, ϖ need not be a monomorphism.

Example 43. Let \mathcal{C} be the free cyclic operad generated by the two-point set

$$\{(u, v), (v, u)\} \subset \mathcal{C}(\{u, v\})$$

with the obvious action of the group $\text{Aut}(\{u, v\})$. Denote by $\mathcal{B} \subset \mathcal{C}$ the cyclic suboperad consisting of \circ -compositions of at least two elements of \mathcal{C} . Let finally

$$a := (u, x')_{x' \circ x''}(v, x'') \in \mathcal{B}(\{u, v\}) \quad \text{and} \quad b := (x', u)_{x' \circ x''}(x'', v) \in \mathcal{B}(\{u, v\}).$$

It follows from the axioms of cyclic operads that

$$\begin{aligned} \circ_{uv}(a) &= \circ_{uv}((u, x')_{x' \circ x''}(v, x'')) \\ &= \circ_{x'x''}((u, x')_u \circ_v(v, x'')) \\ &= \circ_{uv}((x', u)_{x' \circ x''}(x'', v)) = \circ_{uv}(b) \end{aligned}$$

in $\text{Mod}(\mathcal{C})$, while it is simple to check that $\circ_{uv}(a) \neq \circ_{uv}(b)$ in $\text{Mod}(\mathcal{B})$. So the induced map $\varpi: \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{C})$ is not a monomorphism.

The main idea of the example can be illustrated as follows. Represent the generator of \mathcal{C} by the arrow

$$u \longleftarrow v$$

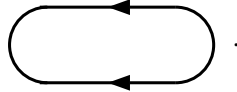
pointing from v to u . In this graphical representation,

$$a = u \longleftrightarrow v, \quad b = u \rightrightarrows v,$$

so we have in $\text{Mod}(\mathcal{B})$

$$\circ_{uv}(a) = \text{[Diagram: A loop with a dashed top arc and a solid bottom arc. A double-headed arrow is on the top arc, pointing left.]}, \quad \circ_{uv}(b) = \text{[Diagram: A loop with a dashed top arc and a solid bottom arc. A double-headed arrow is on the top arc, pointing right.]}.$$

The dashed ovals indicate that the arrows representing the generators cannot be separated in $\text{Mod}(\mathcal{B})$. The ovals however can be erased in $\text{Mod}(\mathcal{C})$ and the arrowheads moved around bringing both pictures in the above display into



The central technical result of this section reads

Proposition 44. *Let $\mathcal{B} \subset \mathcal{C}$ be cyclic operads. Assume that for every $w' \in \mathcal{C}(\{p', q'\} \sqcup R)$ and $w'' \in \mathcal{C}(\{p'', q''\} \sqcup S)$ such that $w'_{q' \circ q''} w'' \in \mathcal{B}(\{p', p''\} \sqcup R \sqcup S)$ either*

- (i) *there is a bijection $\rho: \{p', p''\} \sqcup R \sqcup S \rightarrow \{q', q''\} \sqcup R \sqcup S$ fixing $R \sqcup S$ such that*

$$w'_{p' \circ p''} w'' = \rho(w'_{q' \circ q''} w''),$$

- (ii) *or there are $w'_1 \in \mathcal{B}(\{p', q'\} \sqcup R)$ and $w''_1 \in \mathcal{B}(\{p'', q''\} \sqcup S)$ such that*

$$w'_{p' \circ p''} w'' = w'_1_{p' \circ p''} w''_1 \quad \text{and} \quad w'_{q' \circ q''} w'' = w'_1_{q' \circ q''} w''_1.$$

Then the induced map $\varpi: \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{C})$ is injective.

The assumption of Proposition 44 is in Example 43 violated by $w' := (p', q') \in \mathcal{C}(\{p', q'\})$ and $w'' := (p'', q'') \in \mathcal{C}(\{p'', q''\})$, $S = R := \emptyset$. Proposition 44 will follow from Proposition 47 whose formulation and proof we postpone to the end of this section. It would be interesting to investigate how is the injectivity of Proposition 44 related to the notion of minimal extensions of Feynman categories in the sense of [16, Definition 4.5]. We will also need

Definition 45. Let \mathcal{B} be a cyclic suboperad of a modular operad \mathcal{C} . The ξ -closure of \mathcal{B} in \mathcal{C} is defined as

$$\xi_{\mathcal{C}}(\mathcal{B}) := \{ \circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n} (x) \in \mathcal{C} \mid n \in \mathbb{N}, x \in \mathcal{B}, p'_1, p''_1, \dots, p'_n, p''_n \text{ are some inputs of } x \}.$$

The terminology is inspired by the old-fashioned notation ξ_{uv} for \circ_{uv} . Notice that $\xi_{\mathcal{C}}(\mathcal{B})$ is the smallest modular suboperad of \mathcal{C} containing \mathcal{B} , so it is a modular completion of \mathcal{B} in \mathcal{C} or relative to \mathcal{C} . From this point of view, $\text{Mod}(\mathcal{B})$ is the absolute modular completion of \mathcal{B} . The following statement describes a situation when absolute and relative completions agree.

Proposition 46. *If $\mathcal{B} \subset \mathcal{C}$ are cyclic operads such that the map $\varpi: \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{C})$ is injective, then*

$$\text{Mod}(\mathcal{B}) \cong \xi_{\text{Mod}(\mathcal{C})}(\mathcal{B}).$$

In particular, $\text{Mod}(\mathcal{B}) \cong \xi_{\text{Mod}(\mathcal{B})}(\mathcal{B})$.

Proof. By the universal property of $\text{Mod}(\mathcal{B})$ applied to the inclusions $\mathcal{B} \hookrightarrow \xi_{\text{Mod}(\mathcal{C})}(\mathcal{B})$ and $\mathcal{B} \hookrightarrow \text{Mod}(\mathcal{C})$, there is a modular operad morphism $i: \text{Mod}(\mathcal{B}) \rightarrow \xi_{\text{Mod}(\mathcal{C})}(\mathcal{B})$ such that the diagram

$$\begin{array}{ccc} \text{Mod}(\mathcal{B}) & \xrightarrow{\varpi} & \text{Mod}(\mathcal{C}) \\ & \searrow i & \nearrow \\ & \xi_{\text{Mod}(\mathcal{C})}(\mathcal{B}) & \end{array}$$

commutes. Since $\varpi: \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{C})$ is injective by assumption, so is i . As $\xi_{\text{Mod}(\mathcal{C})}(\mathcal{B})$ is the smallest modular suboperad of $\text{Mod}(\mathcal{C})$ containing \mathcal{B} , i must be an isomorphism. The second isomorphism of the proposition is the particular case when ϖ is the identity morphism $\mathbb{1}: \mathcal{B} \rightarrow \mathcal{B}$. \square

Proposition 44 is a consequence of

Proposition 47. *Every element $x \in \text{Mod}(\mathcal{C})$ in the modular completion of a cyclic operad \mathcal{C} is of the form*

$$x = \circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n}(y) \tag{52}$$

where $y \in \mathcal{C}$ and $p'_1, p''_1, \dots, p'_n, p''_n, n \in \mathbb{N}$, are some of its inputs. On elements in this form, consider the following “moves:”

(i) Let $w' \in \mathcal{C}(\{p', q'\} \sqcup R)$ and $w'' \in \mathcal{C}(\{p'', q''\} \sqcup S)$. Then replace

$$\begin{array}{l} \circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{p' p''}(w' \circ_{q' q''} w'') \\ \text{by} \quad \circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{q' q''}(w' \circ_{p' p''} w''). \end{array}$$

(ii) Let p', p'', q', q'' be some of the inputs of y and ρ a bijection mapping p', p'' to q', q'' in this order which restricts to the identity on the remaining inputs of y . Then replace

$$\begin{array}{l} \circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{p' p''}(y) \\ \text{by} \quad \circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{q' q''}(\rho y). \end{array}$$

(iii) For an arbitrary permutation $\sigma \in \Sigma_n$ replace

$$\begin{array}{l} \circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n}(y) \\ \text{by} \quad \circ_{p'_{\sigma(1)} p''_{\sigma(1)}} \cdots \circ_{p'_{\sigma(n)} p''_{\sigma(n)}}(y). \end{array}$$

Two expressions (52) represent the same element of $\text{Mod}(\mathcal{C})$ if and only if they are related by a finite numbers of the above moves.

Proof. The modular completion $\text{Mod}(\mathcal{C})$ is isomorphic to the quotient $\mathbb{M}(\mathcal{C})/\sim$, where $\mathbb{M}(\mathcal{C})$ is the free modular operad generated by \mathcal{C} and \sim is the equivalence that identifies \circ -operations inside \mathcal{C} with the formal ones in $\mathbb{M}(\mathcal{C})$.

As explained e.g. in [25, II.1.9], $\mathbb{M}(\mathcal{C})$ can be constructed as an explicit colimit whose elements are represented by decorated graphs. Since we are working in Set , every $x \in \mathbb{M}(\mathcal{C})$ has well-defined underlying graph $G(x)$. Choose a contractible, not necessary connected, subgraph T in $G(x)$ and contract x along T using the cyclic operad structure of \mathcal{C} . Denote the result by $C_T(x)$; clearly

$$C_T(x) \sim x.$$

If T is in particular a maximal subtree of $G(x)$, then the underlying graph of $C_T(x)$ has one vertex, call such a graph a brouček.¹¹ The element $C_T(x)$ is obtained by iterated contractions of some $y \in \mathcal{C}$. To describe it in such a way explicitly, i.e. as

$$\circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n}(y), \quad n \in \mathbb{N},$$

with some specific symbols $p'_1, p''_1, \dots, p'_n, p''_n$, one needs to label the half-edges of brouček and choose their order. The ambiguity of these choices is reflected by moves (ii) and (iii) of Proposition 47.

Another ambiguity comes from different choices of a maximal subtree of $G(x)$. Let us analyze this situation. Assume that T_1 and T_2 are different maximal subtrees of $G(x)$. By [28, Chapter 6], T_1 and T_2 are related by a “singular cyclic interchange.” This means that there exists a subgraph $H \subset G(x)$ with precisely one cycle, and two edges e_1, e_2 belonging to this cycle, such that

$$H \setminus \{e_2\} = T_1 \quad \text{and} \quad H \setminus \{e_1\} = T_2.$$

In this situation, $H \setminus \{e_1, e_2\}$ is the disjoint union of two (non-maximal) trees U and V . Let $z := C_{U \sqcup V}(x)$. Obviously, $G(z)$ is a graph with two vertices decorated by some $a, b \in \mathcal{C}$. Let u_1, \dots, u_k are the edges of $G(x)$ that do not belong to $H \setminus \{e_1, e_2\}$. We then have, due to the interchange law between contractions and \circ -operations,

$$z = \circ_{u_1} \cdots \circ_{u_k} \circ_{e_1}(a \circ_{e_2} b) = \circ_{u_1} \cdots \circ_{u_k} \circ_{e_2}(a \circ_{e_1} b)$$

modulo the relations defining $\text{Mod}(\mathcal{C})$. In the above display, \circ_e denotes the contraction along e . Finally, we observe that

$$\circ_{u_1} \cdots \circ_{u_k} \circ_{e_1}(a \circ_{e_2} b) \text{ represents } C_{T_1}(x)$$

while

$$\circ_{u_1} \cdots \circ_{u_k} \circ_{e_2}(a \circ_{e_1} b) \text{ represents } C_{T_2}(x).$$

Equality $C_{T_1}(x) = C_{T_2}(x)$ is therefore reflected by move (i) of Proposition 47. \square

¹¹Czech for little beetle.

The above proof shows that move (i) is the relevant one, the remaining moves only account for different choices of labels.

Proof of Proposition 44. Recall that each element of $\text{Mod}(\mathcal{B})$ is of the form (52). So assume that $y, z \in \mathcal{B}$ and that

$$\circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n}(y) = \circ_{q'_1 q''_1} \cdots \circ_{q'_n q''_n}(z) \quad \text{in } \text{Mod}(\mathcal{C}). \tag{53}$$

All we need is to show that the same equality holds also in $\text{Mod}(\mathcal{B})$. By Proposition 47, (53) holds if and only if there is a finite sequence of moves (i)–(iii) transforming its left hand side into its right hand side. Each move is a replacement of the form

$$\circ_{r'_1 r''_1} \cdots \circ_{r'_n r''_n}(u) \quad \longmapsto \quad \circ_{s'_1 s''_1} \cdots \circ_{s'_n s''_n}(v) \tag{54}$$

with some $u, v \in \mathcal{C}$. The proof will thus be finished if we show that $u \in \mathcal{B}$ in (54) implies that $v \in \mathcal{B}$.

This is obvious if (54) is move (ii) or (iii). Let us analyze move (i), that is, see what happens if we replace

$$\circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{p' p''}(u) \quad \text{by} \quad \circ_{p'_1 p''_1} \cdots \circ_{p'_{n-1} p''_{n-1}} \circ_{q' q''}(v), \tag{55}$$

where $u = w'_{q' \circ q''} w''$ and $v = w'_{p' \circ p''} w''$ with some

$$w' \in \mathcal{C}(\{p', q'\} \sqcup R) \quad \text{and} \quad w'' \in \mathcal{C}(\{p'', q''\} \sqcup S)$$

such that $u = w'_{q' \circ q''} w'' \in \mathcal{B}(\{p', p''\} \sqcup R \sqcup S)$.

In case (i) of Proposition 44, $w'_{p' \circ p''} w' = \rho(w_{q' \circ q''} w')$, i.e.

$$v = \rho(u) \in \mathcal{B}(\{q', q''\} \sqcup R \sqcup S)$$

as required. In case (ii),

$$v = w'_1{}_{p' \circ p''} w''_1$$

for some $w'_1 \in \mathcal{B}(\{p', q'\} \sqcup R)$ and $w''_1 \in \mathcal{B}(\{p'', q''\} \sqcup S)$, thus again

$$v \in \mathcal{B}(\{q', q''\} \sqcup R \sqcup S)$$

and we are done as well. □

4. Modular completions of the stable and Kaufmann–Penner parts

In this section we use the results of Section 3 and derive the stable and Kaufmann–Penner versions of Theorem 38.

4.1. Stable version. Proposition 49 below guarantees that one may use Proposition 34 to describe explicitly the modular completion of the stable part $\underline{\mathcal{O}}\mathcal{C}_{\text{st}} \subset \underline{\mathcal{O}}\mathcal{C}$, as done in Remark 50. The main result of this subsection is Theorem 51.

Lemma 48. *Let \mathcal{C}_{st} be the stable part of a cyclic operad \mathcal{C} as in Definition 22. Then one has the isomorphism*

$$\xi_{\text{Mod}(\mathcal{C})}(\mathcal{C}_{\text{st}}) \cong \text{Mod}(\mathcal{C})_{\text{st}}.$$

Consequently, if the induced map $\varpi: \text{Mod}(\mathcal{C}_{\text{st}}) \rightarrow \text{Mod}(\mathcal{C})$ is a monomorphism, then

$$\text{Mod}(\mathcal{C}_{\text{st}}) \cong \text{Mod}(\mathcal{C})_{\text{st}}.$$

Proof. By Proposition 47, every $x \in \text{Mod}(\mathcal{C})$ is of the form

$$x = \circ_{p'_1 p''_1} \cdots \circ_{p'_n p''_n}(y) \tag{56}$$

for some $y \in \mathcal{C}$ and $n \in \mathbb{N}$. If $x \in \xi_{\text{Mod}(\mathcal{C})}(\mathcal{C}_{\text{st}})$, we may assume that $y \in \mathcal{C}_{\text{st}} \subset \text{Mod}(\mathcal{C})_{\text{st}}$. Since contractions preserve stable parts of modular operads by Lemma 23, $x \in \text{Mod}(\mathcal{C})_{\text{st}}$, which shows that

$$\xi_{\text{Mod}(\mathcal{C})}(\mathcal{C}_{\text{st}}) \subset \text{Mod}(\mathcal{C})_{\text{st}}.$$

If $x \in \text{Mod}(\mathcal{C})_{\text{st}}$, then $y \in \text{Mod}(\mathcal{C})_{\text{st}} \cap \mathcal{C} = \mathcal{C}_{\text{st}}$ by the second part of Lemma 23, hence $x \in \xi_{\text{Mod}(\mathcal{C})}(\mathcal{C}_{\text{st}})$, thus

$$\text{Mod}(\mathcal{C})_{\text{st}} \subset \xi_{\text{Mod}(\mathcal{C})}(\mathcal{C}_{\text{st}}).$$

Having this established, the second part of the lemma follows from Proposition 46. □

Let us turn our attention to the cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}$ from Example 20 and its stable version $\underline{\mathcal{O}}\mathcal{C}_{\text{st}}$ analyzed in Example 29.

Proposition 49. *One has an isomorphism $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}}) \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})_{\text{st}}$.*

Remark 50. An explicit description of $\text{Mod}(\underline{\mathcal{O}}\mathcal{C})_{\text{st}}$ and therefore, by Proposition 49, also of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$, is provided by imposing the stability assumption on the expressions in (23) of Proposition 34. Explicitly, the symbol

$$\left[\begin{array}{c} [\mathcal{O}_1] \cdots [\mathcal{O}_a] \\ [g_1] \cdots [g_a] \\ g \\ C \end{array} \right] \tag{57}$$

is stable if and only if

$$4(g + \sum_{i=1}^a g_i) + 2b + 2|C| + |\mathcal{O}| > 4 \tag{58}$$

where $b := \sum_{i=1}^a b_i$ is the total number of cycles in $\mathcal{O}_1, \dots, \mathcal{O}_a$.

Proof of Proposition 49. We verify that the inclusion $\underline{\mathcal{QC}}_{\text{st}} \subset \underline{\mathcal{QC}}$ satisfies condition (i) of Proposition 44. The proof will then follow from Proposition 46 and Lemma 48.

Let $w', w'' \in \underline{\mathcal{QC}}$ be such that $w'_{q' \circ q''} w'' \in \underline{\mathcal{QC}}_{\text{st}}$. It easily follows from the definition of the stable part if $w', w'' \notin \underline{\mathcal{QC}}_{\text{st}}$, then also $w'_{q' \circ q''} w'' \notin \underline{\mathcal{QC}}$, so we may assume e.g. that $w' \in \underline{\mathcal{QC}}_{\text{st}}$ while $w'' \notin \underline{\mathcal{QC}}_{\text{st}}$. Since w'' has at least 2 inputs, according to (21) it must be either

$$\boxed{3} = \begin{bmatrix} ((p''q'')) \\ 0 \\ \emptyset \end{bmatrix} \quad \text{or} \quad \boxed{6} = \begin{bmatrix} \emptyset \\ 0 \\ \{p'', q''\} \end{bmatrix}.$$

In both cases, $w'_{q' \circ q''} w''$ “replaces q' by p'' ” and $w'_{p' \circ p''} w''$ “replaces p' by q'' ” in w' .

Let us explain what we mean by this when $w'' = \boxed{6}$. Then p', p'', q' and q'' must be “closed” inputs and one has, by definition,

$$\begin{bmatrix} \circ \\ g \\ R \sqcup \{p', q'\} \end{bmatrix} q' \circ q'' \begin{bmatrix} \emptyset \\ 0 \\ \{p'', q''\} \end{bmatrix} = \begin{bmatrix} \circ \\ g \\ R \sqcup \{p', p''\} \end{bmatrix}$$

and

$$\begin{bmatrix} \circ \\ g \\ R \sqcup \{p', q'\} \end{bmatrix} p' \circ p'' \begin{bmatrix} \emptyset \\ 0 \\ \{p'', q''\} \end{bmatrix} = \begin{bmatrix} \circ \\ g \\ R \sqcup \{q', q''\} \end{bmatrix}.$$

Clearly $w'_{q' \circ q''} w' = \rho(w'_{p' \circ p''} w'')$ for a bijection ρ mapping $\{p', p''\}$ to $\{q', q''\}$ and restricting to the identity on R . The case when $w'' = \boxed{3}$ can be discussed similarly. We leave the details to the reader. \square

We have the following stable analog of Theorem 38.

Theorem 51. *Let \mathcal{I} be the ideal in the stable modular hybrid $\underline{\text{Mod}}(\underline{\mathcal{QC}}_{\text{st}}) \cong \underline{\text{Mod}}(\underline{\mathcal{QC}})_{\text{st}}$ generated by the relations*

$$\begin{bmatrix} \begin{bmatrix} ((q)) \\ 0 \\ \emptyset \end{bmatrix} \begin{bmatrix} ((r)) \\ 0 \\ \emptyset \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} ((q))((r)) \\ 0 \\ \emptyset \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \begin{bmatrix} ((q)) \\ 0 \\ \emptyset \end{bmatrix} \begin{bmatrix} ((\circ)) \\ 0 \\ \emptyset \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} ((q))((\circ)) \\ 0 \\ \emptyset \end{bmatrix} \end{bmatrix}. \quad (59)$$

Then one has an isomorphism of stable modular hybrids

$$\underline{\mathcal{QC}}_{\text{st}} \cong \underline{\text{Mod}}(\underline{\mathcal{QC}}_{\text{st}}) / \mathcal{I}.$$

Therefore for any, not necessarily stable, modular hybrid \mathcal{H} and a morphism $F: \underline{\mathcal{QC}}_{\text{st}} \rightarrow \mathcal{H}$ of cyclic hybrids satisfying

$$\circ_{uv} F \begin{bmatrix} ((uqvr)) \\ 0 \\ \emptyset \end{bmatrix} = F \begin{bmatrix} ((q))((r)) \\ 0 \\ \emptyset \end{bmatrix} \quad \text{and} \quad \circ_{uv} F \begin{bmatrix} ((uqv)) \\ 0 \\ \emptyset \end{bmatrix} = F \begin{bmatrix} ((q))((\circ)) \\ 0 \\ \emptyset \end{bmatrix}$$

there is a unique morphism $\hat{F}: \underline{\mathcal{QC}}_{\text{st}} \rightarrow \mathcal{H}$ of modular hybrids making the diagram

$$\begin{array}{ccc} \underline{\mathcal{QC}}_{\text{st}} & \hookrightarrow & \underline{\mathcal{QC}}_{\text{st}} \\ & \searrow F & \downarrow \hat{F} \\ & & \mathcal{H} \end{array}$$

commutative.

The reader may wonder why we have two relations in (59) while the “unstable” Theorem 38 has only one. The explanation is that, in the unstable case, the second relation in (59) is the same as

$$\begin{bmatrix} [(q)] & [(r)] \\ 0 & 0 \\ \emptyset & \emptyset \end{bmatrix} r \circ_s \begin{bmatrix} [(s)] \\ 0 \\ \emptyset \end{bmatrix} = \begin{bmatrix} [(q)(r)] \\ 0 \\ \emptyset \end{bmatrix} r \circ_s \begin{bmatrix} [(s)] \\ 0 \\ \emptyset \end{bmatrix},$$

so it belongs to the ideal generated by the first relation. Since

$$\begin{bmatrix} [(s)] \\ 0 \\ \emptyset \end{bmatrix}$$

is not stable, the same reasoning does not apply to $\text{Mod}(\mathcal{O}\mathcal{C}_{\text{st}})$.

Proof of Theorem 51. The proof is a modification of the proof of Theorem 38 so we mention only the differences. First of all, in addition to (42), we also need to prove that

$$\begin{bmatrix} [()] & [(qr)] \\ 0 & 0 \\ C & C \end{bmatrix} = \begin{bmatrix} [()](qr) \\ 0 \\ C \end{bmatrix}$$

modulo \mathcal{I} . This equality can easily be obtained by replacing, in (43a) and (43b), $((p))$ by $()$.

It might also happen that some terms in (45) which we used to prove (44) are unstable. Let us denote the terms constituting (45) by

$$\boxed{\text{A}} := \begin{bmatrix} [p'o'_1 o'_2 \dots o'_{b'}] \\ g_1 \\ 0 \\ \emptyset \end{bmatrix}, \quad \boxed{\text{B}} := \begin{bmatrix} [q'o'_1 o''_2 \dots o'_{b''}] \\ g_2 \\ 0 \\ \emptyset \end{bmatrix}, \quad \boxed{\text{C}} := \begin{bmatrix} [(p'')] & [(q'r')] \\ 0 & 0 \\ \emptyset & \emptyset \end{bmatrix}$$

and
$$\boxed{\text{D}} := \begin{bmatrix} [(r'')] & [O_3] \dots [O_a] \\ 0 & [g_3] \dots [g_a] \\ & g \\ & C \end{bmatrix}.$$

Term $\boxed{\text{C}}$ is always stable. Term $\boxed{\text{A}}$ is unstable if and only if $g_1 = 0$ and $O'_1 = o'_1 = ((p'))$ or $((p's))$ for some symbol s , in which case

$$\boxed{\text{A}} := \begin{bmatrix} [(p'')] \\ 0 \\ 0 \\ \emptyset \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} [(p's)] \\ 0 \\ 0 \\ \emptyset \end{bmatrix}.$$

Likewise, $\boxed{\text{B}}$ is unstable if and only if

$$\boxed{\text{B}} := \begin{bmatrix} [(q'')] \\ 0 \\ 0 \\ \emptyset \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} [(q't)] \\ 0 \\ 0 \\ \emptyset \end{bmatrix}$$

for a symbol t . Finally, $\boxed{\text{D}}$ is unstable if and only if $a = 2$, in which case

$$\boxed{\text{D}} := \begin{bmatrix} [(r'')] \\ 0 \\ 0 \\ \emptyset \end{bmatrix}.$$

Let us analyze all possible situations.

Term \boxed{A} is unstable but \boxed{B} is stable. The left hand side of (44) takes the form

$$\left[\begin{array}{c} [o'_1][O_2] \dots [O_a] \\ 0 \\ g \\ C \end{array} \right]$$

with $o'_1 = (())$ or $o_1 = ((s))$. We then instead of (45) take

$$\left[\begin{array}{c} [q' o'_1 o'_2 \dots o'_{b'}] \\ g_2 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} [o'_1][((q'' r'))] \\ 0 \\ 0 \\ C \end{array} \right] r' \circ r'' \left[\begin{array}{c} [((r''))][O_3] \dots [O_a] \\ 0 \\ g \\ C \end{array} \right]$$

and proceed as before. The situation when \boxed{B} is unstable but \boxed{A} is stable is similar.

Both \boxed{A} and \boxed{B} are unstable. The left hand side of (44) is of the form

$$\left[\begin{array}{c} [o'_1][o'_1] \dots [O_a] \\ 0 \\ g \\ C \end{array} \right]$$

where $o'_1 = (())$ or $((s))$ and $o''_1 = (())$ or $((t))$. We then instead of (45) take

$$\left[\begin{array}{c} [o'_1][o'_1 r'] \\ 0 \\ 0 \\ \emptyset \end{array} \right] r' \circ r'' \left[\begin{array}{c} [((r''))][O_3] \dots [O_a] \\ 0 \\ g \\ C \end{array} \right]$$

and proceed as in the proof of Theorem 38.

Term \boxed{D} is unstable. Then the left hand side of (44) takes the form

$$\left[\begin{array}{c} [O_1][O_2] \\ g_1 \\ g_2 \\ g \\ C \end{array} \right]$$

In this case, instead of (45), we simply take

$$\left[\begin{array}{c} [p' o'_1 o'_2 \dots o'_{b'}] \\ g_1 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [q' o'_1 o'_2 \dots o'_{b'}] \\ g_2 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} [((p''))][((q''))] \\ 0 \\ 0 \\ \emptyset \end{array} \right].$$

This finishes the proof. □

4.2. Kaufmann–Penner variant. The first result of this subsection explains how to modify Proposition 34 for the modular completion of the KP cyclic hybrid \mathcal{QC}_{KP} . Theorem 54 then describes \mathcal{QC}_{KP} as the quotient of this modular completion by the Cardy condition.

Proposition 52. *The modular completion $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$ is the modular subhybrid of the modular completion $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$ obtained by imposing the stability assumption (58) on symbols in (23) resp. in (57), and further discarding*

- (i) symbols $\left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \\ \emptyset \end{array} \right]$ with $a \geq 3$,¹²
- (ii) symbols $\left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] V \\ \emptyset \end{array} \right]$, $a \geq 2$, where $V \in \underline{\mathcal{O}}\mathcal{O}$ has at least one input, and
- (iii) symbols $\left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \\ \{d\} \end{array} \right]$, $a \geq 2$, where d is a single closed input.

Proof. Denote by \mathcal{M} the subcollection of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$ specified in the proposition. We need to prove that $\mathcal{M} \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$. Our strategy will be first to show that \mathcal{M} is indeed a modular subhybrid of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$, then verify the assumptions of Proposition 44, apply Proposition 46 and finally check directly that the ξ -closure of $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ is \mathcal{M} .

Verification that \mathcal{M} is a modular subhybrid of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$. Let us check first that \mathcal{M} is closed under the \circ -operations. Assume that $x = y \circ_{p'} \circ_{p''} z$ for some $x, y, z \in \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$. We must show that, if $x \notin \mathcal{M}$, then either $y \notin \mathcal{M}$ or $z \notin \mathcal{M}$. Denote by a_x, a_y and a_z the number of nests in x, y and z , respectively. We distinguish three cases.

The element x is of type (i). If p', p'' are open inputs, then clearly the only possibility is that

$$y = \left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] [\begin{array}{c} \textcircled{p'} \\ 0 \end{array}] \\ \emptyset \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] [\begin{array}{c} \textcircled{p''} \\ 0 \end{array}] \\ \emptyset \end{array} \right].$$

The numbers of nests are related by $a_x = a_y + a_z - 1$, therefore $x \notin \mathcal{M}$ if and only if

$$a_y + a_z \geq 4. \tag{60}$$

On the other hand, $y \notin \mathcal{M}$ (resp. $z \notin \mathcal{M}$) if and only if $a_y \geq 2$ (resp. $a_z \geq 2$), so (60) implies that at least one of y, z does not belong to \mathcal{M} . If p', p'' are closed, then obviously

$$y = \left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \\ \{p'\} \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \dots [\begin{array}{c} \textcircled{0} \\ 0 \end{array}] \\ \{p''\} \end{array} \right].$$

Now $a_x = a_y + a_z$, so $x \notin \mathcal{M}$ if and only if $a_y + a_z \geq 3$ and we conclude as in the open case that either $a_y \geq 2$ or $a_z \geq 2$.

¹²Recall that a is the number of nests.

The element x is of type (ii). If p', p'' are open inputs, one has two possibilities. The first one is that

$$y = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] V \\ 0 \\ \emptyset \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \emptyset \end{array} \right]$$

(or the rôles of y and z interchanged). Then $z \notin \mathcal{M}$ and we are done. The second option is

$$y = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] V_y \\ 0 \\ \emptyset \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] V_z \\ 0 \\ \emptyset \end{array} \right]$$

for some $V_y, V_z \in \mathcal{Q}\mathcal{G}$, both having at least one input, such that $V_y \circ_{p'} \circ_{p''} V_z = V$. We easily verify that $x \notin \mathcal{M}$ if and only if (60) holds which implies, as before, that either y or z does not belong to \mathcal{M} . In case of closed inputs, the only possibility is

$$y = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] V \\ 0 \\ \{p'\} \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \{p''\} \end{array} \right]$$

(or y and z interchanged). We see right away that $z \notin \mathcal{M}$.

The element x is of type (iii). If p' and p'' are open inputs, then

$$y = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \{d\} \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \emptyset \end{array} \right]$$

(or vice versa). If they are closed, the only possibility is

$$y = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \{dp'\} \end{array} \right] \quad \text{and} \quad z = \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \{p''\} \end{array} \right]$$

(or vice versa). In both cases $z \notin \mathcal{M}$.

It remains to verify that \mathcal{M} is closed under contractions. Let $x = \circ_{p'} \circ_{p''} y$ for some elements $x, y \in \underline{\text{Mod}}(\mathcal{Q}\mathcal{C}_{\text{st}})$. We must show that $x \notin \mathcal{M}$ implies $y \notin \mathcal{M}$. If x is of type (i), there are only three thinkable candidates for y , namely

$$\left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \emptyset \end{array} \right], \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \emptyset \end{array} \right] \quad \text{or} \quad \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \{p', p''\} \end{array} \right].$$

The respective values of the contraction $\circ_{p'} \circ_{p''} y$ are

$$\left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 1 \\ \emptyset \end{array} \right], \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 0 \\ \emptyset \end{array} \right], \quad \text{and} \quad \left[\begin{array}{c} [(\emptyset)] \dots [(\emptyset)] [(\emptyset)] [(\emptyset)] \\ 1 \\ \emptyset \end{array} \right]$$

which excludes this possibility. The situation when x is of type (iii) is similar.

Assume finally that x is of type (ii). Besides the candidates for y similar to the ones above, there are also

$$\begin{aligned} & \left[\begin{array}{c} [\begin{array}{c} \circ \\ 0 \end{array}] \dots [\begin{array}{c} \circ \\ 0 \end{array}] V_1 V_2 \\ \circ \\ \emptyset \end{array} \right] \quad \text{with } V_1 \circ_{p' \circ p''} V_2 = V, \\ \text{and} \quad & \left[\begin{array}{c} [\begin{array}{c} \circ \\ 0 \end{array}] \dots [\begin{array}{c} \circ \\ 0 \end{array}] W \\ \circ \\ \emptyset \end{array} \right] \quad \text{with } \circ_{p' \circ p''} W = V. \end{aligned}$$

For the first candidate

$$\circ_{p' \circ p''} y = \left[\begin{array}{c} [\begin{array}{c} \circ \\ 0 \end{array}] \dots [\begin{array}{c} \circ \\ 0 \end{array}] V \\ 1 \\ \emptyset \end{array} \right]$$

while the second candidate does not belong to \mathcal{M} . This finishes the verification that \mathcal{M} is a modular subhybrid of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$.

Verifying assumptions of Proposition 44. Let $w', w'' \in \underline{\mathcal{O}}\mathcal{C}_{\text{st}}$ be such that $w' \circ_{q' \circ q''} w'' \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$. If both $w', w'' \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, there is nothing to verify. If both $w', w'' \notin \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, then it is easy to check that also $w' \circ_{q' \circ q''} w'' \notin \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, so the only interesting case is when precisely one of w' and w'' does not belong to $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$.

Assume therefore that $w' \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ but $w'' \notin \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$. Since w'' has to have at least two inputs p'' and q'' , it must be of type (ii) in the classification of Example 30. This leaves us with two possibilities.

Case 1: $w' = \left[\begin{array}{c} \circ_1 \circ_2 \circ_3 \dots \circ_b \\ \circ \\ C \end{array} \right]$, $w'' = \left[\begin{array}{c} \circ \circ \dots \circ \\ \circ \\ \emptyset \end{array} \right]$, $p' \in \circ_1, q' \in \circ_2$ and $p'', q'' \in \circ$. If it is so, then

$$\begin{aligned} w' \circ_{q' \circ q''} w'' &= \left[\begin{array}{c} \circ_1 (\circ_2 q' \circ q'' \circ) \circ_3 \dots \circ_b \circ \dots \circ \\ \circ \\ C \end{array} \right] \\ \text{and} \quad w' \circ_{p' \circ p''} w'' &= \left[\begin{array}{c} (\circ_1 p' \circ p'' \circ) \circ_2 \circ_3 \dots \circ_b \circ \dots \circ \\ \circ \\ C \end{array} \right]. \end{aligned}$$

If $|\circ| \geq 3$, then the assumption (ii) of Proposition 44 is satisfied with

$$w'_1 = \left[\begin{array}{c} \circ_1 \circ_2 \circ_3 \dots \circ_b \circ \dots \circ \\ \circ \\ C \end{array} \right] \quad \text{and} \quad w''_1 = \left[\begin{array}{c} \circ \\ \circ \\ \emptyset \end{array} \right]$$

where w'_1 absorbed all empty cycles of w'' . If $\circ = ((p''q''))$, such w''_1 is not stable. We however have

$$\begin{aligned} w' \circ_{q' \circ q''} w'' &= \left[\begin{array}{c} \circ_1 (\circ_2 q' \circ q'' ((p''q''))) \circ_3 \dots \circ_b \circ \dots \circ \\ \circ \\ C \end{array} \right], \\ w' \circ_{p' \circ p''} w'' &= \left[\begin{array}{c} (\circ_1 p' \circ p'' ((p''q''))) \circ_2 \circ_3 \dots \circ_b \circ \dots \circ \\ \circ \\ C \end{array} \right] \end{aligned}$$

so we notice, as in the proof of Proposition 49, that $w'_{q' \circ q''} w''$ replaces q' by p'' and $w'_{p' \circ p''} w''$ replaces p' by q'' in w' . Therefore $w'_{q' \circ q''} w' = \rho(w'_{p' \circ p''} w'')$ for a bijection ρ mapping $\{p', p''\}$ to $\{q', q''\}$ and restricting to the identity elsewhere. Assumption (i) of Proposition 44 is thus satisfied.

Case 2: $w' = \begin{bmatrix} \circ_1 \circ_2 \dots \circ_b \\ 0 \\ C \end{bmatrix}$, $w'' = \begin{bmatrix} \circ \circ \dots \circ \\ 0 \\ \emptyset \end{bmatrix}$, $p', q' \in \circ_1$, and $p'', q'' \in \circ$. Then we calculate

$$w'_{q' \circ q''} w'' = \begin{bmatrix} (\circ_1 q' \circ q'' \circ) \circ_2 \dots \circ_b \circ \dots \circ \\ 0 \\ C \end{bmatrix}$$

and

$$w'_{p' \circ p''} w'' = \begin{bmatrix} (\circ_1 p' \circ p'' \circ) \circ_2 \dots \circ_b \circ \dots \circ \\ 0 \\ C \end{bmatrix}.$$

If $\circ \neq ((p'' q''))$, the assumption (ii) of Proposition 44 is satisfied with

$$w'_1 = \begin{bmatrix} \circ_1 \circ_2 \dots \circ_b \circ \dots \circ \\ 0 \\ C \end{bmatrix} \quad \text{and} \quad w''_1 = \begin{bmatrix} \circ \\ 0 \\ \emptyset \end{bmatrix}.$$

If $\circ = ((p'' q''))$, we argue precisely as in the first case.

This finishes the verification of assumptions of Proposition 44. Proposition 46 now implies

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) = \xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}).$$

Since we already know that \mathcal{M} is a modular subhybrid of $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$, the minimality of the ξ -closure implies the inclusions

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) = \xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \subset \mathcal{M} \subset \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}}).$$

It therefore remains to:

Verify that $\mathcal{M} \subset \xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$. We know by [24] that $\underline{\mathcal{O}}\mathcal{O} \cong \underline{\text{Mod}}(\underline{\mathcal{A}}_{\text{st}})$. For any $V \in \underline{\mathcal{O}}\mathcal{O}$ therefore exists a (non-unique) $\circ_V \in \underline{\mathcal{A}}_{\text{st}}$ such that

$$V = \circ_{q'_1 q''_1} \dots \circ_{q'_t q''_t}(\circ_V) \tag{61}$$

for some $q'_1, q''_1, \dots, q'_t, q''_t \in \circ_V$. To save space, we will denote the iterated contraction in (61) by ξ_V ; (61) will then read $V = \xi_V(\circ_V)$. With this notation, we have in $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$ the equality

$$\begin{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} = \circ_{p'_1 p''_1} \dots \circ_{p'_g p''_g} \xi_{V_1} \dots \xi_{V_a} \begin{bmatrix} \begin{bmatrix} \circ_{V_1} \\ 0 \end{bmatrix} \dots \begin{bmatrix} \circ_{V_a} \\ 0 \end{bmatrix} \\ 0 \\ C \sqcup \{p'_1, p''_1, \dots, p'_g, p''_g\} \end{bmatrix} \tag{62}$$

along with the identification

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}) \ni \begin{bmatrix} \begin{bmatrix} \circ_{V_1} \\ 0 \end{bmatrix} \dots \begin{bmatrix} \circ_{V_a} \\ 0 \end{bmatrix} \\ 0 \\ C \sqcup \{p'_1, p''_1, \dots, p'_g, p''_g\} \end{bmatrix} = \begin{bmatrix} \circ_{V_1} \dots \circ_{V_a} \\ 0 \\ C \sqcup \{p'_1, p''_1, \dots, p'_g, p''_g\} \end{bmatrix} \in \underline{\mathcal{O}}\mathcal{C} \tag{63}$$

provided by the unit (24).

Denote the left hand side of (62) and the element in (63) by y . If x is stable, then so is y by Lemma 23, so (62) in fact holds in $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})$. We need to show that if $x \in \mathcal{M}$, then $y \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$.

If $g \geq 1$, then y has at least two closed inputs, thus $y \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$. Suppose that $g = 0$ and $|C| \geq 1$. If $y \notin \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, then $|C| = 1$ and $\mathfrak{o}_1 = \cdots = \mathfrak{o}_a = \begin{bmatrix} \binom{0}{0} \\ 0 \end{bmatrix}$, so $V_1 = \cdots = V_a = \begin{bmatrix} \binom{0}{0} \\ 0 \end{bmatrix}$, hence $x \notin \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, which contradicts the assumption. The last case to be analyzed is $g = 0$ and $|C| = 0$. Then at least two V_i 's, say V_1 and V_2 , have at least one input, otherwise $x \notin \mathcal{M}$. So the same is true for \mathfrak{o}_{V_1} and \mathfrak{o}_{V_2} , thus $y \in \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$. \square

To sum up, at this moment we know that the sequence of inclusions (51) induce inclusions

$$\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \hookrightarrow \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}}) \hookrightarrow \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}).$$

We also know that

$$\xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \quad \text{and} \quad \xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})$$

while the isomorphism $\xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{st}})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) \cong \xi_{\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C})}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$ is immediate.

Definition 53. The *Kaufmann–Penner modular hybrid* $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ is the modular subhybrid of $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{st}}$ generated by $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$, i.e. $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}} := \xi_{\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{st}}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$.

A more intelligent description of $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ will be given in Theorem 57 below. The linearization of the Kaufmann–Penner hybrid $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ is in fact isomorphic to the degree 0 homology H_0 of the arc operad Arc of [15, p. 346], whence its name. Notice that $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ contains the genus-graded stable cyclic operad $\mathcal{C}om_{\text{st}}$ from Example 26 as a suboperad of elements with $g = b = 0$, i.e. elements of the form

$$\begin{bmatrix} \emptyset \\ 0 \\ C \end{bmatrix}, \quad C \in \text{Fin}. \tag{64}$$

The following variant of Theorems 38 and 51 for $\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ is implicit in [15, Theorem 4.1].

Theorem 54. *Let \mathcal{I} denote the ideal in the modular hybrid $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$ generated by the relation*

$$\begin{bmatrix} \begin{bmatrix} \binom{(q)}{0} \\ 0 \\ \emptyset \end{bmatrix} \begin{bmatrix} \binom{(r)}{0} \\ 0 \\ \emptyset \end{bmatrix} \\ \emptyset \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \binom{(q)}{0} \\ 0 \\ \emptyset \end{bmatrix} \binom{(r)}{0} \\ \emptyset \end{bmatrix}. \tag{65}$$

Then

$$\underline{\mathcal{Q}}\underline{\mathcal{O}}\mathcal{C}_{\text{KP}} \cong \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}) / \mathcal{I}.$$

Therefore, for any modular hybrid \mathcal{H} and a morphism $F: \underline{\mathcal{O}}\mathcal{C}_{\text{KP}} \rightarrow \mathcal{H}$ such that

$$\circ_{uv} F \begin{bmatrix} \binom{(uqv r)}{0} \\ 0 \\ \emptyset \end{bmatrix} = F \begin{bmatrix} \binom{(q)}{0} \binom{(r)}{0} \\ 0 \\ \emptyset \end{bmatrix},$$

there is a unique morphism $\widehat{F}: \mathcal{Q}\mathcal{Q}\mathcal{C}_{\text{KP}} \rightarrow \mathcal{H}$ of modular hybrids making the diagram

$$\begin{array}{ccc} \mathcal{Q}\mathcal{C}_{\text{KP}} & \hookrightarrow & \mathcal{Q}\mathcal{Q}\mathcal{C}_{\text{KP}} \\ & \searrow F & \downarrow \widehat{F} \\ & & \mathcal{H} \end{array}$$

commutative.

An immediate consequence of this theorem combined with Proposition 52 is that the symbol (17) with $g \geq 1$ belongs to $\mathcal{Q}\mathcal{Q}\mathcal{C}_{\text{KP}}$ if and only if it is stable, i.e. if either of b , $|O|$ or $|C|$ is nonzero.

Proof of Theorem 54. It is clear that $\mathcal{Q}\mathcal{Q}\mathcal{C}_{\text{KP}}$ is isomorphic to the ξ -closure of $\mathcal{Q}\mathcal{C}_{\text{KP}}$ in $\mathcal{Q}\mathcal{Q}\mathcal{C}$, therefore the isomorphism (39) identifies $\underline{\text{Mod}}(\mathcal{Q}\mathcal{C}_{\text{KP}})$ with $\mathcal{Q}\mathcal{Q}\mathcal{C}_{\text{KP}}$. The proof therefore goes along the similar lines as the proof of Theorem 38, so we only highlight the differences.

We must again be aware that some terms in (45) which we used to prove (44) may not belong to $\underline{\text{Mod}}(\mathcal{Q}\mathcal{C}_{\text{KP}})$. In the proof of Theorem 51 we explained how to avoid appearances of unstable terms. The remaining terms outside $\underline{\text{Mod}}(\mathcal{Q}\mathcal{C}_{\text{KP}})$ will be eliminated by absorbing trivial nests $\begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix}$.

By this we mean that, for arbitrary nontrivial nests $V_1, \dots, V_a \neq \begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix}$, we prove the following equality modulo \mathcal{I}

$$\begin{bmatrix} \begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix} \dots \begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} = \begin{bmatrix} \widetilde{V}_1 V_2 \dots V_a \\ g \\ C \end{bmatrix}, \tag{66}$$

where

$$\widetilde{V}_1 := \begin{bmatrix} \textcircled{\circ} \dots \textcircled{\circ} o_1 \dots o_b \\ g_1 \end{bmatrix}$$

if $V_1 = \begin{bmatrix} o_1 \dots o_b \\ g_1 \end{bmatrix}$. Assuming this, it suffices to prove (44) for elements not containing a trivial nest $\begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix}$, which proceeds as in the proof of Theorem 51. To verify (66), it suffices to prove that

$$\begin{bmatrix} \begin{bmatrix} \textcircled{\circ} \\ 0 \end{bmatrix} V_1 \dots V_a \\ g \\ C \end{bmatrix} = \begin{bmatrix} \widetilde{V}_1 V_2 \dots V_a \\ g \\ C \end{bmatrix} \text{ modulo } \mathcal{I} \tag{67}$$

for arbitrary V_1, \dots, V_a such that the left hand side of (67), which we denote by x , belongs to $\underline{\text{Mod}}(\mathcal{Q}\mathcal{C}_{\text{KP}})$. In the following calculation we denote, for $V = \begin{bmatrix} \textcircled{\circ} \\ g \end{bmatrix} \in \mathcal{Q}\mathcal{Q}$ and an independent symbol p , by pV a nest of the form $V = \begin{bmatrix} p\textcircled{\circ} \\ g \end{bmatrix}$, where $p\textcircled{\circ}$ is an extension of the multicycle $\textcircled{\circ}$ introduced in the proof of Proposition 34. We distinguish four cases.

Case 1: $g \geq 1$. If x has exactly two nests, we use the decomposition

$$\left[\begin{array}{c} [(\emptyset)]_V \\ 0 \\ g \\ C \end{array} \right] = \circ_{r'r''} \left(\left[\begin{array}{c} [(r')] [(p')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')] [(q'')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} q' r'' V \\ g-1 \\ C \end{array} \right] \right). \tag{68}$$

Applying relation (65) to the middle term of the right hand side, we get

$$\begin{aligned} \left[\begin{array}{c} [(\emptyset)]_V \\ 0 \\ g \\ C \end{array} \right] &= \circ_{r'r''} \left(\left[\begin{array}{c} [(r')] [(p')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')] [(q'')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} q' r'' V \\ g-1 \\ C \end{array} \right] \right) \\ &= \circ_{r'r''} \left[\begin{array}{c} [(r'')((p'))] [r'' V] \\ 0 \\ g-1 \\ C \end{array} \right] = \left[\begin{array}{c} \tilde{V} \\ g \\ C \end{array} \right]. \end{aligned}$$

The only term in the right hand side of (68) that might not belong to $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$ is the rightmost one. This happens if and only if $g = 1, C = \emptyset, V = [(\emptyset)]$, in which case we verify directly that

$$\begin{aligned} \left[\begin{array}{c} [(\emptyset)] [(\emptyset)] \\ 0 \\ 1 \\ \emptyset \end{array} \right] &= \circ_{q'q''} \left(\left[\begin{array}{c} [(p')] [(q')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')] [(q'')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] \right) \\ &= \circ_{q'q''} \left(\left[\begin{array}{c} [(p'')((q'))] \\ 0 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')((q''))] \\ 0 \\ 0 \\ \emptyset \end{array} \right] \right) \\ &= \circ_{q'q''} \left[\begin{array}{c} [((q'')((q'')))] \\ 0 \\ 0 \\ \emptyset \end{array} \right] = \left[\begin{array}{c} [(\emptyset)(\emptyset)] \\ 0 \\ 1 \\ \emptyset \end{array} \right]. \end{aligned}$$

If x has at least three nests, we use the decomposition

$$\left[\begin{array}{c} [(\emptyset)]_{V_1 \dots V_a} \\ 0 \\ g \\ C \end{array} \right] = \circ_{r'r''} \left(\left[\begin{array}{c} [(r')] [(p')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')] [(q'')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} q'' V_1 r'' V_2 V_3 \dots V_a \\ g-1 \\ C \end{array} \right] \right),$$

apply (65) to the middle term in the right hand side and proceed as before. In this case all terms clearly belong to $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$.

Case 2: $g = 0$ and $|C| \geq 2$. Let $d \in C$ and $C' := C \setminus \{d\}$. Then we use the decomposition

$$\left[\begin{array}{c} [(\emptyset)]_{V_1 \dots V_a} \\ 0 \\ C \end{array} \right] = \left[\begin{array}{c} [(p')] \\ 0 \\ \{d\} \end{array} \right] p' \circ p'' \left[\begin{array}{c} [(p'')] [(q'')] \\ 0 \\ 0 \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} q' V_1 V_2 \dots V_a \\ 0 \\ C' \end{array} \right].$$

All terms in the right hand side obviously belong to $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$.

Case 3: $g = 0$ and $C = \{d\}$. We want to decompose

$$\left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right]_{V_1 \dots V_a} \\ \emptyset \\ \{d\} \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} (p') \\ 0 \end{array} \right] \\ \emptyset \\ \{d\} \end{array} \right] p' \circ p'' \left[\begin{array}{c} \left[\begin{array}{c} (p'') \\ 0 \end{array} \right] \left[\begin{array}{c} (q') \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} q' V_1 V_2 \dots V_a \\ \emptyset \\ \emptyset \end{array} \right]. \quad (69)$$

While the first two terms in the right hand side always belong to $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$, the last one may be problematic. Let us discuss the case when $a \geq 2$ first. Since $x \in \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$, at least one of its nests must differ from $\left[\begin{array}{c} \emptyset \\ 0 \end{array} \right]$; we may assume without loss of generality it is V_2 . Then the rightmost term in (69) belongs to $\underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$.

The same is true if $a = 1$ and if $q' V_1$ is stable. If it is not stable, then V_1 must be of the form $\left[\begin{array}{c} (r) \\ 0 \end{array} \right]$ and we verify directly that

$$\begin{aligned} \left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right] \left[\begin{array}{c} (r) \\ 0 \end{array} \right] \\ \emptyset \\ \{d\} \end{array} \right] &= \left[\begin{array}{c} \left[\begin{array}{c} (p') \\ 0 \end{array} \right] \left[\begin{array}{c} (r) \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} \left[\begin{array}{c} (p'') \\ 0 \end{array} \right] \\ \emptyset \\ \{d\} \end{array} \right] \\ &= \left[\begin{array}{c} \left[\begin{array}{c} (p')(r) \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} \left[\begin{array}{c} (p'') \\ 0 \end{array} \right] \\ \emptyset \\ \{d\} \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right] \left[\begin{array}{c} (r) \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right]. \end{aligned}$$

Case 4: $g = 0$ and $C = \emptyset$. Since $x \in \underline{\text{Mod}}(\underline{\mathcal{O}}\mathcal{C}_{\text{KP}})$, at least two of its nests are nontrivial, so we may assume that

$$x = \left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right]_{V_1 V_2 \dots V_a} \\ \emptyset \\ \emptyset \end{array} \right],$$

where $V_1, V_2 \neq \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right]$. If pV_1 is stable, we decompose x as

$$\left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right]_{V_1 V_2 \dots V_a} \\ \emptyset \\ \emptyset \end{array} \right] = \left[\begin{array}{c} p' V_1 \\ \emptyset \\ \emptyset \end{array} \right] p' \circ p'' \left[\begin{array}{c} \left[\begin{array}{c} (p'') \\ 0 \end{array} \right] \left[\begin{array}{c} (q') \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} \left[\begin{array}{c} (q'') \\ 0 \end{array} \right]_{V_2 \dots V_a} \\ \emptyset \\ \emptyset \end{array} \right]$$

and apply (65) to the middle term in the right hand side as before. If pV_1 is not stable, then V_1 has to be of the form $\left[\begin{array}{c} (r) \\ 0 \end{array} \right]$ and we use instead the decomposition

$$\left[\begin{array}{c} \left[\begin{array}{c} \emptyset \\ 0 \end{array} \right] \left[\begin{array}{c} (r) \\ 0 \end{array} \right]_{V_2 \dots V_a} \\ \emptyset \\ \emptyset \end{array} \right] = \left[\begin{array}{c} \left[\begin{array}{c} (r) \\ 0 \end{array} \right] \left[\begin{array}{c} (q') \\ 0 \end{array} \right] \\ \emptyset \\ \emptyset \end{array} \right] q' \circ q'' \left[\begin{array}{c} \left[\begin{array}{c} (q'') \\ 0 \end{array} \right]_{V_2 \dots V_a} \\ \emptyset \\ \emptyset \end{array} \right].$$

This finishes our verification of (66). □

5. Finitary presentations

The aim of this section is to give an explicit finitary presentation of the Kaufmann–Penner modular hybrid $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ and derive from it a description of its algebras. As

the first step we express $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ in terms of generators and relations. Recall that the components of a cyclic hybrid \mathcal{H} are indexed by couples consisting of a multicycle \mathcal{O} and a finite set C . We will call the symbol $\binom{\mathcal{O}}{C}$ the *biarity* of elements in $\mathcal{H}(\mathcal{O}, C)$.

Theorem 55. *The cyclic hybrid $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ has the following presentation. The generators are:*

(g1) an “open pair of pants” $\mu = \mu_{\emptyset}^{((pqr))}$ of biarity $\binom{((pqr))}{\emptyset}$ with $G = 0$, with the trivial action of cyclic order-preserving automorphisms of $((pqr))$,

(g2) a “closed pair of pants” $\omega = \omega_{\{def\}}^{\emptyset}$ of biarity $\binom{\emptyset}{\{def\}}$ with $G = \frac{1}{2}$, and the trivial action of the group of automorphisms of $\{d, e, f\}$, and

(g3) a “morphism” $\phi = \phi_{\{d\}}^{((p))}$ with $G = \frac{1}{2}$ of biarity $\binom{((p))}{\{d\}}$,

subject to the axioms:

(a1) associativity in open inputs:

$$\mu_{\emptyset}^{((pqr))} \circ_s \mu_{\emptyset}^{((stu))} = \mu_{\emptyset}^{((pru))} \circ_s \mu_{\emptyset}^{((qts))},$$

(a2) associativity in closed inputs:

$$\omega_{\{def\}}^{\emptyset} \circ_g \omega_{\{ghi\}}^{\emptyset} = \omega_{\{dfi\}}^{\emptyset} \circ_g \omega_{\{ehg\}}^{\emptyset},$$

(a3) morphism property:

$$\phi_{\{g\}}^{((p))} \circ_f \omega_{\{def\}}^{\emptyset} = \left(\mu_{\emptyset}^{((pqr))} \circ_s \phi_{\{d\}}^{((s))} \right) \circ_t \phi_{\{e\}}^{((t))}, \text{ and}$$

(a4) centrality:

$$\mu_{\emptyset}^{((pqr))} \circ_s \phi_{\{d\}}^{((s))} = \mu_{\emptyset}^{((prq))} \circ_s \phi_{\{d\}}^{((s))}.$$

In other words, $\underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ is the quotient

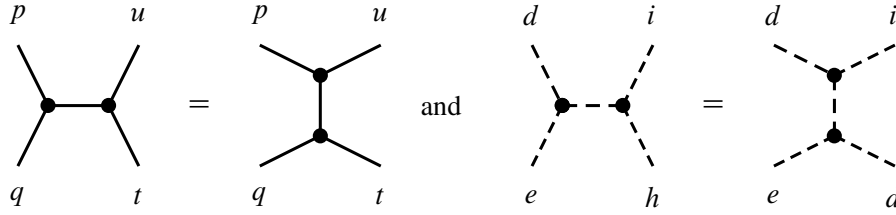
$$\underline{\mathcal{O}}\mathcal{C}_{\text{KP}} \cong \mathbb{F}_{\text{cyc}}(E) / \mathcal{J} \tag{70}$$

of the free cyclic hybrid generated by the collection E consisting of μ , ω and ϕ as above, modulo the ideal \mathcal{J} generated by the relations (a1)–(a4).

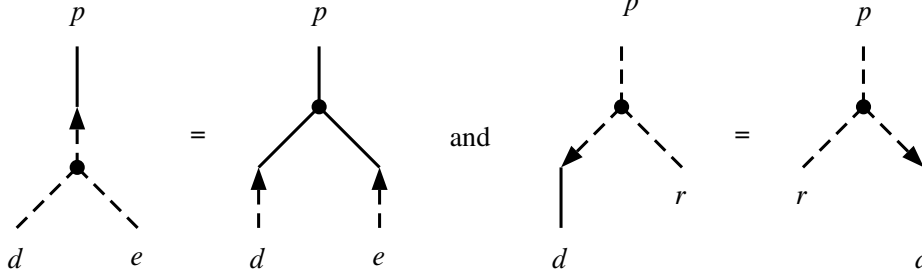
Generators (g1)–(g3) will be depicted as

$$\mu : \begin{array}{c} r \\ | \\ \bullet \\ / \quad \backslash \\ p \quad q \end{array} , \quad \omega : \begin{array}{c} f \\ | \\ \bullet \\ / \quad \backslash \\ d \quad e \end{array} , \quad \phi : \begin{array}{c} p \\ | \\ \blacktriangle \\ | \\ d \end{array} . \tag{71}$$

The pictorial forms of the associativities (a1) and (a2) are the “fusion rules”



while the morphism property (a3) and the centrality (a4) are depicted as



Proof of Theorem 55. We define a morphism $\pi: \mathbb{F}_{\text{cyc}}(E) \rightarrow \underline{\mathcal{O}}\mathcal{C}_{\text{KP}}$ of cyclic hybrids by

$$\pi \left(\mu_{\emptyset}^{((pqr))} \right) := \begin{bmatrix} ((pqr)) \\ 0 \\ \emptyset \end{bmatrix}, \quad \pi \left(\omega_{\{def\}}^{\emptyset} \right) := \begin{bmatrix} \emptyset \\ 0 \\ \{def\} \end{bmatrix} \quad \text{and} \quad \pi \left(\phi_{\{d\}}^{((p))} \right) := \begin{bmatrix} ((p)) \\ 0 \\ \{d\} \end{bmatrix}.$$

Let us verify that π descends to a morphism

$$\alpha: \mathbb{F}_{\text{cyc}}(E) / \mathcal{J} \rightarrow \underline{\mathcal{O}}\mathcal{C}_{\text{KP}} \tag{72}$$

of cyclic hybrids. The compatibility with (a1) means,

$$\begin{bmatrix} ((pqr)) \\ 0 \\ \emptyset \end{bmatrix} r \circ_s \begin{bmatrix} ((stu)) \\ 0 \\ \emptyset \end{bmatrix} = \begin{bmatrix} ((pru)) \\ 0 \\ \emptyset \end{bmatrix} r \circ_s \begin{bmatrix} ((qts)) \\ 0 \\ \emptyset \end{bmatrix},$$

the compatibility with (a2) leads to

$$\begin{bmatrix} \emptyset \\ 0 \\ \{def\} \end{bmatrix} f \circ_g \begin{bmatrix} \emptyset \\ 0 \\ \{ghi\} \end{bmatrix} = \begin{bmatrix} \emptyset \\ 0 \\ \{dfi\} \end{bmatrix} f \circ_g \begin{bmatrix} \emptyset \\ 0 \\ \{ehg\} \end{bmatrix},$$

the compatibility with (a3) amounts to verifying

$$\begin{bmatrix} ((p)) \\ 0 \\ \{g\} \end{bmatrix} g \circ_f \begin{bmatrix} \emptyset \\ 0 \\ \{def\} \end{bmatrix} = \left(\begin{bmatrix} ((pqr)) \\ 0 \\ \emptyset \end{bmatrix} q \circ_s \begin{bmatrix} ((s)) \\ 0 \\ \{d\} \end{bmatrix} \right) r \circ_t \begin{bmatrix} ((t)) \\ 0 \\ \{e\} \end{bmatrix}$$

and, finally, the compatibility with (a4) translates to

$$\begin{bmatrix} ((pqr)) \\ 0 \\ \emptyset \end{bmatrix} q \circ_s \begin{bmatrix} ((s)) \\ 0 \\ \{d\} \end{bmatrix} = \begin{bmatrix} ((prq)) \\ 0 \\ \emptyset \end{bmatrix} q \circ_s \begin{bmatrix} ((s)) \\ 0 \\ \{d\} \end{bmatrix}.$$

All the above equations follow directly from the definition of the \circ -operations in $\underline{\mathcal{O}}\mathcal{C}$.

We are going to prove that (72) is an isomorphism. Let us start with a couple of preliminary remarks. Free operads and operad-like structures are represented by decorated graphs, as explained at several places, see e.g. [23, Sections 6 and 9], [24, Section 4]. We assume that the reader is familiar with this description. In our case, elements of $\mathbb{F}_{\text{cyc}}(E)$ are connected, simply connected graphs with three types of vertices as in (71), and two types of (half)-edges: solid ones representing “open” propagators, and dashed ones representing “closed” propagators. Moreover, half-edges adjacent to a vertex representing the open pair of pants are cyclically ordered.

The associativities (a1) and (a2) enable one to contract propagators connecting two μ -vertices or two ω -vertices. The result will be a graph Γ with vertices

$$\begin{array}{c} \curvearrowright \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \dots \end{array} , \quad \begin{array}{c} \text{---} \star \text{---} \\ \text{---} \star \text{---} \\ \text{---} \star \text{---} \\ \dots \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \blacktriangleright \\ \text{---} \blacktriangleright \\ \dots \end{array} , \quad (73)$$

which represents an element in the quotient $\mathbb{F}_{\text{cyc}}(E)$ modulo the ideal generated by (a1) and (a2). We will call the vertices in (73) the μ -, ω - and ϕ -vertices, respectively. The half-edges adjacent to a μ -vertex are cyclically ordered. When drawn in the plane, we assume they have the implicit anti-clockwise cyclic order.

The case when Γ has only μ -vertices is very special, Γ then must be a corolla formed by an ω -vertex whose all adjacent half-edges are legs¹³ labelled by a finite set C . The equivalence class of Γ in $\mathbb{F}_{\text{cyc}}(E)/\mathcal{I}$ is then an element of biarity $\binom{\emptyset}{C}$.

So assume that Γ has at least one ω - or ϕ -vertex, which happens if and only if it has at least one solid half-edge. Cutting all its internal dashed edges in the middle produces b connected graphs $\Gamma_1, \dots, \Gamma_b$; the non-negative integer $b \in \mathbb{N}_+$ can easily be seen to be the number of boundaries of the equivalence class of Γ in $\mathbb{F}_{\text{cyc}}(E)/\mathcal{I}$. The open legs of Γ_i are cyclically ordered and their labels form for each $1 \leq i \leq b$ a cycle \mathfrak{o}_i . Denoting by C the set of labels of closed legs, the equivalence class of Γ in $\mathbb{F}_{\text{cyc}}(E)/\mathcal{I}$ has biarity $\binom{\mathfrak{O}}{C}$ with $\mathfrak{O} := \mathfrak{o}_1 \cdots \mathfrak{o}_b$.

In both cases, we explicitly assigned to elements of $\mathbb{F}_{\text{cyc}}(E)/\mathcal{I}$ a biarity $\binom{\mathfrak{O}}{C}$ preserved by the map α in (72). We denote by $\alpha \binom{\mathfrak{O}}{C}$ the restriction of this map to subsets of elements with the indicated biarity.

Let $\mathbb{F}_{\text{cyc}}(\omega)$ be the free cyclic operad generated by ω , \mathcal{A} the ideal generated by the associativity (a2), and $\mathcal{C}om_{\text{st}}$ the cyclic stable commutative operad from Example 26 identified with the cyclic suboperad of $\mathcal{Q}\mathcal{C}_{\text{KP}}$ consisting of elements as in (64). It is clear that $\alpha \binom{\emptyset}{C}$ can be identified with the morphism

$$\mathbb{F}_{\text{cyc}}(\omega)/\mathcal{A} \rightarrow \mathcal{C}om_{\text{st}}$$

that sends ω to the generator of $\mathcal{C}om_{\text{st}}$. This map map is an isomorphism since $\mathbb{F}_{\text{cyc}}(\omega)/\mathcal{A}$ is the standard presentation of the cyclic commutative operad [25, Example II.3.33], so the $\mathfrak{O} = \emptyset$ case of Theorem 55 is proven. Therefore,

¹³I.e., by definition, external half-edges.

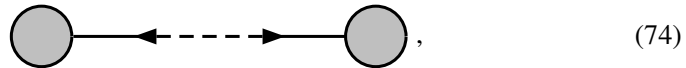
from now on we assume that $O \neq \emptyset$. Notice that for each biarity $\binom{O}{C}$ there is either precisely one element in \mathcal{QC}_{KP} of that biarity, or none. To prove that $\alpha \binom{O}{C}$ is an isomorphism, it is therefore enough to establish

Lemma 56. *Let us denote by $(\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}) \binom{O}{C}$ resp. $(\mathcal{QC}_{KP}) \binom{O}{C}$ the subsets of elements of the indicated biarity. Then*

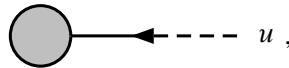
- (i) $(\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}) \binom{O}{C}$ is either empty or a one-point set, and
- (ii) $(\mathcal{QC}_{KP}) \binom{O}{C} \neq \emptyset$ implies that $(\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}) \binom{O}{C} \neq \emptyset$.

Our strategy of the proof will be to modify the graph Γ , bringing it in a ‘‘canonical’’ form (81), and show that this form is uniquely determined by the biarity. Let us start the process of modification of Γ .

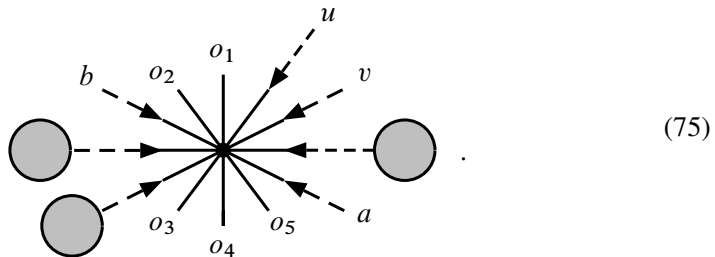
Since Γ has at least one solid (half)-edge, we may use the morphism property (a3) to eliminate all its ω -vertices. The only dashed internal edges will then be of the form



where the two gray cycles indicate (possibly empty) subgraphs. The only dashed legs are of the form



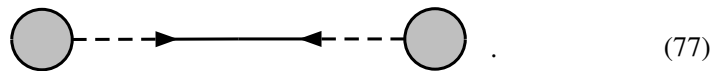
with the label u belonging to the set C of closed inputs. The local structure of Γ around a μ -vertex looks as in



Here the solid legs represent open inputs in the boundary cycle $((o_1, \dots, o_5))$, dashed legs closed inputs labelled by $u, a, b, v \in C$, and the gray circles are some subgraphs. The graph Γ may also have open inputs appearing e.g. as

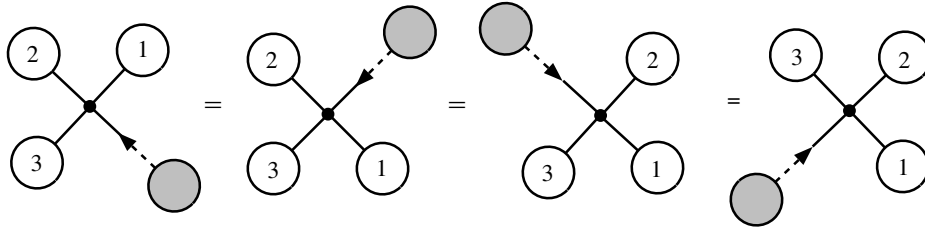


that corresponds to the open boundary component $((p))$. Finally, empty boundary components are introduced by solid edges connecting two ϕ -vertices:

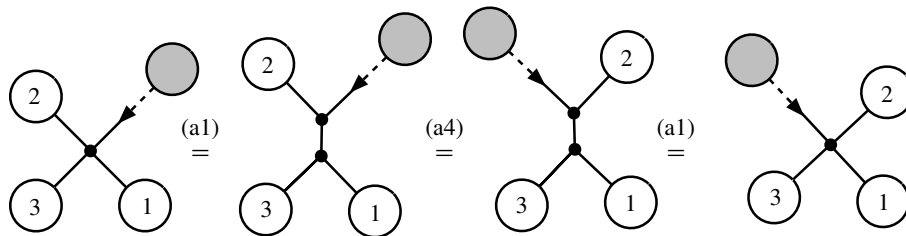


The centrality (a4) implies that the position of an edge connecting a μ -vertex with a ϕ -vertex, call it a (μ, ϕ) -edge, and the positions of other half-edges adjacent to

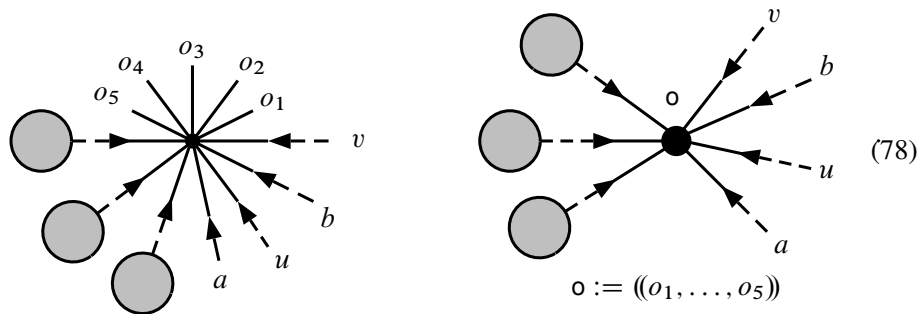
the same μ -vertex can be interchanged, so the (μ, ϕ) -edges are not subjected to the cyclic order. What we mean should be clear from the following particular example of four half-edges adjacent to μ :



where the numbered cycles are arbitrary possibly empty subgraphs. The above equalities can be proved by successive applications of the associativity (a1) and centrality (a4). For instance, the middle equality follows from



We leave the formulation and proof for an arbitrary number of (half)edges adjacent to a μ -vertex as an exercise. In particular, all (μ, ϕ) -edges can be mutually interchanged. Consequently, the half edges adjacent to the vertex in (75) can be rearranged as in the left picture in



The only half-edges subject to the cyclic order are those labelled by o_1, \dots, o_5 . The local structure around the above μ -vertex can therefore be encoded by a “fat” vertex \bullet labelled by the corresponding cycle, as shown in the right picture of (78).

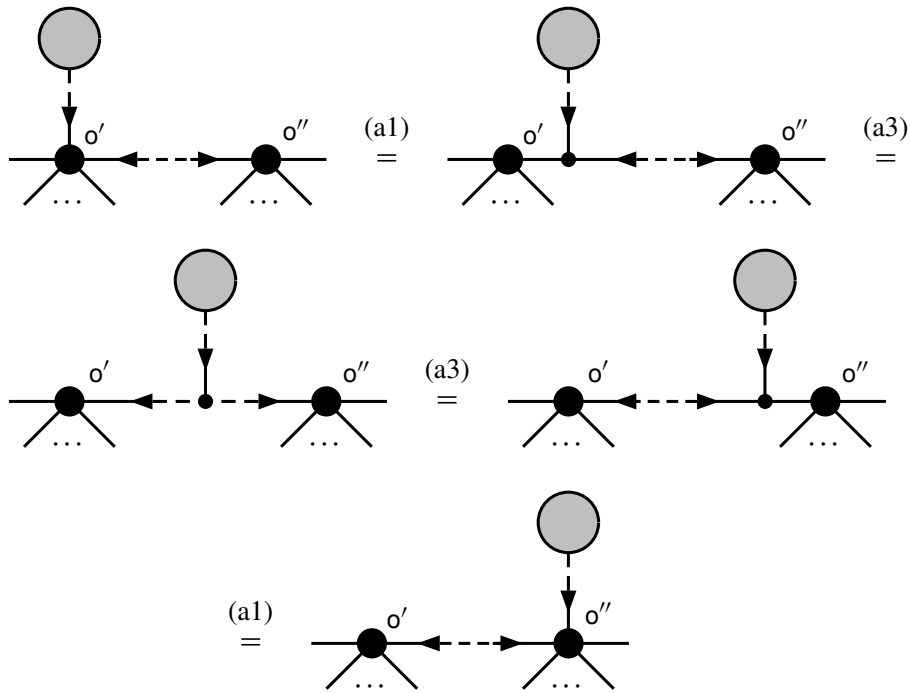
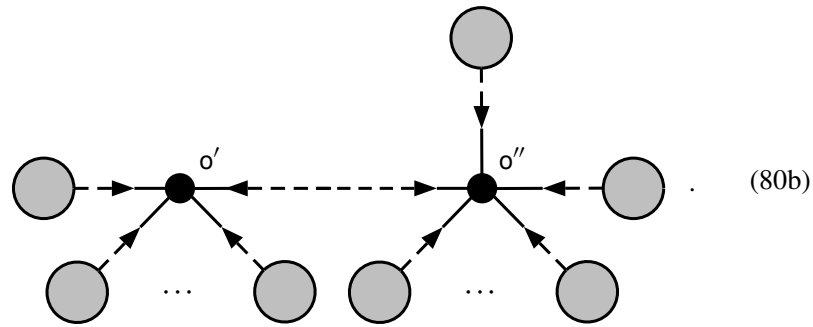


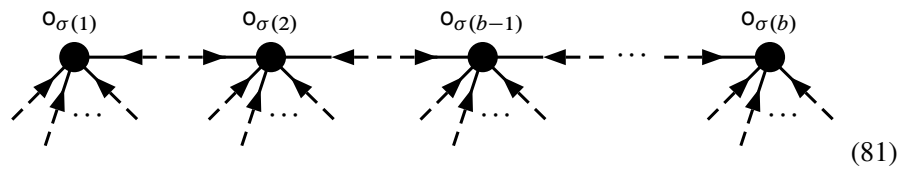
Figure 4. Proof of the sliding rule: subsequent use of axioms (a1) and (a3).

is, modulo \mathcal{J} , the same as

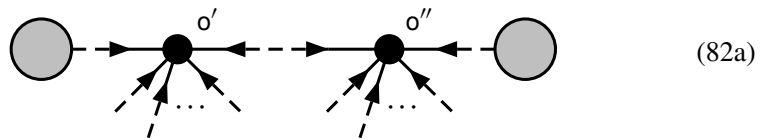


In words, the sliding rule claims that an arbitrary half-edge adjacent to a fat vertex can be amputated, moved along the graph, and attached to another fat vertex. The only restriction is that in doing so we must not create a forbidden fat vertex (79). The proof of the sliding rule is given in Figure 4. It is clear that, using the sliding rule,

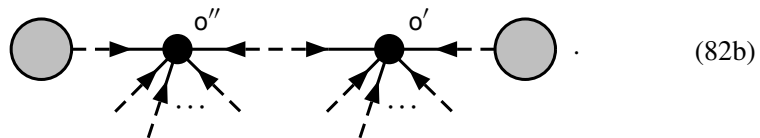
the graph Γ can be brought, modulo \mathcal{J} , to the following linear form:



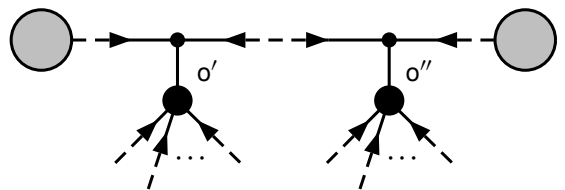
where the open legs are labelled by elements of C and σ is a permutation of the set $\{1, \dots, b\}$. As the next step we show that the order of cycles is (81) is not substantial. Concretely, we show that



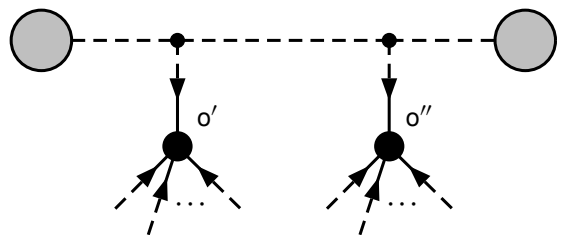
is, modulo \mathcal{J} , the same as



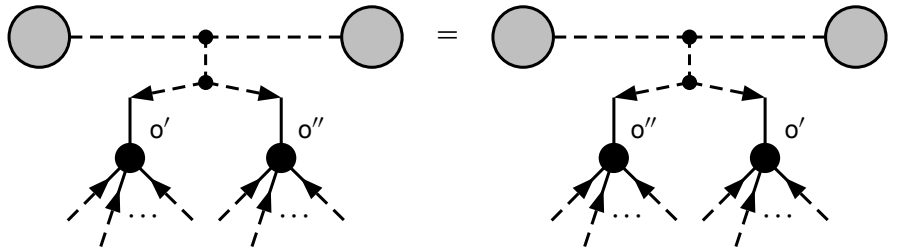
Using the associativity (a1) of μ we modify (82a) into



the morphism property (a4) turns it into



while the associativity (a2) of the “open” multiplication ω together with its commutativity shows that the above graph is, modulo \mathcal{J} , the same as the graphs



Backtracking the above modifications we convert the graph in the right hand side of the above equality into (82b).

We are finally ready to prove Lemma 56. As before, b denotes the number of boundaries of \mathcal{O} and we assume that $b \geq 1$. Let $\circ_1 \cdots \circ_{b'}$ be all nontrivial cycles in \mathcal{O} so that $\mathcal{O} = \circ_1 \cdots \circ_{b'}(()) \cdots (())$ with $b'' := b - b'$ trivial cycles $(())$. We distinguish four cases.

Case $b' \geq 2$. Using the commutativity (82a)–(82b) we can rearrange (81) so that the labels of the fat vertices read from the left to the right are

$$\circ_1, ((), \dots, ((), \circ_2, \dots, \circ_{b'}$$

Since both \circ_1 and $\circ_{b'}$ are nontrivial, the forbidden vertices (79) cannot occur so that, according to the sliding rule (80a)–(80b), the positions of closed legs are not constrained. In other words, for each biarity $\binom{\mathcal{O}}{C}$ with at least two nontrivial cycles in \mathcal{O} there exists exactly one isomorphism class of graphs in $\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}$ with that biarity.

Case $b' = 1, b'' \geq 1$. With the aid of commutativity (82a)–(82b) we order the fat vertices of (81) from the left to the right into

$$\circ_1, ((), \dots, (())$$

To avoid the forbidden ones, the rightmost fat vertex must be adjacent to at least one open leg, which may happen only when $C \neq \emptyset$. All remaining open legs can be then, using the sliding rule (80a)–(80b), transferred to the rightmost fat vertex, so their positions are irrelevant. We conclude that if $C \neq \emptyset$, $\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}$ contains exactly one element of biarity $\binom{\mathcal{O}}{C}$ while there are no elements of this biarity if $C = \emptyset$.

Case $b' = 1, b'' = 0$. The graph Γ is a corolla around a fat vertex which is clearly an allowed one if and only if the stability $2|C| + |O| > 2$ is satisfied.

Case $b' = 0$. Since $b = b' + b'' \neq 0$, $b'' \geq 1$ and all fat vertices in (81) are labelled by the trivial cycle $(())$. To avoid forbidden fat vertices at both extremities,

we need $|C| \geq 2$ otherwise there will be no graphs of biarity $\binom{O}{C}$. If $|C| = 2$, there is precisely one open leg at both sides of (81) and, due to the obvious left-right symmetry of the graph, the labels of these legs can be interchanges. If $|C| \geq 3$, the sliding rule applies so the positions of open legs are irrelevant as well.

We see that in all four cases, (i) of Lemma 56 is satisfied. The second part can be verified easily by comparing the list of elements belonging to $\underline{\mathcal{QC}}_{KP}$ given in Example 30 with the above calculations. This finishes the proof of the lemma and therefore also of the theorem. \square

Theorems 54 and 55 together give:

Theorem 57. *The modular hybrid $\underline{\mathcal{QC}}_{KP}$ has the following presentation. It is generators are (g1)–(g3) of Theorem 55 and the relations are (a1)–(a4) of Theorem 55, together with the Cardy condition*

$$\circ_{uv}(\omega_{\emptyset}^{(uqa)}) \circ_b \omega_{\emptyset}^{(bvr)}) = \phi_{\{c\}}^{(q)} \circ_d \phi_{\{d\}}^{(r)}. \tag{83}$$

Theorem 55 as well as the previous one are explicit as the degree 0 part in [15, Proposition 5.3].

Proof of Theorem 57. It follows from the commutativity of diagrams in Example 17 combined with (70) that

$$\underline{\text{Mod}}(\underline{\mathcal{QC}}_{KP}) \cong \underline{\text{Mod}}(\mathbb{F}_{\text{cyc}}(E)/\mathcal{J}) \cong \underline{\text{Mod}}(\mathbb{F}_{\text{cyc}}(E))/\mathcal{J} \cong \mathbb{F}_{\text{mod}}(E)/\mathcal{J}, \tag{84}$$

where $\mathbb{F}_{\text{mod}}(-)$ is the free modular hybrid functor, and the collection E and the ideal \mathcal{J} have the same generators as in Theorem 55.

Theorem 54 combined with (84) implies that the modular hybrid $\underline{\mathcal{QC}}_{KP}$ is isomorphic to the quotient of $\mathbb{F}_{\text{mod}}(E)$ by \mathcal{J} and relation (65). The proof is finished by observing that the isomorphisms (84) translates (65) into (83). \square

In Example 18 we defined algebras over cyclic hybrids. The finitary presentation of $\underline{\mathcal{QC}}_{KP}$ given in Theorem 55 offers an explicit description of its algebras. Recall that a *Frobenius algebra* on a vector space A equipped with a non-degenerate symmetric bilinear form β_A has an associative multiplication $\mu_A: A \otimes A \rightarrow A$ such that the expression

$$\beta_A(\mu_A(a_1, a_2), a_3) \in \mathbb{k} \tag{85}$$

is cyclically invariant in $a_1, a_2, a_3 \in A$. A Frobenius algebra is *commutative* if (85) is invariant under all permutations of a_1, a_2 and a_3 ; this forces μ_A to be commutative. The following theorem is explicit in the degree 0 part of [15, Theorem 5.4].

Theorem 58. *An algebra over the Kaufmann–Penner cyclic hybrid $\underline{\mathcal{QC}}_{KP}$ on a pair A, B of finite dimensional vector spaces equipped with symmetric non-degenerate bilinear forms β_A, β_B is the same as*

- (i) *a Frobenius algebra on A with the associated form β_A ,*

- (ii) a commutative Frobenius algebra on B with the associated form β_B , and
- (iii) an associative algebra morphism $B \rightarrow A$ with values in the center of A .

Proof of Theorem 55. By definition, an \mathcal{QC}_{KP} -algebra is a morphism of cyclic hybrids $\alpha: \mathcal{OC}_{\text{KP}} \rightarrow \mathcal{E}nd_{A,B}$. Let μ, ω and ϕ be the generators of \mathcal{QC}_{KP} as in (g1)–(g3) of Theorem 55. Since the bilinear forms β_A and β_B are non-degenerate, the equations

$$\begin{aligned}\beta_A(\mu_A(a_1, a_2), a_3) &= \alpha(\mu)(a_1 \otimes a_2 \otimes a_3), \\ \beta_B(\omega_B(c_1, c_2), c_3) &= \alpha(\omega)(c_1 \otimes c_2 \otimes c_3),\end{aligned}\tag{86}$$

$a_i \in A, c_i \in B, i = 1, 2, 3$, define bilinear maps $\mu_A: A \otimes A \rightarrow A$ and $\omega_B: B \otimes B \rightarrow B$ while $f := \alpha(\phi)$ is a linear map $B \rightarrow A$.

It is easy to show that (a1) of Theorem 55 translates to the associativity of μ_A and (a2) to the associativity of ω_B . The symmetry, i.e. the commutativity of ω_B , follows from the invariance of ω under the group of automorphisms of its inputs. Likewise, the morphism property (a3) implies that $f: B \rightarrow A$ is an algebra morphism while the centrality (a4) implies (iii) of the theorem. Finally, the symmetry of the expressions (85) for μ_A resp. ω_B follows from the defining equations (86) and the cyclic symmetry of μ resp. ω . \square

Theorem 57 offers the following description of algebras for the modular hybrid \mathcal{QQ}_{KP} in the spirit of the classical result about 2-dimensional topological field theories [18], see also [15, Theorem 5.4] and [19, Section 4].

Theorem 59. *An algebra for the KP modular hybrid \mathcal{QQ}_{KP} on a pair A, B of vector spaces with symmetric nondegenerate bilinear forms β_A, β_B is the same as*

- (i) a Frobenius algebra (A, μ_A, β_A) ,
- (ii) a commutative Frobenius algebra (B, ω_B, β_B) , and
- (iii) an associative algebra morphism $f: B \rightarrow A$ with values in the center of A , satisfying the Cardy condition

$$\beta_A(\mu_A \otimes \mu_A)(\mathbb{1} \otimes \tau \otimes \mathbb{1})(\mathbb{1} \otimes \mathbb{1} \otimes \beta_A^{-1}) = (\beta_A \otimes \beta_A)(\mathbb{1} \otimes (f \otimes f) \beta_B^{-1} \otimes \mathbb{1}),\tag{87}$$

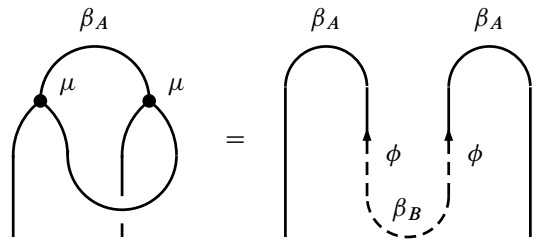
where τ is the standard symmetry in the monoidal category of graded vector spaces.

In (87), β_A^{-1} is the inverse of $\beta_A: A \otimes A \rightarrow \mathbb{k}$, i.e. the unique linear map $\beta_A^{-1}: \mathbb{k} \rightarrow A \otimes A$ satisfying

$$(\beta_A \otimes \mathbb{1})(\mathbb{1} \otimes \beta_A^{-1}) = (\mathbb{1} \otimes \beta_A)(\beta_A^{-1} \otimes \mathbb{1}) = \mathbb{1}_A;$$

the inverse $\beta_B^{-1}: \mathbb{k} \rightarrow B \otimes B$ is defined similarly.

Proof of Theorem 59. It follows the pattern of the proof of Theorem 58 and we leave the details to the reader. A pictorial form of the Cardy condition (87) is



□

References

- [1] M. Atiyah, Topological quantum field theories, *Inst. Hautes Études Sci. Publ. Math.*, (1988), no. 68, 175–186, (1989). [Zbl 0692.53053](#) [MR 1001453](#)
- [2] C. Braun, Moduli spaces of Klein surfaces and related operads, *Algebr. Geom. Topol.*, **12** (2012), no. 3, 1831–1899. [Zbl 1254.30071](#) [MR 2980000](#)
- [3] J. Burke and R. Koytcheff, A colored operad for string link infection, *Algebr. Geom. Topol.*, **15** (2015), no. 6, 3371–3408. [Zbl 1337.57012](#) [MR 3450765](#)
- [4] J. L. Cardy and D. C. Lewellen, Bulk and boundary operators in conformal field theory, *Phys. Lett. B*, **259** (1991), no. 3, 274–278. [MR 1107480](#)
- [5] J. Chuang and A. Lazarev, Dual Feynman transform for modular operads, *Commun. Number Theory Phys.*, **1** (2007), no. 4, 605–649. [MR 2412267](#)
- [6] M. Doubek, The modular envelope of the cyclic operad Ass, *preprint*, 2013. [arXiv:1312.5501](#)
- [7] M. R. Gaberdiel and B. Zwiebach, Tensor constructions of open string theories. I. Foundations, *Nuclear Phys. B*, **505** (1997), no. 3, 569–624. [Zbl 0911.53044](#) [MR 1490781](#)
- [8] E. Getzler and M. M. Kapranov, Cyclic operads and cyclic homology, in *Geometry, topology, & physics*, S.-T. Yau (ed.), 167–201, Conf. Proc. Lect. Notes. Geom. Topol., IV, Int. Press, Cambridge, MA, 1995. [Zbl 0883.18013](#) [MR 1358617](#)
- [9] E. Getzler and M. M. Kapranov, Modular operads, *Compositio Math.*, **110** (1998), no. 1, 65–126. [Zbl 0894.18005](#) [MR 1601666](#)
- [10] V. Ginzburg and M. M. Kapranov, Koszul duality for operads, *Duke Math. J.*, **76** (1994), no. 1, 203–272. [Zbl 0855.18006](#) [MR 1301191](#)
- [11] E. Harrelson, On the homology of open/closed string theory, *preprint*, 2004.
- [12] B. Jurčo and K. Münster, Type II superstring field theory: Geometric approach and operadic description, *J. High Energy Phys.*, (2013), no. 4, 126, 36pp. [Zbl 1342.83383](#) [MR 3065840](#)
- [13] R. M. Kaufmann, Open/closed string topology and moduli space actions via open/closed Hochschild actions, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **6** (2010), Paper 036, 33pp. [Zbl 1197.55005](#) [MR 2647315](#)
- [14] R. M. Kaufmann, M. Livernet, and R. C. Penner, Arc operads and arc algebras, *Geom. Topol.*, **7** (2003), 511–568. [Zbl 1034.18009](#) [MR 2026541](#)
- [15] R. M. Kaufmann and R. C. Penner, Closed/open string diagrammatics, *Nuclear Phys. B*, **748** (2006), no. 3, 335–379. [Zbl 1214.32004](#) [MR 2242677](#)

- [16] R. M. Kaufmann and J. Lucas, Decorated Feynman categories, *J. Noncommut. Geom.*, **11** (2017), no. 4, 1437–1464. [Zbl 06830247](#) [MR 3743229](#)
- [17] R. M. Kaufmann and B. C. Ward, *Feynman categories*, Astérisque, 387, Société Mathématique de France, Paris, 2017. [Zbl 06731251](#) [MR 3636409](#)
- [18] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, 59, Cambridge University Press, Cambridge, 2004. [Zbl 1046.57001](#) [MR 2037238](#)
- [19] A. D. Lauda and H. Pfeiffer, Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras, *Topology Appl.*, **155** (2008), no. 7, 623–666. [Zbl 1158.57038](#) [MR 2395583](#)
- [20] C. Lazaroiu, On the structure of open–closed topological field theory in two dimensions, *Nuclear Physics B*, **603** (2001), no. 3, 497–530. [Zbl 0983.81090](#) [MR 1839382](#)
- [21] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 346, Springer, Heidelberg, 2012. [Zbl 1260.18001](#) [MR 2954392](#)
- [22] M. Markl, Loop homotopy algebras in closed string field theory, *Comm. Math. Phys.*, **221** (2001), no. 2, 367–384. [Zbl 1016.55005](#) [MR 1845329](#)
- [23] M. Markl, Operads and PROPs, in *Handbook of algebra. Vol. 5*, 87–140, Handb. Algebr., 5, Elsevier/North-Holland, Amsterdam, 2008. [Zbl 1211.18007](#) [MR 2523450](#)
- [24] M. Markl, Modular envelopes, OSFT and nonsymmetric (non- Σ) modular operads, *J. Noncommut. Geom.*, **10** (2016), no. 2, 775–809. [Zbl 1375.81205](#) [MR 3519052](#)
- [25] M. Markl, S. Shnider, and J. D. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, 96, American Mathematical Society, Providence, RI, 2002. [Zbl 1017.18001](#) [MR 1898414](#)
- [26] K. Münster and I. Sachs, Quantum open-closed homotopy algebra and string field theory, *Comm. Math. Phys.*, **321** (2013), no. 3, 769–801. [Zbl 1270.81185](#) [MR 3070036](#)
- [27] V. Novák, Cyclically ordered sets, *Czechoslovak Math. J.*, **32(107)** (1982), no. 3, 460–473. [Zbl 0515.06003](#) [MR 669787](#)
- [28] O. Ore, *Theory of Graphs*, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, R.I. 1962. [Zbl 0105.35401](#) [MR 150753](#)
- [29] M. Saadi and B. Zwiebach, Closed string field theory from polyhedra, *Ann. Physics*, **192** (1989), no. 1, 213–227. [Zbl 0688.93010](#) [MR 1006943](#)
- [30] B. Zwiebach, Closed String Field Theory: Quantum action and the Batalin-Vilkovisky master equation, *Nuclear Phys. B*, **390** (1993), no. 1, 33–152. [MR 1203476](#)

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