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Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants

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Abstract. This paper is part of a series of articles on noncommutative geometry and conformal geometry. In this paper, we reformulate the local index formula in conformal geometry in such a way to take into account the action of conformal diffeomorphisms. We also construct and compute a whole new family of geometric conformal invariants associated with conformal diffeomorphisms. This includes conformal invariants associated with equivariant characteristic classes. The approach of this paper involves using various tools from noncommutative geometry, such as twisted spectral triples and cyclic theory. An important step is to establish the conformal invariance of the Connes–Chern character of the conformal Dirac spectral triple of Connes–Moscovici. Ultimately, however, the main results of the paper are stated in a purely differential-geometric fashion.

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1. Introduction

This paper is part of a series of papers applying techniques of noncommutative geometry in the settings of conformal geometry and noncommutative versions of conformal geometry (cf. [78, 79] for other papers of this series). The present paper has two main results.

The first main result is the reformulation of the local index formula in conformal geometry in such a way to take into account the action of conformal diffeomorphisms (Theorem 8.4). This is in the line of research initiated by Alain Connes and Henri Moscovici (see [35, 38, 68]). In particular, the existence of such an index formula was hinted at by Moscovici (see [68, Remark 3.8]).

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The second main result is the construction and explicit computation of a whole new families of geometric conformal invariants attached with conformal diffeomorphisms (Theorem 11.1). This includes conformal invariants associated with equivariant characteristic classes (Theorem 11.4). These conformal invariants are not of the same type of the conformal invariants considered by Alexakis [1] in his proof of the conjecture of Deser–Schwimmer [41] on the characterization of global conformal invariants. Our conformal invariants take into account the action of the group of conformal diffeomorphisms, i.e. the conformal gauge group. This should of be of interest in the context of string theory and conformal gravity. In addition, our conformal invariants also shed new light on the conformal indices of Branson–Ørsted [16, 17] (see end of Section 11).

To put things into context, recall that, given an action of a group G on a manifold M, the orbit space M/G has a natural manifold structure when the action is free and proper. In general, however, M/G need not even be Hausdorff, and so it is difficult to use M/G to extract much geometric or even topological information for general group actions. The solution proposed by noncommutative geometry (in the sense of Connes [32]) is to trade the space M/G for the crossed-product algebra $C^\infty(M) \rtimes G$. This algebra always makes sense independently how wild the action might be; the non-Hausdorffness of the quotient only pertains through the noncommutativity of $C^\infty(M) \rtimes G$.

More generally, noncommutative geometry provides us with a general framework that allows us to deal with a variety of geometric problems whose noncommutative natures prevent us from using classical techniques. In this framework spaces are traded for algebras which formally play the roles of the algebras of functions on (ghost) noncommutative spaces. From this point of view, a noncommutative manifold is usually represented by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an algebra represented by bounded operators on the Hilbert space $\mathcal{H}$ and D is a selfadjoint unbounded operator that commutes with $\mathcal{A}$ up to bounded operators. This operator accounts for the geometric properties of the spectral triple. The paradigm of a spectral triple is provided by a Dirac spectral triple associated with a Dirac operator on a spin manifold.

Much like with Dirac operators, the datum of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ gives rise to an index problem $\mathcal{E} \to \operatorname{ind} D_{\nabla^{\mathcal{E}}}$ in terms of the Fredholm indices of operators obtained by twisting the operator D with connections on noncommutative vector bundles $\mathcal{E}$ (i.e. finitely generated projective modules over $\mathcal{A}$). These indices are K-theory invariants. The relevant noncommutative analogue of de Rham theory is provided by the cyclic homology and cyclic cohomology of Connes [28–30] (see also Tsygan [83]). In particular, Connes [30] established the following index formula,

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \operatorname{Ch}(D), \operatorname{Ch}(\mathcal{E}) \rangle \qquad \forall (\mathcal{E}, \nabla^{\mathcal{E}}), \tag{1.1}$$

where $\operatorname{Ch}(\mathcal{E})$ is the Chern character in cyclic homology and $\operatorname{Ch}(D)$ is a class in cyclic cohomology, called the Connes–Chern character. Under further assumptions, the

Connes–Chern character is represented by the CM cocycle [34,35]. The components of the CM cocycle are given by formulas that are local in the sense they involve a version for spectral triples of the noncommutative residue trace of Guillemin [53] and Wodzicki [84]. Together with (1.1) this provides us with the local index formula in noncommutative geometry. In the case of a Dirac spectral triple this enables us to recover the local index formula of Atiyah–Singer [4,6] (see [35,73]).

Given an arbitrary group of diffeomorphisms G of a manifold M, there is a technical difficulty for constructing a spectral triple over the crossed-product algebra $C^\infty(M) \rtimes G$ due to the lack of geometric structures that are invariant under the full diffeomorphism group (at the notable exception of the manifold structure itself). In particular, there need not exist a G-invariant metric. As observed by Connes [31], up to a Connes-Thom isomorphism, the problem can be solved by passing to the total space of the metric bundle $P \to M$. Following up this idea Connes–Moscovici [35] built a spectral triple over the algebra $C_c^\infty(P) \rtimes G$. The computation of the CM cocycle of that spectral triple involves Hopf cyclic cohomology and expresses the Connes–Chern character in terms of Gelf'and–Fuchs cohomology classes (see [36,37]).

As observed by Connes–Moscovici [38], when G is a group of conformal diffeomorphisms, we actually can have a spectral triple over $C^\infty(M) \rtimes G$ instead of $C_c^\infty(P) \rtimes G$ at the expense of twisting the definition of a spectral triple. More precisely, a *twisted* spectral $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is like an ordinary spectral triple at the exception that the boundedness of commutators $[D, a]$, $a \in \mathcal{A}$, is replaced by that of twisted commutators,

$$[D, a]_\sigma := Da - \sigma(a)D, \qquad a \in \mathcal{A},$$

where σ is a given automorphism of the algebra $\mathcal{A}$. A natural example is given by conformal deformations of ordinary spectral triples,

$$(\mathcal{A}, \mathcal{H}, D) \longrightarrow (\mathcal{A}, \mathcal{H}, kDk)_\sigma, \qquad \sigma(a) = k^2 a k^{-2},$$

where k ranges over positive invertible elements of $\mathcal{A}$ (see [38]). Another example due to Connes–Moscovici [38] is the conformal Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L_g^2(M, \$), \not{D}_g\big)_{\sigma_g}$$

associated with the Dirac operator $\not{D}_g$ on a compact spin Riemannian manifold (M, g) and a group G of diffeomorphisms preserving a given conformal structure $\mathcal{C}$ (see also Section 7).

As for ordinary spectral triples, the datum of a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ gives rise to an index problem by twisting the operator D with σ-connections (see [77]). The resulting Fredholm indices are computed by a Connes–Chern character $\mathrm{Ch}(D)_\sigma$ defined as a class in the (ordinary) cyclic cohomology of $\mathcal{A}$

(see [38, 77]). However, although Moscovici [68] produced an Ansatz for a version of the CM cocycle for twisted spectral triples, this Ansatz has been verified only in a special class of examples (see [68]). In particular, to date it is still not known if Moscovici's Asantz even holds for conformal deformations of ordinary spectral triples.

For our purpose it is important to define the Connes–Chern character in the cyclic cohomology of *continuous* cochains, which is smaller than the cyclic cohomology of arbitrary cochains. We show that for a natural class of twisted spectral triples $(\mathcal{A}, \mathcal{H}, D)_\sigma$ over locally convex algebras, which we call *smooth* twisted spectral triples, the Connes–Chern character descends to a class $\mathbf{Ch}(D)_\sigma \in \mathbf{HP}^0(\mathcal{A})$, where $\mathbf{HP}^\bullet(\mathcal{A})$ is the periodic cyclic cohomology of continuous cochains (Proposition 5.9). We also show the invariance of this class under conformal perturbations of twisted spectral triples (Proposition 6.8).

The construction of the conformal Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}$$

associated with a conformal class $\mathcal{C}$ on a compact spin manifold M alluded to above *a priori* depends on a choice of the metric $g \in \mathcal{C}$. As it turns out, up to equivalences of twisted spectral triples, changing a metric within $\mathcal{C}$ only amounts to make a conformal deformation of twisted spectral triples (see [68]). Combining this with the invariance under conformal deformations of the Connes–Chern character mentioned above then shows that the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g} \in \mathbf{HP}^0(C^\infty(M) \rtimes G)$ is an invariant of the conformal class $\mathcal{C}$ (Theorem 7.7).

The next step is the computation of the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g}$. As mentioned above, we do not know if Moscovici's Ansatz holds for the conformal Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}.$$

Therefore, we cannot make use of the CM cocycle given by that Ansatz. However, we can observe that the conformal invariance of $\mathbf{Ch}(\not{D}_g)_{\sigma_g}$ allows us to use *any* metric in the conformal class $\mathcal{C}$. In particular, we may use a G-invariant metric. This is a convenient choice of metric, since in that case the automorphism σ_g is the identity, and so the associated conformal Dirac spectral triple becomes an *ordinary* spectral triple. The existence of such metrics is ensured by Ferrand–Obata theorem, provided that the conformal structure $\mathcal{C}$ is non-flat, i.e. it is not equivalent to the conformal structure of the round sphere.

Assuming the non-flatness of $\mathcal{C}$ we are reduced to the computation of the Connes–Chern character of an ordinary equivariant Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big),$$

where G is a group of isometries. In this case, the Connes–Chern character is computed in [78] by making use of a CM cocycle representative and by computing

this cocycle by means of heat kernel techniques. This allows us to express the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g}$ in terms of explicit polynomials in curvatures and normal curvatures of the fixed-point manifolds of diffeomorphisms in G (see Theorem 8.4 for the precise statement). These formulas are reminiscent of the local equivariant index theorem for Dirac operators of Atiyah–Segal–Singer [5, 6]. This provides us with a reformulation of the local index formula in conformal-diffeomorphism invariant geometry.

As mentioned above, pairing the Connes–Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ with any (even) periodic cyclic cycle over the algebra $\mathcal{A}_G := C^\infty(M) \rtimes G$ produces an invariant of the conformal class $\mathcal{C}$. This potentially produces a large family of conformal invariants. To understand and compute these invariants we need an explicit construction of (periodic) cyclic cycles over $\mathcal{A}_G$. There is a large amount of work on the cyclic homology of crossed-product algebras, especially in the case of group actions on manifolds or varieties (see, e.g. [10, 18, 20–22, 31, 32, 40, 44, 47, 69, 70, 74, 75]). In his seminal work on cyclic cohomology and foliations in the early 80s, Connes [31] constructed an explicit cochain map and quasi-isomorphism from the equivariant cohomology into the homogeneous component of the periodic cyclic cohomology of $\mathcal{A}_G$. However, since then it was not until the recent notes [74, 75] that further explicit quasi-isomorphisms were exhibited.

We refer to Section 10 for a review of the results of [74, 75] in the case of group actions on manifolds. This involves a large amount of homological algebra. Combining these results with the explicit computation of the Connes–Chern character enables us to define and compute explicitly a whole family of geometric conformal invariants (Theorem 11.1). These conformal invariants are associated with conformal diffeomorphisms and mixed equivariant cycles. The mixed equivariant homology was introduced in [75]. It mixes group homology and de Rham homology. It is also the natural receptacle for the cap product of group homology with equivariant cohomology. As a result, we obtain conformal invariants associated with equivariant characteristic classes (Theorem 11.4). We also relate these conformal invariants to the conformal indices of Branson–Ørsted [16, 17] (see end of Section 11).

The paper is organized as follows. In Section 2, we review the main definitions and examples regarding twisted spectral triples and the construction of their index maps. In Section 3, we review the main facts about cyclic cohomology, cyclic homology and the Chern character in cyclic homology. In Section 4, we review the construction of the Connes–Chern character of a twisted spectral triple. In Section 5, we show that for smooth twisted spectral triples the Connes–Chern character descends to the cyclic cohomology of continuous cochains. In Section 6, we establish the invariance of the Connes–Chern character under conformal deformations. In Section 7, after reviewing the construction of the conformal Dirac spectral triple, we prove that its Connes–Chern Character is a conformal invariant. In Section 8, we compute this Connes–Chern character by using the results of [78]. This provides us with a local index formula in conformal-diffeomorphism invariant geometry. In Section 10, we

review the results of [74, 75] on the construction of explicit quasi-isomorphisms computing the cyclic homology of crossed-product algebras in the case of group actions on manifolds. Finally, in Section 11 we combine the results of the previous sections to construct and compute our conformal invariants.

Throughout this paper we focus exclusively on the even-dimensional case. The main results of this paper have analogues in odd dimension. However, a complete treatment would require a comprehensive account on the index theory for odd twisted spectral triples. This will be dealt with in forthcoming papers. In addition, the conformal invariants that we construct in this paper are associated with conformal diffeomorphisms of finite order (i.e. periodic diffeomorphisms). It is also possible to define and compute conformal invariants associated with non-periodic conformal diffeomorphisms. We also postpone this to a forthcoming paper. Finally, the non-flatness leaves out the important case of the action of the projective Poincaré group $\mathrm{PO}(n,1)$ on the round sphere $\mathbb{S}^n$. The main issue is the computation of the Connes–Chern character in this case. As pointed out in [68] this requires a refinement of the analytical considerations of [78] in order to take into account the contribution of the non-elliptic components of $\mathrm{PO}(n,1)$.

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2. Index theory for twisted spectral triples

In this section, we recall how the datum of a twisted spectral triple naturally gives rise to an index problem ([38, 77]). The exposition closely follows that of [77] (see also [67] for the case of ordinary spectral triples).

In the setting of noncommutative geometry, the role of manifolds is played by spectral triples.

Definition 2.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by

(1) A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

(2) A unital $*$-algebra $\mathcal{A}$ represented by bounded operators on $\mathcal{H}$ preserving its $\mathbb{Z}_2$-grading.

(3) A selfadjoint unbounded operator D on $\mathcal{H}$ such that

(a) D maps $\mathrm{dom}(D) \cap \mathcal{H}^{\pm}$ to $\mathcal{H}^{\mp}$.

(b) The resolvent $(D+i)^{-1}$ is a compact operator.

(c) $a\,\mathrm{dom}(D) \subset \mathrm{dom}(D)$ and $[D,a]$ is bounded for all $a \in \mathcal{A}$.

Remark 2.2. The condition (3)(a) implies that with respect to the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the operator D takes the form,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D^{\pm}\colon \mathrm{dom}(D) \cap \mathcal{H}^{\pm} \to \mathcal{H}^{\mp}.$$

Example 2.3. The paradigm of a spectral triple is given by a Dirac spectral triple,

$$\big(C^\infty(M), L^2_g(M, \not{S}), \not{D}_g\big),$$

where (M^n, g) is a compact spin Riemannian manifold of even dimension n and $\not{D}_g$ is its Dirac operator acting on the spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$.

The definition of a twisted spectral triple is similar to that of an ordinary spectral triple, except for some "twist" given by the conditions (3) and (4)(b) below.

Definition 2.4 ([38]). A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is given by

(1) A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

(2) A unital $*$-algebra $\mathcal{A}$ represented by even bounded operators on $\mathcal{H}$.

(3) An automorphism $\sigma\colon \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.

(4) An odd selfadjoint unbounded operator D on $\mathcal{H}$ such that

(a) The resolvent $(D+i)^{-1}$ is compact.

(b) $a(\mathrm{dom}\, D) \subset \mathrm{dom}\, D$ and $[D,a]_\sigma$ is bounded for all $a \in \mathcal{A}$, where the twisted commutator $[D,a]_\sigma$ is defined by

$$[D,a]_\sigma := Da - \sigma(a)D \qquad \forall a \in \mathcal{A}. \tag{2.1}$$

The relevance of the notion of twisted spectral triples in the setting of conformal geometry stems from the following observation. Let $(C^\infty(M), L^2_g(M, \not{S}), \not{D}_g)$ be a Dirac spectral triple as in Example 2.3, and consider a conformal change of metrics,

$$\widehat{g} = k^{-2} g, \qquad k \in C^\infty(M),\ k > 0.$$

We then can form a Dirac spectral triple $(C^\infty(M), L^2_{\widehat{g}}(M, \not{S}), \not{D}_{\widehat{g}})$ associated with the new metric $\widehat{g}$. As it turns out (see [77]) this spectral triple is equivalent to the following spectral triple,

$$\big(C^\infty(M), L^2_g(M, \not{S}), \sqrt{k}\ \not{D}_g \sqrt{k}\big).$$

We note that the above spectral triple continues to make sense if we only assume k to be a positive Lipschitz function on M.

More generally, let $(\mathcal{A}, \mathcal{H}, D)$ be an ordinary spectral triple and k a positive element of $\mathcal{A}$. If we replace D by its *conformal deformation* kDk then, when $\mathcal{A}$ is noncommutative, the triple $(\mathcal{A}, \mathcal{H}, kDk)$ needs not be an ordinary spectral triple. However, as the following result shows, it always gives rise to a twisted spectral triple.

Proposition 2.5 ([38]). *Let $\sigma\colon \mathcal{A} \to \mathcal{A}$ be the automorphism defined by*

$$\sigma(a) := k^2 a k^{-2} \qquad \forall a \in \mathcal{A}. \tag{2.2}$$

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ is a twisted spectral triple.

Remark 2.6. A more elaborate version of the above example, and the main focus of this paper, is the conformal Dirac spectral triple of Connes–Moscovici [38]. This is a twisted spectral triple taking into account the action of the group of diffeomorphisms preserving a given conformal structure. We refer to Section 7.1 for a review of this example.

Remark 2.7. We refer to [38,39,51,57,68,79] for the constructions of various other examples of twisted spectral triples.

In what folllows, we let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a twisted spectral triple. In addition, we let $\mathcal{E}$ be a finitely generated projective right module over $\mathcal{A}$, i.e. $\mathcal{E}$ is a direct summand of a free module $\mathcal{E}_0 \simeq \mathcal{A}^N$.

Definition 2.8 ([77]). A σ-translation of $\mathcal{E}$ is given by a pair $(\mathcal{E}, \sigma^{\mathcal{E}})$, equipped with the following data:

(i) $\mathcal{E}^\sigma$ is a finitely generated projective right module over $\mathcal{A}$.

(ii) $\sigma^{\mathcal{E}}$ is a $\mathbb{C}$-linear isormophism from $\mathcal{E}$ onto $\mathcal{E}^\sigma$ such that

$$\sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \qquad \text{for all } \xi \in \mathcal{E} \text{ and } a \in \mathcal{A}. \tag{2.3}$$

Remark 2.9. When $\sigma = \mathrm{id}$ the condition (2.3) simply means that $\sigma^{\mathcal{E}}$ is a right-module isomorphism, and so in this case we can take $(\mathcal{E}, \mathrm{id})$ as a σ-translation of $\mathcal{E}$.

Remark 2.10. Suppose that $\mathcal{E} = e\mathcal{A}^N$ with $e = e^2 \in M_N(\mathcal{A})$, $N \geq 1$. Note that the automorphism σ lifts to a $\mathbb{C}$-linear isomorphism $\sigma\colon \mathcal{A}^N \to \mathcal{A}^N$ given by

$$\sigma(\xi) = (\sigma(\xi_j)) \qquad \text{for all } \xi = (\xi_j) \in \mathcal{A}^N.$$

This induces a $\mathbb{C}$-linear isomorphism $\sigma_e\colon e\mathcal{A}^N \to \sigma(e)\mathcal{A}^N$ satisfying (2.3). Therefore, in this case we may take $(\sigma(e)\mathcal{A}^N, \sigma_e)$ as a σ-translation of $\mathcal{E} = e\mathcal{A}^N$.

Throughout the rest of the section we let $(\mathcal{E}^\sigma, \sigma^{\mathcal{E}})$ be a σ-translation of $\mathcal{E}$. In addition, we consider the space of twisted 1-forms,

$$\Omega^1_{D,\sigma}(\mathcal{A}) = \{\Sigma a^i [D, b^i]_\sigma : a^i, b^i \in \mathcal{A}\},$$

where $[D, b]_\sigma$, $b \in \mathcal{A}$, is the twisted commutator (2.1). We note that $\Omega^1_{D,\sigma}(\mathcal{A})$ is a subspace of $\mathcal{L}(\mathcal{H})$. The *twisted* differential $d_\sigma \colon \mathcal{A} \to \Omega^1_{D,\sigma}(\mathcal{A})$ is given by

$$d_\sigma a := [D, a]_\sigma \qquad \forall a \in \mathcal{A}. \tag{2.4}$$

This is a σ-derivation, in the sense that

$$d_\sigma(ab) = (d_\sigma a)b + \sigma(a) d_\sigma b \qquad \forall a, b \in \mathcal{A}. \tag{2.5}$$

In particular, this implies that $\Omega^1_{D,\sigma}(\mathcal{A})$ is a sub-$(\mathcal{A}, \mathcal{A})$-bimodule of $\mathcal{L}(\mathcal{H})$.

Definition 2.11. A σ-connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla^{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E}^\sigma \otimes_{\mathcal{A}} \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = (\nabla^{\mathcal{E}} \xi) a + \sigma^{\mathcal{E}}(\xi) \otimes d_\sigma a \qquad \forall \xi \in \mathcal{E}\ \forall a \in \mathcal{A}. \tag{2.6}$$

Example 2.12. Suppose that $\mathcal{E} = e\mathcal{A}^N$ with $e = e^2 \in M_N(\mathcal{A})$. Then a natural σ-connection on $\mathcal{E}$ is the *Grassmannian σ-connection* $\nabla^{\mathcal{E}}_0$ defined by

$$\nabla^{\mathcal{E}}_0 \xi = \sigma(e)(d_\sigma \xi_j) \qquad \text{for all } \xi = (\xi_j) \text{ in } \mathcal{E}. \tag{2.7}$$

Definition 2.13. A Hermitian metric on $\mathcal{E}$ is a map $(\cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that

(1) $(\cdot, \cdot)$ is $\mathcal{A}$-sesquilinear, i.e. it is $\mathcal{A}$-antilinear with respect to the first variable and $\mathcal{A}$-linear with respect to the second variable.

(2) $(\cdot, \cdot)$ is positive, i.e. $(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{E}$.

(3) $(\cdot, \cdot)$ is nondegenerate, i.e. $\xi \to (\xi, \cdot)$ is an $\mathcal{A}$-antilinear isomorphism from $\mathcal{E}$ onto its $\mathcal{A}$-dual $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$.

Example 2.14. The canonical Hermitian structure on the free module $\mathcal{A}^N$ is given by

$$(\xi, \eta)_0 = \xi_1^* \eta_1 + \cdots + \xi_q^* \eta_q \qquad \text{for all } \xi = (\xi_j) \text{ and } \eta = (\eta_j) \text{ in } \mathcal{A}^N. \tag{2.8}$$

It induces a Hermitian metric on any direct summand $\mathcal{E} = e\mathcal{A}^N$, $e = e^2 \in M_N(\mathcal{A})$, (see, e.g. [77]).

From now on we assume that $\mathcal{E}$ and its σ-translation $\mathcal{E}^\sigma$ carry Hermitian metrics. We denote by $\mathcal{H}(\mathcal{E})$ the pre-Hilbert space consisting of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ equipped with the Hermitian inner product,

$$\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle := \langle \zeta_1, (\xi_1, \xi_2)\zeta_2 \rangle, \qquad \xi_j \in \mathcal{E}, \zeta_j \in \mathcal{H}, \tag{2.9}$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\mathcal{E}$. It can be shown that $\mathcal{H}(\mathcal{E})$ is actually a Hilbert space and its topology is independent of the choice of the Hermitian inner product of $\mathcal{E}$ (see, e.g. [77]). We also note there is a natural $\mathbb{Z}_2$-grading on $\mathcal{H}(\mathcal{E})$ given by

$$\mathcal{H}(\mathcal{E}) = \mathcal{H}^+(\mathcal{E}) \oplus \mathcal{H}^-(\mathcal{E}), \qquad \mathcal{H}^\pm(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}^\pm. \tag{2.10}$$

We denote by $\mathcal{H}(\mathcal{E}^\sigma)$ the similar $\mathbb{Z}_2$-graded Hilbert space associated with $\mathcal{E}^\sigma$ and its Hermitian metric.

Let $\nabla^{\mathcal{E}}$ be a σ-connection on $\mathcal{E}$. Regarding $\Omega^1_{D,\sigma}(\mathcal{A})$ as a subalgebra of $\mathcal{L}(\mathcal{H})$ we have a natural left-action $c\colon \Omega^1_{D,\sigma}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \to \mathcal{H}$ given by

$$c(\omega \otimes \zeta) = \omega(\zeta) \qquad \text{for all } \omega \in \Omega^1_{D,\sigma}(\mathcal{A}) \text{ and } \zeta \in \mathcal{H}.$$

We then denote by $c(\nabla^{\mathcal{E}})$ the composition $(1_{\mathcal{E}^\sigma} \otimes c) \circ (\nabla^{\mathcal{E}} \otimes 1_{\mathcal{H}})\colon \mathcal{E} \otimes \mathcal{H} \to \mathcal{E}^\sigma \otimes \mathcal{H}$. Thus, for $\xi \in \mathcal{E}$ and $\zeta \in \mathcal{H}$, and upon writing $\nabla^{\mathcal{E}}\xi = \sum \xi_\alpha \otimes \omega_\alpha$ with $\xi_\alpha \in \mathcal{E}^\sigma$ and $\omega_\alpha \in \Omega^1_{D,\sigma}(\mathcal{A})$, we have

$$c(\nabla^{\mathcal{E}})(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta). \tag{2.11}$$

In what follows we regard the domain of D as a left $\mathcal{A}$-module, which is possible since the action of $\mathcal{A}$ on $\mathcal{H}$ preserves the domain $\operatorname{dom} D$.

Definition 2.15. The operator $D_{\nabla^{\mathcal{E}}}\colon \mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom}(D) \to \mathcal{H}(\mathcal{E}^\sigma)$ is defined by

$$D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta) := \sigma^{\mathcal{E}}(\xi) \otimes D\zeta + c(\nabla^{\mathcal{E}})(\xi \otimes \zeta) \qquad \text{for all } \xi \in \mathcal{E} \text{ and } \zeta \in \operatorname{dom} D. \tag{2.12}$$

Remark 2.16. Although the operators $\sigma^{\mathcal{E}}$, D, and $\nabla^{\mathcal{E}}$ are not module maps, the operator is well defined as a linear map with domain $\mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom}(D)$ (see [77]).

Remark 2.17. With respect to the $\mathbb{Z}_2$-gradings (2.10) for $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(\mathcal{E}^\sigma)$ the operator $D_{\nabla^{\mathcal{E}}}$ takes the form,

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D^-_{\nabla^{\mathcal{E}}} \\ D^+_{\nabla^{\mathcal{E}}} & 0 \end{pmatrix}, \qquad D^\pm_{\nabla^{\mathcal{E}}}\colon \mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom} D^\pm \longrightarrow \mathcal{H}^\mp(\mathcal{E}^\sigma). \tag{2.13}$$

That is, $D_{\nabla^{\mathcal{E}}}$ is an odd operator.

Example 2.18 (See [77]). Suppose that $\mathcal{E} = e\mathcal{A}^N$ with $e = e^2 \in M_N(\mathcal{A})$ and let $\nabla^{\mathcal{E}}_0$ be the Grassmanian σ-connection of $\mathcal{E}$. Then up to the canonical unitary identifications $\mathcal{H}(\mathcal{E}) \simeq e\mathcal{H}^N$ and $\mathcal{H}(\mathcal{E}^\sigma) \simeq \sigma(e)\mathcal{H}^N$ the operators $D_{\nabla^{\mathcal{E}}_0}$ agrees with

$$\sigma(e)(D \otimes 1_N)\colon e(\operatorname{dom} D)^N \longrightarrow \sigma(e)\mathcal{H}^N.$$

Example 2.19. In the case of a Dirac spectral triple $(C^\infty(M), L^2_g(M, \$), \not{D}_g)$, we may take $\mathcal{E}$ to be the module $C^\infty(M, E)$ of smooth sections of a vector bundle E over M. Any Hermitian metric and connection on E give rise to a Hermitian metric and a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$. Furthermore, if we set $\mathcal{H} = L^2_g(M, E)$, then, under the natural identification $\mathcal{H}(\mathcal{E}) \simeq L^2(M, \$ \otimes E)$, the operator $(\not{D}_g)_{\nabla^{\mathcal{E}}}$ agrees with the usual twisted Dirac operator $\not{D}_{\nabla^E}$ as defined, e.g. in [11].

Proposition 2.20 ([77]). *The operator $D_{\nabla^{\mathcal{E}}}$ is closed and Fredholm.*

Note that the above result implies that the operators $D^{\pm}_{\nabla^{\mathcal{E}}}$ in (2.13) are Fredholm. This leads us to the following definition.

Definition 2.21. The index of the operator $D_{\nabla^{\mathcal{E}}}$ is

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \frac{1}{2}\big(\operatorname{ind} D^{+}_{\nabla^{\mathcal{E}}} - \operatorname{ind} D^{-}_{\nabla^{\mathcal{E}}}\big),$$

where $\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}}$ is the usual Fredholm index of $D^{\pm}_{\nabla^{\mathcal{E}}}$ i.e.

$$\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}} = \dim\ker D^{\pm}_{\nabla^{\mathcal{E}}} - \dim\operatorname{coker} D^{\pm}_{\nabla^{\mathcal{E}}}.$$

Remark 2.22. In general the indices $\pm\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}}$ do not agree, so that it is natural take their mean to define the index of $D_{\nabla^{\mathcal{E}}}$. However, as shown in [77], the index $\operatorname{ind} D_{\nabla^{\mathcal{E}}}$ is an integer when the automorphism σ is *ribbon*, in the sense there is another automorphism $\tau\colon \mathcal{A} \to \mathcal{A}$ such that $\sigma = \tau\circ\tau$ and $\tau(a)^* = \tau^{-1}(a^*)$ for all $a \in \mathcal{A}$. The ribbon condition is satisfied in all main examples of twisted spectral triples (see [77]). When this condition holds, it is shown in [79] that we always can endow $\mathcal{E}$ with a "σ-Hermitian structure" (see [79] for the precise definition). In that case, for any connection $\nabla^{\mathcal{E}}$ compatible with the σ-Hermitian structure,

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \dim\ker D^{+}_{\nabla^{\mathcal{E}}} - \dim\ker D^{-}_{\nabla^{\mathcal{E}}} \in \mathbb{Z}.$$

The above formula is the generalization of the usual formula for the index of a Dirac operator twisted by a Hermitian connection on a Hermitian vector bundle.

As it turns out the index $\operatorname{ind} D_{\nabla^{\mathcal{E}}}$ only depends on the K-theory class of $\mathcal{E}$ (see [77]). More precisely, we have the following result.

Proposition 2.23 ([38, 77]). *There is a unique additive map* $\operatorname{ind}_{D,\sigma}\colon K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$ *such that, for any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and any σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we have*

$$\operatorname{ind}_{D,\sigma}[\mathcal{E}] = \operatorname{ind} D_{\nabla^{\mathcal{E}}}.$$

3. Cyclic homology and cyclic cohomology

Cyclic homology and cyclic cohomology were introduced by Connes [28–30] as the relevant noncommutative analogue of de Rham theory (see also Tsygan [83]). In this section, we review the main background on cyclic homology and cyclic cohomology that is needed for this paper. We refer to the monographs [30, 32, 64] for more comprehensive accounts on these topics.

3.1. Preamble: mixed complexes, cyclic modules, and S-maps. In what follows we let $\mathbf{k}$ be an arbitrary unital ring. According to Burghelea [24] and Kassel [60] a mixed complex of $\mathbf{k}$-modules is given by a datum $(C_\bullet, b, B)$, where C_m, $m \geq 0$, are (left) $\mathbf{k}$-modules and $b: C_\bullet \to C_{\bullet-1}$ and $B: C_\bullet \to C_{\bullet+1}$ are $\mathbf{k}$-module maps such that

$$b^2 = B^2 = bB + Bb = 0.$$

The *cyclic complex* of a mixed complex $C = (C_\bullet, b, B)$ is the chain complex

$$C^\natural = (C^\natural_\bullet, b + SB),$$

where $C^\natural_m = C_m \oplus C_{m-2} \oplus \cdots$, $m \geq 0$, and $S: C^\natural_\bullet \to C^\natural_{\bullet-2}$ is the canonical projection obtained by factorizing out C_m. The homology of this chain complex is called the *cyclic homology* of the mixed complex C and is denoted by $\mathrm{HC}_\bullet(C)$.

The operator $S: C^\natural_\bullet \to C^\natural_{\bullet-2}$ is called the *periodicity operator*. The *periodic cyclic complex* is the $\mathbb{Z}_2$-graded chain complex

$$C^\sharp = (C^\sharp_\bullet, b + B),$$

where $C^\sharp_i = \prod_{q\geq 0} C_{2q+i}$, $i = 0, 1$. Equivalently, we have

$$C^\sharp_\bullet = \varprojlim_S C_{2q+\bullet},$$

where $\varprojlim_S$ is the projective limit of the projective system defined by the operators

$$S: C_{2q+\bullet} \to C_{2q+\bullet-2}, \qquad q \geq 1.$$

The homology of the chain complex $C^\sharp$ is called the *periodic cyclic homology* of the mixed complex C. It is denoted by $\mathrm{HP}_\bullet(C)$.

Following Connes [29], a *cyclic* $\mathbf{k}$*-module* is given by a datum $(C_\bullet, d, s, t)$, where C_m, $m \geq 0$, are $\mathbf{k}$-modules and the $\mathbf{k}$-module maps

$$d: C_\bullet \to C_{\bullet-1}, \quad s: C_\bullet \to C_{\bullet+1}, \quad \text{and} \quad t: C_\bullet \to C_\bullet$$

define a simplicial module structure with faces $d_j = t^j d t^{-(j+1)}$ and degeneracies $s_j = t^{j+1} d t^{-(j+1)}$. In addition, the operator t is required to be cyclic, in the sense that $t^{m+1} = 1$ on C_m. Any cyclic $\mathbf{k}$-module $C = (C_\bullet, d, s, t)$ gives rise to a mixed complex

$$C = (C_\bullet, b, B),$$

where $b = \sum_{j=0}^m (-1)^j d_j$ and $B = (1 - \tau)sN$ on C_m with $\tau = (-1)^m t$ and $N = 1 + \tau + \cdots + \tau^m$. We then define the cyclic and periodic cyclic homologies of C as the the cyclic and periodic cyclic homologies of that mixed complex.

Given mixed complexes $C = (C_\bullet, b, B)$ and $\bar{C} = (C_\bullet, b, B)$, an S-map in the sense of Kassel [60] is a chain map $f: C_\bullet^\natural \to \bar{C}_\bullet^\natural$ which is compatible with the S-operator. Any such map uniquely decomposes as $f = \sum f^{(j)} S^j$, where $f^{(j)}: C_\bullet \to \bar{C}_{\bullet+2j}$ is a $\mathbf{k}$-module map of degree $2j$ such that

$$\left[b, f^{(0)}\right] = 0 \quad \text{and} \quad \left[B, f^{(j)}\right] + \left[b, f^{(j+1)}\right] = 0$$

for $j \geq 0$. In particular, it uniquely extends to a chain map $f^\sharp: C_\bullet^\sharp \to \bar{C}_\bullet^\sharp$ between the periodic cyclic complexes of C and $\bar{C}$.

3.2. Cyclic homology of algebras. From now on we let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. Connes [29] associated with any such algebra a cyclic space (i.e. a cyclic $\mathbb{C}$-module). This is $C(\mathcal{A}) = (C_\bullet(\mathcal{A}), d, s, t)$, where $C_m(\mathcal{A}) = \mathcal{A}^{\otimes(m+1)}$, $m \geq 0$, and the structural operators

$$d: C_\bullet(\mathcal{A}) \to C_{\bullet-1}(\mathcal{A}), \quad s: C_\bullet(\mathcal{A}) \to C_{\bullet+1}(\mathcal{A}), \quad \text{and} \quad t: C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{A})$$

are given by

$$d(a^0 \otimes \cdots \otimes a^m) = (a^m a^0) \otimes a^1 \cdots \otimes a^{m-1}, \tag{3.1}$$

$$s(a^0 \otimes \cdots \otimes a^m) = 1 \otimes a^0 \cdots \otimes a^m, \tag{3.2}$$

$$t(a^0 \otimes \cdots \otimes a^m) = a^m \otimes a^0 \cdots \otimes a^{m-1}, \qquad a^j \in \mathcal{A}. \tag{3.3}$$

We get a mixed complex $(C_\bullet(\mathcal{A}), b, B)$, where the differential $b: C_\bullet(\mathcal{A}) \to C_{\bullet-1}$ is the Hochschild boundary,

$$b(a^0 \otimes \cdots \otimes a^m) = \sum_{j=0}^{m-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^m + (-1)^m a^m a^0 \otimes \cdots \otimes a^{m-1}, \qquad a^j \in \mathcal{A}.$$

This gives rise to the cyclic complex

$$C^\natural(\mathcal{A}) = (C_\bullet^\natural(\mathcal{A}), b + SB)$$

and the periodic cyclic complex

$$C^\sharp(\mathcal{A}) = (C_\bullet^\sharp(\mathcal{A}), b + B).$$

Their respective homologies are called the *cyclic homology* and *periodic cyclic homology* of the algebra $\mathcal{A}$. They are denoted by $\mathrm{HC}_\bullet(\mathcal{A})$ and $\mathrm{HP}_\bullet(\mathcal{A})$, respectively.

The mixed complex $(C_\bullet(\mathcal{A}), b, B)$ is normalized as follows. The degeneracies of the cyclic space $C(\mathcal{A})$ are

$$s_j = t^{j+1} s t^{-(j+1)}, \qquad j = 0, \ldots, m.$$

The space of degenerate m-chains is

$$D_m(\mathcal{A}) = s_0(C_{m-1}(\mathcal{A})) \oplus \cdots \oplus s_{m-1}(C_{m-1}(\mathcal{A})).$$

Here

$$s_j(a^0 \otimes \cdots \otimes a^{m-1}) = a^0 \otimes \cdots \otimes a^j \otimes 1 \otimes a^{j+1} \otimes \cdots \otimes a^{m-1}, \qquad a^l \in \mathcal{A}.$$

Therefore, we see that $D_m(\mathcal{A})$ is spanned by elementary tensor products $a^0 \otimes \cdots \otimes a^m$ where $a^j = 1$ for some $j \in \{1, \ldots, m\}$. The differentials b and B preserve $D_\bullet(\mathcal{A})$, and so the mixed complex $(C_\bullet(\mathcal{A}), b, B)$ descends to a *normalized* mixed complex

$$\bar{C}(\mathcal{A}) = (\bar{C}_\bullet(\mathcal{A}), b, B),$$

with

$$\bar{C}_\bullet(\mathcal{A}) = C_\bullet(\mathcal{A})/D_\bullet(\mathcal{A}).$$

We let

$$\bar{C}^\natural(\mathcal{A}) = (\bar{C}^\natural_\bullet(\mathcal{A}), b + SB) \quad \text{and} \quad \bar{C}^\sharp(\mathcal{A}) = (\bar{C}^\sharp_\bullet(\mathcal{A}), b + B)$$

be the corresponding cyclic and periodic cyclic complexes. The canonical projection $\pi: C_\bullet(\mathcal{A}) \to \bar{C}_\bullet(\mathcal{A})$ is a map of mixed complexes. At the level of Hochschild homology we obtain a quasi-isomorphism. General homological algebra arguments then imply that we also obtain quasi-isomorphisms at the level of cyclic homology and periodic cyclic homology. This enables us to describe $\mathrm{HC}_\bullet(\mathcal{A})$ and $\mathrm{HP}_\bullet(\mathcal{A})$ in terms of *normalized* cycles.

When $\mathcal{A}$ is a locally convex algebra, the space of chains $C_m(\mathcal{A})$ is often completed into $\mathbf{C}_m(\mathcal{A}) := \mathcal{A}^{\hat{\otimes}(m+1)}$, where $\hat{\otimes}$ is the projective topological tensor product [52, 82]. In particular, given any locally convex space X, a linear map $\Phi: \mathbf{C}_m(\mathcal{A}) \to X$ is continuous iff the $(m+1)$-linear map

$$\mathcal{A}^{m+1} \ni (a^0, \ldots, a^m) \to \Phi(a^0 \otimes \cdots \otimes a^m)$$

is jointly continuous. The structural operators (d, s, t) in (3.1)–(3.3) uniquely extend to continuous operators on $\mathbf{C}_\bullet(\mathcal{A})$, so that we obtain a cyclic space $\mathbf{C}(\mathcal{A}) := (\mathbf{C}_\bullet(\mathcal{A}), d, s, t)$. In the same way as above, this cyclic space gives rise to cyclic and periodic cyclic complexes $\mathbf{C}^\natural(\mathcal{A}) := (\mathbf{C}^\natural_\bullet(\mathcal{A}), b + SB)$ and $\mathbf{C}^\sharp(\mathcal{A}) := (\mathbf{C}^\sharp_\bullet(\mathcal{A}), b + B)$. Their respective homologies are denoted by $\mathbf{HC}_\bullet(\mathcal{A})$ and $\mathbf{HP}_\bullet(\mathcal{A})$.

Example 3.1. Let $\mathcal{A} = C^\infty(M)$, where M is a closed manifold. We endow $\mathcal{A}$ with its standard Fréchet algebra topology. In addition, we let $\Omega(M) = (\Omega^\bullet(M), d)$ be the de Rham complex of (smooth) differential forms on M. We shall regard $\Omega(M)$

as a mixed complex $\Omega(M) = (\Omega^\bullet(M), 0, d)$. The Hochschild–Kostant–Rosenberg map $\alpha\colon \Omega^\bullet(M) \to \mathbf{C}_\bullet(\mathcal{A})$ is given by

$$\alpha(f^0 \otimes \cdots \otimes f^m) = \frac{1}{m!} f^0 df^1 \wedge \cdots \wedge df^m, \qquad f^j \in \mathcal{A}. \tag{3.4}$$

This is a map of mixed complexes. Moreover, Connes [30] proved that this is a quasi-isomorphism, and so we can describe $\mathbf{HC}_\bullet(\mathcal{A})$ and $\mathbf{HP}_\bullet(\mathcal{A})$ in terms of de Rham cohomology $H^\bullet(M, \mathbb{C})$. Namely,

$$\mathbf{HC}_m(\mathcal{A}) \simeq \big(\Omega^m(M)/d(\Omega^{m-1}(M))\big) \oplus H^{m-2}(M, \mathbb{C}) \oplus \cdots, \qquad m \geq 0,$$
$$\mathbf{HP}_\bullet(\mathcal{A}) \simeq \bigoplus_{q \geq 0} H^{\mathrm{ev/odd}}(M, \mathbb{C}), \qquad \text{where } H^{\mathrm{ev/odd}}(M, \mathbb{C}) := \bigoplus_{m \text{ even/odd}} H^m(M, \mathbb{C}).$$

3.3. Cyclic cohomology of algebras. Cyclic cohomology is the dual version of cyclic homology. There is a natural duality between $C_m(\mathcal{A}) = \mathcal{A}^{\otimes(m+1)}$ and the space $C^m(\mathcal{A})$ that consists of $(m+1)$-linear maps $\varphi\colon \mathcal{A}^{m+1} \to \mathbb{C}$. We then get a "cyclic co-space" $(C^\bullet(\mathcal{A}), d, s, t)$, where the operators

$$d\colon C^\bullet(\mathcal{A}) \to C^{\bullet+1}(\mathcal{A}), \quad s\colon C^\bullet(\mathcal{A}) \to C^{\bullet-1}(\mathcal{A}), \quad \text{and} \quad t\colon C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{A})$$

are given by

$$(d\varphi)(a^0, \ldots, a^{m+1}) = \varphi((a^{m+1}a^0), a^1, \ldots, a^m), \tag{3.5}$$
$$(s\varphi)(a^0, \ldots, a^{m-1}) = \varphi(1, a^0, \ldots, a^{m-1}), \tag{3.6}$$
$$(t\varphi)(a^0, \ldots, a^m) = \varphi(a^m, a^0, \ldots, a^{m-1}), \qquad \varphi \in C^m(\mathcal{A}),\ a^j \in \mathcal{A}. \tag{3.7}$$

We obtain the Hochschild cochain complex $(C^\bullet(\mathcal{A}), b)$, where $b\colon C^\bullet(\mathcal{A}) \to C^{\bullet+1}(\mathcal{A})$ is given by

$$b\varphi(a^0, \ldots, a^{m+1}) = \sum_{j=0}^{m} (-1)^j \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{m+1}) + (-1)^{m+1} \varphi(a^{m+1}a^0, \ldots, a^m), \qquad a^j \in \mathcal{A}.$$

The dual B-operator

$$B\colon C^\bullet(\mathcal{A}) \to C^{\bullet-1}(\mathcal{A})$$

is $B = Ns(1-T)$, where $T = (-1)^m t$ and $N = 1 + T + \cdots + T^m$ on $C^m(\mathcal{A})$. We then have a duality between the cyclic complex $C^\natural(\mathcal{A})$ and the cochain complex

$$C_\natural(\mathcal{A}) := (C^\bullet_\natural(\mathcal{A}), b + SB),$$

where $C^m_\natural(\mathcal{A}) = C^m(\mathcal{A}) \oplus C^{m-2}(\mathcal{A}) \oplus \cdots$ and $S\colon C^m_\natural(\mathcal{A}) \to C^{m+2}_\natural(\mathcal{A})$ is the inclusion of $C^m_\natural(\mathcal{A})$ into $C^{m+2}_\natural(\mathcal{A})$. We also have a natural duality between the periodic cyclic complex $C^\sharp(\mathcal{A})$ and the periodic cyclic cochain complex

$$C_\sharp(\mathcal{A}) = (C^\bullet_\sharp(\mathcal{A}), b + B),$$

where $C^i_\sharp(\mathcal{A}) = \bigoplus_{q\geq 0} C^{2q+i}(\mathcal{A})$, $i = 0, 1$. The cohomology of $C_\sharp(\mathcal{A})$ is the *periodic cyclic cohomology* of $\mathcal{A}$ and is denoted by $\mathrm{HP}^\bullet(\mathcal{A})$.

The cyclic cohomology of $\mathcal{A}$ can be defined in terms of the cochain complex $C_\natural(\mathcal{A})$. For our purpose it is more convenient to use the original definition of Connes [30] in terms of cyclic cochains. A cochain $\varphi \in C^m(\mathcal{A})$ is called *cyclic* when $t\varphi = (-1)^m\varphi$. We denote by $C^m_\lambda(\mathcal{A})$ the space of cyclic m-cochains. As the operator b preserves $C_\lambda(\mathcal{A})$, we obtain a subcomplex $C_\lambda(\mathcal{A}) := (C^\bullet_\lambda(\mathcal{A}), b)$ of the Hochschild cochain complex of $\mathcal{A}$. The *cyclic cohomology* is defined as the cohomology of $C_\lambda(\mathcal{A})$ and is denoted by $H_\lambda(\mathcal{A})$.

As the operator B is annihilated by cyclic cochains, the natural inclusion of $C^\bullet_\lambda(\mathcal{A}) \subset C^\bullet_\natural(\mathcal{A})$ is a cochain map. This is actually a quasi-isomorphism (this is even a chain homotopy equivalence; see [61]). Under this quasi-isomorphism the operator S is chain homotopic to Connes' periodicity operator

$$S_\lambda\colon C^\bullet_\lambda(\mathcal{A}) \to C^{\bullet+2}_\lambda(\mathcal{A}),$$

which is defined as follows. On $C^m(\mathcal{A})$, $m \geq 0$, it is given by

$$S_\lambda = (m+2)^{-1}(m+1)^{-1}\sum_{j=1}^{m+1}(-1)^j S_j,$$

where

$$\begin{aligned} S_j\varphi(a^0,\dots,a^{m+2}) = \sum_{0\leq l\leq j-2}(-1)^l\varphi(a^0,\dots,a^la^{l+1},\dots,a^ja^{j+1},\dots,a^{m+2}) \\ + (-1)^{j+1}\varphi(a^0,\dots,a^{j-1}a^ja^{j+1},\dots,a^{m+2}). \end{aligned} \tag{3.8}$$

If $m = 2q + i$ with $q \geq 0$ and $i \in \{0, 1\}$, then the inclusion of $C^m_\lambda(\mathcal{A})$ into $C^m_\natural(\mathcal{A})$ gives an inclusion into $C^i_\sharp(\mathcal{A})$. This gives a cochain map from $C^\bullet_\lambda(\mathcal{A})$ into $C^\bullet_\sharp(\mathcal{A})$ which is compatible with the periodicity operator S_λ in the sense that, for any cocycle $\varphi \in C^m_\lambda(\mathcal{A})$, the cocycles φ and $S_\lambda\varphi$ define the same class in $\mathrm{HP}^\bullet(\mathcal{A})$.

A cochain $\varphi \in C^m(\mathcal{A})$ is called *normalized* when $\varphi(a^0,\dots,a^m) = 0$ whenever $a^j = 1$ for some $j \geq 1$. We denote by $\bar{C}^m(\mathcal{A})$ the space of normalized m-cochains. The duality between $C_m(\mathcal{A})$ and $C^m(\mathcal{A})$ descends to a duality between normalized chains and normalized cochains. The operators b and B preserve the space of normalized cochains. Therefore, we get normalized cochain complexes

$$\bar{C}_\lambda(\mathcal{A}) := (\bar{C}^\bullet_\lambda(\mathcal{A}), b) \quad \text{and} \quad \bar{C}_\sharp(\mathcal{A}) := (\bar{C}^\bullet_\sharp(\mathcal{A}), b + B),$$

where

$$\bar{C}^m_\lambda(\mathcal{A}) = C^m_\lambda(\mathcal{A}) \cap \bar{C}^m(\mathcal{A}) \quad \text{and} \quad \bar{C}^i_\sharp(\mathcal{A}) = \bigoplus_{q\geq 0}\bar{C}^{2q+i}(\mathcal{A}).$$

Furthermore, the inclusion of $\bar{C}^\bullet(\mathcal{A})$ into $C^\bullet(\mathcal{A})$ gives rise to a quasi-isomorphism of cochain complexes from $\bar{C}^\bullet_\lambda(\mathcal{A})$ (resp., $\bar{C}^\bullet_\sharp(\mathcal{A})$) into $C^\bullet_\lambda(\mathcal{A})$ (resp., $C^\bullet_\sharp(\mathcal{A})$). This enables us to represent classes in $H^\bullet_\lambda(\mathcal{A})$ and $\mathrm{HP}^\bullet(\mathcal{A})$ by normalized cochains.

When $\mathcal{A}$ is a locally convex algebra we denote by $\mathbf{C}^m(\mathcal{A})$ the space of (jointly) continuous m-cochains $\varphi\colon \mathcal{A}^{m+1} \to \mathbb{C}$. The duality between $C_m(\mathcal{A})$ and $C^m(\mathcal{A})$ induces a duality between $\mathbf{C}_m(\mathcal{A})$ and $\mathbf{C}^m(\mathcal{A})$. The structural operators (d, s, t) in (3.5)–(3.7) preserve the space of continuous cochains $\mathbf{C}^\bullet(\mathcal{A})$. Therefore, we obtain subcomplexes

$$\mathbf{C}_\lambda(\mathcal{A}) := (\mathbf{C}^\bullet_\lambda(\mathcal{A}), b) \quad \text{and} \quad \mathbf{C}_\sharp(\mathcal{A}) := (\mathbf{C}^\bullet_\sharp(\mathcal{A}), b + B),$$

where

$$\mathbf{C}^m_\lambda(\mathcal{A}) = C^m_\lambda(\mathcal{A}) \cap \mathbf{C}^m(\mathcal{A}) \quad \text{and} \quad \mathbf{C}^i_\sharp(\mathcal{A}) = \bigoplus_{q\geq 0} \mathbf{C}^{2q+i}(\mathcal{A}).$$

The cohomologies of these cochain complexes are denoted by $\mathbf{H}^\bullet_\lambda(\mathcal{A})$ and $\mathbf{HP}^\bullet(\mathcal{A})$, respectively. We also obtain a cochain complex

$$\mathbf{C}_\natural(\mathcal{A}) := (\mathbf{C}^\bullet_\natural(\mathcal{A}), b + SB),$$

where $\mathbf{C}^m_\natural(\mathcal{A}) = \mathbf{C}^m(\mathcal{A}) \oplus \mathbf{C}^{m-2}(\mathcal{A}) \oplus \cdots$. The inclusion of $\mathbf{C}^\bullet_\lambda(\mathcal{A})$ into $\mathbf{C}^\bullet_\natural(\mathcal{A})$ is a quasi-isomorphism of cochain complexes.

Example 3.2. As in Example 3.1, let $\mathcal{A} = C^\infty(M)$, where M is a closed manifold. In addition, let $(\Omega_\bullet(M), d)$ be the de Rham complex of currents on M, so that $\Omega_m(M)$ is the topological dual of $\Omega^m(M)$. By duality the HKR map (3.4) gives rise to a cochain map $\alpha\colon (\Omega_\bullet, 0) \to (\mathbf{C}^\bullet(\mathcal{A}), b)$ given by

$$\alpha(C)(f^0, \ldots, f^m) = \frac{1}{m!}\left\langle C, f^0 df^1 \wedge \cdots \wedge d^m \right\rangle, \qquad C \in \Omega_m(M),\ f^j \in \mathcal{A}.$$

This maps intertwines the de Rham boundary d and the operator B, and so it gives rise to cochain maps

$$\alpha\colon \Omega^\natural_\bullet(M) \to \mathbf{C}^\bullet_\natural(\mathcal{A}) \quad \text{and} \quad \alpha\colon \Omega^\bullet_{\mathrm{ev/odd}}(M) \to \mathbf{C}^\bullet_\sharp(\mathcal{A}),$$

where

$$\Omega^\natural_m(M) = \Omega_m(M) \oplus \Omega_{m-2}(M) \oplus \cdots \quad \text{and} \quad \Omega^\bullet_{\mathrm{ev/odd}}(M) = \bigoplus_{m \text{ even/odd}} \Omega^m(M).$$

As shown by Connes [30] these cochain maps are quasi-isomorphisms, and so this allows us to express the cyclic and periodic cyclic homologies of $\mathcal{A} = C^\infty(M)$ in terms of the de Rham homology $H_\bullet(M, \mathbb{C})$. More precisely, we obtain isomorphisms,

$$\mathbf{H}^m_\lambda(\mathcal{A}) \simeq H_m(M, \mathbb{C}) \oplus H_{m-2}(M, \mathbb{C}) \oplus \cdots, \qquad m \geq 0,$$
$$\mathbf{HP}^\bullet(\mathcal{A}) \simeq H_{\mathrm{ev/odd}}(M, \mathbb{C}), \quad \text{where } H_{\mathrm{ev/odd}}(M, \mathbb{C}) := \bigoplus_{m \text{ even/odd}} H_m(M, \mathbb{C}).$$

3.4. The Chern character in cyclic homology. Given $N \in \mathbb{N}$, let $M_N(\mathcal{A}) = \mathcal{A} \otimes M_N(\mathbb{C})$ be the algebra of $N \times N$-matrices with coefficients in $\mathcal{A}$. The standard trace tr: $M_N(\mathcal{A}) \to \mathcal{A}$ gives rise to a linear map tr: $C_\bullet(M_N(\mathcal{A})) \to C_\bullet(\mathcal{A})$ defined by

$$\operatorname{tr}\big[(a^0 \otimes \mu^0) \otimes \cdots \otimes (a^m \otimes \mu^m)\big] = \operatorname{tr}\big[\mu^0 \cdots \mu^m\big] a^0 \otimes \cdots \otimes a^m, \qquad a^j \in \mathcal{A},\ \mu^j \in M_N(\mathbb{C}).$$

This map is compatible with the structural operators (d, s, t) in (3.1)–(3.3), and so it gives rise to a map of cyclic spaces tr: $C_\bullet(M_N(\mathcal{A})) \to C_\bullet(\mathcal{A})$.

The Chern character in cyclic homology [30, 48] is defined as follows. Let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$, so that $\mathcal{E} \simeq e\mathcal{A}^N$ for some idempotent $e \in M_N(\mathcal{A})$. The Chern character of e is the (normalized) even chain $\operatorname{Ch}(e) = (\operatorname{Ch}_{2q}(e))_{q \geq 0} \in \bar{C}_0^\sharp(\mathcal{A})$ defined by

$$\operatorname{Ch}_0(e) = \operatorname{tr}[e], \qquad \operatorname{Ch}_{2q}(e) = (-1)^q \frac{(2q)!}{q!} \operatorname{tr}\big[\big(e - \tfrac{1}{2}\big) \otimes \overbrace{e \otimes \cdots \otimes e}^{2q \text{ times}}\big], \quad q \geq 1.$$

This chain is a cycle in the normalized periodic complex $\bar{C}^\sharp(\mathcal{A})$, and so this defines a class in $\operatorname{HP}_0(\mathcal{A})$. This class only depends on the K-theory class of e. Therefore, we get an additive map Ch: $K_0(\mathcal{A}) \to \operatorname{HP}_0(\mathcal{A})$, which is called the *Chern character* in cyclic homology.

Composing the above Chern character with the duality pairing between $\operatorname{HP}_0(\mathcal{A})$ and $\operatorname{HP}^0(\mathcal{A})$ provides us with a bilinear pairing $\langle \cdot, \cdot \rangle : \operatorname{HP}^0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$. Given any cyclic $2q$-cocycle φ and any idempotent $e \in M_N(\mathcal{A})$, it can be shown (see, e.g. [77, Remark 6.4]) that

$$\langle [\varphi], [\operatorname{Ch}(e)] \rangle = (-1)^q \frac{(2q)!}{q!} \big\langle \varphi, \operatorname{tr}\big[e^{\otimes(2q+1)}\big] \big\rangle. \tag{3.9}$$

Therefore, we recover the original pairing of Connes [30, 32].

When $\mathcal{A}$ is a locally convex algebra, by using the inclusion of chain complexes of $C_\bullet^\sharp(\mathcal{A})$ into $\mathbf{C}_\bullet^\sharp(\mathcal{A})$ we obtain a Chern character $\mathbf{Ch}$: $K_0(\mathcal{A}) \to \mathbf{HP}_0(\mathcal{A})$. We then obtain a bilinear pairing $\langle \cdot, \cdot \rangle : \mathbf{HP}^0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$. If we let $\mathcal{A} = C^\infty(M)$, where M is a closed manifold, then, under the CHKR isomorphism $\mathbf{HP}_0(\mathcal{A}) \simeq H^{\mathrm{ev}}(M, \mathbb{C})$ and the Serre–Swan isomorphism $K_0(\mathcal{A}) \simeq K^0(M)$, we recover the usual Chern character Ch: $K^0(M) \to H^{\mathrm{ev}}(M, \mathbb{C})$.

4. The Connes–Chern character of a twisted spectral triple

In this section, we shall recall the construction the Connes–Chern character of a twisted spectral triple and how this enables us to compute the associated index

map [38,77]. This extends to twisted spectral triples the construction of the Connes–Chern character of an ordinary spectral triple by Connes [30]. The exposition follows closely that of [77].

Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a twisted spectral triple. We assume that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is *p-summable* for some $p \geq 1$, that is,

$$\operatorname{Tr} |D|^{-p} < \infty. \tag{4.1}$$

In what follows, letting $\mathcal{L}^1(\mathcal{H})$ be the ideal of trace-class operators on $\mathcal{H}$, we denote by Str its supertrace, i.e. $\operatorname{Str}[T] = \operatorname{Tr}[\gamma T]$, where $\gamma := \mathrm{id}_{\mathcal{H}^+} - \mathrm{id}_{\mathcal{H}^-}$ is the $\mathbb{Z}_2$-grading of $\mathcal{H}$.

Definition 4.1. Assume D is invertible and let q be an integer $\geq \frac{1}{2}(p-1)$. Then $\tau_{2q}^{D,\sigma}$ is the $2q$-cochain on $\mathcal{A}$ defined by

$$\tau_{2q}^{D,\sigma}(a^0, \ldots, a^{2q}) = c_q \operatorname{Str}\left(D^{-1}[D, a^0]_\sigma \cdots D^{-1}[D, a^{2q}]_\sigma\right) \qquad \forall a^j \in \mathcal{A}, \tag{4.2}$$

where we have set $c_q = \frac{1}{2}(-1)^q \frac{q!}{(2q)!}$.

Remark 4.2. The right-hand side of (4.2) is well defined since the p-summability condition (4.1) and the fact that $q \geq \frac{1}{2}(p-1)$ imply that

$$D^{-1}[D, a^0]_\sigma \cdots D^{-1}[D, a^q]_\sigma \in \mathcal{L}^1(\mathcal{H}) \qquad \forall a^j \in \mathcal{A}.$$

Proposition 4.3 ([38,77]). *Assume D is invertible and let q be an integer $\geq \frac{1}{2}(p-1)$. Then*

(1) *The cochain $\tau_{2q}^{D,\sigma}$ is a normalized cyclic cocycle whose class in $\mathrm{HP}^0(\mathcal{A})$ is independent of the value of q.*

(2) *For any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$,*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \left\langle \tau_{2q}^{D,\sigma}, \operatorname{Ch}(\mathcal{E}) \right\rangle, \tag{4.3}$$

where $\operatorname{Ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ in cyclic homology.

When D is not invertible, we can reduce to the invertible case by passing to the unital invertible double of $(\mathcal{A}, \mathcal{H}, D)_\sigma$ as follows.

Consider the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, which we equip with the $\mathbb{Z}_2$-grading given by

$$\widetilde{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix},$$

where γ is the grading operator of $\mathcal{H}$. On $\widetilde{\mathcal{H}}$ consider the selfadjoint operator,

$$\widetilde{D} = \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix}, \qquad \operatorname{dom}(\widetilde{D}) := \operatorname{dom}(D) \oplus \operatorname{dom}(D).$$

Noting that

$$\widetilde{D}^2 = \begin{pmatrix} D^2+1 & 0 \\ 0 & D^2+1 \end{pmatrix},$$

we see that $\widetilde{D}$ is invertible and $|\widetilde{D}|^{-p}$ is a trace-class operator. Let $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be the unitalization of $\mathcal{A}$ whose product and involution are given by

$$(a,\lambda)(b,\mu) = (ab+\lambda b+\mu a, \lambda\mu), \quad (a,\lambda)^* = (a^*, \overline{\lambda}), \qquad a,b \in \mathcal{A},\ \lambda,\mu \in \mathbb{C}.$$

The unit of $\widetilde{\mathcal{A}}$ is $1_{\widetilde{\mathcal{A}}} = (0,1)$. Thus, identifying any element $a \in \mathcal{A}$ with $(a,0)$, any element $\widetilde{a} = (a,\lambda) \in \widetilde{\mathcal{A}}$ can be uniquely written as $(a,\lambda) = a + \lambda 1_{\widetilde{\mathcal{A}}}$. We represent $\widetilde{\mathcal{A}}$ in $\mathcal{H}$ using the representation $\pi\colon \widetilde{\mathcal{A}} \to \mathcal{L}(\mathcal{H})$ given by

$$\pi(1_{\widetilde{\mathcal{A}}}) = 1, \qquad \pi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \forall a \in \mathcal{A}.$$

In addition, we extend the automorphism σ into the automorphism $\widetilde{\sigma}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ given by

$$\widetilde{\sigma}(a + \lambda 1_{\widetilde{\mathcal{A}}}) = \sigma(a) + \lambda 1_{\widetilde{\mathcal{A}}} \qquad \forall (a,\lambda) \in \mathcal{A} \times \mathbb{C}.$$

It can be verified that any twisted commutator $[\widetilde{D}, \pi(\widetilde{a})]_{\widetilde{\sigma}}$, $\widetilde{a} \in \widetilde{\mathcal{A}}$, is bounded. We then deduce that $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_{\widetilde{\sigma}}$ is a p-summable twisted spectral triple. Moreover, as $\widetilde{D}$ is invertible we may define the normalized cyclic cocycles $\tau_{2q}^{\widetilde{D},\widetilde{\sigma}}$, $q \geq \frac{1}{2}(p-1)$.

Definition 4.4. Let $q \geq \frac{1}{2}(p-1)$. Then $\overline{\tau}_{2q}^{D,\sigma}$ is the $2q$-cochain on $\mathcal{A}$ defined by

$$\begin{aligned} \overline{\tau}_{2q}^{\widetilde{D},\sigma}(a^0,\dots,a^{2q}) &= \tau_{2q}^{\widetilde{D},\sigma}(a^0,\dots,a^{2q}) \\ &= c_q \operatorname{Str}\big(\widetilde{D}^{-1}[\widetilde{D},\pi(a^0)]_\sigma \cdots \widetilde{D}^{-1}[\widetilde{D},\pi(a^{2q})]_\sigma\big) \qquad \forall a^j \in \mathcal{A}. \end{aligned}$$

Remark 4.5. The cochain $\overline{\tau}_{2q}^{\widetilde{D}}$ is the restriction to $\mathcal{A}^{2q+1}$ of the cochain $\tau_{2q}^{\widetilde{D},\widetilde{\sigma}}$. Note that, as the restriction to $\mathcal{A}$ of the representation π is not unital, unlike in the invertible case, we don't have a normalized cochain.

Proposition 4.6 ([77]). *Let q be an integer $\geq \frac{1}{2}(p-1)$.*

(1) *The cochain $\overline{\tau}_{2q}^{D,\sigma}$ is a cyclic cocycle whose class in $\mathrm{HP}^0(\mathcal{A})$ is independent of q.*

(2) *If D is invertible, then the cocycles $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$ are cohomologous in $\mathrm{HP}^0(\mathcal{A})$.*

(3) *For any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$,*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \big\langle \tau_{2q}^{D,\sigma}, \operatorname{Ch}(\mathcal{E}) \big\rangle. \tag{4.4}$$

All this leads us to the following definition.

Definition 4.7. The Connes–Chern character of $(\mathcal{A}, \mathcal{H}, D)_\sigma$, denoted by $\mathrm{Ch}(D)_\sigma$, is defined as follows:

- If D is invertible, then $\mathrm{Ch}(D)_\sigma$ is the common class in $\mathrm{HP}^0(\mathcal{A})$ of the cyclic cocycles $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$, with $q \geq \frac{1}{2}(p-1)$.
- If D is not invertible, then $\mathrm{Ch}(D)_\sigma$ is the common class in $\mathrm{HP}^0(\mathcal{A})$ of the cyclic cocycles $\overline{\tau}_{2q}^{D,\sigma}$, $q \geq \frac{1}{2}(p-1)$.

Remark 4.8. When $\sigma = \mathrm{id}$ the Connes–Chern character is simply denoted $\mathrm{Ch}(D)$; this is the usual Connes–Chern Character of an ordinary spectral triple constructed by Connes [30].

With this definition in hand, the index formulas (4.3) and (4.4) can be merged onto the following result.

Proposition 4.9 ([38,77]). *For any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and any σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$,*

$$\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D)_\sigma, \mathrm{Ch}(\mathcal{E}) \rangle\,, \tag{4.5}$$

where $\mathrm{Ch}(\mathcal{E})$ *is the Chern character of* $\mathcal{E}$ *in cyclic homology.*

Example 4.10 ([12,30,73]). Let (M^n, g) be a compact Riemannian manifold. The Connes–Chern character $\mathrm{Ch}(\not{D}_g)$ of the Dirac spectral triple

$$\big(C^\infty(M), L^2(M, \not{S}), \not{D}_g\big)$$

is cohomologous to the even periodic cocycle $\varphi = (\varphi_{2q})_{q\geq 0}$ given by

$$\varphi_{2q}(f^0, \ldots, f^{2q}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2q)!} \int_M \widehat{A}(R^M) \wedge f^0 df^1 \wedge \cdots \wedge df^{2q}, \quad f^j \in C^\infty(M). \tag{4.6}$$

This allows us to recover the index theorem of Atiyah–Singer [4,6] for Dirac operators.

Remark 4.11. The definitions of the cocycles $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$ involve the usual (super)trace on trace-class operators, but this is not a local functional since it does not vanish on finite rank operators. As a result this cocycle is difficult to compute in practice (see, e.g. [12]). Therefore, it stands for reason to seek for a representatative of the Connes–Chern character which is easier to compute. For ordinary spectral triples, and under further assumptions, such a representative is provided by the CM cocycle of Connes–Moscovici [35]. This cocycle is an even periodic cycle whose components are given by formulas which are *local* in the sense of noncommuative geometry. More precisely, they involve a version for spectral triples of the noncommutative residue trace of Guillemin [53] and Wodzicki [84]. This provides us with the local index formula in noncommutative geometry. In Example 4.10 the cocycle $\varphi = (\varphi_{2q})$ given by (4.6) is precisely the CM cocycle of the Dirac spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_g)$ (see [35, Remark II.1] and [73]).

Remark 4.12. In the case of twisted spectral triples, Moscovici [68] attempted to extend the local index formula to the setting of twisted spectral triples. He devised an Ansatz for such a local index formula and verified it in the special case of a ordinary spectral triples twisted by scaling automorphisms (see [68] for the precise definition). Whether Moscovici's Ansatz holds for a larger class of twisted spectral triples still remains an open question to date. For instance, it is not known if Moscovici Ansatz holds for conformal deformations of ordinary spectral triples satisfying the local index formula in noncommutative geometry.

5. Twisted spectral triples over locally convex algebras

In this section, we shall explain how to refine the construction of the Connes–Chern character for twisted spectral triples over locally convex algebras.

In what follows by *locally convex $*$-algebra* we shall mean a $*$-algebra $\mathcal{A}$ equipped with a locally convex space topology with respect to which its product is a jointly continuous bilinear map from $\mathcal{A}^2 \to \mathcal{A}$ and its involution is a continuous anti-linear map.

Definition 5.1. A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ over a locally convex $*$-algebra $\mathcal{A}$ is called smooth when the following conditions hold:

(1) The representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is continuous.

(2) The map $a \to [D, a]_\sigma$ is continuous from $\mathcal{A}$ to $\mathcal{L}(\mathcal{H})$.

(3) The automorphism $\sigma\colon \mathcal{A} \to \mathcal{A}$ is a homeomorphism.

Example 5.2. Let (M^n, g) be a closed spin Riemannian manifold of even dimension. Then the associated Dirac spectral triple $(C^\infty(M), L^2_g(M, \$), \not{D}_g)$ is smooth.

Example 5.3. Any conformal deformation of a smooth ordinary spectral triple yields a smooth twisted spectral triple.

Remark 5.4. As we shall see in Section 7 the conformal Dirac spectral triple of [38] is smooth.

Throughout the rest of this section we let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a *smooth* twisted spectral triple which is p-summable for some $p \geq 1$. We shall now show that the Connes–Chern character of $(\mathcal{A}, \mathcal{H}, D)_\sigma$, which is originally defined as a class in $\mathrm{HP}^0(\mathcal{A})$, actually descends to a class in $\mathbf{HP}^0(\mathcal{A})$.

Lemma 5.5. *Let q be any integer $\geq \frac{1}{2}(p-1)$.*

(1) *If D is invertible, then the cyclic cocycle $\tau_{2q}^{D,\sigma}$ is continuous and its class in $\mathbf{HP}^0(\mathcal{A})$ is independent of q.*

(2) *The cyclic cocycle $\overline{\tau}_{2q}^{D,\sigma}$ is continuous and its class in $\mathbf{HP}^0(\mathcal{A})$ is independent of q.*

Proof. Assume that D is invertible and let q be an integer $\geq \frac{1}{2}(p-1)$. By assumption the map $a \to [D,a]_\sigma$ is continuous from $\mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. Combining this with Hölder's inequality for Schatten ideals we deduce that the map

$$(a^0,\ldots,a^{2q}) \to \gamma D^{-1}[D,a^0]\cdots D^{-1}[D,a^{2q}]$$

is continuous from $\mathcal{A}^{2q+1}$ to $\mathcal{L}^1(\mathcal{H})$. As $\tau_{2q}^{D,\sigma}$ is (up to a multiple constant) the composition of this map with the operator trace, we deduce that $\tau_{2q}^{D,\sigma}$ is a continuous cochain.

Moreover, by Lemma 7.4 and Lemma 7.5 of [77] we have

$$\tau_{2q+2}^{D,\sigma} - \tau_{2q}^{D,\sigma} = (b+B)(\varphi_{2q+1} - \psi_{2q+1}), \tag{5.1}$$

where, up to normalization constants, the cochains φ_{2q+1} and ψ_{2q+1} are given by

$$\begin{aligned}
\varphi_{2q+1}(a^0,\ldots,a^{2q+1}) &= \operatorname{Str}\big(a^0 D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2q+1}]_\sigma\big),\\
\psi_{2q+1}(a^0,\ldots,a^{2q+1}) &= \operatorname{Str}\big(\sigma(a^0)[D,a^1]_\sigma D^{-1}\cdots[D,a^{2q+1}]_\sigma D^{-1}\big), \quad a^j \in \mathcal{A}.
\end{aligned}$$

Note that φ_{2q+1} and ψ_{2q+1} are normalized cochains. Moreover, in the same way as with the cocycle $\tau_{2q}^{D,\sigma}$ we can show that these cochains are continuous. Therefore (5.1) shows that the images in $\mathbf{C}_\natural^{2q+2}(\mathcal{A})$ of $\tau_{2q+2}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ are cohomologous, and, hence, they define the same class in $\mathbf{HP}^0(\mathcal{A})$. It then follows that the class of $\tau_{2q}^{D,\sigma}$ in $\mathbf{HP}^0(\mathcal{A})$ is independent of q. This also implies that

$$S_\lambda \tau_{2q}^{D,\sigma} = \tau_{2q+2}^{D,\sigma} \quad \text{in } \mathbf{H}_\lambda^{2q+2}(\mathcal{A}). \tag{5.2}$$

When D is not invertible, we observe that the unitalization $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is a locally convex $*$-algebra with respect to the direct sum topology. It then can be checked that the invertible double $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_{\widetilde{\sigma}}$ considered in Section 4 is a smooth twisted spectral triple. Therefore, the first part of the proof shows that the cocycle $\tau_{2q}^{\widetilde{D},\sigma}$ is continuous. As the inclusion of $\mathcal{A}$ into $\widetilde{\mathcal{A}}$ is continuous and $\overline{\tau}_{2q}^{D,\sigma}$ is the restriction to $\mathcal{A}$ of $\tau_{2q}^{\widetilde{D},\sigma}$, we then see that $\overline{\tau}_{2q}^{D,\sigma}$ is a continuous cocycle on $\mathcal{A}$. Moreover, using (3.8) it can be checked that $S_\lambda \overline{\tau}_{2q}^{D,\sigma}$ is the restriction to $\mathcal{A}$ of $S_\lambda \tau_{2q}^{\widetilde{D},\sigma}$. Therefore using (5.2) we deduce that the cocycles $S_\lambda \overline{\tau}_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q+2}^{D,\sigma}$ are cohomologous in $\mathbf{H}_\lambda^{2q+2}(\mathcal{A})$. It then follows that $\overline{\tau}_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q+2}^{D,\sigma}$ define the same class in $\mathbf{HP}^0(\mathcal{A})$, and so the class of $\overline{\tau}_{2q}^{D,\sigma}$ in $\mathbf{HP}^0(\mathcal{A})$ is independent of q. The proof is complete. □

In addition, we will also need the following version for smooth twisted spectral triples of [77, Propositon C.1] on the homotopy invariance of the Connes–Chern character of a twisted spectral triple.

Lemma 5.6. *Assume D is invertible and consider an operator homotopy of the form,*

$$D_t = D + V_t, \qquad 0 \leq t \leq 1,$$

where $(V_t)_{0\leq t\leq 1}$ is a C^1-family of selfadjoint operators in $\mathcal{L}(\mathcal{H})$ anticommuting with the grading γ such that D_t is invertible for all $t \in [0,1]$ and $(D_t^{-1})_{0\leq t\leq 1}$ is a bounded family in $\mathcal{L}^p(\mathcal{H})$. Then

(1) *$(\mathcal{A}, \mathcal{H}, D_t)_\sigma$ is a smooth p-summable twisted spectral triple for all $t \in [0,1]$.*

(2) *For any $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2q}^{D_0,\sigma}$ and $\tau_{2q}^{D_1,\sigma}$ are cohomologous in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$.*

Proof. We know from [77, Propositon C.1] that $(\mathcal{A}, \mathcal{H}, D_t)_\sigma$ is a p-summable twisted spectral triple for all $t \in [0,1]$ and, for any $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2q}^{D_0,\sigma}$ and $\tau_{2q}^{D_1,\sigma}$ are cohomologous in $H_\lambda^{2q}(\mathcal{A})$. Therefore, we only need to show that the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D_t)_\sigma$, $t \in [0,1]$, are smooth and the cocycles $\tau_{2q}^{D_0,\sigma}$ and $\tau_{2q}^{D_1,\sigma}$ differ by the coboundary of a *continuous* cyclic cochain.

By assumption $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is a smooth twisted spectral triple. In particular, the representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is continuous and the automorphism σ is a homeomorphism. Moreover, for all $t \in [0,1]$ and $a \in \mathcal{A}$,

$$[D_t, a]_\sigma = [D, a]_\sigma + V_t a - \sigma(a) V_t.$$

We then see that the map $(t,a) \to [D_t, a]_\sigma$ is continuous from $[0,1] \times \mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. It then follows that $(\mathcal{A}, \mathcal{H}, D_t)_\sigma$ is a smooth twisted spectral triple for all $t \in [0,1]$.

Let q be an integer $\geq \frac{1}{2}(p-1)$. In [77] the explicit homotopy between $\tau_{2q}^{D_0,\sigma}$ and $\tau_{2q}^{D_1,\sigma}$ is realized as follows. For $t \in [0,1]$ and $a \in \mathcal{A}$ set

$$\delta_t(a) = D_t^{-1}[\dot{V}_t D_t^{-1}, \sigma(a)] D_t.$$

In addition, for $j = 0, \ldots, 2q+1$ we set

$$\alpha_j^t(a) = a \text{ if } j \text{ is even} \qquad \text{and} \qquad \alpha_j^t(a) = D_t^{-1}\sigma(a) D_t \text{ if } j \text{ is odd.}$$

We note that

$$\delta_t(a) = D_t^{-1}(\dot{V}_t \alpha_1^t(a) - \sigma(a)\dot{V}_t) \quad \text{and} \quad \alpha_1^t(a) = a - D_t^{-1}[D_t, a]_\sigma.$$

In particular we see that the maps $(t,a) \to \alpha_j^t(a)$ are continuous from $[0,1] \times \mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. Moreover, it is shown in [77, Appendix C] that $(D_t^{-1})_{0\leq t\leq t}$ is a C^1-family in $\mathcal{L}^p(\mathcal{H})$. Therefore, we also see that the map $(t,a) \to \delta_t(a)$ is continuous from $[0,1] \times \mathcal{A}$ to $\mathcal{L}^p(\mathcal{H})$.

Bearing this mind it is shown in [77] that

$$\tau_{2q}^{D_1,\sigma} - \tau_{2q}^{D_0,\sigma} = B\eta, \tag{5.3}$$

where η is the Hochschild $(2q+1)$-cocycle given by

$$\eta(a^0,\dots,a^{2q+1}) = c_q \sum_{j=0}^{2q+1} \int_0^1 \operatorname{Str}\left(\alpha_j(a^0) D_t^{-1}[D_t,a^1]_\sigma \cdots \delta_t(a^j) \cdots D_t^{-1}[D_t,a^{2q+1}]_\sigma\right) dt,$$

where c_q is some normalization constant. It follows from all the previous observations and the fact that $q \geq \frac{1}{2}(p-1)$ that all the maps

$$(t,a^0,\dots,a^{2q+1}) \to \alpha_j(a^0) D_t^{-1}[D_t,a^1]_\sigma \cdots \delta_t(a^j) \cdots D_t^{-1}[D_t,a^{2q+1}]_\sigma$$

are continuous from $[0,1]\times\mathcal{A}^{2q+2}$ to $\mathcal{L}^1(\mathcal{H})$. This ensures us that η is a continuous cochain. Therefore, using (5.2) and the fact that η is a Hochschild cocycle we see that

$$\tau_{2q}^{D_1,\sigma} - \tau_{2q}^{D_0,\sigma} = (b+B)\eta \qquad \text{in } \mathbf{C}_\natural^{2q+2}(\mathcal{A}).$$

This implies that $S(\tau_{2q}^{D_1,\sigma} - \tau_{2q}^{D_0,\sigma})$ is a coboundary in $\mathbf{H}_\lambda^{2q+2}(\mathcal{A})$. Combining this with (5.2) we then deduce that $\tau_{2q+2}^{D_0,\sigma}$ and $\tau_{2q+2}^{D_0,\sigma}$ are cohomologous in $\mathbf{H}_\lambda^{2q+2}(\mathcal{A})$. This proves the 2nd part of the lemma and completes the proof. □

The homotopy lemma is the main ingredient in the proof of the following result.

Lemma 5.7. *Assume D is invertible and let q be an integer $\geq \frac{1}{2}(p+1)$. Then the cyclic cocycles $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$ are cohomologous in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$, and hence define the the same class in $\mathbf{HP}^0(\mathcal{A})$.*

Proof. We know from [77, §7] that, for $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$ define the same class in $H_\lambda^{2q}(\mathcal{A})$. The argument relies on the homotopy of unbounded operators on $\widetilde{\mathcal{H}}$ given by

$$\widetilde{D}_t = \widetilde{D}_0 + tJ, \qquad 0 \leq t \leq 1,$$

where we have set

$$\widetilde{D}_0 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\widetilde{D}_1 = \widetilde{D}$. It is shown in [77, §7] that this homotopy satisfies the assumptions of Lemma 5.6. Therefore, $(\widetilde{\mathcal{A}}, \widetilde{H}, \widetilde{D}_t)_{\widetilde{\sigma}}$ is a smooth spectral triple for all $t \in [0,1]$ and, for $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2q}^{\widetilde{D}_0,\widetilde{\sigma}}$ and $\tau_{2q}^{\widetilde{D}_1,\widetilde{\sigma}}$ are cohomologous in $\mathbf{H}_\lambda^{2q}(\widetilde{\mathcal{A}})$. Incidentally, the restrictions to $\mathcal{A}$ of $\tau_{2q}^{\widetilde{D}_0,\widetilde{\sigma}}$ and $\tau_{2q}^{\widetilde{D}_1,\widetilde{\sigma}}$ are cohomologous cocycles in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$. As $\widetilde{D}_1 = \widetilde{D}$ the restriction of $\tau_{2q}^{\widetilde{D}_1,\widetilde{\sigma}}$ is $\overline{\tau}_{2q}^{D,\sigma}$. Moreover, it is shown

in [77, §7] that $\tau_{2q}^{D,\sigma}$ is the restriction of $\tau_{2q}^{\widetilde{D}_0,\widetilde{\sigma}}$. Therefore, we see that $\tau_{2q}^{D,\sigma}$ and $\overline{\tau}_{2q}^{D,\sigma}$ are cohomologous in $\mathbf{H}_{\lambda}^{2q}(\mathcal{A})$, and hence define the the same class in $\mathbf{HP}^0(\mathcal{A})$. The proof is complete. □

Granted Lemma 5.5 and Lemma 5.7 the Connes–Chern character descends to a class in $\mathbf{HP}^0(\mathcal{A})$ as follows.

Definition 5.8. The Connes–Chern character $\mathbf{Ch}(D)_\sigma \in \mathbf{HP}^0(\mathcal{A})$ defined as follows:

- When D is invertible this the common class in $\mathbf{HP}^0(\mathcal{A})$ of any of the cyclic cocycles $\tau_{2q}^{D,\sigma}$ or $\overline{\tau}_{2q}^{D,\sigma}$, $q \geq \frac{1}{2}(p-1)$.
- When D is not invertible this the common class in $\mathbf{HP}^0(\mathcal{A})$ of the cyclic cocycles $\overline{\tau}_{2q}^{D,\sigma}$, $q \geq \frac{1}{2}(p-1)$.

Proposition 5.9. *Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a p-summable smooth twisted spectral triple. Then*

(1) *The class $\mathbf{Ch}(D)_\sigma$ agrees with the Connes–Chern character $\mathrm{Ch}(D)_\sigma$ under the morphism $\mathbf{HP}^0(\mathcal{A}) \to \mathrm{HP}^0(\mathcal{A})$ induced by the inclusion of $\mathbf{C}_\sharp^\bullet(\mathcal{A})$ into $C_\sharp^\bullet(\mathcal{A})$.*

(2) *For any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we have*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \mathbf{Ch}(D)_\sigma, \mathbf{Ch}(\mathcal{E}) \rangle ,$$

where $\mathbf{Ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ seen as a class in $\mathbf{HP}_0(\mathcal{A})$.

Proof. The first part is immediate since by their very definitions, $\mathbf{Ch}(D)_\sigma$ and $\mathrm{Ch}(D)_\sigma$ are represented by the same cyclic cocycles. As for the 2nd part, consider an idempotent e in some $M_N(\mathcal{A})$, $N \geq 1$ such that $\mathcal{E} \simeq e\mathcal{A}^N$. Then, for any $q \geq \frac{1}{2}(p-1)$,

$$\langle \mathbf{Ch}(D)_\sigma, \mathbf{Ch}(\mathcal{E}) \rangle = \left\langle \overline{\tau}_{2q}^{D,\sigma}, \mathrm{Ch}(e) \right\rangle = \langle \mathrm{Ch}(D)_\sigma, \mathrm{Ch}(\mathcal{E}) \rangle .$$

Combining this with Proposition 4.9 gives the result. □

Remark 5.10. In what follows by the Connes–Chern character of a smooth (p-summable) twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ we shall mean the cohomology class $\mathbf{Ch}(D)_\sigma \in \mathbf{HP}^0(\mathcal{A})$.

6. Invariance of the Connes–Chern character

In preparation for the next section, we prove in this section the invariance of the Connes–Chern character under equivalences and conformal deformations of twisted spectral triples.

6.1. Equivalence of twisted spectral triples. The equivalence of two twisted spectral triples over the same algebra is defined as follows.

Definition 6.1. Let $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ be twisted spectral triples over the same algebra. For $i = 1, 2$ let us denote by π_i the representation of $\mathcal{A}$ into $\mathcal{H}_i$. Then we say that $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ are equivalent when there is a unitary operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$D_1 = U^* D_2 U, \tag{6.1}$$

$$\pi_1(a) = U^*\pi_2(a)U \quad \text{and} \quad \pi_1(\sigma_1(a)) = U^*\pi_2(\sigma_2(a))U \quad \text{for all } a \in \mathcal{A}. \tag{6.2}$$

Remark 6.2. We may also define the equivalence of a pair of spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)_{\sigma_2}$ with $\mathcal{A}_1 \neq \mathcal{A}_2$ by requiring the existence of a $*$-algebra isomorphism $\psi\colon \mathcal{A}_1 \to \mathcal{A}_2$ and replacing the condition (6.2) by

$$\pi_1(a) = U^*\pi_2\left(\psi(a)\right)U \quad \text{and} \quad \pi_1(\sigma_1(a)) = U^*\pi_2\left(\sigma_2\left(\psi(a)\right)\right)U \quad \text{for all } a \in \mathcal{A}_1.$$

Proposition 6.3. *Let $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ be equivalent twisted spectral triples that are p-summable for some $p \geq 1$. In addition, let q be an integer $\geq \frac{1}{2}(p-1)$. Then*

(1) *The cyclic cocycles $\overline{\tau}_{2q}^{D_1,\sigma_1}$ and $\overline{\tau}_{2q}^{D_2,\sigma_2}$ agree.*

(2) *The same result holds for the cocycles $\tau_{2q}^{D_1,\sigma_1}$ and $\tau_{2q}^{D_2,\sigma_2}$ when D_1 and D_2 are invertible.*

(3) *The twisted spectral triples $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ have the same Connes–Chern character.*

Proof. The last part is an immediate consequence of the first two parts, so we only need to prove these two parts. In addition, we note that the invertible doubles of $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ are equivalent, where the equivalence is implemented by the unitary operator $U \oplus U$ acting on $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$. Therefore, it is enough to assume that D_1 and D_2 are invertible and prove the 2nd part.

Under the aforementioned assumption and using (6.1)–(6.2) we see that, for all $a \in \mathcal{A}$,

$$D_1^{-1}[D_1, a]_{\sigma_1} = U^* D_2^{-1}[D_2, a]_{\sigma_2} U.$$

Therefore, for all $a^j \in \mathcal{A}$, we have

$$\begin{aligned}
&\tau_{2q}^{D_1,\sigma_1}(a^0, \dots, a^{2q}) \\
&\qquad = \frac{1}{2}(-1)^q \frac{q!}{(2q)!} \operatorname{Str}\left\{U^* D_2^{-1}[D_2, a^0]_{\sigma_2} U \cdots U^* D_2^{-1}[D_2, a^{2q}]_{\sigma_2} U\right\} \\
&\qquad = \tau_{2q}^{D_2,\sigma_2}(a^0, \dots, a^{2q}).
\end{aligned}$$

This proves the result. □

6.2. Conformal deformations of twisted spectral triples. For future purpose it will be useful to extend to the setting of twisted spectral triples the conformal deformations of ordinary spectral triples. Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a twisted spectral triple and k a positive element of $\mathcal{A}$. We let $\widehat{\sigma}\colon \mathcal{A} \to \mathcal{A}$ be the automorphism of $\mathcal{A}$ given by

$$\widehat{\sigma}(a) = k\sigma(kak^{-1})k^{-1} \qquad \forall a \in \mathcal{A}.$$

Proposition 6.4. *$(\mathcal{A}, \mathcal{H}, kDk)_{\widehat{\sigma}}$ is a twisted spectral triple.*

Proof. We only need to check the boundedness of the twisted commutators

$$[kDk, a]_{\widehat{\sigma}}, \qquad a \in \mathcal{A}.$$

To see this we note that

$$\begin{aligned} [kDk, a]_{\widehat{\sigma}} &= kDka - \widehat{\sigma}(a)kDk \\ &= k\big(D(kak^{-1}) - (k^{-1}\widehat{\sigma}(a)k)D\big)k = k[D, a]_\sigma k. \end{aligned} \tag{6.3}$$

As the twisted commutator $[D, a]_\sigma$ is bounded, it follows that $[kDk, a]_{\widehat{\sigma}}$ is bounded as well. The proof is complete. □

The following shows that the Connes–Chern character is invariant under conformal deformations.

Proposition 6.5. *Assume that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is p-summable for some $p \geq 1$. Then, for any positive element $k \in \mathcal{A}$, we have*

$$\mathrm{Ch}(kDk)_{\widehat{\sigma}} = \mathrm{Ch}(D)_\sigma \in \mathrm{HP}^0(\mathcal{A}).$$

Remark 6.6. The above result is proved in [38] in the special case $\sigma = \mathrm{id}$ and D is invertible.

Proof of Proposition 6.5. Set $D_k = kDk$. We shall first prove the result when D is invertible, as the proof is simpler in that case. Given an integer $q > \frac{1}{2}(p-1)$ let $a^j \in \mathcal{A}$, $j = 0, \ldots, 2q$. Using (4.2) and (6.3) we get

$$\begin{aligned} &\tau_{2q}^{D_k,\widehat{\sigma}}(a^0, \ldots, a^{2q}) \\ &\quad = c_q \operatorname{Str}\big\{D_k^{-1}[D_k, a^0]_{\widehat{\sigma}} \cdots D_k^{-1}[D_k, a^{2q}]_{\widehat{\sigma}}\big\} && (6.4)\\ &\quad = c_q \operatorname{Str}\big\{(k^{-1}D^{-1}[D, ka^0k^{-1}]_\sigma k) \cdots (k^{-1}D^{-1}[D, ka^{2q}k^{-1}]_\sigma k)\big\} && (6.5)\\ &\quad = c_q \operatorname{Str}\big\{D^{-1}[D, ka^0k^{-1}]_\sigma \cdots D^{-1}[D, ka^{2q}k^{-1}]_\sigma\big\} && (6.6)\\ &\quad = \tau_{2q}^{D,\sigma}\big(ka^0k^{-1}, \ldots, ka^{2q}k^{-1}\big). && (6.7) \end{aligned}$$

As cyclic cohomology is invariant under the action of inner automorphisms (see [32, Prop. III.1.8] and [64, Prop. 4.1.3]), we deduce that the cyclic cocycles $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D_k,\widehat{\sigma}}$

are cohomologous in $H_\lambda^{2q}(\mathcal{A})$. Therefore, they define the same class in $\mathrm{HP}^0(\mathcal{A})$, and so the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D)_\sigma$ and $(\mathcal{A}, \mathcal{H}, D_k)_{\widehat{\sigma}}$ have the same Connes–Chern character.

Let us now prove the result when D is not invertible. To this end consider the respective unital invertible doubles $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_\sigma$ and $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_k)_{\widehat{\sigma}}$ of $(\mathcal{A}, \mathcal{H}, D)_\sigma$ and its conformal deformation $(\mathcal{A}, \mathcal{H}, D_k)_{\widehat{\sigma}}$, where by a slight abuse of notation we have denoted by σ and $\widehat{\sigma}$ the extensions to $\widetilde{\mathcal{A}}$ of the automorphisms σ and $\widehat{\sigma}$. As it turns out, $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_k)_{\widehat{\sigma}}$ is not a conformal deformation of $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_\sigma$ so that the proof in the invertible case does not extend to the invertible doubles. Nevertheless, as we shall see, $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_k)_{\widehat{\sigma}}$ is a pseudo-inner twisting in the sense of [79] of a twisted spectral triple which is homotopy equivalent to $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_\sigma$.

To wit consider the selfadjoint unbounded operator on $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ given by

$$\widetilde{D}_1 = \begin{pmatrix} D & k^{-2} \\ k^{-2} & -D \end{pmatrix}, \qquad \mathrm{dom}(\widetilde{D}_1) = \mathrm{dom}(D) \oplus \mathrm{dom}(D).$$

As $\widetilde{D}_1$ agrees with $\widetilde{D}$ up to a bounded operator, we see that $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_1)_\sigma$ is a p-summable twisted spectral triple. Note also that

$$\widetilde{D}_k = \begin{pmatrix} kDk & 1 \\ 1 & -kDk \end{pmatrix} = \omega \widetilde{D}_1 \omega, \qquad \text{where } \omega := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}. \tag{6.8}$$

In particular, we see that $\widetilde{D}_1$ is invertible. An explicit homotopy between $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})_\sigma$ and $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_1)_\sigma$ is given by the family of twisted spectral triples $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}_t)_\sigma$, $t \in [0, 1]$, with

$$\widetilde{D}_t = \begin{pmatrix} D & k^{-2t} \\ k^{-2t} & -D \end{pmatrix} = \widetilde{D} + V_t, \qquad \text{where } V_t = \begin{pmatrix} 0 & k^{-2t} - 1 \\ k^{-2t} - 1 & 0 \end{pmatrix}. \tag{6.9}$$

We observe that $(V_t)_{t\in[0,1]}$ is a C^1-family in $\mathcal{L}(\widetilde{\mathcal{H}})$ and, in the same way as for $\widetilde{D}_1$, it can be shown that the operator $\widetilde{D}_t$ is invertible for all $t \in [0, 1]$. Therefore, we may use the homotopy invariance of the Connes–Chern character in the form of [77, Appendix C] to deduce that, for all $q \geq \frac{1}{2}(p + 1)$, the cyclic cocycles $\tau_{2q}^{\widetilde{D},\sigma}$ and $\tau_{2q}^{\widetilde{D}_1,\sigma}$ are cohomologous in $H_\lambda^{2q}(\widetilde{\mathcal{A}})$. Incidentally, if denote by $\overline{\tau}_{2q}^{\widetilde{D},\sigma}$ and $\overline{\tau}_{2q}^{\widetilde{D}_1,\sigma}$ their respective restrictions to $\mathcal{A}$, then we see that $\overline{\tau}_{2q}^{\widetilde{D},\sigma}$ and $\overline{\tau}_{2q}^{\widetilde{D}_1,\sigma}$ are cohomologous cyclic cocycles in $H_\lambda^{2q}(\mathcal{A})$.

Bearing this in mind, let $a \in \mathcal{A}$. Using (6.8) and noting that $\omega\widetilde{\pi}(a)\omega^{-1} = \widetilde{\pi}(kak^{-1})$, we see that

$$\widetilde{D}_k \widetilde{\pi}(a) = \omega \widetilde{D}_1 (\omega \widetilde{\pi}(a) \omega^{-1}) \omega = \omega \widetilde{D}_1 \widetilde{\pi}(kak^{-1}) \omega.$$

Likewise,

$$\begin{aligned} \widetilde{\pi}(\widehat{\sigma}(a)) \widetilde{D}_k &= \omega(\omega^{-1} \widetilde{\pi}(\widehat{\sigma}(a)) \omega) \widetilde{D}_1 \omega \\ &= \omega \widetilde{\pi}(k^{-1} \widehat{\sigma}(a) k) \widetilde{D}_1 \omega = \omega \widetilde{\pi}(\sigma(kak^{-1})) \widetilde{D}_1 \omega. \end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{D}_k^{-1}[\widetilde{D}_k, \widetilde{\pi}(a)]_{\widehat{\sigma}} &= \big(\omega^{-1}\widetilde{D}_1^{-1}\omega^{-1}\big)\big(\omega[\widetilde{D}_1, \widetilde{\pi}(kak^{-1})]_\sigma\omega\big)\\ &= \omega^{-1}\widetilde{D}_1^{-1}[\widetilde{D}_1, \widetilde{\pi}(kak^{-1})]_\sigma\omega.\end{aligned} \tag{6.10}$$

Given an integer $q > \frac{1}{2}(p-1)$, let $a^j \in \mathcal{A}$, $j = 0, \dots, 2q$. Using (4.2) and (6.10) and arguing as in (6.7) we obtain

$$\begin{aligned}&\overline{\tau}_{2q}^{\widetilde{D}_k,\widehat{\sigma}}(a^0, \dots, a^{2q})\\ &\quad= c_q \operatorname{Str}\big\{\widetilde{D}_k[\widetilde{D}_k, \widetilde{\pi}(a^0)]_{\widehat{\sigma}} \cdots \widetilde{D}_k[\widetilde{D}_k, \widetilde{\pi}(a^{2q})]_{\widehat{\sigma}}\big\} &(6.11)\\ &\quad= c_q \operatorname{Str}\big\{(\omega^{-1}\widetilde{D}_1[\widetilde{D}_1, \widetilde{\pi}(ka^0k^{-1})]_\sigma\omega) \cdots (\omega^{-1}\widetilde{D}_1[\widetilde{D}_1, \widetilde{\pi}(ka^{2q}k^{-1})]_\sigma\omega)\big\} &(6.12)\\ &\quad= c_q \operatorname{Str}\big\{\widetilde{D}_1[\widetilde{D}_1, \widetilde{\pi}(ka^0k^{-1})]_\sigma \cdots \widetilde{D}_1[\widetilde{D}_1, \widetilde{\pi}(ka^{2q}k^{-1})]_\sigma\big\} &(6.13)\\ &\quad= \overline{\tau}_{2q}^{\widetilde{D}_1,\sigma}(ka^0k^{-1}, \dots, ka^{2q}k^{-1}). &(6.14)\end{aligned}$$

Therefore, in the same way as in the invertible case, we deduce that the cocycles $\overline{\tau}_{2q}^{\widetilde{D}_k,\widehat{\sigma}}$ and $\overline{\tau}_{2q}^{\widetilde{D}_1,\sigma}$ are cohomologous in $H_\lambda^{2q}(\mathcal{A})$. It then follows that, for $q \geq \frac{1}{2}(p+1)$, the cocycles $\overline{\tau}_{2q}^{\widetilde{D},\sigma}$ and $\overline{\tau}_{2q}^{\widetilde{D}_k,\widehat{\sigma}}$ are cohomologous in $H_\lambda^{2q}(\mathcal{A})$, and hence define the same class in $\mathrm{HP}^0(\mathcal{A})$. This shows that the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D)_\sigma$ and $(\mathcal{A}, \mathcal{H}, D_k)_{\widehat{\sigma}}$ have same Connes–Chern character. The proof is complete. □

6.3. Smooth twisted spectral triples. We shall now explain how to extend to smooth twisted spectral triples the previous results of this section. First, we have the following result.

Proposition 6.7. *Let $(\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}$ be equivalent smooth twisted spectral triples that are p-summable for some $p \geq 1$. Then $\mathbf{Ch}(D_1)_{\sigma_1} = \mathbf{Ch}(D_2)_{\sigma_2}$ in $\mathbf{HP}^0(\mathcal{A})$.*

Proof. This is an immediate consequence of the first two parts of Proposition 6.3, since they imply that $\mathbf{Ch}(D_1)_{\sigma_1}$ and $\mathbf{Ch}(D_2)_{\sigma_2}$ are represented by the same cocycles. □

Proposition 6.8. *Assume that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is p-summable for some $p \geq 1$. Then, for any positive element $k \in \mathcal{A}$, we have*

$$\mathbf{Ch}(kDk)_{\widehat{\sigma}} = \mathbf{Ch}(D)_\sigma \in \mathbf{HP}^0(\mathcal{A}).$$

Proof. We shall continue using the notation of the proof of Proposition 6.5. This proof shows that, for any $q \geq \frac{1}{2}(p+1)$, the cocycles $\overline{\tau}^{D,\sigma}$ and $\overline{\tau}^{D_k,\widehat{\sigma}}$ define the same class in $H_\lambda^{2q}(\mathcal{A})$ by establishing that they are both cohomologous to the

cocycle $\overline{\tau}^{\widetilde{D}_1,\sigma}$. In order to prove the result we only need to show that $\overline{\tau}^{D,\sigma}$ and $\overline{\tau}^{D_k,\widehat{\sigma}}$ are both cohomologous to $\overline{\tau}^{\widetilde{D}_1,\sigma}$ in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$.

The equality between the classes of $\overline{\tau}^{D,\sigma}$ and $\overline{\tau}^{\widetilde{D}_1,\sigma}$ in $H_\lambda^{2q}(\mathcal{A})$ is a consequence of the homotopy invariance of the Connes–Chern character and the fact that the operators $\widetilde{D}$ and $\widetilde{D}_1$ can be connected by a operator homotopy of the form (6.9). Using Lemma 5.6 we then see that the cocycles $\tau^{\widetilde{D},\widetilde{\sigma}}$ and $\tau^{\widetilde{D}_1,\sigma}$ are cohomologous in $\mathbf{H}_\lambda^{2q}(\widetilde{\mathcal{A}})$. Therefore, their restrictions to $\mathcal{A}$, i.e. the cocycles $\overline{\tau}^{D,\sigma}$ and $\overline{\tau}^{\widetilde{D}_1,\sigma}$, define the same class in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$.

In addition, Eq. (6.14) shows that the cocycle $\overline{\tau}^{D_k,\widehat{\sigma}}$ is obtained from $\overline{\tau}^{\widetilde{D}_1,\sigma}$ by composing with the inner automorphism defined by k. The proof of the invariance of cyclic cohomology by the action of inner automorphisms in [64] holds *verbatim* for the cyclic cohomology of continuous cochains. It then follows that the cocycles $\overline{\tau}^{D_k,\widehat{\sigma}}$ and $\overline{\tau}^{\widetilde{D}_1,\sigma}$ define the same class in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$, and so $\overline{\tau}^{D,\sigma}$ and $\overline{\tau}^{D_k,\widehat{\sigma}}$ are cohomologous in $\mathbf{H}_\lambda^{2q}(\mathcal{A})$. The proof is complete. □

7. The conformal Connes–Chern character

In this section, after recalling the construction of the conformal Dirac spectral triple of [38] associated with any given conformal structure, we show that its Connes–Chern character (defined in continuous cyclic cohomology) is actually a conformal invariant.

7.1. The conformal Dirac spectral triple. Throughout this section and the rest of the paper we let M be a compact (closed) spin oriented manifold of even dimension n. We also let $\mathcal{C}$ be a conformal structure on M, i.e. a conformal class of Riemannian metrics on M. We then denote by G the identity component of the group of (smooth) orientation-preserving diffeomorphisms of M preserving the conformal structure $\mathcal{C}$ and the spin structure of M. Let g be a representative metric in the conformal class $\mathcal{C}$, and consider the associated Dirac operator $\not{D}_g : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ on the sections of the spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$. In addition, we denote by $L_g^2(M, \not{S})$ the corresponding Hilbert space of L^2-spinors.

In the setup of noncommutative geometry, the role of the quotient space M/G is played by the (discrete) crossed-product algebra $C^\infty(M) \rtimes G$. The underlying vector space of $C^\infty(M) \rtimes G$ is $\mathbb{C}G \otimes C^\infty(M)$, where $\mathbb{C}G$ is the group algebra of G over $\mathbb{C}$. Given $\phi \in G$ and $f \in C^\infty(M)$, it is convenient to denote the tensor products $1 \otimes \phi$ and $f \otimes 1$ by u_ϕ and f, respectively. Using this notation, any element of $C^\infty(M) \rtimes G$ has a unique representation as a finite sum $\sum f_\phi u_\phi$, $f_\phi \in C^\infty(M)$,

so that we have the direct-sum decomposition,

$$C^\infty(M) \rtimes G = \bigoplus_{\phi \in G} C^\infty(M) u_\phi. \tag{7.1}$$

The action of G on M gives to an action on $C^\infty(M)$ given by

$$\phi \cdot f = f \circ \phi^{-1}, \qquad \phi \in G,\ f \in C^\infty(M). \tag{7.2}$$

The product and involution of $C^\infty(M) \rtimes G$ are given by

$$(f^0 u_{\phi_0})(f^1 u_{\phi_1}) = f^0(\phi_0 \cdot f^1) u_{\phi_0\phi_1}, \qquad \phi_j \in G,\ f^j \in C^\infty(M), \tag{7.3}$$

$$\big(f u_\phi\big)^* = u_\phi^{-1} \bar{f} = (\phi^{-1} \cdot \bar{f}) u_{\phi^{-1}}, \qquad \phi \in G,\ f \in C^\infty(M). \tag{7.4}$$

In particular, we have the relations,

$$u_\phi f = (\phi \cdot f) u_\phi \qquad \text{and} \qquad u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}.$$

Let $\phi \in G$. As ϕ is a diffeomorphism preserving the conformal class $\mathcal{C}$, there is a unique function $k_\phi \in C^\infty(M)$, $k_\phi > 0$, such that

$$\phi_* g = k_\phi^2 g. \tag{7.5}$$

Moreover, ϕ uniquely lifts to a unitary vector bundle isomorphism $\phi^{\$}: \$ \to \phi_* \$$, i.e. a unitary section of $\mathrm{Hom}(\$, \phi_* \$)$ (see [14]). We then let $V_\phi: L^2_g(M, \$) \to L^2_g(M, \$)$ be the bounded operator given by

$$V_\phi u(x) = \phi^{\$}\big(u \circ \phi^{-1}(x)\big), \qquad \forall u \in L^2_g(M, \$),\ \forall x \in M. \tag{7.6}$$

The map $\phi \to V_\phi$ is a representation of G in $L^2_g(M, \$)$, but this is not a unitary representation. In order to get a unitary representation we need to take into account the Jacobian $|\phi'(x)| = k_\phi(x)^n$ of $\phi \in G$. This is achieved by using the unitary operator $U_\phi: L^2_g(M, \$) \to L^2_g(M, \$)$ given by

$$U_\phi = k_\phi^{\frac{n}{2}} V_\phi, \qquad \phi \in G. \tag{7.7}$$

Then $\phi \to U_\phi$ is a unitary representation of G in $L^2_g(M, \$)$. Combining this with the action of $C^\infty(M)$ by multiplication operators provides with a $*$-representation of the crossed-product algebra $C^\infty(M) \rtimes G$. In addition, we let σ_g be the automorphism of $C^\infty(M) \rtimes G$ given by

$$\sigma_g(f u_\phi) := k_\phi f u_\phi, \qquad \forall f \in C^\infty(M),\ \forall \phi \in G. \tag{7.8}$$

Proposition 7.1 ([38]). *The triple* $(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g)_{\sigma_g}$ *is a twisted spectral triple.*

Remark 7.2. The bulk of the proof is to show the boundedness of the twisted commutators $[\not{D}_g, U_\phi]_{\sigma_g}$, $\phi \in G$. We remark that

$$U_\phi \not{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} (V_\phi \not{D}_g V_\phi^{-1}) k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \not{D}_{\phi_* g} k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \not{D}_{k_\phi^2 g} k_\phi^{-\frac{n}{2}}.$$

Combining this with the conformal invariance of the Dirac operator (see, e.g. [55]) we obtain

$$U_\phi \not{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} \Big(k_\phi^{-\left(\frac{n+1}{2}\right)} \not{D}_g k_\phi^{\frac{n-1}{2}} \Big) k_\phi^{-\frac{n}{2}} = k_\phi^{-\frac{1}{2}} \not{D}_g k_\phi^{-\frac{1}{2}}.$$

Using this we see that the twisted commutator $[\not{D}_g, U_\phi]_{\sigma_g} = \not{D}_g U_\phi - k_\phi U_\phi \not{D}_g$ is equal to

$$\begin{aligned} \Big(\not{D}_g k_\phi^{\frac{1}{2}} - k_\phi (U_\phi \not{D}_g U_\phi^*) k_\phi^{\frac{1}{2}} \Big) k_\phi^{-\frac{1}{2}} U_\phi &= \Big(\not{D}_g k_\phi^{\frac{1}{2}} - k_\phi^{\frac{1}{2}} \not{D}_g \Big) k_\phi^{-\frac{1}{2}} U_\phi \\ &= [\not{D}_g, k_\phi^{\frac{1}{2}}] k_\phi^{-\frac{1}{2}} U_\phi. \end{aligned}$$

This shows that $[\not{D}_g, U_\phi]_{\sigma_g}$ is bounded.

Remark 7.3. We shall refer to $(C^\infty(M) \rtimes G, L_g^2(M, \$), \not{D}_g)_{\sigma_g}$ as the *conformal Dirac spectral triple* associated with the representative metric g.

Remark 7.4. The automorphism σ_g is ribbon in the sense mentioned in Remark 2.22. This is seen by using the automorphism τ_g of $C^\infty(M) \rtimes G$ defined by

$$\tau_g(f u_\phi) = \sqrt{k_\phi} f u_\phi, \qquad \forall f \in C^\infty(M),\ \forall \phi \in G,$$

where k_ϕ is the conformal factor in the sense (7.5).

The crossed-product $C^\infty(M) \rtimes G$ is endowed with a locally convex space topology as follows. The direct-sum decomposition (7.1) means that, as a vector space, $C^\infty(M) \rtimes G$ is the direct union of the subspaces,

$$C^\infty(M) \rtimes F = \bigoplus_{\phi \in F} C^\infty(M) u_\phi, \qquad F \subset G \text{ finite}.$$

Given a finite subset $F \subset G$, we have a canonical vector space identification $C^\infty(M) \rtimes F \simeq C^\infty(M)^F$. Note that $C^\infty(M)^F$ is a Fréchet space with respect to the product topology. Pulling back this topology to $C^\infty(M) \rtimes F$ we obtain a canonical Fréchet space topology on this space. We then endow $C^\infty(M) \rtimes G$ with the coarsest locally convex topology that makes continuous the inclusions of the subspaces $C^\infty(M) \rtimes F$ into $C^\infty(M) \rtimes G$ as F ranges over finite subsets of G. Thus a basis of neighborhoods of the origin in $C^\infty(M) \rtimes G$ consists of convex balanced subsets $\mathcal{U}$ such that, for all every finite subset $F \subset G$, the intersection $\mathcal{U} \cap (C^\infty(M) \rtimes F)$ is a neighborhood of 0 in $C^\infty(M) \rtimes F$. In particular, given a

topological vector space X, a linear map $T: C^\infty(M) \rtimes G \to X$ is continuous if and only if, for all $\phi \in G$, the linear map $f \to T(fu_\phi)$ is continuous from $C^\infty(M)$ to X.

We also observe that, given ϕ_0 and ϕ_1 in G, the multiplication (7.3) induces a jointly continuous bilinear map

$$\big(C^\infty(M)u_{\phi_0}\big) \times \big(C^\infty(M)u_{\phi_1}\big) \to C^\infty(M)u_\phi.$$

In addition, given any $\phi \in G$, the involution (7.4) induces a continuous anti-linear map

$$C^\infty(M)u_\phi \longrightarrow C^\infty(M)u_{\phi^{-1}}.$$

It then follows that the multiplication of the crossed-product algebra $C^\infty(M) \rtimes G$ is a jointly continuous bilinear and its involution is continuous. Therefore, we see that $C^\infty(M) \rtimes G$ is a locally convex $*$-algebra with respect the topology described above.

Lemma 7.5. *For any metric $g \in \mathcal{C}$, the twisted spectral triple*

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}$$

is smooth in the sense of Definition 5.1.

Proof. Let $\phi \in G$. The map $f \to fU_\phi$ is continuous from $C^\infty(M)$ to $\mathcal{L}(L^2_g(M, \$))$. As $\sigma_g(fu_\phi) = k_\phi fu_\phi$ and $\sigma_g^{-1}(fu_\phi) = k_\phi^{-1} fu_\phi$, the maps $\phi \to \sigma_g^{\pm 1}(fu_\phi)$ are continuous from $C^\infty(M)$ to $C^\infty(M) \rtimes G$. In addition, note that, for any $f \in C^\infty(M)$,

$$[\not{D}_g, fU_\phi]_{\sigma_g} = [\not{D}_g, f]U_\phi + f[\not{D}_g, U_\phi]_{\sigma_g} = ic(df)U_\phi + f[\not{D}_g, U_\phi]_{\sigma_g},$$

where $c(df)$ is the Clifford representation of the differential df. Thus, the map $f \to [\not{D}_g, fU_\phi]_{\sigma_g}$ is continuous from $C^\infty(M)$ to $\mathcal{L}(L^2_g(M, \$))$. It follows from all this that the conditions (1)–(3) of Definition 5.1 are satisfied, and so

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}$$

is a smooth twisted spectral triple. The lemma is thus proved. □

7.2. Conformal invariance of the Connes–Chern character. The construction of the conformal Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g)_{\sigma_g}$, depends on the choice of a representative metric g in the conformal class $\mathcal{C}$. The following proposition describes the dependence on this choice.

Proposition 7.6. *Let $\widehat{g}$ be another metric in the conformal class $\mathcal{C}$, i.e. $\widehat{g} = k^2 g$ for some function $k \in C^\infty(M)$, $k > 0$. Let $\widehat{\sigma}$ be the automorphism of $C^\infty(M) \rtimes G$ defined by*

$$\widehat{\sigma}(a) = k^{-\frac{1}{2}}\sigma_g(k^{-\frac{1}{2}}ak^{\frac{1}{2}})k^{\frac{1}{2}} = k^{-1}\sigma_g(a)k, \qquad a \in C^\infty(M) \rtimes G. \tag{7.9}$$

Then the conformal Dirac spectral

$$\big(C^\infty(M) \rtimes G, L^2_{\widehat{g}}(M, \$), \not{D}_{\widehat{g}}\big)_{\sigma_{\widehat{g}}}$$

associated with $\widehat{g}$ is equivalent to the conformal deformation

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), k^{-\frac{1}{2}} \not{D}_g k^{-\frac{1}{2}}\big)_{\widehat{\sigma}}.$$

Proof. Let $U\colon L^2_{\widehat{g}}(M, \$) \to L^2_g(M, \$)$ be the operator given by the multiplication by $k^{\frac{n}{2}}$. Let $u \in L^2_{\widehat{g}}(M, \$)$. As $|dx|_{\widehat{g}} = k(x)^n |dx|_g$, we have

$$\begin{aligned}\|Uu\|^2_{L^2_g(M,\$)} &= \int_M \big(k(x)^{\frac{n}{2}} u(x), k(x)^{\frac{n}{2}} u(x)\big)|dx|_g \\ &= \int_M \big(u(x), u(x)\big)|dx|_{\widehat{g}} = \|u\|^2_{L^2_{\widehat{g}}(M,\$)}.\end{aligned}$$

This shows that U is a unitary operator. Moreover, using the conformal invariance of the Dirac operator [56] we see that

$$\not{D}_{\widehat{g}} = k^{-\frac{1}{2}(n+1)} \not{D}_g k^{\frac{1}{2}(n-1)} = U^* k^{-\frac{1}{2}} \not{D}_g k^{-\frac{1}{2}} U. \tag{7.10}$$

Let $\phi \in G$. We have two representations of ϕ. One is the unitary operator V_ϕ of $L^2_g(M, \$)$ given by (7.6) using the representative metric g. We have another representation of ϕ as a unitary operator $\widehat{V}_\phi$ of $L^2_{\widehat{g}}(M, \$)$ given by the same formula using the metric $\widehat{g}$. That is,

$$\widehat{V}_\phi u = e^{n\widehat{h}_\phi} \phi_* u, \qquad u \in L^2_{\widehat{g}}(M, \$),$$

where $\widehat{h}_\phi \in C^\infty(M, \mathbb{R})$ and satisfies that $\phi_* \widehat{g} = e^{2h_\phi} \widehat{g}$. Set $h = \log k$. Then

$$\phi_* \widehat{g} = \phi_*(k^2 g) = (k \circ \phi^{-1})^2 \phi_* g = e^{2h} e^{2h_\phi} g = e^{2(h\circ\phi^{-1} + h_\phi - h)} \widehat{g},$$

which shows that $\widehat{h}_\phi = h \circ \phi^{-1} + h_\phi - h$. Let $u \in L^2_{\widehat{g}}(M, \$)$. Then

$$\widehat{V}_\phi u = e^{-nh} e^{nh_\phi} e^{-nh\circ\phi} \phi_* u = k^{-\frac{n}{2}} e^{nh_\phi} \phi_*(k^{\frac{n}{2}} u) = U^* V_\phi U u.$$

Likewise,

$$\begin{aligned}\sigma_{\widehat{g}}(\widehat{V}_\phi) u &= e^{-(n+1)\widehat{h}_\phi} \phi_* u \\ &= k^{-(n+1)} e^{(n+1)h_\phi} \phi_*(k^{n+1} u) \\ &= U k^{-1} \sigma_g(V_\phi) k U u = U^{-1} \widehat{\sigma}(V_\phi) U u.\end{aligned}$$

We then deduce that, for all $f \in C^\infty(M)$ and $\phi \in G$,

$$f \widehat{V}_\phi = U^*(f V_\phi) U \qquad \text{and} \qquad \sigma_{\widehat{g}}(f \widehat{V}_\phi) = U^* \widehat{\sigma}(f V_\phi) U.$$

Combining this with (7.10) shows that the twisted spectral triples

$$\left(C^\infty(M) \rtimes G, L^2_{\widehat{g}}(M, \$), \not{D}_{\widehat{g}}\right)_{\sigma_{\widehat{g}}}$$

and

$$\left(C^\infty(M) \rtimes G, L^2_g(M, \$), k^{-\frac{1}{2}} \not{D}_g k^{-\frac{1}{2}}\right)_{\widehat{\sigma}}$$

are equivalent. The proof is complete. □

Given any metric $g \in \mathcal{C}$, the jth eingenvalue of $|\not{D}_g|$ grows like $j^{\frac{1}{n}}$ as $j \to \infty$. Therefore, the associated twisted spectral triple

$$\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$$

is p-summable for all $p > n$. Combining this with Proposition 6.3 shows that the Connes–Chern character of

$$\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$$

is well defined as a class in $\mathbf{HP}^0(C^\infty(M) \rtimes G)$ for every metric g in the conformal class $\mathcal{C}$.

We are now in a position to state the main result of this section.

Theorem 7.7. *The Connes–Chern character* $\mathbf{Ch}(\not{D}_g)_{\sigma_g} \in \mathbf{HP}^0\left(C^\infty(M) \rtimes G\right)$ *is independent of the choice of the metric* $g \in \mathcal{C}$*, i.e, it is an invariant of the conformal structure* $\mathcal{C}$.

Proof. Let $\widehat{g}$ be another metric in the conformal class $\mathcal{C}$, so that $\widehat{g} = k^2 g$ with $k \in C^\infty(M)$, $k > 0$. Combining Proposition 7.6 with Proposition 6.3 and Proposition 6.8 we get

$$\mathbf{Ch}(\not{D}_{\widehat{g}})_{\sigma_{\widehat{g}}} = \mathbf{Ch}\left(k^{-\frac{1}{2}} \not{D}_g k^{-\frac{1}{2}}\right)_{\widehat{\sigma}} = \mathbf{Ch}(\not{D}_g)_{\sigma_g} \qquad \text{in } \mathbf{HP}^0\left(C^\infty(M) \rtimes G\right),$$

where $\widehat{\sigma}$ is defined as in (7.9). This proves the result. □

This leads us to the following definition.

Definition 7.8. The Connes–Chern character of the conformal class $\mathcal{C}$, denoted by $\mathbf{Ch}(\mathcal{C})$, is the common class in $\mathbf{HP}^0\left(C^\infty(M) \rtimes G\right)$ of the Connes–Chern characters of the conformal Dirac spectral triples

$$\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$$

as the metric g ranges over the conformal class $\mathcal{C}$.

Combining this with Proposition 5.9 we obtain the following index formula.

Proposition 7.9. *Let* $\mathcal{E}$ *be a finitely generated projective module over* $C^\infty(M) \rtimes G$*. Then, for any metric* $g \in \mathcal{C}$ *and any* σ_g*-connection on* $\mathcal{E}$*, we have*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \mathbf{Ch}(\mathcal{C}), \mathbf{Ch}(\mathcal{E}) \rangle .$$

8. Local index formula in conformal geometry

In this section, we shall compute the conformal Connes–Chern character $\mathbf{Ch}(\mathcal{C})$ when the conformal structure $\mathcal{C}$ is not flat. Together with Proposition 7.9 this will provide us with the local index formula in conformal-diffeomorphism invariant geometry. We recall that the conformal structure $\mathcal{C}$ is flat when it is equivalent to the conformal structure of the round sphere $\mathbb{S}^n$. In case M is simply connected and has dimension ≥ 4 this is equivalent to the vanishing of the Weyl curvature tensor of M (see [63]).

As pointed out in Remark 4.12, the Ansatz of Moscovici [68] is not known to hold for conformal deformations of ordinary spectral triples sastisfying the local index formula in noncommutative geometry of [35]. As a result, unless the conformal structure $\mathcal{C}$ is flat and G is a maximal parabolic subgroup of $\mathrm{PO}(n+1,1)$, we cannot claim that for a general metric in $\mathcal{C}$ the corresponding conformal Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}$$

satisfies Moscovici's Ansatz.

Having said this, the conformal invariance of the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g}$ provided by Theorem 7.7 allows us to choose *any* metric in the conformal class $\mathcal{C}$ to compute $\mathbf{Ch}(\mathcal{C})$. In particular, as we shall see below (and as also observed by Moscovici [68]), the computation is greatly simplified by choosing a G-invariant metric in the conformal class $\mathcal{C}$. When the conformal structure $\mathcal{C}$ is non-flat, the existence of such a metric is ensured by the following result.

Proposition 8.1 (Ferrand–Obata [9, 45, 81]). *If the conformal structure $\mathcal{C}$ is not flat, then the group of smooth diffeomorphisms of M preserving $\mathcal{C}$ is a compact Lie group and there is a metric in $\mathcal{C}$ that is invariant by this group.*

The relevance of using a G-invariant metric $g \in \mathcal{C}$ stems from the observation that in this case $\phi_* g = g$ for all $\phi \in G$, and so, for every $\phi \in G$, the conformal factor k_ϕ is always the constant function 1. This implies that, for any $\phi \in G$, the unitary operator U_ϕ agrees with the pushforward by ϕ. This also implies that the automorphism σ_g given by (7.8) is trivial, so that the conformal Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)_{\sigma_g}$$

is actually an *ordinary* spectral triple. Therefore, we arrive at the following statement.

Proposition 8.2. *The conformal Connes–Chern character $\mathbf{Ch}(\mathcal{C})$ agrees with the ordinary Connes–Chern character of the equivariant Dirac spectral triple*

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\big)$$

associated with any G-invariant metric g in the conformal class $\mathcal{C}$.

The computation of the Connes–Chern character of an equivariant Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\big)$$

associated with any G-invariant metric $g \in \mathcal{C}$ is carried out in [78]. In order to present the results of [78] we need to introduce some notation.

In what follows we let g be a G-invariant metric in the conformal class $\mathcal{C}$. Given $\phi \in G$ we denote by M^ϕ its fixed-point set. As ϕ preserves the orientation and the metric g, we have the following disjoint-union decomposition,

$$M^\phi = \bigsqcup_{n-a \in 2\mathbb{N}_0} M^\phi_a, \qquad \text{where } M^\phi_a = \big\{x \in M^\phi;\ \mathrm{rk}(\phi'(x) - \mathrm{id}) = a\big\},$$

and each component M^ϕ_a is a submanifold of M of dimension a. For $a = 0, 2, \ldots, n$ we let $\mathcal{N}^\phi_a = (TM^\phi)^\perp$ be the normal bundle of M^ϕ_a; this is a smooth vector bundle over M^ϕ_a. We denote by $\phi^{\mathcal{N}}$ the isometric vector bundle isomorphism induced on $\mathcal{N}^\phi_a$ by ϕ'. We note that the eigenvalues of $\phi^{\mathcal{N}}$ are either -1 (which has even multiplicity) or complex conjugates $e^{\pm i\theta}$, $\theta \in (0, \pi)$, with same multiplicity. In addition, we shall orient M^ϕ_a like in [11, Prop. 6.14], so that the vector bundle isomorphism $\phi^{\not{S}}: \not{S} \to \phi_* \not{S}$ gives rise to a section of $\Lambda^{n-a}(\mathcal{N}^\phi_a)^*$ which is positive with respect to the orientation of $\mathcal{N}^\phi$ defined by the orientations of M and M^ϕ_a.

For $a = 0, 2, \ldots, n$, we let $\Omega(M^\phi_a) = (\Omega^\bullet(M^\phi_a), \wedge, d)$ be the differential graded algebra (DGA) of differential forms on M^ϕ_a. We then define $\Omega(M^\phi)$ as the DGA obtained as the direct sum of the DGAs $\Omega(M^\phi_a)$, $a = 0, 2, \ldots, n$. We shall refer to elements of $\Omega^\bullet(M^\phi)$ as differential forms on M^ϕ. Given $\omega \in \Omega^\bullet(M^\phi)$ we shall denote by $\omega|_{M^\phi_a}$ its component in $\Omega^\bullet(M^\phi_a)$. Note that

$$(\omega^1 \wedge \omega^2)\big|_{M^\phi_a} = \big(\omega^1\big|_{M^\phi_a}\big) \wedge \big(\omega^2\big|_{M^\phi_a}\big), \qquad \omega^j \in \Omega(M^\phi),\ a = 0, 2, \ldots, n.$$

For $a = 0, 2, \ldots, n$, we let $\iota^*_{M^\phi_a}: \Omega(M) \to \Omega(M^\phi_a)$ be the DGA map given by pulling-pack by the smooth inclusion $\iota_{M^\phi_a}: M^\phi_a \hookrightarrow M$. Taking direct sums of these DGA maps then provides us with a DGA map $\iota^*_{M^\phi}: \Omega(M) \to \Omega(M^\phi)$. We also define the integration $\int_{M^\phi}: \Omega(M^\phi) \to \mathbb{C}$ by

$$\int_{M^\phi} \omega := \sum_{n-a \in 2\mathbb{N}_0} \int_{M^\phi_a} \big(\omega|_{M^\phi_a}\big)^{(a)}, \qquad \omega \in \Omega(M^\phi), \tag{8.1}$$

where $(\omega|_{M^\phi_a})^{(a)} \in \Omega^a(M^\phi_a)$ is the top degree component of $\omega|_{M^\phi_a}$.

In addition, as the Levi-Civita connection ∇^{TM} is preserved by ϕ, for $a = 0, 2, \ldots, n$, it preserves the orthogonal splitting $TM|_{M^\phi_a} = TM^\phi_a \oplus \mathcal{N}^\phi_a$, and so it induces a connection $\nabla^{\mathcal{N}^\phi_a}$ on $\mathcal{N}^\phi_a$ in such a way that

$$\nabla^{TM}\big|_{|TM^\phi_a} = \nabla^{TM^\phi_a} \oplus \nabla^{\mathcal{N}^\phi_a},$$

where $\nabla^{TM_a^\phi}$ is the Levi-Civita connection of TM_a^ϕ. Let $R^{TM_a^\phi}$ and $R^{\mathcal{N}_a^\phi}$ be the respective curvatures of $\nabla^{TM_a^\phi}$ and $\nabla^{\mathcal{N}_a^\phi}$. We associate with these curvatures the following characteristic forms,

$$\begin{aligned} \widehat{A}\big(R^{TM_a^\phi}\big) &:= \det{}^{\frac{1}{2}}\left(\frac{R^{TM_a^\phi}/2}{\sinh(R^{TM_a^\phi}/2)}\right) \\ \text{and}\quad \mathcal{V}_\phi\big(R^{\mathcal{N}_a^\phi}\big) &:= \det{}^{-\frac{1}{2}}\big(1-\phi^{\mathcal{N}}e^{-R^{\mathcal{N}_a^\phi}}\big), \end{aligned} \tag{8.2}$$

where $\det^{-\frac{1}{2}}(1-\phi^{\mathcal{N}}e^{-R^{\mathcal{N}_a^\phi}})$ is defined in the same way as in [11, Section 6.3]. In particular, $\widehat{A}(R^{TM_a^\phi})$ is the $\widehat{A}$-form of the curvature $R^{TM_a^\phi}$ and the zeroth degree component of the characteristic form $\mathcal{N}_\phi(R^{\mathcal{N}_a^\phi})$ is the Lefschetz number $\det^{-\frac{1}{2}}(1-\phi^{\mathcal{N}})$. We then let $\Upsilon^\phi(R)\in\Omega(M^\phi)$ be the differential form on M^ϕ defined by

$$\Upsilon^\phi(R)\big|_{M_a^\phi} = (-i)^{\frac{n}{2}}(2\pi)^{-\frac{a}{2}}\widehat{A}\big(R^{TM_a^\phi}\big)\wedge\mathcal{V}_\phi\big(R^{\mathcal{N}_a^\phi}\big), \qquad a=0,2,\dots,n. \tag{8.3}$$

Note that $\Upsilon^\phi(R)$ is a closed differential form on M^ϕ.

Theorem 8.3 ([78, Theorem 7.8]). *Let g be a G-invariant metric. Then the Connes–Chern character* $\mathbf{Ch}(D)$ *of the equivariant Dirac spectral triple*

$$\big(C^\infty(M)\rtimes G, L^2_g(M,\$), \not{D}_g\big)$$

is represented in $\mathbf{HP}^0(C^\infty(M)\rtimes G)$ *by the cocycle* $\varphi^{\mathrm{CM}}=(\varphi_{2q})_{q\geq 0}$ *defined by*

$$\begin{aligned} &\varphi_{2q}(f^0u_{\phi_0},\dots,f^{2q}u_{\phi_{2q}}) \\ &\qquad = \frac{1}{(2q)!}\int_{M^\phi}\Upsilon^\phi(R)\wedge\iota^*_{M^\phi}\big(f^0d\widehat{f}^1\wedge\cdots\wedge d\widehat{f}^{2q}\big), \end{aligned} \tag{8.4}$$

where we have set $\phi:=\phi_0\circ\cdots\circ\phi_{2q}$ *and* $\widehat{f}^j:=f^j\circ\phi_{j-1}^{-1}\circ\cdots\circ\phi_0^{-1}$, $j=1,\dots,2q$.

Combining this with Proposition 8.2 we obtain the following index formula in conformal-diffeomorphism invariant geometry.

Theorem 8.4. *Assume that the conformal structure $\mathcal{C}$ is non-flat.*

(1) *For any G-invariant metric $g\in\mathcal{C}$, the conformal Connes–Chern character* $\mathbf{Ch}(\mathcal{C})$ *is represented in* $\mathbf{HP}^0(C^\infty(M)\rtimes G)$ *by the cocycle φ associated with g by (8.4).*

(2) *Let $\mathcal{E}$ be a finitely generated projective module over $C^\infty(M)\rtimes G$. Then, for any metric $g\in\mathcal{C}$ and any σ_g-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we have*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle\varphi,\mathbf{Ch}(\mathcal{E})\rangle,$$

where $\mathbf{Ch}(\mathcal{E})$ *is the Chern character of $\mathcal{E}$ and φ is the cocycle (8.4) associated with any G-invariant metric in $\mathcal{C}$.*

Remark 8.5 (See also [68]). For $q = \frac{1}{2}n$ the right-hand side of (8.4) reduces to an integral over M_n^ϕ. This submanifold is a disjoint union of connected components of M and on each of those components ϕ is the identity. Therefore, if M is connected, then M^ϕ is empty unless $\phi = \mathrm{id}$. In this case we then have

$$\varphi_n(f^0 u_{\phi_0}, \dots, f^n u_{\phi_n}) = \begin{cases} \frac{(2i\pi)^{-\frac{n}{2}}}{n!} \int_M f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^n, & \text{if } \phi_0 \circ \cdots \circ \phi_n = \mathrm{id}, \\ 0, & \text{otherwise.} \end{cases} \tag{8.5}$$

That is, φ_n agrees with the transverse fundamental class cocycle of Connes [31].

Remark 8.6. The computation in [78] of the Connes–Chern character in $\mathbf{HP}(C^\infty(M) \rtimes G)$ of an equivariant Dirac spectral triple

$$\big(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\big)$$

is divided into two main steps. The first step consists in showing that the Connes–Chern character is represented in $\mathbf{HP}^0(C^\infty(M) \rtimes G)$ (and not just in $\mathrm{HP}^0(C^\infty(M) \rtimes G)$) by the CM cocycle. This requires extending the local index formula of [34, 35] to the setting of smooth spectral triples. The second step is the explicit computation of that CM cocycle as an immediate byproduct of a new proof of the equivariant local index theorem of Patodi [72], Donnelly–Patodi [42], and Gilkey [49] (see also [8, 27]).

9. Equivariant cohomology and mixed equivariant homology

In this section, we recall the main background on group homology and equivariant cohomology that is needed in this paper. We also review the mixed equivariant homology of [75]. For a more comprehensive accounts on group homology and equivariant cohomology we refer to the monographs [19, 54].

Throughout this section we let Γ be a group. By a Γ-space we shall mean a vector space equipped with a left action of Γ, i.e. a left $\mathbb{C}\Gamma$-module.

9.1. Group homology and group cohomology. The standard cyclic Γ-space (or $\mathbb{C}\Gamma$-module) of Γ is $C(\Gamma) = (C_\bullet(\Gamma), d, s, t)$, where $C_m(\Gamma) := \mathbb{C}\Gamma^{m+1}$, $m \geq 0$, and the structural operators

$$d: C_\bullet(\Gamma) \to C_{\bullet-1}(\Gamma), \qquad s: C_\bullet(\Gamma) \to C_{\bullet+1}(\Gamma), \qquad t: C_\bullet(\Gamma) \to C_\bullet(\Gamma)$$

are given by

$$d(\psi_0, \dots, \psi_m) = (\psi_0, \dots, \psi_{m-1}), \quad s(\psi_0, \dots, \psi_m) = (\psi_m, \psi_0, \dots, \psi_m), \tag{9.1}$$

$$t(\psi_0, \dots, \psi_m) = (\psi_m, \psi_0, \dots, \psi_{m-1}), \quad \psi_j \in \Gamma. \tag{9.2}$$

Each space $C_m(\Gamma)$ is equipped with the Γ-action such that

$$\psi \cdot (\psi_0, \dots, \psi_m) = (\psi\psi_0, \dots, \psi\psi_m)$$

for all $\psi, \psi_0, \dots, \psi_m$ in Γ. The operators (d, s, t) are Γ-equivariant with respect to this action so that we get a cyclic Γ-space. At the simplicial level we obtain the standard simplicial Γ-space of Γ whose geometric realization is the universal Γ-bundle $E\Gamma$ (a.k.a. Milnor construction). This gives rise to a chain complex $(C_\bullet(\Gamma), \partial)$, where the differential $\partial\colon C_\bullet(\Gamma) \to C_{\bullet-1}(\Gamma)$ is given by

$$\partial(\psi_0, \dots, \psi_m) = \sum_{0 \le j \le m} (-1)^j (\psi_0, \dots, \widehat{\psi}_j, \dots, \psi_m), \qquad \psi_j \in \Gamma. \tag{9.3}$$

In what follows, given Γ-spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ we equip the tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ (over $\mathbb{C}$) with the diagonal action of Γ. We then denote by $\mathcal{E}_1 \otimes_\Gamma \mathcal{E}_2$ their tensor product over Γ, i.e. the quotient of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by the action of Γ. This ensures us that $(\psi\xi_1) \otimes_\Gamma (\psi\xi_2) = \xi_1 \otimes_\Gamma \xi_2$ for all $\xi_j \in \Gamma$ and $\psi \in \Gamma$. Bearing this in mind, for any Γ-space $\mathcal{E}$, we form the chain complex $C(\Gamma, \mathcal{E}) := (C_\bullet(\Gamma, \mathcal{E}), \partial)$, where $C_m(\Gamma, \mathcal{E}) = C_m(\Gamma) \otimes_\Gamma \mathcal{E}$. The homology of this complex is the *homology of* Γ *with coefficients in* $\mathcal{E}$ and is denoted by $H_\bullet(\Gamma, \mathcal{E})$. In the case of the trivial Γ-space $\mathcal{E} = \mathbb{C}$ we recover the homology of the classifying space $B\Gamma = E\Gamma/\Gamma$ (with coefficients in $\mathbb{C}$).

There is a natural duality between $C_m(\Gamma, \mathcal{E})$ and the space $C^m(\Gamma, \mathcal{E})$ that consists of all Γ-equivariant maps $u\colon \Gamma^{m+1} \to \mathcal{E}$. We thus obtain a cochain complex $(C^\bullet(\Gamma, \mathcal{E}), \partial)$, where $\partial\colon C^\bullet(\Gamma, \mathcal{E}) \to C^{\bullet+1}(\Gamma, \mathcal{E})$ is given by

$$(\partial u)(\psi_0, \dots, \psi_{m+1}) = \sum_{0 \le j \le m+1} (-1)^j u(\psi_0, \dots, \widehat{\psi}_j, \dots, \psi_{m+1}), \qquad \psi_j \in \Gamma.$$

The cohomology of this cochain complex is the *cohomology of* Γ *with coefficients in* $\mathcal{E}$ and is denoted by $H^\bullet(\Gamma, \mathcal{E})$. When $\mathcal{E} = \mathbb{C}$ we recover the cohomology of $B\Gamma$.

We have a natural cup product

$$\smile\colon C^p(\Gamma, \mathcal{E}_1) \times C^q(\Gamma, \mathcal{E}_2) \to C^{p+q}(\Gamma, \mathcal{E}_1 \otimes \mathcal{E}_2)$$

given by

$$(u \smile v)(\psi_0, \dots, \psi_{p+q}) = u(\psi_q, \dots, \psi_{p+q}) \otimes v(\psi_0, \dots, \psi_q), \qquad \psi_j \in \Gamma.$$

This cup product is associative and compatible with the group coboundary ∂. Therefore, it induces a (graded associative) cup product on group cohomology. In particular, this turns $H^\bullet(\Gamma, \mathbb{C})$ into a graded ring. By duality we get a cap product

$$\frown\colon C^p(\Gamma, \mathcal{E}_1) \times C_{p+q}(\Gamma, \mathcal{E}_2) \to C_q(\Gamma, \mathcal{E}_1 \otimes \mathcal{E}_2)$$

given by

$$u \frown \big[(\psi_0, \dots, \psi_{p+q}) \otimes_\Gamma \xi\big] = (\psi_p, \dots, \psi_{p+q}) \otimes_\Gamma \big[u(\psi_0, \dots, \psi_p) \otimes \xi\big], \qquad \psi_j \in \Gamma,\ \xi \in \mathcal{E}_2. \quad (9.4)$$

This cap product is compatible with the group boundary and coboundary, as well as with the cup product. Therefore, it descends to a cap product between group cohomology and group homology which is compatible with the cup product. In particular, when $\mathcal{E}_1 = \mathbb{C}$ we obtain a graded (left) action of the cohomology ring $H^\bullet(\Gamma, \mathbb{C})$ on the group homology $H_\bullet(\Gamma, \mathcal{E}_2)$.

9.2. Equivariant cohomology. From now on we assume that Γ acts on a manifold M by (smooth) diffeomorphisms. The equivariant cohomology of M is the cohomology of the homotopy quotient $E\Gamma \times_\Gamma M$ (a.k.a. Borel construction). This cohomology can be defined by using the bicomplex of Bott [13] as follows. This bicomplex is $C_\Gamma(M) := (C_\Gamma^{\bullet,\bullet}(M), \partial, d)$, where $C_\Gamma^{p,q}(M) = C^p(\Gamma, \Omega^q(M))$ and we regard the de Rham differential as an operator $d\colon C_\Gamma^{\bullet,\bullet}(M) \to C_\Gamma^{\bullet,\bullet+1}(M)$. It gives rise to the total complex

$$\mathrm{Tot}(C_\Gamma(M)) = \big(\mathrm{Tot}^\bullet(C_\Gamma(M)), d^\dagger\big),$$

where

$$\mathrm{Tot}^m(C_\Gamma(M)) = \bigoplus_{p+q=m} C^p(\Gamma, \Omega^q(M))$$

and $d^\dagger = \partial + (-1)^p d$ on $C^p(\Gamma, \Omega^q(M))$. The *equivariant cohomology* of the Γ-manifold M is the cohomology of the cochain complex $\mathrm{Tot}(C_\Gamma(M))$. It is denoted by $H_\Gamma^\bullet(M)$.

We also define the *even/odd equivariant cohomology* $H_\Gamma^{\mathrm{ev/odd}}(M)$ as the cohomology of the cochain complex $(C_\Gamma^{\mathrm{ev/odd}}(M), d^\dagger)$, where

$$C_\Gamma^{\mathrm{ev/odd}}(M) = \prod_{p+q \text{ even/odd}} C^p(\Gamma, \Omega^q(M)).$$

This is a natural receptacle for the construction of equivariant characteristic classes. In particular, given any equivariant vector bundle E over M, we have a uniquely defined equivariant Chern character $\mathrm{Ch}_\Gamma(E) \in H_\Gamma^{\mathrm{ev}}(M)$. Following Getzler [46], given any connection ∇^E on E, we can represent this equivariant Chern character by an explicit equivariant cochain $\mathrm{Ch}_\Gamma(\nabla^E) \in C_\Gamma^{\mathrm{ev}}(M)$ as follows.

As E is a Γ-equivariant bundle, any $\psi \in \Gamma$ lifts to a vector bundle isomorphism ψ^E on E. In particular, this enables us to pushforward ∇_0^E to the connection

$$\psi_* \nabla^E := (\psi_* \otimes \psi_*^E) \circ \nabla^E \circ (\psi_*^E)^{-1},$$

where ψ_* and ψ_*^E are the respective actions of ψ on $C^\infty(M, T^*M)$ and $C^\infty(M, E)$. The equivariant Chern form is $\mathrm{Ch}_\Gamma(\nabla^E) = (\mathrm{Ch}_\Gamma^p(\nabla^E))_{p\geq 0}$, where the components $\mathrm{Ch}_\Gamma^p(\nabla^E) \in C^p(\Gamma, \Omega^\bullet(M))$ are given by

$$\mathrm{Ch}_\Gamma^0(\nabla^E)(\psi_0) = \mathrm{Ch}\left((\psi_0)_*\nabla^E\right), \tag{9.5}$$

$$\mathrm{Ch}_\Gamma^p(\nabla^E)(\psi_0, \dots, \psi_p) = (-1)^p \mathrm{CS}\big((\psi_0)_*\nabla^E, \dots, (\psi_p)_*\nabla^E\big), \quad \psi_j \in \Gamma. \tag{9.6}$$

Here $\mathrm{Ch}((\psi_0)_*\nabla^E)$ is the Chern form of the connection $(\psi_0)_*\nabla^E$ and

$$\mathrm{CS}\big((\psi_0)_*\nabla^E, \dots, (\psi_p)_*\nabla^E\big)$$

is the Chern–Simons form of the connections $(\psi_0)_*\nabla^E, \dots, (\psi_p)_*\nabla^E$ as defined in [46]. Note that $\mathrm{Ch}_\Gamma^p(\nabla^E)$ is an element of $C^p(\Gamma, \Omega^{\mathrm{ev}}(M))$ (resp., $C^p(\Gamma, \Omega^{\mathrm{odd}}(M))$) when p is even (resp., odd).

When Γ is a compact connected Lie group, the equivariant cohomology $H_\Gamma^\bullet(M)$ can be described equivalently in terms of the Cartan model of equivariant differentiel forms and the Weil model (see [25, 54, 66]).

9.3. Mixed equivariant homology. We also can define the equivariant homology of the Γ-manifold M by using a chain version of Bott's bicomplex. For our purpose it is more convenient to use the mixed equivariant homology of [75]. Following [75] the equivariant mixed bicomplex of M is obtained as the tensor product over Γ of the mixed complexes $(C_\bullet(\Gamma), \partial, 0)$ and $(\Omega^\bullet(M), 0, d)$. Thus, this is

$$C(\Gamma, M) := \big(C_{\bullet,\bullet}(\Gamma, M), \partial, 0, 0, d\big),$$

where $C_{p,q}(\Gamma, M) = C_p(\Gamma, \Omega^q(M))$, $p, q \geq 0$. This gives rise to the "total" mixed complex

$$\mathrm{Tot}(C(\Gamma, M)) = \big(\mathrm{Tot}_\bullet(C(\Gamma, M)), \partial, (-1)^p d\big),$$

where

$$\mathrm{Tot}_m(C(\Gamma, M)) = \bigoplus_{p+q=m} C_p(\Gamma, \Omega^q(M)).$$

The cyclic homology of this mixed complex is called the *mixed equivariant homology* of the Γ-manifold M and is denoted by $H_\bullet^\Gamma(M)^\sharp$. A mixed equivariant chain in $\mathrm{Tot}_m(C(\Gamma, M))^\natural$ is of the form $\omega = (\omega_{p,q})_{m-(p+q)\in 2\mathbb{N}_0}$ where

$$\omega_{p,q} \in \mathbb{C}\Gamma^{p+1} \otimes_\Gamma \Omega^q(M).$$

This gives a cycle iff $\partial\omega_{p,q} = 0$ when $p+q = m$ and $\partial\omega_{p+1,q} + (-1)^p d\omega_{p,q-1} = 0$ when $m - (p+q) \in 2\mathbb{N}$.

The periodic cyclic homology of the equivariant mixed complex $\mathrm{Tot}(C(\Gamma, M))$ is called the *even/odd mixed equivariant homology* of M and is denoted by $H_{\mathrm{ev/odd}}^\Gamma(M)^\sharp$. Note that the periodic cyclic complex is

$$(C_{\mathrm{ev/odd}}(\Gamma, M), \partial + (-1)^p d),$$

where

$$C_{\mathrm{ev/odd}}(\Gamma, M) = \prod_{p+q \text{ even/odd}} C_p(\Gamma, \Omega^q(M)).$$

The mixed equivariant homology is the natural receptacle for the cap product between equivariant cohomology and group homology. More precisely, the cap product (9.4) gives a cap product

$$\frown: C^p(\Gamma, \Omega^q(M)) \times C_m(\Gamma, \mathbb{C}) \to C_{m-p}(\Gamma, \Omega^q(M)).$$

This is a differential bilinear map which is compatible with the group differentials and the de Rham differential. Therefore, it gives rise to a (graded) cap product,

$$\frown: H_\Gamma^{\mathrm{ev/odd}}(M) \times H_\bullet(\Gamma, \mathbb{C}) \longrightarrow H^\Gamma_{\mathrm{ev/odd}}(M)^\sharp. \tag{9.7}$$

In particular, the cap products of equivariant cohomology classes with group homology classes naturally give rise to classes in $H^\Gamma_{\mathrm{ev/odd}}(M)^\sharp$.

10. Cyclic homology and group actions on manifolds

There is a large amount of work on the cyclic homology of crossed-product algebras, especially in the case of group actions on manifolds or varieties (see, e.g. [10, 18, 20–22, 31, 32, 40, 44, 47, 69, 70, 74, 75]). In his seminal work on cyclic cohomology and foliations Connes [31] constructed an explicit cochain map and quasi-isomorphism from equivariant cohomology into the homogeneous component of the periodic cyclic cohomology. However, since then it was not until the recent notes [74, 75] that further explicit quasi-isomorphisms were exhibited. In this section, we briefly describe the results of [74, 75] in the case of group actions on manifolds.

Throughout this section we let Γ be a group acting by smooth diffeomorphisms on a closed manifold M^n. We let $\mathcal{A}$ be the Fréchet algebra $C^\infty(M)$. The action of Γ gives rise to an action on $\mathcal{A}$ given by (7.2). We also denote by $\mathcal{A}_\Gamma$ the crossed-product algebra $C^\infty(M) \rtimes \Gamma$, which we endow with the locally convex algebra topology described in Section 7.

10.1. Preamble 1: Paracyclic and cylindrical modules. In the following two preambles we let $\mathbf{k}$ be an arbitrary unital ring. In the context of group actions on algebras, we are naturally lead to go beyond the scope of the mixed complexes and cyclic modules described in Section 3. According to Getzler–Jones [47], a *parachain complex* is given by the datum of $(C_\bullet, b, B)$, where C_m, $m \geq 0$, are $\mathbf{k}$-modules and $b: C_\bullet \to C_{\bullet-1}$ and $B: C_\bullet \to C_{\bullet+1}$ are $\mathbf{k}$-module maps such that $b^2 = B^2 = 0$ and $bB + Bb = 1 - T$, where $T: C_\bullet \to C_\bullet$ is some invertible $\mathbf{k}$-module map. Given a parachain complex $C = (C_\bullet, b, B)$ we also can form a cyclic complex $C^\natural = (C^\natural_\bullet, b + SB)$ as in the case of mixed complexes. This need not

be a chain complex (unless C is a mixed complex), but we obtain a para-S-module (cf. Preamble 2).

Following [47], a *paracyclic* **k**-*module* is given by the datum of $(C_\bullet, d, s, t)$, where C_m, $m \geq 0$, are **k**-modules and the **k**-module maps $d: C_\bullet \to C_{\bullet-1}$, $s: C_\bullet \to C_{\bullet+1}$ and $t: C_\bullet \to C_\bullet$ define a simplicial module structure in the same way as cyclic modules, but the cyclicity of the operator t is replaced by the relation $t = ds$ and the requirement that t is invertible. Any paracyclic **k**-module gives rise to a parachain complex $(C_\bullet, b, B)$, where b is defined in the same way as with cyclic **k**-modules and $B = (1-\tau)s'N$ with $s' = sb's$ and $b' = \sum_{j=0}^{m-1}(-1)^j d_j = b - d$ on C_m. In that case $T = 1 - (bB + Bb) = t^{m+1}$ on C_m. (When C is a cyclic module we obtain a mixed complex which is isomorphic to its usual mixed complex.)

A *parachain bicomplex* is given by the datum of $(C_{\bullet,\bullet}, \bar{b}, \bar{B}, b, B)$, where $C_{p,q}$, $p, q \geq 0$, are **k**-modules, $(C_{\bullet,q}, \bar{b}, \bar{B})$ and $(C_{p,\bullet}, b, B)$ are parachain complexes for all $p, q \geq 0$, and the horizontal differentials $(\bar{b}, \bar{B})$ both commute with each of the vertical differentials (b, B). We have a *cylindrical complex* when the operators $\bar{T} = 1 - (\bar{b}\bar{B} + \bar{B}\bar{b})$ and $T = 1 - (bB + Bb)$ are inverses of each other (cf. [47]). As observed in [47], any cylindrical complex gives rise to a total mixed complex

$$\mathrm{Tot}(C) = \left(\mathrm{Tot}_\bullet(C), b^\dagger, B^\dagger\right),$$

where

$$\mathrm{Tot}_m(C) = \bigoplus_{p+q=m} C_{p,q}$$

and $b^\dagger = \bar{b} + (-1)^p b$ and $B^\dagger = \bar{B} + (-1)^p \bar{T} B$ on $C_{p,q}$.

A *bi-paracyclic* **k**-*module* is given by the datum of $(C_{\bullet,\bullet}, \bar{d}, \bar{s}, \bar{t}, d, s, t)$, where $C_{p,q}$, $p, q \geq 0$, are **k**-modules, $(C_{\bullet,q}, \bar{d}, \bar{s}, \bar{t})$ and $(C_{p,\bullet}, d, s, t)$ are paracyclic **k**-modules for all $p, q \geq 0$, and all the horizontal operators $(\bar{d}, \bar{s}, \bar{t})$ commute with each of the vertical operators (d, s, t). We have a *cylindrical* **k**-*module* when $\bar{t}^{p+1} t^{q+1} = 1$ on $C_{p,q}$. Any cyclindrical **k**-module

$$C = \left(C_{\bullet,\bullet}, \bar{d}, \bar{s}, \bar{t}, d, s, t\right)$$

gives rise to a cyclindrical complex $(C_{\bullet,\bullet}, \bar{b}, \bar{B}, b, B)$, where $(\bar{b}, \bar{B})$ (resp., (b, B)) are defined as above by using $(\bar{d}, \bar{s}, \bar{t})$ (resp., (d, s, t)). This gives rise to a total mixed complex $\mathrm{Tot}(C)$. We also obtain a diagonal cyclic **k**-module

$$\mathrm{Diag}(C) = (\mathrm{Diag}_\bullet(C), \bar{d}d, \bar{s}s, \bar{t}t),$$

with $\mathrm{Diag}_m(C) = C_{m,m}$, $m \geq 0$.

10.2. Preamble 2: Triangular S-modules. The S-modules of Jones–Kassel [59, 61] encapsulate various approaches to cyclic homology. More generally, according to [74] a *para-S-module* is given by the datum of $(C_\bullet, b, S)$, where C_m, $m \geq 0$, are

$\mathbf{k}$-modules and $d: C_\bullet \to C_{\bullet-1}$ and $S: C_\bullet \to C_{\bullet-2}$ are $\mathbf{k}$-module maps commuting with each other such that $d^2 = (1-T)S$ where $T: C_\bullet \to C_\bullet$ is some $\mathbf{k}$-module map commuting with both d and S. When $d^2 = 0$ we obtain an S-module. For instance, if $C = (C_\bullet, b, B)$ is a parachain complex, then we can define its cyclic complex of the para-S-module $C^\natural = (C^\natural_\bullet, b + SB, S)$.

According to [74], a (left) *triangular para-S-module* is given by the datum of $(C_{\bullet,\bullet}, d, b, B, S)$, where $C_{p,q}$, $p, q \geq 0$, are $\mathbf{k}$-modules, $(C_{\bullet,q}, d, S)$ is a para-S-module and $(C_{p,\bullet}, b, B)$ is a parachain complex for all $p, q \geq 0$, the horizontal operators (d, S) both commute with each of the vertical differentials (b, B). We say that we have a *(left) triangular S-module* when $d^2 + S(bB + Bb) = 0$. Any cylindrical complex gives rise to a triangular S-module (cf. [74]). Triangular (para-)S-modules provide us with a natural framework for defining the tensor product of (para-)S-modules with mixed and parachain complexes (see Subsection 10.7 below).

Any triangular para-S-module $C = (C_{\bullet,\bullet}, \bar{d}, b, B, S)$ gives rise to a *total para-S-module*

$$\mathrm{Tot}(C) = \big(\mathrm{Tot}_\bullet(C), d^\dagger, S\big),$$

where

$$\mathrm{Tot}_m(C) = \bigoplus_{p+q=m} C_{p,q}$$

and $d^\dagger = d + (-1)^p(b + SB)$ on $C_{p,q}$. When C is a triangular S-module the condition $d^2 + S(bB + Bb) = 0$ precisely ensures us that $(d^\dagger)^2 = 0$, and so we actually obtain an S-module. Furthermore, the filtration by columns of $\mathrm{Tot}_\bullet(C)$ is a filtration of chain complexes, and so it gives rise to a spectral sequence converging to $H_\bullet(\mathrm{Tot}(C))$. The spectral sequence of Getzler–Jones [47] is an instance of such a spectral sequence (cf. [74]).

10.3. The cylindrical space $\mathbf{C}^\phi(\Gamma, \mathcal{A})$. As we shall see, a central role in the computation of the cyclic homology of $\mathcal{A}_\Gamma$ is played by some cylindrical spaces $\mathbf{C}^\phi(\Gamma, \mathcal{A})$ that are defined as follows.

Suppose that ϕ is a *central* element of Γ. We can use ϕ to twist the cyclic structure of the cyclic Γ-space $C(\Gamma)$. We obtain the paracyclic Γ-space

$$C^\phi(\Gamma) = (C_\bullet(\Gamma), d, s_\phi, t_\phi),$$

where d is as in (9.1) and the operators

$$s_\phi: C_\bullet(\Gamma) \to C_{\bullet+1}(\Gamma) \quad \text{and} \quad t_\phi: C_\bullet(\Gamma) \to C_\bullet(\Gamma)$$

are given by

$$\begin{aligned} s_\phi(\psi_0, \ldots, \psi_m) &= (\phi^{-1}\psi_m, \psi_0, \ldots, \psi_m), \\ t_\phi(\psi_0, \ldots, \psi_m) &= (\phi^{-1}\psi_m, \psi_0, \ldots, \psi_{m-1}), \qquad \psi_j \in \Gamma. \end{aligned}$$

The simplicial space structures of $C^\phi(\Gamma)$ and $C(\Gamma)$ agree, and so the b-differential of $C^\phi(\Gamma)$ is the group differential ∂ given by (9.3). We also observe that

$$t_\phi^{m+1}(\psi_0,\dots,\psi_m) = (\phi^{-1}\psi_0,\dots,\phi^{-1}\psi_m),$$

i.e. t_ϕ^{m+1} agrees with the action of ϕ^{-1} on $C_m(\Gamma)$.

We also can twist the cyclic structure of the cyclic Γ-space $\mathbf{C}(\mathcal{A})$. We get the paracyclic Γ-space

$$\mathbf{C}^\phi(\mathcal{A}) = (\mathbf{C}_\bullet(\mathcal{A}), d_\phi, s, t_\phi),$$

where s is given by (3.2) and the operators

$$d_\phi\colon \mathbf{C}_\bullet(\mathcal{A}) \to \mathbf{C}_{\bullet-1}(\mathcal{A}) \quad \text{and} \quad t_\phi\colon \mathbf{C}_\bullet(\mathcal{A}) \to \mathbf{C}_\bullet(\mathcal{A})$$

are given by

$$\begin{aligned} d_\phi(f^0\otimes\cdots\otimes f^m) &= [(\phi^{-1}\cdot f^m)f^0]\otimes f^1\otimes\cdots\otimes f^{m-1},\\ t_\phi(f^0\otimes\cdots\otimes f^m) &= (\phi^{-1}\cdot f^m)\otimes f^0\otimes\cdots\otimes f^{m-1}, \qquad f^j\in\mathcal{A}. \end{aligned}$$

Note also that t_ϕ^{m+1} agrees with the action of ϕ^{-1} on $\mathbf{C}_m(\mathcal{A}_\Gamma)$.

The cylindrical space $\mathbf{C}^\phi(\Gamma,\mathcal{A})$ is the tensor product over Γ of the paracyclic Γ-spaces $C^\phi(\Gamma)$ and $\mathbf{C}^\phi(\mathcal{A})$. Namely,

$$\mathbf{C}^\phi(\Gamma,\mathcal{A}) = \big(\mathbf{C}_{\bullet,\bullet}(\Gamma,\mathcal{A}), \bar{d}, \bar{s}_\phi, \bar{t}_\phi, d_\phi, s, t_\phi\big),$$

where

$$\mathbf{C}_{p,q}(\Gamma,\mathcal{A}) = C_p(\Gamma)\otimes_\Gamma \mathbf{C}_q(\mathcal{A}),$$

and the horizontal (resp., vertical) structural operators $(\bar{d},\bar{s}_\phi,\bar{t}_\phi)$ (resp., (d_ϕ,s,t_ϕ)) are given by the structural operators of $C^\phi(\Gamma)$ (resp., $\mathbf{C}^\phi(\mathcal{A})$). We obtain a cyclindrical space because in degree m the operator $\bar{t}_\phi^{m+1}t_\phi^{m+1}$ arises from the diagonal action of ϕ^{-1} on $C_m(\Gamma)\otimes\mathbf{C}_m(\mathcal{A})$, and hence it agrees with the identity map on $C_m(\Gamma)\otimes_\Gamma\mathbf{C}_m(\mathcal{A})$. This gives rise to a diagonal cyclic space $\mathrm{Diag}(\mathbf{C}^\phi(\Gamma,\mathcal{A}))$ and a total mixed complex $\mathrm{Tot}(\mathbf{C}^\phi(\Gamma,\mathcal{A}))$. By the Eilenberg–Zilber theorem for bi-paracyclic spaces [62,76,85], at the level of their cyclic complexes we have an explicit chain homotopy equivalence

$$\mathrm{Tot}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\big)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\big)^\natural,$$

where $\sqcup\!\sqcup^\natural$ and $\mathrm{AW}^\natural$ are S-maps whose zeroth degree components are the usual shuffle and Alexander–Whitney maps. At the periodic level we also have a chain homotopy equivalence

$$\mathrm{Tot}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\big)^\sharp \underset{\mathrm{AW}^\sharp}{\overset{\sqcup\!\sqcup^\sharp}{\rightleftarrows}} \mathrm{Diag}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\big)^\sharp,$$

where $\sqcup\!\sqcup^\sharp$ and $\mathrm{AW}^\sharp$ are the respective extensions of $\sqcup\!\sqcup^\natural$ and $\mathrm{AW}^\natural$ to periodic chains.

10.4. Splitting along conjugacy classes. Given any $\phi \in \Gamma$, we denote by $[\phi]$ its conjugacy class. In addition, we let $[\Gamma]$ be the set of conjugacy classes of Γ. Given any $\phi \in \Gamma$ and $m \geq 0$, we let $\mathbf{C}_m(\mathcal{A}_\Gamma)_{[\phi]}$ be the closure in $\mathbf{C}(\mathcal{A}_\Gamma)$ of the subspace spanned by elementary tensor products $f^0 u_{\phi_0} \otimes \cdots \otimes f^m u_{\phi_m}$, with $f^0, \ldots, f^m$ in $\mathcal{A}$ and $\phi_0, \ldots, \phi_m$ in Γ such that $\phi_0 \cdots \phi_m \in [\phi]$. The subspace $\mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ is preserved by the structural operators (d, s, t) in (3.1)–(3.3). We thus obtain a cyclic subspace

$$\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]} = \big(\mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}, d, s, t\big).$$

We then have a direct-sum decomposition of cyclic spaces,

$$\mathbf{C}_\bullet(\mathcal{A}_\Gamma) = \bigoplus_{[\phi]\in[\Gamma]} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}, \tag{10.1}$$

where the summation goes over all conjugacy classes.

In what follows, given $\phi \in \Gamma$, we denote by $\mathbf{C}^\natural(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{C}^\sharp(\mathcal{A}_\Gamma)_{[\phi]}$ the respective cyclic and periodic complexes of the cyclic space $\mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. We also denote by $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ their respective homologies. At the cyclic level the decomposition (10.1) gives rise to direct-sum decompositions,

$$\mathbf{C}^\natural_\bullet(\mathcal{A}_\Gamma) = \bigoplus_{[\phi]\in[\Gamma]} \mathbf{C}^\natural_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \quad \text{and} \quad \mathbf{HC}_\bullet(\mathcal{A}_\Gamma) = \bigoplus_{[\phi]\in[\Gamma]} \mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}.$$

At the periodic level $\mathbf{C}^\sharp(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ are direct summands of $\mathbf{C}^\sharp(\mathcal{A}_\Gamma)$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)$, respectively. Therefore, we have direct-summand inclusions,

$$\bigoplus_{[\phi]\in[\Gamma]} \mathbf{C}^\sharp_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \subset \mathbf{C}^\sharp_\bullet(\mathcal{A}_\Gamma) \quad \text{and} \quad \bigoplus_{[\phi]\in[\Gamma]} \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \subset \mathbf{HP}_\bullet(\mathcal{A}_\Gamma).$$

These inclusions are onto when there is a finite number of conjugacy classes.

Given $\phi \in \Gamma$, let us denote by Γ_ϕ its centralizer in Γ. As ϕ is a central element of Γ_ϕ, we may form the cylindrical complex $\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})$ as above. We have a natural embedding of cyclic spaces

$$\mu_\phi \colon \mathrm{Diag}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big) \to \mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$$

given by

$$\begin{aligned}&\mu_\phi\big((\psi_0, \ldots, \psi_m) \otimes_{\Gamma_\phi} (f^0 \otimes \cdots \otimes f^m)\big)\\ &= [(\psi_m^{-1}\phi) \cdot f^0] u_{\phi\psi_m^{-1}\psi_0} \otimes (\psi_0^{-1} \cdot f^1) u_{\psi_0^{-1}\psi_1} \otimes \cdots \otimes (\psi_{m-1}^{-1} \cdot f^m) u_{\psi_{m-1}^{-1}\psi_m}.\end{aligned}$$

This cyclic space embedding gives rise to quasi-isomorphisms at the level the cyclic and periodic complexes. We thus obtain explicit quasi-isomorphisms,

$$\mathrm{Tot}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural \xrightarrow{\mu_\phi} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{10.2}$$

There are analogous quasi-isomorphisms between the corresponding periodic cyclic complexes.

All this shows that the computation of the cyclic homology and periodic cyclic homology of $\mathcal{A}_\Gamma$ reduces to that of the mixed complexes $\mathrm{Tot}_\bullet(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A}))$, $\phi \in \Gamma$.

10.5. A twisted Connes–Hochschild–Kostant–Rosenberg theorem. Let $\phi \in \Gamma$ and denote by M^ϕ its fixed-point set in M. We shall say that the action of ϕ on M is *clean* when, for every $x_0 \in M^\phi$, the following two conditions are satisfied:

(i) The fixed-point set M^ϕ is a submanifold of M near x_0.

(ii) We have $T_{x_0}M^\phi = \ker(\phi'(x_0) - 1)$ and $T_{x_0}M = T_{x_0}M^\phi \oplus \mathrm{ran}(\phi'(x_0) - 1)$.

We have a clean action whenever ϕ preserves an affine connection or even a Riemannian metric. In particular, we always have a clean action when ϕ has finite order.

From now on we assume that the action of ϕ on M is clean. For $a = 0, 1, \ldots, \dim M$, set

$$M_a^\phi := \{x \in M^\phi;\ \dim\ker(\phi'(x) - 1) = a\}.$$

Each subset M_a^ϕ is a submanifold of M, and so we have a disjoint union stratification $M^\phi = \bigsqcup_a M_a^\phi$. This enables us to define the de Rham complex $\Omega(M^\phi) = (\Omega^\bullet(M^\phi), d)$ as the direct sum of the de Rham complexes $\Omega(M_a^\phi)$. Note also that each component M_a^ϕ is preserved by the action of the centralizer Γ_ϕ. Let $(\mathbf{C}^\phi(\mathcal{A}), b_\phi, B_\phi)$ be the parachain complex of the paracyclic Γ-space. Regarding $\Omega(M^\phi)$ as a mixed complex $(\Omega^\bullet(M^\phi), 0, d)$ we then have a map of parachain complexes $\alpha^\phi\colon \mathbf{C}_\bullet^\phi(\mathcal{A}) \to \Omega^\bullet(M^\phi)$ given by

$$\alpha^\phi(f^0 \otimes \cdots \otimes f^m) = \frac{1}{m}\sum_a (f^0 df^1 \wedge \cdots \wedge df^m)_{|M_a^\phi}, \qquad f^j \in \mathcal{A}. \tag{10.3}$$

Proposition 10.1 (Brylinski [20], Brylinski–Nistor [22]). *Let $\phi \in \Gamma$ acts cleanly on M. Then the chain map*

$$\alpha^\phi\colon (\mathbf{C}_\bullet^\phi(\mathcal{A}), b_\phi) \to (\Omega^\bullet(M^\phi), 0)$$

is a quasi-isomorphism.

10.6. The cyclic space $\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$ when ϕ has finite order. Let $\phi \in \Gamma$ have finite order r. Let $C^\flat(\Gamma_\phi)$ be the mixed complex $(C_\bullet(\Gamma_\phi), \partial, 0)$. We have a Γ_ϕ-equivariant map of parachain complexes $(\varepsilon\nu_\phi)\colon C_\bullet^\phi(\Gamma_\phi) \to C_\bullet^\flat(\Gamma_\phi)$, where

$$\nu_\phi\colon C_\bullet^\phi(\Gamma_\phi) \to C_\bullet(\Gamma_\phi) \quad \text{and} \quad \varepsilon\colon C_\bullet(\Gamma_\phi) \to C_\bullet^\flat(\Gamma_\phi)$$

are projections given by

$$\nu_\phi(\psi_0,\dots,\psi_m) = \frac{1}{r^{m+1}} \sum_{0\le j\le m} \sum_{0\le \ell_j\le r-1} (\phi^{\ell_0}\psi_0,\dots,\phi^{\ell_m}\psi_m), \tag{10.4}$$

$$\varepsilon(\psi_0,\dots,\psi_m) = \frac{1}{(m+1)!} \sum_{\sigma\in\mathfrak{S}_m} (\psi_{\sigma^{-1}(0)},\dots,\psi_{\sigma^{-1}(m)}), \quad \psi_j \in \Gamma_\phi, \tag{10.5}$$

where $\mathfrak{S}_m$ is the group of permutations of $\{0,\dots,m\}$. This gives rise to explicit chain homotopy equivalences between the cyclic complexes of $C^\phi(\Gamma_\phi)$ and $C^\flat(\Gamma_\phi)$, as well as between their periodic cyclic complexes ([23, 65, 74]).

As ϕ has finite order, its action on M is clean. We define the equivariant bicomplex $C_{\Gamma_\phi}(M^\phi)$ and the equivariant mixed bicomplex $C(\Gamma_\phi, M^\phi)$ as the direct sums of the bicomplexes $C_{\Gamma_\phi}(M_a^\phi)$ and $C(\Gamma_\phi, M_a^\phi)$, respectively. This enables us to define the equivariant cohomology $H^\bullet_{\Gamma_\phi}(M^\phi)$ and mixed equivariant homology $H_\bullet^{\Gamma_\phi}(M^\phi)^\natural$ of M^ϕ. We observe that the mixed bicomplex $C(\Gamma_\phi, M^\phi)$ is the tensor product over Γ_ϕ of the mixed complexes $C^\flat(\Gamma_\phi)$ and $(\Omega(M^\phi), 0, d)$. Therefore, we obtain a mixed complex map

$$(\varepsilon\nu_\phi)\otimes\alpha^\phi\colon \mathrm{Tot}_\bullet(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})) \to \mathrm{Tot}_\bullet(C(\Gamma_\phi, M^\phi)),$$

which is shown to be a quasi-isomorphism [75]. Combining this with the quasi-isomorphisms (10.2) we then obtain the following result.

Theorem 10.2 ([75]). *Let $\phi\in\Gamma$ have finite order.*

(1) *The following chain maps are quasi-isomorphisms,*

$$\mathrm{Tot}_\bullet\left(C(\Gamma_\phi, M^\phi)\right)^\natural \xleftarrow{(\varepsilon\nu_\phi)\otimes\alpha^\phi} \mathrm{Tot}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi,\mathcal{A})\right)^\natural \xrightarrow{\mu_\phi\circ\sqcup^\natural} \mathbf{C}_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{10.6}$$

(2) *There are similar quasi-isomorphisms between the respective periodic cyclic complexes.*

(3) *This gives rise to isomorphisms,*

$$\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq H_\bullet^{\Gamma_\phi}(M^\phi)^\natural, \qquad \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq H^{\Gamma_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp.$$

Remark 10.3. Brylinski–Nistor [22] expressed $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ and $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ in terms of the equivariant homology of M^ϕ. We obtain explicit quasi-isomorphisms with the equivariant homology complex by combining the quasi-isomorphisms (10.6) with the Poincaré duality for the de Rham complex $\Omega(M^\phi)$. In particular, this enables us to recover the results of [22].

Remark 10.4. When $\phi = 1$ the cyclic space embedding μ_1 actually is an isomorphism and has an explicit inverse $\mu_1^{-1}\colon \mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[1]} \to \mathrm{Diag}_\bullet(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A}))$. Therefore, in this case we obtain an explicit quasi-isomorphism,

$$\mu_1^{-1} \circ \mathrm{AW}^\natural \circ (\varepsilon \otimes \alpha)\colon \mathbf{C}_\bullet(\mathcal{A}_\Gamma)_{[1]} \longrightarrow \mathrm{Tot}_\bullet\,(C(\Gamma, M))^\natural\,.$$

There is a similar quasi-isomorphism at the periodic cyclic level. By duality this gives rise to a map from the Γ-equivariant cohomology of M into the cyclic cohomology of $\mathcal{A}_\Gamma$. This provides us with an alternative to the cochain map of Connes [31, 32].

Remark 10.5. When Γ_ϕ is finite, we have an explicit quasi-isomorphism of mixed complexes from $\mathrm{Tot}(C(\Gamma_\phi, M^\phi))$ to the Γ_ϕ-invariant de Rham mixed complex $(\Omega^\bullet(M^\phi)^{\Gamma_\phi}, 0, d)$. This allows us to express the cyclic homology and periodic cyclic homology of $\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$ in terms of the invariant de Rham cohomology $H^\bullet(M^\phi)^{\Gamma_\phi}$. When Γ is finite, by using the splitting (10.1) we obtain an explicit quasi-isomorphisms of mixed complexes between $\mathbf{C}(\mathcal{A}_\Gamma)$ and the orbifold de Rham mixed complex $(\Omega^\bullet(M/\Gamma), 0, d)$, where $\Omega^\bullet(M/\Gamma) := \bigoplus_a \Omega^\bullet(M^\phi)^{\Gamma_\phi}$. This allows us to recover the identification by Baum–Connes [10] of $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)$ with the orbifold cohomology $H^{\mathrm{ev/odd}}(M/\Gamma)$.

Let $\eta^\phi\colon H^{\Gamma_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp \to \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ be the isomorphism given by Theorem 10.2. Composing it with the cap product (9.7) arrives at the following statement.

Corollary 10.6 ([75]). *Let $\phi \in \Gamma$ have finite order. Then we have a graded bilinear graded map,*

$$\eta^\phi(- \frown -)\colon H^{\mathrm{ev/odd}}_{\Gamma_\phi}(M^\phi) \times H_{\mathrm{ev/odd}}(\Gamma_\phi, \mathbb{C}) \longrightarrow \mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}.$$

In particular, equivariant characteristic classes naturally give rise to classes in $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$.

The definition of the isomorphism η^ϕ involve a cylindrical version of the shuffle map. Although explicit, this map involves numerous terms. As it turns out, we actually obtain a very simple and concise formula when we pair η^ϕ with cochains arising from equivariant currents as follows.

Let $\Omega^\Gamma(M) = (\Omega^\Gamma_\bullet(M), d)$ be the cochain complex of equivariant currents, where $\Omega^\Gamma_m(M)$, $m \geq 0$, consists of maps $C\colon \Gamma \to \Omega_m(M)$ that are Γ-equivariant in the sense that

$$C(\psi^{-1}\phi_0\psi) = \psi_*[C(\phi_0)] \qquad \text{for all } \phi_0, \psi \in \Gamma.$$

Any equivariant current $C \in \Omega^\Gamma_m(M)$ defines a cochain $\varphi_C \in C^m(\mathcal{A}_\Gamma)$ given by

$$\varphi_C(f^0 u_{\psi_0}, \dots, f^m u_{\psi_m}) = \frac{1}{m!}\langle C(\psi), f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^m\rangle, \qquad f^j \in \mathcal{A},\ \psi_j \in \Gamma, \quad (10.7)$$

where we have set $\psi = \psi_0 \cdots \psi_m$ and $\widehat{f}^j = f^j \circ (\psi_0 \cdots \psi_{j-1})^{-1}$. This provides us with a map of mixed complexes from $(\Omega^\Gamma_\bullet(M), d, 0)$ to $(C^\bullet(\mathcal{A}_\Gamma), B, b)$. Therefore, we obtain cochain maps between their respective cyclic and periodic cochain complexes. Note that the periodic cyclic complex of $\Omega^\Gamma(M)$ is just $(\Omega^\Gamma_{\mathrm{ev/odd}}(M), d)$, with

$$\Omega^\Gamma_{\mathrm{ev/odd}}(M) = \bigoplus_{m \text{ even/odd}} \Omega^\Gamma_m(M).$$

In what follows, given any equivariant chain $\omega \in C^{\mathrm{ev/odd}}(\Gamma_\phi, M^\phi)$, we denote by ω_0 its component in $C_{0,\bullet}(\Gamma_\phi, M^\phi) \simeq \Omega^\bullet(M^\phi)$. In addition, when ω is a cycle we shall denote by $[\omega]$ its class in $H^{\mathrm{ev/odd}}_{\Gamma_\phi}(M^\phi)^\sharp$.

Proposition 10.7 ([75]). *Let $\phi \in \Gamma$ have finite order. Then, for any closed equivariant current $C \in \Omega^\Gamma_{\mathrm{ev/odd}}(M)$ and any mixed equivariant cycle $\omega \in C_{\mathrm{ev/odd}}(\Gamma_\phi, M^\phi)$, we have*

$$\langle \varphi_C, \eta^\phi([\omega]) \rangle = \langle C(\phi), \widetilde{\omega}_0 \rangle ,$$

where $\widetilde{\omega}_0 \in \Omega^{\mathrm{ev}}(M)$ is such that $\iota^*_{M^\phi} \widetilde{\omega}_0 = \omega_0$.

10.7. The cyclic space $\mathbf{C}(\mathcal{A}_\Gamma)_{[\phi]}$ when ϕ has infinite order. Let $\phi \in \Gamma$ have infinite order. Set $\overline{\Gamma}_\phi = \Gamma_\phi / \langle \phi \rangle$, where $\langle \phi \rangle$ is the subgroup generated by ϕ. The canonical projection $\overline{\pi}: \Gamma_\phi \to \overline{\Gamma}_\phi$ gives rise to a $(\Gamma_\phi, \overline{\Gamma}_\phi)$-equivariant paracyclic Γ_ϕ-space map

$$\overline{\pi}: C^\phi_\bullet(\Gamma_\phi) \to C_\bullet(\overline{\Gamma}_\phi).$$

Composing this map with the projection $\pi^\natural: C_\bullet(\overline{\Gamma}_\phi)^\natural \to C_\bullet(\overline{\Gamma}_\phi)$ and the antisymmetrization map $\varepsilon: C_\bullet(\overline{\Gamma}_\phi) \to C_\bullet(\overline{\Gamma}_\phi)$ given by (10.5), we then obtain a $(\Gamma_\phi, \overline{\Gamma}_\phi)$-equivariant chain map

$$\overline{\varepsilon}^\natural: C^\phi_\bullet(\Gamma_\phi)^\natural \to C_\bullet(\overline{\Gamma}_\phi).$$

In addition, let $u_\phi \in C^2(\overline{\Gamma}_\phi, \mathbb{Z})$ be a 2-cocycle representing the Euler class $e_\phi \in H^2(\overline{\Gamma}_\phi, \mathbb{Z})$ of the Abelian extension (cf. [19, 80])

$$1 \to \langle \phi \rangle \to \overline{\Gamma}_\phi \to \Gamma_\phi \to 1.$$

The cap product with u_ϕ then gives rise to a chain map

$$u_\phi \frown -: C_\bullet(\overline{\Gamma}_\phi) \to C_{\bullet-2}(\overline{\Gamma}_\phi).$$

Then there is an explicit chain homotopy

$$h_\phi: C^\phi_\bullet(\Gamma_\phi)^\natural \to C_{\bullet-1}(\overline{\Gamma}_\phi)$$

such that

$$\overline{\varepsilon}^\natural S - (u_\phi \frown -)\overline{\varepsilon}^\natural = \partial h_\phi + h_\phi(\partial + B_\phi S),$$

where B_ϕ is the B-differential of $C^\phi(\Gamma_\phi)$ (see [58, 74]).

Assume further that ϕ acts cleanly on M. As ϕ acts trivially on the fixed-point set M^ϕ, the action of Γ_ϕ on $\Omega^\bullet(M^\phi)$ descends to an action of $\overline{\Gamma}_\phi$. Moreover, as $u_\phi \frown -$ is a chain map of degree -2, we get an S-module

$$C^\sigma(\overline{\Gamma}_\phi) = (C_\bullet(\overline{\Gamma}_\phi), \partial, u_\phi \frown -).$$

The triangular S-module $C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))$ is obtained as the tensor product over $\overline{\Gamma}_\phi$ of $C^\sigma(\overline{\Gamma}_\phi)$ with the de Rham mixed complex $(\Omega^\bullet(M^\phi), 0, d)$. Namely,

$$C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi)) = \big(C^\sigma_{\bullet,\bullet}(\overline{\Gamma}_\phi, \Omega(M^\phi)), \partial, 0, d, u_\phi \frown -\big),$$

where

$$C^\sigma_{p,q}(\overline{\Gamma}_\phi, \Omega(M^\phi)) = C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi).$$

Its total S-module is

$$\big(\mathrm{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))), d^\dagger, u_\phi \frown -\big),$$

where

$$\mathrm{Tot}_m\big(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))\big) = \bigoplus_{p+q=m} C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi)$$

and $d^\dagger = \partial + (-1)^p d(u_\phi \frown -)$ on $C_p(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi)$.

We observe that

$$\begin{aligned}
\mathrm{Tot}_m\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural &= \mathrm{Tot}_m\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big) \oplus \mathrm{Tot}_{m-2}\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural \cdots \\
&= \bigoplus_{m-(p+q)\in 2\mathbb{N}_0} C_p(\Gamma_\phi) \otimes_{\Gamma_\phi} \mathbf{C}_q(\mathcal{A}) \\
&= \bigoplus_{p+q=m} \big(C_p(\Gamma_\phi) \oplus C_{p-2}(\Gamma_\phi) \oplus \cdots\big) \otimes_{\Gamma_\phi} \mathbf{C}_q(\mathcal{A}) \\
&= \bigoplus_{p+q=m} C^\phi_p(\Gamma_\phi)^\natural \otimes_{\Gamma_\phi} \mathbf{C}_q(\mathcal{A}).
\end{aligned}$$

Using this we get a chain map

$$\theta(1 \otimes \alpha^\phi)\colon \mathrm{Tot}_\bullet\big(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\big)^\natural \to \mathrm{Tot}_\bullet\big(C^\sigma(\overline{\Gamma}_\phi, \Omega(M^\phi))\big),$$

where α^ϕ is given by (10.3) and

$$\theta\colon C^\phi_\bullet(\Gamma_\phi) \otimes \Omega^\bullet(M^\phi) \to C_\bullet(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^\bullet(M^\phi)$$

is given by

$$\theta = \overline{\varepsilon}^\natural \otimes 1 + (-1)^{p-1}(h_\phi \otimes d) \quad \text{on } C^\phi_p(\Gamma_\phi) \otimes_{\overline{\Gamma}_\phi} \Omega^q(M^\phi).$$

We then have the following result.

Theorem 10.8 ([75]). *Let $\phi \in \Gamma$ have infinite order and act cleanly on M.*

(1) *The following chain maps are quasi-isomorphisms,*

$$\mathrm{Tot}_\bullet\left(C^\sigma(\bar{\Gamma}_\phi, \Omega(M^\phi))\right) \xleftarrow{\theta(1\otimes\alpha)} \mathrm{Tot}_\bullet\left(\mathbf{C}^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi \circ \sqcup^\natural} C_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{10.8}$$

(2) *This gives rise to an isomorphism,*

$$\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq H_\bullet\big(\mathrm{Tot}\left(C^\sigma(\bar{\Gamma}_\phi, \Omega(M^\phi))\right)\big). \tag{10.9}$$

(3) *Under this isomorphism, the periodicity operator of $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ is given by the cap product,*

$$e_\phi \frown - : H_\bullet\left(\mathrm{Tot}(C^\sigma(\bar{\Gamma}_\phi, \Omega(M^\phi)))\right) \longrightarrow H_{\bullet-2}\big(\mathrm{Tot}\left(C^\sigma(\bar{\Gamma}_\phi, \Omega(M^\phi))\right)\big),$$

where $e_\phi \in H^2(\bar{\Gamma}_\phi, \mathbb{Z})$ is the Euler class of the extension

$$1 \to \langle\phi\rangle \to \bar{\Gamma}_\phi \to \Gamma_\phi \to 1.$$

In particular, $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} = 0$ whenever e_ϕ is nilpotent in $H^\bullet(\bar{\Gamma}_\phi, \mathbb{C})$.

Remark 10.9. The quasi-isomorphisms (10.8) and the filtration by columns of $\mathrm{Tot}_\bullet(C^\sigma(\bar{\Gamma}_\phi, \Omega(M^\phi)))$ give rise to a spectral sequence,

$$E^2_{p,q} = H_p\big(\bar{\Gamma}_\phi, \Omega^q(M^\phi)\big) \Longrightarrow \mathbf{HC}_{p+q}(\mathcal{A}_\Gamma)_{[\phi]},$$

where the d^2-differential is

$$(-1)^p d(u_\phi \frown -): H_p\big(\bar{\Gamma}_\phi, \Omega^q(M^\phi)\big) \to H_{p-2}\big(\bar{\Gamma}_\phi, \Omega^{q+1}(M^\phi)\big).$$

Crainic [40] obtained such a spectral sequence. He inferred from this that

$$\mathbf{HC}_m(\mathcal{A}_\Gamma)_{[\phi]} \simeq \oplus_{p+q=m} H_p\big(\bar{\Gamma}_\phi, \Omega^q(M^\phi)\big)$$

(see [40, Corollary 4.15]). What we really have is the isomorphism (10.9).

Remark 10.10. When $\phi \in \Gamma$ has infinite order and does not act cleanly on M, we still have a spectral sequence

$$E^2_{p,q} = H_p\big(\bar{\Gamma}_\phi, H^\phi_q(\mathcal{A})\big) \Longrightarrow \mathbf{HC}_{p+q}(\mathcal{A}_\Gamma)_{[\phi]},$$

where $H^\phi_\bullet(\mathcal{A})$ is the homology of the twisted Hochschild complex $(\mathbf{C}^\phi_\bullet(\mathcal{A}), b_\phi)$ (see [40, 75]). In addition, it can be shown that $\mathbf{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ is a module over the cohomology ring $H^\bullet(\bar{\Gamma}_\phi, \mathbb{C})$ and its periodicity operator is given by the action of the Euler class e_ϕ (see [40, 70, 75]). In particular, $\mathbf{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} = 0$ whenever e_ϕ is nilpotent in $H^\bullet(\bar{\Gamma}_\phi, \mathbb{C})$. In [75] this action arises from the construction in [74] of an explicit coproduct at the level of chains.

Remark 10.11. The nilpotence of the Euler class e_ϕ in $H^\bullet(\Gamma, \mathbb{Q})$ is closely related to the Bass and idempotent conjectures (see, e.g. [43,58]). In particular, e_ϕ is rationally nilpotent for every infinite order element $\phi \in \Gamma$ whenever Γ belongs to one of the following classes of groups: free products of Abelian groups, hyperbolic groups, and arithmetic groups.

11. Conformal invariants

In this section, we shall use the results of the previous section to compute the conformal invariants produced by Theorem 7.7. This will provide us with a new family of geometric conformal invariants associated with equivariant classes and conformal diffeomorphisms.

Throughout this section we let M^n be an even dimensional spin oriented compact manifold equipped with a non-flat conformal structure $\mathcal{C}$. In addition, we let G be the (identity component) of the group of orientation-preserving diffeomorphisms of M preserving the conformal structure $\mathcal{C}$ and the spin structure of M.

11.1. Conformal invariants from mixed equivariant cycles. Let $\phi \in G$ have finite order. We denote by G_ϕ its centralizer in G. As in Section 10, given any mixed equivariant chain $\omega \in C_{\mathrm{ev/odd}}(G_\phi, M^\phi)$ we denote by ω_0 its component in

$$C_0(G_\phi, \mathbb{C}) \otimes_{G_\phi} \Omega^{\mathrm{ev/odd}}(M^\phi) \simeq \Omega^{\mathrm{ev/odd}}(M^\phi).$$

By Theorem 10.2 we have an isomorphism

$$\eta^\phi \colon H^{G_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp \to \mathbf{HP}_\bullet(\mathcal{A}_G)_{[\phi]}.$$

As $\mathbf{HP}_\bullet(\mathcal{A}_G)_{[\phi]}$ is a direct summand of $\mathbf{HP}_\bullet(\mathcal{A}_G)$ we obtain a linear embedding

$$\eta^\phi \colon H^{G_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp \hookrightarrow \mathbf{HP}_\bullet(\mathcal{A}_G).$$

Given any metric $g \in \mathcal{C}$ and any even mixed equivariant cycle $\omega \in C^{G_\phi}_{\mathrm{ev}}(G_\phi, M^\phi)$, we set

$$I^\phi_g(\omega) := \left\langle \mathbf{Ch}(\not{D}_g)_{\sigma_g}, \eta^\phi[\omega] \right\rangle,$$

where $[\omega]$ is the class of ω in $H^{G_\phi}_{\mathrm{ev/odd}}(M^\phi)^\sharp$, and $\mathbf{Ch}(\not{D}_g)_{\sigma_g} \in \mathbf{HP}^0(\mathcal{A}_G)$ is the Connes–Chern character of the conformal twisted spectral triple

$$\left(C^\infty(M), L^2_g(M, \not{S}), \not{D}_g\right)_{\sigma_g}$$

associated with g.

Theorem 11.1. *Let $\phi \in G$ have finite order. In addition, let $\omega \in C_{\mathrm{ev}}^{G_\phi}(G_\phi, M^\phi)$ be a mixed equivariant cycle.*

(1) *The scalar $I_g^\phi(\omega)$ is an invariant of the conformal class $\mathcal{C}$. Moreover, it depends only on the class of ω in the even mixed equivariant homology $H_{\mathrm{ev}}^{G_\phi}(M^\phi)^\sharp$.*

(2) *For any G-invariant metric $g \in \mathcal{C}$, we have*

$$I_g^\phi(\omega) = \int_{M^\phi} \Upsilon^\phi(R) \wedge \omega_0, \tag{11.1}$$

where $\Upsilon^\phi(R)$ is defined by (8.3) and the integral has the same meaning as in (8.1).

Proof. By Theorem 7.7 the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g} \in \mathbf{HP}^0(\mathcal{A}_G)$ is an invariant of the conformal class $\mathcal{C}$. Therefore, $I_g^\phi(\omega) = \langle \mathbf{Ch}(\not{D}_g)_{\sigma_g}, \eta^\phi[\omega]\rangle$ depends only on the conformal class $\mathcal{C}$ and the class of ω in $H_{\mathrm{ev}}^{G_\phi}(M^\phi)^\sharp$. This proves the first part.

As for the second part, let g be a G-invariant metric in $\mathcal{C}$. By Theorem 8.4 the Connes–Chern character $\mathbf{Ch}(\not{D}_g)_{\sigma_g} \in \mathbf{HP}^0(\mathcal{A}_G)$ is represented by the CM cocycle $\varphi^{\mathrm{CM}} \in \mathbf{C}^0(\mathcal{A}_G)^\sharp$ associated with g by (8.4). Thus,

$$I_g^\phi(\omega) = \langle [\varphi^{\mathrm{CM}}], \eta^\phi[\omega]\rangle. \tag{11.2}$$

Recall that, given any $\phi_0 \in G$, the differential form $\Upsilon^{\phi_0}(R) \in \Omega^{\mathrm{ev}}(M^{\phi_0})$ is defined by

$$\Upsilon^{\phi_0}(R)\big|_{M_a^{\phi_0}} = (-i)^{\frac{n}{2}}(2\pi)^{-\frac{a}{2}}\widehat{A}\big(R^{TM_a^{\phi_0}}\big) \wedge \mathcal{V}_{\phi_0}\big(R^{\mathcal{N}_a^{\phi_0}}\big), \qquad a = 0, 2, \ldots, n,$$

where the differential forms $\widehat{A}(R^{TM_a^{\phi_0}})$ and $\mathcal{V}_{\phi_0}(R^{\mathcal{N}_a^{\phi_0}})$ are given by (8.2). Note that $\widehat{A}(R^{TM_a^{\phi_0}})$ and $\mathcal{V}_{\phi_0}(R^{\mathcal{N}_a^{\phi_0}})$ are characteristic differential forms for the vector bundles $TM_a^{\phi_0}$ and $\mathcal{N}_a^{\phi_0}$, respectively. In particular, these differential forms are closed, and hence, $\Upsilon^{\phi_0}(R)$ is closed differential form on M^{ϕ_0}.

Let $\psi \in G$ and set $\phi_1 = \psi^{-1}\phi_0\psi$. For $a = 0, 2, \ldots, n$, the diffeomorphism ψ induces an isometric diffeomorphism from $M_a^{\phi_0}$ onto $M_a^{\phi_1}$, and, hence,

$$\psi_*\widehat{A}\big(R^{TM_a^{\phi_0}}\big) = \widehat{A}\big(R^{TM_a^{\phi_1}}\big).$$

As the action of ψ on TM is isometric, it also induces a vector bundle isomorphism $\psi_*\colon \mathcal{N}_a^{\phi_0} \to \mathcal{N}_a^{\phi_1}$ such that $\psi_*\nabla^{\mathcal{N}_a^{\phi_0}} = \nabla^{\mathcal{N}_a^{\phi_1}}$ and $(\psi_*)^{-1}\phi_0^{\mathcal{N}}\psi_* = \phi_1^{\mathcal{N}}$ on $\mathcal{N}_a^{\phi_1}$. Thus,

$$\begin{aligned}\psi_*\mathcal{V}_{\phi_0}(R^{\mathcal{N}_a^{\phi_0}}) &= \psi_*\det{}^{-\frac{1}{2}}\big(1 - \phi_0^{\mathcal{N}}e^{-R^{\mathcal{N}_a^{\phi_0}}}\big)\\ &= \det{}^{-\frac{1}{2}}\big(1 - \phi_1^{\mathcal{N}}e^{-R^{\mathcal{N}_a^{\phi_1}}}\big) = \mathcal{V}_{\phi_1}(R^{\mathcal{N}_a^{\phi_1}}).\end{aligned}$$

It then follows that

$$\psi_* \Upsilon^{\phi_0}(R) = \Upsilon^{\psi^{-1}\phi_0\psi}(R), \qquad \forall \psi \in G. \tag{11.3}$$

Let $\Omega^G(M) = (\Omega^G_\bullet(M), d)$ be the G-equivariant de Rham complex of M as defined in Section 10. Let $C: G \to \Omega_{\mathrm{ev}}(M)$ be the map defined by

$$\langle C(\phi_0), \xi \rangle = \int_{M^{\phi_0}} \Upsilon^{\phi_0}(R) \wedge \iota^*_{M^{\phi_0}} \xi, \qquad \xi \in \Omega^{\mathrm{ev}}(M),\ \phi_0 \in G.$$

For each $\phi_0 \in G$, this defines a current $C(\phi_0) \in \Omega_{\mathrm{ev}}(M)$ which is the image of the Poincaré dual of $\Upsilon^{\phi_0}(R)$ under the canonical embedding

$$(\iota_{M^{\phi_0}})_*: \Omega_\bullet(M^{\phi_0}) \to \Omega_\bullet(M).$$

In particular, we see that $C(\phi_0)$ is a closed de Rham current. In addition, let $\psi \in G$ and $\phi_1 = \psi^{-1}\psi_0\psi$. Then the G-equivariance (11.3) of C implies that

$$\begin{aligned}\langle C(\phi_0), \psi_* \xi \rangle &= \int_{M^{\phi_0}} \psi^*\big(\Upsilon^{\phi_0}(R) \wedge \iota^*_{M^{\phi_0}} \xi\big) \\ &= \int_{M^{\phi_1}} \Upsilon^{\phi_1}(R) \wedge \iota^*_{M^{\phi_1}} \xi = \langle C(\phi_1), \xi \rangle .\end{aligned}$$

This shows that $C: G \to \Omega_{\mathrm{ev}}(M)$ is a closed equivariant current in $\Omega^G_{\mathrm{ev}}(M)$.

We observe that it follows from (10.7) and (8.4) that $\varphi^{\mathrm{CM}} = \varphi_C$. Combining this with Proposition 10.7 and (11.2) we then get

$$I^\phi_g(\omega) = \langle \varphi_C, \eta^\phi[\omega] \rangle = \langle C(\phi), \widetilde{\omega}_0 \rangle ,$$

where $\widetilde{\omega}_0 \in \Omega^{\mathrm{ev}}(M)$ is such that $\iota^*_{M^\phi} \widetilde{\omega}_0 = \omega_0$. In view of the definition of $C(\phi)$ we obtain

$$I^\phi_g(\omega) = \int_{M^\phi} \Upsilon^\phi(R) \wedge \iota^*_{M^\phi} \widetilde{\omega}_0 = \int_{M^\phi} \Upsilon^\phi(R) \wedge \omega_0.$$

This proves (11.1) and completes the proof. □

Combining Theorem 11.1 with Corollary 10.6 we immediately get the following corollary.

Corollary 11.2. *Let $\phi \in G$ have finite order. In addition, let ω be a closed equivariant cocycle in $C^{\mathrm{ev}}_{G_\phi}(M^\phi)$ (resp., $C^{\mathrm{odd}}_{G_\phi}(M^\phi)$) and γ a group cycle in $C_p(G_\phi, \mathbb{C})$ with p even (resp., odd).*

(1) *The scalar $I^\phi_g(\omega \frown \gamma)$ is an invariant of the conformal class $\mathcal{C}$ which depends only on the class of ω in the equivariant cohomology $H^{\mathrm{ev/odd}}_{G_\phi}(M^\phi)$ and on the class of γ in the group homology $H_p(G_\phi, \mathbb{C})$.*

(2) *For any G-invariant metric $g \in \mathcal{C}$, we have*

$$I^\phi_g(\omega \frown \gamma) = \int_{M^\phi} \Upsilon^\phi(R) \wedge (\omega \frown \gamma)_0, \tag{11.4}$$

11.2. Conformal invariants and equivariant characteristic classes. Given $a \in \{0, 2, \dots, n\}$, let E be a G_ϕ-equivariant vector bundle over M_a^ϕ equipped with a connection ∇^E. Let $\mathrm{Ch}_{G_\phi}(\nabla^E) \in C^{\mathrm{ev}}_{G_\phi}(M_a^\phi)$ be the equivariant Chern form associated with ∇^E (cf. Section 9). As mentioned in Section 9, it represents the equivariant Chern character of E in $H^{\mathrm{ev}}_{G_\phi}(M_a^\phi)$. By construction

$$\mathrm{Ch}_{G_\phi}(\nabla^E) = (\mathrm{Ch}^p_{G_\phi}(\nabla^E))_{p\geq 0},$$

where $\mathrm{Ch}^p_{G_\phi}(\nabla^E) \in C^p(G_\phi, \Omega^\bullet(M^\phi))$ is given by (9.5) for $p = 0$ and by (9.6) for $p \geq 1$. Thus,

$$\big(\mathrm{Ch}_{G_\phi}(\nabla^E) \frown 1\big)_0 = \mathrm{Ch}_{G_\phi}(\nabla^E)(1) = \mathrm{Ch}(\nabla^E).$$

Combining this with (11.4) we then arrive at the following statement.

Proposition 11.3. *Let $\phi \in G$ have finite order. In addition, let E be a G_ϕ-equivariant vector bundle over M_a^ϕ for some $a \in \{0, 2, \dots, n\}$. Then, for every G-invariant metric $g \in \mathcal{C}$ and every connection ∇^E on E, we have*

$$I_g^\phi\big(\mathrm{Ch}_{G_\phi}(\nabla^E) \frown 1\big) = \int_{M_a^\phi} \Upsilon^\phi(R) \wedge \mathrm{Ch}(\nabla^E). \tag{11.5}$$

The right-hand side of (11.5) has a natural interpretation as a generalized Lefschtez number as follows. Let E be a G-equivariant vector bundle over M equipped with a connection ∇^E. For $a = 0, 2, \dots, n$, we denote by E_a^ϕ the restriction of E to M_a^ϕ. This is a G_ϕ-equivariant vector bundle over M_a^ϕ. The connection ∇^E induces a G_ϕ-invariant connection $\nabla^{E_a^\phi}$ on each vector bundle E_a^ϕ. We also set $E^\phi = \bigsqcup_a E_a^\phi$. This is a G_ϕ-equivariant (topological) vector bundle over M^ϕ. We equip E^ϕ with the connection $\nabla^E = \bigoplus_a \nabla^{E_a^\phi}$. Recall that

$$C^{\mathrm{ev}}_{G_\phi}(M^\phi) = \bigoplus_a C^{\mathrm{ev}}_{G_\phi}(M_a^\phi) \quad \text{and} \quad H^{\mathrm{ev}}_{G_\phi}(M^\phi) = \bigoplus_a H^{\mathrm{ev}}_{G_\phi}(M_a^\phi).$$

We then define the G_ϕ-equivariant Chern character of E^ϕ by

$$\mathrm{Ch}_{G_\phi}(E^\phi) := \sum_a \mathrm{Ch}_{G_\phi}(E^\phi) \in H^{\mathrm{ev}}_{G_\phi}(M^\phi).$$

It is represented by the equivariant Chern form,

$$\mathrm{Ch}_{G_\phi}(\nabla^{E^\phi}) := \sum_a \mathrm{Ch}_{G_\phi}(\nabla^{E_a^\phi}) \in C^{\mathrm{ev}}_{G_\phi}(M^\phi).$$

We have

$$I_g\big(\mathrm{Ch}_{G_\phi}(\nabla^{E^\phi}) \frown 1\big) = \sum_a I_g\big(\mathrm{Ch}_{G_\phi}(\nabla^{E_a^\phi}) \frown 1\big).$$

Therefore, by using (11.5) we get

$$I_g\big(\mathrm{Ch}_{G_\phi}(\nabla^{E^\phi}) \frown 1\big) = \sum_a \int_{M_a^\phi} \Upsilon^\phi(R) \wedge \mathrm{Ch}(\nabla^{E_a^\phi})$$
$$= \int_{M^\phi} \Upsilon^\phi(R) \wedge \iota_{M^\phi}^* \mathrm{Ch}(\nabla^E). \tag{11.6}$$

Let us further assume that E is equipped with a G-invariant Hermitian metric and ∇^E is a G-invariant Hermitian connection on E. In particular, any diffeomorphism $\phi \in G$ lifts to a unitary vector bundle isomorphism $\phi^E : E \to \phi_* E$. Given any G-invariant metric $g \in \mathscr{C}$, we then get a unitary representation $\phi \to U_\phi^E$ of G into $L_g^2(M, \$ \otimes E)$ given by

$$(U_\phi^E \xi)(x) = (\phi^\$ \otimes \phi^E)[\xi(\phi^{-1}(x))] \qquad \text{for all } \xi \in L_g^2(M, \$ \otimes E).$$

Note also that U_ϕ preserves the $\mathbb{Z}_2$-grading,

$$L_g^2(M, \$ \otimes E) = L_g^2(M, \$^+ \otimes E) \oplus L_g^2(M, \$^- \otimes E). \tag{11.7}$$

Let $\not{D}_{\nabla^E} : C^\infty(M, \$ \otimes E) \to C^\infty(M, \$ \otimes E)$ be the Dirac operator associated with the Hermitian connection ∇^E. Namely,

$$\not{D}_{\nabla^E} = \not{D}_g \otimes 1_E + (c \otimes 1_E) \circ \nabla^E,$$

where $\not{D}_g : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of M and $c : T^*M \times \$ \to \$$ is the Clifford action of T^*M on $\$$. With respect to the splitting (11.7) the operator $\not{D}_{\nabla^E}$ takes the form,

$$\not{D}_{\nabla^E} = \begin{pmatrix} 0 & \not{D}_{\nabla^E}^- \\ \not{D}_{\nabla^E}^+ & 0 \end{pmatrix}.$$

As $\not{D}_{\nabla^E}$ is G-invariant, the nullspaces $\ker \not{D}_{\nabla^E}^\pm$ are preserved by the action of G associated with the representation $\phi \to U_\phi^E$. The *equivariant index* $\mathrm{ind}\, \not{D}_{\nabla^E} : G \to \mathbb{R}$ (a.k.a. generalized Lefschetz number) is then defined by

$$\mathrm{ind}\, \not{D}_{\nabla^E}(\phi) = \mathrm{Tr}\big[U^E_{\phi|\ker \not{D}_{\nabla^E}^+}\big] - \mathrm{Tr}\big[U^E_{\phi|\ker \not{D}_{\nabla^E}^-}\big] \qquad \forall \phi \in G.$$

When $\phi = \mathrm{id}_M$ we recover the Fredholm index of $\not{D}_{\nabla^E}$. When all the fixed-points of ϕ are isolated this agrees with the Lefschetz number of ϕ with respect to the Dirac complex with coefficients in E in the sense of Atiyah–Bott [2, 3]. The equivariant index theorem of Atiyah–Segal–Singer [5, 6] produces a geometric formula for the equivariant index. Namely,

$$\mathrm{ind}\, \not{D}_{\nabla^E}(\phi) = \int_{M^\phi} \Upsilon^\phi(R) \wedge \iota_{M^\phi}^* \mathrm{Ch}(\nabla^E).$$

Combining this formula with (11.6) we then arrive at the following result.

Theorem 11.4. *Let $\phi \in G$ have finite order. In addition, let E be a G-equivariant Hermitian vector bundle over M. Then, for every G-invariant metric $g \in \mathcal{C}$ and every G-invariant Hermitian connection ∇^E on E, we have*

$$I_g\big(\mathrm{Ch}_{G_\phi}(\nabla^{E^\phi}) \frown 1\big) = \int_{M^\phi} \Upsilon^\phi(R) \wedge \iota^*_{M^\phi} \mathrm{Ch}(\nabla^E) = \mathrm{ind}\, \not{D}_{\nabla^E}(\phi).$$

In particular, when the fixed-points of ϕ are isolated, the conformal invariant $I_g(\mathrm{Ch}_{G_\phi}(\nabla^{E^\phi}) \frown 1)$ is given by the Lefschetz number of ϕ with respect to the Dirac complex with coefficients in E.

11.3. Conformal indices. The conformal invariants $I_g^\phi(\omega)$ arising from Theorem 11.1 shed a new light on the conformal indices of Branson–Ørsted [16, 17]. To put things into context, let $P_g: C^\infty(M, E) \to C^\infty(M, E)$ be a natural differential operator of (even) order $m \leq n + 2$ with positive-definite leading symbol which is the power of some conformally covariant operator. Examples include the Yamabe and Paneitz operators, and more generally the conformal powers of the Laplacian of Graham–Jenne–Mason–Sparling [50]. Non-scalar examples include the square of the Dirac operator and the conformal powers of the Hodge Laplacian of Branson–Gover [15]. Such operators then admit a short-time heat-semigroup expansion,

$$\mathrm{Tr}\big[e^{-tP_g}\big] \sim \sum_{j \geq 0} t^{j-\widehat{m}} \int_M b_j(P_g)(x)|dx| \qquad \text{as } t \to 0^+.$$

Here $\widehat{m} = n + 2 - m$, $|dx|$ is the Riemmanian density, and $b_j(P_g)(x)$ is given by a universal expression in terms of the metric g and its derivatives. Consider the constant term in the asymptotic expansion above, i.e.

$$J(P_g) := \int_M b_{\widehat{m}}(P_g)(x)|dx|.$$

Branson–Ørsted [16] (see also Parker–Rosenberg [71]) showed that $J(P_g)$ is an invariant of the conformal class of g. Branson–Ørsted called it the *conformal index* of P_g.

More generally, let $\phi: M \to M$ be a smooth isometry which lifts to a unitary vector bundle isomorphism $\phi^E: E \to E$. Then, in the same way as in Section 7, this gives rise to a unitary operator $U_\phi: L^2(M, E) \to L^2(M, E)$. In addition, the fixed-point set of ϕ is a disjoint union $M^\phi = \bigsqcup M_a^\phi$, where M_a^ϕ is a submanifold of M of dimension a. It can be shown (see, e.g., [49, 78]) that we have a short-time asymptotic,

$$\mathrm{Tr}\big[e^{-tP_g} U_\phi\big] \sim \sum_{j \geq 0} \sum_{0 \leq a \leq n} t^{j-\widehat{m}(a)} \int_{M_a^\phi} b_j^{(a)}(P_g)(x)|dx| \qquad \text{as } t \to 0^+.$$

Here $\widehat{m}(a) = a + 2 - m$ and $b_j^{(a)}(P_g)(x)$ is given by a universal expression in terms of the metric g and the diffeomorphism ϕ and their partial derivatives. For $a = 0, \dots, n$, define

$$J_\phi^a(P_g) := \int_{M_a^\phi} b_{\widehat{m}(a)}^{(a)}(P_g)(x)|dx|.$$

Let us denote by $\mathcal{C}$ the conformal class of g and by $\mathcal{C}_\phi$ the sub-class consisting of ϕ-invariant metrics in $\mathcal{C}$. Thus, $\widehat{g} \in \mathcal{C}_\phi$ if and only if $\widehat{g} = k^{-2}g$ with $k \in C^\infty(M)$, $k > 0$ such that $k \circ \phi = k$.

Proposition 11.5 (Branson–Ørsted [17]). *For $a = 0, \dots, n$, the scalar $J_\phi^a(P_g)$ is an invariant of the sub-class $\mathcal{C}_\phi$.*

This result provides us with invariants of the sub-class $\mathcal{C}_\phi$ only. As we shall now see, in the case of the square of the Dirac operator, we actually obtain invariants of the whole conformal class $\mathcal{C}$ as follows.

We shall work in the original setup of this section, where M is a compact spin oriented Riemannian manifold of even dimension n and G is a group of diffeomorphisms preserving the spin structure and a given (non-flat) conformal structure $\mathcal{C}$. Let $\omega = \omega_0 + \omega_2 + \cdots + \omega_n$ be an even differential form on M, where $\omega_{2q} \in \Omega^{2q}(M)$. For $t > 0$ set

$$c_t(\omega) := c(\omega_0) + tc(\omega_2) + \cdots + t^n c(\omega_n),$$

where $c\colon \Omega^\bullet(M) \to C^\infty(M, \mathrm{End}(\not{S}))$ is the Clifford representation. The results of [78] then imply that, for any G-invariant metric $g \in \mathcal{C}$, we have

$$J_g^\phi(\omega) := \lim_{t\to 0^+} \mathrm{Str}\left[c_t(\omega)e^{-\not{D}_g} U_\phi\right] = \int_{M^\phi} \Upsilon^\phi(R) \wedge \iota_{M^\phi}^* \omega.$$

Note that if we take $\omega = 1$ on M_a^ϕ and $\omega = 0$ on M_b^ϕ with $b \neq a$, then we obtain a "super-symmetric" version of the conformal index $J_\phi^a(\not{D}_g^2)$. In fact, an extension of the arguments of [17] shows that $J_g^\phi(\omega)$ also is an invariant of the subclass $\mathcal{C}_\phi$.

Bearing this in mind, Theorem 11.1 asserts that, given any mixed equivariant cycle $\omega \in C_{\mathrm{ev}}^{G_\phi}(G_\phi, M^\phi)$, for every G-invariant metric $g \in \mathcal{C}$, we have

$$I_g^\phi(\omega) = \int_{M^\phi} \Upsilon^\phi(R) \wedge \omega_0 = J_g^\phi(\widetilde{\omega}_0),$$

where $\widetilde{\omega}_0$ is any differential form in $\Omega^{\mathrm{ev}}(M)$ such that $\iota_{M^\phi}^* \widetilde{\omega}_0$ is the component of ω in

$$C_0(G_\phi) \otimes_{G_\phi} \Omega^{\mathrm{ev}}(M^\phi) \simeq \Omega^{\mathrm{ev}}(M^\phi).$$

Therefore, we see that the (generalized) conformal indices $J_g^\phi(\omega)$ actually lead us to invariants of the *full* conformal class $\mathcal{C}$.

In addition, let $\mathcal{E}$ be a finitely generated product module over $\mathcal{A}_\Gamma$ and $\nabla^{\mathcal{E}}$ a connection on $\mathcal{E}$. Assume further that ω is such that $\eta^\phi(\omega) = \mathrm{Ch}(\mathcal{E})$. Then by the very definition of $I_g^\phi(\omega)$ we have

$$J_g^\phi(\widetilde{\omega}_0) = I_g^\phi(\omega) = \left\langle \mathbf{Ch}(\not{D}_g)_{\sigma_g}, \eta^\phi(\omega)\right\rangle = \left\langle \mathbf{Ch}(\not{D}_g)_{\sigma_g}, \mathrm{Ch}(\mathcal{E})\right\rangle = \mathrm{ind}\, \not{D}_{\nabla^{\mathcal{E}}}.$$

This shows that the generalized conformal index $J_g^\phi(\widetilde{\omega}_0)$ is a true Fredholm index. This confirms the intuition of Branson–Ørsted that their conformal indices could be interpreted as some sort of Fredholm indices.

11.4. Deser–Schwimmer conjecture. The conformal invariants arising from Theorem 11.1 are not the same kind of conformal invariants as those considered by Alexakis [1] in his long solution of the conjecture of Deser–Schwimmer [41]. This conjectures gives a description of *all* the global conformal invariants. Here by a global conformal invariant it is meant an invariant of the form,

$$\mathcal{I}_g = \int_M P(g)(x)|dx|, \tag{11.8}$$

where $P(g)(x)$ is a universal polynomial in the metric g and its derivatives in such a way that the above integral is an invariant of the conformal class of the metric g. The difference with the invariants from Theorem 11.1 lies in the fact that the latter takes into account the action of the conformal diffeomorphisms, i.e. the conformal gauge group. As mentioned in the introduction, this of some importance in the context of string theory and conformal gravity.

We obtain from Theorem 11.1 conformal invariants of the form (11.8) when $\phi = 1$. In this case, given any mixed equivariant cycle $\omega \in C_{\mathrm{ev}}(G, M)$, for every G-invariant metric $g \in \mathcal{C}$, we have

$$I_g^1(\omega) = \int_M \widehat{A}(R^M) \wedge \omega_0.$$

Note that the conformal invariance of the above integral is also an immediate consequence of the well-known conformal invariance of the $\widehat{A}$-form $\widehat{A}(R^M)$ (see Avez [7] and Chern–Simons [26]).

References

[1] A. Alexakis, *The decomposition of global conformal invariants*, Annals of Mathematics Studies, 182, Princeton University Press, Princeton, NJ, 2012. Zbl 1258.53003 MR 2918125

[2] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes. I, *Ann. of Math. (2)*, **86** (1967), 374–407. Zbl 0161.43201 MR 0212836

[3] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes. II. Applications, *Ann. of Math. (2)*, **88** (1968), 451–491. Zbl 0167.21703 MR 0232406

[4] M. F. Atiyah and I. M. Singer, The index of elliptic operators. I, *Ann. of Math. (2)*, **87** (1968),484–530. Zbl 0164.24001 MR 0236950

[5] M. F. Atiyah and G. B. Segal, The index of elliptic operators. II, *Ann. of Math. (2)*, **87** (1968), 531–545. Zbl 0164.24201 MR 0236951

[6] M. F. Atiyah and I. M. Singer, The index of elliptic operators. III, *Ann. of Math. (2)*, **87** (1968), 546–604. Zbl 0164.24301 MR 0236952

[7] A. Avez, Characteristic classes and Weyl tensor: Applications to general relativity, *Proc. Nat. Acad. Sci. U.S.A.*, **66** (1970), 265–268. Zbl 0197.26503 MR 263098

[8] F. Azmi, The equivariant Dirac cyclic cocycle, *Rocky Mountain J. Math.*, **30** (2000) 1171–1206. Zbl 0982.58017 MR 1810162

[9] A. Banyaga, On essential conformal groups and a conformal invariant, *J. Geom.*, **68** (2000), no. 1-2, 10–15. Zbl 0977.53027 MR 1779834

[10] P. Baum and A. Connes, Chern character for discrete groups, in *A fête of topology*, 163–232, Academic Press, Boston, MA, 1988. Zbl 0656.55005 MR 928402

[11] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 298, Springer-Verlag, Berlin, 1992. Zbl 0744.58001 MR 1215720

[12] J. Block and J. Fox, Asymptotic pseudodifferential operators and index theory, in *Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988)*, 1–32, Contemp. Math., 105, Amer. Math. Soc., Providence, RI, 1990. Zbl 0704.58048 MR 1047274

[13] R. Bott, On some formulas for the characteristic classes of group actions, in *Differential topology, foliations and Gelfand–Fuks cohomology*, 25–61, Lecture Notes in Math., 652, Springer-Verlag, 1978. Zbl 0393.57011 MR 505649

[14] J.-P. Bourguignon and P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, *Comm. Math. Phys.*, **144** (1992), no. 3, 581–599. Zbl 0755.53009 MR 1158762

[15] T. Branson and A. R. Gover, Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature, *Comm. Partial Differential Equations*, **30** (2005), no. 10-12, 1611–1669. Zbl 1226.58011 MR 2182307

[16] T. P. Branson and B. Ørsted, Conformal indices of Riemannian manifolds, *Compositio Math.*, **60** (1986), 261–293. Zbl 0608.58039 MR 869104

[17] T. P. Branson and B. Ørsted, Conformal geometry and global invariants, *Differential Geom. Appl.*, **1** (1991), no. 3, 279–308. Zbl 0785.53025 MR 1244447

[18] J. Brodzki, S. Dave, and V. Nistor, The periodic cyclic homology of crossed products of finite type algebras, *Adv. Math.*, **306** (2017), 494–523. Zbl 06666036 MR 3581309

[19] K. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87, Springer, 1982. Zbl 0584.20036 MR 672956

[20] J.-L. Brylinski, Cyclic homology and equivariant theories, *Ann. Inst. Fourier (Grenoble)*, **37** (1987), 15–28. Zbl 0625.55003 MR 927388

[21] J.-L. Brylinski, *Algebras associated with group actions and their homology*, preprint, Brown University, 1987.

[22] J.-L. Brylinski and V. Nistor, Cyclic cohomology of étale groupoids, *K-Theory*, **8** (1994), 341–365. Zbl 0812.19003 MR 1300545

[23] D. Burghelea, The cyclic homology of the group rings, *Comment. Math. Helv.*, **60** (1985), 354–365. Zbl 0595.16022 MR 814144

[24] D. Burghelea, Cyclic homology and the algebraic K-theory of spaces. I, in *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, 89–115, Contemp. Math., 55, Amer. Math. Soc., Providence, RI, 1986. Zbl 0615.55009 MR 862632

[25] H. Cartan, La transgression dans un groupe de lie et dans un espace fibré principal, in *Colloque de topologie (espaces fibrés) (Bruxelles, 1950)*, 57–71, Georges Thone, Liège, 1951. Zbl 0045.30701 MR 42427

[26] S. S. Chern and J. Simons, Characteristic forms and geometric invariants, *Ann. of Math. (2)*, **99** (1974), 48–69. Zbl 0283.53036 MR 353327

[27] S. Chern and X. Hu, Equivariant Chern character for the invariant Dirac operator, *Michigan Math. J.*, **44** (1997), no. 3, 451–473. Zbl 0911.58035 MR 1481113

[28] A. Connes, *Spectral sequence and homology of currents for operator algebras*, Math. Forschungsinstitut Oberwolfach Tagungsbericht 42/81, Funktionalanalysis und C*-Algebren, 27.9.–3.10, 1981.

[29] A. Connes, Cohomologie cyclique et foncteur Ext^n, *C. R. Acad. Sci. Paris Ser. I Math.*, **296** (1983), 953–958. Zbl 0534.18009 MR 777584

[30] A. Connes, Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.*, **62** (1985), 257–360. Zbl 0592.46056 MR 0823176

[31] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in *Geometric methods in operator algebras (Kyoto, 1983)*, 52–144, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986. Zbl 0647.46054 MR 0866491

[32] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. Zbl 0818.46076 MR 1303779

[33] A. Connes, Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules, *K-Theory*, **1** (1988), no. 6, 519–548. Zbl 0657.46049 MR 0953915

[34] A. Connes and H. Moscovici, Transgression and the Chern character of finite-dimensional K-cycles, *Comm. Math. Phys.*, **155** (1993), no. 1, 103–122. Zbl 0793.58006 MR 1228528

[35] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, *Geom. Funct. Anal.*, **5** (1995), no. 2, 174–243. Zbl 0960.46048 MR 1334867

[36] A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, *Comm. Math. Phys.*, **198** (1998), no. 1, 199–246. Zbl 0940.58005 MR 1657389

[37] A. Connes and H. Moscovici, Differentiable cyclic cohomology Hopf algebraic structures in transverse geometry, in *Essays on geometry and related topics, Vol. 1, 2*, 217–255, Monogr. Enseign. Math., 38, Enseignement Math., Geneva, 2001. Zbl 1018.57013 MR 1929328

[38] A. Connes and H. Moscovici, Type III and spectral triples, in *Traces in number theory, geometry and quantum fields*, 57–71, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008. Zbl 1159.46041 MR 2427588

[39] A. Connes and P. Tretkoff, The Gauss-Bonnet theorem for the noncommutative two torus, in *Noncommutative geometry, arithmetic, and related topics*, 141–158, Johns Hopkins Univ. Press, Baltimore, MD, 2011. Zbl 1251.46037 MR 2907006

[40] M. Crainic, Cyclic cohomology of étale Groupoids: the general case, *K-Theory*, **17** (1999), 319–362. Zbl 0937.19007 MR 1706117

[41] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, *Phys. Lett. B*, **309** (1993), 279–284. MR 1227281

[42] H. Donnelly and V. K. Patodi, Spectrum and the fixed point sets of isometries. II, *Topology*, **16** (1977), no. 1, 1–11. Zbl 0341.53023 MR 0433513

[43] I. Emmanouil, On a class of groups satisfying Bass' conjecture, *Invent. Math.*, **132** (1998), 307–330. Zbl 0913.20002 MR 1621420

[44] B. P. Feigin and B. L. Tsygan, Additive K-theory, in *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 97–209, Lecture Notes in Math., 1289, Springer, 1987. Zbl 0635.18008 MR 923136

[45] J. Ferrand, The action of conformal transformations on a Riemannian manifold, *Math. Ann.*, **304** (1996), 277–291. Zbl 0866.53027 MR 1371767

[46] E. Getzler, The equivariant Chern character for noncompact Lie groups, *Adv. Math.*, **109** (1994), no. 1, 88–107. Zbl 0853.57034 MR 1302758

[47] E. Getzler and J. D. Jones, The cyclic cohomology of crossed product algebras, *J. Reine Angew. Math.*, **445** (1993), 161–174. Zbl 0795.46052 MR 1244971

[48] E. Getzler and A. Szenes, On the Chern character of a theta-summable Fredholm module, *J. Funct. Anal.*, **84** (1989), no. 2, 343–357. Zbl 0686.46044 MR 1001465

[49] P. B. Gilkey, Lefschetz fixed point formulas and the heat equation, in *Partial differential equations and geometry (Proc. Conf., Park City, Utah, 1977)*, 91–147, Lecture Notes in Pure and Appl. Math., 48, Dekker, New York, 1979. Zbl 0405.58044 MR 0535591

[50] C. R. Graham, R. Jenne, L. Mason, and G. A. Sparling, Conformally invariant powers of the Laplacian. I. Existence, *J. London Math. Soc. (2)*, **46** (1992), no. 3, 557–565. Zbl 0726.53010 MR 1190438

[51] M. Greenfield, M. Marcolli, and K. Teh, Twisted spectral triples and quantum statistical mechanical systems, *p-Adic Numbers Ultrametric Anal. Appl.*, **6** (2014), no. 2, 81–104. Zbl 1315.58006 MR 3200942

[52] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.*, (1955), no. 16, 140pp. Zbl 0123.30301 MR 75539

[53] V. Guillemin, A new proof of Weyl's formula on the asymptotic distribution of eigenvalues, *Adv. in Math.*, **55** (1985), no. 2, 131–160. Zbl 0559.58025 MR 0772612

[54] V. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present. Springer-Verlag, Berlin, 1999. Zbl 0934.55007 MR 1689252

[55] N. Higson, The residue index theorem of Connes and Moscovici, in *Surveys in noncommutative geometry*, 71–126, Clay Math. Proc., 6, Amer. Math. Soc., Providence, RI, 2006. Zbl 1120.58017 MR 2277669

[56] N. Hitchin, Harmonic spinors, *Adv. Math.*, **14** (1974), 1–55. Zbl 0284.58016 MR 358873

[57] B. Iochum and T. Masson, Crossed product extensions of spectral triples, *J. Noncommut. Geom.*, **10** (2016), 65–133. Zbl 1348.46070 MR 3500817

[58] R. Ji, Nilpotency of Connes' periodicity operator and the idempotent conjectures, *K-Theory*, **9** (1995), 59–76. Zbl 0826.46063 MR 1340840

[59] J. D. S. Jones and C. Kassel, Bivariant cyclic theory, *K-Theory*, **3** (1989), 339–365. Zbl 0755.18008 MR 1047192

[60] C. Kassel, Cyclic homology, comodules and mixed complexes, *J. Algebra*, **107** (1987), 195–216. Zbl 0617.16015 MR 883882

[61] C. Kassel, Homologie cyclique, caractère de Chern et lemme de perturbation, *J. reine angew. Math.*, **408** (1990), 159–180. Zbl 0691.18002 MR 1058987

[62] M. Khalkhali and B. Rangipour, On the generalized cyclic Eilenberg–Zilber Theorem, *Canad. Math. Bull.*, **47** (2004), no. 1, 38–48. Zbl 1048.19004 MR 2032267

[63] N. Kuiper, Conformally flat spaces in the large, *Ann. of Math.*, **50** (1949), 916–924. Zbl 0041.09303 MR 31310

[64] J.-L. Loday, *Cyclic homology*, Springer, Berlin, 1992. Zbl 0780.18009 MR 1217970

[65] Z. Marciniak, Cyclic homology of group rings, in *Geometric and algebraic topology*, 305–312, Banach Center Publ., 18, PWN, Warsaw, 1986. Zbl 0637.16015 MR 925872

[66] V. Mathai and D. Quillen, Superconnections, Thom classes and equivariant differential forms, *Topology*, **25** (1986), 85–110. Zbl 0592.55015 MR 836726

[67] H. Moscovici, Eigenvalue inequalities and Poincaré duality in noncommutative geometry, *Comm. Math. Phys.*, **184** (1997), no. 3, 619–628. Zbl 0874.46044 MR 1462513

[68] H. Moscovici, Local index formula and twisted spectral triples, in *Quanta of maths*, 465–500, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010. Zbl 1232.58013 MR 2732069

[69] N. Neumaier, M. Pflaum, H. Posthuma, and X. Tang, Homology of formal deformations of proper étale Lie groupoids, *J. Reine Angew. Math.*, **593** (2006), 117–168. Zbl 1246.53120 MR 2227141

[70] V. Nistor, Group cohomology and the cyclic cohomology of crossed products, *Invent. Math.*, **99** (1990), 411–424. Zbl 0692.46065 MR 1031908

[71] T. Parker and S. Rosenberg, Invariants of conformal Laplacians, *J. Differential Geom.*, **25** (1987), no. 2, 199–222. Zbl 0644.53038 MR 880183

[72] V. K. Patodi, Holomorphic Lefschetz fixed point formula, *Bull. Amer. Math. Soc.*, **79** (1973), 825–828. Zbl 0279.32015 MR 317359

[73] R. Ponge, A new short proof of the local index formula and some of its applications, *Comm. Math. Phys.*, **241** (2003), no. 2-3, 215–234. Zbl 1191.58006 MR 2013798

[74] R. Ponge, The Cyclic homology of crossed-product algebras. I, *C. R. Acad. Sci. Paris, Sér. I*, **355** (2017), no. 6, 618–622. Zbl 06748828 MR 3661543

[75] R. Ponge, The Cyclic homology of crossed-product algebras. II, *C. R. Acad. Sci. Paris, Sér. I*, **355** (2017), no. 6, 623–627. Zbl 06748829 MR 3661544

[76] R. Ponge, Perturbation lemma, para-S-modules, and cup product for paracyclic modules, in preparation.

[77] R. Ponge and H. Wang, Index map, σ-connections and Connes–Chern character in the setting of twisted spectral triples, *Kyoto J. Math.*, **56** (2016), no. 2, 347–399. Zbl 1361.46055 MR 3500845

[78] R. Ponge and H. Wang, Noncommutative geometry and conformal geometry. II. Connes–Chern character and the local equivariant index theorem, *J. Noncommut. Geom.*, **10** (2016), 307–378. Zbl 1343.58004 MR 3500823

[79] R. Ponge and H. Wang, Noncommutative geometry and conformal geometry. III. Vafa-Witten inequality and Poincaré duality, *Adv. Math.*, **272** (2015), 761–819. Zbl 1311.53037 MR 3303248

[80] J. Rotman, *An Introduction to Homological Algebra*, 2nd edition, Universitext, Springer, New York, 2009. Zbl 1157.18001 MR 2455920

[81] R. Schoen, On the conformal and CR automorphisms groups, *Geom. Funct. Anal.*, **5** (1995), 464–481. Zbl 0835.53015 MR 1334876

[82] F. Tréves, *Topological vector spaces, distributions, and kernels*, Pure and Applied Mathematics, 25, Academic Press, New-York, NY, 1967. Zbl 0171.10402 MR 225131

[83] B. L. Tsygan, Homology of matrix Lie algebras over rings and Hochschild homology, *Uspekhi Math. Nawk.*, **38** (1983), 217–218. Zbl 0518.17002 MR 695483

[84] M. Wodzicki, Local invariants of spectral asymmetry, *Invent. Math.*, **75** (1984), no. 1, 143–177. Zbl 0538.58038 MR 0728144

[85] J. Zhang and Q.-W. Wang, An explicit formula for the generalised cyclic shuffle map, *Canad. Math. Bull.*, **57** (2014), 210–223. Zbl 1286.19005 MR 3150735

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