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Homotopy Poisson algebras, Maurer–Cartan elements and Dirac structures of CLWX 2-algebroids

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Abstract. In this paper, we construct a homotopy Poisson algebra of degree 3 associated to a split Lie 2-algebroid, by which we give a new approach to characterize a split Lie 2-bialgebroid. We develop the differential calculus associated to a split Lie 2-algebroid and establish the Manin triple theory for split Lie 2-algebroids. More precisely, we give the notion of a strict Dirac structure and define a Manin triple for split Lie 2-algebroids to be a CLWX 2-algebroid with two transversal strict Dirac structures. We show that there is a one-to-one correspondence between Manin triples for split Lie 2-algebroids and split Lie 2-bialgebroids. We further introduce the notion of a weak Dirac structure of a CLWX 2-algebroid and show that the graph of a Maurer–Cartan element of the homotopy Poisson algebra of degree 3 associated to a split Lie 2-bialgebroid is a weak Dirac structure. Various examples including the string Lie 2-algebra, split Lie 2-algebroids constructed from integrable distributions and left-symmetric algebroids are given.

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1. Introduction

The notion of a Lie algebroid was introduced by Pradines in 1967, which is a generalization of Lie algebras and tangent bundles. Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. See [40] for the general theory about Lie algebroids. The notion of a Lie bialgebroid was introduced by Mackenzie and Xu in [41] as the infinitesimal of a Poisson groupoid. To study the double of a Lie bialgebroid, Liu, Weinstein and Xu introduced the notion of a Courant algebroid in [38] and established the Manin triple theory for Lie algebroids. There are many applications of Courant algebroids. See [11, 19, 20, 28, 33, 48, 50–52] for more details. In particular, the relation between Dirac structures and Maurer–Cartan elements were studied in detail in [35, 38]. The

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notion of a Dirac structure was originally introduced by Courant in [17] to unify symplectic structures and Poisson structures. Then it was widely studied and has many applications, e.g. in generalized complex geometry [19, 20], in the theory of D-branes for the Wess–Zumino–Witten model [13], in moment map theories [12], in Poisson geometry [34, 44, 45], in Dixmier–Douady bundles [3], in reduction theory [22], in L_∞-algebras [60] and in integrable systems [14]. See [10] for more details.

Recently, people have paid more attention to higher categorical structures by reasons in both mathematics and physics. A Lie 2-algebra is the categorification of a Lie algebra [4]. The 2-category of Lie 2-algebras is equivalent to the 2-category of 2-term L_∞-algebras. Due to this reason, an n-term L_∞-algebra will be called a Lie n-algebra. See [30,31,57] for more details of L_∞-algebras. Usually an NQ-manifold of degree n is considered as a Lie n-algebroid [59]. In [56], a split Lie n-algebroid is defined using the language of graded vector bundles. The equivalence between the category of split Lie n-algebroids and the category of NQ-manifolds of degree n was given in [7].

The notion of a CLWX 2-algebroid was introduced in [37] as the categorification of a Courant algebroid. There is a one-to-one correspondence between CLWX 2-algebroids and symplectic NQ-manifolds of degree 3, and the later can be used to construct 4D topological field theory [1,21]. There is also a close connection between CLWX 2-algebroids and the first Pontryagin classes of quadratic Lie 2-algebroids introduced in [53], which are represented by closed 5-forms. The notion of a split Lie 2-bialgebroid was also introduced in [37] and it is shown that the double of a split Lie 2-bialgebroid is a CLWX 2-algebroid. The split Lie 2-bialgebroid used here is a direct geometric generalization of the Lie 2-bialgebra introduced in [5, 16]. See [29] and [6] for the more general notions of an L_∞-bialgebra and an L_∞-bialgebroid.

The first purpose of this paper is to establish the Manin triple theory for split Lie 2-algebroids, i.e. to show that two transversal Dirac structures of a CLWX 2-algebroid constitute a split Lie 2-bialgebroid. The second purpose of this paper is to study the homotopy Poisson algebra associated to a split Lie 2-algebroid, and establish the relation between its Maurer–Cartan elements and Dirac structures of the CLWX 2-algebroid. Upon careful study, we found that we need to take two kinds of Dirac structures into account, one is called a strict Dirac structure, served for the first purpose, and the other is called a weak Dirac structure, served for the second purpose. It is the weak Dirac structures that reflect properties of higher structures and make the paper containing more meaningful contents. Due to aforementioned important applications of Dirac geometry, it is natural to explore similar applications of Dirac structures introduced in this paper, which will be studied in the future.

To study the Manin triple theory for split Lie 2-algebroids, we develop the differential calculus for split Lie 2-algebroids in Section 3. In particular, we define the coboundary operator, Lie derivatives, the contraction operator and give their properties.

In Section 4, first we define bracket operations $[\cdot]_S$, $[\cdot,\cdot]_S$, $[\cdot,\cdot,\cdot]_S$ on $\mathsf{Sym}(\mathcal{A}[-3])$ associated to a split Lie 2-algebroid $\mathcal{A}$ using the derived bracket [18,25,26], and show that

$$\big(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S,\ [\cdot,\cdot]_S,\ [\cdot,\cdot,\cdot]_S\big)$$

is a homotopy Poisson algebra of degree 3. The notion of a homotopy Poisson manifold of degree n was given in [32] in the study of the dual structure of a Lie 2-algebra. See also [8, 9, 15, 25, 42, 58] for more applications of similar structures. Then we use the usual differential geometry language to give a new characterization of a split Lie 2-bialgebroid.

In Section 5, we introduce the notion of a strict Dirac structure of a CLWX 2-algebroid and establish the Manin triple theory for split Lie 2-algebroids. More precisely, we show that there is a one-to-one correspondence between Manin triples of split Lie 2-algebroids and split Lie 2-bialgebroids. Note that a strict Dirac structure of a CLWX 2-algebroid is defined to be a maximal isotropic graded subbundle whose section space is closed under the multiplication.

In Section 6, first we introduce the notion of a weak Dirac structure of a CLWX 2-algebroid, which is a Lie 2-algebroid such that there is a Leibniz 2-algebra morphism from the underlying Lie 2-algebra to the underlying Leibniz 2-algebra of the original CLWX 2-algebroid satisfying some compatibility conditions. Note that the image is not closed under the multiplication in the CLWX 2-algebroid anymore, and this is the main difference between strict Dirac structures and weak Dirac structures. Such ideas had already been used in [55] to integrate semidirect product Lie 2-algebras. Then we show that a Maurer–Cartan element $H + K$, where $H \in A_{-1} \odot A_{-2}$ and $K \in \wedge^3 A_{-2}$, of the homotopy Poisson algebra

$$\big(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S, [\cdot,\cdot]_S, [\cdot,\cdot,\cdot]_S\big)$$

associated to a split Lie 2-algebroid

$$\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$$

gives rise to a split Lie 2-algebroid structure on $\mathcal{A}^*[3]$ such that $(H^\sharp, -H^\natural, -K^\flat)$ is a morphism from the split Lie 2-algebroid $\mathcal{A}^*[3]$ to the split Lie 2-algebroid $\mathcal{A}$. Consequently, the graph of $(H^\sharp, -H^\natural)$ is a weak Dirac structure of the CLWX 2-algebroid $\mathcal{A} \oplus \mathcal{A}^*[3]$. Finally, we generalize the above result to the case of split Lie 2-bialgebroids. We also give various examples including the string Lie 2-algebra, integrable distributions and the split Lie 2-algebroids constructed from left-symmetric algebroids.

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2. Leibniz 2-algebras

The notion of a strongly homotopy Leibniz algebra, or a Lod_∞-algebra was given in [39] by Livernet, which was further studied by Ammar and Poncin in [2]. In [54], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and proved that the category of Leibniz 2-algebras and the category of 2-term Lod_∞-algebras are equivalent. Here we use a shift 1 version of Leibniz 2-algebras.

Definition 2.1. A *Leibniz* 2-*algebra* $\mathcal{V}$ consists of the following data:

- a complex of vector spaces $\mathcal{V}\colon V_{-2} \xrightarrow{\mathrm{d}} V_{-1}$,
- bilinear maps $l_2\colon V_i \times V_j \longrightarrow V_{i+j+1}$, where $-3 \le i + j \le -2$,
- a trilinear map $l_3\colon V_{-1} \times V_{-1} \times V_{-1} \longrightarrow V_{-2}$,

such that for all $w, x, y, z \in V_{-1}$ and $m, n \in V_{-2}$, the following equalities are satisfied:

(a) $\mathrm{d}l_2(x, m) = l_2(x, \mathrm{d}m)$,

(b) $\mathrm{d}l_2(m, x) = -l_2(\mathrm{d}m, x)$,

(c) $l_2(\mathrm{d}m, n) = -l_2(m, \mathrm{d}n)$,

(d) $\mathrm{d}l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(l_2(x, y), z) - l_2(y, l_2(x, z))$,

(e_1) $l_3(x, y, \mathrm{d}m) = l_2(x, l_2(y, m)) - l_2(l_2(x, y), m) - l_2(y, l_2(x, m))$,

(e_2) $-l_3(x, \mathrm{d}m, y) = l_2(x, l_2(m, y)) - l_2(l_2(x, m), y) - l_2(m, l_2(x, y))$,

(e_3) $-l_3(\mathrm{d}m, x, y) = l_2(m, l_2(x, y)) + l_2(l_2(m, x), y) - l_2(x, l_2(m, y))$,

(f) the Jacobiator identity:

$$\begin{aligned} l_2(x, l_3(y, z, w)) - l_2(y, l_3(x, z, w)) + l_2(z, l_3(x, y, w)) - l_2(l_3(x, y, z), w) \\ - l_3(l_2(x, y), z, w) - l_3(y, l_2(x, z), w) - l_3(y, z, l_2(x, w)) \\ + l_3(x, l_2(y, z), w) + l_3(x, z, l_2(y, w)) - l_3(x, y, l_2(z, w)) = 0. \end{aligned}$$

We usually denote a Leibniz 2-algebra by $(V_{-2}, V_{-1}, \mathrm{d}, l_2, l_3)$, or simply by $\mathcal{V}$. In particular, if l_2 is graded symmetric and l_3 is totally skew-symmetric, we call it a *Lie 2-algebra*. A Lie 2-algebra used in this paper is equivalent to a 2-term $L_\infty[1]$-algebra.

Definition 2.2. Let $\mathcal{V}$ and $\mathcal{V}'$ be Leibniz 2-algebras. A *morphism* F from $\mathcal{V}$ to $\mathcal{V}'$ consists of:

- linear maps $F_1\colon V_{-1} \longrightarrow V'_{-1}$ and $F_2\colon V_{-2} \longrightarrow V'_{-2}$ commuting with the differential, i.e.
$$F_1 \circ \mathrm{d} = \mathrm{d}' \circ F_2; \tag{1}$$
- a bilinear map $F_3\colon V_{-1} \times V_{-1} \longrightarrow V'_{-2}$,

such that for all $x, y, z \in V_{-1},\ m \in V_{-2}$, we have

$$\begin{cases} F_1 l_2(x,y) - l_2'(F_1(x), F_1(y)) = \mathrm{d}' F_3(x,y), \\ F_2 l_2(x,m) - l_2'(F_1(x), F_2(m)) = F_3(x, \mathrm{d}m), \\ F_2 l_2(m,x) - l_2'(F_2(m), F_1(x)) = -F_3(\mathrm{d}m, x), \end{cases} \tag{2}$$

and

$$\begin{aligned} &- F_2(l_3(x,y,z)) + l_2'(F_1(x), F_3(y,z)) - l_2'(F_1(y), F_3(x,z)) \\ &\quad + l_2'(F_3(x,y), F_1(z)) - F_3(l_2(x,y),z) + F_3(x, l_2(y,z)) \\ &\qquad - F_3(y, l_2(x,z)) + l_3'(F_1(x), F_1(y), F_1(z)) = 0. \end{aligned} \tag{3}$$

In particular, if $\mathcal{V}$ and $\mathcal{V}'$ are Lie 2-algebras and F_3 is skew-symmetric, we obtain the definition of a morphism between Lie 2-algebras.

3. Differential calculus on split Lie 2-algebroids

3.1. Characterization of split Lie 2-algebroids via the big bracket. The notion of a split Lie n-algebroid was introduced in [56] using graded vector bundles. The equivalence between the category of split Lie n-algebroids and the category of NQ-manifolds of degree n was given in [7].

Definition 3.1. A split Lie 2-algebroid is a graded vector bundle $\mathcal{A} = A_{-1} \oplus A_{-2}$ over a manifold M equipped with a bundle map $a\colon A_{-1} \longrightarrow TM$, and brackets $l_i\colon \Gamma(\wedge^i \mathcal{A}) \longrightarrow \Gamma(\mathcal{A})$ of degree 1 for $i = 1, 2, 3$, such that:

(1) $(\Gamma(A_{-2}), \Gamma(A_{-1}), l_1, l_2, l_3)$ is a Lie 2-algebra;

(2) l_2 satisfies the Leibniz rule with respect to a:

$$l_2(X^1, fY) = f l_2(X^1, Y) + a(X^1)(f)Y,$$

for all $X^1 \in \Gamma(A_{-1}),\ f \in C^\infty(M),\ Y \in \Gamma(\mathcal{A})$;

(3) for $i \neq 2$, l_i are $C^\infty(M)$-linear.

Denote a split Lie 2-algebroid by $(A_{-2}, A_{-1}, l_1, l_2, l_3, a)$, or simply by $\mathcal{A}$.

Split Lie 2-algebroids become active research objects recently. See [23, 24, 43, 54, 56] for more examples and applications of split Lie 2-algebroids.

Lemma 3.2. *Let $(A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid. Then we have:*

$$a \circ l_1 = 0, \tag{4}$$

$$a(l_2(X^1, Y^1)) = [a(X^1), a(Y^1)], \quad \forall\ X^1, Y^1 \in \Gamma(A_{-1}). \tag{5}$$

Definition 3.3. Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ and $\mathcal{A}' = (A'_{-2}, A'_{-1}, l'_1, l'_2, l'_3, a')$ be split Lie 2-algebroids over the same base manifold M. A morphism F from $\mathcal{A}$ to $\mathcal{A}'$ consists of bundle maps

$$F^1\colon A_{-1} \longrightarrow A'_{-1}, \quad F^2\colon A_{-2} \longrightarrow A'_2, \quad \text{and} \quad F^3\colon \wedge^2 A_{-1} \longrightarrow A'_{-2}$$

such that $a' \circ F^1 = a$ and (F^1, F^2, F^3) is a morphism between the underlying Lie 2-algebras.

In the sequel, we describe a split Lie 2-algebroid structure on $\mathcal{A} = A_{-1} \oplus A_{-2}$ using the graded Poisson bracket $\{\cdot,\cdot\}$ on the symplectic manifold $\mathcal{M}{:=}T^*[3](A_{-1}{\oplus}A_{-2})$. Denote by $(x^i, \xi^j, \theta^k, p_i, \xi_j, \theta_k)$ a canonical Darboux coordinate on $\mathcal{M}$, where x^i is a coordinate on M, (ξ^j, θ^k) is the fiber coordinate on $A_{-1} \oplus A_{-2}$, (p_i, ξ_j, θ_k) is the momentum coordinate on $\mathcal{M}$ for (x^i, ξ^j, θ^k). The degrees of variables $(x^i, \xi^j, \theta^k, p_i, \xi_j, \theta_k)$ are respectively $(0, 1, 2, 3, 2, 1)$.

We introduce the *tridegree* of the coordinates $(x^i, \xi^j, \theta^k, p_i, \xi_j, \theta_k)$ as follows[1]:

$$((0,0,0), (0,1,0), (1,1,0), (1,1,1), (1,0,1), (0,0,1)).$$

It is straightforward to check that the tridegree is globally well-defined. Denote by $C^n(\mathcal{M})$ the space of functions on $\mathcal{M}$ of degree n. Then the space $C^n(\mathcal{M})$ is uniquely decomposed into the homogeneous subspaces with respect to the tridegree,

$$C^n(\mathcal{M}) = \sum_{i+j+k=n} C^{(i,j,k)}(\mathcal{M}).$$

The degree and tridegree of the symplectic structure

$$\omega = dx^i dp_i + d\xi^j d\xi_j + d\theta_k d\theta^k$$

is 3 and $(1,1,1)$ respectively and the degree and tridegree of the corresponding graded Poisson structure $\{\cdot,\cdot\}$ is -3 and $(-1,-1,-1)$ respectively.

Now we consider the following fiberwise linear function μ of degree 4 on $\mathcal{M}$:

$$\mu = \mu_1{}^i_j(x) p_i \xi^j + \mu_2{}^i_j(x) \xi_i \theta^j + \tfrac{1}{2}\mu_3{}^k_{ij}(x)\xi_k \xi^i \xi^j + \mu_4{}^k_{ij}\theta_k \xi^i \theta^j + \tfrac{1}{6}\mu_5{}^l_{ijk}(x)\theta_l \xi^i \xi^j \xi^k, \quad (6)$$

where $\mu_1{}^i_j$, $\mu_2{}^i_j$, $\mu_3{}^k_{ij}$, $\mu_4{}^k_{ij}$, $\mu_5{}^l_{ijk}$ are functions on M. The function μ can be uniquely decomposed into

$$\mu = \mu^{(2,1,1)} + \mu^{(1,2,1)} + \mu^{(0,3,1)},$$

[1] We thank the referee for pointing that the third entry is the fiberwise polynomial degree, the second entry is the fiberwise polynomial degree after applying the Legendre transformation and the first entry is chosen so that the sum is the total degree.

where

$$\mu^{(2,1,1)} = \mu_2{}^i_j(x)\xi_i\theta^j, \quad \mu^{(1,2,1)} = \mu_1{}^i_j(x)p_i\xi^j + \tfrac{1}{2}\mu_3{}^k_{ij}(x)\xi_k\xi^i\xi^j + \mu_4{}^k_{ij}\theta_k\xi^i\theta^j,$$
$$\mu^{(0,3,1)} = \tfrac{1}{6}\mu_5{}^l_{ijk}(x)\theta_l\xi^i\xi^j\xi^k.$$

Define

$$\begin{aligned} &l_1\colon A_{-2} \longrightarrow A_{-1},\\ &l_2\colon \Gamma(A_i)\times\Gamma(A_j) \longrightarrow \Gamma(A_{i+j+1}), \quad -3 \le i+j \le -2,\\ &l_3\colon \wedge^3 A_{-1} \longrightarrow A_{-2} \end{aligned}$$

and a bundle map $a\colon A_{-1} \longrightarrow TM$ by

$$\begin{cases} l_1(X^2) = -\{\mu^{(2,1,1)}, X^2\},\\ l_2(X^1,Y^1) = -\{\{\mu^{(1,2,1)}, X^1\}, Y^1\},\\ l_2(X^1,Y^2) = -\{\{\mu^{(1,2,1)}, X^1\}, Y^2\},\\ l_3(X^1,Y^1,Z^1) = -\{\{\{\mu^{(0,3,1)}, X^1\}, Y^1\}, Z^1\},\\ a(X^1)(f) = -\{\{\mu^{(1,2,1)}, X^1\}, f\}, \end{cases} \tag{7}$$

for all $X^1, Y^1, Z^1 \in \Gamma(A_{-1})$, $X^2, Y^2 \in \Gamma(A_{-2})$, and $f \in C^\infty(M)$.

Theorem 3.4. *Let $\mathcal{A} = A_{-2} \oplus A_{-1}$ be a graded vector bundle and μ a degree 4 function given by (6). If $\{\mu,\mu\} = 0$, then $(A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ is a split Lie 2-algebroid, where l_1, l_2, l_3 and a are given by (7).*

Conversely, if $(A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ is a split Lie 2-algebroid, then we have $\{\mu,\mu\} = 0$, where μ is given by (6), in which $\mu_1{}^i_j$, $\mu_2{}^i_j$, $\mu_3{}^k_{ij}$, $\mu_4{}^k_{ij}$, $\mu_5{}^l_{ijk}$ are given by:

$$a(\xi_j) = \mu_1{}^i_j\frac{\partial}{\partial x^i}, \quad l_1(\theta_j) = \mu_2{}^i_j\xi_i, \quad l_2(\xi_i,\xi_j) = \mu_3{}^k_{ij}\xi_k,$$
$$l_2(\xi_i,\theta_j) = \mu_4{}^k_{ij}\theta_k, \quad l_3(\xi_i,\xi_j,\xi_k) = \mu_5{}^l_{ijk}\theta_l.$$

3.2. Differential calculus on Lie 2-algebroids. Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid with the structure function μ. Then we have the generalized Chevalley–Eilenberg cochain complex $(\mathsf{Sym}(\mathcal{A}^*) = \bigoplus_k \mathsf{Sym}^k(\mathcal{A}^*), \delta)$, where the set of k-cochains $\mathsf{Sym}^k(\mathcal{A}^*)$ is given by

$$\mathsf{Sym}^k(\mathcal{A}^*) = \sum_{p+2q=k} \mathsf{Sym}^{p,q}(\mathcal{A}^*)$$

with

$$\mathsf{Sym}^{p,q}(\mathcal{A}^*) = \Gamma(\wedge^p A^*_{-1}) \odot \Gamma(\mathsf{Sym}^q(A^*_{-2}))$$

and the differential $\delta\colon \mathsf{Sym}^k(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{k+1}(\mathcal{A}^*)$ is defined by[2]

$$\delta(\cdot) = \{\mu, \cdot\}. \tag{8}$$

If there is a Lie 2-algebroid structure on $\mathcal{A}^*[3]$, we use δ_* to denote the corresponding differential. The differential δ can be written as

$$\delta = \overline{\delta} + \mathrm{d} + \widehat{\delta},$$

where

$$\overline{\delta}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p-1,q+1}(\mathcal{A}^*),$$
$$\mathrm{d}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p+1,q}(\mathcal{A}^*),$$

and

$$\widehat{\delta}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p+3,q-1}(\mathcal{A}^*)$$

are given by

$$\overline{\delta}(\cdot) = \{\mu^{(2,1,1)}, \cdot\}, \quad \mathrm{d}(\cdot) = \{\mu^{(1,2,1)}, \cdot\}, \quad \widehat{\delta}(\cdot) = \{\mu^{(0,3,1)}, \cdot\}.$$

In particular, for $\alpha^1 \in \Gamma(A^*_{-1})$, we have

$$\overline{\delta}\alpha^1(X^2) = -\langle \alpha^1, l_1(X^2)\rangle, \quad \forall\, X^2 \in \Gamma(A_{-2}).$$

For all $f \in C^\infty(M), \alpha^1 \in \Gamma(A^*_{-1}), \alpha^2 \in \Gamma(A^*_{-2}), X^1, Y^1 \in \Gamma(A_{-1}), Y^2 \in \Gamma(A_{-2})$, we have

$$\begin{cases} \mathrm{d}(f)(X^1) = a(X^1)(f), \\ \mathrm{d}(\alpha^1)(X^1, Y^1) = a(X^1)\langle \alpha^1, Y^1\rangle - a(Y^1)\langle \alpha^1, X^1\rangle - \langle \alpha^1, l_2(X^1, Y^1)\rangle, \\ \mathrm{d}(\alpha^2)(X^1, Y^2) = a(X^1)\langle \alpha^2, Y^2\rangle - \langle \alpha^2, l_2(X^1, Y^2)\rangle. \end{cases}$$

For all $\alpha^2 \in \Gamma(A^*_{-2})$, we have

$$\widehat{\delta}\alpha^2(X^1, Y^1, Z^1) = -\langle l_3(X^1, Y^1, Z^1), \alpha^2\rangle, \quad \forall\, X^1, Y^1, Z^1 \in \Gamma(A_{-1}).$$

By the properties of graded Poisson bracket, for all $\phi_1 \in \mathsf{Sym}^k(\mathcal{A}^*)$ and $\phi_2 \in \mathsf{Sym}^l(\mathcal{A}^*)$, we have

$$\delta(\phi_1 \odot \phi_2) = \delta(\phi_1) \odot \phi_2 + (-1)^k \phi_1 \odot \delta(\phi_2). \tag{9}$$

Define the Lie derivative $L^0\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p-1,q+1}(\mathcal{A}^*)$ by

$$L^0(\phi) = -\{\mu^{(2,1,1)}, \phi\}, \quad \forall \phi \in \mathsf{Sym}^{p,q}(\mathcal{A}^*). \tag{10}$$

[2] $\mathsf{Sym}(\mathcal{A}^*)$ can be embedded into the Poisson algebra $C^\infty(\mathcal{M})$ through the pullback of the canonical map $T^*[3](A_{-1} \oplus A_{-2}) \to A_{-1} \oplus A_{-2}$. On the other hand, by the Legendre transformation, $T^*[3](A_{-1} \oplus A_{-2})$ is isomorphic to $T^*[3](A^*_{-1} \oplus A^*_{-2})[3]$ as graded symplectic manifolds. Thus, $\mathsf{Sym}(\mathcal{A})$ can also be embedded into the Poisson algebra $C^\infty(\mathcal{M})$. See [48] for more details on the Legendre transformation.

In particular, for all $\alpha^1 \in \Gamma(A^*_{-1})$, we have

$$\langle L^0(\alpha^1), X^2\rangle = \langle \alpha^1, l_1(X^2)\rangle, \quad \forall\, X^2 \in \Gamma(A_{-2}).$$

It is obvious that $L^0(\alpha^1) = l_1^*(\alpha^1)$.

For all $X^1 \in \Gamma(A_{-1})$, define the Lie derivative

$$L^1_{X^1}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p,q}(\mathcal{A}^*)$$

by

$$L^1_{X^1}\phi = -\{\{\mu^{(1,2,1)}, X^1\}, \phi\}, \quad \forall\, \phi \in \mathsf{Sym}^{p,q}(\mathcal{A}^*). \tag{11}$$

In particular, for all $\alpha^i \in \Gamma(A^*_{-i}),\ i = 1, 2$, we have

$$\langle L^1_{X^1}\alpha^i, Y^i\rangle = a(X^1)\langle Y^i, \alpha^i\rangle - \langle \alpha^i, l_2(X^1, Y^i)\rangle, \quad \forall\, Y^i \in \Gamma(A_{-i}).$$

It is straightforward to deduce that

$$L^1_{X^1}(\phi\odot\psi) = L^1_{X^1}\phi\odot\psi + \phi\odot L^1_{X^1}\psi, \quad \forall\, \phi \in \mathsf{Sym}^k(\mathcal{A}^*),\ \psi \in \mathsf{Sym}^l(\mathcal{A}^*). \tag{12}$$

For all $X^2 \in \Gamma(A_{-2})$, define the Lie derivative

$$L^2_{X^2}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p+1,q-1}(\mathcal{A}^*)$$

by

$$L^2_{X^2}\phi = -\{\{\mu^{(1,2,1)}, X^2\}, \phi\}, \quad \forall\, \phi \in \mathsf{Sym}^{p,q}(\mathcal{A}^*). \tag{13}$$

In particular, for all $\alpha^2 \in \Gamma(A^*_{-2})$, we have

$$\langle L^2_{X^2}\alpha^2, Y^1\rangle = -\langle \alpha^2, l_2(X^2, Y^1)\rangle, \quad \forall\, Y^1 \in \Gamma(A_{-1}).$$

It is easy to see that

$$L^2_{X^2}(\phi\odot\psi) = L^2_{X^2}\phi\odot\psi + (-1)^k\phi\odot L^2_{X^2}\psi, \quad \forall\, \phi \in \mathsf{Sym}^k(\mathcal{A}^*),\ \psi \in \mathsf{Sym}^l(\mathcal{A}^*). \tag{14}$$

For all $X^1, Y^1 \in \Gamma(A_{-1})$, define the Lie derivative

$$L^3_{X^1,Y^1}\colon \mathsf{Sym}^{p,q}(\mathcal{A}^*) \longrightarrow \mathsf{Sym}^{p+1,q-1}(\mathcal{A}^*)$$

by

$$L^3_{X^1,Y^1}\phi = -\{\{\{\mu^{(0,3,1)}, X^1\}, Y^1\}, \phi\}, \quad \forall\, \phi \in \mathsf{Sym}^{p,q}(\mathcal{A}^*). \tag{15}$$

In particular, for all $\alpha^2 \in \Gamma(A^*_{-2})$, we have

$$\langle L^3_{X^1,Y^1}\alpha^2, Z^1\rangle = -\langle \alpha^2, l_3(X^1, Y^1, Z^1)\rangle, \quad \forall\, Z^1 \in \Gamma(A_{-1}).$$

It is easy to see that

$$L^3_{X^1,Y^1}(\phi\odot\psi)=L^3_{X^1,Y^1}\phi\odot\psi+(-1)^k\phi\odot L^3_{X^1,Y^1}\psi, \tag{16}$$

for all $\phi\in\mathsf{Sym}^k(\mathcal{A}^*)$, $\psi\in\mathsf{Sym}^l(\mathcal{A}^*)$.

For all $X\in\Gamma(\mathcal{A})$, $\alpha^1_i\in\Gamma(A^*_{-1})$, $\alpha^2_j\in\Gamma(A^*_{-2})$, define the contraction operator $\iota_X\colon\mathsf{Sym}^{p,q}(\mathcal{A}^*)\to\mathsf{Sym}^{p-1,q}(\mathcal{A}^*)\oplus\mathsf{Sym}^{p,q-1}(\mathcal{A}^*)$ by

$$\begin{aligned}\iota_X&(\alpha^1_1\odot\cdots\odot\alpha^1_p\odot\alpha^2_1\odot\cdots\odot\alpha^2_q)\\&=\sum_{i=1}^p(-1)^{i+1}\langle X,\alpha^1_i\rangle\alpha^1_1\odot\cdots\odot\widehat{\alpha^1_i}\odot\cdots\odot\alpha^1_p\odot\alpha^2_1\odot\cdots\odot\alpha^2_q\\&\qquad+\sum_{j=1}^q\langle X,\alpha^2_j\rangle\alpha^1_1\odot\cdots\odot\alpha^1_p\odot\alpha^2_1\odot\cdots\odot\widehat{\alpha^2_j}\odot\cdots\odot\alpha^2_q.\end{aligned}$$

For any $\phi\in\mathsf{Sym}^k(\mathcal{A}^*)$, let us denote by

$$\phi(X_1,X_2,\ldots,X_k)=\iota_{X_k}\iota_{X_{k-1}}\cdots\iota_{X_1}\phi,\quad\forall\ X_i\in\Gamma(\mathcal{A}).$$

Thus, for all $X^1\in\Gamma(A_{-1})$, $\phi\in\mathsf{Sym}^k(\mathcal{A}^*)$, and $Y_i\in\Gamma(\mathcal{A})$, we have

$$(L^1_{X^1}\phi)(Y_1,\ldots,Y_k)=a(X^1)\phi(Y_1,\ldots,Y_k)-\sum_{i=1}^{p+q}\phi(Y_1,\ldots,l_2(X^1,Y_i),\ldots,Y_k).$$

The following lemmas list some properties of the above operators.

Lemma 3.5. *For all $X^1\in\Gamma(A_{-1})$, $X^2\in\Gamma(A_{-2})$, $f\in C^\infty(M)$, and $\phi\in\mathsf{Sym}^k(\mathcal{A}^*)$, we have*

$$\begin{aligned}L^1_{X^1}f\phi&=f(L^1_{X^1}\phi)+a(X^1)(f)\phi, & L^1_{fX^1}\phi&=f(L^1_{X^1}\phi)+\mathrm{d}f\odot\iota_{X^1}\phi,\\L^2_{X^2}f\phi&=f(L^2_{X^2}\phi), & L^2_{fX^2}\phi&=f(L^2_{X^2}\phi)-\mathrm{d}f\odot\iota_{X^2}\phi,\\L^1_{X^1}\phi&=\iota_{X^1}\mathrm{d}\phi+\mathrm{d}\iota_{X^1}\phi, & L^2_{X^2}\phi&=\iota_{X^2}\mathrm{d}\phi-\mathrm{d}\iota_{X^2}\phi.\end{aligned}$$

Lemma 3.6. *For all $X^1,Y^1\in\Gamma(A_{-1})$, $X^2\in\Gamma(A_{-2})$, $\phi\in\mathsf{Sym}^k(\mathcal{A}^*)$, we have*

$$L^1_{l_2(X^1,Y^1)}\phi-L^1_{X^1}L^1_{Y^1}\phi+L^1_{Y^1}L^1_{X^1}\phi=-L^3_{X^1,Y^1}L^0(\phi)-L^0(L^3_{X^1,Y^1}\phi), \tag{17}$$

$$L^2_{l_2(X^1,Y^2)}\phi-L^1_{X^1}L^2_{Y^2}\phi+L^2_{Y^2}L^1_{X^1}\phi=-L^3_{l_1(Y^2),X^1}\phi. \tag{18}$$

Lemma 3.7. *For all $X^1,Y^1\in\Gamma(A_{-1})$, $X^2\in\Gamma(A_{-2})$, $\alpha^1\in\Gamma(A^*_{-1})$, $\alpha^2\in\Gamma(A^*_{-2})$, we have*

$$\iota_{l_2(X^1,Y^1)}\mathrm{d}\alpha^1-L^1_{X^1}\iota_{Y^1}\mathrm{d}\alpha^1+\iota_{Y^1}L^1_{X^1}\mathrm{d}\alpha^1=-L^3_{X^1,Y^1}l_1^*\alpha^1, \tag{19}$$

$$\iota_{l_2(X^1,Y^1)}\mathrm{d}\alpha^2-L^1_{X^1}\iota_{Y^1}\mathrm{d}\alpha^2+\iota_{Y^1}L^1_{X^1}\mathrm{d}\alpha^2=-l_1^*L^3_{X^1,Y^1}\alpha^2, \tag{20}$$

$$\iota_{l_2(X^1,Y^2)}\mathrm{d}\alpha^2-L^1_{X^1}\iota_{Y^2}\mathrm{d}\alpha^2+\iota_{Y^2}L^2_{X^1}\mathrm{d}\alpha^2=-L^3_{l_1(Y^2),X^1}\alpha^2. \tag{21}$$

If $\mathcal{A}^*[3]$ is a split Lie 2-algebroid, we use $\mathcal{L}^0$, $\mathcal{L}^1$, $\mathcal{L}^2$, and $\mathcal{L}^3$ to denote the corresponding Lie derivatives.

4. Homotopy Poisson algebras of degree 3 associated to split Lie 2-algebroids and split Lie 2-bialgebroids

4.1. Homotopy Poisson algebras of degree 3 associated to split Lie 2-algebroids. Associated to a Lie algebroid A, we have the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot])$, which is also a Poisson algebra of degree -1, where $\wedge^\bullet A = \sum_k \wedge^k A$ and $[\cdot,\cdot]$ is the Schouten bracket on $\Gamma(\wedge^\bullet A)$. This algebra plays very important role in the theory of Lie algebroids. Note that $\Gamma(\wedge^\bullet A)$ can be understood as the symmetric algebra of $A[-1]$ and elements in $\Gamma(\wedge^k A)$ are of degree k. This algebra can be obtained by the derived bracket as follows. Consider the shifted cotangent bundle $T^*[2]A[1]$, which is a symplectic manifold of degree 2. The corresponding Poisson structure is of degree -2. The Lie algebroid structure is equivalent to a degree 3 function κ on $T^*[2]A[1]$ which is fiberwise linear. Then the Schouten bracket on $\Gamma(\wedge^\bullet A)$ can be obtained by

$$[P,Q] = -\{\{\kappa, P\}, Q\}, \quad \forall\ P \in \Gamma(\wedge^k A),\ Q \in \Gamma(\wedge^l A).$$

By the fact that the degree of κ is 3 and the degree of the Poisson bracket is -2, we deduce that the degree of the Schouten bracket is $3-2-2=-1$.

Now for a split Lie 2-algebroid, using the above idea, we define higher bracket operations on its symmetric algebra using the canonical Poisson bracket on the shifted cotangent bundle.

Given a split Lie 2-algebroid $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ with the structure function μ given by (6), denote by $(\mathsf{Sym}(\mathcal{A}[-3]) = \bigoplus_k \mathsf{Sym}^k(\mathcal{A}[-3]), \odot)$ the symmetric algebra of $\mathcal{A}[-3]$, in which $\mathsf{Sym}^k(\mathcal{A}[-3])$ is given by[3]

$$\mathsf{Sym}^k\big(\mathcal{A}[-3]\big) = \sum_{p+2q=k} \mathsf{Sym}^{p,q}\big(\mathcal{A}[-3]\big),$$

where

$$\mathsf{Sym}^{p,q}\big(\mathcal{A}[-3]\big) = \Gamma(\wedge^p A_{-2}) \odot \Gamma\big(\mathsf{Sym}^q(A_{-1})\big).$$

We will use $|\cdot|$ to denote the degree of a homogeneous element in $\mathsf{Sym}(\mathcal{A}[-3])$.

For all $P \in \mathsf{Sym}^k(\mathcal{A}[-3])$, $Q \in \mathsf{Sym}^l(\mathcal{A}[-3])$, $R \in \mathsf{Sym}^m(\mathcal{A}[-3])$, define

$$[\cdot]_S : \mathsf{Sym}^k\big(\mathcal{A}[-3]\big) \longrightarrow \mathsf{Sym}^{k+1}\big(\mathcal{A}[-3]\big)$$

[3]For the shifted vector bundle $\mathcal{A}[-3]$, elements in $\Gamma(A_{-2})$ are of degree 1 and elements in $\Gamma(A_{-1})$ are of degree 2.

by

$$[P]_S = -\{\mu^{(2,1,1)}, P\}. \tag{22}$$

Define

$$[\cdot,\cdot]_S \colon \mathsf{Sym}^k\big(\mathcal{A}[-3]\big) \times \mathsf{Sym}^l\big(\mathcal{A}[-3]\big) \longrightarrow \mathsf{Sym}^{k+l-2}\big(\mathcal{A}[-3]\big)$$

by

$$[P, Q]_S = -\{\{\mu^{(1,2,1)}, P\}, Q\}. \tag{23}$$

Define

$$[\cdot,\cdot,\cdot]_S \colon \mathsf{Sym}^k\big(\mathcal{A}[-3]\big) \times \mathsf{Sym}^l\big(\mathcal{A}[-3]\big) \times \mathsf{Sym}^m\big(\mathcal{A}[-3]\big) \longrightarrow \mathsf{Sym}^{k+l+m-5}\big(\mathcal{A}[-3]\big)$$

by

$$[P, Q, R]_S = -\{\{\{\mu^{(0,3,1)}, P\}, Q\}, R\}. \tag{24}$$

Comparing with (7), it is straightforward to obtain that

Lemma 4.1. *With the above notations, for all $X^1, Y^1, Z^1 \in \Gamma(A_{-1})$, $X^2 \in \Gamma(A_{-2})$, we have*

$$[X^2]_S = l_1(X^2), \quad [X^1, f]_S = a(X^1)(f), \quad [X^1, Y^1]_S = l_2(X^1, Y^1),$$
$$[X^1, X^2]_S = l_2(X^1, X^2), \quad [X^1, Y^1, Z^1]_S = l_3(X^1, Y^1, Z^1).$$

For $\alpha \in \Gamma(\mathcal{A}^*)$, define

$$\iota_\alpha \colon \mathsf{Sym}^{p,q}\big(\mathcal{A}[-3]\big) \longrightarrow \mathsf{Sym}^{p-1,q}\big(\mathcal{A}[-3]\big) \oplus \mathsf{Sym}^{p,q-1}\big(\mathcal{A}[-3]\big)$$

by

$$\begin{aligned}
\iota_\alpha&\big(X_1^1 \odot \cdots \odot X_p^1 \odot X_1^2 \odot \cdots \odot X_q^2\big)\\
&= \sum_{i=1}^{p} \langle \alpha, X_i^1 \rangle X_1^1 \odot \cdots \odot \widehat{X_i^1} \odot \cdots \odot X_p^1 \odot X_1^2 \odot \cdots \odot X_q^2\\
&+ \sum_{j=1}^{q} (-1)^{j+1} \langle \alpha, X_j^2 \rangle X_1^1 \odot \cdots \odot X_p^1 \odot X_1^2 \odot \cdots \odot \widehat{X_j^2} \odot \cdots \odot X_q^2,
\end{aligned}$$

where $X_i^1 \in \Gamma(A_{-1}[-3])$, $X_j^2 \in \Gamma(A_{-2}[-3])$.

For all $P \in \mathsf{Sym}^p(\mathcal{A}[-3])$, let us denote by

$$P(\alpha_1, \ldots, \alpha_p) = \iota_{\alpha_p} \cdots \iota_{\alpha_1} P, \quad \forall\, \alpha_i \in \Gamma(\mathcal{A}^*).$$

Using the properties of the graded Poisson bracket $\{\cdot,\cdot\}$, we get the following formulas.

Proposition 4.2. *For all* $P\in \mathsf{Sym}^{|P|}(\mathcal{A}[-3])$*,* $Q\in \mathsf{Sym}^{|Q|}(\mathcal{A}[-3])$*,* $R\in \mathsf{Sym}^{|R|}(\mathcal{A}[-3])$*, and* $W\in \mathsf{Sym}^{|W|}(\mathcal{A}[-3])$*, we have*

$$\begin{aligned}
[P, fQ]_S &= f[P,Q]_S + (-1)^{|P|}\iota_{\delta(f)}P\odot Q,\\
[P\odot Q]_S &= [P]_S\odot Q + (-1)^{|P|}P\odot [Q]_S,\\
[P,Q]_S &= (-1)^{(|P|-1)(|Q|-1)}[Q,P]_S,\\
[P, Q\odot R]_S &= [P,Q]_S\odot R + (-1)^{|P||Q|}Q\odot [P,R]_S,\\
[P,Q,R]_S &= (-1)^{(|P|-3)(|Q|-3)}[Q,P,R]_S = (-1)^{(|R|-3)(|Q|-3)}[P,R,Q]_S,\\
[P,Q,R\odot W]_S &= [P,Q,R]_S\odot W + (-1)^{(|P|+|Q|-5)|R|}R\odot [P,Q,W]_S.
\end{aligned}$$

It is easy to see that for all $X\in\Gamma(\mathcal{A})$, we have

$$[X, Y_1\odot\cdots\odot Y_n]_S = \sum_{i=1}^{n} Y_1\odot\cdots\odot l_2(X,Y_i)\odot\cdots\odot Y_n,\qquad \forall\, Y_i\in\Gamma(\mathcal{A}).$$

For all $X^1\in\Gamma(A_{-1})$, $D\in \mathsf{Sym}^{p,q}(\mathcal{A}[-3])$ and $\alpha_i\in\Gamma(\mathcal{A}^*)$, we have

$$\begin{aligned}
\big([X^1, D]_S\big)(\alpha_1,\ldots,\alpha_{p+q}) &= a(X^1)D(\alpha_1,\ldots,\alpha_{p+q})\\
&\quad - \sum_{i=1}^{p+q} D(\alpha_1,\ldots, L^1_{X^1}\alpha_i,\ldots,\alpha_{p+q}).
\end{aligned}$$

Proposition 4.3. *For all* $P\in \mathsf{Sym}^{|P|}(\mathcal{A}[-3])$*,* $Q\in \mathsf{Sym}^{|Q|}(\mathcal{A}[-3])$*,* $R\in \mathsf{Sym}^{|R|}(\mathcal{A}[-3])$*, and* $W\in \mathsf{Sym}^{|W|}(\mathcal{A}[-3])$*, we have*

$$\begin{aligned}
&[[P,Q]_S]_S = -[[P]_S, Q]_S + (-1)^{|P|}[P,[Q]_S]_S,\\
&[P,[Q,R]_S]_S - (-1)^{|P|}[[P,Q]_S,R]_S - (-1)^{|P||Q|}[Q,[P,R]_S]_S\\
&\quad = (-1)^{|P|}[[P,Q,R]_S]_S + (-1)^{|Q|}[P,Q,[R]_S]_S - [P,[Q]_S,R]_S\\
&\qquad + (-1)^{|P|}[[P]_S,Q,R]_S,
\end{aligned}$$

and

$$\begin{aligned}
&-(-1)^{|P|}[P,[Q,R,W]_S]_S + (-1)^{|P|(|Q|-1)}[Q,[P,R,W]_S]_S\\
&\quad + (-1)^{(|P|+|Q|-1)(|R|-1)}[R,[P,Q,W]_S]_S + [[P,Q,R]_S,W]_S\\
&\quad - (-1)^{(|R|-1)(|W|-1)}[[P,Q]_S,R,W]_S + (-1)^{|P|(|Q|-1)}[Q,[P,R]_S,W]_S\\
&\quad - (-1)^{|P|(|R|+|Q|)}[Q,R,[P,W]_S]_S - (-1)^{|P|}[P,[Q,R]_S,W]_S\\
&\quad - (-1)^{|P|+|Q|}[P,Q,[R,W]_S]_S + (-1)^{|R||Q|-|P|-|Q|}[P,R,[Q,W]_S]_S = 0.
\end{aligned}$$

Proof. It follows from the graded Jacobi identity for the Poisson bracket $\{\cdot,\cdot\}$. We omit details. □

To understand the meaning of the above brackets, we need the notion of a homotopy Poisson algebra ([32,42]).

Definition 4.4. (i) A *homotopy Poisson algebra* of degree n is a graded commutative algebra $\mathfrak{a}$ over a field of characteristic zero with an $L_\infty[1]$-algebra structure $\{l_m\}_{m\geq 1}$ on $\mathfrak{a}[n]$, such that the map

$$x \longrightarrow l_m(x_1,\ldots,x_{m-1},x), \quad x_1,\ldots,x_{m-1},x \in \mathfrak{a}$$

is a derivation of $\mathfrak{a}$ of degree $\kappa := \sum_{i=1}^{m-1}|x_i| + 1 - n(m-1)$, i.e. for all $x, y \in \mathfrak{a}$, we have

$$l_m(x_1,\ldots,x_{m-1},xy) = l_m(x_1,\ldots,x_{m-1},x)y + (-1)^{\kappa|x|}x l_m(x_1,\ldots,x_{m-1},y).$$

Here, $|x_i|$ denotes the degree of a homogeneous element $x_i \in \mathfrak{a}$.

(ii) A homotopy Poisson algebra of degree n is of *finite type* if there exists a q such that $l_m = 0$ for all $m > q$.

(iii) A *homotopy Poisson manifold* of degree n is a graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is equipped with a degree n homotopy Poisson algebra structure of finite type.

(iv) A *Maurer–Cartan element* of a homotopy Poisson algebra of degree n is a degree n element m satisfying

$$l_1(m) + \frac{1}{2}l_2(m,m) + \frac{1}{6}l_3(m,m,m) + \cdots = 0.$$

The only difference between the above definition and the one provided in [32] is that we use $L_\infty[1]$-algebra here, while in [32] the authors used L_∞-algebra. Since $L_\infty[1]$-algebras and L_∞-algebras are equivalent, there is no intrinsic difference.

The following theorem can be proved quickly using the derived bracket construction by Voronov in [58]. Here we give another proof using the properties of $[\cdot]_S$, $[\cdot,\cdot]_S$, and $[\cdot,\cdot,\cdot]_S$.

Theorem 4.5. *Let $\mathcal{A}$ be a split Lie 2-algebroid. Then* $(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S, [\cdot,\cdot]_S, [\cdot,\cdot,\cdot]_S)$ *is a homotopy Poisson algebra of degree* 3.

Proof. By Proposition 4.3, it is obvious that $[\cdot]_S$, $[\cdot,\cdot]_S$, $[\cdot,\cdot,\cdot]_S$ define an $L_\infty[1]$-algebra structure on $\mathsf{Sym}(\mathcal{A}[-3])[3]$. By Proposition 4.2, the graded derivation conditions for $[\cdot]_S$, $[\cdot,\cdot]_S$, $[\cdot,\cdot,\cdot]_S$ are satisfied. Thus, $(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S, [\cdot,\cdot]_S, [\cdot,\cdot,\cdot]_S)$ is a homotopy Poisson algebra of degree 3. □

Corollary 4.6. *Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid. Then $\mathcal{A}^*[3] = A^*_{-2}[3] \oplus A^*_{-1}[3]$ is a homotopy Poisson manifold of degree* 3.

The method of defining higher bracket operations using the Poisson bracket of the shifted cotangent bundle can be easily generalized to a split Lie n-algebroid. Consequently, one can obtain a homotopy Poisson algebra of degree $n+1$ on the symmetric algebra $\mathsf{Sym}(\mathcal{A}[-n-1])$ of a split Lie n-algebroid $\mathcal{A}$ and $\mathcal{A}^*[n+1]$ is a homotopy Poisson manifold of degree $n+1$. In [32, Example 3.6], it is stated (but not proved) that the dual of a Lie n-algebroid is a homotopy Poisson manifold of degree n. The difference is generated by the difference between an $L_\infty[1]$-algebra and an L_∞-algebra, which is not intrinsic.

4.2. A new approach to split Lie 2-bialgebroids. In this subsection, we extend some results given in [41, 48] to Lie 2-bialgebroids. The notion of a split Lie 2-bialgebroid was introduced in [37] using the canonical Poisson bracket of the shifted cotangent bundle.

Now assume that there is a split Lie 2-algebroid structure on the dual bundle $\mathcal{A}^*[3] = A^*_{-1}[3] \oplus A^*_{-2}[3]$. The two cotangent bundles $\mathcal{M} = T^*[3](A_{-1} \oplus A_{-2})$ and $T^*[3](A^*_{-1} \oplus A^*_{-2})[3]$ are naturally isomorphic as graded symplectic manifold by the Legendre transformation. By Theorem 3.4, the split Lie 2-algebroid $(\mathcal{A}^*[3], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ gives rise to a degree 4 function γ on $\mathcal{M}$ satisfying $\{\gamma, \gamma\} = 0$. It is given in local coordinates $(x^i, \xi^j, \theta^k, p_i, \xi_j, \theta_k)$ by

$$\gamma = {\gamma_1}^{ij}(x) p_j \theta_i + {\gamma_2}_i^j(x)\xi_j\theta^i + \tfrac{1}{2}{\gamma_3}_k^{ij}(x)\theta^k\theta_i\theta_j + {\gamma_4}_k^{ij}\xi_i\theta_j\xi^k + \tfrac{1}{6}{\gamma_5}_l^{ijk}(x)\xi^l\theta_i\theta_j\theta_k. \quad (25)$$

According to the tridegree, γ can be decomposed into

$$\gamma = \gamma^{(2,1,1)} + \gamma^{(1,1,2)} + \gamma^{(0,1,3)}.$$

Definition 4.7. ([37]) Let $\mathcal{A}$ and $\mathcal{A}^*[3]$ be split Lie 2-algebroids with the structure functions μ and γ respectively. The pair $(\mathcal{A}, \mathcal{A}^*[3])$ is called a split Lie 2-bialgebroid if $\mu^{(2,1,1)} = \gamma^{(2,1,1)}$ and

$$\{\mu + \gamma - \mu^{(2,1,1)}, \mu + \gamma - \mu^{(2,1,1)}\} = 0, \quad (26)$$

where $\{\cdot,\cdot\}$ is the graded Poisson bracket on the shifted cotangent bundle

$$T^*[3](A_{-1} \oplus A_{-2}).$$

Denote a split Lie 2-bialgebroid by $(\mathcal{A}, \mathcal{A}^*[3])$.

Remark 4.8. (1) The condition $\mu^{(2,1,1)} = \gamma^{(2,1,1)}$ is due to the invariant condition (iii) in the definition of a CLWX 2-algebroid (see Definition 5.1).

(2) Note that the function $\mu + \gamma$ contains two copies of the term $\mu^{(2,1,1)}$, which is of the tridegree $(2, 1, 1)$. Thus, we use the degree 4 function $\mu + \gamma - \mu^{(2,1,1)}$ in the definition of a split Lie 2-bialgebroid.

Now using the homotopy Poisson algebra associated to a split Lie 2-algebroid, we can describe a split Lie 2-bialgebroid using the usual language of differential geometry similar as the case of a Lie bialgebroid ([41]).

Theorem 4.9. *Let $\mathcal{A}$ and $\mathcal{A}^*[3]$ be split Lie 2-algebroids with the structure functions μ and γ respectively, such that $\mu^{(2,1,1)} = \gamma^{(2,1,1)}$. Then $(\mathcal{A}, \mathcal{A}^*[3])$ is a split Lie 2-bialgebroid if and only if the following two conditions hold:*

$$\delta_*[X,Y]_S = -[\delta_*(X),Y]_S + (-1)^{|X|}[X,\delta_*(Y)]_S, \quad \forall\, X,Y \in \Gamma\big(\mathcal{A}[-3]\big), \tag{27}$$

$$\delta[\alpha,\beta]_S = -[\delta(\alpha),\beta]_S + (-1)^{|\alpha|}[\alpha,\delta(\beta)]_S, \quad \forall\, \alpha,\beta \in \Gamma(\mathcal{A}^*), \tag{28}$$

where δ_* and δ are coboundary operators associated to split Lie 2-algebroids $\mathcal{A}^*[3]$ and $\mathcal{A}$, respectively.

Proof. Let $\mathcal{A}$ and $\mathcal{A}^*[3]$ be split Lie 2-algebroids. Then by the tridegree reason, the following equalities are automatically satisfied:

$$\{\mu^{(1,2,1)},\mu^{(2,1,1)}\} = 0, \quad \{\mu^{(1,2,1)},\mu^{(1,2,1)}\} + 2\{\mu^{(2,1,1)},\mu^{(0,3,1)}\} = 0, \quad \{\mu^{(1,2,1)},\mu^{(0,3,1)}\} = 0; \tag{29}$$

$$\{\gamma^{(1,1,2)},\gamma^{(2,1,1)}\} = 0, \quad \{\gamma^{(1,1,2)},\gamma^{(1,1,2)}\} + 2\{\gamma^{(2,1,1)},\gamma^{(0,1,3)}\} = 0, \quad \{\gamma^{(1,1,2)},\gamma^{(0,1,3)}\} = 0. \tag{30}$$

If $(\mathcal{A}, \mathcal{A}^*[3])$ is a split Lie 2-bialgebroid, by the tridegree reason,

$$\{\mu+\gamma-\mu^{(2,1,1)},\mu+\gamma-\mu^{(2,1,1)}\} = 0$$

is equivalent to

$$\{\mu^{(1,2,1)},\gamma^{(1,1,2)}\} = 0, \quad \{\mu^{(1,2,1)},\gamma^{(0,1,3)}\} = 0, \quad \{\gamma^{(1,1,2)},\mu^{(0,3,1)}\} = 0. \tag{31}$$

By (29) and (30), (31) is equivalent to

$$\{\gamma,\mu^{(1,2,1)}\} = 0, \quad \{\mu,\gamma^{(1,1,2)}\} = 0.$$

For all $X, Y \in \Gamma(\mathcal{A})$, by the graded Jacobi identity and (23), we have

$$\begin{aligned}
&\{\{\{\mu^{(1,2,1)},\gamma\},X\},Y\} \\
&= \{\{\mu^{(1,2,1)},\{\gamma,X\}\},Y\} + (-1)^{(|X|-3)}\{\{\{\mu^{(1,2,1)},X\},\gamma\},Y\} \\
&= \{\mu^{(1,2,1)},\{\{\gamma,X\},Y\}\} + (-1)^{(|Y|-3)(|X|)}\{\{\mu^{(1,2,1)},Y\},\{\gamma,X\}\} \\
&\quad + (-1)^{(|X|-3)}\{\{\mu^{(1,2,1)},X\},\{\gamma,Y\}\} + (-1)^{(|X|+|Y|)}\{\{\{\mu^{(1,2,1)},X\},Y\},\gamma\} \\
&= \{\{\mu^{(1,2,1)},\{\gamma,X\}\},Y\} + (-1)^{(|X|-3)}\{\{\mu^{(1,2,1)},X\},\{\gamma,Y\}\} \\
&\qquad + \{\gamma,\{\{\mu^{(1,2,1)},X\},Y\}\} \\
&= -\delta_*[X,Y]_S - [\delta_*(X),Y]_S + (-1)^{|X|}[X,\delta_*(Y)]_S.
\end{aligned}$$

Thus, $\{\gamma, \mu^{(1,2,1)}\} = 0$ if and only if (27) holds. Similarly, we can show that $\{\mu, \gamma^{(1,1,2)}\} = 0$ if and only if (28) holds. We finish the proof. □

At the end of this section, we give some useful formulas that will be used in the next section.

Proposition 4.10. *Let* $(\mathcal{A}, \mathcal{A}^*[3])$ *be a split Lie 2-bialgebroid. Then we have*

$$\mathcal{L}^2_{\mathrm{d}f} X^1 = -[X^1, \mathrm{d}_* f]_S, \quad L^2_{\mathrm{d}_* f}\alpha^2 = -[\alpha^2, \mathrm{d}f]_S,$$

for all $X^1 \in \Gamma(A_{-1})$, $\alpha^2 \in \Gamma(A^*_{-2})$, $f \in C^\infty(M)$.

Proof. For all $X, Y \in \Gamma(\mathcal{A}[-3])$ and $f \in C^\infty(M)$, by (27), we have

$$\begin{aligned}\mathrm{d}_*[X, fY]_S &= \mathrm{d}_*(f[X,Y]_S) + \mathrm{d}_*(a(X)(f)Y) \\ &= f\mathrm{d}_*[X,Y]_S + \mathrm{d}_*(f) \odot [X,Y]_S \\ &\quad + \big(\mathrm{d}_*(a(X)(f)) \odot Y + a(X)(f)\mathrm{d}_*(Y)\big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathrm{d}_*[X, fY]_S &= -[\mathrm{d}_*(X), fY]_S + (-1)^{|X|}[X, \mathrm{d}_*(fY)]_S \\ &= -f[\mathrm{d}_*(X), Y]_S - (-1)^{|X|+1}\iota_{\mathrm{d}f}\mathrm{d}_* X \odot Y \\ &\quad + (-1)^{|X|}[X, \mathrm{d}_* f]_S \odot Y + \mathrm{d}_*(f) \odot [X,Y]_S \\ &\quad + (-1)^{|X|} f[X, \mathrm{d}_* Y]_S + a(X)(f)\mathrm{d}_*(Y).\end{aligned}$$

Therefore, we have

$$\mathrm{d}_*(a(X)(f)) \odot Y = (-1)^{|X|}\iota_{\mathrm{d}f}\mathrm{d}_* X \odot Y + (-1)^{|X|}[X, \mathrm{d}_* f]_S \odot Y.$$

For $X = X^1$, we have

$$\mathrm{d}_*(a(X^1)(f)) - \iota_{\mathrm{d}f}\mathrm{d}_* X^1 = -[X^1, \mathrm{d}_* f]_S,$$

which implies that $\mathcal{L}^2_{\mathrm{d}f} X^1 = -[X^1, \mathrm{d}_* f]_S$.

The other one can be proved similarly. We omit details. □

5. Manin triples of split Lie 2-algebroids

The notion of a CLWX 2-algebroid (named after Courant–Liu–Weinstein–Xu) was introduced in [37] as the categorification of a Courant algebroid [38, 48].

Definition 5.1. A CLWX 2-*algebroid* is a graded vector bundle $\mathcal{E} = E_{-2} \oplus E_{-1}$ over M equipped with a non-degenerate graded symmetric bilinear form S on $\mathcal{E}$, a bilinear operation

$$\diamond\colon \Gamma(E_i) \times \Gamma(E_j) \longrightarrow \Gamma(E_{i+j+1}), \quad -3 \le i + j \le -2,$$

which is skewsymmetric on $\Gamma(E_{-1})\times\Gamma(E_{-1})$, an E_{-2}-valued 3-form Ω on E_{-1}, two bundle maps $\partial\colon E_{-2}\longrightarrow E_{-1}$ and $\rho\colon E_{-1}\longrightarrow TM$, such that E_{-2} and E_{-1} are isotropic and the following axioms are satisfied:

(i) $(\Gamma(E_{-2}),\Gamma(E_{-1}),\partial,\diamond,\Omega)$ is a Leibniz 2-algebra;

(ii) for all $e^1\in\Gamma(E_{-1})$, $e^2\in\Gamma(E_{-2})$, we have

$$e^1\diamond e^2-e^2\diamond e^1=\mathcal{D}S(e^1,e^2),$$

where $\mathcal{D}\colon C^\infty(M)\longrightarrow\Gamma(E_{-2})$ is defined by

$$S(\mathcal{D}f,e^1)=\rho(e^1)(f),\quad \forall\, e^1\in\Gamma(E_{-1});$$

(iii) for all $e^2_1,e^2_2\in\Gamma(E_{-2})$, we have

$$S(\partial(e^2_1),e^2_2)=S(e^2_1,\partial(e^2_2));$$

(iv) for all $e_1,e_2,e_3\in\Gamma(\mathcal{E})$, we have

$$\rho(e_1)S(e_2,e_3)=S(e_1\diamond e_2,e_3)+S(e_2,e_1\diamond e_3);$$

(v) for all $e^1_1,e^1_2,e^1_3,e^1_4\in\Gamma(E_{-1})$, we have

$$S(\Omega(e^1_1,e^1_2,e^1_3),e^1_4)=-S(e^1_3,\Omega(e^1_1,e^1_2,e^1_4)).$$

Denote a CLWX 2-algebroid by $(E_{-2},E_{-1},\partial,\rho,S,\diamond,\Omega)$, or simply by $\mathcal{E}$. The following lemma lists some properties of a CLWX 2-algebroid.

Lemma 5.2. *Let* $(E_{-2},E_{-1},\partial,\rho,S,\diamond,\Omega)$ *be a* CLWX 2*-algebroid. Then for all* $e_1,e_2\in\Gamma(\mathcal{E})$, $e^1,e^1_1,e^1_2\in\Gamma(E_{-1})$, *and* $f\in C^\infty(M)$, *we have*

$$e_1\diamond fe_2=f(e_1\diamond e_2)+\rho(e_1)(f)e_2,\quad (fe_1)\diamond e_2=f(e_1\diamond e_2)+\rho(e_2)(f)e_1+S(e_1,e_2)\mathcal{D}f,$$

$$\rho\diamond\partial=0,\qquad \partial\diamond\mathcal{D}=0,$$

$$e^1\diamond\mathcal{D}f=\mathcal{D}S(e^1,\mathcal{D}f),\qquad \mathcal{D}f\diamond e^1=0.$$

Definition 5.3. Let $\mathcal{E}=(E_{-2},E_{-1},\partial,\rho,S,\diamond,\Omega)$ be a CLWX 2-algebroid.

(a) A graded subbundle $L=L_{-2}\oplus L_{-1}$ of $\mathcal{E}$ is called *isotropic* if $S(X,Y)=0$, for all $X,Y\in\Gamma(L)$.

(b) A graded subbundle $L=L_{-2}\oplus L_{-1}$ of $\mathcal{E}$ is called *integral* if

(i) $\partial\big(\Gamma(L_{-2})\big)\subseteq\Gamma(L_{-1})$;

(ii) $\Gamma(L)$ is closed under the operation $\diamond$;

(iii) $\Omega\big(\Gamma(L_{-1}),\Gamma(L_{-1}),\Gamma(L_{-1})\big)\subseteq\Gamma(L_{-2})$.

(c) A maximal isotropic and integral graded subbundle L of $\mathcal{E}$ is called a *strict Dirac structure* of a CLWX 2-algebroid.

The following proposition follows immediately from the definition.

Proposition 5.4. *Let L be a strict Dirac structure of a* CLWX 2*-algebroid* $\mathcal{E} = (E_{-2}, E_{-1}, \partial, \rho, S, \diamond, \Omega)$. *Then* $(L_{-2}, L_{-1}, \partial|_L, \diamond|_L, \Omega|_L, \rho|_L)$ *is a split Lie* 2*-algebroid.*

Definition 5.5. A *Manin triple of split Lie* 2*-algebroids* $(\mathcal{E}; \mathcal{A}, \mathcal{B})$ consists of a CLWX 2-algebroid $\mathcal{E} = (E_{-2}, E_{-1}, \partial, \rho, S, \diamond, \Omega)$ and two transversal strict Dirac structures $\mathcal{A}$ and $\mathcal{B}$.

Theorem 5.6. *There is a one-to-one correspondence between Manin triples of split Lie* 2*-algebroids and split Lie* 2*-bialgebroids.*

Proof. It follows from the following Proposition 5.7 and Proposition 5.8. □

Assume that $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ is a split Lie 2-algebroid with structure function μ and $\mathcal{A}^*[3] = (A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ a split Lie 2-algebroid with the structure function γ. Let $E_{-1} = A_{-1} \oplus A^*_{-2}$ and $E_{-2} = A_{-2} \oplus A^*_{-1}$, and $\mathcal{E} = E_{-2} \oplus E_{-1}$.

Let $\partial\colon E_{-2} \longrightarrow E_{-1}$ and $\rho\colon E_{-1} \longrightarrow TM$ be bundle maps defined by

$$\partial(X^2 + \alpha^1) = l_1(X^2) + \mathfrak{l}_1(\alpha^1), \tag{32}$$
$$\rho(X^1 + \alpha^2)(f) = a(X^1)(f) + \mathfrak{a}(\alpha^2)(f). \tag{33}$$

On $\Gamma(\mathcal{E})$, there is a natural symmetric bilinear form $(\cdot\,,\cdot)_+$ given by

$$(X^1 + \alpha^2 + X^2 + \alpha^1, Y^1 + \beta^2 + Y^2 + \beta^1)_+ \\ = \langle X^1, \beta^1\rangle + \langle Y^1, \alpha^1\rangle + \langle X^2, \beta^2\rangle + \langle Y^2, \alpha^2\rangle. \tag{34}$$

On $\Gamma(\mathcal{E})$, we introduce operations

$$\diamond\colon E_i \times E_j \longrightarrow E_{i+j+1}, \quad -3 \le i + j \le -2,$$

by

$$\begin{cases} (X^1 + \alpha^2) \diamond (Y^1 + \beta^2) = l_2(X^1, Y^1) + L^1_{X^1}\beta^2 - L^1_{Y^1}\alpha^2 + \mathfrak{l}_2(\alpha^2, \beta^2) \\ \qquad\qquad + \mathcal{L}^1_{\alpha^2} Y^1 - \mathcal{L}^1_{\beta^2} X^1, \\ (X^1 + \alpha^2) \diamond (X^2 + \alpha^1) = l_2(X^1, X^2) + L^1_{X^1}\alpha^1 + \iota_{X^2}\mathrm{d}(\alpha^2) + \mathfrak{l}_2(\alpha^2, \alpha^1) \\ \qquad\qquad + \mathcal{L}^1_{\alpha^2} X^2 + \iota_{\alpha^1}\mathrm{d}_*(X^1), \\ (X^2 + \alpha^1) \diamond (X^1 + \alpha^2) = l_2(X^2, X^1) + L^2_{X^2}\alpha^2 + \iota_{X^1}\mathrm{d}(\alpha^1) + \mathfrak{l}_2(\alpha^1, \alpha^2) \\ \qquad\qquad + \mathcal{L}^2_{\alpha^1} X^1 + \iota_{\alpha^2}\mathrm{d}_*(X^2). \end{cases} \tag{35}$$

An E_{-2}-valued 3-form Ω on E_{-1} is defined by

$$\Omega(X^1+\alpha^2, Y^1+\beta^2, Z^1+\gamma^2) = l_3(X^1, Y^1, Z^1) + L^3_{X^1,Y^1}\gamma^2 + L^3_{Y^1,Z^1}\alpha^2 \\ + L^3_{Z^1,X^1}\beta^2 + \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2) + \mathfrak{L}^3_{\alpha^2,\beta^2}Z^1 + \mathfrak{L}^3_{\beta^2,\gamma^2}X^1 + \mathfrak{L}^3_{\gamma^2,\alpha^2}Y^1, \quad (36)$$

for all $X^1, Y^1, Z^1 \in \Gamma(A_{-1}), X^2, Y^2 \in \Gamma(A_{-2}), \alpha^1, \beta^1 \in \Gamma(A^*_{-1}), \alpha^2, \beta^2, \gamma^2 \in \Gamma(A^*_{-2})$.

It is proved in [37] that:

Proposition 5.7. *Let $(\mathcal{A}, \mathcal{A}^*[3])$ be a split Lie 2-bialgebroid. Then*

$$(E_{-2}, E_{-1}, \partial, \rho, (\cdot,\cdot)_+, \diamond, \Omega)$$

is a CLWX *2-algebroid, where $E_{-1} = A_{-1} \oplus A^*_{-2}$, $E_{-2} = A_{-2} \oplus A^*_{-1}$, ∂ is given by (32), ρ is given by (33), $(\cdot,\cdot)_+$ is given by (34), $\diamond$ is given by (35), and Ω is given by (36).*

Conversely, we have:

Proposition 5.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two transversal strict Dirac structures of a* CLWX *2-algebroid $\mathcal{E} = (E_{-2}, E_{-1}, \partial, \rho, S, \diamond, \Omega)$, i.e. $\mathcal{E} = \mathcal{A} \oplus \mathcal{B}$ as graded vector bundles. Then $(\mathcal{A}, \mathcal{B})$ is a split Lie 2-bialgebroid, where $\mathcal{B}$ is considered as the shifted dual bundle of $\mathcal{A}$ under the bilinear form S.*

The proof is a long and tedious calculation and we include it in the appendix.

6. Weak Dirac structures and Maurer–Cartan elements

Recall that given a Lie algebroid $(A, [\cdot,\cdot], \rho)$, there is naturally a Courant algebroid $A \oplus A^*$. Given an element $\pi \in \Gamma(\wedge^2 A)$ such that $[\pi, \pi] = 0$, one can define a Lie bracket $[\cdot,\cdot]_\pi$ on $\Gamma(A^*)$ such that $(A^*, [\cdot,\cdot]_\pi, \rho \circ \pi^\sharp)$ is a Lie algebroid, which we denote by A^*_π. Moreover, $\pi^\sharp$ is a Lie algebroid morphism from A^*_π to A and (A, A^*_π) is a triangular Lie bialgebroid ([41]). The graph of $\pi^\sharp$, which we denote by $\mathcal{G}_\pi \subset A \oplus A^*$ is a Dirac structure of the Courant algebroid $A \oplus A^*$. In this section, we generalize the above story to Lie 2-algebroids. First we introduce the notion of a weak Dirac structure of a CLWX 2-algebroid. Then we study Maurer–Cartan elements of the homotopy Poisson algebra associated to a Lie 2-algebroid $\mathcal{A}$ given in Section 4. We show that a Maurer–Cartan element gives rise to a split Lie 2-algebroid structure on the shifted dual bundle $\mathcal{A}^*[3]$ as well as a morphism from $\mathcal{A}^*[3]$ to $\mathcal{A}$. We also study Maurer–Cartan elements associated to a Lie 2-bialgebroid and show that the graph of such a Maurer–Cartan element is a weak Dirac structure of the corresponding CLWX 2-algebroid. Finally we give various examples including the string Lie 2-algebra, integrable distributions and left-symmetric algebroids.

Definition 6.1. A split Lie 2-algebroid $(L_{-2}, L_{-1}, l_1, l_2, l_3, a)$ is called a *weak Dirac structure* of a CLWX 2-algebroid $(E_{-2}, E_{-1}, \partial, \rho, S, \diamond, \Omega)$ if there exist bundle maps

$$F_1\colon L_{-1} \longrightarrow E_{-1}, \quad F_2\colon L_{-2} \longrightarrow E_{-2}, \quad \text{and} \quad F_3\colon \wedge^2 L_{-1} \longrightarrow E_{-2},$$

such that:

(i) F_1 and F_2 are injective such that the image $\mathrm{im}(F_2) \oplus \mathrm{im}(F_1)$ is a maximal isotropic graded subbundle of $E_{-2} \oplus E_{-1}$;

(ii) (F_1, F_2, F_3) is a morphism from the Lie 2-algebra $(\Gamma(L_{-2}), \Gamma(L_{-1}), l_1, l_2, l_3)$ to the Leibniz 2-algebra $(\Gamma(E_{-2}), \Gamma(E_{-1}), \partial, \diamond, \Omega)$;

(iii) $\rho \circ F_1 = a$.

It is obvious that a strict Dirac structure L given in Definition 5.3 is a weak Dirac structure, in which F_1 and F_2 are inclusion maps and $F_3 = 0$.

6.1. Maurer–Cartan elements associated to a split Lie 2-algebroid.

Definition 6.2. Let $\mathcal{A}$ be a split Lie 2-algebroid. A *Maurer–Cartan element* of the associated homotopy Poisson algebra $(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S, [\cdot,\cdot]_S, [\cdot,\cdot,\cdot]_S)$ given in Theorem 4.5 is an element

$$m \in \mathsf{Sym}^3\big(\mathcal{A}[-3]\big) = A_{-1}[-3] \odot A_{-2}[-3] \oplus \wedge^3 A_{-2}[-3]$$

such that

$$[m]_S + \tfrac{1}{2}[m,m]_S + \tfrac{1}{6}[m,m,m]_S = 0. \tag{37}$$

An element $m \in \mathsf{Sym}^3(\mathcal{A}[-3])$ consists of an $H \in \Gamma(A_{-1} \odot A_{-2})$ and a $K \in \Gamma(\wedge^3 A_{-2})$. For $H \in \Gamma(A_{-1} \odot A_{-2})$, define

$$H^{\natural}\colon \Gamma(A^*_{-1}) \to \Gamma(A_{-2}) \quad \text{and} \quad H^{\sharp}\colon \Gamma(A^*_{-2}) \to \Gamma(A_{-1})$$

by

$$\langle H^{\natural}(\alpha^1), \alpha^2\rangle = H(\alpha^1, \alpha^2), \quad \langle H^{\sharp}(\alpha^2), \alpha^1\rangle = H(\alpha^2, \alpha^1),$$

for all $\alpha^1 \in \Gamma(A^*_{-1}), \alpha^2 \in \Gamma(A^*_{-2})$. We have $\{\alpha^1, H\} = H^{\natural}(\alpha^1), \{\alpha^2, H\} = -H^{\sharp}(\alpha^2)$. For $K \in \Gamma(\wedge^3 A_{-2})$, define

$$K^{\flat}\colon \wedge^2 A^*_{-2} \longrightarrow A_{-2}$$

by

$$\langle K^{\flat}(\alpha^2, \beta^2), \gamma^2\rangle = K(\alpha^2, \beta^2, \gamma^2), \quad \forall\, \alpha^2, \beta^2, \gamma^2 \in \Gamma(A^*_{-2}). \tag{38}$$

It is not hard to see that

$$\{\{K, \alpha^2\}, \beta^2\} = -K^{\flat}(\alpha^2, \beta^2), \quad \{\{\{K, \alpha^2\}, \beta^2\}, \gamma^2\} = -K(\alpha^2, \beta^2, \gamma^2).$$

Let μ be the degree 4 function on $T^*[3](A_{-1} \oplus A_{-2})$ corresponding to the split Lie 2-algebroid $\mathcal{A}$. For $H \in \Gamma(A_{-1} \odot A_{-2})$ and $K \in \Gamma(\wedge^3 A_{-2})$, define

$$\begin{cases} \mathfrak{l}_1^H(\alpha^1) = -\{\mu^{(2,1,1)}, \alpha^1\}, \\ \mathfrak{l}_2^H(\alpha^2, \beta^2) = -\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^2\}, \\ \mathfrak{l}_2^H(\alpha^2, \beta^1) = -\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^1\}, \\ \mathfrak{l}_3^{H,K}(\alpha^2, \beta^2, \gamma^2) = -\{\{\{\{\mu^{(1,2,1)}, K\}, \alpha^2\}, \beta^2\}, \gamma^2\} \\ \qquad - \frac{1}{2}\{\{\{\{\{\mu^{(0,3,1)}, H\}, H\}, \alpha^2\}, \beta^2\}, \gamma^2\}, \\ \mathfrak{a}_H(\alpha^2)(f) = -\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, f\}, \end{cases} \tag{39}$$

for all $\alpha^2, \beta^2, \gamma^2 \in \Gamma(A_{-2}^*)$, $\alpha^1, \beta^1 \in \Gamma(A_{-1}^*)$, $f \in C^\infty(M)$. We use the Lie derivatives introduced in Section 3 to give a precise description of the above operations.

Lemma 6.3. *For all $\alpha^2, \beta^2, \gamma^2 \in \Gamma(A_{-2}^*)$, $\beta^1 \in \Gamma(A_{-1}^*)$, we have*

$$\mathfrak{l}_1^H = l_1^*, \tag{40}$$

$$\mathfrak{l}_2^H(\alpha^2, \beta^2) = L^1_{H^\sharp(\alpha^2)}\beta^2 - L^1_{H^\sharp(\beta^2)}\alpha^2, \tag{41}$$

$$\mathfrak{l}_2^H(\alpha^2, \beta^1) = L^1_{H^\sharp(\alpha^2)}\beta^1 - L^2_{H^\natural(\beta^1)}\alpha^2 - \mathrm{d}H(\alpha^2, \beta^1), \tag{42}$$

$$\begin{aligned} \mathfrak{l}_3^{H,K}(\alpha^2, \beta^2, \gamma^2) = & -L^2_{K^\flat(\alpha^2,\beta^2)}\gamma^2 - L^2_{K^\flat(\gamma^2,\alpha^2)}\beta^2 - L^2_{K^\flat(\beta^2,\gamma^2)}\alpha^2 \\ & - 2\mathrm{d}K(\alpha^2, \beta^2, \gamma^2) + L^3_{H^\sharp(\alpha^2), H^\sharp(\beta^2)}\gamma^2 \\ & + L^3_{H^\sharp(\beta^2), H^\sharp(\gamma^2)}\alpha^2 + L^3_{H^\sharp(\gamma^2), H^\sharp(\alpha^2)}\beta^2, \end{aligned} \tag{43}$$

$$\mathfrak{a}_H = a \circ H^\sharp. \tag{44}$$

We need the following preparation before we give the main result in this subsection.

Lemma 6.4. *Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid with the structure function μ and $H \in \Gamma(A_{-1} \odot A_{-2})$, $K \in \Gamma(\wedge^3 A_{-2})$. Then for all $\alpha^2, \beta^2, \gamma^2 \in \Gamma(A_{-2}^*)$, $\alpha^1, \beta^1 \in \Gamma(A_{-1}^*)$, we have*

$$[H]_S(\alpha^1) = l_1(H^\natural(\alpha^1)) + H^\sharp(\mathfrak{l}_1^H(\alpha^1)), \tag{45}$$

$$[K]_S(\alpha^2, \beta^2) = l_1 K^\flat(\alpha^2, \beta^2), \tag{46}$$

$$[K]_S(\alpha^2, \beta^1) = -K^\flat(\alpha^2, \mathfrak{l}_1^H(\beta^1)), \tag{47}$$

$$\tfrac{1}{2}[H, H]_S(\alpha^2, \beta^2, \cdot) = H^\sharp(\mathfrak{l}_2^H(\alpha^2, \beta^2)) - l_2(H^\sharp(\alpha^2), H^\sharp(\beta^2)), \tag{48}$$

$$\tfrac{1}{2}[H, H]_S(\alpha^2, \beta^1, \cdot) = H^\natural(\mathfrak{l}_2^H(\alpha^2, \beta^1)) - l_2(H^\sharp(\alpha^2), H^\natural(\beta^1)), \tag{49}$$

$$
\begin{aligned}
[H, K]_S(\alpha^2, \beta^2, \gamma^2) = {} & K^\flat(\mathfrak{l}_2^H(\alpha^2, \beta^2), \gamma^2) + K^\flat(\mathfrak{l}_2^H(\gamma^2, \alpha^2), \beta^2) \\
& + K^\flat(\mathfrak{l}_2^H(\beta^2, \gamma^2), \alpha^2) - l_2(H^\sharp(\alpha^2), K^\flat(\beta^2, \gamma^2)) \\
& - l_2(H^\sharp(\gamma^2), K^\flat(\alpha^2, \beta^2)) - l_2(H^\sharp(\beta^2), K^\flat(\gamma^2, \alpha^2)) \\
& - H^\natural(L^2_{K^\flat(\alpha^2,\beta^2)}\gamma^2 + L^2_{K^\flat(\gamma^2,\alpha^2)}\beta^2 + L^2_{K^\flat(\beta^2,\gamma^2)}\alpha^2 \\
& + 2\mathrm{d}K(\alpha^2, \beta^2, \gamma^2)), \qquad (50) \\
\tfrac{1}{6}[H, H, H]_S(\alpha^2, \beta^2, \gamma^2) = {} & H^\natural(L^3_{H^\sharp(\alpha^2),H^\sharp(\beta^2)}\gamma^2 + L^3_{H^\sharp(\beta^2),H^\sharp(\gamma^2)}\alpha^2 \\
& + L^3_{H^\sharp(\gamma^2),H^\sharp(\alpha^2)}\beta^2) + l_3(H^\sharp(\alpha^2), H^\sharp(\beta^2), H^\sharp(\gamma^2)). \qquad (51)
\end{aligned}
$$

Proof. By the graded Jacobi identity for the canonical Poisson bracket on $T^*[3](A_{-1} \oplus A_{-2})$, we have

$$
\begin{aligned}
[H]_S(\alpha^1) &= \{\{\mu^{(2,1,1)}, H\}, \alpha^1\} \\
&= -\{\mu^{(2,1,1)}, H^\natural(\alpha^1)\} + \{\{\mu^{(2,1,1)}, \alpha^1\}, H\} \\
&= l_1(H^\sharp(\alpha^1)) + H^\sharp(\mathfrak{l}_1^H(\alpha_1)), \\
[K]_S(\alpha^2, \beta^2) &= \{\{\{\mu^{(2,1,1)}, K\}, \alpha^2\}, \beta^2\} = \{\{\mu^{(2,1,1)}, \{K, \alpha^2\}\}, \beta^2\} \\
&= \{\mu^{(2,1,1)}, \{\{K, \alpha^2\}, \beta^2\}\} = -\{\mu^{(2,1,1)}, K^\flat(\alpha^2, \beta^2)\} \\
&= l_1 K^\flat(\alpha^2, \beta^2), \\
[K]_S(\alpha^2, \beta^1) &= -\{\{\{\mu^{(2,1,1)}, K\}, \alpha^2\}, \beta^1\} = -\{\{\mu^{(2,1,1)}, \{K, \alpha^2\}\}, \beta^1\} \\
&= -\{\{\mu^{(2,1,1)}, \beta^1\}, \{K, \alpha^2\}\} = \{\mathfrak{l}_1^H(\beta^1), \{K, \alpha^2\}\} \\
&= -K^\flat(\alpha^2, \mathfrak{l}_1^H(\beta^1)), \\
[H, H]_S(\alpha^2, \beta^2, \cdot) &= \{\{\{\{\mu^{(1,2,1)}, H\}, H\}, \alpha^2\}, \beta^2\} \\
&= \{\{\{\mu^{(1,2,1)}, H\}, \{H, \alpha^2\}, \beta^2\} \\
&\qquad + \{\{\{\{\mu^{(1,2,1)}, H\}, H\}, \alpha^2\}\}, \beta^2\} \\
&= -\{\{\{\mu^{(1,2,1)}, H\}, \beta^2\}, \{H, \alpha^2\}\} \\
&\qquad + \{\{\{\mu^{(1,2,1)}, H\}, \{H, \alpha^2\}\}, \beta^2\} \\
&\qquad + \{\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, H\}, \beta^2\} \\
&= -\{\{\mu^{(1,2,1)}, \{H, \beta^2\}\}, \{H, \alpha^2\}\} \\
&\qquad - \{\{\{\mu^{(1,2,1)}, \beta^2\}, H\}, \{H, \alpha^2\}\} \\
&\qquad + \{\{\mu^{(1,2,1)}, \{H, \alpha^2\}\}, \{H, \beta^2\}\} \\
&\qquad + \{\{\{\mu^{(1,2,1)}, \alpha^2\}, H\}, \{H, \beta^2\}\} \\
&\qquad + \{\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^2\}, H\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\{\{\mu^{(1,2,1)}, \{H, \alpha^2\}\}, \{H, \beta^2\}\} \\
&\qquad + \{\{\{\mu^{(1,2,1)}, \{H, \alpha^2\}\}, \beta^2\}, H\} \\
&\qquad - \{\{\{\mu^{(1,2,1)}, \{H, \beta^2\}\}, \alpha^2\}, H\} \\
&\qquad + \{\{\{\{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^2\}, H\} \\
&= 2H^\sharp \mathfrak{l}_2^H(\alpha^2, \beta^2) - 2l_2(H^\sharp(\alpha^2), H^\sharp(\beta^2)),
\end{aligned}
$$

which imply that (45)–(48) hold.

By direct calculation, we have

$$
\begin{aligned}
&\langle H^\natural(\mathfrak{l}_2^H(\alpha^2, \beta^1)) - l_2(H^\sharp(\alpha^2), H^\natural(\beta^1)), \beta^2\rangle \\
&\qquad = \langle H^\sharp(\mathfrak{l}_2^H(\alpha^2, \beta^2)) - l_2(H^\sharp(\alpha^2), H^\sharp(\beta^2)), \beta^1\rangle.
\end{aligned}
$$

By (48), (49) follows immediately.

(50) and (51) can be proved similarly. We omit the details. □

Now we are ready to give the main result in this subsection. Consider the following function[4] $\gamma_{H,K}$ of degree 4 on $\mathcal{M} = T^*[3](A_{-1} \oplus A_{-2})$:

$$
\gamma_{H,K} = \mu^{(2,1,1)} + \{\mu^{(1,2,1)}, H\} + \{\mu^{(1,2,1)}, K\} + \tfrac{1}{2}\{\{\mu^{(0,3,1)}, H\}, H\}. \tag{52}
$$

Theorem 6.5. *Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid and $H + K$ is a Maurer–Cartan element of the associated homotopy Poisson algebra $(\mathsf{Sym}(\mathcal{A}[-3]), [\cdot]_S, [\cdot,\cdot]_S, [\cdot,\cdot,\cdot]_S)$, i.e.*

$$
[H + K]_S + \tfrac{1}{2}[H + K, H + K]_S + \tfrac{1}{6}[H + K, H + K, H + K]_S = 0. \tag{53}
$$

Then we have:

(i) $\mathcal{A}^*[3] = (A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H)$ *is a split Lie 2-algebroid, where $\mathfrak{l}_1^H$, $\mathfrak{l}_2^H$, $\mathfrak{l}_3^{H,K}$, and $\mathfrak{a}_H$ are given by (40)–(44) respectively;*

(ii) $(H^\sharp, -H^\natural, -K^\flat)$ *is a morphism from the split Lie 2-algebroid*

$$
\mathcal{A}^*[3] = (A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H)
$$

to the split Lie 2-algebroid $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$;

[4] The function $\gamma_{H,K}$ is obtained in the following intrinsic way. The map

$$
\{H + K, \cdot\}\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})
$$

is an inner derivation of $C^\infty(\mathcal{M})$. It follows that $e^{\{H+K,\cdot\}}$ is an automorphism of $C^\infty(\mathcal{M})$. Thus, $\mu' := e^{\{H+K,\cdot\}}\mu$ is also a degree 4 function satisfying $\{\mu', \mu'\} = 0$. $\gamma_{H,K}$ is exactly the projection of μ' to the subspace

$$
C^{(2,1,1)}(\mathcal{M}) \oplus C^{(1,1,2)}(\mathcal{M}) \oplus C^{(0,1,3)}(\mathcal{M})
$$

of $C^\infty(\mathcal{M})$. See [49] for a similar discussion in Lie algebroids.

(iii) $(\mathcal{A}, \mathcal{A}^*[3])$ *is a split Lie 2-bialgebroid if and only if*

$$\{\mu^{(2,1,1)}, \{\mu^{(0,3,1)}, H\}\} = 0, \quad \{\mu^{(1,2,1)}, \{\mu^{(0,3,1)}, H\}\} = 0,$$
$$\{\mu^{(0,3,1)}, \{\mu^{(2,1,1)}, K\}\} = 0.$$

Proof. It is straightforward to deduce that (53) is equivalent to the following equations

$$[H]_S = 0, \tag{54}$$
$$[K]_S + \tfrac{1}{2}[H, H]_S = 0, \tag{55}$$
$$[H, K]_S + \tfrac{1}{6}[H, H, H]_S = 0. \tag{56}$$

(i) To show that

$$\mathcal{A}^*[3] = (A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H)$$

is a split Lie 2-algebroid, we only need to prove

$$\{\gamma_{H,K}, \gamma_{H,K}\} = 0,$$

which is equivalent to the following equations:

$$\{\{\mu^{(1,2,1)}, H\}, \mu^{(2,1,1)}\} = 0, \tag{57}$$
$$\begin{aligned}\{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, H\}\} + 2\{\mu^{(2,1,1)}, \{\mu^{(1,2,1)}, K\}\}&\\ + \{\mu^{(2,1,1)}, \{\{\mu^{(0,3,1)}, H\}, H\}\} &= 0,\end{aligned} \tag{58}$$
$$\{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, K\}\} + \tfrac{1}{2}\{\{\mu^{(1,2,1)}, H\}, \{\{\mu^{(0,3,1)}, H\}, H\}\} = 0. \tag{59}$$

By (54) and the fact that $\{\mu^{(1,2,1)}, \mu^{(2,1,1)}\} = 0$, we have

$$\{\{\mu^{(1,2,1)}, H\}, \mu^{(2,1,1)}\} = \{\mu^{(1,2,1)}, \{H, \mu^{(2,1,1)}\}\} + \{\{\mu^{(1,2,1)}, \mu^{(2,1,1)}\}, H\} = 0,$$

which implies that (57) holds.

By (54), (55) and the fact that $\frac{1}{2}\{\mu^{(1,2,1)}, \mu^{(1,2,1)}\} + \{\mu^{(2,1,1)}, \mu^{(0,3,1)}\} = 0$, we have

$$\begin{aligned}
&\{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, H\}\} + 2\{\mu^{(2,1,1)}, \{\mu^{(1,2,1)}, K\}\}\\
&\qquad + \{\mu^{(2,1,1)}, \{\{\mu^{(0,3,1)}, H\}, H\}\}\\
&= \{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, H\}\} + \{\mu^{(1,2,1)}, \{\{\mu^{(1,2,1)}, H\}, H\}\}\\
&\qquad + \{\{\{\mu^{(2,1,1)}, \mu^{(0,3,1)}\}, H\}, H\}\\
&= \{\{\tfrac{1}{2}\{\mu^{(1,2,1)}, \mu^{(1,2,1)}\} + \{\mu^{(2,1,1)}, \mu^{(0,3,1)}\}, H\}, H\} = 0,
\end{aligned}$$

which implies that (58) holds.

By (55) and (56), we have

$$
\begin{aligned}
&\{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, K\}\} \\
&\quad = \{\{\{\mu^{(1,2,1)}, H\}, \mu^{(1,2,1)}\}, K\} - \{\mu^{(1,2,1)}, \{\{\mu^{(1,2,1)}, H\}, K\}\} \\
&\quad = \{\{\tfrac{1}{2}\{\mu^{(1,2,1)}, \mu^{(1,2,1)}\}, H\}, K\} - \{\mu^{(1,2,1)}, \{\{\mu^{(1,2,1)}, H\}, K\}\} \\
&\quad = -\{\{\{\mu^{(2,1,1)}, \mu^{(0,3,1)}\}, H\}, K\} - \{\mu^{(1,2,1)}, \{\{\mu^{(1,2,1)}, H\}, K\}\} \\
&\quad = -\{\{\mu^{(2,1,1)}, K\}, \{\mu^{(0,3,1)}, H\}\} - \{\mu^{(1,2,1)}, \{\{\mu^{(1,2,1)}, H\}, K\}\} \\
&\quad = \tfrac{1}{2}\{\{\{\mu^{(1,2,1)}, H\}, H\}, \{\mu^{(0,3,1)}, H\}\} + \tfrac{1}{6}\{\mu^{(1,2,1)}, \{\{\{\mu^{(0,3,1)}, H\}, H\}, H\}\}.
\end{aligned}
$$

On the other hand, by the fact that $\{\mu^{(1,2,1)}, \mu^{(0,3,1)}\} = 0$, we have

$$
\begin{aligned}
\tfrac{1}{6}\{\mu^{(1,2,1)}, \{\{\{\mu^{(0,3,1)}, H\}, H\}, H\}\} = &-\tfrac{1}{2}\{\{\{\mu^{(1,2,1)}, H\}, H\}, \{\mu^{(0,3,1)}, H\}\} \\
&- \tfrac{1}{2}\{\{\mu^{(1,2,1)}, H\}, \{\{\mu^{(0,3,1)}, H\}, H\}\}.
\end{aligned}
$$

Therefore we have

$$
\{\{\mu^{(1,2,1)}, H\}, \{\mu^{(1,2,1)}, K\}\} + \tfrac{1}{2}\{\{\mu^{(1,2,1)}, H\}, \{\{\mu^{(0,3,1)}, H\}, H\}\} = 0,
$$

which implies that (59) holds. Thus, $(A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H = a \circ H^\sharp)$ is a split Lie 2-algebroid.

(ii) By (54) and Lemma 6.4, we obtain

$$
-l_1 \circ H^\natural = H^\sharp \circ \mathfrak{l}_1^H. \tag{60}
$$

By (55) and Lemma 6.4, we have

$$
H^\sharp \mathfrak{l}_2^H(\alpha^2, \beta^2) - l_2(H^\sharp(\alpha^2), H^\sharp(\beta^2)) = -l_1 K^\flat(\alpha^2, \beta^2), \tag{61}
$$

$$
(-H^\natural)\mathfrak{l}_2^H(\alpha^2, \beta^1) - l_2(H^\sharp(\alpha^2), -H^\natural(\beta^1)) = -K^\flat(\alpha^2, \mathfrak{l}_1^H(\beta^1)). \tag{62}
$$

By (56) and Lemma 6.4, we have

$$
\begin{aligned}
&K^\flat(\mathfrak{l}_2^H(\alpha^2, \beta^2), \gamma^2) + K^\flat(\mathfrak{l}_2^H(\gamma^2, \alpha^2), \beta^2) + K^\flat(\mathfrak{l}_2^H(\beta^2, \gamma^2), \alpha^2) \\
&- l_2(H^\sharp(\alpha^2), K^\flat(\beta^2, \gamma^2)) - l_2(H^\sharp(\gamma^2), K^\flat(\alpha^2, \beta^2)) - l_2(H^\sharp(\beta^2), K^\flat(\gamma^2, \alpha^2)) \\
&+ H^\natural\big(- L^2_{K^\flat(\alpha^2,\beta^2)}\gamma^2 - L^2_{K^\flat(\gamma^2,\alpha^2)}\beta^2 - L^2_{K^\flat(\beta^2,\gamma^2)}\alpha^2 - 2\mathrm{d}K(\alpha^2, \beta^2, \gamma^2) \\
&\qquad + L^3_{H^\sharp(\alpha^2),H^\sharp(\beta^2)}\gamma^2 + L^3_{H^\sharp(\beta^2),H^\sharp(\gamma^2)}\alpha^2 + L^3_{H^\sharp(\gamma^2),H^\sharp(\alpha^2)}\beta^2\big) \\
&+ l_3(H^\sharp(\alpha^2), H^\sharp(\beta^2), H^\sharp(\gamma^2)) = 0,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&-K^\flat(\mathfrak{l}_2^H(\alpha^2,\beta^2),\gamma^2)-K^\flat(\mathfrak{l}_2^H(\gamma^2,\alpha^2),\beta^2)-K^\flat(\mathfrak{l}_2^H(\beta^2,\gamma^2),\alpha^2)\\
&\qquad\qquad -H^\natural(\mathfrak{l}_3^{H,K}(\alpha^2,\beta^2,\gamma^2)\\
&=l_3(H^\sharp(\alpha^2),H^\sharp(\beta^2),H^\sharp(\gamma^2))+l_2(H^\sharp(\alpha^2),-K^\flat(\beta^2,\gamma^2))\\
&\quad +l_2(H^\sharp(\gamma^2),-K^\flat(\alpha^2,\beta^2))+l_2(H^\sharp(\beta^2),-K^\flat(\gamma^2,\alpha^2)). \qquad (63)
\end{aligned}
$$

Thus, $(H^\sharp,-H^\natural,-K^\flat)$ is a morphism from the split Lie 2-algebroid

$$(A^*_{-1}[3],A^*_{-2}[3],\mathfrak{l}_1^H,\mathfrak{l}_2^H,\mathfrak{l}_3^{H,K},\mathfrak{a}_H)$$

to the split Lie 2-algebroid $(A_{-2},A_{-1},l_1,l_2,l_3,a)$.

(iii) Note that $(\mathcal{A},\mathcal{A}^*[3])$ is a split Lie 2-bialgebroid if and only if

$$\{\mu+\gamma_{H,K}-\mu^{(2,1,1)},\mu+\gamma_{H,K}-\mu^{(2,1,1)}\}=0.$$

Since $(\mathcal{A},\mu)$ and $(\mathcal{A}^*[3],\gamma_{H,K})$ are split Lie 2-algebroids, the above equality is equivalent to

$$
\begin{aligned}
\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},H\}\}&=0,\\
\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},K\}\}+\tfrac12\{\mu^{(1,2,1)},\{\{\mu^{(0,3,1)},H\},H\}\}&=0,\\
\{\{\mu^{(1,2,1)},H\},\mu^{(0,3,1)}\}&=0.
\end{aligned}
$$

By the fact that $\frac12\{\mu^{(1,2,1)},\mu^{(1,2,1)}\}+\{\mu^{(2,1,1)},\mu^{(0,3,1)}\}=0$ and $\{H,\mu^{(2,1,1)}\}=0$, we have

$$\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},H\}\}=-\{\mu^{(2,1,1)},\{\mu^{(0,3,1)},H\}\}.$$

By the fact that $\{\mu^{(1,2,1)},\mu^{(0,3,1)}\}=0$, we have

$$\{\{\mu^{(1,2,1)},H\},\mu^{(0,3,1)}\}=-\{\mu^{(1,2,1)},\{\mu^{(0,3,1)},H\}\}.$$

By the fact that $\{\mu^{(1,2,1)},\mu^{(0,3,1)}\}=0$ and (55), we have

$$
\begin{aligned}
&\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},K\}\}+\tfrac12\{\mu^{(1,2,1)},\{\{\mu^{(0,3,1)},H\},H\}\}\\
&\qquad=\{\{\mu^{(1,2,1)},\{\mu^{(0,3,1)},H\}\},H\}-2\{\mu^{(0,3,1)},\{\mu^{(2,1,1)},K\}\}.
\end{aligned}
$$

Thus, $(\mathcal{A},\mathcal{A}^*[3])$ is a split Lie 2-bialgebroid if and only if

$$
\begin{gathered}
\{\mu^{(2,1,1)},\{\mu^{(0,3,1)},H\}\}=0,\quad \{\mu^{(1,2,1)},\{\mu^{(0,3,1)},H\}\}=0,\\
\{\mu^{(0,3,1)},\{\mu^{(2,1,1)},K\}\}=0.
\end{gathered}
$$

We finish the proof. □

Remark 6.6. According to Proposition 5.7, given a Lie 2-algebroid

$$(A_{-2}, A_{-1}, l_1, l_2, l_3, a),$$

then

$$(A_{-2} \oplus A^*_{-1}, A_{-1} \oplus A^*_{-2}, \partial, \rho, (\cdot,\cdot)_+, \diamond, \Omega)$$

is a CLWX 2-algebroid. It is natural to expect that the graph of $(H^\sharp, -H^\natural)$ is a "Dirac structure". However, it is straightforward to see that the graph of $(H^\sharp, -H^\natural)$ is not closed under the operation $\diamond$ anymore. Thus, it is not a strict Dirac structure defined in Definition 5.3. But by Theorem 6.5, we can deduce that the graph of $(H^\sharp, -H^\natural)$ is a weak Dirac structure defined in Definition 6.1. We will prove this result for the more general case of split Lie 2-bialgebroids in the next subsection.

6.2. Weak Dirac structures and Maurer–Cartan elements associated to a split Lie 2-bialgebroid. Let $\mathcal{A} = (A_{-2}, A_{-1}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid with the structure function μ and $\mathcal{A}^*[3] = (A^*_{-1}[3], A^*_{-2}[3], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ a split Lie 2-algebroid with the structure function γ such that $(\mathcal{A}, \mathcal{A}^*[3])$ is a split Lie 2-bialgebroid. Let $H \in \Gamma(A_{-1} \odot A_{-2})$ and $K \in \Gamma(\wedge^3 A_{-2})$. Define Λ to be the degree 4 function on $T^*[3](A_{-1} \oplus A_{-2})$ by[5]

$$\Lambda = \gamma + \gamma_{H,K} - \mu^{(2,1,1)},$$

where $\gamma_{H,K}$ is given by (52). Write $\Lambda = \Lambda^{(2,1,1)} + \Lambda^{(1,1,2)} + \Lambda^{(0,1,3)}$, where

$$\begin{aligned}
\Lambda^{(2,1,1)} &= \gamma^{(2,1,1)} = \mu^{(2,1,1)},\\
\Lambda^{(1,1,2)} &= \gamma^{(1,1,2)} + \{\mu^{(1,2,1)}, H\},\\
\Lambda^{(0,1,3)} &= \gamma^{(0,1,3)} + \{\mu^{(1,2,1)}, K\} + \tfrac{1}{2}\{\{\mu^{(0,3,1)}, H\}, H\}.
\end{aligned}$$

For all $\alpha^2, \beta^2, \gamma^2 \in \Gamma(A^*_{-2})$, $\alpha^1, \beta^1 \in \Gamma(A^*_{-1})$, and $f \in C^\infty(M)$, define

$$\begin{cases}
\widetilde{\mathfrak{l}}^H_1(\alpha^1) = -\{\mu^{(2,1,1)}, \alpha^1\} = l^*_1(\alpha^1),\\
\widetilde{\mathfrak{a}}_H(\alpha^2)(f) = -\{\{\gamma^{(1,1,2)} + \{\mu^{(1,2,1)}, H\}, \alpha^2\}, f\} = (\mathfrak{a} + \mathfrak{a}_H)(\alpha^2)(f),\\
\widetilde{\mathfrak{l}}^H_2(\alpha^2, \beta^2) = -\{\{\gamma^{(1,1,2)} + \{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^2\} = (\mathfrak{l}_2 + \mathfrak{l}^H_2)(\alpha^2, \beta^2),\\
\widetilde{\mathfrak{l}}^H_2(\alpha^2, \beta^1) = -\{\{\gamma^{(1,1,2)} + \{\mu^{(1,2,1)}, H\}, \alpha^2\}, \beta^1\} = (\mathfrak{l}_2 + \mathfrak{l}^H_2)(\alpha^2, \beta^1),\\
\widetilde{\mathfrak{l}}^{H,K}_3(\alpha^2, \beta^2, \gamma^2) = -\{\{\{\gamma^{(0,1,3)} + \{\mu^{(1,2,1)}, K\}, \alpha^2\}, \beta^2\}, \gamma^2\}\\
\qquad - \tfrac{1}{2}\{\{\{\{\{\mu^{(0,3,1)}, H\}, H\}, \alpha^2\}, \beta^2\}, \gamma^2\}\\
\qquad = (\mathfrak{l}_3 + \mathfrak{l}^{H,K}_3)(\alpha^2, \beta^2, \gamma^2),
\end{cases} \tag{64}$$

where $\mathfrak{a}_H$, $\mathfrak{l}^H_2$ and $\mathfrak{l}^{H,K}_3$ are given in Lemma 6.3.

[5]The function Λ is obtained by taking the projection of $e^{\{H+K,\cdot\}}(\mu + \gamma - \mu^{(2,1,1)})$ to the subspace

$$C^{(2,1,1)}(\mathcal{M}) \oplus C^{(1,1,2)}(\mathcal{M}) \oplus C^{(0,1,3)}(\mathcal{M})$$

of $C^\infty(\mathcal{M})$. See the footnote 4 for more explanation of $\gamma_{H,K}$.

The following result is a higher analogue of [38, Section 6].

Proposition 6.7. *With the above notations, if $H+K$ satisfies the following Maurer–Cartan type equation:*

$$-\delta_*(H+K)+\frac{1}{2}[H+K,H+K]_S+\frac{1}{6}[H+K,H+K,H+K]_S=0, \quad (65)$$

where δ_* is the differential corresponding to the split Lie 2-algebroid $\mathcal{A}^*[3]$, then we have $\{\Lambda,\Lambda\}=0$. Consequently,

$$\mathcal{A}^*[3]=(A^*_{-1}[3],A^*_{-2}[3],\widetilde{\mathfrak{l}}^H_1,\widetilde{\mathfrak{l}}^H_2,\widetilde{\mathfrak{l}}^{H,K}_3,\widetilde{\mathfrak{a}}_H)$$

is a split Lie 2-algebroid.

Proof. First, note that (65) is equivalent to the following equations

$$\overline{\delta}_*H=0, \quad (66)$$

$$-\overline{\delta}_*K-\mathrm{d}_*H+\frac{1}{2}[H,H]_S=0, \quad (67)$$

$$-\widehat{\delta}_*H-\mathrm{d}_*K+[H,K]_S+\frac{1}{6}[H,H,H]_S=0. \quad (68)$$

On the other hand, $\{\Lambda,\Lambda\}=0$ is equivalent to

$$\{\Lambda^{(1,1,2)},\Lambda^{(2,1,1)}\}=0,\quad \{\Lambda^{(1,1,2)},\Lambda^{(1,1,2)}\}+2\{\Lambda^{(2,1,1)},\Lambda^{(0,1,3)}\}=0,$$
$$\{\Lambda^{(1,1,2)},\Lambda^{(0,1,3)}\}=0.$$

By $\{\gamma^{(1,1,2)},\gamma^{(2,1,1)}\}=0$, $\{\mu^{(1,2,1)},\gamma^{(2,1,1)}\}=0$ and (66), we have

$$\begin{aligned}\{\Lambda^{(1,1,2)},\Lambda^{(2,1,1)}\}&=\{\gamma^{(1,1,2)}+\{\mu^{(1,2,1)},H\},\gamma^{(2,1,1)}\}=\{\{\mu^{(1,2,1)},H\},\gamma^{(2,1,1)}\}\\&=\{\mu^{(1,2,1)},\{H,\gamma^{(2,1,1)}\}\}=-\{\mu^{(1,2,1)},\overline{\delta}_*H\}=0.\end{aligned}$$

By $\{\gamma^{(1,1,2)},\gamma^{(1,1,2)}\}+2\{\gamma^{(2,1,1)},\gamma^{(0,1,3)}\}=0$, $\{\mu^{(1,2,1)},\mu^{(1,2,1)}\}+2\{\mu^{(2,1,1)},\mu^{(0,3,1)}\}=0$, $\{\gamma^{(1,1,2)},\mu^{(1,2,1)}\}=0$, (66) and (67), we have

$$\begin{aligned}&\{\Lambda^{(1,1,2)},\Lambda^{(1,1,2)}\}+2\{\Lambda^{(2,1,1)},\Lambda^{(0,1,3)}\}\\
&=\{\gamma^{(1,1,2)},\gamma^{(1,1,2)}\}+2\{\gamma^{(2,1,1)},\gamma^{(0,1,3)}\}+\frac{1}{2}\{\gamma^{(1,1,2)},\{\mu^{(1,2,1)},H\}\}\\
&\qquad+\{\{\mu^{(1,2,1)},H\},\{\mu^{(1,2,1)},H\}\}+2\{\gamma^{(2,1,1)},\{\mu^{(1,2,1)},K\}\}\\
&\qquad+\{\gamma^{(2,1,1)},\{\{\mu^{(0,3,1)},H\},H\}\}\\
&=-2\{\mu^{(1,2,1)},\mathrm{d}_*H\}+\frac{1}{2}\{\{\{\mu^{(1,2,1)},\mu^{(1,2,1)}\},H\},H\}\\
&\qquad-\{\mu^{(1,2,1)},\{\{\mu^{(1,2,1)},H\},H\}\}-2\{\mu^{(1,2,1)},\overline{\delta}_*K\}\\
&\qquad+\{\{\{\gamma_2,\mu^{(0,3,1)}\},H\},H\}\\
&=2\{\mu^{(1,2,1)},\overline{\delta}_*K\}+\{\mu^{(1,2,1)},\{\{\mu^{(1,2,1)},H\},H\}\}\\
&\qquad+\{\{\frac{1}{2}\{\mu^{(1,2,1)},\mu^{(1,2,1)}\}+\{\mu^{(2,1,1)},\mu^{(0,3,1)}\},H\},H\}\\
&\qquad-\{\mu^{(1,2,1)},\{\{\mu^{(1,2,1)},H\},H\}\}-2\{\mu^{(1,2,1)},\overline{\delta}_*K\}\\
&=0.\end{aligned}$$

By $\{\mu^{(1,2,1)},\mu^{(1,2,1)}\}+2\{\mu^{(2,1,1)},\mu^{(0,3,1)}\}=0$, $\{\gamma^{(1,1,2)},\mu^{(0,3,1)}\}=0$ and (67), we have

$$\begin{aligned}\{\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},K\}\},H\} &= \{\{\tfrac{1}{2}\{\mu^{(1,2,1)},\mu^{(1,2,1)}\},K\},H\}\\ &= -\{\{\{\mu^{(2,1,1)},\mu^{(0,3,1)}\},K\},H\} = -\{\{\mu^{(2,1,1)},K\},\{\mu^{(0,3,1)},H\}\}\\ &= \tfrac{1}{2}\{\{\{\mu^{(1,2,1)},H\},H\},\{\mu^{(0,3,1)},H\}\}+\{\{\gamma^{(1,1,2)},H\},\{\mu^{(0,3,1)},H\}\}\\ &= \tfrac{1}{2}\{\{\{\mu^{(1,2,1)},H\},H\},\{\mu^{(0,3,1)},H\}\}-\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}.\end{aligned}$$

By the fact that $\{\mu^{(1,2,1)},\mu^{(0,3,1)}\}=0$, we have

$$\begin{aligned}\tfrac{1}{6}\{\mu^{(1,2,1)},\{\{\{\mu^{(0,3,1)},H\},H\},H\}\} = &-\tfrac{1}{2}\{\{\{\mu^{(1,2,1)},H\},H\},\{\mu^{(0,3,1)},H\}\}\\ &-\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}.\end{aligned}$$

By $\{\gamma^{(1,1,2)},\gamma^{(0,1,3)}\}=0$, $\{\gamma^{(1,1,2)},\mu^{(1,2,1)}\}=0$, $\{\mu^{(1,2,1)},\gamma^{(0,1,3)}\}=0$ and (68), we have

$$\begin{aligned}&\{\Lambda^{(1,1,2)},\Lambda^{(0,1,3)}\}\\ &= \{\gamma^{(1,1,2)},\{\mu^{(1,2,1)},K\}\}+\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &\quad +\{\{\mu^{(1,2,1)},H\},\gamma^{(0,1,3)}\}+\{\{\mu^{(1,2,1)},H\},\{\mu^{(1,2,1)},K\}\}\\ &\quad +\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &= -\{\mu^{(1,2,1)},\mathrm{d}_*K\}-\{\mu^{(1,2,1)},\widehat{\delta}_*H\}\\ &\quad +\{\{\mu^{(1,2,1)},H\},\{\mu^{(1,2,1)},K\}\}+\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &\quad +\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &= \{\mu^{(1,2,1)},\{\{\mu^{(1,2,1)},H\},K\}\}\\ &\quad +\tfrac{1}{6}\{\mu^{(1,2,1)},\{\{\{\mu^{(0,3,1)},H\},H\},H\}\}-\{\mu^{(1,2,1)},\{\{\mu^{(1,2,1)},H\},K\}\}\\ &\quad +\{\{\mu^{(1,2,1)},\{\mu^{(1,2,1)},K\}\},H\}+\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &\quad +\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &= -\tfrac{1}{2}\{\{\{\mu^{(1,2,1)},H\},H\},\{\mu^{(0,3,1)},H\}\}\\ &\quad -\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}+\tfrac{1}{2}\{\{\{\mu^{(1,2,1)},H\},H\},\{\mu^{(0,3,1)},H\}\}\\ &\quad -\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}+\tfrac{1}{2}\{\gamma^{(1,1,2)},\{\{\mu^{(0,3,1)},H\},H\}\}\\ &\quad +\tfrac{1}{2}\{\{\mu^{(1,2,1)},H\},\{\{\mu^{(0,3,1)},H\},H\}\}=0.\end{aligned}$$

Thus, we have $\{\Lambda,\Lambda\}=0$. By Theorem 3.4,

$$\mathcal{A}^*[3]=(A^*_{-1}[3],A^*_{-2}[3],\widetilde{\mathfrak{l}}_1^H,\widetilde{\mathfrak{l}}_2^H,\widetilde{\mathfrak{l}}_3^H,\widetilde{\mathfrak{a}}_H)$$

is a split Lie 2-algebroid. □

Define vector bundles G_{-1} and G_{-2} by

$$G_{-1} = \{\alpha^2 + H^\sharp(\alpha^2) \mid \alpha^2 \in \Gamma(A^*_{-2})\}, \quad G_{-2} = \{\alpha^1 - H^\natural(\alpha^1) \mid \alpha^1 \in \Gamma(A^*_{-1})\},$$

which are subbundles of $A_{-1} \oplus A^*_{-2}$ and $A_{-2} \oplus A^*_{-1}$ respectively. Denote by $\mathcal{G} = G_{-1} \oplus G_{-2}$. Define

$$l_1^{\mathcal{G}}: G_{-2} \longrightarrow G_{-1}, \quad l_2^{\mathcal{G}}: \Gamma(G_i) \times \Gamma(G_j) \longrightarrow \Gamma(G_{i+j+1}),$$
$$l_3^{\mathcal{G}}: \wedge^3 G_{-1} \longrightarrow G_{-2}, \quad \text{and} \quad a^{\mathcal{G}}: G_{-1} \longrightarrow TM$$

by

$$\begin{cases} l_1^{\mathcal{G}}(\alpha^1 - H^\natural(\alpha^1)) = l_1^*(\alpha^1) + H^\sharp(l_1^*(\alpha^1)), \\ l_2^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2)) \\ \qquad = (\mathfrak{l}_2 + \mathfrak{l}_2^H)(\alpha^2, \beta^2) + H^\sharp\big((\mathfrak{l}_2 + \mathfrak{l}_2^H)(\alpha^2, \beta^2)\big), \\ l_2^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^1 - H^\natural(\beta^1)) \\ \qquad = (\mathfrak{l}_2 + \mathfrak{l}_2^H)(\alpha^2, \beta^1) - H^\natural\big((\mathfrak{l}_2 + \mathfrak{l}_2^H)(\alpha^2, \beta^1)\big), \\ l_3^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2), \gamma^2 + H^\sharp(\gamma^2)) \\ \qquad = (\mathfrak{l}_3 + \mathfrak{l}_3^{H,K})(\alpha^2, \beta^2, \gamma^2) - H^\natural\big((\mathfrak{l}_3 + \mathfrak{l}_3^{H,K})(\alpha^2, \beta^2, \gamma^2)\big), \\ a^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2)) = (\mathfrak{a} + \mathfrak{a}_H)(\alpha^2), \end{cases} \tag{69}$$

for all $\alpha^2, \beta^2, \gamma^2 \in \Gamma(A^*_{-2})$, $\alpha^1, \beta^1 \in \Gamma(A^*_{-1})$.

By Proposition 6.7, we have

Corollary 6.8. *Let $(\mathcal{A}, \mathcal{A}^*[3])$ be a split Lie 2-bialgebroid, $H \in \Gamma(A_{-1} \odot A_{-2})$ and $K \in \Gamma(\wedge^3 A_{-2})$. If $H + K$ satisfies the Maurer–Cartan type equation (65), then $\mathcal{G}_H = (G_{-2}, G_{-1}, l_1^{\mathcal{G}}, l_2^{\mathcal{G}}, l_3^{\mathcal{G}}, a^{\mathcal{G}})$ is a split Lie 2-algebroid.*

It is not hard to see that

$$\delta_* H \in \Gamma(\odot^2 A_{-1}) \oplus \Gamma(\wedge^2 A_{-2} \odot A_{-1}) \oplus \Gamma(\wedge^4 A_{-2}),$$
$$\delta_* K \in \Gamma(\wedge^2 A_{-2} \odot A_{-1}) \oplus \Gamma(\wedge^4 A_{-2}).$$

We need the following preparation before we give the main result in this subsection.

Lemma 6.9. *For all $\alpha^2, \beta^2, \gamma^2, \zeta^2 \in \Gamma(A^*_{-2})$ and $\alpha^1, \beta^1, \gamma^1 \in \Gamma(A^*_{-1})$, we have*

$$\overline{\delta}_* H(\alpha^1, \beta^1) = \langle -[H]_S(\alpha^1), \beta^1 \rangle; \tag{70}$$

$$\overline{\delta}_* K(\alpha^2, \beta^2, \gamma^1) = \langle -[K]_S(\alpha^2, \beta^2), \gamma^1 \rangle = \langle -[K]_S(\alpha^2, \gamma^1), \beta^2 \rangle; \tag{71}$$

$$\mathrm{d}_* H(\alpha^2, \beta^2, \gamma^1) = \langle \mathcal{L}^1_{\alpha^2} H^\sharp(\beta^2) - \mathcal{L}^1_{\beta^2} H^\sharp(\alpha^2) - H^\sharp(\mathfrak{l}_2(\alpha^2, \beta^2)), \gamma^1 \rangle \tag{72}$$

$$= \langle \mathcal{L}^1_{\alpha^2} H^\natural(\gamma^1) - \iota_{\gamma^1} \delta_* H^\sharp(\alpha^2) - H^\natural(\mathfrak{l}_2(\alpha^2, \gamma^1)), \beta^2 \rangle; \tag{73}$$

$$\widehat{\delta}_* H(\alpha^2,\beta^2,\gamma^2,\zeta^2) = \langle -\mathcal{L}^3_{\alpha^2,\beta^2} H^\sharp(\gamma^2) - \mathcal{L}^3_{\beta^2,\gamma^2} H^\sharp(\alpha^2) - \mathcal{L}^3_{\gamma^2,\alpha^2} H^\sharp(\beta^2) - H^\natural(l_3(\alpha^2,\beta^2,\gamma^2)),\zeta^2\rangle; \quad (74)$$

$$\begin{aligned}\mathrm{d}_* K(\alpha^2,\beta^2,\gamma^2,\zeta^2) = \langle &\mathcal{L}^1_{\alpha^2} K^\flat(\beta^2,\gamma^2) + \iota_{\gamma^2}\delta_*(K^\flat(\alpha^2,\beta^2)) - \mathcal{L}^1_{\beta^2} K^\flat(\alpha^2,\gamma^2)\\ &- K^\flat(\mathfrak{l}_2^H(\alpha^2,\beta^2),\gamma^2) - K^\flat(\beta^2,\mathfrak{l}_2^H(\alpha^2,\gamma^2))\\ &+ K^\flat(\alpha^2,\mathfrak{l}_2^H(\beta^2,\gamma^2)),\zeta^2\rangle. \end{aligned}\quad (75)$$

Proof. The proof is similar to that of Lemma 6.4. We omit the details. □

Now we are ready to give the main result in this paper, which says that the graph of a Maurer–Cartan element is a weak Dirac structure defined in Definition 6.1.

Theorem 6.10. *Let $(\mathcal{A},\mathcal{A}^*[3])$ be a split Lie 2-bialgebroid, $H \in \Gamma(A_{-1}\odot A_{-2})$ and $K \in \Gamma(\wedge^3 A_{-2})$. If $H+K$ satisfies the Maurer–Cartan type equation (65), then $(\mathbb{i}_1,\mathbb{i}_2,-\widetilde{K^\flat})$ is a morphism from the Lie 2-algebra*

$$(\Gamma(G_{-2}),\Gamma(G_{-1}),l_1^{\mathcal{G}},l_2^{\mathcal{G}},l_3^{\mathcal{G}})$$

to the Leibniz 2-algebra

$$(\Gamma(E_{-2}),\Gamma(E_{-1}),\partial,\diamond,\Omega)$$

underlying the CLWX *2-algebroid given in Proposition 5.7, where $\mathbb{i}_1$ and $\mathbb{i}_2$ are inclusion maps from G_{-1} and G_{-2} to E_{-1} and E_{-2} respectively and*

$$\widetilde{K^\flat}\colon \wedge^2 G_{-1} \longrightarrow E_{-2}$$

is defined by

$$\widetilde{K^\flat}(\alpha^2 + H^\sharp(\alpha^2),\beta^2 + H^\sharp(\beta^2)) = K^\flat(\alpha^2,\beta^2).$$

Consequently, the split Lie 2-algebroid $\mathcal{G}_H = (G_{-2},G_{-1},l_1^{\mathcal{G}},l_2^{\mathcal{G}},l_3^{\mathcal{G}},a^{\mathcal{G}})$ is a weak Dirac structure of the CLWX *2-algebroid given in Proposition 5.7.*

Proof. First by the fact that H is symmetric, i.e. $H(\alpha^2,\alpha^1) = H(\alpha^1,\alpha^2)$, it is obvious that the graded subbundle $\mathcal{G}_H$ is maximal isotropic.

Then by (66), we have $-l_1\circ H^\natural = H^\sharp\circ l_1^*$, which implies that

$$\mathbb{i}_1\circ l_1^{\mathcal{G}} = \partial\circ\mathbb{i}_2. \quad (76)$$

By (48) and (72), we have

$$\begin{aligned}(-\overline{\delta}_* K - \mathrm{d}_* H + \tfrac{1}{2}[H,H]_S)(\alpha^2,\beta^2,-)\\ &= -\mathrm{d}_* H(\alpha^2,\beta^2,-) + \big([K]_S + \tfrac{1}{2}[H,H]_S(\alpha^2,\beta^2,-)\big)\\ &= H^\sharp(\mathfrak{l}_2(\alpha^2,\beta^2)) - \mathcal{L}^1_{\alpha^2} H^\sharp(\beta^2) + \mathcal{L}^1_{\beta^2} H^\sharp(\alpha^2) + H^\sharp \mathfrak{l}_2^H(\alpha^2,\beta^2)\\ &\qquad - l_2(H^\sharp(\alpha^2),H^\sharp(\beta^2)) + l_1 K^\flat(\alpha^2,\beta^2).\end{aligned}$$

Thus, by (67), we deduce that

$$\begin{aligned} & l_2^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2)) - (\alpha^2 + H^\sharp(\alpha^2)) \diamond (\beta^2 + H^\sharp(\beta^2)) \\ &\quad = H^\sharp(\mathfrak{l}_2(\alpha^2, \beta^2) + \mathfrak{l}_2^H(\alpha^2, \beta^2)) - \mathfrak{L}^1_{\alpha^2} H^\sharp(\beta^2) \\ &\qquad\qquad + \mathfrak{L}^1_{\beta^2} H^\sharp(\alpha^2) - l_2(H^\sharp(\alpha^2), H^\sharp(\beta^2)) \\ &\quad = -l_1 K^\flat(\alpha^2, \beta^2) = -l_1 \widetilde{K}^\flat(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2)). \end{aligned} \tag{77}$$

Similarly, by (49) and (73), we have

$$\begin{aligned} & (-\overline{\delta}_* K - \mathrm{d}_* H + \tfrac{1}{2}[H, H]_S)(\alpha^2, \beta^1, -) \\ &\quad = -\mathrm{d}_* H(\alpha^2, \beta^1, -) + ([K]_S + \tfrac{1}{2}[H, H]_S(\alpha^2, \beta^1, -)) \\ &\quad = -\mathfrak{L}^1_{\alpha^2} H^\natural(\beta^1) + \iota_{\beta^1} \mathrm{d}_* H^\sharp(\alpha^2) + H^\natural(\mathfrak{l}_2(\alpha^2, \beta^1)) \\ &\qquad + H^\natural(\mathfrak{l}_2^H(\alpha^2, \beta^1)) - l_2(H^\sharp(\alpha^2), H^\natural(\beta^1)) - K^\flat(\alpha^2, l_1^*(\beta^1)). \end{aligned}$$

Thus, by (67), we deduce that

$$\begin{aligned} & l_2^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^1 - H^\natural(\beta^1)) - (\alpha^2 + H^\sharp(\alpha^2)) \diamond (\beta^1 - H^\natural(\beta^1)) \\ &\qquad = -\widetilde{K}^\flat(\alpha^2 + H^\sharp(\alpha^2), l_1^{\mathcal{G}}(\beta^1 - H^\natural(\beta^1))). \end{aligned} \tag{78}$$

By (50), (51), (74) and (75), we have

$$\begin{aligned} & (-\widehat{\delta}_* H - \mathrm{d}_* K + [H, K]_S + \tfrac{1}{6}[H, H, H]_S)(\alpha^2, \beta^2, \gamma^2, -) \\ &\quad = \mathfrak{L}^3_{\alpha^2,\beta^2} H^\sharp(\gamma^2) + \mathfrak{L}^3_{\beta^2,\gamma^2} H^\sharp(\alpha^2) + \mathfrak{L}^3_{\gamma^2,\alpha^2} H^\sharp(\beta^2) + H^\natural(\mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2)) \\ &\qquad - \mathfrak{L}^1_{\alpha^2} K^\flat(\beta^2, \gamma^2) - \iota_{\gamma^2} \mathrm{d}_*(K^\flat(\alpha^2, \beta^2)) + \mathfrak{L}^1_{\beta^2} K^\flat(\alpha^2, \gamma^2) \\ &\qquad + K^\flat(\mathfrak{l}_2(\alpha^2, \beta^2), \gamma^2) + K^\flat(\beta^2, \mathfrak{l}_2(\alpha^2, \gamma^2)) - K^\flat(\alpha^2, \mathfrak{l}_2(\beta^2, \gamma^2)) \\ &\qquad + K^\flat(\mathfrak{l}_2^H(\alpha^2, \beta^2), \gamma^2) + K^\flat(\mathfrak{l}_2^H(\gamma^2, \alpha^2), \beta^2) + K^\flat(\mathfrak{l}_2^H(\beta^2, \gamma^2), \alpha^2) \\ &\qquad - l_2(H^\sharp(\alpha^2), K^\flat(\beta^2, \gamma^2)) - l_2(H^\sharp(\gamma^2), K^\flat(\alpha^2, \beta^2)) \\ &\qquad - l_2(H^\sharp(\beta^2), K^\flat(\gamma^2, \alpha^2)) + H^\natural(\mathfrak{l}_3^{H,K}(\alpha^2, \beta^2, \gamma^2)) \\ &\qquad + l_3(H^\sharp(\alpha^2), H^\sharp(\beta^2), H^\sharp(\gamma^2)). \end{aligned}$$

Thus, by (68), we deduce that

$$\begin{aligned} & -l_3^{\mathcal{G}}(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2), \gamma^2 + H^\sharp(\gamma^2)) \\ &\qquad - (\alpha^2 + H^\sharp(\alpha^2)) \diamond \widetilde{K}^\flat(\beta^2 + H^\sharp(\beta^2), \gamma^2 + H^\sharp(\gamma^2)) \\ &\qquad + (\beta^2 + H^\sharp(\beta^2)) \diamond \widetilde{K}^\flat(\alpha^2 + H^\sharp(\alpha^2), \gamma^2 + H^\sharp(\gamma^2)) \\ &\qquad - \widetilde{K}^\flat(\alpha^2 + H^\sharp(\alpha^2), \beta^2 + H^\sharp(\beta^2)) \diamond (\gamma^2 + H^\sharp(\gamma^2)) \end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{K}^\flat(l_2^{\mathcal{G}}(\alpha^2,\beta^2)+H^\sharp l_2^{\mathcal{G}}(\alpha^2,\beta^2),\gamma^2+H^\sharp(\gamma^2))\\
&- \widetilde{K}^\flat(\alpha^2+H^\sharp(\alpha^2),l_2^{\mathcal{G}}(\beta^2,\gamma^2)+H^\sharp l_2^{\mathcal{G}}(\beta^2,\gamma^2))\\
&+ \widetilde{K}^\flat(\beta^2+H^\sharp(\beta^2),l_2^{\mathcal{G}}(\alpha^2,\gamma^2)+H^\sharp l_2^{\mathcal{G}}(\alpha^2,\gamma^2))\\
&+ \Omega(\alpha^2+H^\sharp(\alpha^2),\beta^2+H^\sharp(\beta^2),\gamma^2+H^\sharp(\gamma^2))=0. \qquad (79)
\end{aligned}
$$

By (76)–(79), we deduce that $(\mathbb{i}_1,\mathbb{i}_2,-\widetilde{K}^\flat)$ is a morphism from the Lie 2-algebra

$$
(\Gamma(G_{-2}),\Gamma(G_{-1}),l_1^{\mathcal{G}},l_2^{\mathcal{G}},l_3^{\mathcal{G}})
$$

to the Leibniz 2-algebra $(\Gamma(E_{-2}),\Gamma(E_{-1}),\partial,\diamond,\Omega)$.

Finally, it is obvious that

$$
\rho(\alpha^2+H^\sharp(\alpha^2))=a(H^\sharp(\alpha^2))+\mathfrak{a}(\alpha^2)=a^{\mathcal{G}}(\alpha^2+H^\sharp(\alpha^2)).
$$

Therefore, the split Lie 2-algebroid

$$
(G_{-2},G_{-1},l_1^{\mathcal{G}},l_2^{\mathcal{G}},l_3^{\mathcal{G}},a^{\mathcal{G}})
$$

is a weak Dirac structure of the CLWX 2-algebroid $(E_{-2},E_{-1},\partial,\rho,(\cdot,\cdot)_+,\diamond,\Omega)$ given in Proposition 5.7. □

6.3. Examples. In this subsection, we give some examples of Theorem 6.5 including the string Lie 2-algebra and split Lie 2-algebroids constructed from integrable distributions and left-symmetric algebroids.

Example 6.11. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ be a semisimple Lie algebra and $B(\cdot,\cdot)$ its killing form. Recall that the string Lie 2-algebra $(\mathbb{R}[2],\mathfrak{g}[1],l_1,l_2,l_3)$ is given by

$$
l_1=0,\quad l_2(x,y)=[x,y]_{\mathfrak{g}},\quad l_2(x,v)=0,\quad l_3(x,y,z)=B\big(x,[y,z]_{\mathfrak{g}}\big),
$$

for all $x,y,z\in\mathfrak{g}$, $v\in\mathbb{R}$. We have $\mathfrak{g}\odot\mathbb{R}=\mathfrak{g}$ and $\wedge^3\mathbb{R}=0$. It is straightforward to see that any $h\in\mathfrak{g}$ is a Maurer–Cartan element of the associated homotopy Poisson algebra. Furthermore,

$$
h^\sharp\colon\mathbb{R}^*\cong\mathbb{R}\longrightarrow\mathfrak{g}\quad\text{and}\quad h^\natural\colon\mathfrak{g}^*\longrightarrow\mathbb{R}
$$

are given by

$$
h^\sharp(s)=sh,\quad h^\natural(\alpha)=\langle h,\alpha\rangle,\quad \forall s\in\mathbb{R},\ \alpha\in\mathfrak{g}^*.
$$

The Lie 2-algebra $(\mathfrak{g}^*[2],\mathbb{R}[1],\mathfrak{l}_1^h,\mathfrak{l}_2^h,\mathfrak{l}_3^h)$ given in Theorem 6.5 (i) is given by

$$
\mathfrak{l}_1^h=0,\quad \mathfrak{l}_2^h(s,t)=0,\quad \mathfrak{l}_2^h(s,\beta)=\mathrm{ad}^*_{sh}\beta,\quad \mathfrak{l}_3^h(s,t,w)=0,
$$

for all $\beta\in\mathfrak{g}^*$, $s,t,w\in\mathbb{R}$.

Moreover, the Lie 2-algebras $(\mathfrak{g}^*, \mathbb{R}, \mathfrak{l}_1^h, \mathfrak{l}_2^h, \mathfrak{l}_3^h)$ and $(\mathbb{R}, \mathfrak{g}, l_1, l_2, l_3)$ define a Lie 2-bialgebra, whose double is the Lie 2-algebra $(\mathbb{R} \oplus \mathfrak{g}^*[2], \mathfrak{g} \oplus \mathbb{R}[1], \partial, \diamond, \Omega)$ given by

$$\begin{cases} \partial = 0, \\ (x,s) \diamond (y,t) = \big([x,y]_{\mathfrak{g}}, 0\big), \\ (x,s) \diamond (u,\alpha) = (u,\alpha) \diamond (x,s) = (0, \mathrm{ad}_x^* \alpha + \mathrm{ad}_{sh}^* \alpha), \\ \Omega((x,s),(y,t),(z,r)) = \big(B\big(x,[y,z]_{\mathfrak{g}}\big), -rB^\sharp\big([x,y]_{\mathfrak{g}}\big) \\ \qquad - sB^\sharp\big([y,z]_{\mathfrak{g}}\big) - tB^\sharp\big([z,x]_{\mathfrak{g}}\big)\big), \end{cases}$$

for all $(x,s),(y,t),(z,r) \in \mathfrak{g} \oplus \mathbb{R}$ and $(u,\alpha) \in \mathbb{R} \oplus \mathfrak{g}^*$ and $B^\sharp : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by $\langle B^\sharp(x), y\rangle = B(x,y)$.

Let $(A, [\cdot,\cdot]_A, a_A)$ be a Lie algebroid and $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ a representation of Lie A on a vector bundle E. Then it is straightforward to see that

$$\mathcal{A} = (E[2], A[1], l_1, l_2, l_3, a)$$

is a split Lie 2-algebroid, where $l_1 = 0$, $l_3 = 0$, $a = a_A$, and

$$l_2(X,Y) = [X,Y]_A, \quad l_2(X,e) = \nabla_X e, \quad \forall\, X, Y \in \Gamma(A),\ e \in \Gamma(E).$$

Proposition 6.12. *With above notations, let $H \in \Gamma(A \odot E)$ and $K \in \Gamma(\wedge^3 E)$.*

(i) *If $[H,H]_S = 0$, then $(E^*, \mathfrak{l}_2^H, \mathfrak{a}_H)$ is a Lie algebroid, where $\mathfrak{a}_H = a_A \circ H^\sharp$ and $\mathfrak{l}_2^H$ is given by*

$$\mathfrak{l}_2^H(e_1^*, e_2^*) = \nabla^*_{H^\sharp(e_1^*)} e_2^* - \nabla^*_{H^\sharp(e_2^*)} e_1^*, \quad \forall e_1^*, e_2^* \in \Gamma(E^*), \tag{80}$$

Here ∇^* is the dual representation of ∇ on E^*.

(ii) *If $[H,H]_S = 0$, then $[H,K]_S = 0$ if and only if K is a 3-cocycle on the Lie algebroid $(E^*, \mathfrak{l}_2^H, \mathfrak{a}_H)$.*

Proof. (i) By (41), it is straightforward to deduce that $\mathfrak{l}_2^H$ is given by (80). Since $l_1 = 0$, we obtain

$$\mathfrak{l}_1^H = l_1^* = 0.$$

Thus, $\mathfrak{l}_2^H$ satisfies the Jacobi identity and $(E^*, \mathfrak{l}_2^H, \mathfrak{a}_H)$ is a Lie algebroid.

(ii) For all $e_1^*, e_2^*, e_3^*, e_4^* \in \Gamma(E^*)$, we have

$$\begin{aligned} &\langle [H,K]_S(e_1^*, e_2^*, e_3^*), e_4^*\rangle \\ &= \langle K^\flat(\mathfrak{l}_2^H(e_1^*, e_2^*), e_3^*) + K^\flat(\mathfrak{l}_2^H(e_3^*, e_1^*), e_2^*) + K^\flat(\mathfrak{l}_2^H(e_2^*, e_3^*), e_1^*) \\ &\quad - l_2(H^\sharp(e_1^*), K^\flat(e_2^*, e_3^*)) - l_2(H^\sharp(e_3^*), K^\flat(e_1^*, e_2^*)) - l_2(H^\sharp(e_2^*), K^\flat(e_3^*, e_1^*)) \\ &\quad - H^\natural(L^2_{K^\flat(e_1^*, e_2^*)} e_3^* + L^2_{K^\flat(e_3^*, e_1^*)} e_2^* + L^2_{K^\flat(e_2^*, e_3^*)} e_1^* + 2\mathrm{d}K(e_1^*, e_2^*, e_3^*)), e_4^*\rangle \end{aligned}$$

$$
\begin{aligned}
&= K(\mathfrak{l}_2^H(e_1^*, e_2^*), e_3^*, e_4^*) + K(\mathfrak{l}_2^H(e_3^*, e_1^*), e_2^*, e_4^*) + K(\mathfrak{l}_2^H(e_2^*, e_3^*), e_1^*, e_4^*) \\
&\quad - \langle \nabla^*_{H^\sharp(e_1^*)} K^\flat(e_2^*, e_3^*), e_4^* \rangle - \langle \nabla^*_{H^\sharp(e_3^*)} K^\flat(e_1^*, e_2^*), e_4^* \rangle - \langle \nabla^*_{H^\sharp(e_2^*)} K^\flat(e_3^*, e_1^*), e_4^* \rangle \\
&\quad + \langle \nabla^*_{H^\sharp(e_4^*)} K^\flat(e_1^*, e_2^*), e_3^* \rangle + \langle \nabla^*_{H^\sharp(e_4^*)} K^\flat(e_3^*, e_1^*), e_2^* \rangle + \langle \nabla^*_{H^\sharp(e_4^*)} K^\flat(e_2^*, e_3^*), e_1^* \rangle \\
&\quad - 2\mathfrak{a}_H(e_4^*) K(e_1^*, e_2^*, e_3^*) \\
&= K(\mathfrak{l}_2^H(e_1^*, e_2^*), e_3^*, e_4^*) + K(\mathfrak{l}_2^H(e_3^*, e_1^*), e_2^*, e_4^*) + K(\mathfrak{l}_2^H(e_2^*, e_3^*), e_1^*, e_4^*) \\
&\qquad - \mathfrak{a}_H(e_1^*) K(e_2^*, e_3^*, e_4^*) + K(e_2^*, e_3^*, \nabla^*_{H^\sharp(e_1^*)} e_4^*) - \mathfrak{a}_H(e_3^*) K(e_1^*, e_2^*, e_4^*) \\
&\qquad + K(e_1^*, e_2^*, \nabla^*_{H^\sharp(e_3^*)} e_4^*) - \mathfrak{a}_H(e_2^*) K(e_3^*, e_1^*, e_4^*) + K(e_3^*, e_1^*, \nabla^*_{H^\sharp(e_2^*)} e_4^*) \\
&\qquad + \mathfrak{a}_H(e_4^*) K(e_1^*, e_2^*, e_3^*) - K(e_1^*, e_2^*, \nabla^*_{H^\sharp(e_4^*)} e_3^*) + \mathfrak{a}_H(e_4^*) K(e_2^*, e_3^*, e_1^*) \\
&\qquad - K(e_3^*, e_1^*, \nabla^*_{H^\sharp(e_4^*)} e_2^*) + \mathfrak{a}_H(e_4^*) K(e_2^*, e_3^*, e_1^*) - K(e_2^*, e_3^*, \nabla^*_{H^\sharp(e_4^*)} e_1^*) \\
&\qquad - 2\mathfrak{a}_H(e_4^*) K(e_1^*, e_2^*, e_3^*) \\
&= K(\mathfrak{l}_2^H(e_1^*, e_2^*), e_3^*, e_4^*) + K(\mathfrak{l}_2^H(e_3^*, e_1^*), e_2^*, e_4^*) + K(\mathfrak{l}_2^H(e_2^*, e_3^*), e_1^*, e_4^*) \\
&\qquad + K(e_2^*, e_3^*, \mathfrak{l}_2^H(e_1^*, e_4^*)) + K(e_1^*, e_2^*, \mathfrak{l}_2^H(e_3^*, e_4^*)) + K(e_3^*, e_1^*, \mathfrak{l}_2^H(e_2^*, e_4^*)) \\
&\qquad - \mathfrak{a}_H(e_1^*) K(e_2^*, e_3^*, e_4^*) - \mathfrak{a}_H(e_3^*) K(e_1^*, e_2^*, e_4^*) - \mathfrak{a}_H(e_2^*) K(e_3^*, e_1^*, e_4^*) \\
&\qquad + \mathfrak{a}_H(e_4^*) K(e_1^*, e_2^*, e_3^*) \\
&= -(\mathrm{d}^H K)(e_1^*, e_2^*, e_3^*, e_4^*),
\end{aligned}
$$

where d^H is the coboundary operator on the Lie algebroid $(E^*, \mathfrak{l}_2^H, \mathfrak{a}_H)$ with the coefficient in the trivial representation. Thus, $[H, K]_S = 0$ if and only if K is a 3-cocycle. □

Corollary 6.13. *Let $H \in \Gamma(A \odot E)$ and $K \in \Gamma(\wedge^3 E)$ such that $[H, H]_S = 0$, $[H, K]_S = 0$. Then*

$$
\mathcal{A}^*[3] = (A^*, E^*, \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H)
$$

is a split Lie 2-algebroid, where $\mathfrak{l}_1^H = 0$, and

$$
\begin{aligned}
\mathfrak{l}_2^H(e_1^*, e_2^*) &= \nabla^*_{H^\sharp(e_1^*)} e_2^* - \nabla^*_{H^\sharp(e_2^*)} e_1^*, \\
\mathfrak{l}_2^H(e_1^*, \beta) &= \mathfrak{L}_{H^\sharp(e_1^*)} \beta + \langle \nabla . H^\natural(\beta), e_1^* \rangle - \mathrm{d}^A H(e_1^*, \beta), \\
\mathfrak{l}_3^{H,K}(e_1^*, e_2^*, e_3^*) &= \langle \nabla . K^\flat(e_1^*, e_2^*), e_3^* \rangle + \langle \nabla . K^\flat(e_3^*, e_1^*), e_2^* \rangle \\
&\quad + \langle \nabla . K^\flat(e_2^*, e_3^*), e_1^* \rangle - 2\mathrm{d}^A K(e_1^*, e_2^*, e_3^*),
\end{aligned}
$$

for all $e_1^*, e_2^*, e_3^* \in \Gamma(E^*)$, $\beta \in \Gamma(A^*)$. Here $\mathfrak{L}_X \colon \Gamma(A^*) \longrightarrow \Gamma(A^*)$ and d^A are the Lie derivative and the differential for the Lie algebroid A, respectively. Furthermore, $(H^\sharp, -H^\natural, -K^\flat)$ is a morphism form the split Lie 2-algebroid $\mathcal{A}^*[3]$ to the split Lie 2-algebroid $\mathcal{A}$ and $(\mathcal{A}, \mathcal{A}^*[3])$ is a split Lie 2-bialgebroid.

According to Proposition 6.12 and Corollary 6.13, we can give the following example in which the Lie algebroid is given by an integral distribution and the representation is given by the Lie derivative on its normal bundle.

Example 6.14. Let $\mathscr{F} \subset TM$ be an integral distribution on a manifold M and $\mathscr{F}^\perp \subset T^*M$ its conormal bundle. Then

$$(\mathscr{F}^\perp[2], \mathscr{F}[1], l_1, l_2, l_3, a)$$

is a Lie 2-algebroid, where $l_1 = 0$, $l_3 = 0$, $a: \mathscr{F} \longrightarrow TM$ is the inclusion map and l_2 is given by

$$l_2(X_1, X_2) = [X_1, X_2], \quad l_2(X, \xi) = \mathfrak{L}_X \xi, \quad \forall X_1, X_2, X \in \Gamma(\mathscr{F}),\ \xi \in \Gamma(\mathscr{F}^\perp),$$

where $\mathfrak{L}$ is the usual Lie derivative on M. Then $H \in \mathscr{F} \odot \mathscr{F}^\perp$ satisfies $[H, H]_S = 0$ if and only if

$$H^\sharp(L^1_{H^\sharp(\alpha_1)}\alpha_2 - L^1_{H^\sharp(\alpha_2)}\alpha_1) = [H^\sharp(\alpha_1), H^\sharp(\alpha_2)], \quad \forall\, \alpha_1, \alpha_2 \in \Gamma((\mathscr{F}^\perp)^*),$$

where $L^1: \mathscr{F} \times (\mathscr{F}^\perp)^* \longrightarrow (\mathscr{F}^\perp)^*$ is the Lie derivative on the Lie 2-algebroid defined by (11). In fact, there is a natural isomorphism between $TM/\mathscr{F}$ and $(\mathscr{F}^\perp)^*$. For any $Y \in \Gamma(TM)$, we denote by $\overline{Y}$ its image in $\Gamma(TM/\mathscr{F})$ of the natural projection $\mathrm{pr}: TM \longrightarrow TM/\mathscr{F}$. Then we have

$$L^1_X \overline{Y} = \overline{[X, Y]}, \quad \forall X \in \Gamma(\mathscr{F}), Y \in \Gamma(TM).$$

Such an $H \in \mathscr{F} \odot \mathscr{F}^\perp$ induces a Lie algebroid $((\mathscr{F}^\perp)^*, [\cdot,\cdot]_H, \mathfrak{a}_H)$, where

$$[\alpha_1, \alpha_2]_H = L^1_{H^\sharp(\alpha_1)}\alpha_2 - L^1_{H^\sharp(\alpha_2)}\alpha_1, \quad \mathfrak{a}_H = H^\sharp, \quad \forall\, \alpha_1, \alpha_2 \in \Gamma((\mathscr{F}^\perp)^*).$$

Let $K \in \Gamma(\wedge^3 \mathscr{F}^\perp)$ be a 3-cocycle on the Lie algebroid $((\mathscr{F}^\perp)^*, [\cdot,\cdot]_H, \mathfrak{a}_H)$. Then there is an induced Lie 2-algebroid

$$(\mathscr{F}^*[2], (\mathscr{F}^\perp)^*[1], \mathfrak{l}_1^H = 0, [\cdot,\cdot]_H, [\cdot,\cdot,\cdot]_{H,K}, \mathfrak{a}_H = H^\sharp),$$

where

$$\begin{aligned}
[\alpha_1, \alpha_2]_H &= L^1_{H^\sharp(\alpha_1)}\alpha_2 - L^1_{H^\sharp(\alpha_2)}\alpha_1, \\
[\alpha, \theta]_H &= \mathfrak{L}^{\mathscr{F}}_{H^\sharp(\alpha)}\theta - L^2_{H^\natural(\theta)}\alpha - d^{\mathscr{F}} H(\alpha, \theta), \\
[\alpha_1, \alpha_2, \alpha_3]_{H,K} &= -2d^{\mathscr{F}}(K(\alpha_1, \alpha_2, \alpha_3)) - L^2_{K^\flat(\alpha_1,\alpha_2)}\alpha_3 \\
&\qquad - L^2_{K^\flat(\alpha_3,\alpha_1)}\alpha_2 - L^2_{K^\flat(\alpha_2,\alpha_3)}\alpha_1,
\end{aligned}$$

for all $\alpha, \alpha_1, \alpha_2, \alpha_3 \in \Gamma((\mathscr{F}^\perp)^*)$, $\theta \in \Gamma(\mathscr{F}^*)$. Here $\mathfrak{L}^{\mathscr{F}}: \mathscr{F} \times \mathscr{F}^* \longrightarrow \mathscr{F}^*$ and $d^{\mathscr{F}}: \wedge^k \mathscr{F}^* \longrightarrow \wedge^{k+1}\mathscr{F}^*$ are the Lie derivative and the differential for the Lie algebroid $\mathscr{F}$ respectively. Furthermore, it is straightforward to deduce that the relation between L^2 and the Lie derivative $\mathfrak{L}$ is given by

$$\langle L^2_\xi \alpha, X\rangle = -\langle \mathfrak{L}_X \xi, \alpha\rangle, \quad \forall X \in \Gamma(\mathscr{F}), \xi \in \Gamma(\mathscr{F}^\perp), \alpha \in \Gamma((\mathscr{F}^\perp)^*).$$

The notion of a left-symmetric algebroid, also called Koszul–Vinberg algebroid, was introduced in [36, 46, 47] as a geometric generalization of a left-symmetric algebra (pre-Lie algebra). Let $(A, \cdot_A, a_A)$ be a left-symmetric algebroid. Define a skew-symmetric bilinear bracket operation $[\cdot,\cdot]_A$ on $\Gamma(A)$ by

$$[x, y]_A = x \cdot_A y - y \cdot_A x, \quad \forall\, x, y \in \Gamma(A).$$

Then, $(A, [\cdot,\cdot]_A, a_A)$ is a Lie algebroid, and denoted by A^c, called the *sub-adjacent Lie algebroid* of $(A, \cdot_A, a_A)$. Furthermore, $L\colon A \longrightarrow \mathfrak{D}(A)$ defined by $L_X Y = X \cdot_A Y$ gives a representation of the Lie algebroid A^c on A, where $\mathfrak{D}(A)$ denotes the first order covariant differential operator bundle of the vector bundle A.

Example 6.15. Let $(A, \cdot_A, a_A)$ be a left-symmetric algebroid. Then

$$\mathcal{A} = (A^*[2], A[1], l_1, l_2, l_3, a_A)$$

is a split Lie 2-algebroid, where $l_1 = 0$, $l_3 = 0$, and l_2 is given by

$$l_2(X, Y) = [X, Y]_A, \quad l_2(X, \eta) = L^*_X \eta, \quad \forall\, X, Y \in \Gamma(A),\ \eta \in \Gamma(A^*).$$

Here L^* is the dual representation of L. Let $H \in A \odot A^*$ be given by

$$H(X, \xi) = \langle X, \xi \rangle, \quad \forall X \in \Gamma(A),\ \xi \in \Gamma(A^*).$$

That is, $H^\sharp = \mathrm{id}_A$ and $H^\natural = \mathrm{id}_{A^*}$. Then we have $[H, H]_S = 0$. In fact, it follows from

$$\begin{aligned}\tfrac{1}{2}[H, H]_S(X, Y, \cdot) &= H^\sharp(X \cdot_A Y - Y \cdot_A X) - l_2(H^\sharp(X), H^\sharp(Y))\\ &= [X, Y]_A - [X, Y]_A = 0.\end{aligned}$$

Furthermore, the induced Lie algebroid structure on $A = (A^*)^*$ is exactly the sub-adjacent Lie algebroid A^c. For a $K \in \Gamma(\wedge^3 A^*)$, by Proposition 6.12, $[H, K]_S = 0$ if and only if K is a 3-cocycle on the sub-adjacent Lie algebroid A^c. Under these conditions,

$$\mathcal{A}^*[3] = (A^*[2], A[1], \mathfrak{l}_1^H, \mathfrak{l}_2^H, \mathfrak{l}_3^{H,K}, \mathfrak{a}_H)$$

is a split Lie 2-algebroid, where $\mathfrak{l}_1^H = 0$, and

$$\begin{aligned}\mathfrak{l}_2^H(X, Y) &= [X, Y]_A, \quad \mathfrak{l}_2^H(X, \xi) = L^*_X \xi,\\ \mathfrak{l}_3^{H,K}(X, Y, Z) &= R^*_Z K^b(X, Y) + R^*_Y K^b(Z, X)\\ &\quad + R^*_X K^b(Y, Z) + \mathrm{d}^A(K(X, Y, Z)).\end{aligned}$$

for all $X, Y, Z \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$. Here $R^*_X\colon A^* \longrightarrow A^*$ is the dual map of the right multiplication R_X, i.e. $\langle R^*_X \xi, Y \rangle = -\langle \xi, Y \cdot_A X \rangle$.

Since left-symmetric algebras are left-symmetric algebroids naturally, the following example is a special case of the above example.

Example 6.16. Let $(\mathfrak{g},\cdot)$ be a 3-dimensional left-symmetric algebra generated by the following relations

$$e_1\cdot e_1=2e_1,\quad e_1\cdot e_2=e_2,\quad e_1\cdot e_3=e_3,\quad e_2\cdot e_3=e_3\cdot e_2=e_1,$$

where $\{e_1,e_2,e_3\}$ is a basis of $\mathfrak{g}$. The corresponding sub-adjacent Lie algebra structure is given by

$$[e_1,e_2]_{\mathfrak{g}}=e_2,\quad [e_1,e_3]_{\mathfrak{g}}=e_3.$$

The dual representation L^* of the sub-adjacent Lie algebra $\mathfrak{g}^c$ on $\mathfrak{g}^*$ is given by

$$\begin{gathered}L^*_{e_1}e_1^*=-2e_1^*,\quad L^*_{e_1}e_2^*=-e_2^*,\quad L^*_{e_1}e_3^*=-e_3^*,\\ L^*_{e_2}e_1^*=-e_3^*,\quad L^*_{e_3}e_1^*=-e_2^*,\end{gathered}$$

where $\{e_1^*,e_2^*,e_3^*\}$ is the dual basis.

The dual map of the right multiplication R is given by

$$\begin{gathered}R^*_{e_1}e_1^*=-2e_1^*,\quad R^*_{e_2}e_1^*=-e_3^*,\quad R^*_{e_2}e_2^*=-e_1^*,\\ R^*_{e_3}e_1^*=-e_2^*,\quad R^*_{e_3}e_3^*=-e_1^*.\end{gathered}$$

Let $H\in\mathfrak{g}\odot\mathfrak{g}^*$ be given by

$$H(x,\xi)=\langle x,\xi\rangle,\quad \forall x\in\mathfrak{g},\ \xi\in\mathfrak{g}^*.$$

That is, $H=\sum_{i=1}^3 e_i\odot e_i^*$. For any constant number k_0, set $K=k_0e_1^*\wedge e_2^*\wedge e_3^*$ and then K is a 3-cocycle on the sub-adjacent Lie algebra $\mathfrak{g}^c$. Thus,

$$(\mathfrak{g}^*[2],\mathfrak{g}[1],\mathfrak{l}_1^H,\mathfrak{l}_2^H,\mathfrak{l}_3^{H,K})$$

is a Lie 2-algebra, where $\mathfrak{l}_1^H=0$, and for all $x,y\in\mathfrak{g},\ \xi\in\mathfrak{g}^*$,

$$\mathfrak{l}_2^H(x,y)=[x,y]_{\mathfrak{g}},\quad \mathfrak{l}_2^H(x,\xi)=L^*_x\xi,\quad \mathfrak{l}_3^{H,K}(e_1,e_2,e_3)=-4k_0e_1^*.$$

A. The proof of Proposition 5.8

Proof. Let $\mathcal{A}=A_{-1}\oplus A_{-2}$ and $\mathcal{B}=B_{-1}\oplus B_{-2}$. Since the pairing S is nondegenerate, B_{-1} is isomorphic to A_{-2}^*, the dual bundle of A_{-2}, via $\langle\alpha^2,X^2\rangle=S(\alpha^2,X^2)$ for all $X^2\in\Gamma(A_{-2})$, $\alpha^2\in\Gamma(B_{-1})$, and B_{-2} is isomorphic to A_{-1}^*, the dual bundle of A_{-1}, via $\langle\alpha^1,X^1\rangle=S(\alpha^1,X^1)$ for all $X^1\in\Gamma(A_{-1})$, $\alpha^1\in\Gamma(B_{-2})$. Under this isomorphism, the graded symmetric bilinear form S is given by (34). By Proposition 5.4, both $\mathcal{A}$ and $\mathcal{B}$ are split Lie 2-algebroids, and denoted by $(\mathcal{A};l_1,l_2,l_3,a)$ and $(\mathcal{B};\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{a})$. We use δ and δ_* to denote their differentials, respectively.

By (iii) in Definition 5.1, we deduce that $\mathfrak{l}_1 = l_1^*$. By (ii) and (iv) in Definition 5.1, we deduce that the brackets between $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$ are given by (35). By (v) in Definition 5.1, we deduce that the $(A_{-2} \oplus B_{-2})$-valued 3-form Ω is given by (36).

Next, we will use the following two steps to show that (27) in Theorem 4.9 holds.

Step 1. We will show that

$$\delta_*[X^1, Y^1]_S = -[\delta_*(X^1), Y^1]_S + [X^1, \delta_*(Y^1)]_S, \quad \forall X^1, Y^1 \in \Gamma(A_{-1}). \tag{81}$$

In fact, since for all $X^1 \in \Gamma(A_{-1})$, $\overline{\delta}_*(X^1) = 0$, we have

$$\overline{\delta}_*[X^1, Y^1]_S = -[\overline{\delta}_*(X^1), Y^1]_S + [X^1, \overline{\delta}_*(Y^1)]_S. \tag{82}$$

For all $X^1, Y^1 \in \Gamma(A_{-1})$ and $\alpha^1 \in \Gamma(B_{-2})$, by (e_1) in Definition 2.1, we have

$$X^1 \diamond (Y^1 \diamond \alpha^1) - (X^1 \diamond Y^1) \diamond \alpha^1 - Y^1 \diamond (X^1 \diamond \alpha^1) = \Omega(X^1, Y^1, l_1^*(\alpha^1)).$$

By (17), this condition is equivalent to

$$\begin{aligned} &\iota_{L^1_{Y^1}\alpha^1}\mathrm{d}_* X^1 + l_2(X^1, \iota_{\alpha^1}\mathrm{d}_* Y^1) - \iota_{\alpha^1}\mathrm{d}_* l_2(X^1, Y^1) \\ &\qquad - \iota_{L^1_{Y^1}\alpha^1}\mathrm{d}_* X^1 - l_2(X^1, \iota_{\alpha^1}\mathrm{d}_* Y^1) = 0. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} \langle \iota_{L^1_{Y^1}\alpha^1}\mathrm{d}_* X^1, \alpha^2\rangle &= \mathfrak{a}(\alpha^2)a(Y^1)\langle X^1, \alpha^1\rangle - \mathfrak{a}(\alpha^2)\langle \alpha^1, l_2(X^1, Y^1)\rangle \\ &\quad - \langle X^1, \mathfrak{l}_2(L^1_{Y^1}\alpha^1, \alpha^2)\rangle; \\ \langle l_2(X^1, \iota_{\alpha^1}\mathrm{d}_* Y^1), \alpha^2\rangle &= a(X^1)\mathfrak{a}(\alpha^2)\langle Y^1, \alpha^1\rangle - a(X^1)\langle \mathfrak{l}_2(\alpha^1, \alpha^2), Y^1\rangle \\ &\quad - \mathfrak{a}(L^1_{X^1}\alpha^2)\langle Y^1, \alpha^1\rangle + \langle Y^1, \mathfrak{l}_2(L^1_{X^1}\alpha^2, \alpha^1)\rangle; \\ \langle \iota_{\alpha^1}[\mathrm{d}_* X^1, Y^1]_S, \alpha^2\rangle &= a(Y^1)\mathrm{d}_* X^1(\alpha^1, \alpha^2) - \mathrm{d}_* X^1(L^1_{Y^1}\alpha^1, \alpha^2) \\ &\quad - \mathrm{d}_* X^1(\alpha^1, L^1_{Y^1}\alpha^2) \\ &= a(Y^1)\mathfrak{a}(\alpha^2)\langle X^1, \alpha^1\rangle - a(Y^1)\langle \mathfrak{l}_2(\alpha^1, \alpha^2), X^1\rangle \\ &\quad - \mathfrak{a}(\alpha^2)a(Y^1)\langle X^1, \alpha^1\rangle + \mathfrak{a}(\alpha^2)\langle \alpha^1, l_2(Y^1, X^1)\rangle \\ &\quad + \langle X^1, \mathfrak{l}_2(L^1_{Y^1}\alpha^1, \alpha^2)\rangle - \mathfrak{a}(L^1_{Y^1}\alpha^2)\langle X^1, \alpha^1\rangle \\ &\quad + \langle X^1, \mathfrak{l}_2(L^1_{Y^1}\alpha^2, \alpha^1)\rangle. \end{aligned}$$

Then by the above formulas, we have

$$\begin{aligned} 0 &= \iota_{L^1_{Y^1}\alpha^1}\mathrm{d}_* X^1 + l_2(X^1, \iota_{\alpha^1}\mathrm{d}_* Y^1) - \iota_{\alpha^1}\mathrm{d}_* l_2(X^1, Y^1) \\ &\qquad - \iota_{L^1_{Y^1}\alpha^1}\mathrm{d}_* X^1 - l_2(X^1, \iota_{\alpha^1}\mathrm{d}_* Y^1) \\ &= \iota_{\alpha^1}\big(-[\mathrm{d}_*(X^1), Y^1]_S + [X^1, \mathrm{d}_*(Y^1)]_S - \mathrm{d}_*[X^1, Y^1]_S\big), \end{aligned}$$

which implies that

$$\mathrm{d}_*[X^1, Y^1]_S = -[\mathrm{d}_*(X^1), Y^1]_S + [X^1, \mathrm{d}_*(Y^1)]_S. \tag{83}$$

For all $X^1 \in \Gamma(A_{-1})$ and $\alpha^2, \beta^2, \gamma^2 \in \Gamma(B_{-1})$, by (f) in Definition 2.1, we have

$$\begin{aligned}
&\alpha^2 \diamond \Omega(\beta^2, \gamma^2, X^1) - \beta^2 \diamond \Omega(\alpha^2, \gamma^2, X^1) + \gamma^2 \diamond \Omega(\alpha^2, \beta^2, X^1) - \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2) \diamond X^1 \\
&\quad - \Omega(\mathfrak{l}_2(\alpha^2, \beta^2), \gamma^2, X^1) - \Omega(\beta^2, \mathfrak{l}_2(\alpha^2, \gamma^2), X^1) - \Omega(\beta^2, \gamma^2, \alpha^2 \diamond X^1) \\
&\quad + \Omega(\alpha^2, \mathfrak{l}_2(\beta^2, \gamma^2), X^1) + \Omega(\alpha^2, \gamma^2, \beta^2 \diamond X^1) - \Omega(\alpha^2, \beta^2, \gamma^2 \diamond X^1) = 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\iota_{\mathfrak{L}^3_{\beta^2,\gamma^2} X^1} \mathrm{d}\alpha^2 - \iota_{\mathfrak{L}^3_{\alpha^2,\gamma^2} X^1} \mathrm{d}\beta^2 + \iota_{\mathfrak{L}^3_{\alpha^2,\beta^2} X^1} \mathrm{d}\gamma^2 - \iota_{X^1} \mathrm{d}\mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2) \\
&\quad + \mathfrak{l}_3(\alpha^2, \beta^2, L^1_{X^1}\gamma^2) + \mathfrak{l}_3(\gamma^2, \alpha^2, L^1_{X^1}\beta^2) + \mathfrak{l}_3(\beta^2, \gamma^2, L^1_{X^1}\alpha^2) = 0.
\end{aligned} \tag{84}$$

By direct calculation, we have

$$\begin{aligned}
&\langle \iota_{\mathfrak{L}^3_{\beta^2,\gamma^2} X^1} \mathrm{d}\alpha^2 - \iota_{\mathfrak{L}^3_{\alpha^2,\gamma^2} X^1} \mathrm{d}\beta^2 + \iota_{\mathfrak{L}^3_{\beta^2,\beta^2} X^1} \mathrm{d}\gamma^2 - \iota_{X^1} \mathrm{d}\mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), Y^1 \rangle \\
&\quad = \langle -\mathfrak{l}_3(\gamma^2, \alpha^2, L^1_{Y^1}\beta^2) - \mathfrak{l}_3(\alpha^2, \beta^2, L^1_{Y^1}\gamma^2) - \mathfrak{l}_3(\beta^2, \gamma^2, L^1_{Y^1}\alpha^2), X^1 \rangle \\
&\qquad - a(X^1)\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), Y^1 \rangle + a(Y^1)\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), X^1 \rangle \\
&\qquad + \langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), l_2(X^1, Y^1) \rangle.
\end{aligned} \tag{85}$$

On the other hand, we have

$$\begin{aligned}
&[\widehat{\delta}_* X^1, Y^1]_S(\alpha^2, \beta^2, \gamma^2) \\
&\quad = -a(Y^1)\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), X^1 \rangle + \langle \mathfrak{l}_3(\gamma^2, \alpha^2, L^1_{Y^1}\beta^2) + \mathfrak{l}_3(\alpha^2, \beta^2, L^1_{Y^1}\gamma^2) \\
&\qquad + \mathfrak{l}_3(\beta^2, \gamma^2, L^1_{Y^1}\alpha^2), X^1 \rangle, \widehat{\delta}_*[X^1, Y^1]_S(\alpha^2, \beta^2, \gamma^2) \\
&\quad = -\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), l_2(X^1, Y^1) \rangle.
\end{aligned}$$

Then by (84) and (85), we have

$$\begin{aligned}
&\big(-[\widehat{\delta}_*(X^1), Y^1]_S + [X^1, \widehat{\delta}_*(Y^1)]_S - \widehat{\delta}_*[X^1, Y^1]_S\big)(\alpha^2, \beta^2, \gamma^2) \\
&\quad = a(Y^1)\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), X^1 \rangle - \langle \mathfrak{l}_3(\gamma^2, \alpha^2, L^1_{Y^1}\beta^2) + \mathfrak{l}_3(\alpha^2, \beta^2, L^1_{Y^1}\gamma^2) \\
&\qquad + \mathfrak{l}_3(\beta^2, \gamma^2, L^1_{Y^1}\alpha^2), X^1 \rangle - a(X^1)\langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), Y^1 \rangle \\
&\qquad + \langle \mathfrak{l}_3(\gamma^2, \alpha^2, L^1_{X^1}\beta^2) + \mathfrak{l}_3(\alpha^2, \beta^2, L^1_{X^1}\gamma^2) \\
&\qquad + \mathfrak{l}_3(\beta^2, \gamma^2, L^1_{X^1}\alpha^2), Y^1 \rangle + \langle \mathfrak{l}_3(\alpha^2, \beta^2, \gamma^2), l_2(X^1, Y^1) \rangle = 0,
\end{aligned}$$

which implies that

$$\widehat{\delta}_*[X^1, Y^1]_S = -[\widehat{\delta}_*(X^1), Y^1]_S + [X^1, \widehat{\delta}_*(Y^1)]_S. \tag{86}$$

Thus, by (82), (83), and (86), (81) follows immediately.

Step 2. We will show that

$$\delta_*[X^1,Y^2]_S=-[\delta_*(X^1),Y^2]_S+[X^1,\delta_*(Y^2)]_S,\quad \forall X^1\in\Gamma(A_{-1}),Y^2\in\Gamma(A_{-2}). \tag{87}$$

For $X^1\in\Gamma(A_{-1}),\ Y^2\in\Gamma(A_{-2})$, by (a) in Definition 2.1, we have

$$\begin{aligned}\langle\overline{\delta}_*[X^1,Y^2]_S+[\overline{\delta}_*(X^1),Y^2]_S-[X^1,\overline{\delta}_*(Y^2)]_S,\alpha^1\rangle\\ =\langle -l_1(l_2(X^1,Y^2)+l_2(X^1,l_1(Y^2)),\alpha^1\rangle=0,\end{aligned}$$

which implies that

$$\overline{\delta}_*[X^1,Y^2]_S=-[\overline{\delta}_*(X^1),Y^2]_S+[X^1,\overline{\delta}_*(Y^2)]_S. \tag{88}$$

For all $X^1\in\Gamma(A_{-1}),\ Y^2\in\Gamma(A_{-2})$ and $\alpha^2\in\Gamma(B_{-1})$, by ($e_1$) in Definition 2.1, we have

$$X^1\diamond(\alpha^2\diamond Y^2)-(X^1\diamond\alpha^2)\diamond Y^2-\alpha^2\diamond(X^1\diamond Y^2)=\Omega(X^1,\alpha^2,l_1(Y^2)).$$

By (21), this condition is equivalent to

$$\begin{aligned}l_2(X^1,\mathfrak{L}^1_{\alpha^2}Y^2)+\iota_{\iota_{Y^2}\mathrm{d}\alpha^2}\mathrm{d}_*X^1-\mathfrak{L}^1_{L^1_{X^1}\alpha^2}Y^2\\ +l_2(\mathfrak{L}^1_{\alpha^2}X^1,Y^2)-\mathfrak{L}^1_{\alpha^2}l_2(X^1,Y^2)=0.\end{aligned}$$

By direct calculation, on the one hand, we have

$$\begin{aligned}\langle l_2(X^1,\mathfrak{L}^1_{\alpha^2}Y^2),\beta^2\rangle&=a(X^1)\mathfrak{a}(\alpha^2)\langle Y^2,\beta^2\rangle-a(X^1)\langle Y^2,\mathfrak{l}_2(\alpha^2,\beta^2)\rangle\\&\quad-\mathfrak{a}(\alpha^2)a(X^1)\langle Y^2,\beta^2\rangle+\mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle\\&\quad+\langle Y^2,\mathfrak{l}_2(\alpha^2,L^1_{X^1}\beta^2)\rangle;\\ \langle\iota_{\iota_{Y^2}\mathrm{d}\alpha^2}\mathrm{d}_*X^1,\beta^2\rangle&=a(\mathfrak{L}^1_{\beta^2}X^1)\langle Y^2,\alpha^2\rangle-\mathfrak{a}(\beta^2)\langle\alpha^2,l_2(X^1,Y^2)\rangle\\&\quad-\langle X^1,\mathfrak{l}_2(\beta^2,L^2_{Y^2}\alpha^2)\rangle;\\ \langle\mathfrak{L}^1_{L^1_{X^1}\alpha^2}Y^2,\beta^2\rangle&=\mathfrak{a}(L^1_{X^1}\alpha^2)\langle Y^2,\beta^2\rangle-\langle Y^2,\mathfrak{l}_2(L^1_{X^1}\alpha^2,\beta^2)\rangle;\\ \langle l_2(\mathfrak{L}^1_{\alpha^2}X^1,Y^2),\beta^2\rangle&=\mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle+\langle X^1,\mathfrak{l}_2(\alpha^2,L^2_{Y^2}\beta^2)\rangle;\\ \langle\mathfrak{L}^1_{\alpha^2}l_2(X^1,Y^2),\beta^2\rangle&=\mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle-\langle l_2(X^1,Y^2),\mathfrak{l}_2(\alpha^2,\beta^2)\rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle\iota_{\alpha^2}\mathrm{d}_*[X^1,Y^2]_S,\beta^2\rangle&=\mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle-\mathfrak{a}(\beta^2)\langle\alpha^2,l_2(X^1,Y^2)\rangle\\&\quad-\langle l_2(X^1,Y^2),\mathfrak{l}_2(\alpha^2,\beta^2)\rangle;\\ \langle\iota_{\alpha^2}[\mathrm{d}_*(X^1),Y^2]_S,\beta^2\rangle&=-\mathrm{d}_*X^1(L^2_{Y^2}\alpha^2,\beta^2)+\mathrm{d}_*X^1(L^2_{Y^2}\beta^2,\alpha^2)\\&=\mathfrak{a}(\beta^2)\langle\alpha^2,l_2(X^1,Y^2)\rangle+\langle X^1,\mathfrak{l}_2(\beta^2,L^2_{Y^2}\alpha^2)\rangle\\&\quad-\mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle-\langle X^1,\mathfrak{l}_2(\alpha^2,L^2_{Y^2}\beta^2)\rangle;\end{aligned}$$

$$\begin{aligned}\langle\iota_{\alpha^2}[X^1,\mathrm{d}_*(Y^2)]_S,\beta^2\rangle = {} & a(X^1)\mathfrak{a}(\alpha^2)\langle Y^2,\beta^2\rangle - a(X^1)\mathfrak{a}(\beta^2)\langle Y^2,\alpha^2\rangle \\ & - a(X^1)\langle Y^2,\mathfrak{l}_2(\alpha^2,\beta^2)\rangle - \mathfrak{a}(L^1_{X^1}\alpha^2)\langle Y^2,\beta^2\rangle \\ & + \mathfrak{a}(\beta^2)a(X^1)\langle Y^2,\alpha^2\rangle - \mathfrak{a}(\beta^2)\langle\alpha^2,l_2(X^1,Y^2)\rangle \\ & + \langle Y^2,\mathfrak{l}_2(L^1_{X^1}\alpha^2,\beta^2)\rangle - \mathfrak{a}(\alpha^2)a(X^1)\langle Y^2,\beta^2\rangle \\ & + \mathfrak{a}(\alpha^2)\langle\beta^2,l_2(X^1,Y^2)\rangle + \mathfrak{a}(L^1_{X^1}\beta^2)\langle Y^2,\alpha^2\rangle \\ & + \langle Y^2,\mathfrak{l}_2(\alpha^2,L^1_{X^1}\beta^2)\rangle.\end{aligned}$$

Then by the above formulas and $\rho(X^1\diamond\alpha^2)=[\rho(X^1),\rho(\alpha^2)]$, we have

$$\begin{aligned}0 &= l_2(X^1,\mathcal{L}^1_{\alpha^2}Y^2) + \iota_{\iota_{Y^2}\mathrm{d}\alpha^2}\mathrm{d}_*X^1 - \mathcal{L}^1_{L^1_{X^1}\alpha^2}Y^2 \\ &\qquad + l_2(\mathcal{L}^1_{\alpha^2}X^1,Y^2) - \mathcal{L}^1_{\alpha^2}l_2(X^1,Y^2) \\ &= \iota_{\alpha^2}\big(-[\mathrm{d}_*(X^1),Y^2]_S + [X^1,\mathrm{d}_*(Y^2)]_S - \mathrm{d}_*[X^1,Y^2]_S\big),\end{aligned}$$

which implies that

$$\mathrm{d}_*[X^1,Y^2]_S = -[\mathrm{d}_*(X^1),Y^2]_S + [X^1,\mathrm{d}_*(Y^2)]_S. \tag{89}$$

By a direct calculation, we have

$$\begin{aligned}[\widehat{\delta}_*(X^1),Y^2]_S(\alpha^2,\beta^2,\gamma^2) &= [X^1,\widehat{\delta}_*(Y^2)]_S(\alpha^2,\beta^2,\gamma^2) \\ &= \widehat{\delta}_*[X^1,Y^2]_S(\alpha^2,\beta^2,\gamma^2) = 0,\end{aligned}$$

which implies that

$$\widehat{\delta}_*[X^1,Y^2]_S = -[\widehat{\delta}_*(X^1),Y^2]_S + [X^1,\widehat{\delta}_*(Y^2)]_S. \tag{90}$$

Thus, by (88), (89) and (90), (87) follows immediately.

By (81) and (87), for all $X,Y\in\Gamma(\mathcal{A}[-3])$, we have

$$\delta_*[X,Y]_S = -[\delta_*(X),Y]_S + (-1)^{|X|}[X,\delta_*(Y)]_S,$$

which implies that (27) in Theorem 4.9 holds.

Similarly, we can show that (28) also holds. Therefore, $(\mathcal{A},\mathcal{B})$ is a split Lie 2-bialgebroid. □

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