

Equivariant vector bundles over even quantum spheres

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Abstract. We quantize $SO(2n + 1)$ -equivariant vector bundles on an even complex sphere \mathbb{S}^{2n} as one-sided projective modules over its quantized coordinate ring. We realize them in two different ways: as linear maps between pseudo-parabolic modules and as induced modules of the orthogonal quantum group. Based on this alternative, we study representations of a quantum symmetric pair related to \mathbb{S}_q^{2n} and prove their complete reducibility.

Mathematics Subject Classification (2020). 17B10, 17B37, 53D55.

Keywords. Quantum groups, quantum spheres, equivariant vector bundles, symmetric pairs.

1. Introduction

In this paper we construct one-sided projective modules over the quantized coordinate ring of the even sphere \mathbb{S}^{2n} , $n \geq 1$, that are equivariant with respect to the quantized orthogonal group $SO(2n + 1)$. Such modules can be viewed as quantized equivariant vector bundles naturally extending the context of the conventional quantization programme for Poisson manifolds [1].

Spheres are important spaces in the classical geometry and topology [13], as well as for applications in other areas [30]. There has been a persistent interest to their non-commutative counterparts [4, 5, 11, 22, 28, 33] since the invention of non-commutative geometry [3] and quantum groups [6]. Although in some aspects all quantum spheres admit a uniform description, e.g. as subvarieties of quantum Euclidean planes [11], the even-dimensional case has been less covered in the literature. The main reason is a general effect when a pair (group, subgroup) determining a classical homogeneous space may be destroyed in quantization. This is observed for spheres as homogeneous spaces of orthogonal groups, however the odd dimensional case allows for reduced symmetries that survive quantization. Along these lines, \mathbb{S}_q^{2n-1} were approached via the quantum unitary groups in [33] and \mathbb{S}_q^{4n-1} via the quantum symplectic group in [21]. Projective modules over odd quantum spheres were recently studied in [31]. On the contrary, \mathbb{S}^{2n} lacks its quantum stabilizer subgroup but benefits from being a conjugacy class of a connected group, $G = SO(2n + 1)$. This enables one with the

machinery of the category \mathcal{O} to construct and study \mathbb{S}_q^{2n} via its faithful representation on a highest weight module, [24]. In the present paper we extend that study to equivariant vector bundles over \mathbb{S}_q^{2n} .

Semi-simple conjugacy classes of a simple Lie group fall into two classes depending on the type of their stabilizer subgroup. A Levi subalgebra \mathfrak{k} of a simple Lie algebra \mathfrak{g} has a basis of simple roots that is a part of the total root basis. As a consequence, its universal enveloping algebra $U(\mathfrak{k})$ is quantized as a Hopf subalgebra in the total quantum group $U_q(\mathfrak{g})$. A regular approach to equivariant star product quantization of vector bundles over conjugacy classes with such stabilizer subgroups was developed in [8]. However, if some simple roots of \mathfrak{k} are not simple with respect to the fixed polarization of \mathfrak{g} , it cannot be quantized as a subgroup, which disadvantage makes this case rather challenging. Nevertheless it is known that the trivial vector bundles (algebras of functions) on such classes are quantizable, [24]. This fact suggests that a uniform quantization scheme should exist for all vector bundles on semi-simple classes of all types (this conjecture is also supported by a local star product on \mathbb{S}^{2n} constructed in [23], similarly to the Levi case [10]). In the present paper we implement this programme for all equivariant vector bundles on \mathbb{S}^{2n} .

The Poisson structure on \mathbb{S}^{2n} under study is restricted from the Semenov–Tian–Shansky bracket on G , which is related to the Reflection Equation [17]. It makes G a Poisson–Lie manifold over the Poisson group G endowed with the Drinfeld–Sklyanin bracket, with respect to the conjugation action. The quantum sphere is treated as a subvariety of G_q and assumes an (conjugation) action of the corresponding quantum group $U_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{so}(2n + 1)$, on the quantized polynomial ring $\mathbb{C}_q[\mathbb{S}^{2n}]$ (we mean the Poisson group G that gives rise to standard $U_q(\mathfrak{g})$, [2]).

One possible realization of $\mathbb{C}_q[\mathbb{S}^{2n}]$ represents it by linear operators on a special $U_q(\mathfrak{g})$ -module of highest weight M which we call base module [24]. To construct an equivariant projective $\mathbb{C}_q[\mathbb{S}^{2n}]$ -module, one should find an invariant idempotent \hat{P} in the algebra $\text{End}(V) \otimes \mathbb{C}_q[\mathbb{S}^{2n}]$ for some finite-dimensional $U_q(\mathfrak{g})$ -module V . Such an idempotent projects $V \otimes M$ onto a direct summand submodule and therefore can be obtained from an irreducible decomposition of $V \otimes M$, provided certain technical conditions are fulfilled. In the case of Levi \mathfrak{k} , submodules of highest weight in $V \otimes M$ are parabolically induced from irreducible $U_q(\mathfrak{k})$ -submodules in V , which in the classical limit turn to the fibers over the initial point. In spite of no $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$ for non-Levi \mathfrak{k} , one still may expect that the classical irreducible \mathfrak{k} -submodules in V parameterize $U_q(\mathfrak{g})$ -submodules of highest weight in $V \otimes M$ as in the Levi case.

Thus the problem is to decompose $V \otimes M$ into a direct sum of highest weight modules, and describe the irreducible summands. The first question is discussed in general in [25], where a criterion for complete reducibility of $V \otimes M$ is formulated in terms of a canonical contravariant form on $V \otimes M$. This form is the product of unique, up to scalar factors, contravariant forms on V and M . Then $V \otimes M$ is completely reducible if and only if the restriction of the canonical form to the span

of singular vectors in $V \otimes M$ is non-degenerate. We apply this criterion to the base module M and all finite-dimensional modules V and prove complete reducibility of $V \otimes M$ for all q not a root of unity. To answer the second question, we define pseudo-parabolic modules that generalize parabolic modules for Levi subalgebras. We prove that all highest weight submodules in $V \otimes M$ are pseudo-parabolic, at generic q .

As spheres are symmetric spaces, we take advantage of an alternative presentation of $\mathbb{C}_q[\mathbb{S}^{2n}]$ via quantum symmetric pairs. That part of the paper is analogous to [26] devoted to quantum projective spaces. There is a 1-parameter family of solutions of the Reflection Equation [17] associated with $U_q(\mathfrak{g})$. Every solution defines a one-dimensional representation of $\mathbb{C}_q[\mathbb{S}^{2n}]$ (a “point” in \mathbb{S}_q^{2n}) and facilitates its realization as a subalgebra in the Hopf dual \mathcal{T} to $U_q(\mathfrak{g})$. At the same time, it defines a left coideal subalgebra $\mathcal{B} \subset U_q(\mathfrak{g})$ such that $\mathbb{C}_q[\mathbb{S}^{2n}]$ is identified with the subalgebra of \mathcal{B} -invariants in \mathcal{T} under the action by right translations. The subalgebra \mathcal{B} is a deformation of $U(\mathfrak{k}')$, where $\mathfrak{k}' \simeq \mathfrak{k}$ is the isotropy Lie algebra of the classical limit of the “quantum point”.

We prove that every finite-dimensional $U_q(\mathfrak{g})$ -module V is completely reducible over \mathcal{B} and that simple \mathcal{B} -submodules are deformations of irreducible \mathfrak{k}' -submodules. They are constructed via invariant projectors from $V \otimes M$ onto the corresponding pseudo-parabolic Verma submodules. Finally, we identify projective left $\mathbb{C}_q[\mathbb{S}^{2n}]$ -modules with \mathcal{B} -invariants in $\mathcal{T} \otimes X$, where X is a right \mathcal{B} -module and \mathcal{T} is equipped with the \mathcal{B} -action by right translations. It carries a $U_q(\mathfrak{g})$ -action via the left translations on \mathcal{T} . This is a deformation of the classical construction of the equivariant vector bundle associated with the fiber X .

The paper is organized as follows. Section 2 is devoted to certain properties of extremal projectors that we need for construction of singular vectors. We use these results for description of irreducible submodules of $V \otimes M$ in Section 3, where we also prove its complete reducibility. Therein we describe the pseudo-parabolic subcategory associated with the quantum sphere. Section 4 is devoted to equivariant quantum vector bundles. We present them as projective modules over $\mathbb{C}_q[\mathbb{S}^{2n}]$ using an equivalence with the pseudo-parabolic category and compute their equivariant Grothendieck group. Finally, we give a description of vector bundles via invariants of the coideal subalgebra from the quantum symmetric pair. Appendix A proves that parabolic modules are locally finite over the underlying quantum Levi subgroups. Appendix B contains some useful facts about tensor products of lowest and highest weight vectors.

2. Singular vectors of $U_q(\mathfrak{so}(5))$

This is a technical section devoted to certain aspects of representation theory of the quantum group $U_q(\mathfrak{so}(5))$. It is preparatory for the subsequent section, where we

define and study the main tool of our approach to quantum vector bundles: generalized parabolic Verma modules.

For a detailed exposition of general quantum groups the reader is referred to [2, 6]. A specific reminder of the odd orthogonal quantum group of rank $n > 0$ will be given later in Section 3.1. Here we start with the simplest interesting case of $U_q(\mathfrak{so}(5))$ and study singular vectors in its modules with the help of extremal projectors shifted by a “spectral parameter” [19, 34]. We present them in a form that is convenient for our applications and establish some useful facts that we have not found in the literature.

2.1. Shifted extremal projector. In what follows, we use the shortcuts

$$\bar{q} = q^{-1}, \quad [z]_q = \frac{q^z - q^{-z}}{q - q^{-1}} \quad \text{and} \quad [x, y]_a = xy - ayx \quad \text{for any } a \in \mathbb{C}.$$

The complex parameter $q \neq 0$ is assumed to be not a root of unity.

Let $e, f, q^{\pm h}$ denote generators of $U_q(\mathfrak{sl}(2))$ satisfying

$$q^{\pm h} e = q^{\pm 2} e q^{\pm h}, \quad q^{\pm h} f = q^{\mp 2} f q^{\pm h}, \quad [e, f] = [h]_q.$$

To accommodate certain operators of interest, we need to extend $U_q(\mathfrak{sl}(2))$ to an algebra $\widehat{U}_q(\mathfrak{sl}(2))$. First of all, we extend the Cartan subalgebra to the field $\mathbb{C}(q^h)$ of rational functions in q^h . Secondly, we include formal series in $f^k e^m$ of the same weight with coefficients from $\mathbb{C}(q^h)$. Such an extension can be done for an arbitrary quantum group, see [19] for details.

Define a one-parameter family $\pi(s) \in \widehat{U}_q(\mathfrak{sl}(2))$ by

$$\pi(s) = \sum_{k=0}^{\infty} f^k e^k \frac{(-1)^k q^{k(s-h-1)}}{[k]_q! \prod_{i=1}^k [s+i]_q}, \quad s \in \mathbb{C}. \quad (2.1)$$

More exactly, it belongs to the zero weight subalgebra $\widehat{U}_q^0(\mathfrak{sl}(2)) \subset \widehat{U}_q(\mathfrak{sl}(2))$. The element $\pi(s)$ features the following properties, which can be checked by a direct calculation:

$$e\pi(s) = \pi(s+1) \frac{q^{-h}[s+1-h]_q}{[s+1]_q} e, \quad (2.2)$$

$$f\pi(s) = \pi(s-1) \frac{q^{h+2}[s]_q}{[s-2-h]_q} f. \quad (2.3)$$

It follows that $\pi(h+1)$ is the extremal projector, i.e. an idempotent satisfying $e\pi(h+1) = 0$, $\pi(h+1)f = 0$, cf. [19]. Another corollary is that $\pi(s)$ is well defined on every $U_q(\mathfrak{sl}(2))$ -module of highest weight, where it returns

$$q^{-m\mu(h)+m(m-3)} \prod_{i=1}^m \frac{[-s+\mu(h)+i]_q}{[-s-i]_q}, \quad (2.4)$$

on a vector v of weight μ such that $e^m v$ is the highest vector.

We consider finite-dimensional $U_q(\mathfrak{sl}(2))$ -modules whose weights belong to $q^{\mathbb{Z}}$. Then the following fact is important for our exposition.

Proposition 2.1. *For all $s \in \mathbb{C}$ such that $q^{2s} \in -q^{\mathbb{Q}}$, the operator $\pi(s)$ is invertible on every finite-dimensional module.*

Proof. Immediate corollary of (2.4) since $\mu(h) \in \mathbb{Z}$ and q is not a root of unity. \square

Let σ denote an algebra automorphism of $U_q(\mathfrak{sl}(2))$ acting by the assignment $f \mapsto e$, $e \mapsto f$, $h \mapsto -h$. Observe that the operator $\sigma(\pi(s))$ is well defined on every finite-dimensional module. We are going to relate it to $\pi(s)$.

To that end, we extend $\mathfrak{sl}(2)$ to $\mathfrak{g} = \mathfrak{sl}(3)$ and use a fact that $\pi(s)$ is essentially a unique element satisfying a certain identity in $\hat{U}_q(\mathfrak{g})$. We assume that our $\mathfrak{sl}(2)$ corresponds to a simple root α . Let β be the other simple root and put $\gamma = \alpha + \beta$.

In the algebra $\hat{U}_q(\mathfrak{g})$, define

$$\begin{aligned} f_\gamma &= f_\beta f - q f f_\beta, & e_\gamma &= e e_\beta - q^{-1} e_\beta e, \\ \hat{f}_\gamma(s) &= f_\beta f - f f_\beta \frac{[s]_q}{[s+1]_q}, & \hat{e}_\gamma(s) &= e e_\beta - e_\beta e \frac{[s]_q}{[s+1]_q}. \end{aligned}$$

The vectors e_γ and f_γ form a quantum $\mathfrak{sl}(2)$ -pair in $U_q(\mathfrak{g})$.

Proposition 2.2. *The element $\pi(s)$ satisfies the equality*

$$\pi(s) f_\gamma = \hat{f}_\gamma(s) \pi(s+1).$$

Furthermore, $\pi(h-s)$ is a unique element from $\hat{U}_q^0(\mathfrak{sl}(2))$, up to a factor $a(h-s) \in \hat{U}_q(\mathfrak{h})$, satisfying

$$\hat{e}_\gamma(s) \pi(h-s) = \pi(h-s-1) e_\gamma. \quad (2.5)$$

Proof. Direct calculation. \square

Define a family $C(m, s)$ of rational trigonometric functions of s parameterized by $m \in \mathbb{Z}_+$ (the set of non-negative integers):

$$C(m, s) = \sum_{k=0}^{\infty} (-1)^k q^{k(s+m-1)} \prod_{i=1}^k \frac{[m-i+1]_q}{[s+i]_q}.$$

Note that all terms with $k > m$ vanish, so that the sum is finite.

Lemma 2.3. *For all $m \in \mathbb{Z}_+$,*

$$C(m, s) = q^{-m} \frac{[s]_q}{[s+m]_q}.$$

Proof. The statement is true for $m = 0$, as the summation over $k > 0$ turns zero. It is easy to check that the right-hand side satisfies the identity

$$C(m + 1, s) = 1 - \frac{q^{s+m}[m + 1]_q}{[s + 1]_q} C(m, s + 1).$$

Then induction on m employing this equality proves the formula for $m > 0$. \square

Proposition 2.4. *The operator identity*

$$\sigma(\pi(s)) = q^{-h} \frac{[s]_q}{[s + h]_q} \pi(h + s)$$

holds true in every finite-dimensional representation of $U_q(\mathfrak{sl}(2))$.

Proof. Put $\tilde{\pi}(s) = \sigma(\pi(-s))$. Applying σ to the first equality in Proposition 2.2 and changing the sign of the parameter s we get

$$\begin{aligned} \hat{e}_\gamma(s) \frac{1}{[s]_q} q^h \tilde{\pi}(s) &= \frac{1}{[s + 1]_q} q^h \tilde{\pi}(s + 1) e_\gamma \\ \Rightarrow \frac{1}{[s]_q} q^h \tilde{\pi}(s) &= a(h - s) \pi(h - s), \end{aligned} \quad (2.6)$$

due to uniqueness, cf. Proposition 2.2. To complete the proof, we need to calculate the factor $a(h - s)$. We do it by evaluating (2.6) on a subspace of weight $\mu(h) = m \in \mathbb{Z}$.

Since the equality

$$\tilde{\pi}(s) = C(h, -s) \bmod \hat{U}_q(\mathfrak{sl}(2))e$$

holds on every non-negative weight, we have

$$a(m - s) = -q^m \frac{1}{[s]_q} \frac{q^{-m}[s]_q}{[m - s]_q} = \frac{1}{[s - m]_q}, \quad m \geq 0.$$

Applying σ to $\tilde{\pi}(s)$ and evaluating the result on $m < 0$, one arrives at

$$\pi(s) = C(-h, s) \bmod \hat{U}_q(\mathfrak{sl}(2))f,$$

and therefore at

$$\pi(h - s) = C(-h, h - s) \bmod \hat{U}_q(\mathfrak{sl}(2))f.$$

So, for negative m , the equality (2.6) turns to

$$\frac{1}{[s]_q} q^m = a(m - s) C(-m, m - s)$$

producing the same result. Therefore $a(h - s) = \frac{1}{[s - h]_q}$ on all weight vectors. \square

2.2. Dynamical root vectors of $U_q(\mathfrak{so}(5))$. To the end of the section we assume $\mathfrak{g} = \mathfrak{so}(5)$. We express its simple roots $\alpha = \varepsilon_1$ and $\beta = \varepsilon_2 - \varepsilon_1$ through the orthogonal basis of short roots $\{\varepsilon_1, \varepsilon_2\} \in \mathfrak{h}^*$ normalizing the inner product so that $(\varepsilon_i, \varepsilon_i) = 1$, $i = 1, 2$. We introduce ‘‘compound root vectors’’ for $\gamma = \varepsilon_2$ and $\delta = \varepsilon_1 + \varepsilon_2$ by

$$\begin{aligned} f_\gamma &= [f_\alpha, f_\beta]_{\bar{q}}, & e_\gamma &= [e_\beta, e_\alpha]_q, \\ f_\delta &= [[f_\alpha, f_\beta]_{\bar{q}}, f_\alpha], & e_\delta &= [e_\alpha, [e_\beta, e_\alpha]_q]. \end{aligned}$$

Along with $q^{\pm h_\gamma}$ and $q^{\pm h_\delta}$, these pairs generate subalgebras isomorphic to $U_q(\mathfrak{sl}(2))$. We denote the corresponding Lie subalgebras by $\mathfrak{g}^{(\eta)}$ for each positive root η . Let \mathfrak{g}_+ denote the linear span of $e_\alpha, e_\beta, e_\gamma, e_\delta$, and define \mathfrak{g}_- similarly.

We will also need a ‘‘dynamical’’ version of the vectors f_δ and e_δ :

$$\widehat{f}_\delta(s) = \bar{p}^2 \left(f_\alpha^2 f_\beta \frac{[s]_p}{[s+2]_p} - f_\alpha f_\beta f_\alpha [2]_p \frac{[s]_p}{[s+1]_p} + f_\beta f_\alpha^2 \right), \quad (2.7)$$

$$\widehat{e}_\delta(s) = e_\alpha^2 e_\beta - e_\alpha e_\beta e_\alpha [2]_p \frac{[s+2]_p}{[s+1]_p} + e_\beta e_\alpha^2 \frac{[s+2]_p}{[s]_p}, \quad (2.8)$$

where $p = q^{\frac{1}{2}}$ and $\bar{p} = p^{-1}$. Introduce a one-parameter family of elements $\pi_\alpha(s) \in \widehat{U}_q(\mathfrak{g})$ by

$$\pi_\alpha(s) = \sum_{k=0}^{\infty} f_\alpha^k e_\alpha^k \frac{[2]_p^k (-1)^k p^{k(s-2h_\alpha-1)}}{[k]_p! \prod_{i=1}^k [s+i]_p}.$$

It is the image of $\pi(s)$ under the embedding $\mathfrak{sl}(2) \subset \mathfrak{g}$ corresponding to the short root α (the parameter q in the preceding subsections is to be replaced with $q_\alpha = p$).

Proposition 2.5. *The following relations hold true:*

$$\pi_\alpha(s) f_\delta = \widehat{f}_\delta(s) \pi_\alpha(s+2), \quad (2.9)$$

$$\pi_\alpha(2h_\alpha + s) e_\delta = \widehat{e}_\delta(s) \pi_\alpha(2h_\alpha + s + 2) q. \quad (2.10)$$

Proof. Equality (2.9) is checked through a straightforward calculation. The automorphism σ takes f_δ to $\bar{q} e_\delta$ and $\widehat{f}_\delta(s)$ to $\bar{q} \frac{[s]_p}{[s+2]_p} \widehat{e}_\delta(s)$. Applying it to (2.9) we get

$$\sigma(\pi_\alpha(s)) e_\delta = \frac{[s]_p}{[s+2]_p} \widehat{e}_\delta(s) \sigma(\pi_\alpha(s+2)).$$

Substitution of $\sigma(\pi_\alpha(s)) = \frac{p^{-2h_\alpha} [s]_p}{[s+2h_\alpha]_p} \pi_\alpha(2h_\alpha + s)$, thanks to Proposition 2.4, proves (2.10) in all finite-dimensional representations and therefore in $\widehat{U}_q(\mathfrak{g})$. \square

2.3. On singular vectors in Verma modules. Recall that a weight vector in a $U_q(\mathfrak{g})$ -module is called singular if it is annihilated by generators of $U_q(\mathfrak{g}_+)$. In this section we derive explicit formulas for singular vectors of weights $\mu - m\delta$, $m \in \mathbb{Z}_+$, in a Verma module of highest weight μ subject to certain conditions.

Lemma 2.6. *One has $f_\alpha \widehat{f}_\delta(s+1) = \widehat{f}_\delta(s) \frac{[s+1]_p}{[s+3]_p} f_\alpha$.*

Proof. The Serre relations imply $f_\alpha f_\delta = p^2 f_\delta f_\alpha$, cf. Section 3.1. Applying Proposition 2.5 and then formula (2.3) to $f_\alpha \widehat{f}_\delta(s) \pi_\alpha(s+2)$ we write it as

$$\begin{aligned} f_\alpha \pi_\alpha(s) f_\delta &= \frac{p^{2h_\alpha+2} [s]_p}{[s-2-2h_\alpha]_p} \pi_\alpha(s-1) p^2 f_\delta f_\alpha \\ &= \frac{p^{2h_\alpha+4} [s]_p}{[s-2-2h_\alpha]_p} \widehat{f}_\delta(s-1) \pi_\alpha(s+1) f_\alpha \\ &= \frac{p^{2h_\alpha+4} [s]_p}{[s-2-2h_\alpha]_p} \widehat{f}_\delta(s-1) \frac{[s-2h_\alpha]_p^{-2h_\alpha-2}}{[s+2]_p} f_\alpha \pi_\alpha(s+2) \\ &= \frac{[s]_p}{[s+2]_p} \widehat{f}_\delta(s-1) f_\alpha \pi_\alpha(s+2). \end{aligned}$$

Evaluating these on the highest vector of a generic Verma module and replacing s with $s+1$ we prove the lemma. \square

As a consequence, we obtain the formula

$$\widehat{f}_\delta(s) f_\alpha^2 = \frac{[s+4]_p [s+3]_p}{[s+2]_p [s+1]_p} f_\alpha^2 \widehat{f}_\delta(s+2). \quad (2.11)$$

We will also need the commutation relation

$$[e_\beta, \widehat{f}_\delta(s)] = f_\alpha^2 \frac{[2]_p [s+h_\beta+2]_q \bar{q}}{[s+2]_p [s+1]_p}, \quad (2.12)$$

which readily follows from the defining relations of $U_q(\mathfrak{g})$.

Proposition 2.7. *Let \widehat{M}_μ be the Verma module of highest weight μ with the highest vector 1_μ . Suppose that μ satisfies the condition $[(\mu, \delta) - m + 2]_q = 0$ for some $m \in \mathbb{N}$. Then the vector $\pi_\alpha(2h_\alpha+1) f_\delta^m 1_\mu$ is singular.*

Proof. We mentioned that $\pi_\alpha(2h_\alpha+1)$ is the extremal projector $\widehat{U}_q(\mathfrak{g}^{(\alpha)})$, therefore e_α kills $\pi_\alpha(2h_\alpha+1) f_\delta^m 1_\mu$. Explicitly, we write it as $\pi_\alpha(s) f_\delta^m 1_\mu$ with $s = 2(\mu, \alpha) - 2m + 1$. Using (2.3) we transform $e_\alpha \pi_\alpha(s) f_\delta^m$ to

$$\pi_\alpha(s+1) q^{-2h_\alpha} e_\alpha f_\delta^m \frac{[s+1-2h_\alpha+2m-2]_p}{[s+1]_p}.$$

Therefore $\pi_\alpha(s) f_\delta^m 1_\mu$ is annihilated by e_α indeed. Furthermore, notice that

$$e_\beta \pi_\alpha(s + 2m) \in \widehat{U}_q(\mathfrak{g}) \mathfrak{g}_+$$

since $\pi_\alpha(s + 2m)$ has zero weight. Using the factorization

$$\pi_\alpha(s) f_\delta^m = q^{-m} \widehat{f}_\delta(s) \widehat{f}_\delta(s + 2) \dots \widehat{f}_\delta(s + 2m - 2) \pi_\alpha(s + 2m) \quad (2.13)$$

that follows from (2.9) we present $e_\beta \pi_\alpha(s) f_\delta^m$ as

$$\sum_{k=1}^m q^{-k} \prod_{i=1}^{k-1} \widehat{f}_\delta(s + 2i - 2) [e_\beta, \widehat{f}_\delta(s + 2k - 2)] \pi_\alpha(s + 2k) f_\delta^{m-k} \text{ mod } \widehat{U}_q(\mathfrak{g}) \mathfrak{g}_+.$$

Using (2.12) for the commutator we rewrite this sum as

$$[2]_p \sum_{k=1}^m q^{-k-1} \prod_{i=1}^{k-1} \widehat{f}_\delta(s + 2i - 2) f_\alpha^2 \frac{[s + h_\beta + 2k]_q}{[s + 2k]_p [s + 2k - 1]_p} \pi_\alpha(s + 2k) f_\delta^{m-k}.$$

Pushing f_α^2 to the left with the help of (2.11) we get

$$[2]_p q^{-2} f_\alpha^2 \sum_{k=1}^m \frac{[s + h_\beta + 2k]_q}{[s + 1]_p [s + 2]_p} \prod_{i=1}^{k-1} q^{-1} \widehat{f}_\delta(s + i + 1) \pi_\alpha(s + 2k) f_\delta^{m-k}.$$

Now pushing π_α back to the left with the help of (2.9) we restore the factor f_δ^{m-1} on the right and arrive to

$$[2]_p q^{-2} f_\alpha^2 \frac{\pi_\alpha(s + 1)}{[s + 1]_p [s + 2]_p} \sum_{k=1}^m [s + h_\beta + 2k]_q f_\delta^{m-1}.$$

The sum here is equal to $[s + h_\beta + m + 1]_q [m]_q$. Pushing $\pi_\alpha(s)$ to the right with the use of (2.13) and applying to the highest vector we eventually obtain

$$f_\alpha^2 \widehat{f}_\delta(s + 1) \widehat{f}_\delta(s + 3) \dots \widehat{f}_\delta(s + 2m - 3) \frac{[2]_p [m]_q [s + h_\beta + m + 1]_q}{[s + 1]_p [s + 2]_p} q^{-m-1} 1_\mu.$$

Therefore, the vector in question is annihilated by e_β if the factor $[s + (\mu, \beta) + m + 1]_q$ is zero. Substitution of $s = 2(\mu, \alpha) - 2m + 1$ translates this to the condition on the value of (μ, δ) . \square

In the modules of our interest appearing in the next section, the highest weight μ satisfies

$$q^{2(\mu, \alpha)} = q_\alpha^{2(\mu, \alpha^\vee)} \in -q^{\mathbb{Z}},$$

where α^\vee is the coroot $\frac{2\alpha}{(\alpha, \alpha)}$. Hence the right-hand side in (2.13) is regular at such weights by Proposition 2.1.

Remark that the assumption on μ of Proposition 2.7 can be written as the condition $[(\mu + \rho, \delta) - m]_q = 0$, where ρ is the half-sum of positive roots of $\mathfrak{so}(5)$. This is a pole of the quantum Shapovalov determinant of the module \widehat{M}_μ , [7].

3. Generalized parabolic modules

In this section we introduce and study a class of modules over the odd orthogonal quantum group that play a role of “representations” for quantized vector bundles. Some of their properties are derived from corresponding properties of parabolic modules, which are established in Appendix A. We show that generalized parabolic modules form a nice category that is equivalent to the category of classical finite-dimensional modules over the even orthogonal group which is the stabilizer of the initial point on the even sphere. In the classical geometry, its modules are fibers over the initial point. They will also parameterize quantized vector bundles in the subsequent section.

3.1. The odd orthogonal quantum group. From now on \mathfrak{g} is the Lie algebra $\mathfrak{so}(2n+1)$, $n \in \mathbb{N}$, and $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. An ad-invariant form identifies \mathfrak{g} with \mathfrak{g}^* and \mathfrak{h} with \mathfrak{h}^* . Let (\cdot, \cdot) designate its restriction to \mathfrak{h} . Denote by R its root system and fix the subset of positive roots R^+ with basis Π^+ . We normalize the inner product so that the length of short roots is 1. Then R^+ contains an orthogonal basis $\{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^*$ of short roots such that $\alpha_1 = \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$, $i = 2, \dots, n$, constitute Π^+ . For all $\lambda \in \mathfrak{h}^*$ we denote by h_λ the element of \mathfrak{h} such that $\mu(h_\lambda) = (\mu, \lambda)$ for arbitrary $\mu \in \mathfrak{h}^*$.

By $U_q(\mathfrak{g})$ we understand the standard orthogonal quantum group [2] over the complex field with the set of generators e_α , f_α , and invertible q^{h_α} satisfying

$$q^{h_\alpha} e_\beta = q^{(\alpha, \beta)} e_\beta q^{h_\alpha}, \quad [e_\alpha, f_\beta] = \delta_{\alpha, \beta} [h_\alpha]_q, \quad q^{h_\alpha} f_\beta = q^{-(\alpha, \beta)} f_\beta q^{h_\alpha},$$

for $\alpha, \beta \in \Pi^+$. The elements e_α and f_α , $\alpha \in \Pi^+$, obey the Serre relations,

$$[e_\alpha, [e_\alpha, e_\beta]_q]_{\bar{q}} = 0, \quad [f_\alpha, [f_\alpha, f_\beta]_q]_{\bar{q}} = 0,$$

for all $\alpha, \beta \in \Pi^+$ such that $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = -1$, and

$$\begin{aligned} [e_\alpha, e_\beta] &= 0, \quad [f_\alpha, f_\beta] = 0, \quad \forall \alpha, \beta \in \Pi^+ \text{ s.t. } (\alpha, \beta) = 0, \\ [e_{\alpha_1}, \hat{e}_\delta] &= 0, \quad [f_{\alpha_1}, \hat{f}_\delta] = 0, \end{aligned}$$

where $\hat{e}_\delta = [e_{\alpha_1}, [e_{\alpha_1}, e_{\alpha_2}]_q]_{\bar{q}}$ and $\hat{f}_\delta = [f_{\alpha_1}, [f_{\alpha_1}, f_{\alpha_2}]_q]_{\bar{q}}$. Note that \hat{e}_δ and \hat{f}_δ are deformations of classical root vectors. Although they do not form a quantum $\mathfrak{sl}(2)$ -pair, they play a role in what follows.

Fix the comultiplication on the generators of $U_q(\mathfrak{g})$ as

$$\begin{aligned} \Delta(f_\alpha) &= f_\alpha \otimes 1 + q^{-h_\alpha} \otimes f_\alpha, \quad \Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha, \\ \Delta(q^{h_\alpha}) &= q^{h_\alpha} \otimes q^{h_\alpha}. \end{aligned}$$

Then the antipode γ acts by the assignment

$$\gamma(f_\alpha) = -q^{h_\alpha} f_\alpha, \quad \gamma(e_\alpha) = -e_\alpha q^{-h_\alpha}, \quad \gamma(q^{h_\alpha}) = q^{-h_\alpha}.$$

The counit ϵ returns $\epsilon(e_\alpha) = 0 = \epsilon(f_\alpha)$, and $\epsilon(q^{h_\alpha}) = 1$.

Denote by $U_q(\mathfrak{h})$, $U_q(\mathfrak{g}_+)$, and $U_q(\mathfrak{g}_-)$ the subalgebras generated by

$$\{q^{\pm h_\alpha}\}_{\alpha \in \Pi^+}, \quad \{e_\alpha\}_{\alpha \in \Pi^+}, \quad \text{and} \quad \{f_\alpha\}_{\alpha \in \Pi^+},$$

respectively. The Lie subalgebra

$$\mathfrak{l} = \mathfrak{gl}(n) \supset \mathfrak{h}$$

with the basis of simple roots $\Pi_{\mathfrak{l}}^+ = \{\alpha_i\}_{i=2}^n$ is a Levi subalgebra in \mathfrak{g} . Its universal enveloping algebra is quantized as a quantum subgroup $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$. On the contrary, the Lie subalgebra

$$\mathfrak{k} = \mathfrak{so}(2n) \subset \mathfrak{g}$$

with the basis of simple roots $\Pi_{\mathfrak{k}}^+ = \{\delta, \alpha_2, \dots, \alpha_n\}$ does not have a natural quantum analog of $U(\mathfrak{k}) \subset U(\mathfrak{g})$.

For each $\alpha \in \mathbb{R}^+$ we denote by $\mathfrak{g}^{(\alpha)}$ the corresponding $\mathfrak{sl}(2)$ -subalgebra in \mathfrak{g} . If α is simple, then $U_q(\mathfrak{g}^{(\alpha)})$ is a quantum subgroup in $U_q(\mathfrak{g})$. For compound α , there is a triple of elements (not unique) in $U_q(\mathfrak{g})$ generating a $U_q(\mathfrak{sl}(2))$ -subalgebra (but not a Hopf one), which is a deformation of $U(\mathfrak{g}^{(\alpha)})$. Its generators of weights $\pm\alpha$ enter a Poincaré–Birkhoff–Witt (PBW) system delivering a basis in $U_q(\mathfrak{g}_\pm)$, [2]. We denote this subalgebra by $U_q(\mathfrak{g}^{(\alpha)})$ assuming its root vectors fixed.

Given a diagonalizable $U_q(\mathfrak{h})$ -module W with weight subspaces $W[\mu]$, denote by $\text{ch}(W)$ the formal sum

$$\sum_{\mu} \dim W[\mu] q^\mu,$$

where q^μ is a one-dimensional representation of $U_q(\mathfrak{h})$ acting by $q^\mu: q^{h_\alpha} \mapsto q^{(\mu, \alpha)}$. We also use this definition for classical \mathfrak{h} -modules (replacing e^μ with q^μ) with the obvious definition of weight subspaces.

For two $U_q(\mathfrak{h})$ -modules W_i , $i = 1, 2$, we write

$$\text{ch}(W_1) \leq \text{ch}(W_2) \quad \text{if } \dim W_1[\mu] \leq \dim W_2[\mu] \text{ for all } \mu \in \mathfrak{h}^*.$$

The usual rules for characters hold true: given a short exact sequence

$$0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_3 \rightarrow 0$$

one has

$$\text{ch}(W_2) = \text{ch}(W_1) + \text{ch}(W_3).$$

Obviously,

$$\text{ch}(W_1) \leq \text{ch}(W_2) \quad \text{and} \quad \text{ch}(W_3) \leq \text{ch}(W_2).$$

Also,

$$\text{ch}(W_1 \otimes W_2) = \text{ch}(W_1)\text{ch}(W_2)$$

for any W_1 and W_2 when the left-hand side exists.

Recall that finite-dimensional $U_q(\mathfrak{g})$ -modules are $U_q(\mathfrak{h})$ -diagonalizable, and the eigenvalues of $q^{h\alpha}$ belong to $\pm q^{\frac{(\alpha, \alpha)}{2}} \mathbb{Z}$. Throughout the text, we mean by finite-dimensional $U_q(\mathfrak{g})$ -modules only those whose $q^{h\alpha}$ -eigenvalues belong to $q^{\frac{(\alpha, \alpha)}{2}} \mathbb{Z}$ and denote their category by $\text{Fin}_q(\mathfrak{g})$. Such modules are in one-to-one correspondence with finite-dimensional $U(\mathfrak{g})$ -modules and have the same characters. This weight convention does not apply to finite-dimensional $U_q(\mathfrak{l})$ -modules and infinite-dimensional $U_q(\mathfrak{g})$ -modules.

The algebra $U_q(\mathfrak{g})$ and its modules appearing in this exposition rationally depend on the parameter q which can be specialized to complex numbers except for a certain set of values. By *all* q we mean all q that are not a root of unity. Saying *almost all* we exclude a finite set of q distinct from 1. We say that a property holds for *generic* q if it is true for almost all q when restricted to every finite-dimensional subspace. That can be formalized by passing to the local ring of rational functions regular at $q = 1$.

3.2. Extremal twist and complete reducibility of tensor products. Let us recall the construction of extremal twist, which is responsible for irreducible decomposition of tensor product $V \otimes Z$ of two highest weight modules [25].

Consider an irreducible $U_q(\mathfrak{g})$ -module Z of highest weight as a $U_q(\mathfrak{g}_-)$ -module and denote by I_Z^- the left ideal in $U_q(\mathfrak{g}_-)$ that annihilates the highest vector. Denote by

$$I_Z^+ \subset U_q(\mathfrak{g}_+)$$

the left ideal $\sigma(I_Z^-)$, where σ is an involutive automorphism of $U_q(\mathfrak{g})$ defined on the generators by the assignment

$$e_\alpha \rightarrow f_\alpha, \quad f_\alpha \rightarrow e_\alpha, \quad q^{\pm h\alpha} \rightarrow q^{\mp h\alpha}$$

for all $\alpha \in \mathbb{R}^+$.

Now let V and Z be a pair of irreducible $U_q(\mathfrak{g})$ -modules of highest weights ν and ζ , respectively, and highest vectors 1_ν and 1_ζ . Denote by V_Z^+ the kernel of the left ideal I_Z^+ in V and similarly define Z_V^+ . There are linear isomorphisms between V_Z^+ , Z_V^+ , and the subspace $(V \otimes Z)^+$ spanned by singular vectors in $V \otimes Z$. To describe this correspondence, present $u \in (V \otimes Z)^+$ as

$$u = 1_\nu \otimes z_{\mu-\nu} + \cdots + v_{\mu-\zeta} \otimes 1_\zeta,$$

where the terms with factors of other weights are suppressed. The isomorphisms are given by the assignments

$$\bar{\delta}_l(u) = v_{\nu-\zeta} \quad \text{and} \quad \bar{\delta}_r(u) = v_{\mu-\nu}.$$

We denote by δ_l and δ_r their inverse isomorphisms.

Let ω denote an anti-algebra involution on $U_q(\mathfrak{g})$ defined as $\gamma^{-1} \circ \sigma$. This map preserves the comultiplication on $U_q(\mathfrak{g})$. Choose a weight basis in V and present a singular vector u as

$$u = \sum_i v_i \otimes f_i 1_\xi$$

for some $\{f_i\} \subset U_q(\mathfrak{g}_-)$. With $v = \bar{\delta}_l(u)$, define a map

$$\theta_{V,Z}(v): V \rightarrow V/\omega(I_Z^+)V$$

sending v to the image of

$$\sum_i \gamma^{-1}(f_i)(v_i) \quad \text{in } V/\omega(I_Z^+)V.$$

This map is independent of the choice of f_i in the presentation of u and called extremal twist.

Define a canonical symmetric bilinear form on $V \otimes Z$ as the product of the ω -contravariant forms on V and Z (such forms do exist, are unique up to a scalar multiplier and non-degenerate). As ω is a coalgebra map, the canonical form is also contravariant. If $\langle \cdot, \cdot \rangle$ is the contravariant form on V , then the form $\langle \theta_{V,Z}(\cdot), \cdot \rangle$ is the pullback of the canonical form via the isomorphism $\delta_l: V_Z^+ \rightarrow (V \otimes Z)^+$. Similarly one can consider $\theta_{Z,V}$ instead of $\theta_{V,Z}$ due to the obvious symmetry between V and Z .

Theorem 3.1 ([25]). *The following are equivalent:*

- (i) $V \otimes Z$ is completely reducible.
- (ii) $\theta_{V,Z}$ is bijective.
- (iii) All highest weight submodules in $V \otimes Z$ are irreducible.
- (iv) $V \otimes Z$ is the sum of submodules of highest weight.

Practically we use (ii) and (iii) in order to check (i); the criterion (iv) is included for completeness.

As an example, consider the case with $\mathfrak{g} = \mathfrak{sl}(2)$ and Z a Verma module of highest weight λ . Choose a vector $v = f_\alpha^m 1_v \in V$ of weight $\xi = v - m\alpha$. It is easy to check that

$$\theta_{V,Z}(v) \propto \prod_{k=1}^m \frac{[(\lambda + \rho + \xi, \alpha^\vee) + k]_{q_\alpha}}{[(\lambda + \rho, \alpha^\vee) - k]_{q_\alpha}} v, \quad (3.1)$$

up to a non-vanishing factor. This can be done via presenting $\delta_l(v)$ in the form $\mathcal{F}(v \otimes 1_\xi)$, where

$$\mathcal{F} = 1 \otimes 1 + \cdots \in U_q(\mathfrak{g}_+) \widehat{\otimes} U_q(\mathfrak{g}_-)$$

is a lift of the (unique) invariant element

$$*1_{-\xi} \otimes 1_\xi + \cdots \in {}^*Z \widehat{\otimes} Z.$$

This lift can be easily computed for the $\mathfrak{sl}(2)$ -case. For Z a Verma module, $\theta_{V,Z}$ coincides with $\gamma^{-1}(\mathcal{F}_2)\mathcal{F}_1$ represented in $\text{End}(V)$. The latter can be expressed through the operator $\pi(s)$, then (2.4) translates to (3.1).

3.3. Description of base module and its generalized extremal spaces. The base module M for quantum \mathbb{S}^{2n} has a PBW basis that makes it isomorphic to the vector space of polynomials in n variables. To describe this basis, we need root vectors of weights $-\varepsilon_i$ for $i = 1, \dots, n$. Define $e_{\varepsilon_1} = e_{\alpha_1}$ and $f_{\varepsilon_1} = f_{\alpha_1}$ and proceed recurrently as

$$e_{\varepsilon_{i+1}} = [e_{\alpha_{i+1}}, e_{\varepsilon_i}]q, \quad f_{\varepsilon_{i+1}} = [f_{\varepsilon_i}, f_{\alpha_{i+1}}]\bar{q}$$

for $i > 1$. Fix $\lambda \in \mathfrak{h}^*$ by the condition $q^{2(\lambda, \varepsilon_i)} = -q^{-1}$ for all $i = 1, \dots, n$ and $(\alpha_i, \lambda) = 0$ for $i > 1$. We call it base weight. The Verma module \widehat{M}_λ of highest weight λ with the canonical generator 1_λ has singular vectors $f_{\alpha_i}1_\lambda$ with $i > 1$ and $\widehat{f}_\delta 1_\lambda$. Define M as the quotient of \widehat{M}_λ by the sum of submodules generated by these vectors. As proved in [23], it has a basis of weight vectors $f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda$ where m_i take all possible values in \mathbb{Z}_+ (we use the same notation for the image of 1_λ in M). In particular,

$$\text{ch}(M) = \prod_{i=1}^n (1 - q^{-\varepsilon_i})^{-1} q^\lambda.$$

Lemma 3.2. *The quantum group $U_q(\mathfrak{g})$ acts on M by*

$$\begin{aligned} e_{\alpha_i} f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda &\propto f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_{i-1}}^{m_{i-1}+1} f_{\varepsilon_i}^{m_i-1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda, \\ f_{\alpha_i} f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda &\propto f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_{i-1}}^{m_{i-1}-1} f_{\varepsilon_i}^{m_i+1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda, \quad i = 1, \dots, n, \end{aligned}$$

where the suppressed numerical factors are non-zero once q is not a root of unity.

Proof. By weight arguments the action has the specified form up to numerical factors. That they are not zero can be proved by the following calculation. First of all, they are given explicitly for the f_{α_i} -action in Lemma 3.2 in [23]. One can check that f_{ε_i} satisfy the relation

$$f_{\varepsilon_{i+1}} f_{\varepsilon_i} = q^{-1} f_{\varepsilon_i} f_{\varepsilon_{i+1}}$$

as operators in $\text{End}(M)$. Using these relations, one can easily prove that the factors arising from the e_{α_i} -action are non-zero too. \square

Remark that irreducibility of M readily follows from these formulas since all weights in M are multiplicity free, and M clearly has no singular vectors. Alternatively, that can be proved via the contravariant form on M , which is found to be non-degenerate in [23].

Denote by $\rho \in \mathfrak{h}^*$ the half-sum of positive roots and put $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ for all $\alpha \in \mathfrak{h}^*$. Fix a finite-dimensional $U_q(\mathfrak{g})$ -module V of highest weight ν and introduce

$$\ell_i = (\nu + \rho, \alpha_i^\vee) - 1 \in \mathbb{Z}_+ \quad \text{for } i = 1, \dots, n.$$

It is known that V is a quotient of the Verma module \widehat{M}_ν by the submodule $I_V^- 1_\nu$, where I_V^- is a left ideal in $U_q(\mathfrak{g}_-)$ generated by $\{f_{\alpha_i}^{\ell_i+1}\}_{i=1}^n$, [15]. Set $I_V^+ = \sigma(I_V^-)$. Then singular vectors in $V \otimes M$ are parameterized by

$$M_V^+ = \ker I_V^+ \subset M,$$

as explained above.

Proposition 3.3. *For all V , the contravariant form on M is non-degenerate when restricted to M_V^+ . The module M splits into the orthogonal sum*

$$M = M_V^+ \oplus \omega(I_V^+)M$$

with

$$\begin{aligned} M_V^+ &= \text{Span}\{f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda\}_{m_1 \leq \ell_1, \dots, m_n \leq \ell_n}, \\ \omega(I_V^+)M &= \text{Span}\{f_{\varepsilon_1}^{k_1} \dots f_{\varepsilon_n}^{k_n} 1_\lambda\}_{k_1, \dots, k_n}, \end{aligned}$$

where $k_i > \ell_i$ for some $i = 1, \dots, n$.

Proof. The first equality follows from Lemma 3.2. Furthermore, observe that

$$\omega(I_V^+)M = \sum_{i=1}^n f_{\alpha_i}^{\ell_i+1} M,$$

which proves the second equality. Since $\omega(I_V^+)M$ is orthogonal to M_V^+ and the form is non-degenerate due to irreducibility of M , the proposition is proved. \square

Singular vectors in $V \otimes M$ can be alternatively parameterized by the subspace $V_M^+ \subset V$, which is the joint kernel of $\{e_{\alpha_i}\}_{i=2}^n$ and \widehat{e}_δ . The weight subspaces in V_M^+ have dimension 1 and carry weights

$$\nu - \sum_{i=1}^n m_i \varepsilon_i, \quad \text{with } 0 \leq m_i \leq \ell_i.$$

In the classical limit, V_M^+ is spanned by highest vectors of irreducible \mathfrak{k} -modules, according to the Gelfand–Zeitlin reduction.

Consider, for example, the case of $\mathfrak{g} = \mathfrak{so}(5)$ and the $U_q(\mathfrak{g})$ -module V of highest weight $\nu = 3\nu_1 + 2\nu_2$, where ν_i are fundamental weights, $(\nu_i, \alpha_j^\vee) = \delta_{ij}$, $i, j = 1, 2$. The subspace M_V^+ is spanned by $f_{\varepsilon_1}^{m_1} f_{\varepsilon_2}^{m_2} 1_\lambda$ with $m_1 \leq 3$ and $m_2 \leq 2$. Its reciprocal

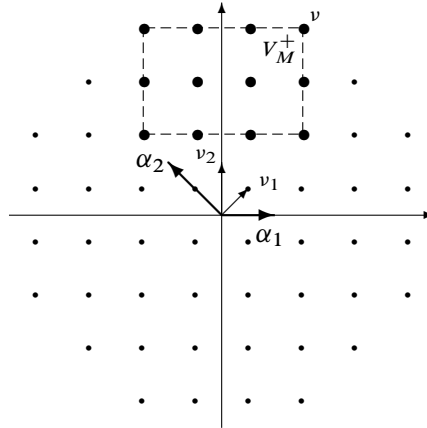


Figure 1. Gelfand–Zetlin reduction.

space V_M^+ is spanned by vectors of weights $\nu - m_1\varepsilon_1 - m_2\varepsilon_2$ of multiplicity 1, which are depicted on the weight lattice of V on Figure 1 by the fat circles:

An interesting problem is evaluation of the quantum reduction algebra of the pair $(\mathfrak{g}, \mathfrak{k})$ or, more exactly, of the left ideal $U_q(\mathfrak{g})I_M^+$. Its positive part is generated by a pair of commuting elements

$$z_{\varepsilon_1} = e_{\varepsilon_1}[h_{\alpha_2}]_q - qe_{\varepsilon_2}f_{\alpha_2} \quad \text{and} \quad z_{\varepsilon_2} = e_{\varepsilon_2}.$$

It allows to restore V_M^+ from the vector of lowest weight in V_M^+ . Negative generators, which are unknown to us, would deliver V_M^+ out of the highest vector of V .

The subspace V_M^+ is spanned by “singular vectors” of a “subalgebra” $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$, which is also unknown (and whose existence is under question). We believe that such a subalgebra does exist, maybe in an appropriate extension of $U_q(\mathfrak{g})$. One indication in favor of this conjecture is the presence of a coideal subalgebra, which is a quantization of the stabilizer of a different point in \mathbb{S}^{2n} , cf. Section 4.3.

3.4. Pseudo-parabolic Verma modules. Suppose that $\mathfrak{k} \subset \mathfrak{g}$ is a reductive subalgebra of maximal rank. Suppose that its polarization is induced by the polarization of \mathfrak{g} , so there are inclusions $R_{\mathfrak{k}}^{\pm} \subset R_{\mathfrak{g}}^{\pm}$ of their root subsystems. Take a weight $\xi \in \mathfrak{h}^*$ that is integral dominant with respect to \mathfrak{k} and let X be the irreducible finite-dimensional \mathfrak{k} -module X of highest weight ξ . For each $\alpha \in \Pi_{\mathfrak{k}}^+$ the sum of ξ and the base weight λ satisfies the condition

$$q^{2(\xi+\lambda+\rho, \alpha)-m_{\alpha}(\alpha, \alpha)} = 1, \tag{3.2}$$

where $m_{\alpha} = (\xi, \alpha^{\vee}) + 1$ a positive integer. Then there is a submodule $\widehat{M}_{\xi+\lambda-m_{\alpha}\alpha}$ in the Verma module $\widehat{M}_{\xi+\lambda}$, [7]. Let $M_{X, \lambda}$ denote the quotient of $\widehat{M}_{\xi+\lambda}$ by the

sum of $\widehat{M}_{\xi+\lambda-m\alpha}$ over all $\alpha \in \Pi_{\mathfrak{k}}^+$. If \mathfrak{k} were a Levi subalgebra in \mathfrak{g} , that is, if $\Pi_{\mathfrak{k}}^+ \subset \Pi_{\mathfrak{g}}^+$, then $M_{X,\lambda}$ would be a parabolic Verma module. That justifies the name of pseudo-parabolic Verma module for $M_{X,\lambda}$ when $\Pi_{\mathfrak{k}}^+ \not\subset \Pi_{\mathfrak{g}}^+$.

For non-Levi \mathfrak{k} , $M_{X,\lambda}$ is factored through a parabolic module $M_{\check{X}_\lambda}$ relative to the maximal Levi subalgebra \mathfrak{l} among those sitting inside \mathfrak{k} . The finite-dimensional $U_q(\mathfrak{l})$ -module $\check{X}_\lambda = \check{X} \otimes \mathbb{C}_\lambda$ is of the same highest weight $\xi + \lambda$. As a $U_q(\mathfrak{l})$ -module, $M_{\check{X}_\lambda}$ is isomorphic to the tensor product $U_q(\mathfrak{n}_-) \otimes \check{X}_\lambda$, where $U_q(\mathfrak{n}_-) \subset U_q(\mathfrak{g})$ is a subalgebra invariant under the adjoint action of $U_q(\mathfrak{l})$. It is a deformation of $U(\mathfrak{n}_-)$, where $\mathfrak{n}_- = \mathfrak{g}_- \ominus \mathfrak{l}_-$, see Section A.3 for details.

In the case of concern, $\mathfrak{l} = \mathfrak{gl}(n)$, and $M_{X,\lambda}$ is a quotient of $M_{\check{X}_\lambda}$ by the submodule generated by the image of $\pi_{\alpha_1}(2h_\alpha + 1)f_\delta^m 1_{\xi+\lambda}$ with $m = m_\delta$. By Proposition 2.7, it is singular unless it is zero.

Lemma 3.4. *For all $m \in \mathbb{Z}_+$, the vector $\pi_{\alpha_1}(2h_\alpha + 1)f_\delta^m 1_{\xi+\lambda}$ does not vanish in $M_{\check{X}_\lambda}$.*

Proof. We can assume that $n = 2$, then \widehat{f}_δ is central in $U_q(\mathfrak{g}_-)$. It is easy to check that the ordered monomials in $f_{\alpha_1}, \widehat{f}_\delta, f_{\alpha_1+\alpha_2}, f_{\alpha_2}$ whose degree of f_{α_2} is less than m_{α_2} deliver a basis in $M_{\check{X}_\lambda}$. Factorization (2.13) implies

$$\pi_{\alpha_1}(2h_\alpha + 1)f_\delta^m 1_{\xi+\lambda} = q^{-m} \prod_{i=0}^{m-1} \widehat{f}_\delta(s + 2i) 1_{\xi+\lambda} = q^{-m} \widehat{f}_{m\delta}(s) 1_{\xi+\lambda},$$

where $s = 2(\xi + \lambda, \alpha) - 2m + 2$ and $\widehat{f}_{m\delta}(s) \in U_q(\mathfrak{g}_-)$. Observe that $\widehat{f}_{m\delta}(s)$ is regular as $q^{2(\lambda, \alpha_1)} = -q^{-1}$. Induction on m proves that the right-hand side expands as

$$c \widehat{f}_\delta^m 1_{\xi+\lambda} + \dots,$$

where $c \neq 0$ and the omitted terms involve different basis elements. This proves the statement. \square

Thus $M_{X,\lambda}$ is a quotient of $M_{\check{X}_\lambda}$ by a proper submodule. Since $M_{\check{X}_\lambda}$ is locally finite over $U_q(\mathfrak{l})$ by Proposition A.3, so is $M_{X,\lambda}$.

Corollary 3.5. *For generic q , $\text{ch}(M_{X,\lambda}) \leq \text{ch}(M)\text{ch}(X)$.*

Proof. In the classical limit $q \rightarrow 1$, the vector $\widehat{f}_{m\delta}(s)$ goes over to f_δ^m . Observe that the $U_q(\mathfrak{g}_-)$ -module $M_{X,\lambda}$ turns into the induced module $U(\mathfrak{g}_-) \otimes_{U(\mathfrak{k}_-)} X$. Therefore

$$\text{ch}(M_{X,\lambda}) \leq \text{ch}(M)\text{ch}(X)$$

for generic q , since deformation does not increase quotients. \square

3.5. Pseudo-parabolic category $\mathcal{O}(\mathbb{S}^{2n})$. In this section we prove that generalized parabolic modules of highest weight are irreducible, and tensor products of the base module with finite-dimensional modules are completely reducible. We end up showing that the corresponding subcategory in the category \mathcal{O} is equivalent to the category of finite-dimensional \mathfrak{k} -modules.

Fix a finite-dimensional $U_q(\mathfrak{g})$ -module V . The subspace $V_M^+ \subset V$ is the kernel of the left ideal $I_M^+ \subset U_q(\mathfrak{g}_+)$ generated by $e_{\alpha_i}, i > 1$, and by $\widehat{e}_\delta(s)$, with $s = 2(\lambda, \alpha_1) - 1$ (note that $\widehat{e}_\delta(s) \propto \widehat{e}_\delta \pmod{U_q(\mathfrak{g}_+)e_{\alpha_2}}$ for such s). In the classical limit, V_M^+ is the kernel of \mathfrak{k}_+ .

Proposition 3.6. *For each non-zero $v \in V_M^+[\xi]$, the homomorphism $\widehat{M}_{\xi+\lambda} \rightarrow V \otimes M$, $1_{\xi+\lambda} \mapsto \delta_l(v)$, factors through a homomorphism $M_{X,\lambda} \rightarrow V \otimes M$.*

Proof. Since M is irreducible, the right dual N to M is a module of lowest weight, cf. Appendix B. It is a quotient of the lower Verma module by the submodule generated by $\widehat{e}_\delta 1_{-\lambda}$ and $e_\alpha 1_{-\lambda}$, $\alpha \in \Pi_1^+$. Furthermore, the singular vector

$$\widehat{f}_{m\delta}(x)1_{\xi+\lambda} \in \widehat{M}_{\xi+\lambda}$$

with $m = m_\delta$ and $x = 2(\xi + \lambda, \alpha) - 2m + 1$ vanishes in $M_{X,\lambda}$. Therefore, by Corollary B.2, $\text{Hom}(M_{X,\lambda}, V \otimes M)$ is isomorphic to $V_M^+[\xi] \cap \ker \widehat{f}_{m\delta}(x)$. The statement will be proved if we show that

$$V_M^+[\xi] \subset \ker \widehat{f}_{m\delta}(x).$$

Set $\alpha = \alpha_1$. The operator $\pi_\alpha(x + 2m)$ is invertible on V by Proposition 2.1, so put

$$v_0 = \pi_\alpha^{-1}(x + 2m)v.$$

Similarly, the operator $\pi_\alpha(2h_\alpha + s)$, with $s = 2(\lambda, \alpha) - 1$ is invertible on V . Then, by (2.10),

$$0 = \widehat{e}_\delta(s)\pi_\alpha(x + 2m)v_0 = \widehat{e}_\delta(s)\pi_\alpha(2h_\alpha + s + 2)v_0 = q^{-1}\pi_\alpha(2h_\alpha + s)e_\delta v_0,$$

hence $e_\delta v_0 = 0$. The vector v_0 generates a $U_q(\mathfrak{g}^{(\delta)})$ -submodule with $f_\delta^m v_0 = 0$. Then, by (2.9),

$$\widehat{f}_{m\delta}(x)\pi_\alpha(x + 2m)v_0 = \pi_\alpha(x)f_\delta^m v_0 = 0$$

as required. \square

Denote by $M'_{X_i,\lambda}$ the image of the module $M_{X_i,\lambda}$ in $V \otimes M$, then

$$\text{ch}(M'_{X_i,\lambda}) \leq \text{ch}(M_{X_i,\lambda}).$$

We aim to prove that $V \otimes M$ is a direct sum of $M'_{X_i,\lambda}$ and essentially $M'_{X_i,\lambda} = M_{X_i,\lambda}$. The proof will be done by induction on the rank of \mathfrak{g} . For each $k = 1, \dots, n$, we

identify the Lie subalgebra $\mathfrak{g}^{(k)} = \mathfrak{so}(2k+1) \subset \mathfrak{g}$ whose root basis is $\{\alpha_1, \dots, \alpha_k\}$. The module M contains base modules for \mathbb{S}^{2k} , which we denote by $M^{(k)}$. Obviously, $M^{(k)} \subset M^{(k+1)}$ for all $k < n$.

Theorem 3.7. *Let V be a finite-dimensional $U_q(\mathfrak{g})$ -module. Then*

- (1) $V \otimes M$ is completely reducible for all q and splits into a direct sum $\oplus_i M'_{X_i, \lambda}$ for an irreducible decomposition $V = \oplus_i X_i$ over \mathfrak{k} ,
- (2) all modules $M'_{X_i, \lambda}$ are irreducible and $\text{ch}(M'_{X_i, \lambda}) = \text{ch}(M)\text{ch}(X_i)$ for all q ,
- (3) $M_{X, \lambda} \simeq M'_{X, \lambda}$ for generic q .

Note that irreducibility of $M'_{X_i, \lambda}$ is equivalent to complete reducibility of $V \otimes M$, by Theorem 3.1 (iii). Below we give a full proof only for generic q . For all q , complete reducibility of $V \otimes M$ follows from a direct calculation of $\theta_{M, V}$ that employs a relation of $\theta_{M, V}$ with extremal projectors, in generalization of (3.1). That is based on a more advanced development of [25] and done in [27].

Proof. Consider the case of $n = 1$. Then $\mathfrak{g} \simeq \mathfrak{sl}(2)$, $\mathfrak{k} = \mathfrak{h}$, and M is a Verma module. Let ν be the highest weight of V . The extremal twist acts on a vector $f_{\varepsilon_1}^m 1_\nu \in V$ of weight $\xi = \nu - m\varepsilon_1$ as in (3.1), where $\alpha = \varepsilon_1$. It does not vanish once $q^{2(\lambda, \varepsilon_1)} = -q^{-1}$, therefore

$$V \otimes M = \oplus_i M'_{X_i, \lambda},$$

where all $M'_{X_i, \lambda}$ are irreducible Verma modules. This proves the theorem for $n = 1$ and all q .

Suppose that the theorem is proved for $n \geq 1$. An irreducible $U_q(\mathfrak{g}^{(n+1)})$ -module V is parameterized by the highest weight

$$\nu = l_1 \varepsilon_1 + \dots + l_{n+1} \varepsilon_{n+1} \quad \text{with } 0 \leq l_i \leq l_{i+1}.$$

Observe through the Gelfand–Zeitlin reduction that all irreducible $U_q(\mathfrak{g}^{(n)})$ -modules can be obtained by considering the special case of $l_n = l_{n+1}$. This corresponds to the zero $n + 1$ -th coordinate in the expansion of ν over the fundamental weights. Then V_M^+ is contained in the $U_q(\mathfrak{g}^{(n)})$ -submodule $V^{(n)}$ generated by the highest vector of V , and the corresponding singular vector lies in $V^{(n)} \otimes M^{(n)}$. By the assumption, it is completely reducible over $U_q(\mathfrak{g}^{(n)})$, therefore $\theta_{V^{(n)}, M^{(n)}}$ is bijective. Since the extremal twist is independent of the choice of lift for singular vectors,

$$\theta_{V^{(n)}, M^{(n)}} = \theta_{V, M},$$

and thus $\theta_{V, M}$ is bijective. Therefore $V \otimes M$ is completely reducible for special V (and for all q).

Furthermore, Corollary 3.5 implies

$$\begin{aligned} \text{ch}(V \otimes M) &= \sum_i \text{ch}(M'_{X_i, \lambda}) \leq \sum_i \text{ch}(M_{X_i, \lambda}) \\ &\leq \sum_i \text{ch}(M)\text{ch}(X_i) = \text{ch}(M)\text{ch}(V) \end{aligned}$$

for generic q . Since $\text{ch}(V)\text{ch}(M) = \text{ch}(V \otimes M)$, we conclude that

$$\sum_i \text{ch}(M'_{X_i, \lambda}) = \sum_i \text{ch}(M_{X_i, \lambda}) = \sum_i \text{ch}(M)\text{ch}(X_i)$$

and hence $M'_{X_i, \lambda} = M_{X_i, \lambda}$ for all i . Since $M'_{X_i, \lambda}$ are rational submodules in $V \otimes M$, their characters do not exceed their values at generic points. But they cannot be less than $\text{ch}(M)\text{ch}(X_i)$ either because they sum up to $\text{ch}(V)\text{ch}(M)$. This proves (3) and also proves (2) for all q and special V .

Now suppose that V is arbitrary. Proposition 3.6 implies that its all highest weight submodules in $V \otimes M$ are irreducible, for generic q . This proves (1), by Theorem 3.1 (iii). □

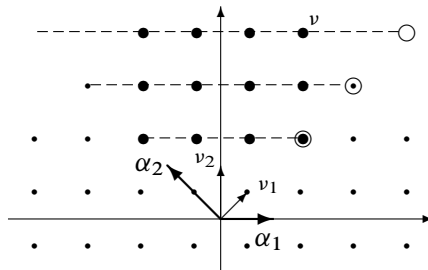


Figure 2. Reduction $\mathbb{S}_q^2 \subset \mathbb{S}_q^4$.

We illustrate the induction transition in the proof with the upper part of the weight diagram of the module $V = (3, 2)$ for $\mathfrak{g} = \mathfrak{so}(5)$, Figure 2. The weights of V_M^+ lie on the three horizontal dashed lines. These lines can be obtained from three special modules of highest weights $\frac{\ell}{2}(\varepsilon_1 + \varepsilon_2)$ with $\ell = 3, 5, 7$. They are marked on the diagram with large circles. Their weight subspaces are in $U_q(\mathfrak{g}^{(1)})$ -submodules $V^{(1)}$ of dimensions 4, 6, 8, therefore we can restrict to $U_q(\mathfrak{g}^{(1)})$ when calculating singular vectors.

Conjecture 1. *For any irreducible \mathfrak{k} -module X , the module $M_{X, \lambda}$ is isomorphic to $M'_{X, \lambda}$ for all q .*

Let us comment that its proof is equivalent to computing $\text{ch}(M_{X, \lambda})$. In the light of Theorem 3.7 (3), the inequality of Corollary 3.5 becomes equality at generic q and can be rectified as $\text{ch}(M_{X, \lambda}) \geq \text{ch}(M)\text{ch}(X)$ for all q .

Recall that the category \mathcal{O} is formed by finitely generated $U_q(\mathfrak{h})$ -diagonalizable $U_q(\mathfrak{g})$ -modules that are locally finite over $U_q(\mathfrak{g}_+)$. Define pseudo-parabolic category $\mathcal{O}(\mathbb{S}_q^{2n})$ as the full subcategory in \mathcal{O} whose objects are submodules in $V \otimes M$, where $V \in \text{Fin}_q(\mathfrak{g})$. Theorem 3.7 can be reworded as:

Proposition 3.8. $\mathcal{O}(\mathbb{S}_q^{2n})$ is a semi-simple Abelian category equivalent to $\text{Fin}(\mathfrak{k})$. It is a $\text{Fin}_q(\mathfrak{g})$ -module category generated by M .

Proof. The category $\mathcal{O}(\mathbb{S}_q^{2n})$ is additive semi-simple, hence it is Abelian. By construction, it is a module category over $\text{Fin}_q(\mathfrak{g})$.

The functor $\mathcal{O}(\mathbb{S}_q^{2n}) \rightarrow \text{Fin}(\mathfrak{k})$ is constructed as follows. Given a module M_1 from $\mathcal{O}(\mathbb{S}_q^{2n})$ define \check{X}_λ to be its finite-dimensional $U_q(\mathfrak{l})$ -submodule generated by the span of singular vectors in M_1 . Let \check{X}_0 be the classical limit of $\check{X}_\lambda \otimes \mathbb{C}_{-\lambda}$ equipped with the corresponding action of $U(\mathfrak{l})$. Proceed to the \mathfrak{k} -module parabolically induced from \check{X}_0 and then to its finite-dimensional quotient X by the maximal submodule that has zero intersection with \check{X}_0 . These steps, apart from maybe the last one, respect morphisms. But morphisms of parabolic \mathfrak{k} -modules induced from the \mathfrak{l} -modules descent to morphisms of their finite-dimensional quotients. Now let M_2 be another module from $\mathcal{O}(\mathbb{S}_q^{2n})$ and define $\check{Y}_\lambda, \check{Y}_0$, and Y accordingly. We have

$$\begin{aligned} \text{Hom}_{U_q(\mathfrak{h})}(M_1^+, M_2^+) &= \text{Hom}_{U_q(\mathfrak{h})}(\check{X}_\lambda^+, \check{Y}_\lambda^+) \simeq \text{Hom}_{U(\mathfrak{h})}(\check{X}_0^+, \check{Y}_0^+) \\ &= \text{Hom}_{U(\mathfrak{h})}(X^+, Y^+), \end{aligned}$$

where $+$ designates span of singular vectors. These imply the isomorphisms

$$\begin{aligned} \text{Hom}_{U_q(\mathfrak{g})}(M_1, M_2) &\simeq \text{Hom}_{U_q(\mathfrak{l})}(\check{X}_\lambda, \check{Y}_\lambda) \simeq \text{Hom}_{U(\mathfrak{l})}(\check{X}_0, \check{Y}_0) \\ &\simeq \text{Hom}_{U(\mathfrak{k})}(X, Y) \end{aligned}$$

due to semi-simplicity. Remark that equivalence of $\text{Fin}(\mathfrak{l})$ and $\text{Fin}_q(\mathfrak{l})$ as Abelian categories follows from twist equivalence between $U_q(\mathfrak{l})$ and $U(\mathfrak{l})$ equipped with the standard quasi-Hopf structure, cf. [9].

Finally, every module from $\text{Fin}(\mathfrak{k})$ is isomorphic to one constructed this way because it can be realized as a submodule of a module from $\text{Fin}(\mathfrak{g})$. \square

In the next section we consider two other $\text{Fin}_q(\mathfrak{g})$ -module categories: equivariant projective modules over $\mathbb{C}_q[\mathbb{S}^{2n}]$ and representations of a coideal subalgebra in $U_q(\mathfrak{g})$. These categories will be shown equivalent to $\mathcal{O}(\mathbb{S}_q^{2n})$.

4. Equivariant vector bundles over quantum spheres

In some respects, this section is analogous to the corresponding section of [26], so we refrain from giving detailed proofs where they repeat similar arguments of the proofs

therein. For the sake of compatibility with [24], we change the comultiplication in $U_q(\mathfrak{g})$ to

$$\begin{aligned}\Delta(f_\alpha) &= f_\alpha \otimes q^{-h_\alpha} + 1 \otimes f_\alpha, & \Delta(e_\alpha) &= e_\alpha \otimes 1 + q^{h_\alpha} \otimes e_\alpha, \\ \Delta(q^{h_\alpha}) &= q^{h_\alpha} \otimes q^{h_\alpha},\end{aligned}$$

for all $\alpha \in \Pi^+$. Since the two coproducts are conjugated via an R -matrix, this modification does not affect the conclusions of the previous sections.

In the deformation context, we pass to the $\mathbb{C}[[\hbar]]$ -extension of $U_q(\mathfrak{g})$ completed in the \hbar -adic topology, which we denote by $U_\hbar(\mathfrak{g})$. Note that the $\mathbb{C}[[\hbar]]$ -extension of the base module does not support the action of $U_\hbar(\mathfrak{g})$. Nevertheless, the $U_\hbar(\mathfrak{g})$ -action on $\text{End}(M)$ is well defined.

Given a \mathbb{C} -algebra A we call a $\mathbb{C}[[\hbar]]$ -algebra A_\hbar quantization of A if it is a free $\mathbb{C}[[\hbar]]$ -module and $A_\hbar/\hbar A_\hbar$ is isomorphic to A as a \mathbb{C} -algebra. The quantization is called equivariant if A_\hbar is a $U_\hbar(\mathfrak{g})$ -module algebra such that the $U_\hbar(\mathfrak{g})$ -action is a deformation of a $U(\mathfrak{g})$ -action on A . If a $U_\hbar(\mathfrak{g})$ -algebra A_\hbar is a $\mathbb{C}[[\hbar]]$ -extension of a $U_q(\mathfrak{g})$ -algebra A_q , then A_q is also called (equivariant) quantization of A . This should not cause a confusion as the ring of scalars is always clear from the context.

Suppose that Γ is a projective A -module with an $U(\mathfrak{g})$ -action such that the multiplication $A \otimes \Gamma \rightarrow \Gamma$ is equivariant. Suppose that A_\hbar is an equivariant quantization of A . We call a projective left A_\hbar -module Γ_\hbar quantization of A if Γ_\hbar is also a $U_\hbar(\mathfrak{g})$ -module such that the multiplication $A_\hbar \otimes \Gamma_\hbar \rightarrow \Gamma_\hbar$ is equivariant and the action of $U_\hbar(\mathfrak{g})$ is a deformation of the action of $U(\mathfrak{g})$. Equally we consider the version of right A_\hbar -modules Γ_\hbar . Our convention about indexing with q instead of \hbar applies to A_q -modules as well.

By the Serre–Swan theorem [29, 32], finitely generated projective A -modules are sections of vector bundles in the classical algebraic geometry. Such modules over A_\hbar can be regarded as quantized vector bundles.

4.1. Projective modules over $\mathbb{C}_q[\mathbb{S}^{2n}]$. By a classical equivariant vector bundle over \mathbb{S}^{2n} with fiber X we understand the (left or right) finitely generated projective $\mathbb{C}[\mathbb{S}^{2n}]$ -module $\Gamma(\mathbb{S}^{2n}, X)$ of its global sections. It can be realized as the subspace of \mathfrak{k} -invariants in $\mathbb{C}[G] \otimes X$, where G is either $SO(2n + 1)$ or its simply connected covering if X is a spin representation of \mathfrak{k} .

The quantum polynomial algebra $\mathbb{C}_q[\mathbb{S}^{2n}]$ is represented as a subalgebra $\mathcal{A} \subset \text{End}(M)$, cf. [24]. In accordance with the above definitions, quantization of an equivariant vector bundle on \mathbb{S}^{2n} is a $U_q(\mathfrak{g})$ -equivariant deformation of $\Gamma(\mathbb{S}^{2n}, X)$ in the class of right $\mathbb{C}_q[\mathbb{S}^{2n}]$ -modules. It is realized as $\hat{P}(V \otimes \mathcal{A})$, where $\hat{P} \in \text{End}(V) \otimes \mathcal{A}$ is a $U_q(\mathfrak{g})$ -invariant idempotent. Such idempotents can be constructed via decomposition of $V \otimes M$ due to the following.

Proposition 4.1. *Every invariant map $V \otimes M \rightarrow W \otimes M$ belongs to $\text{Hom}_{\mathbb{C}}(V, W) \otimes \mathcal{A}$.*

One can prove it using the fact that \mathcal{A} exhausts all of the locally finite part of the $U_q(\mathfrak{g})$ -module $\text{End}(M)$, which can be checked with the help of Corollary B.2. This is the answer to a Kostant's problem for quantum groups, [18].

For each non-zero weight vector $v \in V_M^+$ there is a projector $\hat{P} \in \text{End}(V) \otimes \mathcal{A}$ to a particular copy of $M'_{X,\lambda}$ generated by the singular vector $\delta_l(v) \in V \otimes M$.

Theorem 4.2. *The right \mathcal{A} -module $\hat{P}(V \otimes \mathcal{A})$ is a quantization of $\Gamma(\mathbb{S}^{2n}, X)$.*

Proof. The $U_q(\mathfrak{g})$ -module $\hat{P}(V \otimes \mathcal{A})$ is isomorphic to the locally finite part of $\text{Hom}_{\mathbb{C}}(M, M'_{X,\lambda})$. We will prove that it has a similar $U_q(\mathfrak{g})$ -module structure as the $U(\mathfrak{g})$ -module $\Gamma(\mathbb{S}^{2n}, X)$.

First of all, observe that the preceding considerations can be conducted in the opposite category \mathcal{O} of modules whose weights are bounded from below, leading to similar conclusions with regard to semi-simplicity *etc.* Let ξ be the highest weight of X and denote by $N_{X,\lambda}$ the module of lowest weight $-\xi - \lambda$ that is opposite to $M_{X,\lambda}$. The annihilator of its lowest vector is obtained from $I_{M_{X,\lambda}}^-$ by applying the involution σ . Let $N'_{X,\lambda} \simeq (M'_{X,\lambda})^*$ stand for its irreducible image in $V^* \otimes N$.

Since the modules $M'_{X,\lambda}$ and M are irreducible, the vector space

$$\text{Hom}(W, \text{Hom}_{\mathbb{C}}(M, M'_{X,\lambda}))$$

is isomorphic to

$$\text{Hom}(N'_{X,\lambda} \otimes M, W^*)$$

for any finite-dimensional module W . Using Corollary B.2, one can show that

$$\text{Hom}(N'_{X,\lambda} \otimes M, W^*) \subset \text{Hom}(N_{X,\lambda} \otimes M, W^*)$$

as vector subspaces in W^* . The latter is isomorphic to

$$\text{Hom}_{\mathfrak{k}}(X^*, W^*) \simeq \text{Hom}_{\mathfrak{k}}(W, X),$$

by a version of Theorem 3.7 for modules of lowest weight. Include X in a \mathfrak{k} -irreducible decomposition $V = \oplus_i X_i$ and write

$$\oplus_i \text{Hom}(W, \text{Hom}_{\mathbb{C}}(M, M'_{X_i,\lambda})) \simeq \text{Hom}(W, \text{Hom}_{\mathbb{C}}(M, V \otimes M)).$$

The latter is isomorphic to

$$\text{Hom}_{\mathfrak{k}}(W, V) \simeq \oplus_i \text{Hom}_{\mathfrak{k}}(W, X_i).$$

This way we arrive at isomorphisms

$$\text{Hom}(W, \text{Hom}_{\mathbb{C}}(M, M'_{X_i,\lambda})) \simeq \text{Hom}_{\mathfrak{k}}(W, X_i)$$

for all X_i by dimensional reasons. □

Denote by $\text{Proj}(\mathcal{A}, \mathfrak{g})$ the additive category of equivariant finitely generated projective right \mathcal{A} -modules. The morphisms in $\text{Proj}(\mathcal{A}, \mathfrak{g})$ commute with the left action of $U_q(\mathfrak{g})$ and the right action of \mathcal{A} . It is a module category over $\text{Fin}_q(\mathfrak{g})$, since tensor product with a module from $\text{Fin}_q(\mathfrak{g})$ preserves direct sums.

Proposition 4.3. *The two $\text{Fin}_q(\mathfrak{g})$ -module categories $\text{Proj}(\mathcal{A}, \mathfrak{g})$ and $\mathcal{O}(\mathbb{S}_q^{2n})$ are equivalent.*

Proof. The functor $\mathcal{O}(\mathbb{S}_q^{2n}) \rightarrow \text{Proj}(\mathcal{A}, \mathfrak{g})$ is facilitated by Proposition 4.1: for a submodule $M_{X,\lambda} \subset V \otimes M_\lambda$ and an invariant projector $\hat{P} \in V \otimes M \rightarrow M_{X,\lambda}$, the \mathcal{A} -module $\hat{P}(V \otimes \mathcal{A})$ depends only on the image of \hat{P} . Clearly every object from $\text{Proj}(\mathcal{A}, \mathfrak{g})$ is isomorphic to one obtained this way. All $U_q(\mathfrak{g})$ -module homomorphisms intertwine projectors and give rise to morphisms in $\text{Proj}(\mathcal{A}, \mathfrak{g})$. Conversely, the functor $\text{Proj}(\mathcal{A}, \mathfrak{g}) \rightarrow \mathcal{O}(\mathbb{S}_q^{2n})$ is implemented by the assignment

$$\Gamma = \hat{P}(V \otimes \mathcal{A}) \mapsto \Gamma \otimes_{\mathcal{A}} M = \hat{P}(V \otimes M).$$

It is straightforward to see that the functor $\mathcal{O}(\mathbb{S}_q^{2n}) \rightarrow \text{Proj}(\mathcal{A}, \mathfrak{g})$ is bijective on morphisms. It also preserves left tensor multiplication by modules from $\text{Fin}_q(\mathfrak{g})$. \square

Remark 4.4. *Let us comment on the star product on \mathbb{S}^{2n} constructed in [23]. It is utilizing the isomorphisms*

$$\text{Hom}(M, V \otimes M) \simeq V_M^+[0]$$

established in [23] by different methods. The collection of these isomorphisms for all irreducible V describes the module structure of the trivial bundle \mathcal{A} as the locally finite part of $\text{End}(M)$. This fact becomes a special case of Theorem 3.7 implying

$$\text{Hom}(M_{X,\lambda}, V \otimes M) \simeq V_M^+[\xi]$$

with ξ the highest weight of a \mathfrak{k} -submodule $X \subset V$ in the classical limit.

In conclusion of this subsection, we compute the equivariant group K_0 of \mathbb{S}_q^{2n} . Let $R(\mathfrak{g})$ denote the representation ring generated by isomorphism classes of finite-dimensional \mathfrak{g} -modules. It is known to be isomorphic to $\mathbb{Z}(\Lambda)^W$, where Λ is the integral weight lattice, $\mathbb{Z}(\Lambda)$ the group ring, and W the Weyl group [12]. This isomorphism is implemented by the character ch . The ring $R(\mathfrak{k})$ has an additional structure of (left) $R(\mathfrak{g})$ -module via the homomorphism $R(\mathfrak{g}) \rightarrow R(\mathfrak{k})$ induced by the restriction functor $\mathfrak{g} \downarrow \mathfrak{k}$.

The representation ring of finite-dimensional $U_q(\mathfrak{g})$ -modules of quasi-classical type is isomorphic to the classical representation ring $R(\mathfrak{g})$. Let $K_0(\mathcal{A}, \mathfrak{g})$ denote the Grothendieck group of the category $\text{Proj}(\mathcal{A}, \mathfrak{g})$.

Theorem 4.5. *The group $K_0(\mathcal{A}, \mathfrak{g})$ is a left $R(\mathfrak{g})$ -module isomorphic to $R(\mathfrak{k})$.*

Proof. The isomorphism of Abelian groups follows from the equivalence of categories

$$\text{Proj}(\mathcal{A}, \mathfrak{g}) \sim \mathcal{O}(\mathbb{S}_q^{2n}) \sim \text{Fin}(\mathfrak{k}),$$

established by Propositions 3.8 and 4.3. The isomorphism of $R(\mathfrak{g})$ -modules is a consequence of the isomorphisms

$$\text{Hom}_{\mathfrak{k}}(X, V) \simeq \text{Hom}_{U_q(\mathfrak{g})}(M_{X,\lambda}, V \otimes M)$$

for all \mathfrak{k} -modules X and all $U_q(\mathfrak{g})$ -modules V , by Theorem 3.7 (1). □

In terms of formal characters, $K_0(\Gamma_X) = \text{ch}(X)\text{ch}(M)$, by Theorem 3.7 (2).

4.2. Coideal stabilizer subalgebra. As for projective spaces in [26], we give an alternative realization of quantum vector bundles over spheres in terms of quantum symmetric pairs. Here we pass to the left \mathcal{A} -module version of vector bundles.

Let \mathcal{T} denote the Hopf dual of $U_q(\mathfrak{g})$ that is a quantization of the function algebra on the spin group covering $SO(2n + 1)$. It contains the quantum function algebra of the orthogonal group, which is generated by matrix elements T_{ij} , $i, j = 1, \dots, 2n + 1$, of the natural representation in \mathbb{C}^{2n+1} . The matrix T is invertible with $(T^{-1})_{ij} = \gamma(T_{ij})$, where γ stands for the antipode of \mathcal{T} . There are two commuting left and right translation actions of $U_q(\mathfrak{g})$ on \mathcal{T} expressed through the Hopf pairing and the comultiplication in \mathcal{T} by

$$h \triangleright a = a^{(1)}(h, a^{(2)}), \quad a \triangleleft h = (a^{(1)}, h)a^{(2)}, \quad a \in \mathcal{T}, h \in U_q(\mathfrak{g}).$$

They are compatible with multiplication on \mathcal{T} .

Let \mathcal{R} be a universal R-matrix of $U_q(\mathfrak{g})$. Fix a representation of $U_q(\mathfrak{g})$ such that the image $R \in \text{End}(\mathbb{C}^{2n+1}) \otimes \text{End}(\mathbb{C}^{2n+1})$ of \mathcal{R} is proportional to the orthogonal R-matrix from [11]. The element $\mathcal{R}_{21}\mathcal{R}$ commutes with the coproduct $\Delta(x)$ for all $x \in U_q(\mathfrak{g})$. Its image \mathcal{Q} in $\text{End}(\mathbb{C}^{2n+1}) \otimes \text{End}(M)$ is a matrix whose entries generate \mathcal{A} . The matrix

$$A = \begin{pmatrix} q^{-2n} - q^{-1} & 0 & \dots & 0 & q^{-n-\frac{1}{2}}c \\ 0 & -q^{-1} & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & -q^{-1} & 0 \\ q^{-n-\frac{1}{2}}c^{-1} & 0 & \dots & 0 & 0 \end{pmatrix} \in \text{End}(\mathbb{C}^{2n+1})$$

with $c \in \mathbb{C} \setminus \{0\}$, solves the Reflection Equation

$$R_{21}A_1R_{12}A_2 = A_2R_{21}A_1R_{12} \in \text{End}(\mathbb{C}^{2n+1}) \otimes \text{End}(\mathbb{C}^{2n+1}),$$

where the indices mark the tensor factors. It also satisfies other equations of the quantum sphere, cf. [24], and defines a one-dimensional representation

$$\chi: \mathcal{A} \rightarrow \mathbb{C}, \quad \mathcal{Q}_{ij} \mapsto A_{ij}.$$

The assignment $\mathcal{Q} \mapsto T^{-1}AT$ extends to an equivariant embedding $\mathcal{A} \subset \mathcal{T}$, where \mathcal{T} is regarded as a $U_q(\mathfrak{g})$ -module under the left translation action. The character χ factors to a composition of this embedding and the counit ϵ . The entries of the matrix

$$\mathcal{R}_{21}A_1\mathcal{R}_{12} \in \text{End}(\mathbb{C}^{2n+1}) \otimes U_q(\mathfrak{g})$$

generate a left coideal subalgebra $\mathcal{B} \subset U_q(\mathfrak{g})$, such that $a \triangleleft b = \epsilon(b)a$ for all $b \in \mathcal{B}$ and $a \in \mathcal{A}$. It is a deformation of $U(\mathfrak{k}')$, where $\mathfrak{k}' \simeq \mathfrak{so}(2n)$.

Let $\hat{P} \in V \otimes \mathcal{A}$ be an invariant idempotent. The projector $P = \hat{P}_1 \chi(\hat{P}_2) \in \text{End}(V)$ commutes with \mathcal{B} (see [26]).

Proposition 4.6. (1) *Every finite-dimensional right $U_q(\mathfrak{g})$ -module V is completely reducible over \mathcal{B} .*

(2) *Every irreducible \mathcal{B} -submodule in V is a deformation of a classical $U(\mathfrak{k}')$ -submodule.*

(3) *Every \mathcal{B} -submodule in V is the image of a \mathcal{B} -invariant projector $(\text{id} \otimes \chi)(\hat{P})$, where $\hat{P} \in \text{End}(V) \otimes \mathcal{A}$ is a $U_q(\mathfrak{g})$ -invariant idempotent.*

The proof is similar to [26]. Note that Proposition 4.6 holds for almost all q for each V .

Proposition 4.6 also can be given a category theoretical interpretation. Denote by $\text{Fin}_q(\mathfrak{k}')$ the additive category of \mathcal{B} -modules whose objects are submodules in some V from $\text{Fin}_q(\mathfrak{g})$ and morphisms are \mathcal{B} -invariant maps. Since it is semi-simple, it is Abelian and it is also a module category over $\text{Fin}_q(\mathfrak{g})$. Given an equivariant projective \mathcal{A} -module Γ , the assignment $\Gamma \mapsto \Gamma \otimes_{\mathcal{A}} \mathbb{C}$ via the character χ is a functor $\text{Proj}(\mathcal{A}, \mathfrak{g}) \rightarrow \text{Fin}_q(\mathfrak{k}')$. This functor is an equivalence of module categories.

4.3. Vector bundles via symmetric pairs. The realization of quantum vector bundles as linear maps between pseudo-parabolic Verma modules has no classical analog. In this section we follow a different approach presenting an associated vector bundle by \mathcal{B} -invariants in the tensor product of \mathcal{T} and a \mathcal{B} -module. This construction is quasi-classical: in the limit $q \rightarrow 1$ we recover the standard construction of equivariant vector bundles as induced modules. It is convenient to pass to left \mathcal{A} -modules, which corresponds to the right coset picture.

It is known that every finite-dimensional right $U_q(\mathfrak{g})$ -module V is a left \mathcal{T} -co-module. We use a Sweedler-like notation for the left coaction

$$V \rightarrow \mathcal{T} \otimes V, \quad v \mapsto v^{(1)} \otimes v^{[2]}.$$

Then

$$v \triangleleft h = (v^{(1)}, h)v^{[2]}$$

for $v \in V$, $h \in U_q(\mathfrak{g})$.

We define a left $U_q(\mathfrak{g})$ -action on V by

$$h \blacktriangleright v = v \triangleleft \gamma(h), \quad h \in U_q(\mathfrak{g}), \quad v \in V,$$

and consider $\mathcal{A} \otimes V$ as a left $U_q(\mathfrak{g})$ -module. The tensor product $\mathcal{T} \otimes V$ is also a left $U_q(\mathfrak{g})$ -module with respect to the left translations on \mathcal{T} and the trivial action on V .

Let $\hat{P} \in \text{End}(V) \otimes \mathcal{A}$ be a $U_q(\mathfrak{g})$ -invariant idempotent and put $P = (\text{id} \otimes \chi)(\hat{P})$ for its \mathcal{B} -invariant image. Denote by $X = PV$ the corresponding \mathcal{B} -submodule in V . The subspaces

$$(\mathcal{A} \otimes V)\hat{P}_{21} \subset \mathcal{T} \otimes V \quad (\text{the diagonal } U_q(\mathfrak{g})\text{-action})$$

and $(\mathcal{T} \otimes X)^{\mathcal{B}} \subset \mathcal{T} \otimes V$ (trivial $U_q(\mathfrak{g})$ -action on V)

are isomorphic as left \mathcal{A} -modules and $U_q(\mathfrak{g})$ -modules, [26].

Theorem 4.7. *The \mathcal{A} -module $(\mathcal{T} \otimes X)^{\mathcal{B}}$ is a quantization of the vector bundle $\Gamma(\mathbb{S}^{2n}; X)$.*

Proof. Similar to [26]. □

By considering vector (integer-spin) modules, one can restrict consideration to \mathcal{T} the function algebra on the quantum orthogonal group.

A. Parabolic Verma modules

It is known that a parabolic Verma module over a classical universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra \mathfrak{g} relative to a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is locally finite over $U(\mathfrak{l})$, cf. [14]. The goal of this auxiliary section is to prove an analogous statement for quantum groups. We are not aware if this topic is covered in the literature and include it for completeness.

We start with a few general definitions. Let A be an associative algebra with unit. Fix an element $a \in A$ and define its quasi-normalizer $\text{QNorm}(a) \subset A$ as the subset of elements $b \in A$ such that $a^{m+k}b = b_k a^k$ for some $m \in \mathbb{Z}_+$, $b_k \in A$ and all positive $k \in \mathbb{Z}_+$. Clearly it is a unital subalgebra in A .

Here is a way of checking if $b \in \text{QNorm}(a)$. Consider a sequence of complex numbers $(q_i)_{i=1}^{\infty}$ and define by induction

$$D_a^0(b) = b, \quad D_a^n(b) = [a, D_a^{n-1}(b)]_{q_n}$$

for $n > 0$.

Proposition A.1. *Suppose that $D_a^{m+1}(b) = 0$ for some $m \geq 0$. Then $b \in \text{QNorm}(a)$.*

Proof. One can prove by induction that

$$a^n b = \sum_{k=0}^n c_{n,k} D_a^k(b) a^{n-k}$$

for all positive integer n , where $c_{n,k}$ are some complex coefficients. It follows that $D_a^l(b) = 0$ for $l > m$, and

$$a^n b = \left(\sum_{k=0}^m c_{n,k} D_a^k(b) a^{m-k} \right) a^{n-m}$$

for all $n > m$, as required. \square

We call an element $a \in A$ quasi-normal if $\text{QNorm}(a) = A$. Their set is stable under the group of automorphism of A . It is straightforward that invertible elements of A are quasi-normal, as well as elements of the center of A . We aim to prove that Lusztig root vectors in $U_q(\mathfrak{g})$ are quasi-normal.

Recall that a total order in \mathbb{R}^+ is called normal if every sum of positive roots is between the summands. Fix such an order and set $\alpha^i \in \mathbb{R}^+$, $i = 1, \dots, N = \#\mathbb{R}^+$. Then there exists a system of “root vectors” $\{e_i\}_{i=1}^N$ of weights α^i such that the monomials $\{e_1^{m_1} \dots e_N^{m_N}\}_{m_i \in \mathbb{Z}_+}$ deliver a PBW basis in $U_q(\mathfrak{g}_+)$. Similarly there are negative root vectors $f_i \in U_q(\mathfrak{g}_-)$ giving rise to a PBW basis of ordered monomials in $U_q(\mathfrak{g}_-)$.

Proposition A.2. *The Lusztig root vectors are quasi-normal.*

Proof. It is sufficient to prove that generators of $U_q(\mathfrak{g})$ belong to $\text{QNorm}(e_i)$ for each e_i . First suppose that $e_i = e_\alpha$ for a simple root α . Then clearly $U_q(\mathfrak{h}) \subset \text{QNorm}(e_\alpha)$. Furthermore,

$$e_\alpha^n f_\beta = (f_\beta e_\alpha + \delta_{\alpha\beta} [n]_q [h_\alpha + n - 1]_q) e_\alpha^{n-1},$$

so $U_q(\mathfrak{g}_-) \subset \text{QNorm}(e_\alpha)$. Finally, the Serre relations can be written as $D_{e_\alpha}^m(e_\beta) = 0$ for certain integer m and a sequence $(q_i)_{i=1}^m$ depending on α and β . Then by Proposition A.1, $U_q(\mathfrak{g}_+) \subset \text{QNorm}(e_\alpha)$, and therefore e_α is quasi-normal. Now observe that compound root vectors are obtained from the Chevalley generators via automorphisms of $U_q(\mathfrak{g})$, [2]. This completes the proof. \square

Let \mathfrak{l} be a Levi subalgebra in \mathfrak{g} with a basis of simple roots $\Pi_{\mathfrak{l}}^+ \subset \Pi_{\mathfrak{g}}^+$. One can choose a normal order so that $\mathbb{R}^+ \setminus \mathbb{R}_{\mathfrak{l}}^+ < \mathbb{R}_{\mathfrak{l}}^+$. Then

$$\text{Span}\{f_1^{m_1} \dots f_{\mathfrak{l}}^{m_{\mathfrak{l}}}\}_{m_i \in \mathbb{Z}_+},$$

where $l = \#(\mathbb{R}^+ \setminus \mathbb{R}_l^+)$, is a subalgebra, cf. [20, Proposition 3.3] (see also [16]). It is a quantization of the $U(\mathfrak{n}_-)$, where \mathfrak{n}_\pm are the nil-radicals of the parabolic Lie algebras $\mathfrak{p}_\pm = \mathfrak{l} + \mathfrak{g}_\pm$. Similarly

$$\text{Span}\{e_l^{m_l} \dots e_1^{m_1}\}_{m_i \in \mathbb{Z}_+}$$

is a subalgebra, $U_q(\mathfrak{n}_+)$. The PBW factorization then yields

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_-)U_q(\mathfrak{p}_+) \quad \text{with } U_q(\mathfrak{p}_+) = U_q(\mathfrak{l})U_q(\mathfrak{n}_+).$$

Now we use an alternative construction of parabolic Verma modules via parabolic induction. Given $\check{X} \in \text{Fin}_q(\mathfrak{l})$ (relax the quasi-classical weight convention) make it a $U_q(\mathfrak{p}_+)$ -module by setting it trivial on $U_q(\mathfrak{n}_+)$. Then $M_{\check{X}} = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_+)} \check{X}$ is said to be parabolically induced from \check{X} . In the special case of irreducible \check{X} , this construction recovers the parabolic Verma module as the quotient of the ordinary Verma module whose highest weight is that of \check{X} . The PBW factorization of $U_q(\mathfrak{g}_-)$ implies that $M_{\check{X}}$ is a free $U_q(\mathfrak{n}_-)$ -module generated by \check{X} .

Recall that an A -module V is called locally finite if for each $v \in V$ the cyclic submodule $Av \subset V$ is finite-dimensional. An element $a \in A$ is called locally nilpotent if $a^n v = 0$ for all $v \in V$ and some positive $n \in \mathbb{Z}_+$ depending on v .

Proposition A.3. *The parabolic module $M_{\check{X}}$ is locally finite over $U_q(\mathfrak{l})$.*

Proof. It is sufficient to prove that $M_{\check{X}}$ is locally finite over $U_q(\mathfrak{l}_-)$ since it is clearly so over $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_+)$. Indeed, every f_α with $\alpha \in \mathbb{R}_l^+$ is locally nilpotent on \check{X} and therefore on $M_{\check{X}} \simeq U_q(\mathfrak{n}_-)\check{X}$ since f_α is quasi-normal. Therefore all but a finite number of PBW-monomials in $U_q(\mathfrak{l}_-)$ vanish on every $v \in M_{\check{X}}$. This implies the statement. \square

It follows that $M_{\check{X}}$ is completely reducible over $U_q(\mathfrak{l})$ and is an infinite direct sum of irreducible finite-dimensional modules with finite multiplicities. As another consequence, $U_q(\mathfrak{n}_-)$ is a locally finite adjoint $U_q(\mathfrak{l})$ -module because it is isomorphic to the parabolic Verma module induced from the trivial representation of $U_q(\mathfrak{p}_+)$.

B. Tensor product of modules of highest and lowest weight

For reader's convenience, we prove some facts that we use in the proof of Proposition 3.6. Recall that a lower Verma module is freely generated over $U_q(\mathfrak{g}_-)$ by its lowest weight vector.

Lemma B.1. *Let \widehat{M}_μ be a Verma module of highest weight μ and \widehat{N}_ν a Verma module of lowest weight μ . Then the tensor product $\widehat{M}_\mu \otimes \widehat{N}_\nu$ is isomorphic to the $U_q(\mathfrak{g})$ -module $\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_{\mu+\nu})$ induced from the $U_q(\mathfrak{h})$ -module $\mathbb{C}_{\mu+\nu}$.*

Proof. Let us prove that the homomorphism

$$\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_{\mu+\nu}) \rightarrow \widehat{M}_{\mu} \otimes \widehat{N}_{\nu}$$

determined by the assignment

$$d: x \mapsto \Delta(x)(1_{\mu} \otimes 1_{\nu}), \quad x \in U_q(\mathfrak{g}),$$

is an isomorphism. Introduce a \mathbb{Z} -grading on $U_q(\mathfrak{g}_{\pm})$ by assigning degree 1 to the Chevalley generators and denote by U_{\pm}^k their homogeneous component of degree k . This makes the vector spaces

$$\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_{\mu+\nu}) = U_q(\mathfrak{g}_{-})U_q(\mathfrak{g}_{+})1_{\mu+\nu}$$

and $\widehat{M}_{\mu} \otimes \widehat{N}_{\nu}$ graded by the semi-group \mathbb{Z}_{+}^2 . The map d is an isomorphism on the components $(0, j)$ for all $j \in \mathbb{Z}_{+}$. Since $f_{\alpha}, \alpha \in \Pi_{\alpha}^{+}$, are quasi-primitive, the map d sends $U_{-}^i U_{+}^j 1_{\mu+\nu}$ onto $U_{-}^i 1_{\mu} \otimes U_{+}^j 1_{\nu}$ modulo $\sum_{l < i, k \leq j} U_{-}^l 1_{\mu} \otimes U_{+}^k 1_{\nu}$. Induction on i proves that d is surjective and therefore injective on $\sum_{l \leq i, k \leq j} U_{-}^l U_{+}^k 1_{\mu+\nu}$ for all j , as the graded components are finite-dimensional. \square

It follows from Lemma B.1 that

$$\text{Hom}(\widehat{M}_{\mu} \otimes \widehat{N}_{\nu}, V) \simeq V[\mu + \nu]$$

for any V . Suppose $\{f_i\} \subset U_q(\mathfrak{g}_{-})$ are such that $f_i 1_{\mu}$ are singular vectors in \widehat{M}_{μ} and let M be the quotient of \widehat{M}_{μ} by the sum of submodules they generate. Similarly let $\{e_j\} \subset U_q(\mathfrak{g}_{+})$ be such that $e_j 1_{\nu}$ are killed by all Chevalley generators f_{α} and let N denote the corresponding quotient of \widehat{N}_{ν} .

Corollary B.2. *For any finite-dimensional module V , the vector space*

$$\text{Hom}(M \otimes N, V)$$

is isomorphic to

$$(\cap_{i,j} \ker_V(f_i) \cap \ker_V(e_j))[\mu + \nu].$$

Proof. The module $M \otimes N$ is a quotient of $\widehat{M}_{\mu} \otimes \widehat{N}_{\nu}$ by the sum of $U_q(\mathfrak{g}_{-})f_i 1_{\mu} \otimes \widehat{N}_{\nu}$ and $\widehat{M}_{\mu} \otimes U_q(\mathfrak{g}_{+})e_j 1_{\nu}$, which disappear in a homomorphism to V . By Lemma B.1, all these submodules are induced from one-dimensional $U_q(\mathfrak{h})$ -modules. Therefore $\text{Hom}(M \otimes N, V)$ is in bijection with vectors in $V[\mu + \nu]$ that are killed by all f_i and e_j . \square

Acknowledgements. This study is supported by the RFBR grant 15-01-03148.

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Received 07 December, 2018

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