Cup product on A_∞-cohomology and deformations

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Abstract. We propose a simple method for constructing formal deformations of differential graded algebras in the category of minimal A_∞-algebras. The basis for our approach is provided by the Gerstenhaber algebra structure on A_∞-cohomology, which we define in terms of the brace operations. As an example, we construct a minimal A_∞-algebra from the Weyl-Moyal $*$-product algebra of polynomial functions.

1. Introduction

The concept of homotopy associative algebras (or A_∞-algebras), which first appeared in the context of algebraic topology [21], has now evolved into a mature algebraic theory with numerous applications in theoretical and mathematical physics [13, 22]. String field theory [2, 12], the deformation quantization of gauge systems [15], non-commutative field theory [1], and higher-spin gravity [14, 18, 19] are just a few examples where these algebras play a dominant role. It turns out that many of A_∞-algebras encountered in applications are obtained by deforming differential graded algebras (DGA) or their families. The general deformation problem for A_∞-algebras has been considered in [3, 11, 16, 20].

In this paper, we propose a simple formula for the deformation of families of DGA's in the category of minimal A_∞-algebras. The basis for our construction is provided by a cup product on A_∞-cohomology. As was first shown by Getzler [7], each A_∞-structure $m \in \mathrm{Hom}(T(V), V)$ on a graded vector space V gives rise to an A_∞-structure M on the vector space $\mathrm{Hom}(T(V), V)$. As with any A_∞-algebra, the second structure map M_2 induces a multiplication operation, called cup product, in the A_∞-cohomology defined by the differential M_1 and we use this operation to deform the original family of A_∞-structures m.

The main results of our paper can be summarized in the following:

Theorem 1.1. *Given a one-parameter family* $A = \bigoplus A_n$ *of DGA's with differential* $\partial\colon A_n \to A_{n-1}$, *one can define a minimal* A_∞-*algebra deforming the associative product in* A *in the direction of an (inhomogeneous) Hochschild cocycle* Δ *given by any linear combination of*

$$\Delta_n(a_1, a_2, \ldots, a_n) = (a_1 \cdot a_2)' \cdot \partial a_3 \cdots \partial a_n, \quad \forall a_i \in A.$$

Here $[\Delta_n] \in HH^n(A, A)$ *and the prime stands for the derivative of the dot product in* A *w.r.t. the parameter. Each solution to the equation* $a \cdot a = 0$ *for* $a \in A_1$ *can be deformed to a Maurer–Cartan element of the* A_∞-*algebra above.*

The theorem above admits various interesting specializations, of which we mention only one. Let $\mathcal{M}$ be a one-parameter family of bimodules over $\mathcal{A}$. Then one can define the family of graded algebras $A = A_0 \bigoplus A_1$, where $A_0 = \mathcal{A}$, $A_1 = \mathcal{M}$, and the product is given by

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2), \quad \forall a_1, a_2 \in \mathcal{A}, \ \forall m_1, m_2 \in \mathcal{M}. \quad (1.1)$$

This is known as the trivial extension of the algebra $\mathcal{A}$ by the bimodule $\mathcal{M}$. In order to endow the algebra A with a differential ∂, we consider the $\mathcal{A}$-dual bimodule

$$\mathcal{M}^* = \mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{M}, \mathcal{A}).$$

Each element $h \in \mathcal{M}^*$ extends to an $\mathcal{A}$-bimodule homomorphism $\widetilde{h}\colon A \to \mathcal{A}$ by setting $\widetilde{h}(a) = 0$, $\forall a \in \mathcal{A}$. In case $\ker h = 0$, one can easily see that $\widetilde{h}$ is a derivation of the algebra A of degree -1. Furthermore, it follows from the definition that $\widetilde{h}^2 = 0$. Hence, we can put $\partial = \widetilde{h}$. The deformation of the algebra $A = A_0 \bigoplus A_1$ stated by Theorem 1.1 yields then a deformation of the $\mathcal{A}$-bimodule $\mathcal{M}$ in the category of minimal A_∞-algebras.

Notice that in the above construction $h(\mathcal{M})$ is a two-sided ideal in $\mathcal{A}$. Conversely, given a two-sided ideal $\mathcal{J} \subset \mathcal{A}$, we can set $\mathcal{M} = \mathcal{J}$ and take h to be the inclusion map $\mathcal{M} \hookrightarrow \mathcal{A}$. This allows one to canonically associate an A_∞-algebra to any pair $(\mathcal{J}, \mathcal{A})$. In the particular case $\mathcal{J} = \mathcal{A}$, we get a deformation of the family $\mathcal{A}$ itself. For this reason it is natural to term these and the other deformations following from Theorem 1.1 the *inner deformations of families*.

The rest of the paper is organized as follows. In Section 2, we review some background material on A_∞-algebras and braces. In Section 3, we define the cohomology groups associated to an A_∞-structure and endow them with a commutative and associative cup product. This product operation is then used in Section 4 for constructing inner deformations of multi-parameter families of A_∞-algebras and, in particular, DGA's. Here we also introduce the concept of local finiteness for families and show that each inner deformation of a locally finite family of A_∞-algebras induces a deformation of the corresponding Maurer–Cartan elements. By way of

illustration, we finally construct a minimal A_∞-algebra that deforms the algebra of polynomial functions regarded as a bimodule over itself. The deformation is completely determined by a canonical Poisson bracket and can be viewed as a certain generalization of the Weyl–Moyal $*$-product.

2. A_∞-algebras and braces

Throughout the paper we work over a fixed ground field k of characteristic zero. All tensor products and Hom's are defined over k unless otherwise indicated. We begin by recalling some basic definitions and constructions related to A_∞-algebras.

Let $V = \bigoplus V^l$ be a $\mathbb{Z}$-graded vector space over k and let $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ denote its tensor algebra; it is understood that $T^0(V) = k$. The k-vector spaces $T(V)$ and $\mathrm{Hom}(T(V), V)$ naturally inherit the grading of V. The vector space

$$\mathrm{Hom}(T(V), V) = \bigoplus_l \mathrm{Hom}^l(T(V), V) \tag{2.1}$$

is known to carry the structure of a graded Lie algebra. This is defined as follows. For any two homogeneous homomorphisms $f \in \mathrm{Hom}(T^n(V), V)$ and $g \in \mathrm{Hom}(T^m(V), V)$, one first defines a (non-associative) *composition* product [5] as

$$(f \circ g)(v_1 \otimes v_2 \otimes \cdots \otimes v_{m+n-1}) = \sum_{i=0}^{n-1} (-1)^{|g| \sum_{j=1}^{i} |v_j|}$$
$$\cdot f(v_1 \otimes \cdots \otimes v_i \otimes g(v_{i+1} \otimes \cdots \otimes v_{i+m}) \otimes \cdots \otimes v_{m+n-1}). \tag{2.2}$$

Here $|g|$ denotes the degree of g as a linear map of graded vector spaces[1]. Then the graded Lie bracket on (2.1) is given by the *Gerstenhaber bracket* [4]

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f. \tag{2.3}$$

One can see that the Gerstenhaber bracket is graded skew-symmetric,

$$[f, g] = -(-1)^{|f||g|} [g, f],$$

and obeys the graded Jacobi identity

$$[[f, g], h] = [f, [g, h]] - (-1)^{|f||g|} [g, [f, h]].$$

In particular, $[f, f] = 2f \circ f$ for any odd f.

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[1] We define the degree of multi-linear maps as in [12]. A more conventional $\mathbb{Z}$-grading [4, 23] on $\mathrm{Hom}(T(V), V)$ is related to ours by *suspension*: $V \to V[-1]$, where $V[-1]^l = V^{l-1}$.

Definition 2.1. An A_∞-structure on a $\mathbb{Z}$-graded vector space V is given by an element $m \in \mathrm{Hom}^1(T(V), V)$ obeying the Maurer–Cartan (MC) equation

$$m \circ m = 0. \qquad (2.4)$$

The pair (V, m) is called the A_∞-algebra.

By definition, each A_∞-structure m is given by an (infinite) sum

$$m = m_0 + m_1 + m_2 + \cdots$$

of multi-linear maps $m_n \in \mathrm{Hom}(T^n(V), V)$. Expanding (2.4) into homogeneous components yields an infinite collection of quadratic relations on the m_n's, which are known as the Stasheff identities [21]. An A_∞-algebra is called *flat* if $m_0 = 0$. For flat algebras, the first structure map

$$m_1 : V^l \to V^{l+1}$$

squares to zero, $m_1 \circ m_1 = m_1^2 = 0$; hence, it makes V into a cochain complex. An A_∞-algebra is called *minimal* if $m_0 = m_1 = 0$. In the minimal case, the second structure map

$$m_2 : V \otimes V \to V$$

endows the space $V[-1]$ with the structure of a graded associative algebra w.r.t. the dot product

$$u \cdot v = (-1)^{|u|-1} m_2(u \otimes v), \qquad (2.5)$$

associativity being provided by the Stasheff identity $m_2 \circ m_2 = 0$. This allows one to regard a graded associative algebra as a 'very degenerate' A_∞-algebra with $m = m_2$. More generally, an A_∞-structure $m = m_1 + m_2$ gives rise to a DGA $(V[-1], d, \cdot)$ with the product (2.5) and the differential $d = m_1$. The graded Leibniz rule

$$d(u \cdot v) = du \cdot v + (-1)^{|u|-1} u \cdot dv$$

follows from the Stasheff identity $[m_1, m_2] = 0$.

The composition product (2.2) is a representative of the infinite sequence of multi-linear operations on $\mathrm{Hom}(T(V), V)$ known as *braces*. The braces first appeared in the work of Kadeishvili [10] and were then studied by several authors [6–8]. To simplify subsequent formulas, let us denote $W = \mathrm{Hom}(T(V), V)$.

Definition 2.2. Given homogeneous elements $A, A_1, \ldots, A_m \in W$ and $v_1, \ldots, v_n \in V$, define the braces $A\{A_1, \ldots, A_m\} \in W$, $m = 0, 1, 2, \ldots$, by the formula

$$A\{A_1, \ldots, A_m\}(v_1, \ldots, v_n) = \sum_{0 \le k_1 \le \cdots \le k_m \le n} (-1)^\epsilon A\big(v_1, \ldots, v_{k_1}, A_1(v_{k_1+1}, \ldots), \ldots$$

$$\ldots, v_{k_m}, A_m(v_{k_m+1}, \ldots), \ldots, v_n\big), \quad (2.6)$$

where $\epsilon = \sum_{i=1}^m |A_i| \sum_{j=1}^{k_i} |v_j|$. It is assumed that $A\{\varnothing\} = A$.

It follows from the definition that

$$A\{A_1\} = A \circ A_1 \,. \tag{2.7}$$

The braces obey the so-called higher pre-Jacobi identities [6]

$$A\{A_1, \ldots, A_m\}\{B_1, \ldots, B_n\} = \sum_{AB\text{-shuffles}} (-1)^\epsilon A\{B_1, \ldots, B_{k_1}, A_1\{B_{k_1+1}, \ldots\}, \ldots$$

$$\ldots, B_{k_m}, A_m\{B_{k_m+1}, \ldots\}, \ldots, B_n\}, \quad (2.8)$$

where $\epsilon = \sum_{i=1}^m |A_i| \sum_{j=1}^{k_i} |B_j|$. Here summation is over all shuffles of the A's and B's (i.e., the order of elements in either group is preserved under permutations) and the case of empty braces $A_k\{\varnothing\}$ is not excluded.

In [7], Getzler have shown that any A_∞-structure m on V can be lifted to a flat A_∞-structure M on W by setting

$$M_0(\varnothing) = 0\,,$$
$$M_1(A) = m \circ A - (-1)^{|A|} A \circ m\,, \tag{2.9}$$
$$M_k(A_1, \ldots, A_k) = m\{A_1, \ldots, A_k\}\,, \qquad k > 1\,.$$

Indeed, by the definition of the composition product (2.2)

$$(M \circ M)(A_1, \ldots, A_n)$$
$$= \sum_{0 \le i \le j \le n} (-1)^\varepsilon M(A_1, \ldots, A_{i-1}, M(A_i, \ldots, A_j), A_{j+1}, \ldots, A_n)\,, \quad (2.10)$$

where $\varepsilon = \sum_{j=1}^{i-1} |A_j|$. This gives

$$(M \circ M)(A_1, \ldots, A_n)$$
$$= \sum_{0 \le i \le j \le n} (-1)^\varepsilon m\{A_1, \ldots, A_{i-1}, m\{A_i, \ldots, A_j\}, A_{j+1}, \ldots, A_n\}$$
$$\sum_{1 \le i \le n} (-1)^\varepsilon m\{A_1, \ldots, A_{i-1}, A_i \circ m, A_{i+1}, \ldots, A_n\}$$
$$+ (-1)^{\sum_{i=1}^n |A_i|} m\{A_1, \ldots, A_n\} \circ m\,. \quad (2.11)$$

Using the pre-Jacobi identities (2.8), one can rewrite the last term as

$$(-1)^{\sum_{i=1}^n |A_i|} m\{A_1, \ldots, A_n\} \circ m$$
$$= \sum_{1 \le i \le n} (-1)^\varepsilon \big(m\{A_1, \ldots, A_{i-1}, A_i \circ m, A_{i+1}, \ldots, A_n\}$$
$$+ m\{A_1, \ldots, A_i, m, A_{i+1}, \ldots, A_n\} \big)\,.$$

Then the r.h.s. of Eq. (2.11) takes the form of

$$(m \circ m)\{A_1, \ldots, A_n\} = m\{m\}\{A_1, \ldots, A_n\}$$
$$= \sum_{m A\text{-shuffles}} (-1)^{\varepsilon} m\{A_1, \ldots, A_{i-1}, m\{A_i, \ldots, A_j\}, A_{j+1}, \ldots, A_n\}.$$

Hence,

$$(M \circ M)(A_1, \ldots, A_n) = (m \circ m)\{A_1, \ldots, A_n\} = 0.$$

In what follows we will refer to (2.9) as the *derived* A_∞-structure.[2]

Remark 2.3. There is also another way to define an A_∞-structure on W by that on V. The corresponding structure maps are given by

$$M_k(A_1, \ldots, A_k) = m\{A_1, \ldots, A_k\} \quad k \geq 0,$$

see [9, Thm. 7] or [23, Prop. 2.5]. Contrary to Getzler's construction (2.9) this A_∞-structure is not flat.

3. A_∞-cohomology

If (V, m) is an A_∞-algebra, then the first map of the derived A_∞-structure (2.9) makes the graded vector space $W = \bigoplus W^n$ into a cochain complex w.r.t. the differential

$$M_1 \colon W^n \to W^{n+1}.$$

Let $H^n(W)$ denote the corresponding cohomology groups.[3] Following [16], we refer to them as A_∞-cohomology groups. The most interesting for us are the groups $H^1(W)$ and $H^2(W)$, which control the formal deformations of the underlying A_∞-structure m. Let us give some relevant definitions.

When dealing with formal deformations of algebras, one first extends the ground field k to the algebra $k[\![t]\!]$, with the formal variable t playing the role of a *deformation parameter*. Since $k[\![t]\!]$ is commutative, the graded Lie algebra structure on W extends naturally to $W \otimes k[\![t]\!]$ and then to its completion $\mathcal{W} = W \,\widehat{\otimes}\, k[\![t]\!]$ w.r.t. the t-adic topology. By definition, the elements of $\mathcal{W}$ are given by the formal power series

$$m_t = m^{(0)} + m^{(1)} t + m^{(2)} t^2 + \cdots, \quad m^{(i)} \in W. \tag{3.1}$$

The natural augmentation $\varepsilon \colon k[\![t]\!] \to k$ induces the k-homomorphism $\pi \colon \mathcal{W} \to W$, which sends the deformation parameter to zero. We say that an MC element $m_t \in \mathcal{W}^1$ is a deformation of $m \in W^1$ if $\pi(m_t) = m$ or, what is the same, $m_{(0)} = m$ in (3.1).

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[2]Do not confuse with the derived A_∞-algebras in the sense of Sagave [17].

[3]For an associative algebra A, this cohomology is simply the Hochschild cohomology of the algebra. In that case one normally uses a more standard notation $HH^{n+1}(A, A)$ for the groups $H^n(W)$.

Extending now the homomorphism m_t from V to $\mathcal{V} = V \otimes k[\![t]\!]$ by $k[\![t]\!]$-linearity and t-adic continuity, we get an A_∞-algebra $(\mathcal{V}, m_t)$ which is referred to as the deformation of the algebra (V, m). The element $m_{(1)}$ in (3.1) is called the *first-order deformation* of m.

Two formal deformations m_t and $\widetilde{m}_t$ of one and the same A_∞-structure m are considered as equivalent if there exists an element $w \in \mathcal{W}^0$ such that

$$\widetilde{m}_t = e^{tw} m_t e^{-tw} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathrm{ad}_w)^n (m_t) \,.$$

This induces an equivalence relation on the space of first-order deformations and it is the standard fact of algebraic deformation theory (see e.g. [16]) that the space of nonequivalent first-order deformations is isomorphic to $H^1(W)$. If in addition $H^2(W) = 0$, then each first-order deformation extends to all orders.

Since the differential M_1 is, by definition, an inner derivation of the graded Lie algebra W, the Gerstenhaber bracket (2.3) induces a Lie bracket on the cohomology space $H^\bullet(W)$, for which we use the same bracket notation. The graded Lie algebra structure on $H^\bullet(W)$ can further be extended to the structure of a graded Poisson (or Gerstenhaber) algebra w.r.t. a cup product. The latter is defined as follows.

By definition, the second structure map

$$M_2 \colon W \otimes W \to W$$

of the derived A_∞-algebra obeys the identity

$$M_1(M_2(A, B)) + M_2(M_1(A), B) + (-1)^{|A|} M_2(A, M_1(B)) = 0 \qquad (3.2)$$

for all $A, B \in W$. From this relation we conclude that (i) $M_2(A, B) \in W$ is an M_1-cocycle whenever A and B are so, and (ii) the cocycle $M_2(A, B)$ is trivial whenever one of the cocycles A and B is an M_1-coboundary. To put this another way, the map M_2 descends to the cohomology inducing a homomorphism

$$M_2^* \colon H^n(W) \otimes H^m(W) \to H^{n+m+1}(W) \,.$$

We can interpret this homomorphism as a multiplication operation making the suspended vector space $H^{\bullet-1}(W)$ into a $\mathbb{Z}$-graded algebra. More precisely, we set

$$a \cup b = (-1)^{|A|-1} M_2(A, B) \,, \qquad (3.3)$$

where $A, B \in W$ are cocycles representing the cohomology classes $a, b \in H^{\bullet-1}(W)$. The properties of the cup product are described by the following proposition.

Proposition 3.1. *The cup product* (3.3) *endows the space* $H^{\bullet-1}(W)$ *with the structure of an associative and graded commutative algebra.*

Proof. Associativity follows immediately from the Stasheff identity

$$M_2 \circ M_2 = -[M_1, M_3] \,.$$

The r.h.s. obviously vanishes when evaluated on M_1-cocycles modulo coboundaries, while the l.h.s. takes the form of the associativity condition

$$(a \cup b) \cup c - a \cup (b \cup c) = 0 \,.$$

The proof of graded commutativity is a bit more cumbersome. Consider the cochain

$$D(A, B) = M_1(A \circ B) - M_1(A) \circ B - (-1)^{|A|} A \circ M_1(B) \,,$$

which measures the deviation of M_1 from being a derivation of the composition product. Using the definitions (2.7) and (2.9), we can write

$$D(A, B) = m\{A\{B\}\} - (-1)^{|A|+|B|} A\{B\}\{m\} - m\{A\}\{B\} + (-1)^{|A|} A\{m\}\{B\}$$
$$- (-1)^{|A|} A\{m\{B\}\} + (-1)^{|A|+|B|} A\{B\{m\}\} \,. \quad (3.4)$$

It follows from the pre-Jacobi identities (2.8) that

$$m\{A\}\{B\} = m\{A, B\} + m\{A\{B\}\} + (-1)^{|A||B|} m\{B, A\} \,.$$

Applying similar transformations to the other terms in (3.4), we find that all but two terms cancel leaving

$$D(A, B) = -M_2(A, B) - (-1)^{|A||B|} M_2(B, A) \,.$$

It remains to note that for any pair of cocycles A and B the cochain $D(A, B)$ is a coboundary, whence

$$a \cup b = (-1)^{(|a|-1)(|b|-1)} b \cup a \,. \qquad \square$$

Notice that for graded associative algebras the associativity of the cup product (3.3) takes place at the level of cochains. This product, however, may not be graded commutative until passing to the Hochschild cohomology.

Proposition 3.2. *The cup product and the Gerstenhaber bracket satisfy the graded Poisson relation*

$$[a, b \cup c] = [a, b] \cup c + (-1)^{|a|(|b|+1)} b \cup [a, c] \quad \forall a, b, c \in H^\bullet(W) \,.$$

Proof. The Poisson relation follows from the identity

$$[A, M_2(B, C)] - (-1)^{|A|} M_2([A, B], C) - (-1)^{|A|(|B|+1)} M_2(B, [A, C])$$
$$= (-1)^{|A|} \big(M_1(A\{B, C\}) - M_1(A)\{B, C\} - (-1)^{|A|} A\{M_1(B), C\}$$
$$- (-1)^{|A|+|B|} A\{B, M_1(C)\} \big) \,,$$

which holds for all $A, B, C \in W$. One can verify it directly by making use of the pre-Jacobi identities (2.8). $\qquad \square$

Thus, the cup product and the Gerstenhaber bracket define the structure of a graded Poisson algebra on the A_∞-cohomology $H^\bullet(W)$.

Remark 3.3. The structure of a graded Poisson algebra on the Hochschild cohomology $HH^\bullet(A, A)$ of an associative algebra A was first observed by Gerstenhaber [4]. One can view the two propositions above as a straightforward extension of Gerstenhaber's results to the case of A_∞-algebras.

4. Inner deformations of families

4.1. Families of algebras. Let $\mathcal{A}_t$ be an n-parameter, formal deformation of an A_∞-algebra $\mathcal{A}$, i.e., the A_∞-structure on $\mathcal{A}_t$ is given by an element

$$m \in \mathcal{W} = W[\![t_1, \ldots, t_n]\!]$$

such that $m \circ m = 0$ and $m|_{t=0}$ gives the products in $\mathcal{A}$. Here we allow the deformation parameters t_i to have non-zero $\mathbb{Z}$-degrees contributing to the total degree $|m| = 1$ of m as an element of $\mathcal{W}^1$. For the sake of simplicity, however, we restrict ourselves to the case where all the degrees $|t_i|$ are *even*. Extension to the general case is straightforward (see Remark 4.3 below). In the following we will refer to $\mathcal{A}_t$ as a family of A_∞-algebras.

Let us denote

$$m_{(i_1 i_2 \cdots i_k)} = \frac{\partial^k m}{\partial t_{i_1} \partial t_{i_2} \cdots \partial t_{i_k}} \in \mathcal{W}.$$

Clearly,

$$|m_{(i_1 i_2 \cdots i_k)}| = 1 - |t_{i_1}| - \cdots - |t_{i_k}|.$$

Taking the partial derivative of the defining relation $m \circ m = 0$ w.r.t. the parameter t_i, we get

$$[m, m_{(i)}] = M_1(m_{(i)}) = 0.$$

In other words, the cochain $m_{(i)}$ is a cocycle of the differential M_1 associated to the A_∞-structure m. So, $m_{(i)}$ defines a cohomology class of $H^{1-|t_i|}(\mathcal{W})$.

Denote by $\mathcal{D}_m$ the subalgebra in the graded Poisson algebra $H^\bullet(\mathcal{W})$ generated by the cocycles $m_{(i)}$.

Proposition 4.1. *The Gerstenhaber bracket on $\mathcal{W}$ induces the trivial Lie bracket on $\mathcal{D}_m$.*

Proof. Differentiating the relation $m \circ m = 0$ twice, we get

$$[m, m_{(i,j)}] = -[m_{(i)}, m_{(j)}].$$

Hence, the bracket $[m_{(i)}, m_{(j)}]$ is an M_1-coboundary. By Proposition (3.2) this result is extended to arbitrary cup products of $m_{(i)}$'s. $\qquad \square$

4.2. Inner deformations. We see that the algebra $\mathcal{D}_m$ is generated by the cup products of the partial derivatives $m_{(i)}$, so that the elements of $\mathcal{D}_m$ are represented by cup polynomials[4]

$$\Delta = \sum_{l=0}^{L} c^{i_1 \cdots i_l} m_{(i_1)} \cup m_{(i_2)} \cup \cdots \cup m_{(i_l)}, \tag{4.1}$$

where $c^{i_1 \cdots i_l} \in k[\![t_1, \ldots, t_n]\!]$. Note that with our restriction on the degrees of t's the graded associative algebra $\mathcal{D}_m = \bigoplus \mathcal{D}_m^l$ is purely commutative as it consists only of even elements.

Proposition 4.2. *Let $m \in \mathcal{W}^1$ be an n-parameter family of A_∞-structures and let Δ be a cocycle representing an element of $\mathcal{D}_m^l$. Then we can define an $(n+1)$-parameter family of A_∞-structures $\tilde{m} \in W[\![t_0, t_1, \ldots, t_n]\!]$ as a unique formal solution to the differential equation*

$$\tilde{m}_{(0)} = \Delta[\tilde{m}] \tag{4.2}$$

subject to the initial condition $\tilde{m}|_{t_0=0} = m$. Here the new formal parameter t_0 has degree $1 - l$.

Proof. It is clear that Eq. (4.2) has a unique formal solution that starts as

$$\tilde{m}(t_0) = m + t_0 \Delta[m] + o(t_0^2).$$

Differentiating now the cochain $\lambda(t_0) = [\tilde{m}, \tilde{m}]$ by t_0, we get

$$\frac{\partial \lambda}{\partial t_0} = 2[\tilde{m}_{(0)}, \tilde{m}] = [\Delta[\tilde{m}], \tilde{m}] = 0.$$

With account of the initial condition $\lambda(0) = [m, m] = 0$ this means that $\lambda(t_0) = 0$; and hence, $\tilde{m}$ defines an $(n + 1)$-parameter family of A_∞-structures. $\square$

We call the deformations of Proposition 4.2 the *inner deformations* of families of A_∞-algebras.

Remark 4.3. Geometrically, we can think of the r.h.s. of (4.2) as a vector field Δ on the infinite-dimensional space $\mathcal{W}$. Then the A_∞-structures form a submanifold $\mathcal{M} \subset \mathcal{W}$ defined by the quadratic equation $[m, m] = 0$. The cocycle condition $[m, \Delta] = 0$ means that the vector field Δ is tangent to $\mathcal{M}$ and generates a flow $\Phi_{t_0}^\Delta$ on $\mathcal{W}$, which leaves $\mathcal{M}$ invariant. Therefore,

$$\tilde{m} = \Phi_{t_0}^\Delta(m) \subset \mathcal{M}.$$

Proceeding with this geometrical interpretation, we can consider the commutator $[\![\Delta, \Delta']\!]$ of two vector fields Δ and Δ' associated with some elements of $\mathcal{D}_m$. The

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[4]By abuse of notation, we write the cup product for the cocycles rather than their cohomology classes.

vector field $[\![\Delta, \Delta']\!]$, being tangent to $\mathcal{M}$, defines an M_1-cocycle. It would be interesting to study the Lie algebra of vector fields generated by the elements of $\mathcal{D}_m$ in more detail.

If we now allow some of the parameters t_i to have odd degrees, then an odd vector field Δ may not be integrable in the sense that $[\![\Delta, \Delta]\!] \neq 0$. In this case Eq. (4.2) for the flow should be modified as

$$\widetilde{m}_{(0)} = \Delta[\widetilde{m}] - \frac{1}{2} t_0 [\![\Delta, \Delta]\!] [\widetilde{m}].$$

Since $(t_0)^2 = 0$, the solution is given by

$$\widetilde{m} = m + t_0 \Delta[m]$$

and it is obvious that $\widetilde{m} \circ \widetilde{m} = 0$.

4.3. Deformation of MC elements. Let (V, m) be a flat A_∞-algebra. Then, whenever it is defined, the MC equation reads

$$m(a) := \sum_{n=1}^\infty m_n(a, \ldots, a) = 0 \tag{4.3}$$

for $|a| = 0$. A solution $a \in V^0$ to this equation is called an MC element of the A_∞-algebra (V, m) and the set of all MC elements, called the MC space, is denoted by $\mathcal{MC}(V, m)$.

In order to ensure the convergence of the series (4.3) one or another assumption about (V, m) is needed. For example, one may assume that $m_n = 0$ for all $n > p$, so that the series (4.3) is actually finite. This is the case of DGA's. Another possibility is to consider the scalar extension $V \otimes \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of an Artinian algebra A; the multi-linear operations on V extend to those on $V \otimes \mathfrak{m}_A$ by A-linearity. Neither of these approaches, however, is appropriate to our purposes. What suits us is, in a sense, a combination of both.

Definition 4.4. We say that an n-parameter family of A_∞-structures $m \in W[\![t_1, \ldots, t_n]\!]$ is *locally finite* if for each k there exists a finite N such that

$$m_{(i_1 \cdots i_k)}|_{t=0} \in \bigoplus_{n=0}^N \mathrm{Hom}(T^n(V), V).$$

With the supposition of local finiteness, the MC equation (4.3) for an element $a \in V[\![t_1, \ldots, t_n]\!]$ gives an infinite collection of well-defined equations on the Taylor coefficients of a. Furthermore, we have the following statement, whose proof is left to the reader.

Proposition 4.5. *Any inner deformation of a locally finite family of A_∞-algebra is locally finite.*

The next proposition shows that inner deformations of locally finite families are always accompanied by deformations of their MC spaces.

Proposition 4.6. *Let $\tilde{m}$ be an inner deformation of a locally finite family of A_∞-structures m. Then each MC element for m can be deformed to that for $\tilde{m}$, establishing thus a monomorphism $\mathcal{MC}(V, m) \to \mathcal{MC}(V, \tilde{m})$.*

Proof. In order to simplify the formulas below we restrict ourselves to inner deformations that are generated by monomials

$$\Delta[\tilde{m}] = \tilde{m}_{(i_1)} \cup \tilde{m}_{(i_2)} \cup \cdots \cup \tilde{m}_{(i_l)} \,.$$

The generalization to arbitrary cup polynomials (4.1) will be obvious.

Suppose that $a \in \mathcal{MC}(V, \tilde{m})$, then

$$\tilde{m}(a) = 0 \,. \tag{4.4}$$

Differentiating this identity by t_i, we get

$$\tilde{m}_{(i)}(a) + \tilde{m}\{D_i\}(a) = 0 \,.$$

Here by D_i we denoted the operator of partial derivative, $D_i a = \partial a / \partial t_i$. Therefore, when evaluated on MC elements, the deformation equation (4.2) can be written as

$$\tilde{m}\{D_0\}(a) = -\Delta[\tilde{m}](a) \,. \tag{4.5}$$

More explicitly, the r.h.s. of this equation is defined by

$$\begin{aligned} \Delta[\tilde{m}](a) &= \Delta(\tilde{m}_{(i_1)}, \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)})(a) \\ &= M_2(\dots (M_2(M_2(M_2(\tilde{m}_{(i_1)}, \tilde{m}_{(i_2)}), \tilde{m}_{(i_3)}), \dots, \tilde{m}_{(i_l)})(a) \,. \end{aligned}$$

Here we used the definition of the cup product (3.3).

Again, when evaluated on MC elements, the expression $\tilde{m}\{D_i\}(a)$ can be replaced with $M_1(D_i)(a)$. This allows us to write

$$\Delta[\tilde{m}](a) = -\Delta(M_1(D_{i_1}), \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)})(a) \,.$$

Since $M_1(\tilde{m}_{(i)}) = 0$, the repeated use of Rel. (3.2) allows us to rewrite the last expression as

$$\Delta[\tilde{m}](a) = (-1)^l M_1(\Delta(D_{i_1}, \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)}))(a)$$

or, equivalently,

$$\Delta[\tilde{m}](a) = (-1)^l m\{\Delta(D_{i_1}, \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)})\}(a) \,.$$

Thus, Eq. (4.5) takes the form

$$m\{D_0\}(a) = -(-1)^l m\{\Delta(D_{i_1}, \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)})\}(a) \,.$$

We will definitely satisfy this equation if require that

$$D_0 a = -(-1)^l \Delta(D_{i_1}, \tilde{m}_{(i_2)}, \ldots, \tilde{m}_{(i_l)})(a). \tag{4.6}$$

This gives a differential equation for $a \in V[\![t_0, t_1, \ldots, t_n]\!]$ w.r.t. the formal 'evolution parameter' t_0 of degree $1 - |\Delta|$.

Now we can evaluate $\tilde{m}$ on a formal solution to Eq. (4.6). It follows from the course of the proof above that the partial derivative $D_0(\tilde{m}(a))$ depends on $\tilde{m}(a)$ linearly thereby vanishes on (4.4). This means that the vector $\tilde{m}(a) \in V[\![t_0, t_1, \ldots, t_n]\!]$ is zero whenever it vanishes at $t_0 = 0$. But the last condition is just the definition of an MC element $a|_{t_0=0} \in \mathcal{MC}(V, m)$. □

4.4. Minimal deformations of DGA's.

We now apply the above machinery of inner deformations to the case of DGA's. Recall that a DGA $\mathcal{A}$ is given by a triple $(V, \partial, \cdot)$, where $V = \bigoplus V^l$ is a graded vector space endowed with an associative dot product and a differential $\partial \colon V^l \to V^{l-1}$.

Remark 4.7. Here we equip a DGA with a differential of degree -1. From the perspective of A_∞-algebras, it is more natural to consider differentials of degree 1. As was discussed in Section 2, a DGA structure on V can then be interpreted as a 'degenerate' A_∞-structure on $V[1]$ involving only a linear map m_1 and a bilinear map m_2, both of degree 1. Actually, there is not much difference between the two definitions as one can always relate them by the degree reversion functor ι. By definition, ιV is a graded vector space with $(\iota V)^l = V^{-l}$. Clearly, the k-linear map $V \to \iota V$ respects the product while reverting the degree of the differential.

Let $\mathcal{A}_t$ be a one-parameter deformation of $\mathcal{A}$, with t being a formal parameter of degree zero. In order to make the DGA $\mathcal{A}_t$ into a family of A_∞-algebras, we define the tensor product algebra $\mathcal{A}_t \otimes k[\![u]\!]$, where u is an auxiliary formal variable of degree 2. Here we consider $k[\![u]\!]$ as a DGA with trivial differential. Multiplying now the differential ∂ in $\mathcal{A}_t$ by u yields the differential $d = u\partial$ in $\mathcal{A}_t \otimes k[\![u]\!]$ of degree 1. This allows us to treat the DGA $\mathcal{A}_t \otimes k[\![u]\!]$ as a 2-parameter family of A_∞-algebras with $m_1 = d$ and m_2 defined by (2.5). On the other hand, given a two-parameter family of A_∞-structures m with the parameters t and u of degrees 0 and 2, respectively, we can define the sequence of cocycles

$$\Delta_n = m_{(t)} \cup \underbrace{m_{(u)} \cup m_{(u)} \cup \cdots \cup m_{(u)}}_{n}, \quad n = 0, 1, 2, \ldots. \tag{4.7}$$

Here the subscripts t and u stand for the partial derivatives of m w.r.t. t and u. Keeping in mind that the cup product has degree 1 while $|m_{(u)}| = -1$, we conclude that $|\Delta_n| = 1$ for all n. By Proposition 4.2, each cocycle Δ_n gives rise to a formal deformation of m with a new deformation parameter s of degree zero. The deformed A_∞-structure $\tilde{m}$ is defined by the differential equation

$$\tilde{m}_{(s)} = \Delta_n[\tilde{m}] \tag{4.8}$$

with the initial condition $\widetilde{m}|_{s=0} = m$. The parameter u plays an auxiliary role in our construction. Setting $u = 0$, we finally get, for each n, a family $\bar{m} = \widetilde{m}|_{u=0}$ of A_∞-structures parameterized by t and s; both the parameters are of degree zero. By construction, $\bar{m}$ starts with m_2 and the first-order deformation in s is given by

$$\bar{m}^{(1)}(a_1, a_2, \ldots, a_{n+2}) = (a_1 \cdot a_2)' \cdot \partial(a_3) \cdot \partial(a_4) \cdots \partial(a_{n+2}),$$

where the prime denotes the partial derivative of the dot product in $\mathcal{A}_t$ by t. Evaluating $\bar{m} \circ \bar{m} = 0$ at the first order in s, we get

$$
\begin{aligned}
[m_2, \bar{m}^{(1)}](a_0, a_1, \ldots, a_{n+2}) &= -a_0 \cdot \bar{m}^{(1)}(a_1, a_2, \ldots, a_{n+2}) \\
&\quad - \sum_{k=0}^{n-1} (-1)^{|a_0|+\cdots+|a_k|} \bar{m}^{(1)}(a_0, \ldots, a_{k-1}, a_k \cdot a_{k+1}, a_{k+2}, \ldots, a_{n+2}) \\
&\quad + (-1)^{|a_0|+\cdots+|a_{n+1}|} \bar{m}^{(1)}(a_0, a_1, \ldots, a_{n+1}) \cdot a_{n+2} = 0.
\end{aligned}
$$

Therefore, $\bar{m}^{(1)}$ is a Hochschild cocycle of the algebra $\mathcal{A}_t$ representing an element of $HH^{n+2}(\mathcal{A}_t, \mathcal{A}_t)$. If the cocycle $m^{(1)}$ is nontrivial, then it defines a nontrivial deformation of the algebra $\mathcal{A}_t$ in the category of A_∞-algebras. Notice that the resulting A_∞-structure $\bar{m}$ is minimal as, by construction, $\bar{m}_1 = 0$. For this reason we refer to $\bar{m}$ as a *minimal deformation* of the DGA structure m. In such a way we arrive at the first statement of Theorem 1.1.

In the special case that the differential ∂ does not depend on t, the r.h.s. of Eq. (4.8) is independent of u, so that the whole dependence of $\widetilde{m}$ of u is concentrated in the first structure map $\widetilde{m}_1 = u\partial$. This means that all the structure maps constituting $\widetilde{m}$ or $\bar{m}$ are differentiated by ∂.

Being determined only by the first and second structure maps, the A_∞-algebra $\mathcal{A}_t \otimes k[\![u]\!]$ is evidently locally finite in the sense of Definition 4.4 and so is its minimal deformation defined by $\bar{m} \in W[\![t, s]\!]$. At $s = 0$, the A_∞-structure $\bar{m}$ reduces to the product in $\mathcal{A}_t$ and the MC equation takes the form[5] $a \cdot a = 0$. Applying Proposition 4.6 to a solution a yields then an MC element

$$\bar{a} = a + \sum_{k>0} a_k s^k$$

for the minimal A_∞-algebra $(V, \bar{m})$, that is, $\bar{m}(\bar{a}) = 0$. This proves the rest part of Theorem 1.1.

As a final remark we note that the above construction of minimal deformations carries over verbatim to the case of smooth (i.e., not formal) families of DGA's $\mathcal{A}_t$. An interesting example of a smooth family of algebras is considered below.

---

[5]Notice that the differential ∂ does not contribute to the MC equation, contrary to what one might expect. From the viewpoint of the DGA the element a has degree 1, so that ∂a is of degree 0, and not 2.

4.5. Example. Let us illustrate the construction of the previous subsection by the example of the polynomial Weyl algebra $A_m[t]$. As a vector space $A_m[t]$ coincides with the space $k[t, x^1, \ldots, x^{2m}]$ of polynomials in $2m + 1$ variables. Multiplication in $A_m[t]$ is given by the $*$-product

$$a * b = a \cdot b + \sum_{k=1}^\infty t^k (a \overset{k}{*} b), \qquad (4.9)$$

where

$$a \overset{k}{*} b = \frac{1}{k!} \omega^{i_1 j_1} \cdots \omega^{i_k j_k} \frac{\partial^k a}{\partial x^{i_1} \cdots \partial x^{i_k}} \frac{\partial^k b}{\partial x^{j_1} \cdots \partial x^{j_k}}$$

and ω^{ij} is a skew-symmetric, non-degenerate matrix with entries in k. The $*$-product is known to be associative but non-commutative. Clearly, one may regard $A_m[t]$ as a one-parameter deformation of the usual polynomial algebra $k[x^1, \ldots, x^{2m}]$ with the commutative dot product.

As was explained in the Introduction, we can turn the polynomial Weyl algebra into a family of DGA's $\mathcal{A}_t$ simply treating $A_m[t]$ as a bimodule over itself. The family $\mathcal{A}_t$ is concentrated in degrees 0 and 1, so that the underlying k-vector space is

$$V = V^0 \oplus V^1 \quad \text{with } V^0 = A_m[t] = V^1.$$

Multiplication is defined by the rule (1.1) and the differential ∂ is completely specified by declaring $\partial \colon V^1 \to V^0$ to be the identity homomorphism of $A_m[t]$ onto itself.

Following prescriptions of Subsection 4.4, we can now produce a two-parameter family of A_∞-structures $\bar{m}$ generated, for example, by the cocycle Δ_1 of (4.7). The family is parameterized by the initial parameter t and a new deformation parameter s of degree zero. The explicit expression for $\bar{m}$ resulting from the deformation equation (4.8) appears to be rather complicated. The situation is slightly simplified if we put $t = 0$. This corresponds to a deformation of the polynomial algebra $k[x^1, \ldots, x^{2m}]$, considered as a bimodule over itself, in the category of minimal A_∞-algebras.

A direct, albeit tedious, calculation shows that the only nonzero maps $m_n \in \mathrm{Hom}(T^n(V[1]), V[1])$ constituting the A_∞-structure $m = \bar{m}|_{t=0}$ are given by

$$m_n(a, b, u_3, \ldots, u_n) = s^{n-2} f_n(a, b, u_3, \ldots, u_{n-1}) \cdot u_n,$$
$$m_n(a, u_2, \ldots, u_n) = s^{n-2} f_n(a, u_2, \ldots, u_{n-1}) \cdot u_n, \qquad (4.10)$$
$$m_n(u_2, b, u_3, \ldots, u_n) = -s^{n-2} f_n(u_2, b, \ldots, u_{n-1}) \cdot u_n,$$

where $a, b \in V^0$, $u_2, \ldots, u_n \in V^1$, and the f_n's are defined by

$$f_{n+1}(a_1, a_2, \ldots, a_n)$$
$$= \sum a_1 \overset{k_1}{*} \underbrace{a_2 \cdot a_3 \cdots a_{l_1+1}}_{l_1} \overset{k_2}{*} \underbrace{a_{l_1+2} \cdots a_{l_1+l_2+2}}_{l_2} \overset{k_3}{*} \cdots \overset{k_p}{*} \underbrace{a_{n-l_p+1} \cdots a_{n-1} \cdot a_n}_{l_p}$$

for all $a_i \in k[x^1, \ldots, x^{2m}]$. Here, to save space, we omit parentheses specifying the order of multiplication; it is understood that all the multiplication operations are performed from left to right and summation runs over all k's and l's obeying the (in)equalities

$$\sum_{j=1}^{p} l_j = \sum_{j=1}^{p} k_j = n - 2, \quad l_j \geq 1, \quad k_j \geq 1, \quad p \geq 0,$$

$$l_p \geq k_p, \quad l_{p-1} + l_p \geq k_{p-1} + k_p, \quad \ldots, \quad l_2 + \cdots + l_p \geq k_2 + \cdots + k_p.$$

In particular, for $n = 2$ (which means $p = 0$) we recover the original bimodule structure for the polynomial algebra $k[x^1, \ldots, x^{2m}]$:

$$m_2(a, b) = a \cdot b, \quad m_2(a, u) = a \cdot u, \quad m_2(u, b) = -u \cdot b,$$

and the first-order deformation is given by

$$m_3(a, b, u) = s(a \overset{1}{*} b) \cdot u,$$
$$m_3(a, u_1, u_2) = s(a \overset{1}{*} u_1) \cdot u_2,$$
$$m_3(u_1, b, u_2) = -s(u_1 \overset{1}{*} b) \cdot u_2.$$

It is not hard to see that this deformation is nontrivial. Finally, for all n

$$m_{n+2}(a, b, 1, \ldots, 1) = s^n a \overset{n}{*} b,$$

which allows us to regard (4.10) as a certain A_∞ generalization of the Weyl–Moyal $*$-product (4.9) to the case of a 'non-constant deformation parameter' u.

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