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Vector bundles over multipullback quantum complex projective spaces

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Abstract. We work on the classification of isomorphism classes of finitely generated projective modules over the C*-algebras $C(\mathbb{P}^n(\mathcal{T}))$ and $C(\mathbb{S}_H^{2n+1})$ of the quantum complex projective spaces $\mathbb{P}^n(\mathcal{T})$ and the quantum spheres $\mathbb{S}_H^{2n+1}$, and the quantum line bundles L_k over $\mathbb{P}^n(\mathcal{T})$, studied by Hajac and collaborators. Motivated by the groupoid approach of Curto, Muhly, and Renault to the study of C*-algebraic structure, we analyze $C(\mathbb{P}^n(\mathcal{T}))$, $C(\mathbb{S}_H^{2n+1})$, and L_k in the context of groupoid C*-algebras, and then apply Rieffel's stable rank results to show that all finitely generated projective modules over $C(\mathbb{S}_H^{2n+1})$ of rank higher than $\lfloor \frac{n}{2} \rfloor + 3$ are free modules. Furthermore, besides identifying a large portion of the positive cone of the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$, we also explicitly identify L_k with concrete representative elementary projections over $C(\mathbb{P}^n(\mathcal{T}))$.

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1. Introduction

Since the concept of noncommutative geometry first popularized by Connes [5], many interesting examples of a C*-algebra $\mathfrak{A}$ viewed as the algebra $C(X_q)$ of continuous functions on a virtual quantum space X_q have been constructed with a topological or geometrical motivation, and analyzed in comparison with their classical counterpart. For example, quantum odd-dimensional spheres and associated complex projective spaces have been introduced and studied by Soibelman, Vaksman, Meyer, and others [14, 32] as $\mathbb{S}_q^{2n+1}$ and $\mathbb{C}P_q^n$ via a quantum universal enveloping algebra approach, and by Hajac and his collaborators including Baum, Kaygun, Matthes, Nest, Pask, Sims, Szymański, Zieliński, and others [2, 9, 10, 12] as $\mathbb{S}_H^{2n+1}$ and $\mathbb{P}^n(\mathcal{T})$ via a multi-pullback and Toeplitz algebra approach. Actually $\mathbb{S}_H^{2n+1}$ is

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the untwisted special case of the more general version of θ-twisted spheres $\mathbb{S}^{2n+1}_{H,\theta}$ introduced in [12].

Motivated by Swan's work [30], the concept of a noncommutative vector bundle E_q over a quantum space X_q can be reformulated as a finitely generated projective (left) module $\Gamma(E_q)$ over $C(X_q)$. Based on the strong connection approach to quantum principal bundles [8] for compact quantum groups [34, 35], Hajac and his collaborators introduced quantum line bundles L_k of degree k over $\mathbb{P}^n(\mathcal{T})$ as some rank-one projective modules realized as spectral subspaces $C(\mathbb{S}^{2n+1}_H)_k$ of $C(\mathbb{S}^{2n+1}_H)$ under a $U(1)$-action [12]. Besides having the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$ computed, they found that L_k is not stably free unless $k = 0$, extending earlier results for the case of $n = 1$ [10, 11].

It has always been an interesting but challenging task to classify finitely generated projective modules over an algebra up to isomorphism, which goes beyond their classification up to stable isomorphism by K_0-group and appears in the form of so-called cancellation problem. Classically it is known that the cancellation law holds for complex vector bundles of rank no less than $\frac{d}{2}$ over a d-dimensional CW-complex, which implies that all complex vector bundles over $\mathbb{S}^{2n+1}$ of rank $n + 1$ or above are trivial.

The study of such classification problem for C*-algebras was popularized by Rieffel [21, 22] who introduced useful versions of stable ranks for C*-algebras to facilitate the analysis involved. Some successes have been achieved for certain quantum algebras [1, 19, 22, 23, 25]. In particular, Peterka showed that all finitely generated projective modules over the θ-deformed 3-spheres S^3_θ are free, and constructed all those over S^4_θ up to isomorphism [19]. With more effort, the result of Bach [1] on the cancellation law for $\mathbb{S}^{2n+1}_q$ and $\mathbb{C}P^n_q$ can be strengthened to a complete classification of finitely generated projective modules over them, which we will address elsewhere.

With the K_0-group of $C(\mathbb{P}^n(\mathcal{T}))$ known [12], it is natural to try to classify finitely generated projective modules over $C(\mathbb{P}^n(\mathcal{T}))$ and identify the line bundles L_k among them. In [29], a complete solution was obtained for the special case of $n = 1$.

In this paper, we use the powerful groupoid approach to C*-algebras initiated by Renault [20] and popularized by Curto, Muhly, and Renault [6, 15] to study multi-variable Toeplitz C*-algebras $\mathcal{T}^{\otimes n}$, quantum spheres $C(\mathbb{S}^{2n+1}_H)$, and quantum complex projective spaces $C(\mathbb{P}^n(\mathcal{T}))$. Utilizing results on stable ranks of C*-algebras obtained by Rieffel [21], we analyze finitely generated projective modules over $\mathcal{T}^{\otimes n+1}$ and $C(\mathbb{S}^{2n+1}_H)$, and get those of rank higher than $\lfloor \frac{n}{2} \rfloor + 3$ and also a large class of "standard" modules classified up to isomorphism. Furthermore, besides identifying a large portion of the positive cone of the K_0-group $K_0(C(\mathbb{P}^n(\mathcal{T})))$, we explicitly identify the quantum line bundles L_k with concrete representative elementary projections.

On the other hand, there are still a lot of questions to be further investigated, e.g. whether the cancellation law holds for low-ranked finitely generated projective modules, and whether the more general case of θ-twisted multipullback quantum sphere $\mathbb{S}^{2n+1}_{H,\theta}$ brings in new phenomena. Finally, it is of interest to note the recent work of Farsi, Hajac, Maszczyk, and Zieliński [7] on $K_0(C(\mathbb{P}^2(\mathcal{T})))$, identifying its free generators arising from Milnor modules as sums of L_k, which are also expressed in terms of elementary projections, showing a perfect consistency with our result.

2. Notations

Taking the groupoid approach to C*-algebras initiated by Renault [20] and popularized by the work of Curto, Muhly, and Renault [6, 15], we give a description of the C*-algebras $C(\mathbb{S}^{2n-1}_H)$ and $C(\mathbb{P}^{n-1}(\mathcal{T}))$ of [12] as some concrete groupoid C*-algebras. We refer to [15,20] for the concepts and theory of groupoid C*-algebras used freely in the following discussion.

By abuse of notation, for any C*-algebra homomorphism $\phi\colon \mathfrak{A} \to \mathfrak{B}$, we denote the C*-algebra homomorphism

$$M_k(\phi)\colon M_k(\mathfrak{A}) \to M_k(\mathfrak{B}) \quad \text{for } k \in \mathbb{N} \equiv \{1, 2, 3, \ldots\}$$

also by ϕ. We use $\mathfrak{A}^\times$ to denote the set of all invertible elements of an algebra $\mathfrak{A}$, and use $\mathfrak{A}^+$ to denote the minimal unitization of $\mathfrak{A}$. For any topological group G, we use G^0 to denote the identity component of G, i.e. the connected component that contains the identity element of G.

We denote by $M_\infty(\mathfrak{A})$ the direct limit (or the union as sets) of the increasing sequence of matrix algebras $M_n(\mathfrak{A})$ over $\mathfrak{A}$ with the canonical inclusion $M_n(\mathfrak{A}) \subset M_{n+1}(\mathfrak{A})$ identifying $x \in M_n(\mathfrak{A})$ with $x \boxplus 0 \in M_{n+1}(\mathfrak{A})$ for any algebra $\mathfrak{A}$, where $\boxplus$ denotes the standard diagonal concatenation (sum) of two matrices. So the size of an element in $M_\infty(\mathfrak{A})$ can be taken arbitrarily large. We also use $\mathrm{GL}_\infty(\mathfrak{A})$ to denote the direct limit of the general linear groups $\mathrm{GL}_n(\mathfrak{A})$ over a unital C*-algebra $\mathfrak{A}$ with $\mathrm{GL}_n(\mathfrak{A})$ embedded in $\mathrm{GL}_{n+1}(\mathfrak{A})$ by identifying $x \in \mathrm{GL}_n(\mathfrak{A})$ with $x \boxplus 1 \in \mathrm{GL}_{n+1}(\mathfrak{A})$.

By an idempotent P over a unital C*-algebra $\mathfrak{A}$, we mean an element $P \in M_\infty(\mathfrak{A})$ with $P^2 = P$, and a self-adjoint idempotent in $M_\infty(\mathfrak{A})$ is called a projection over $\mathfrak{A}$. Two idempotents $P, Q \in M_\infty(\mathfrak{A})$ are called equivalent, denoted as $P \sim Q$, if there exists $U \in \mathrm{GL}_\infty(\mathfrak{A})$ such that $UPU^{-1} = Q$. Each idempotent $P \in M_n(\mathfrak{A})$ over $\mathfrak{A}$ defines a finitely generated left projective module $E := \mathfrak{A}^n P$ over $\mathfrak{A}$ where elements of $\mathfrak{A}^n$ are viewed as row vectors. The mapping $P \mapsto \mathfrak{A}^n P$ induces a bijective correspondence between the equivalence classes of idempotents over $\mathfrak{A}$ and the isomorphism classes of finitely generated left projective modules over $\mathfrak{A}$ [3]. From now on, by a module over $\mathfrak{A}$, we mean a left $\mathfrak{A}$-module, unless otherwise specified.

Two finitely generated projective modules E, F over $\mathfrak{A}$ are called stably isomorphic if they become isomorphic after being augmented by the same finitely generated free $\mathfrak{A}$-module, i.e. $E \oplus \mathfrak{A}^k \cong F \oplus \mathfrak{A}^k$ for some $k \geq 0$. Correspondingly, two idempotents P and Q are called stably equivalent if $P \boxplus I_k$ and $Q \boxplus I_k$ are equivalent for some identity matrix I_k. The K_0-group $K_0(\mathfrak{A})$ classifies idempotents over $\mathfrak{A}$ up to stable equivalence. The classification of idempotents over a C*-algebra up to equivalence, appearing as the so-called cancellation problem, was popularized by Rieffel's pioneering work [21, 22] and is in general an interesting but difficult question.

The set of all equivalence classes of idempotents over a C*-algebra $\mathfrak{A}$ is an abelian monoid $\mathfrak{P}(\mathfrak{A})$ with its binary operation provided by the diagonal sum $\boxplus$. The image of the canonical homomorphism from $\mathfrak{P}(\mathfrak{A})$ into $K_0(\mathfrak{A})$ is the so-called positive cone of $K_0(\mathfrak{A})$.

Furthermore, it is well known [3] that in the above descriptions of $\mathfrak{P}(\mathfrak{A})$ and $K_0(\mathfrak{A})$, one can restrict to the self-adjoint idempotents, called projections over $\mathfrak{A}$, and their unitary equivalence classes, which faithfully represent the elements of $\mathfrak{P}(\mathfrak{A})$ and $K_0(\mathfrak{A})$.

In this paper, we use freely the basic techniques and manipulations for K-theory found in [3, 31].

For a Hilbert space $\mathcal{H}$, we denote the C*-algebra consisting of all compact linear operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$, or simply by $\mathcal{K}$ if $\mathcal{H}$ is the essentially unique separable infinite-dimensional Hilbert space.

In the following, we use the notations

$$\mathbb{Z}_{\geq k} := \{n \in \mathbb{Z} \mid n \geq k\} \quad \text{and} \quad \mathbb{Z}_{\geq} := \mathbb{Z}_{\geq 0}.$$

In particular, $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We use I to denote the identity operator canonically contained in $\mathcal{K}^+ \subset \mathcal{B}(\ell^2(\mathbb{Z}_{\geq}))$, and

$$P_m := \sum_{i=1}^{m} e_{ii} \in M_m(\mathbb{C}) \subset \mathcal{K}$$

to denote the standard $m \times m$ identity matrix in $M_m(\mathbb{C}) \subset \mathcal{K}$ for any integer $m \geq 0$ (with $M_0(\mathbb{C}) = 0$ and $P_0 = 0$ understood). We also use the notation

$$P_{-m} := I - P_m \in \mathcal{K}^+$$

for integers $m > 0$, and take symbolically $P_{-0} \equiv I - P_0 = I \neq P_0$.

3. Quantum spaces as groupoid C*-algebras

Let $\mathfrak{T}_n := (\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\overline{\mathbb{Z}}^n_{\geq}}$ with $n \geq 1$ be the transformation group groupoid $\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n$ restricted to the positive "cone" $\overline{\mathbb{Z}}^n_{\geq}$, where

$$\overline{\mathbb{Z}} := \mathbb{Z} \cup \{+\infty\} \text{ containing } \mathbb{Z}_{\geq} \equiv \{n \in \mathbb{Z} | n \geq 0\}$$

carries the standard topology, and $\mathbb{Z}^n$ acts on $\overline{\mathbb{Z}}^n$ componentwise in the canonical way. From the groupoid isomorphism

$$(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\overline{\mathbb{Z}}^n_{\geq}} \cong \times^n\big((\mathbb{Z} \ltimes \overline{\mathbb{Z}})\big|_{\overline{\mathbb{Z}}_{\geq}}\big)$$

and the well known C*-algebra isomorphism $C^*((\mathbb{Z} \ltimes \overline{\mathbb{Z}})|_{\overline{\mathbb{Z}}_{\geq}}) \cong \mathcal{T}$ for the Toeplitz C*-algebra $\mathcal{T}$, we get the groupoid C*-algebra

$$C^*(\mathfrak{T}_n) \equiv C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\overline{\mathbb{Z}}^n_{\geq}}\big) \cong \mathcal{T}^{\otimes n} \equiv \otimes^n \mathcal{T}.$$

We consider two important nontrivial invariant open subsets of the unit space $\overline{\mathbb{Z}}^n_{\geq}$ of $\mathfrak{T}_n$, namely, $\mathbb{Z}^n_{\geq}$ the smallest one and $\overline{\mathbb{Z}}^n_{\geq}\backslash\{\infty^n\}$ the largest one, where

$$\infty^n := (\infty, \ldots, \infty) \in \overline{\mathbb{Z}}^n_{\geq}.$$

By the theory of groupoid C*-algebras developed in Renault's book [20], they give rise to two short exact sequences of C*-algebras

$$0 \to C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\mathbb{Z}^n_{\geq}}\big) \cong \mathcal{K}\big(\ell^2(\mathbb{Z}^n_{\geq})\big) \to C^*(\mathfrak{T}_n) \equiv \mathcal{T}^{\otimes n} \to C^*(\mathfrak{G}_n) \to 0$$

with $\mathcal{K}(\ell^2(\mathbb{Z}^n_{\geq})) \cong \otimes^n \mathcal{K} \equiv \mathcal{K}^{\otimes n}$, where

$$\mathfrak{G}_n := (\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\overline{\mathbb{Z}}^n_{\geq}\backslash\mathbb{Z}^n_{\geq}}$$

is $\mathfrak{T}_n$ restricted to the "limit boundary" of its unit space, and

$$0 \to C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\overline{\mathbb{Z}}_{\geq}{}^n\backslash\{\infty^n\}}\big) \to C^*(\mathfrak{T}_n) \equiv \mathcal{T}^{\otimes n}$$
$$\xrightarrow{\sigma_n} C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\{\infty^n\}}\big) \cong C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n) \to 0,$$

where the quotient map σ_n extends the notion of the well known symbol map σ on $\mathcal{T}$ in the case of $n = 1$.

Note that the open invariant set $\mathbb{Z}^n_{\geq}$ being dense in the unit space $\overline{\mathbb{Z}}^n_{\geq}$ of $\mathfrak{T}_n$ induces a faithful representation π_n of $C^*(\mathfrak{T}_n)$ on $\ell^2(\mathbb{Z}^n_{\geq})$ that realizes the groupoid C*-algebra $C^*(\mathfrak{T}_n)$ and its closed ideal $C^*((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\mathbb{Z}^n_{\geq}})$, respectively, as a C*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}^n_{\geq}))$ and the closed ideal $\mathcal{K}(\ell^2(\mathbb{Z}^n_{\geq}))$ consisting of all compact operators on $\ell^2(\mathbb{Z}^n_{\geq})$.

In this paper, we freely identify elements of $C^*(\mathfrak{T}_n) \equiv \mathcal{T}^{\otimes n}$ with operators on $\ell^2(\mathbb{Z}^n_{\geq})$ via the faithful representation π_n and use these two conceptually different notions interchangeably.

In [12], Hajac, Nest, Pask, Sims, and Zieliński defined the (untwisted) *multipullback* or *Heegaard* quantum odd-dimensional sphere S_H^{2n-1} as the quantum space of the multipullback C*-algebra [18] determined by homomorphisms of the form

$$\mathrm{id}^{\otimes j} \otimes \sigma \otimes \mathrm{id}^{\otimes n-j-1}$$

from

$$\mathfrak{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathfrak{T}^{\otimes n-i-1} \quad \text{with } i \neq j,$$

to some

$$\mathfrak{T}^{\otimes m} \otimes C(\mathbb{T}) \otimes \mathfrak{T}^{\otimes k} \otimes C(\mathbb{T}) \otimes \mathfrak{T}^{\otimes n-m-k-2}.$$

(Actually more general θ-twisted quantum spheres $S^{2n-1}_{H,\theta}$ are studied there.) They showed that

$$C(\mathbb{S}^{2n-1}_H) \cong (\otimes^n \mathfrak{T})/(\otimes^n \mathfrak{K}),$$

and hence we have

$$C(\mathbb{S}^{2n-1}_H) \cong C^*(\mathfrak{G}_n)$$

identified as a groupoid C*-algebra.

With the ideal $C^*((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\overline{\mathbb{Z}}^n_{\geq} \setminus \{\infty^n\}})$ containing the ideal $C^*((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\mathbb{Z}^n_{\geq}})$, the quotient map σ_n induces a well-defined quotient map τ_n in the short exact sequence

$$0 \to C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\overline{\mathbb{Z}}^n_{\geq} \setminus (\mathbb{Z}^n_{\geq} \cup \{\infty^n\})}\big) \to C(\mathbb{S}^{2n-1}_H) \cong (\otimes^n \mathfrak{T})/(\otimes^n \mathfrak{K}) \xrightarrow{\tau_n} C(\mathbb{T}^n) \to 0.$$

4. Stable ranks of quantum spaces

In his seminal paper [21], Rieffel introduced and popularized the notions of topological stable rank tsr($\mathfrak{A}$) and connected stable rank csr($\mathfrak{A}$) of a C*-algebra $\mathfrak{A}$, which are useful tools in the study of cancellation problems for finitely generated projective modules. Later, Herman and Vaserstein [13] showed that for C*-algebras $\mathfrak{A}$, Rieffel's topological stable rank coincides with the Bass stable rank used in algebraic K-theory. So we will denote tsr($\mathfrak{A}$) simply as sr($\mathfrak{A}$) in our discussion.

In this section, we review an estimate of the stable ranks of the Toeplitz algebras $\mathfrak{T}^{\otimes n}$ and quantum spheres $C(\mathbb{S}^{2n-1}_H)$, which will be used in our study of their finitely generated projective modules. For the case of $n = 1$, it is known [21] that

$$\mathrm{sr}(\mathfrak{T}) = \mathrm{csr}(C(\mathbb{T})) = 2.$$

As an illustration of the groupoid approach to C*-algebras, we first establish some composition sequence structure for $\mathfrak{T}^{\otimes n}$ and $C(\mathbb{S}^{2n-1}_H)$, which leads to an easy estimate of their stable ranks.

Proposition 1. *There is a finite composition sequence of closed ideals*

$$\mathfrak{T}^{\otimes n} \equiv C^*(\mathfrak{T}_n) \equiv \mathfrak{I}_n \rhd \mathfrak{I}_{n-1} \rhd \cdots \rhd \mathfrak{I}_1 \rhd \mathfrak{I}_0 \rhd \mathfrak{I}_{-1} \equiv \{0\}$$

such that $\mathfrak{T}^{\otimes n}/\mathfrak{I}_0 \cong C(\mathbb{S}^{2n-1}_H)$, *and for* $0 \leq j \leq n$,

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \big(\mathfrak{K}(\ell^2(\mathbb{Z}^{n-j}_{\geq})) \otimes C(\mathbb{T}^j)\big),$$

where $\mathbb{T}^0$ *and* $\mathbb{Z}^0_{\geq}$ *denote a singleton.*

Proof. For $0 \le j \le n$, let X_j be the set consisting of $z \in \overline{\mathbb{Z}}^n_{\ge}$ with exactly j of the components $z_1, z_2, \ldots, z_n$ being equal to ∞, and hence $X_n = \{\infty^n\}$. Then the sets

$$Y_j := X_0 \sqcup X_1 \sqcup \cdots \sqcup X_j$$

are open invariant subsets of the unit space $\overline{\mathbb{Z}}^n_{\ge}$ of $\mathfrak{T}_n$ with

$$\mathbb{Z}^n_{\ge} = Y_0 \subset Y_1 \subset \cdots \subset Y_n = \overline{\mathbb{Z}}^n_{\ge},$$

which determines an increasing chain of closed ideals $\mathfrak{L}_0 \lhd \mathfrak{L}_1 \lhd \cdots \lhd \mathfrak{L}_n$ of $C^*(\mathfrak{T}_n)$ defined by

$$\mathfrak{L}_j := C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{Y_j}\big) \equiv C^*(\mathfrak{T}_n\big|_{Y_j}).$$

Note that $Y_j \backslash Y_{j-1} = X_j$ with $Y_{-1} := \emptyset$ is a disjoint union of $\frac{n!}{j!(n-j)!}$ copies of $\mathbb{Z}^{n-j}_{\ge} \times \{\infty^j\}$ each of which is gotten from one of the $\frac{n!}{j!(n-j)!}$ possible selections of exactly j of the n components of $\overline{\mathbb{Z}}^n_{\ge}$. With each such copy of $\mathbb{Z}^{n-j}_{\ge} \times \{\infty^j\}$ clearly a closed invariant subset of $Y_j \backslash Y_{j-1}$, these $\frac{n!}{j!(n-j)!}$ copies of $\mathbb{Z}^{n-j}_{\ge} \times \{\infty^j\}$ are open invariant subsets of $Y_j \backslash Y_{j-1}$, and hence

$$\begin{aligned} C^*(\mathfrak{T}_n\big|_{Y_j \backslash Y_{j-1}}) &= \oplus^{\frac{n!}{j!(n-j)!}} C^*\big((\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\mathbb{Z}^{n-j}_{\ge} \times \{\infty^j\}}\big) \\ &= \oplus^{\frac{n!}{j!(n-j)!}} C^*\Big(\big((\mathbb{Z}^{n-j} \ltimes \mathbb{Z}^{n-j})\big|_{\mathbb{Z}^{n-j}_{\ge}}\big) \times \mathbb{Z}^j\Big) \\ &= \oplus^{\frac{n!}{j!(n-j)!}} \big(\mathcal{K}(\ell^2(\mathbb{Z}^{n-j}_{\ge})) \otimes C(\mathbb{T}^j)\big). \end{aligned}$$

Thus with $\mathfrak{L}_j = C^*(\mathfrak{T}_n\big|_{Y_j})$ and $\mathfrak{L}_{j-1} = C^*(\mathfrak{T}_n\big|_{Y_{j-1}})$, we get

$$\mathfrak{L}_j / \mathfrak{L}_{j-1} \cong C^*(\mathfrak{T}_n\big|_{Y_j \backslash Y_{j-1}}) \cong \oplus^{\frac{n!}{j!(n-j)!}} \big(\mathcal{K}(\ell^2(\mathbb{Z}^{n-j}_{\ge})) \otimes C(\mathbb{T}^j)\big). \qquad \square$$

Corollary 1. *There is a finite composition sequence of closed ideals*

$$C(\mathbb{S}^{2n-1}_H) \equiv C^*(\mathfrak{G}_n) \equiv \mathfrak{J}_n \rhd \mathfrak{J}_{n-1} \rhd \cdots \rhd \mathfrak{J}_1 \rhd \mathfrak{J}_0 \equiv \{0\}$$

such that for $1 \le j \le n$*,*

$$\mathfrak{J}_j / \mathfrak{J}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \big(\mathcal{K}(\ell^2(\mathbb{Z}^{n-j}_{\ge})) \otimes C(\mathbb{T}^j)\big).$$

Proof. With $\mathfrak{L}_0 = \mathcal{K}(\ell^2(\mathbb{Z}^n_{\ge}))$ and hence $C^*(\mathfrak{T}_n)/\mathfrak{L}_0 \cong C(\mathbb{S}^{2n-1}_H)$, we simply take $\mathfrak{J}_j := \mathfrak{L}_j / \mathfrak{L}_0$. $\square$

The above composition sequences lead to the straightforward estimates

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \le \operatorname{sr}\big(C(\mathbb{S}^{2n-1}_H)\big) \le \operatorname{sr}(\mathcal{T}^{\otimes n}) \le \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

and

$$\operatorname{csr}(\mathcal{T}^{\otimes n}) \le \operatorname{csr}\big(C(\mathbb{S}^{2n-1}_H)\big) \le \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

for all $n \ge 1$, based on the general rules established in [21] that:

(i) $\mathrm{sr}(\mathfrak{A} \otimes \mathcal{K}) = \min\{2, \mathrm{sr}(\mathfrak{A})\}$;

(ii) for any closed ideal $\mathfrak{I}$ of a C*-algebra $\mathfrak{A}$,

$$\max\{\mathrm{sr}(\mathfrak{A}/\mathfrak{I}), \mathrm{sr}(\mathfrak{I})\} \leq \mathrm{sr}(\mathfrak{A}) \leq \max\{\mathrm{sr}(\mathfrak{A}/\mathfrak{I}), \mathrm{sr}(\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\};$$

and

(iii) $\mathrm{sr}(C(X)) = \lfloor \frac{n}{2} \rfloor + 1$ for any n-dimensional CW-complex X,

and the rule [16, 17, 25] that, for any closed ideal $\mathfrak{I}$ of a C*-algebra $\mathfrak{A}$,

(iv) $\mathrm{csr}(\mathfrak{A} \otimes \mathcal{K}) \leq 2$ (with $\mathrm{csr}(\mathcal{K}) = 1$); and

(v) $\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{I})\}$.

Indeed, for $n > 1$, applying (i)–(ii) and (iv)–(v) to the short exact sequences

$$0 \to \mathfrak{I}_{j-1} \to \mathfrak{I}_j \to \mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \oplus^{\frac{n!}{j!(n-j)!}} \big(\mathcal{K}(\ell^2(\mathbb{Z}_{\geq}^{n-j})) \otimes C(\mathbb{T}^j)\big) \to 0$$

inductively for j increasing from 1 to $n-1$. Starting with the exact sequence

$$0 \to \mathcal{K}(\ell^2(\mathbb{Z}_{\geq}^{n})) \cong \mathfrak{I}_0 \to \mathfrak{I}_1 \to \mathfrak{I}_1/\mathfrak{I}_0 \cong \oplus^n \big(\mathcal{K}(\ell^2(\mathbb{Z}_{\geq}^{n-1})) \otimes C(\mathbb{T})\big) \to 0$$

for $j = 1$, we get

$$\mathrm{csr}(\mathfrak{I}_j), \mathrm{sr}(\mathfrak{I}_j) \leq 2$$

for all $1 \leq j \leq n-1$. In particular,

$$\mathrm{csr}(\mathfrak{I}_{n-1}), \mathrm{sr}(\mathfrak{I}_{n-1}) \leq 2,$$

which is also valid for $n = 1$ since $\mathfrak{I}_0 \cong \mathcal{K}(\ell^2(\mathbb{Z}_{\geq}^{n}))$. Then with

$$\mathrm{csr}\big(C(\mathbb{T}^n)\big) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

by homotopy theory [33], we get

$$\mathrm{csr}(\mathcal{T}^{\otimes n}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

and

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \mathrm{sr}(\mathcal{T}^{\otimes n}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

by further applying (ii)–(iii) and (v) to the short exact sequence

$$0 \to \mathfrak{I}_{n-1} \to \mathfrak{I}_n \equiv \mathcal{T}^{\otimes n} \to \mathfrak{I}_n/\mathfrak{I}_{n-1} \cong C(\mathbb{T}^n) \to 0.$$

A similar argument yields

$$\mathrm{csr}\big(C(\mathbb{S}_H^{2n-1})\big) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

and

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \operatorname{sr}\left(C(\mathbb{S}_H^{2n-1})\right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1,$$

with the inequality

$$\operatorname{sr}\left(C(\mathbb{S}_H^{2n-1})\right) \leq \operatorname{sr}(\mathfrak{T}^{\otimes n})$$

obviously valid by (ii). Also,

$$\operatorname{csr}(\mathfrak{T}^{\otimes n}) \leq \operatorname{csr}\left(C(\mathbb{S}_H^{2n-1})\right)$$

by (iv)–(v).

Such an estimate determining $\operatorname{sr}(\mathfrak{T}^{\otimes n})$ sharply for even n, and up to an error of 1 for odd $n > 1$, as stated above, was first obtained by G. Nagy in [16] and then sharpened to the exact value

$$\operatorname{sr}(\mathfrak{T}^{\otimes n}) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \left(\text{and hence } \operatorname{sr}\left(C(\mathbb{S}_H^{2n-1})\right) = \left\lfloor \frac{n}{2} \right\rfloor + 1\right)$$

for general $n > 1$ by Nistor in [17] which also gives $\operatorname{csr}(\mathfrak{T}^{\otimes n}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$. We summarize these results as follows.

Proposition 2. *For all $n > 1$,*

$$\operatorname{sr}\left(C(\mathbb{S}_H^{2n-1})\right) = \operatorname{sr}(\mathfrak{T}^{\otimes n}) = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

and

$$\operatorname{csr}(\mathfrak{T}^{\otimes n}) \leq \operatorname{csr}\left(C(\mathbb{S}_H^{2n-1})\right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Corollary 2. *For any $n > 1$ and any $k \geq \left\lfloor \frac{n}{2} \right\rfloor + 3$, the topological group $\mathrm{GL}_k(\mathfrak{T}^{\otimes n})$ is connected.*

Proof. By the Künneth formula [3] for K-groups, we get $K_1(\mathfrak{T}^{\otimes n}) = 0$, since $K_1(\mathfrak{T}) = 0$ is well known. So, by the theorem [21] that

$$K_1(\mathfrak{A}) \cong \mathrm{GL}_k(\mathfrak{A}) / \mathrm{GL}_k^0(\mathfrak{A})$$

for any unital C*-algebra $\mathfrak{A}$ with $k \geq \operatorname{sr}(\mathfrak{A}) + 2$, we get

$$\mathrm{GL}_k(\mathfrak{T}^{\otimes n}) = \mathrm{GL}_k^0(\mathfrak{T}^{\otimes n})$$

for any $k \geq \left\lfloor \frac{n}{2} \right\rfloor + 3 \geq \operatorname{sr}(\mathfrak{T}^{\otimes n}) + 2$. □

Note that the above statement holds for the case of $n = 1$, since $\mathrm{GL}_k(\mathfrak{T})$ is connected for all $k \geq 1$ in the case of $n = 1$ by the index theory of Toeplitz operators for the unit disk $\mathbb{D}$.

5. Projective modules over $\mathfrak{T}^{\otimes n}$

Before proceeding to study finitely generated projective modules over $\mathfrak{T}^{\otimes n}$, we now point out a structure of $\mathfrak{T}^{\otimes n}$ which facilitates some inductive procedures for the study of such modules.

For all $n \in \mathbb{N}$, the topological groupoid $\mathfrak{T}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1}\times\{\infty\}}$ is isomorphic to the product topological groupoid $\mathfrak{T}_{n-1}\times\mathbb{Z}$, while the topological groupoid $\mathfrak{T}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1}\times\mathbb{Z}_{\geq}}$ is isomorphic to the product topological groupoid $\mathfrak{T}_{n-1}\times(\mathbb{Z}\ltimes\mathbb{Z})|_{\mathbb{Z}_{\geq}}$, where the closed subset $\overline{\mathbb{Z}}_{\geq}^{n-1}\times\{\infty\}$ and its open complement $\overline{\mathbb{Z}}_{\geq}^{n-1}\times\mathbb{Z}_{\geq}$ in the unit space $\overline{\mathbb{Z}}_{\geq}^{n}$ of $\mathfrak{T}_n$ are invariant. (Here it is understood that when $n-1=0$, the first factor $\overline{\mathbb{Z}}_{\geq}^{n-1}$ is dropped.) Hence we get the short exact sequence of C*-algebras

$$0\to C^*\big(\mathfrak{T}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1}\times\mathbb{Z}_{\geq}}\big)\cong\mathfrak{T}^{\otimes n-1}\otimes\mathcal{K}\big(\ell^2(\mathbb{Z}_{\geq})\big)$$
$$\to C^*(\mathfrak{T}_n)\equiv\mathfrak{T}^{\otimes n}\xrightarrow{\kappa_n}C^*\big(\mathfrak{T}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1}\times\{\infty\}}\big)\cong\mathfrak{T}^{\otimes n-1}\otimes C(\mathbb{T})\to 0$$

with $\mathfrak{T}^{\otimes 0}:=\mathbb{C}$. Furthermore, the quotient maps κ_n for $n\in\mathbb{N}$ resulting from a groupoid restriction satisfy the commuting diagram

$$\begin{array}{ccccc}
M_k(\mathfrak{T}^{\otimes n}) & \overset{\kappa_n}{\to} & M_k\big(\mathfrak{T}^{\otimes n-1}\otimes C(\mathbb{T})\big) & \equiv & M_k(\mathfrak{T}^{\otimes n-1})\otimes C(\mathbb{T}) \\
\downarrow\sigma_n & \circlearrowright & \downarrow\sigma_{n-1}\otimes\mathrm{id} & & \downarrow\sigma_{n-1}\otimes\mathrm{id} \\
M_k\big(C(\mathbb{T}^n)\big) & \overset{\equiv}{\to} & M_k\big(C(\mathbb{T}^{n-1})\otimes C(\mathbb{T})\big) & \equiv & M_k\big(C(\mathbb{T}^{n-1})\big)\otimes C(\mathbb{T})
\end{array}$$

where $\equiv$ stands for a canonical isomorphism and $\sigma_0:=\mathrm{id}_{\mathbb{C}}$.

To classify the isomorphism classes of finitely generated projective $\mathfrak{T}^{\otimes n}$-modules E, or equivalently the equivalence classes of idempotents $P\in M_\infty(\mathfrak{T}^{\otimes n})$ over $\mathfrak{T}^{\otimes n}$, we first define the rank of (the class of) E or P as the classical rank of (the isomorphism class of) the vector bundle corresponding to (the class of) the $C(\mathbb{T}^n)$-module $C(\mathbb{T}^n)\otimes_{\mathfrak{T}^{\otimes n}}E$ or the idempotent $\sigma_n(P)$ over $C(\mathbb{T}^n)$.

The set of equivalence classes of idempotents $P\in M_\infty(\mathfrak{T}^{\otimes n})$ equipped with the binary operation $\boxplus$ becomes an abelian graded monoid

$$\mathfrak{P}(\mathfrak{T}^{\otimes n})=\sqcup_{m=0}^{\infty}\mathfrak{P}_m(\mathfrak{T}^{\otimes n}),$$

where $\mathfrak{P}_m(\mathfrak{T}^{\otimes n})$ is the set of all (equivalence classes of) idempotents over $\mathfrak{T}^{\otimes n}$ of rank m, and

$$\mathfrak{P}_m(\mathfrak{T}^{\otimes n})\boxplus\mathfrak{P}_l(\mathfrak{T}^{\otimes n})\subset\mathfrak{P}_{m+l}(\mathfrak{T}^{\otimes n})$$

for $m,l\geq 0$. Clearly $\mathfrak{P}_0(\mathfrak{T}^{\otimes n})$ is a submonoid of $\mathfrak{P}(\mathfrak{T}^{\otimes n})$.

Next we define a submonoid of $\mathfrak{P}(\mathfrak{T}^{\otimes n})$ generated by "standard" type of idempotents, which turns out to contain (equivalence classes of) all idempotents of sufficiently high ranks, and then classify its elements.

Note that each permutation Θ on $\{1,2,\ldots,n\}$ induces canonically a C*-algebra automorphism, still denoted as Θ by abuse of notation, on $\mathcal{T}^{\otimes n}$ by permuting the indices of the factors in $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \mathcal{T}^{\otimes n}$ for $a_i \in \mathcal{T}$. A permutation Θ on $\{1,2,\ldots,n\}$ is called a $(j, n-j)$-shuffle on $\{1,2,\ldots,n\}$ if

$$\Theta(1) < \Theta(2) < \cdots < \Theta(j) \quad \text{and} \quad \Theta(j+1) < \Theta(j+2) < \cdots < \Theta(n).$$

Some basic projections over $\mathcal{T}^{\otimes n}$ are given by $\Theta(P_{j,l})$ where

$$P_{j,l} := \boxplus^l \big((\otimes^j I) \otimes (\otimes^{n-j} P_1)\big) \in M_l(\mathcal{T}^{\otimes n})$$

for $l \geq 0$ and $0 \leq j \leq n$ (in particular, $P_{n,m} \equiv \boxplus^m (\otimes^n I) \equiv \boxplus^m \widetilde{I}$ for the unit $\widetilde{I}$ of $\mathcal{T}^{\otimes n}$), and Θ is (the automorphism defined by) a $(j, n-j)$-shuffle on $\{1,2,\ldots,n\}$. Note that $\Theta(P_{j,l}) = \Theta(\boxplus^l P_{j,1}) = \boxplus^l \Theta(P_{j,1})$,

$$\Theta(P_{j,l}) \boxplus \Theta(P_{j,l'}) \sim \Theta(P_{j,l+l'}),$$

and $(\otimes^j I) \otimes (\otimes^{n-j-1} P_1) \otimes P_l \sim P_{j,l}$ over $\mathcal{T}^{\otimes n}$ since $P_l \sim \boxplus^l P_1$ over $\mathcal{K}^+ \subset \mathcal{T}$. Furthermore,

$$\sigma_n(\Theta(P_{j,l})) = \begin{cases} 0, & \text{if } 0 \leq j \leq n-1, \\ \boxplus^l 1, & \text{if } j = n, \end{cases}$$

and hence $\Theta(P_{j,l}) \in \mathfrak{P}_0(\mathcal{T}^{\otimes n})$ if $j < n$ and $\Theta(P_{n,l}) = P_{n,l} \in \mathfrak{P}_l(\mathcal{T}^{\otimes n})$, where $1 \in C(\mathbb{T}^n)$ is the constant function 1 on $\mathbb{T}^n$. So the set $\mathfrak{P}'_0(\mathcal{T}^{\otimes n}) \subset \mathfrak{P}_0(\mathcal{T}^{\otimes n})$ consisting of (the equivalence classes of) all possible $\boxplus$-sums of $\Theta(P_{j,l})$ with $l \geq 0$ and Θ a $(j, n-j)$-shuffle on $\{1,2,\ldots,n\}$ for $0 \leq j \leq n-1$ is a submonoid of $\mathfrak{P}_0(\mathcal{T}^{\otimes n})$. For $m \geq 1$, we define a singleton

$$\mathfrak{P}'_m(\mathcal{T}^{\otimes n}) := \{P_{n,m} \equiv \boxplus^m \widetilde{I}\} \subset \mathfrak{P}_m(\mathcal{T}^{\otimes n}),$$

where $\widetilde{I}$ denotes the identity element of $\mathcal{T}^{\otimes n}$. Clearly $\sqcup_{m=1}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ is also a submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$.

We define a partial ordering $\prec$ on the collection

$$\Omega := \{(j, \Theta) : 0 \leq j \leq n \text{ and } \Theta \text{ is a } (j, n-j)\text{-shuffle}\}$$

by the condition that $(j', \Theta') \prec (j, \Theta)$ if and only if

$$\Theta\big(\{1,2,\ldots,j\}\big) \supsetneqq \Theta'\big(\{1,2,\ldots,j'\}\big)$$

(and hence $j > j'$). Here $\{1,2,\ldots,0\} \equiv \emptyset$ is understood. Note that $\mathrm{id}_{\{1,2,\ldots,n\}}$ is a $(j, n-j)$-shuffle for every j, and $(n, \mathrm{id}_{\{1,2,\ldots,n\}})$ is the greatest element while $(0, \mathrm{id}_{\{1,2,\ldots,n\}})$ is the smallest element in Ω with respect to $\prec$.

Proposition 3. *$\mathfrak{P}'(\mathcal{T}^{\otimes n}) = \sqcup_{m=0}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ is a graded submonoid of $\mathfrak{P}(\mathcal{T}^{\otimes n})$ and its monoid structure is explicitly determined by that for any $l, l' > 0$ and any $(j', \Theta') \prec (j, \Theta)$ in Ω,*

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l}).$$

Proof. Note that since $\mathfrak{P}'_0(\mathcal{T}^{\otimes n})$ and $\boxplus_{m=1}^{\infty}\mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ are submonoids of $\mathfrak{P}(\mathcal{T}^{\otimes n})$, the set $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ is a submonoid if

$$\Theta(P_{n,m}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{n,m})$$

holds for all $m > 0$ and all $\Theta'(P_{j',l'})$ with $j' \leq n-1$. Since $(n, \mathrm{id}_{\{1,2,\ldots,n\}})$ is the greatest element in Ω, it remains to show that

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l})$$

for $n \geq j > j' \geq 0$ with $\Theta\big(\{1,2,\ldots,j\}\big) \supset \Theta'\big(\{1,2,\ldots,j'\}\big)$ and $l, l' > 0$.

Note that for $\Theta\big(\{1,2,\ldots,j\}\big) \supset \Theta'\big(\{1,2,\ldots,j'\}\big)$, there exists a permutation Θ'' (not necessarily a shuffle) on $\{1,2,\ldots,n\}$ such that

$$\Theta''\big(\Theta(P_{j,l})\big) = P_{j,l} \quad \text{and} \quad \Theta''\big(\Theta'(P_{j',l'})\big) = P_{j',l'}.$$

(In fact, one can find a permutation Θ'' such that $\Theta''\Theta$ fixes each of $j+1,\ldots,n$, and $\Theta''\Theta'$ is each of $1,2,\ldots,j'$.) So it suffices to prove that

$$P_{j,l} \boxplus P_{j',l'} \sim P_{j,l},$$

whenever $j > j'$ and $l, l' > 0$. Furthermore, since $P_{j,l} = \boxplus^l P_{j,1}$, we only need to show that $P_{j,1} \boxplus P_{j',1} \sim P_{j,1}$ for $j > j'$.

Note that $U(P_1 \boxplus I)U^* = 0 \boxplus I$ in $M_2(\mathcal{T})$ for the unitary

$$U := e_{11} \otimes \mathcal{S}^* + e_{22} \otimes \mathcal{S} + e_{21} \otimes e_{11} \in M_2(\mathbb{C}) \otimes \mathcal{T} \equiv M_2(\mathcal{T}),$$

where $\mathcal{S} \in \mathcal{T}$ is the (forward) unilateral shift on $\ell^2(\mathbb{Z}_{\geq})$. So,

$$\begin{aligned} P_{j,1} \boxplus P_{j-1,1} &= \big((\otimes^j I) \otimes (\otimes^{n-j} P_1)\big) \boxplus \big((\otimes^{j-1} I) \otimes (\otimes^{n-j+1} P_1)\big) \\ &= (\otimes^{j-1} I) \otimes (I \boxplus P_1) \otimes (\otimes^{n-j} P_1) \sim (\otimes^{j-1} I) \otimes I \otimes (\otimes^{n-j} P_1) \\ &= P_{j,1}. \end{aligned}$$

Thus by iteration of this result, we can "expand" $P_{j,1}$ to get for any $0 \leq k < j$,

$$P_{j,1} \sim P_{j,1} \boxplus P_{j-1,1} \boxplus \cdots \boxplus P_{k,1},$$

and hence

$$P_{j,1} \boxplus P_{j',1} \sim P_{j,1} \boxplus P_{j-1,1} \boxplus \cdots \boxplus P_{j'+1,1} \boxplus P_{j',1} \sim P_{j,1}. \qquad \square$$

For each $(j, \Theta) \in \Omega$, let $X_\Theta \subset \overline{\mathbb{Z}}^n_{\geq}$ be the invariant closed subset of the unit space of $\mathfrak{T}_n$ consisting of $z \in \overline{\mathbb{Z}}^n_{\geq}$ with $z_k = \infty$ for all $k \in \Theta\big(\{1,2,\ldots,j\}\big)$, and let

$$\sigma_{(j,\Theta)}\colon C^*(\mathfrak{T}_n) \to C^*(\mathfrak{T}_n|_{X_\Theta}) \cong C(\mathbb{T}^j) \otimes \mathcal{T}^{\otimes n-j} \subset C(\mathbb{T}^j) \otimes \mathcal{B}\big(\ell^2(\mathbb{Z}^{n-j}_{\geq})\big)$$

be the canonical quotient map, where the isomorphism implicitly involves a rearrangement of factors by the inverse permutation Θ^{-1}. Here as before, $\mathbb{T}^0$ is a singleton. Defining $\rho_{(j,\Theta)}(P)$ for an idempotent P over $C^*(\mathfrak{T}_n)$ as the rank of the projection operator $\sigma_{(j,\Theta)}(P)(t) \in \mathcal{B}(\ell^2(\mathbb{Z}_{\geq}^{n-j}))$ for any $t \in \mathbb{T}^j$, which depends only on the equivalence class of P, we get a well-defined monoid homomorphism

$$\rho_{(j,\Theta)}\colon \big(\mathfrak{P}(\mathcal{T}^{\otimes n}), \boxplus\big) \to \big(\mathbb{Z}_{\geq} \cup \{\infty\}, +\big).$$

A (finite) $\boxplus$-sum of (the equivalence classes of) projections $\Theta(P_{j,l})$ indexed by some $(j,\Theta) \in \Omega$ that are mutually unrelated by $\prec$ with $l \equiv l_{(j,\Theta)} > 0$ depending on (j,Θ) is called a reduced $\boxplus$-sum of standard projections over $\mathcal{T}^{\otimes n}$. It is understood that an "empty" $\boxplus$-sum represents the zero projection and is a reduced $\boxplus$-sum. Two reduced $\boxplus$-sums are called different when they have different sets of (mutually $\prec$-unrelated) indices $(j,\Theta) \in \Omega$ or have different weight functions l of (j,Θ). We are going to show that different reduced $\boxplus$-sums are inequivalent projections. Clearly each projection $\Theta(P_{j,l})$ with $(j,\Theta) \in \Omega$ and $l > 0$ is a reduced $\boxplus$-sum.

Theorem 1. *The submonoid $\mathfrak{P}'(\mathcal{T}^{\otimes n}) = \sqcup_{m=0}^{\infty} \mathfrak{P}'_m(\mathcal{T}^{\otimes n})$ of $\mathfrak{P}(\mathcal{T}^{\otimes n})$ consists exactly of reduced $\boxplus$-sums of standard projections over $\mathcal{T}^{\otimes n}$, and different reduced $\boxplus$-sums are mutually inequivalent projections. Furthermore, the monoid homomorphism*

$$\rho\colon P \in \mathfrak{P}'(\mathcal{T}^{\otimes n}) \mapsto \prod_{(j,\Theta)\in\Omega} \rho_{(j,\Theta)}(P) \in \prod_{(j,\Theta)\in\Omega} \overline{\mathbb{Z}}_{\geq}$$

is injective, with $\rho_{(j,\Theta)}(\Theta(P_{j,l})) = l \in \mathbb{N}$.

Proof. By definition, $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ consists of $\boxplus$-sums of (the equivalence classes of) projections $\Theta(P_{j,l})$ with $(j,\Theta) \in \Omega$ and $l > 0$. Since

$$\Theta(P_{j,l}) + \Theta(P_{j,l'}) \sim \Theta(P_{j,l+l'}),$$

we only need to consider in the following those $\boxplus$-sums, in which all summands $\Theta(P_{j,l})$ are indexed by distinct $(j,\Theta) \in \Omega$ with l depending on (j,Θ). For any such a $\boxplus$-sum, using the property that

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l})$$

for any $(j',\Theta') \prec (j,\Theta)$, we can remove one by one those $\boxplus$-summands $\Theta'(P_{j',l'})$ with (j',Θ') dominated by the index of another summand, without changing the equivalence class, until we reach a $\boxplus$-sum of $\Theta(P_{j,l})$ with $(j,\Theta) \in \Omega$ mutually unrelated by $\prec$, i.e. a reduced $\boxplus$-sum. So $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ consists of the reduced $\boxplus$-sums.

Note that for $(j,\Theta) \in \Omega$ and $l > 0$,

$$\begin{aligned}\sigma_{(j,\Theta)}\big(\Theta(P_{j,l})\big) &= \sigma_{(j,\Theta)}\big(\boxplus^l\, \Theta\big((\otimes^j I) \otimes (\otimes^{n-j} P_1)\big)\big)\\ &= 1 \otimes \big(\boxplus^l\, (\otimes^{n-j} P_1)\big) \in C(\mathbb{T}^j) \otimes \big(\boxplus^l\, \mathcal{B}(\ell^2(\mathbb{Z}_{\geq}^{n-j}))\big),\end{aligned}$$

and hence,

$$\rho_{(j,\Theta)}\big(\Theta(P_{j,l})\big) = l \in \mathbb{N}$$

the operator rank of $\boxplus^l(\otimes^{n-j} P_1) \in \mathfrak{B}\big(\oplus^l \ell^2(\mathbb{Z}_{\geq}^{n-j})\big)$. But for $(j', \Theta') \neq (j, \Theta)$,

$$\rho_{(j,\Theta)}\big(\Theta'(P_{j',l'})\big) := \begin{cases} \infty, & \text{if } (j,\Theta) \prec (j',\Theta'), \\ 0, & \text{otherwise,} \end{cases}$$

because either $\sigma_{(j,\Theta)}\big(\Theta'(P_{j',l'})\big) = 0$ when

$$\Theta\big(\{1,2,\ldots,j\}\big) \backslash \Theta'\big(\{1,2,\ldots,j'\}\big) \neq \emptyset,$$

or $\sigma_{(j,\Theta)}\big(\Theta'(P_{j',l'})\big)$ is an infinite-dimensional projection when

$$\Theta'\big(\{1,2,\ldots,j'\}\big) \supset \Theta\big(\{1,2,\ldots,j\}\big)$$

(but $\Theta\big(\{1,2,\ldots,j\}\big) \neq \Theta'\big(\{1,2,\ldots,j'\}\big)$ since $(j',\Theta') \neq (j,\Theta)$), i.e. when

$$(j,\Theta) \prec (j',\Theta').$$

For a reduced $\boxplus$-sum P of $\Theta'(P_{j',l'})$ indexed by (j',Θ') in some subset $A \subset \Omega$, the (j,Θ)-component of $\rho(P)$ is

$$\sum_{(j',\Theta')\in A} \rho_{(j,\Theta)}\big(\Theta'(P_{j',l'})\big) \begin{cases} = l \in \mathbb{N} & \text{if } (j,\Theta) \in A \text{ with } \Theta(P_{j,l}) \text{ a summand of } P, \\ \in \{0,\infty\} & \text{otherwise,} \end{cases}$$

for any $(j,\Theta) \in \Omega$, since if $(j,\Theta) \in A$ then (j,Θ) is $\prec$-unrelated to any other $(j',\Theta') \in A$. So $\rho(P)$ completely determines the summands of a reduced $\boxplus$-sum P, namely, P is the $\boxplus$-sum of exactly those $\Theta(P_{j,l})$ with l equal to the (j,Θ)-component of $\rho(P)$ that is a strictly positive integer. Since $\mathfrak{P}'(\mathcal{T}^{\otimes n})$ consists of reduced $\boxplus$-sums, this also shows that the clearly well-defined monoid homomorphism ρ is injective.

Thus if $P \sim P'$ for two reduced $\boxplus$-sums P and P' and hence $\rho(P) = \rho(P')$, then the summands of P and P' are exactly the same, i.e. P and P' are the same reduced $\boxplus$-sum. So different reduced $\boxplus$-sums are mutually inequivalent projections. □

Proposition 4. $\mathfrak{P}(\mathcal{T}) = \mathfrak{P}'(\mathcal{T})$. *More concretely,*

$$\mathfrak{P}(\mathcal{T}) \cong \{(0,l) : l \in \mathbb{Z}_{\geq}\} \cup \{(m,\infty) : m > 0\} \subset \overline{\mathbf{Z}}_{\geq}^2,$$

where $\overline{\mathbf{Z}}_{\geq}^2$ *is equipped with the canonical monoid structure.*

Proof. It suffices to show that any element of $\mathfrak{P}_0(\mathcal{T}) \equiv \mathfrak{P}_0(\mathcal{T}^{\otimes 1})$ is of the form $P_{0,l}$ (realized as $(0,l) \in \overline{\mathbf{Z}}_{\geq}^2$) and any element of $\mathfrak{P}_m(\mathcal{T}) \equiv \mathfrak{P}_m(\mathcal{T}^{\otimes 1})$ for $m \in \mathbb{N}$ is of the form $P_{1,m}$ (realized as $(m,\infty) \in \overline{\mathbf{Z}}_{\geq}^2$).

The argument sketched below is similar to one used in [29].

Since any complex vector bundle over $\mathbb{T}$ is trivial, any idempotent over $C(\mathbb{T})$ is equivalent to the standard projection $1 \otimes P_m \in C(\mathbb{T}) \otimes M_\infty(\mathbb{C})$ for some $m \in \mathbb{Z}_\geq$. So for any idempotent $P \in M_\infty(\mathcal{T})$ over $\mathcal{T}$, there is some $U \in \mathrm{GL}_\infty(C(\mathbb{T}))$ such that

$$U\sigma(P)U^{-1} = 1 \otimes P_m = \sigma(\boxplus^m I)$$

for some $m \in \mathbb{Z}_\geq$, where I is the identity of $\mathcal{K}^+ \subset \mathcal{T}$. Hence,

$$VPV^{-1} - \boxplus^m I \in M_\infty(\mathcal{K})$$

for any lift $V \in \mathrm{GL}_\infty(\mathcal{T})$ (which exists) of $U \boxplus U^{-1} \in \mathrm{GL}^0_\infty(C(\mathbb{T}))$ along σ. Replacing P by the equivalent VPV^{-1}, we may assume that

$$P \in (\boxplus^m I) + M_{k-1}(\mathcal{K}) \subset M_{k-1}(\mathcal{K}^+)$$

for some large $k \geq m+1$. Now since $M_\infty(\mathbb{C})$ is dense in $\mathcal{K}$, there is an idempotent

$$Q \in (\boxplus^m I) + M_{k-1}(M_N(\mathbb{C}))$$

sufficiently close to and hence equivalent to P for some large N. So replacing P by Q, we may assume that

$$K := P - \boxplus^m I \in M_{k-1}(M_N(\mathbb{C})).$$

Rearranging the entries of $P \equiv K + \boxplus^m I \in M_{k-1}(\mathcal{T}) \subset M_k(\mathcal{T})$ via conjugation by the unitary

$$U_{k,N} := \sum_{j=1}^{k-1} \left(e_{jj} \otimes (\mathcal{S}^*)^N + e_{kj} \otimes \left(\mathcal{S}^{(j-1)N} P_N\right)\right) + e_{kk} \otimes \mathcal{S}^{(k-1)N} \in M_k(\mathbb{C}) \otimes \mathcal{T} \equiv M_k(\mathcal{T})$$

we get

$$U_{k,N} P U_{k,N}^{-1} \equiv U_{k,N}(P \boxplus 0)U_{k,N}^{-1} = \left((\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0)\right) \boxplus R$$

for some $R \in M_{(k-1)N}(\mathbb{C}) \subset \mathcal{K} \subset \mathcal{T}$ which must be an idempotent. Since any idempotent in $\mathcal{K}$ is equivalent over $\mathcal{K}^+$ to a standard projection P_l, we get

$$P \sim \left((\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0)\right) \boxplus P_l$$

for some $l \in \mathbb{Z}_\geq$.

If $m = 0$, then clearly $P \sim P_l$. Since it is well known that P_l and $\boxplus^l P_1 \equiv P_{0,l}$ are equivalent over $\mathcal{K}^+$ and hence over $\mathcal{T} \supset \mathcal{K}^+$, we get $P \sim P_{0,l}$.

If $m \in \mathbb{N}$, then we can rearrange entries via conjugation by the unitary

$$U_l := e_{11} \otimes \mathcal{S}^l + e_{1k} \otimes P_l + \sum_{j=2}^{k-1} e_{jj} \otimes I + e_{kk} \otimes (\mathcal{S}^*)^l \in M_k(\mathbb{C}) \otimes \mathcal{T} \equiv M_k(\mathcal{T})$$

to get

$$U_l\left(\left((\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0)\right) \boxplus P_l\right)U_l^{-1} = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0) \equiv \boxplus^m I \equiv P_{1,m}. \ \square$$

Theorem 2. *For $n > 1$ and $m > 0$, if*

$$\mathfrak{P}_m(\mathfrak{T}^{\otimes n-1}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n-1}) \equiv \{\boxplus^m (\otimes^{n-1} I)\}$$

and $\mathrm{GL}_m(\mathfrak{T}^{\otimes n-1})$ *is connected, then* $\mathfrak{P}_m(\mathfrak{T}^{\otimes n}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n})$.

Proof. In this proof, we use I and $\widetilde{I}$ to denote respectively the identity elements of $\mathfrak{T}^{\otimes n-1}$ and $\mathfrak{T}^{\otimes n}$.

Let $P \in \mathfrak{P}_m(\mathfrak{T}^{\otimes n})$. The idempotent $\kappa_n(P)$ over $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ satisfies that for any $z \in \mathbb{T}$,

$$\sigma_{n-1}\big(\kappa_n(P)(z)\big) = \sigma_n(P)(\cdot, z) \in M_\infty\big(C(\mathbb{T}^{n-1})\big),$$

which is of rank m pointwise, and hence

$$\kappa_n(P)(z) \in \mathfrak{P}_m(\mathfrak{T}^{\otimes n-1}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n-1}),$$

i.e. $\kappa_n(P)(z) \sim \boxplus^m I$ over $\mathfrak{T}^{\otimes n-1}$. In particular, there is a continuous idempotent-valued path

$$\gamma\colon [0, 1] \to M_k(\mathfrak{T}^{\otimes n-1})$$

for k sufficiently large going from the idempotent $\kappa_n(P)(1)$ to $(\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$. Clearly we may assume that γ is locally constant at 1, say, $\gamma(t) = \boxplus^m I$ for $t \geq 1/2$. The concatenation of the path γ^{-1}, the loop $\kappa_n(P)$, and the path γ defines an idempotent-valued continuous loop

$$\Gamma\colon \mathbb{T} \to M_k(\mathfrak{T}^{\otimes n-1})$$

starting and ending at $\boxplus^m I$ with

$$\Gamma(e^{i\theta}) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0),$$

say, for all $\theta \in [3\pi/2, 2\pi]$ (and $[0, \pi/2]$), and is homotopic to the loop $\kappa_n(P)$ via idempotents, i.e. there is a path of idempotents in $M_k(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ from $\kappa_n(P)$ to Γ. Consequently, there is a continuous path of invertibles $U_t \in \mathrm{GL}_k(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ with $U_0 = I_k$ such that

$$U_1 \kappa_n(P) U_1^{-1} = \Gamma,$$

which can be lifted along κ_n to a continuous path of invertible $V_t \in \mathrm{GL}_k(\mathfrak{T}^{\otimes n})$ with $V_0 = I_k$ such that

$$\kappa_n(V_1 P V_1^{-1}) = \Gamma.$$

Replacing P by the equivalent idempotent $V_1 P V_1^{-1}$, we may now assume directly that the idempotent $\kappa_n(P)$ over $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is a continuous loop of idempotents in $M_k(\mathfrak{T}^{\otimes n-1})$ such that

$$\kappa_n(P)(e^{i\theta}) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$$

for all $\theta \in [3\pi/2, 2\pi]$. So there is a continuous path

$$\theta \in [0, 3\pi/2] \mapsto W_\theta \in \mathrm{GL}_k(\mathcal{T}^{\otimes n-1})$$

with $W_0 = I_k$ such that

$$W_\theta\big(\kappa_n(P)(e^{i\theta})\big)W_\theta^{-1} = \kappa_n(P)(1) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$$

for all $\theta \in [0, 3\pi/2]$. In particular,

$$W_{3\pi/2}\big((\boxplus^m I) \boxplus (\boxplus^{k-m} 0)\big) = \big((\boxplus^m I) \boxplus (\boxplus^{k-m} 0)\big) W_{3\pi/2},$$

and hence

$$W_{3\pi/2} = W' \boxplus W''$$

for some invertibles $W' \in \mathrm{GL}_m(\mathcal{T}^{\otimes n-1})$ and $W'' \in \mathrm{GL}_{k-m}(\mathcal{T}^{\otimes n-1})$.

By the connectedness assumption on $\mathrm{GL}_m(\mathcal{T}^{\otimes n-1})$, there is a continuous path

$$\alpha: [3\pi/2, 2\pi] \to \mathrm{GL}_m(\mathcal{T}^{\otimes n-1})$$

with $\alpha(3\pi/2) = W'$ and $\alpha(2\pi) = I_m$. Since by Künneth formula, $K_1(\mathcal{T}^{\otimes n-1}) = 0$ and hence $\mathrm{GL}_N(\mathcal{T}^{\otimes n-1})$ is connected for N sufficiently large, we may suitably increase the value of k by adding diagonal $\boxplus$-summands 0 to idempotents and diagonal $\boxplus$-summands I to invertibles, so that $\mathrm{GL}_{k-m}(\mathcal{T}^{\otimes n-1})$ is also connected and hence there is a continuous path

$$\beta: [3\pi/2, 2\pi] \to \mathrm{GL}_{k-m}(\mathcal{T}^{\otimes n-1})$$

with $\beta(3\pi/2) = W''$ and $\beta(2\pi) = I_{k-m}$. Now the function $\theta \mapsto W_\theta$ can be continuously extended to the whole interval $[0, 2\pi]$ by setting

$$W_\theta := \alpha(\theta) \boxplus \beta(\theta) \in \mathrm{GL}_k(\mathcal{T}^{\otimes n-1})$$

for $\theta \in [3\pi/2, 2\pi]$, giving rise to a well-defined continuous loop

$$W: e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in \mathrm{GL}_k(\mathcal{T}^{\otimes n-1}),$$

i.e. $W \in \mathrm{GL}_k(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$, satisfying

$$W(\kappa_n(P))W^{-1} = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0).$$

So the idempotent $\kappa_n(P)$ over $\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is equivalent to the idempotent $\boxplus^m I$.

Replacing P by the equivalent idempotent $\widetilde{W}(P \boxplus (\boxplus^k 0))\widetilde{W}^{-1}$ for any fixed lifting $\widetilde{W} \in \mathrm{GL}^0_{2k}(\mathcal{T}^{\otimes n})$ of $W \boxplus W^{-1} \in \mathrm{GL}^0_{2k}(\mathcal{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ along κ_n, we may now assume that

$$\kappa_n(P) = (\boxplus^m I) \boxplus (\boxplus^{2k-m} 0) = \kappa_n\big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big)$$

and proceed to show that $P \sim \boxplus^m \widetilde{I}$.

Note that

$$P - \big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big) \in M_{2k}(\mathcal{T}^{\otimes n-1} \otimes \mathcal{K}).$$

Since $M_\infty(\mathbb{C})$ is dense in $\mathcal{K}$, we may replace P by a suitable equivalent idempotent and assume that

$$K := P - \big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big) \in M_{2k}\big(\mathcal{T}^{\otimes n-1} \otimes M_N(\mathbb{C})\big) \subset M_{2k}(\mathcal{T}^{\otimes n}),$$

for some $N \in \mathbb{N}$.

Rearranging the entries of $P \equiv P \boxplus 0 \in M_{2k+1}(\mathcal{T}^{\otimes n-1} \otimes M_N(\mathbb{C}))$ via conjugation by the unitary

$$\begin{aligned} U_{k,N} := \sum_{j=1}^{2k} \big(e_{jj} \otimes (I \otimes \mathcal{S}^*)^N + e_{2k+1,j} \otimes (I \otimes \mathcal{S}^{(j-1)N} P_N)\big) \\ + e_{2k+1,2k+1} \otimes (I \otimes \mathcal{S}^{2kN}) \\ \in M_{2k+1}(\mathbb{C}) \otimes \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} \equiv M_{2k+1}(\mathcal{T}^{\otimes n}) \end{aligned}$$

we get

$$P \sim U_{k,N} P U_{k,N}^{-1} \equiv U_{k,N}(P \boxplus 0) U_{k,N}^{-1} = \big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big) \boxplus R$$

for some

$$R \in M_{2kN}(\mathcal{T}^{\otimes n-1}) \equiv \mathcal{T}^{\otimes n-1} \otimes M_{2kN}(\mathbb{C}) \subset \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} \equiv \mathcal{T}^{\otimes n},$$

which must be an idempotent over $\mathcal{T}^{\otimes n-1}$.

Since $K_0(\mathcal{T}^{\otimes n-1}) = \mathbb{Z}$ by the Künneth formula,

$$R \boxplus (\boxplus^r I) \sim \big(\boxplus^{r+[R]} I\big)$$

for a sufficiently large $r \in \mathbb{N}$ where $[R] \in \mathbb{Z}$ denotes the class of R in $K_0(\mathcal{T}^{\otimes n-1})$. So there is an invertible $U \in \mathrm{GL}_d(\mathcal{T}^{\otimes n-1})$ for some large $d \geq \max\{2kN + r, r + [R]\}$ such that

$$U\big(R \boxplus \big(\boxplus^{d-2kN-r} 0\big) \boxplus (\boxplus^r I)\big)U^{-1} = \big(\boxplus^{d-r-[R]} 0\big) \boxplus \big(\boxplus^{r+[R]} I\big).$$

With $m > 0$, we can rearrange entries via conjugation by the unitary

$$\begin{aligned} U_{d-r} := e_{11} \otimes (I \otimes \mathcal{S}^{d-r}) + e_{1,2k+1} \otimes I \otimes P_{d-r} \\ + \sum_{j=2}^{2k} e_{jj} \otimes \widetilde{I} + e_{2k+1,2k+1} \otimes (I \otimes \mathcal{S}^*)^{d-r} \\ \in M_{2k+1}(\mathbb{C}) \otimes \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} \equiv M_{2k+1}(\mathcal{T}^{\otimes n}) \end{aligned}$$

to get

$$P \sim U_{d-r}\big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0) \boxplus R\big)U_{d-r}^{-1} = R' \boxplus (\boxplus^{m-1}\widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

where

$$R' = \big(R \boxplus (\boxplus^{d-2kN-r} 0)\big) + (\widetilde{I} - I \otimes P_{d-r}) \in \widetilde{I} + \big(\mathfrak{T}^{\otimes n-1} \otimes M_{d-r}(\mathbb{C})\big) \subset \widetilde{I} + (\mathfrak{T}^{\otimes n-1} \otimes \mathfrak{K}) \subset \mathfrak{T}^{\otimes n-1} \otimes \mathfrak{T} = \mathfrak{T}^{\otimes n}.$$

Note that R' can be interpreted as

$$R \boxplus (\boxplus^{d-2kN-r} 0) \boxplus (\boxplus^{\infty} I) \in \mathfrak{T}^{\otimes n-1} \otimes \mathfrak{K}^+ \subset \mathfrak{T}^{\otimes n},$$

which when conjugated by the invertible $U \equiv U \boxplus (\boxplus^{\infty} I) \in \mathfrak{T}^{\otimes n-1} \otimes \mathfrak{K}^+ \subset \mathfrak{T}^{\otimes n}$ becomes

$$\big(\boxplus^{d-r-[R]} 0\big) \boxplus \big(\boxplus^{r+[R]} I\big) \boxplus (\boxplus^{\infty} I) = \widetilde{I} - I \otimes P_{d-r-[R]} \in \widetilde{I} + (\mathfrak{T}^{\otimes n-1} \otimes \mathfrak{K}) \subset \mathfrak{T}^{\otimes n}.$$

So we get

$$P \sim \big(\widetilde{I} - I \otimes P_{d-r-[R]}\big) \boxplus (\boxplus^{m-1}\widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

the latter of which when conjugated by $U_{d-r-[R]}^{-1}$ yields

$$\widetilde{I} \boxplus (\boxplus^{m-1}\widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

where $U_{d-r-[R]}$ is defined as U_{d-r} by replacing $d-r$ by $d-r-[R]$. Thus we get

$$P \sim (\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0) \equiv \boxplus^m \widetilde{I}. \qquad \square$$

Corollary 3. $\mathfrak{P}_m(\mathfrak{T}^{\otimes n}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n}) \equiv \{\boxplus^m \widetilde{I}\}$ *for all* $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$ *and any* $n \in \mathbb{N}$, *where* $\widetilde{I}$ *is the identity element of* $\mathfrak{T}^{\otimes n}$.

Proof. We prove the corollary by induction on $n \in \mathbb{N}$.

For $n = 1$, we already know that

$$\mathfrak{P}'_m(\mathfrak{T}^{\otimes n}) \equiv \mathfrak{P}_m(\mathfrak{T}^{\otimes n})$$

for all $m > 0$.

Now assume by the induction hypothesis that

$$\mathfrak{P}'_m(\mathfrak{T}^{\otimes n}) = \mathfrak{P}_m(\mathfrak{T}^{\otimes n})$$

for all $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$ for an $n \in \mathbb{N}$.

Since we know that $GL_m(\mathfrak{T}^{\otimes n})$ is connected for all $m \geq \lfloor \frac{n}{2} \rfloor + 3$, the above theorem implies that

$$\mathfrak{P}'_m(\mathfrak{T}^{\otimes n+1}) = \mathfrak{P}_m(\mathfrak{T}^{\otimes n+1})$$

for all $m \geq \lfloor \frac{n}{2} \rfloor + 3$. $\square$

It remains open the problem of classification of low-rank idempotents over $\mathfrak{T}^{\otimes n}$. In particular, it is not clear whether there are idempotents of non-standard (equivalence) type.

6. Projective modules over $C(\mathbb{S}_H^{2n-1})$

Most of the arguments and results in the above study of projective modules over $\mathfrak{T}^{\otimes n}$ can be adapted to the case of the quantum spheres $C(\mathbb{S}_H^{2n-1})$.

Let $\partial_n: \mathfrak{T}^{\otimes n} \to C(\mathbb{S}_H^{2n-1})$ be the canonical quotient map by restricting the groupoid $\mathfrak{T}_n$ to the closed invariant set $\overline{\mathbb{Z}}_{\geq}^n \backslash \mathbb{Z}_{\geq}^n$ in its unit space.

First we note that there is a short exact sequence of C*-algebras

$$0 \to C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K}(\ell^2(\mathbb{Z}_{\geq})) \to C(\mathbb{S}_H^{2n-1}) \stackrel{\lambda_n}{\to} C^*(\mathfrak{T}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1} \times \{\infty\}}) \cong \mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}) \to 0$$

for all $n > 1$. Indeed, since $(\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}) \times \mathbb{Z}_{\geq}$ is an open invariant subset of the unit space $\overline{\mathbb{Z}}_{\geq}^n \backslash \mathbb{Z}_{\geq}^n$ of the groupoid $\mathfrak{G}_n \equiv (\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)|_{\overline{\mathbb{Z}}_{\geq}^n \backslash \mathbb{Z}_{\geq}^n}$ with the invariant complement

$$(\overline{\mathbb{Z}}_{\geq}^n \backslash \mathbb{Z}_{\geq}^n) \backslash ((\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}) \times \mathbb{Z}_{\geq}) = \overline{\mathbb{Z}}_{\geq}^{n-1} \times \{\infty\},$$

the groupoid C*-algebra

$$\begin{aligned} C^*(\mathfrak{G}_n|_{(\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}) \times \mathbb{Z}_{\geq}}) &= C^*((\mathbb{Z}^{n-1} \ltimes \overline{\mathbb{Z}}^{n-1})|_{\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}} \times (\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_{\geq}}) \\ &\cong C^*((\mathbb{Z}^{n-1} \ltimes \overline{\mathbb{Z}}^{n-1})|_{\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}}) \otimes C^*((\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_{\geq}}) \\ &= C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K}(\ell^2(\mathbb{Z}_{\geq})) \end{aligned}$$

is a closed ideal of $C^*(\mathfrak{G}_n) = C(\mathbb{S}_H^{2n-1})$ with quotient

$$\begin{aligned} &C^*(\mathfrak{G}_n)/C^*(\mathfrak{G}_n|_{(\overline{\mathbb{Z}}_{\geq}^{n-1} \backslash \mathbb{Z}_{\geq}^{n-1}) \times \mathbb{Z}_{\geq}}) \\ &\quad \cong C^*(\mathfrak{G}_n|_{\overline{\mathbb{Z}}_{\geq}^{n-1} \times \{\infty\}}) = C^*((\mathbb{Z}^{n-1} \ltimes \overline{\mathbb{Z}}^{n-1})|_{\overline{\mathbb{Z}}_{\geq}^{n-1}} \times \mathbb{Z}) \\ &\quad \cong C^*((\mathbb{Z}^{n-1} \ltimes \overline{\mathbb{Z}}^{n-1})|_{\overline{\mathbb{Z}}_{\geq}^{n-1}}) \otimes C(\mathbb{T}) = \mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}). \end{aligned}$$

So we get the above short exact sequence with λ_n being the canonical map from $C^*(\mathfrak{G}_n)$ to its quotient $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ resulting from restricting the groupoid $\mathfrak{G}_n$ to the closed invariant set $\overline{\mathbb{Z}}_{\geq}^{n-1} \times \{\infty\}$.

Clearly $\kappa_n = \lambda_n \circ \partial_n$. Furthermore, all the quotient maps $\sigma_{(j,\Theta)}$ on $\mathfrak{T}^{\otimes n}$ with $j > 0$ factors through ∂_n and induces a quotient map

$$\tau_{(j,\Theta)}: C(\mathbb{S}_H^{2n-1}) \to C(\mathbb{T}^j) \otimes \mathfrak{B}(\ell^2(\mathbb{Z}_{\geq}^{n-j}))$$

such that $\sigma_{(j,\Theta)} = \tau_{(j,\Theta)} \circ \partial_n$.

Note that the quotient maps λ_n for $n \in \mathbb{N}$ satisfy the commutative diagram

$$\begin{array}{ccccc} M_k(C(\mathbb{S}_H^{2n-1})) & \stackrel{\lambda_n}{\to} & M_k(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})) & \equiv & M_k(\mathfrak{T}^{\otimes n-1}) \otimes C(\mathbb{T}) \\ \downarrow \tau_n & \circlearrowright & \downarrow \sigma_{n-1} \otimes \mathrm{id} & & \downarrow \sigma_{n-1} \otimes \mathrm{id} \\ M_k(C(\mathbb{T}^n)) & \stackrel{\equiv}{\to} & M_k(C(\mathbb{T}^{n-1}) \otimes C(\mathbb{T})) & \equiv & M_k(C(\mathbb{T}^{n-1})) \otimes C(\mathbb{T}). \end{array}$$

We define the rank of (the equivalence class of) an idempotent $Q \in M_\infty\big(C(\mathbb{S}_H^{2n-1})\big)$ over $C(\mathbb{S}_H^{2n-1})$ as the rank of the matrix value $\tau_n(Q)(z) \in M_\infty(\mathbb{C})$ at any $z \in \mathbb{T}^n$ (independent of z since $\mathbb{T}^n$ is connected). Then the set of equivalence classes of idempotents $Q \in M_\infty\big(C(\mathbb{S}_H^{2n-1})\big)$ equipped with the binary operation $\boxplus$ becomes an abelian graded monoid

$$\mathfrak{P}\big(C(\mathbb{S}_H^{2n-1})\big) = \sqcup_{m=0}^\infty \mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big),$$

where $\mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big)$ is the set of all (equivalence classes of) idempotents over $C(\mathbb{S}_H^{2n-1})$ of rank m, with clearly

$$\mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big) \boxplus \mathfrak{P}_l\big(C(\mathbb{S}_H^{2n-1})\big) \subset \mathfrak{P}_{m+l}\big(C(\mathbb{S}_H^{2n-1})\big)$$

for $m, l \geq 0$.

Since $\sigma_n = \tau_n \circ \partial_n$, the rank of an idempotent P over $C(\mathfrak{T}^{\otimes n})$ equals the rank of the idempotent $\partial_n P$ over $C(\mathbb{S}_H^{2n-1})$. We now define

$$\mathfrak{P}'_m\big(C(\mathbb{S}_H^{2n-1})\big) := \partial_n\big(\mathfrak{P}'_m(\mathfrak{T}^{\otimes n})\big) \subset \mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big),$$

and the projections

$$Q_{j,\Theta,l} := \partial_n\big(\Theta(P_{j,l})\big)$$

over $C(\mathbb{S}_H^{2n-1})$. Note that $\mathfrak{P}'_m\big(C(\mathbb{S}_H^{2n-1})\big) = \{\boxplus^m \widetilde{I}\}$ for $m > 0$, where $\widetilde{I}$ denotes the identity element of $C(\mathbb{S}_H^{2n-1})$.

Also note that $Q_{0,\mathrm{id},l} = 0$ for all l, where $\mathrm{id} \equiv \mathrm{id}_{\{1,2,\ldots,n\}}$ is the only $(0,n)$-shuffle. The monoid homomorphism

$$\rho_0 \colon P \in \mathfrak{P}'(\mathfrak{T}^{\otimes n}) \mapsto \prod_{(j,\Theta)\in\Omega_0} \rho_{(j,\Theta)}(P) \in \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}}_\geq \,,$$

with

$$\Omega_0 := \Omega\backslash\{(0,\mathrm{id})\} \equiv \{(j,\Theta) : \ 0 < j \leq n \text{ and } \Theta \text{ is a } (j, n-j)\text{-shuffle}\},$$

"truncated" from ρ induces a well-defined monoid homomorphism

$$\rho_\partial \colon \mathfrak{P}'\big(C(\mathbb{S}_H^{2n-1})\big) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}}_\geq,$$

in the sense that $\rho = \rho_\partial \circ \partial_n$. Indeed for $(j,\Theta) \in \Omega_0$, i.e. with $j > 0$, the quotient map

$$\sigma_{(j,\Theta)} \colon \mathfrak{T}^{\otimes n} \equiv C^*(\mathfrak{T}_n) \to C^*(\mathfrak{T}_n\big|_{X_\Theta})$$

factors through ∂_n since the unit space $\overline{\mathbb{Z}}^n_\geq \backslash \mathbb{Z}^n_\geq$ of $\mathfrak{G}_n$ contains X_Θ, and hence the map $\rho_{(j,\Theta)}$ factors through ∂_n.

We call a $\boxplus$-sum of $Q_{j,\Theta,l}$ indexed by $\prec$-unrelated $(j,\Theta) \in \Omega_0$ (i.e. $1 \leq j \leq n$) and $l \equiv l_{(j,\Theta)} > 0$ depending on (j,Θ) to be a reduced $\boxplus$-sum of standard projections over $C(\mathbb{S}_H^{2n-1})$. (The degenerate empty $\boxplus$-sum 0 is taken as a reduced $\boxplus$-sum.) Two such reduced $\boxplus$-sums are called different when they have different sets of (mutually $\prec$-unrelated) indices $(j,\Theta) \in \Omega_0$ or have different weight functions l of (j,Θ). Each $Q_{j,\Theta,l}$ with $j,l > 0$ is a reduced $\boxplus$-sum of standard projections over $C(\mathbb{S}_H^{2n-1})$.

Proposition 5. *Different reduced $\boxplus$-sums of standard projections over $C(\mathbb{S}_H^{2n-1})$ are mutually inequivalent projections over $C(\mathbb{S}_H^{2n-1})$, and they form a graded submonoid*

$$\mathfrak{P}'\big(C(\mathbb{S}_H^{2n-1})\big) = \sqcup_{m=0}^{\infty}\mathfrak{P}'_m\big(C(\mathbb{S}_H^{2n-1})\big)$$

of the monoid $\mathfrak{P}(C(\mathbb{S}_H^{2n-1}))$, with its monoid structure explicitly determined by $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ for $(j',\Theta') \prec (j,\Theta)$ with $j,j',l,l' > 0$. Furthermore, the monoid homomorphism

$$\rho_\partial: \mathfrak{P}'\big(C(\mathbb{S}_H^{2n-1})\big) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}}_{\geq}$$

is injective.

Proof. The submonoid

$$\mathfrak{P}'\big(C(\mathbb{S}_H^{2n-1})\big) = \partial_n\big(\mathfrak{P}'\big(C(\mathcal{T}^{\otimes n})\big)\big)$$

consists of reduced $\boxplus$-sums of $Q_{j,\Theta,l} = \partial_n(\Theta(P_{j,l}))$ with $j > 0$, since $Q_{0,\mathrm{id},l} = 0$.

Let $\mathfrak{M}$ be the subset of $\mathfrak{P}'(C(\mathcal{T}^{\otimes n}))$ consisting of all reduced $\boxplus$-sums P of $\Theta(P_{j,l})$ with $j > 0$. Then

$$\partial_n|_{\mathfrak{M}}: \mathfrak{M} \to \mathfrak{P}'\big(C(\mathbb{S}_H^{2n-1})\big)$$

is still surjective, and $\rho_0|_{\mathfrak{M}}$ still factors through ρ_∂, i.e. $\rho_0|_{\mathfrak{M}} = \rho_\partial \circ \partial_n|_{\mathfrak{M}}$. These imply that ρ_∂ is injective if $\rho_0|_{\mathfrak{M}}$ is injective.

For any reduced $\boxplus$-sum $P \in \mathfrak{M}$ of $\Theta(P_{j,l})$ with $j > 0$, the (j,Θ)-component of $\rho(P)$ is the same as that of $\rho_0(P)$ for all $(j,\Theta) \in \Omega_0$, while the only other component, namely, the $(0,\mathrm{id})$-component of $\rho(P)$ is ∞ since $\rho_{(0,\mathrm{id})}(\Theta(P_{j,l})) = \infty$ for any $j > 0$. Thus we get

$$\rho(P) = (\infty, \rho_0(P))$$

for all $P \in \mathfrak{M}$. Hence the injectivity of $\rho|_{\mathfrak{M}}$ implies the injectivity of $\rho_0|_{\mathfrak{M}}$ on $\mathfrak{M}$, and hence the injectivity of ρ_∂.

Since two different reduced $\boxplus$-sums Q, Q' over $C(\mathbb{S}_H^{2n-1})$ are of the form $\partial_n(P)$, $\partial_n(P')$, respectively, for two different reduced $\boxplus$-sums $P, P' \in \mathfrak{M}$ over $C(\mathcal{T}^{\otimes n})$, which are inequivalent over $C(\mathcal{T}^{\otimes n})$, and hence

$$\rho_0(P) \neq \rho_0(P').$$

We get

$$\rho_\partial(Q) \neq \rho_\partial(Q')$$

showing that Q, Q' are different equivalence classes in $\mathfrak{P}'(C(\mathbb{S}_H^{2n-1}))$.

The property that

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l})$$

over $\mathfrak{T}^{\otimes n}$ for $(j', \Theta') \prec (j, \Theta)$ is clearly preserved under the quotient map ∂_n, i.e.

$$Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$$

over $C(\mathbb{S}_H^{2n-1})$. □

Theorem 3. *For $n > 1$ and $m \in \mathbb{N}$, if $\mathfrak{P}_m(\mathfrak{T}^{\otimes n-1}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n-1})$ and $\mathrm{GL}_m(\mathfrak{T}^{\otimes n-1})$ is connected, then*

$$\mathfrak{P}'_m\big(C(\mathbb{S}_H^{2n-1})\big) = \mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big).$$

Proof. Many arguments used to prove a similar theorem for $\mathfrak{T}^{\otimes n}$ instead of $C(\mathbb{S}_H^{2n-1})$ can be used again here with minor modifications. In this proof, I and $\widetilde{I}$ denote respectively the identity element of $\mathfrak{T}^{\otimes n-1}$ and $C(\mathbb{S}_H^{2n-1})$.

Let $P \in \mathfrak{P}_m(C(\mathbb{S}_H^{2n-1}))$. The idempotent $\lambda_n(P)$ over $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ satisfies that for any $z \in \mathbb{T}$,

$$\sigma_{n-1}\big(\lambda_n(P)(z)\big) = \tau_n(P)(\cdot, z) \in M_\infty\big(C(\mathbb{T}^{n-1})\big),$$

which is of rank m pointwise, and hence

$$\lambda_n(P)(z) \in \mathfrak{P}_m(\mathfrak{T}^{\otimes n-1}) = \mathfrak{P}'_m(\mathfrak{T}^{\otimes n-1}),$$

i.e. $\lambda_n(P)(z) \sim \boxplus^m I$ over $\mathfrak{T}^{\otimes n-1}$. As before, for some large k, there is an idempotent-valued continuous loop

$$\Gamma: \mathbb{T} \to M_k(\mathfrak{T}^{\otimes n-1})$$

starting and ending at $\boxplus^m I$ with $\Gamma(e^{i\theta}) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$, say, for all $\theta \in [3\pi/2, 2\pi]$, and homotopic to the loop $\lambda_n(P)$ via idempotents. Consequently, there is a continuous path of invertibles $U_t \in \mathrm{GL}_k(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ with $U_0 = I_k$ such that

$$U_1 \lambda_n(P) U_1^{-1} = \Gamma,$$

which can be lifted along λ_n to a continuous path of invertible $V_t \in \mathrm{GL}_k(C(\mathbb{S}_H^{2n-1}))$ with $V_0 = I_k$ such that

$$\lambda_n(V_1 P V_1^{-1}) = \Gamma.$$

Replacing P by the equivalent idempotent $V_1 P V_1^{-1}$, we may now assume directly that the idempotent $\lambda_n(P)$ over $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is a continuous loop of idempotents in $M_k(\mathfrak{T}^{\otimes n-1})$ such that

$$\lambda_n(P)(e^{i\theta}) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$$

for all $\theta \in [3\pi/2, 2\pi]$. As before, by the connectedness assumption on $\mathrm{GL}_m(\mathfrak{T}^{\otimes n-1})$, after suitably increasing the size k, we can find a well-defined continuous loop

$$W : e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in \mathrm{GL}_k(\mathfrak{T}^{\otimes n-1}),$$

i.e. $W \in \mathrm{GL}_k(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}))$, satisfying

$$W\big(\lambda_n(P)\big)W^{-1} = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0).$$

So the idempotent $\lambda_n(P)$ over $\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T})$ is equivalent to the idempotent $\boxplus^m I$.

Replacing P by the equivalent idempotent $\widetilde{W}(P \boxplus (\boxplus^k 0))\widetilde{W}^{-1}$ for any fixed lifting $\widetilde{W} \in \mathrm{GL}^0_{2k}(C(\mathbb{S}_H^{2n-1}))$ of $W \boxplus W^{-1} \in \mathrm{GL}^0_{2k}(\mathfrak{T}^{\otimes n-1} \otimes C(\mathbb{T}))$ along λ_n, we may now assume that

$$\lambda_n(P) = (\boxplus^m I) \boxplus (\boxplus^{2k-m} 0) = \lambda_n\big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big)$$

and proceed to show that $P \sim \boxplus^m \widetilde{I}$ over $C(\mathbb{S}_H^{2n-1})$, where we use $\widetilde{I}$ to denote the identity element in $C(\mathbb{S}_H^{2n-1})$ so as to distinguish it from the identity element I of $\mathfrak{T}^{\otimes n-1}$.

With $P - ((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)) \in M_{2k}(C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K})$ and $M_\infty(\mathbb{C})$ dense in $\mathcal{K}$, we may replace P by a suitable equivalent idempotent and assume that

$$P = K + \big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big) \in M_{2k}\big((C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K})^+\big) \subset M_{2k}\big(C(\mathbb{S}_H^{2n-1})\big)$$

for some $K \in M_{2k}(C(\mathbb{S}_H^{2n-3}) \otimes M_N(\mathbb{C}))$ and some $N \in \mathbb{N}$.

As before, by rearranging entries via conjugation, we get

$$\begin{aligned} P \sim \partial_n(U_{k,N}) P \partial_n(U_{k,N}^{-1}) &\equiv \partial_n(U_{k,N})(P \boxplus 0)\partial_n(U_{k,N}^{-1}) \\ &= \big((\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k-m} 0)\big) \boxplus R \end{aligned}$$

for some

$$R \in M_{2kN}\big(C(\mathbb{S}_H^{2n-3})\big) \equiv C(\mathbb{S}_H^{2n-3}) \otimes M_{2kN}(\mathbb{C}) \subset \big(C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K}\big)^+ \subset C(\mathbb{S}_H^{2n-1})$$

which must be an idempotent over $C(\mathbb{S}_H^{2n-3})$. More precisely, we can lift P to

$$\widehat{P} = \widehat{K} + \big((\boxplus^m I_{\mathfrak{T}^{\otimes n}}) \boxplus (\boxplus^{2k-m} 0)\big) \in M_{2k}\big((\mathfrak{T}^{\otimes n-1} \otimes \mathcal{K})^+\big) \subset M_{2k}(\mathfrak{T}^{\otimes n})$$

for some $\widehat{K} \in M_{2k}(\mathfrak{T}^{\otimes n-1} \otimes M_N(\mathbb{C}))$ and conjugate it by the unitary $U_{k,N}$ over $\mathfrak{T}^{\otimes n}$ to get the form

$$\big((\boxplus^m I_{\mathfrak{T}^{\otimes n}}) \boxplus (\boxplus^{2k-m} 0)\big) \boxplus \widehat{R}$$

with $\widehat{R} \in M_{2kN}(\mathfrak{T}^{\otimes n-1})$ as we did for the case of $\mathfrak{T}^{\otimes n}$. Then the above R is $\partial_n(\widehat{R})$.

Note that even though $\widehat{P}$ and $\widehat{R}$ are not necessarily idempotents, R is since it is the idempotent P conjugated by the unitary $\partial_n(U_{k,N})$ over $C(\mathbb{S}_H^{2n-1})$.

Since $K_0(C(\mathbb{S}_H^{2n-3})) = \mathbb{Z}$ (see [12]),

$$R \boxplus \big(\boxplus^r \widehat{I} \big) \sim \big(\boxplus^{r+[R]} \widehat{I} \big)$$

for a sufficiently large $r \in \mathbb{N}$, where $[R] \in \mathbb{Z}$ denotes the class of R in $K_0(C(\mathbb{S}_H^{2n-3}))$ and $\widehat{I}$ is the identity element of $C(\mathbb{S}_H^{2n-3})$. So there is an invertible $U \in \mathrm{GL}_d(C(\mathbb{S}_H^{2n-3}))$ for some large $d \geq \max\{2kN + r, r + [R]\}$ such that

$$U\big(R \boxplus (\boxplus^{d-2kN-r} 0) \boxplus (\boxplus^r \widehat{I})\big)U^{-1} = (\boxplus^{d-r-[R]} 0) \boxplus (\boxplus^{r+[R]} \widehat{I}).$$

As before, with $m > 0$, by rearranging entries via conjugation, we can get

$$P \sim R' \boxplus (\boxplus^{m-1} \widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

where the idempotent

$$R' = \big(R \boxplus (\boxplus^{d-2kN-r} 0)\big) + (\widetilde{I} - \widehat{I} \otimes P_{d-r}) \in \big(C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K}\big)^+ \subset C(\mathbb{S}_H^{2n-1})$$

when conjugated by the invertible $U \equiv U \boxplus (\widetilde{I} - \widehat{I} \otimes P_d) \in (C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K})^+$ becomes

$$(\boxplus^{d-r-[R]} 0) \boxplus (\widetilde{I} - \widehat{I} \otimes P_{d-r-[R]}) \in \big(C(\mathbb{S}_H^{2n-3}) \otimes \mathcal{K}\big)^+ \subset C(\mathbb{S}_H^{2n-1}).$$

So we get

$$P \sim \big((\boxplus^{d-r-[R]} 0) \boxplus (\widetilde{I} - \widehat{I} \otimes P_{d-r-[R]})\big) \boxplus (\boxplus^{m-1} \widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

the latter of which as before is equivalent to $\widetilde{I} \boxplus (\boxplus^{m-1} \widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0)$ by a further conjugation by $U_{d-r-[R]}^{-1}$. Thus

$$P \sim (\boxplus^m \widetilde{I}) \boxplus (\boxplus^{2k+1-m} 0) \equiv \boxplus^m \widetilde{I}. \qquad \square$$

Corollary 4. $\mathfrak{P}_m(C(\mathbb{S}_H^{2n-1})) = \mathfrak{P}'_m(C(\mathbb{S}_H^{2n-1})) \equiv \{\boxplus^m \widetilde{I}\}$ *for all* $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$ *and any* $n \in \mathbb{N}$*, where* $\widetilde{I}$ *is the identity element of* $C(\mathbb{S}_H^{2n-1})$.

Proof. The case of $n = 1$ is well known. For $n > 1$, since $\mathfrak{P}'_m(\mathfrak{T}^{\otimes n-1}) = \mathfrak{P}_m(\mathfrak{T}^{\otimes n-1})$ for all $m \geq \lfloor \frac{n-2}{2} \rfloor + 3$ and $\mathrm{GL}_m(\mathfrak{T}^{\otimes n-1})$ is connected for all $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$, the above theorem implies that

$$\mathfrak{P}'_m\big(C(\mathbb{S}_H^{2n-1})\big) = \mathfrak{P}_m\big(C(\mathbb{S}_H^{2n-1})\big)$$

for all $m \geq \lfloor \frac{n-1}{2} \rfloor + 3$. $\square$

It is not clear whether there are (low-rank) idempotents over $C(\mathbb{S}_H^{2n-1})$ of non-standard (equivalence) type and whether the cancellation law holds for them.

7. Projective modules over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$

In this section we study the problem of classification of finitely generated projective modules over the multipullback quantum complex projective space $\mathbb{P}^{n-1}(\mathfrak{T})$ that was introduced and studied by Hajac, Kaygun, Zieliński in [9].

In [12],

$$K_0\big(C\big(\mathbb{P}^{n-1}(\mathfrak{T})\big)\big) = \mathbb{Z}^n \quad \text{and} \quad K_1\big(C\big(\mathbb{P}^{n-1}(\mathfrak{T})\big)\big) = 0$$

are computed, and $\mathbb{P}^{n-1}(\mathfrak{T})$ is shown to be a quantum quotient space of $\mathbb{S}_H^{2n-1}$. More precisely, the C*-algebra $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ is isomorphic to the invariant C*-subalgebra $(C(\mathbb{S}_H^{2n-1}))^{U(1)}$ of $C(\mathbb{S}_H^{2n-1})$ under the canonical diagonal $U(1)$-action on $C(\mathbb{S}_H^{2n-1}) \cong \mathfrak{T}^{\otimes n}/\mathfrak{K}^{\otimes n}$, which in the groupoid context can be implemented by the multiplication operator

$$U_\zeta : f \in C_c(\mathfrak{G}_n) \mapsto h_\zeta f \in C_c(\mathfrak{G}_n)$$

for $\zeta \in U(1) \equiv \mathbb{T}$, where

$$h_\zeta : (m, p) \in \mathfrak{G}_n \subset \mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n \mapsto \zeta^{\Sigma m} \in \mathbb{T} \text{ with } \Sigma m := \sum_{i=1}^{n} m_i$$

is a groupoid character. Then $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ is realized as the groupoid C*-algebra $C^*((\mathfrak{G}_n)_0)$ of the subgroupoid $(\mathfrak{G}_n)_0$ of $\mathfrak{G}_n$, where

$$(\mathfrak{G}_n)_k := \{(m, p) \in \mathfrak{G}_n : \Sigma m = k\}$$

for $k \in \mathbb{Z}$. Furthermore, $C^*(\mathfrak{G}_n)$ becomes a (completion of the) graded algebra

$$\oplus_{k\in\mathbb{Z}} \overline{C_c((\mathfrak{G}_n)_k)}$$

with the component $\overline{C_c((\mathfrak{G}_n)_k)}$ being the quantum line bundle $C(\mathbb{S}_H^{2n-1})_k$ [12] of degree k over the quantum space $\mathbb{P}^{n-1}(\mathfrak{T})$.

It is easy to see that the standard projections $Q_{j,\Theta,l} \equiv \partial_n(\Theta(P_{j,l}))$ over $C(\mathbb{S}_H^{2n-1})$ with $j, l > 0$ found in the previous section lie in $M_\infty(C^*((\mathfrak{G}_n)_0))$ since

$$P_{j,l} = \boxplus^l\big((\otimes^j I) \otimes (\otimes^{n-j} P_1)\big)$$

is in $C^*((\mathfrak{T}_n)_0)$, and hence are also projections over $C^*((\mathfrak{G}_n)_0) \equiv C(\mathbb{P}^{n-1}(\mathfrak{T}))$. Furthermore, with $C(\mathbb{P}^{n-1}(\mathfrak{T})) \subset C(\mathbb{S}_H^{2n-1})$, inequivalent $\boxplus$-sums of standard projections over $C(\mathbb{S}_H^{2n-1})$ must be inequivalent over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ as well.

Proposition 6. *Different reduced $\boxplus$-sums of standard projections $Q_{j,\Theta,l}$ over $C(\mathbb{S}_H^{2n-1})$ with $j, l > 0$ when viewed as projections over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ are mutually inequivalent over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$, and they form a graded submonoid*

$$\mathfrak{P}'\big(C\big(\mathbb{P}^{n-1}(\mathfrak{T})\big)\big) = \sqcup_{m=0}^{\infty} \mathfrak{P}'_m\big(C\big(\mathbb{P}^{n-1}(\mathfrak{T})\big)\big)$$

of the monoid $\mathfrak{P}(C(\mathbb{P}^{n-1}(\mathcal{T})))$. *Furthermore, the monoid homomorphism*

$$\mathfrak{P}'\big(C\big(\mathbb{P}^{n-1}(\mathcal{T})\big)\big) \to \prod_{(j,\Theta)\in\Omega_0} \overline{\mathbb{Z}}_{\geq}$$

inherited from ρ_∂ *is injective.*

However, for $(j',\Theta') \prec (j,\Theta)$ with $j, j', l, l' > 0$, it is no longer true in general that

$$Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$$

over $C(\mathbb{P}^{n-1}(\mathcal{T}))$, even though $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ over $C(\mathbb{S}_H^{2n-1})$ since the invertible matrix over $C(\mathbb{S}_H^{2n-1})$ intertwining $Q_{j,\Theta,l} \boxplus Q_{j',\Theta',l'}$ and $Q_{j,\Theta,l}$ may not be replaced by one over the subalgebra $C(\mathbb{P}^{n-1}(\mathcal{T}))$ of $C(\mathbb{S}_H^{2n-1})$.

In the following, we show that the standard projections $Q_{j,\mathrm{id},1}$ with $j > 0$ provide a set of representatives of K_0-classes that freely generate the abelian K_0-group of $C(\mathbb{P}^{n-1}(\mathcal{T}))$.

The subgroupoid

$$\mathfrak{H}_j := \mathfrak{G}_j \times (\mathbb{Z}^{n-j} \ltimes \mathbb{Z}_{\geq}^{n-j})$$

of $\mathfrak{G}_n$ for $1 \leq j \leq n$ is the groupoid $\mathfrak{G}_n$ restricted to the open invariant subset $(\overline{\mathbb{Z}}_{\geq}^{j} \times \mathbb{Z}_{\geq}^{n-j})\backslash\mathbb{Z}_{\geq}^{n}$ and inherits the grading of $\mathfrak{G}_n$. The grade-0 part $(\mathfrak{H}_j)_0$ of $\mathfrak{H}_j$ is the groupoid $(\mathfrak{G}_n)_0$ restricted to $(\overline{\mathbb{Z}}_{\geq}^{j} \times \mathbb{Z}_{\geq}^{n-j})\backslash\mathbb{Z}_{\geq}^{n}$, and from the increasing chain of $(\mathfrak{H}_j)_0$, we get an increasing composition sequence of closed ideals of $C^*((\mathfrak{G}_n)_0)$ as

$$0 =: C^*((\mathfrak{H}_0)_0) \lhd C^*((\mathfrak{H}_1)_0) \lhd \cdots \lhd C^*((\mathfrak{H}_{n-1})_0) \lhd C^*((\mathfrak{H}_n)_0) = C^*((\mathfrak{G}_n)_0)$$

such that with $\big(\overline{\mathbb{Z}}_{\geq}^{j} \times \mathbb{Z}_{\geq}^{n-j}\big)\backslash\big(\overline{\mathbb{Z}}_{\geq}^{j-1} \times \mathbb{Z}_{\geq}^{n-j+1}\big) = \overline{\mathbb{Z}}_{\geq}^{j-1} \times \{\infty\} \times \mathbb{Z}_{\geq}^{n-j}$,

$$\begin{aligned} C^*\big((\mathfrak{H}_j)_0\big)/C^*\big((\mathfrak{H}_{j-1})_0\big) &\cong C^*\Big(\big(\mathfrak{G}_n|_{\overline{\mathbb{Z}}_{\geq}^{j-1}\times\{\infty\}\times\mathbb{Z}_{\geq}^{n-j}}\big)_0\Big) \\ &\cong C^*\big(\mathfrak{T}_{n-1}|_{\overline{\mathbb{Z}}_{\geq}^{j-1}\times\mathbb{Z}_{\geq}^{n-j}}\big) \cong \mathcal{T}^{\otimes j-1} \otimes \mathcal{K}(\mathbb{Z}_{\geq}^{n-j}) \end{aligned}$$

since the groupoid $\big(\mathfrak{G}_n|_{\overline{\mathbb{Z}}_{\geq}^{j-1}\times\{\infty\}\times\mathbb{Z}_{\geq}^{n-j}}\big)_0$ is isomorphic to the groupoid $\mathfrak{T}_{n-1}|_{\overline{\mathbb{Z}}_{\geq}^{j-1}\times\mathbb{Z}_{\geq}^{n-j}}$ via the groupoid isomorphism

$$(m,k,l,p,\infty,q) \mapsto (m,l,p,q),$$

where

$$(m,k,l,p,\infty,q) \in \mathfrak{G}_n|_{\overline{\mathbb{Z}}_{\geq}^{j-1}\times\{\infty\}\times\mathbb{Z}_{\geq}^{n-j}} \subset \mathbb{Z}^{j-1}\times\mathbb{Z}\times\mathbb{Z}^{n-j}\times\overline{\mathbb{Z}}_{\geq}^{j-1}\times\{\infty\}\times\mathbb{Z}_{\geq}^{n-j}$$

with $\sum_{i=1}^{j-1} m_i + k + \sum_{i=1}^{n-j} l_i = 0$ and hence $k = -\sum m - \sum l$ determined by m, l.

Since

$$K_1\big(\mathfrak{T}^{\otimes j-1} \otimes \mathcal{K}(\mathbb{Z}_{\geq}^{n-j})\big) = 0 \quad \text{and} \quad K_0\big(\mathfrak{T}^{\otimes j-1} \otimes \mathcal{K}(\mathbb{Z}_{\geq}^{n-j})\big) = \mathbb{Z},$$

it is easy to conclude from the cyclic six-term exact sequence of K-groups for the pair $C^*((\mathfrak{H}_{j-1})_0) \lhd C^*((\mathfrak{H}_j)_0)$ that the following sequence is exact and splits

$$0 \to K_0\big(C^*\big((\mathfrak{H}_{j-1})_0\big)\big) \to K_0\big(C^*\big((\mathfrak{H}_j)_0\big)\big) \to K_0\big(\mathfrak{T}^{\otimes j-1} \otimes \mathcal{K}(\mathbb{Z}_{\geq}^{n-j})\big) \cong \mathbb{Z} \to 0,$$

where the projection $(\otimes^{j-1} I) \otimes (\otimes^{n-j} P_1)$ is a generator of $K_0(\mathfrak{T}^{\otimes j-1} \otimes \mathcal{K}(\mathbb{Z}_{\geq}^{n-j}))$. Note that this $(\otimes^{j-1} I) \otimes (\otimes^{n-j} P_1)$ lifts to the projection element

$$\chi_{A_j} \in C_c\big((\mathfrak{H}_j)_0\big) \subset C^*\big((\mathfrak{H}_j)_0\big)$$

given by the characteristic function of the set

$$A_j := \{0\} \times \{0\} \times (\overline{\mathbb{Z}}_{\geq}^{j} \backslash \mathbb{Z}_{\geq}^{j}) \times \{0\} \subset \mathfrak{H}_j \subset \mathbb{Z}^j \times \mathbb{Z}^{n-j} \times \overline{\mathbb{Z}}_{\geq}^{j} \times \mathbb{Z}_{\geq}^{n-j}.$$

Furthermore, $\chi_{A_j} = Q_{j,\mathrm{id},1}$ in $C(\mathbb{P}^{n-1}(\mathfrak{T})) \subset C(\mathbb{S}_H^{2n-1})$. So we get

$$K_0\big(C^*\big((\mathfrak{H}_j)_0\big)\big) \cong K_0\big(C^*\big((\mathfrak{H}_{j-1})_0\big)\big) \oplus \mathbb{Z}[Q_{j,\Theta,1}]$$

with $K_0(C^*((\mathfrak{H}_{j-1})_0))$ canonically embedded in $K_0(C^*((\mathfrak{H}_j)_0))$.

Putting together these results for all j, we get

$$K_0\big(C(\mathbb{P}^{n-1}(\mathfrak{T}))\big) \equiv K_0\big(C^*\big((\mathfrak{H}_n)_0\big)\big) \cong \oplus_{j=1}^{n} \mathbb{Z}[Q_{j,\mathrm{id},1}] \cong \mathbb{Z}^n,$$

and hence $Q_{j,\mathrm{id},1}$ freely generate the abelian group $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))$. Note that

$$Q_{j,\mathrm{id},l} = \boxplus^l Q_{j,\mathrm{id},1} \quad \text{and} \quad [Q_{j,\mathrm{id},l}] = l[Q_{j,\mathrm{id},1}]$$

in $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))$ for any $l \in \mathbb{N}$.

We now summarize the above discussion.

Theorem 4. *The projections $Q_{j,\Theta,l}$ over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ with $l \in \mathbb{N}$ and Θ a $(j, n-j)$-shuffle for $0 < j \leq n$ are mutually inequivalent, and the projections $Q_{j,\mathrm{id},1}$ with $0 < j \leq n$ freely generate the abelian group $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))$, such that if*

$$[p] = \sum_{j=1}^{n} m_j [Q_{j,\mathrm{id},1}]$$

for a projection p over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$, then the coefficient m_n of $[Q_{n,\mathrm{id},1}]$ is the rank of p.

Proof. We only need to note that the rank of $Q_{n,\mathrm{id},1}$ is 1 and the rank of any other $Q_{j,\mathrm{id},1}$ is 0. □

Remark. Since any permutation Θ on $\{1, 2, \ldots, n\}$ canonically induces a $U(1)$-equivariant (outer) C*-algebra automorphism of $\mathcal{T}^{\otimes n}$ permuting its tensor factors and preserving its ideal $\mathcal{K}^{\otimes n}$, the above expression of free generators

$$\left[\partial_n(\otimes^j I \otimes \otimes^{n-j} P_1)\right]$$

with $0 < j \leq n$ of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T})))$ can be changed by a permutation to yield some other free generators. For example, both

$$\{[\partial_3(1 \otimes P_1 \otimes P_1)], [\partial_3(1 \otimes 1 \otimes P_1)], [\partial_3(1 \otimes 1 \otimes 1)]\}$$

and

$$\{[\partial_3(P_1 \otimes P_1 \otimes 1)], [\partial_3(P_1 \otimes 1 \otimes 1)], [\partial_3(1 \otimes 1 \otimes 1)]\}$$

are sets of free generators of $K_0(C(\mathbb{P}^2(\mathcal{T})))$.

The above theorem shows that for $(j', \mathrm{id}) \prec (j, \mathrm{id})$ in Ω_0, i.e. $0 < j' < j$, it is not true that $Q_{j,\mathrm{id},1} \boxplus Q_{j',\mathrm{id},1} \sim Q_{j,\mathrm{id},1}$ over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ because

$$[Q_{j,\mathrm{id},1} \boxplus Q_{j',\mathrm{id},1}] = [Q_{j,\mathrm{id},1}] + [Q_{j',\mathrm{id},1}] \neq [Q_{j,\mathrm{id},1}]$$

in $K_0(C(\mathbb{P}^{n-1}(\mathcal{T})))$.

Next we consider the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathcal{T})))$. In the following, we use $\widehat{I}$ and $\widetilde{I}$ to denote the identity elements of $\mathcal{T}^{\otimes n-1}$ and $\mathcal{T}^{\otimes n}$ respectively.

First, it is easy to see that for $k > 0$, the projection $\widehat{I} \otimes P_k$ is a sum of k mutually orthogonal projections $\widehat{I} \otimes e_{jj}$, each equivalent to $\widehat{I} \otimes P_1$ over $(\mathcal{T}^{\otimes n-1} \otimes \mathcal{K})^+ \subset \mathcal{T}^{\otimes n}$, and hence the projection $\partial_n(\widehat{I} \otimes P_k)$ is a sum of k mutually orthogonal projections $\partial_n(\widehat{I} \otimes e_{jj})$, each equivalent to $\partial_n(\widehat{I} \otimes P_1)$ over $C(\mathbb{S}_H^{2n-1})$. So,

$$\widehat{I} \otimes P_k \sim \boxplus^k(\widehat{I} \otimes P_1) \equiv \boxplus^k P_{n-1,1} \equiv P_{n-1,k} \text{ over } (\mathcal{T}^{\otimes n-1} \otimes \mathcal{K})^+ \subset \mathcal{T}^{\otimes n}$$

and

$$\partial_n(\widehat{I} \otimes P_k) \sim \boxplus^k Q_{n-1,\mathrm{id},1} \equiv Q_{n-1,\mathrm{id},k} \text{ over } C(\mathbb{S}_H^{2n-1}).$$

Similarly, by rearranging entries via conjugation by shifts, the projection $\widehat{I} \otimes P_{-k}$ is equivalent to $\widetilde{I}$ over $\mathcal{T}^{\otimes n}$, and hence

$$\partial_n(\widehat{I} \otimes P_{-k}) \sim \partial_n(\widetilde{I}) \text{ over } C(\mathbb{S}_H^{2n-1}).$$

However, such equivalences no longer hold over the algebra $C(\mathbb{P}^{n-1}(\mathcal{T})) \subset C(\mathbb{S}_H^{2n-1})$. For example,

$$\partial_n(\widehat{I} \otimes P_{-k}) \boxplus \partial_n(\widehat{I} \otimes P_k) \sim \partial_n(\widetilde{I}) \text{ over } C(\mathbb{P}^{n-1}(\mathcal{T}))$$

since $\partial_n(\widehat{I} \otimes P_{-k})$ and $\partial_n(\widehat{I} \otimes P_k)$ are orthogonal projections in $C(\mathbb{P}^{n-1}(\mathcal{T}))$, which add up to $\widetilde{I}$. So,

$$\begin{aligned}\left[\partial_n(\widehat{I} \otimes P_{-1})\right] &= \left[\partial_n(\widetilde{I})\right] - \left[\partial_n(\widehat{I} \otimes P_1)\right] \\ &= [Q_{n,\mathrm{id},1}] - [Q_{n-1,\mathrm{id},1}] \text{ in } K_0\big(C(\mathbb{P}^{n-1}(\mathcal{T}))\big),\end{aligned}$$

which shows that

$$\left[\partial_n(\widehat{I} \otimes P_{-1})\right] \in \mathbb{Z}^{n-2} \times \{-1\} \times \{1\} \subset \mathbb{Z}^n \cong K_0\big(C(\mathbb{P}^{n-1}(\mathfrak{T}))\big)$$

and $\partial_n(\widehat{I} \otimes P_{-1})$ is not even stably equivalent over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ to any $\boxplus$-sum of the K_0-generating projections $Q_{j,\mathrm{id},1}$ with $0 < j \leq n$.

From now on, we include all projections of the form

$$\partial_n\big((\otimes^{j-1} I) \otimes P_k \otimes (\otimes^{n-j} P_1)\big)$$

with $k \in \mathbb{Z}$ as elementary projections over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$, where it is understood that for $k = 0$, we take $P_k := P_{-0} \equiv I$ instead of $P_0 \equiv 0$.

Theorem 5. *The positive cone of* $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T}))) \cong \mathbb{Z}^n \equiv \bigoplus_{j=1}^n \mathbb{Z}[Q_{j,\mathrm{id},1}]$ *contains*

$$\mathbb{Z}^n \backslash \{z \in \mathbb{Z}^n : z_j < 0 = z_{j+1} = \cdots = z_n \text{ for some } 1 \leq j \leq n\}$$

which is the part of the cone generated/spanned by the equivalence classes of the elementary projections $\partial_n((\otimes^{j-1} I) \otimes P_k \otimes (\otimes^{n-j} P_1))$ *with* $k \in \mathbb{Z}$ *and* $1 \leq j \leq n$, *where for* $k = 0$, *we take* $P_k := P_{-0} \equiv I$.

Proof. In [29], it has been established that in the case of $n = 2$, the positive cone of

$$\begin{aligned} K_0\big(C(\mathbb{P}^1(\mathfrak{T}))\big) &= \mathbb{Z}[Q_{1,\mathrm{id},1}] \oplus \mathbb{Z}[Q_{2,\mathrm{id},1}] \\ &\equiv \mathbb{Z}\big[\partial_2(I \otimes P_1)\big] \oplus \mathbb{Z}\big[\partial_2(I \otimes I)\big] \cong \mathbb{Z}^2 \end{aligned}$$

consists of $(k, m) \in \mathbb{Z}^2$ with either $k \geq 0$ or the rank $m > 0$, such that

$$\begin{aligned} &\big[\partial_2(I \otimes P_k)\big] = k\big[\partial_2(I \otimes P_1)\big] = (k, 0) \\ \text{and} \qquad &\big[\partial_2(I \otimes P_{-k})\big] = \big[\partial_2(I \otimes I)\big] - k\big[\partial_2(I \otimes P_1)\big] = (-k, 1) \end{aligned}$$

in $K_0(C(\mathbb{P}^1(\mathfrak{T})))$ for all $k > 0$.

By induction on n, we can show that the positive cone of

$$K_0\big(C(\mathbb{P}^{n-1}(\mathfrak{T}))\big) = \mathbb{Z}[Q_{1,\mathrm{id},1}] \oplus \cdots \oplus \mathbb{Z}[Q_{n,\mathrm{id},1}] \cong \mathbb{Z}^n$$

contains the set $(\mathbb{Z}_{\geq}^{n-1} \times \{0\}) \cup (\mathbb{Z}^{n-1} \times \mathbb{N})$ consisting of $(k_1, \ldots, k_{n-1}, m) \in \mathbb{Z}^n$ with either $k_j \geq 0$ for all j or the rank $m > 0$.

Indeed, under the canonical embedding

$$\iota\colon C(\mathbb{P}^{n-2}(\mathfrak{T})) \equiv C^*((\mathfrak{G}_{n-1})_0) \to C(\mathbb{P}^{n-1}(\mathfrak{T})) \equiv C^*((\mathfrak{G}_n)_0)$$

due to the degree-preserving groupoid embedding of

$$(\mathbb{Z}^{n-1} \ltimes \overline{\mathbb{Z}}^{n-1})\big|_{\overline{\mathbb{Z}}_{\geq}^{n-1}} \text{ in } (\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n)\big|_{\overline{\mathbb{Z}}_{\geq}}^{n}$$

as

$$\big((\mathbb{Z}^{n-1}\times\{0\})\ltimes(\overline{\mathbb{Z}}^{n-1}\times\{0\})\big)\big|_{\overline{\mathbb{Z}}^{n-1}_{\geq}\times\{0\}},$$

a projection p (for example, $\partial_{n-1}(P_{k_1}\otimes\cdots\otimes P_{k_{n-1}})$) over $C(\mathbb{P}^{n-2}(\mathfrak{T}))$ becomes the projection $p\otimes P_1$ (for example, $\partial_n(P_{k_1}\otimes\cdots\otimes P_{k_{n-1}}\otimes P_1)$) over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$. Furthermore, if $p\sim q$ over $C(\mathbb{P}^{n-2}(\mathfrak{T}))$, say, $upu^{-1}=q$ for some $u\in\mathrm{GL}_\infty(C(\mathbb{P}^{n-2}(\mathfrak{T})))$ then the equivalence $p\otimes P_1\sim q\otimes P_1$ over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ can be explicitly constructed as

$$\big((u\otimes P_1)+\partial_n(I\otimes P_{-1})\big)(p\otimes P_1)\big((u\otimes P_1)+\partial_n(I\otimes P_{-1})\big)^{-1}=q\otimes P_1$$

with $(u\otimes P_1)+\partial_n(I\otimes P_{-1})\in\mathrm{GL}_\infty(C(\mathbb{P}^{n-1}(\mathfrak{T})))$. Now consider the well-defined group homomorphism

$$K_0(\iota)\colon K_0\big(C(\mathbb{P}^{n-2}(\mathfrak{T}))\big)\cong\mathbb{Z}^{n-1}\to K_0\big(C(\mathbb{P}^{n-1}(\mathfrak{T}))\big)\cong\mathbb{Z}^n$$

mapping the positive cone of $K_0(C(\mathbb{P}^{n-2}(\mathfrak{T})))$ into that of $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))$. Since under ι, the projection $Q_{j,\mathrm{id},1}$ over $C(\mathbb{P}^{n-2}(\mathfrak{T}))$ for $0<j\leq n-1$ is sent to the projection $Q_{j,\mathrm{id},1}$ over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$, by induction hypothesis, we get that the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))\cong\mathbb{Z}^n$ contains

$$\big(\mathbb{Z}^{n-2}_{\geq}\times\{0\}\times\{0\}\big)\cup\big(\mathbb{Z}^{n-2}\times\mathbb{N}\times\{0\}\big),$$

and hence

$$\big(\mathbb{Z}^{n-1}_{\geq}\times\{0\}\big)\cup\big(\mathbb{Z}^{n-2}\times\mathbb{N}\times\mathbb{Z}_{\geq}\big).$$

On the other hand, for $k>0$,

$$\begin{aligned}\hat{I}\otimes P_{-k}&=(\hat{I}\otimes P_{-(k+1)})\boxplus(\hat{I}\otimes e_{kk})\\&\qquad\sim(\hat{I}\otimes P_{-(k+1)})\boxplus(I'\otimes P_{-k}\otimes P_1)\text{ over }\mathfrak{T}^{\otimes n},\end{aligned}$$

where I' denotes the identity element of $\mathfrak{T}^{\otimes n-2}$ and e_{ij} with $i,j\in\mathbb{Z}_{\geq}$ represents a matrix unit projection, because $\hat{I}\otimes P_{-k}$ is the sum of orthogonal projections $(\hat{I}\otimes P_{-(k+1)})$ and $(\hat{I}\otimes e_{kk})$, and $(\hat{I}\otimes e_{kk})\boxplus 0$ when conjugated by

$$u_k:=\begin{pmatrix}I'\otimes I\otimes P_k & I'\otimes(\mathcal{S}^k)^*\otimes\mathcal{S}^k\\ I'\otimes\mathcal{S}^k\otimes(\mathcal{S}^k)^* & I'\otimes P_k\otimes I\end{pmatrix}\in\mathrm{GL}_2(\mathfrak{T}^{\otimes n})$$

becomes $0\boxplus(I'\otimes P_{-k}\otimes P_1)$. Since $\partial_n(u_k)$ of total degree 0 is in $M_2(C(\mathbb{P}^{n-1}(\mathfrak{T})))$, we get

$$\partial_n(\hat{I}\otimes P_{-k})\sim\partial_n\big((\hat{I}\otimes P_{-(k+1)})\big)\boxplus\iota\big(\partial_{n-1}(I'\otimes P_{-k})\big)\text{ over }C(\mathbb{P}^{n-1}(\mathfrak{T})),$$

and hence

$$\big[\partial_n(\hat{I}\otimes P_{-k})\big]-\big[\partial_n\big((\hat{I}\otimes P_{-(k+1)})\big)\big]\in\mathbb{Z}^{n-2}\times\{1\}\times\{0\}\text{ in }\mathbb{Z}^n,$$

because $[\partial_{n-1}(I' \otimes P_{-k})] \in \mathbb{Z}^{n-2} \times \{1\}$ for the rank-one projection $I' \otimes P_{-k}$ over $\mathfrak{T}^{\otimes n-1}$. With

$$\left[\partial_n(\widehat{I} \otimes P_{-1})\right] = \left[\partial_n(\widetilde{I})\right] - \left[\partial_n(\widehat{I} \otimes P_1)\right] = (0, \ldots, 0, -1, 1) \in \mathbb{Z}^n,$$

we get inductively

$$\left[\partial_n(\widehat{I} \otimes P_{-k})\right] \in \mathbb{Z}^{n-2} \times \{-k\} \times \{1\} \subset \mathbb{Z}^n \cong K_0\big(C(\mathbb{P}^{n-1}(\mathfrak{T}))\big)$$

for all $k > 0$. Thus, the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T}))) \cong \mathbb{Z}^n$ contains

$$\big(\mathbb{Z}_{\geq}^{n-1} \times \{0\}\big) \cup \big(\mathbb{Z}^{n-1} \times \{1\}\big),$$

and hence

$$\big(\mathbb{Z}_{\geq}^{n-1} \times \{0\}\big) \cup \big(\mathbb{Z}^{n-1} \times \mathbb{N}\big).$$

On the other hand, the positive cone of $K_0(C(\mathbb{P}^{n-2}(\mathfrak{T})))$ is mapped into the positive cone of $K_0(C(\mathbb{P}^{n-1}(\mathfrak{T})))$ by the homomorphism $\cdot \times \{0\} \equiv K_0(\iota)$. So it is easy to get inductively the conclusion. □

We note that for the case of $n = 2$, the finitely generated projective modules over $C(\mathbb{P}^1(\mathfrak{T}))$ are completely classified with the positive cone of $K_0(C(\mathbb{P}^1(\mathfrak{T})))$ explicitly identified in [29].

8. Quantum line bundles

In this section, we identify the quantum line bundles $L_k := C(\mathbb{S}_H^{2n-1})_k$ of degree k over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$ with a concrete (equivalence class of) projection described in terms of the elementary projections defined in the previous section. We continue to use $\widehat{I}$ and $\widetilde{I}$ to denote the identity elements of $\mathfrak{T}^{\otimes n-1}$ and $\mathfrak{T}^{\otimes n}$ respectively, and we start to use $0^{(l)}$ to denote the zero of $\mathbb{Z}^l$.

To distinguish between ordinary function product and convolution product, we denote the groupoid C*-algebraic (convolution) multiplication of elements in $C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ by $*$, while omitting $*$ when the elements are presented as operators or when they are multiplied together pointwise as functions on $\mathcal{G}$. We also view $C_c(\mathfrak{G}_n)$ or $C_c((\mathfrak{G}_n)_k)$ (also abbreviated as $C_c(\mathfrak{G}_n)_k$) as left $C_c(\mathfrak{G}_n)_0$-modules with $C_c(\mathfrak{G}_n)$ carrying the convolution algebra structure as a subalgebra of the groupoid C*-algebra $C^*(\mathfrak{G}_n)$. Similarly, for a closed subset X of the unit space of $\mathfrak{G}_n$, the inverse image $\mathfrak{G}_n \restriction_X$ of X under the source map of $\mathfrak{G}_n$ or its grade-k component $(\mathfrak{G}_n \restriction_X)_k$ gives rise to a left $C_c(\mathfrak{G}_n)_0$-module $C_c(\mathfrak{G}_n \restriction_X)$ or $C_c(\mathfrak{G}_n \restriction_X)_k$.

We define a partial isometry in $C(\mathbb{S}_H^{2n-1}) \equiv C^*(\mathfrak{G}_n)$ for each $k \in \mathbb{Z}$ as the characteristic function χ_{B_k} of the compact open set

$$B_k := \big\{(0, k, p, q) \in \mathfrak{G}_n \subset \mathbb{Z}^{n-1} \times \mathbb{Z} \times \overline{\mathbb{Z}}_{\geq}^{n-1} \times \overline{\mathbb{Z}}_{\geq} : q + k \geq 0\big\} \subset \mathfrak{G}_n.$$

It is easy to verify that $\chi_{B_k} \in C_c(\mathfrak{G}_n)_k$ and $(\chi_{B_k})^* \in C_c(\mathfrak{G}_n)_{-k}$ such that

$$(\chi_{B_k})^* * \chi_{B_k} = \begin{cases} \chi_{\{0^{(n)}\}\times(\overline{\mathbb{Z}}^n_{\geq}\setminus\mathbb{Z}^n_{\geq})} = 1_{C^*(\mathfrak{G}_n)} \equiv 1_{C^*(\mathfrak{G}_n)_0} & \text{if } k \geq 0, \\ \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq|k|})\setminus\mathbb{Z}^n_{\geq})} = \partial_n(\widehat{I} \otimes P_{-|k|}) & \text{if } k < 0, \end{cases}$$

and

$$\chi_{B_k} * (\chi_{B_k})^* = \begin{cases} \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq k})\setminus\mathbb{Z}^n_{\geq})} = \partial_n(\widehat{I} \otimes P_{-k}) & \text{if } k \geq 0, \\ \chi_{\{0^{(n)}\}\times(\overline{\mathbb{Z}}^n_{\geq}\setminus\mathbb{Z}^n_{\geq})} = 1_{C^*(\mathfrak{G}_n)} \equiv 1_{C^*(\mathfrak{G}_n)_0} & \text{if } k < 0. \end{cases}$$

For $k \geq 0$, we have $C_c(\mathfrak{G}_n)_k * (\chi_{B_k})^* \subset C_c(\mathfrak{G}_n)_0$ and

$$\big(C_c(\mathfrak{G}_n)_k * (\chi_{B_k})^*\big) * \chi_{B_k} = C_c(\mathfrak{G}_n)_k,$$

which implies that the convolution operator $\cdot * \chi_{B_k}$ maps $C_c(\mathfrak{G}_n)_0$ onto $C_c(\mathfrak{G}_n)_k$. Since

$$\chi_{B_k} * (\chi_{B_k})^* = \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq k})\setminus\mathbb{Z}^n_{\geq})},$$

we get $\cdot * \chi_{B_k}$ mapping

$$C_c(\mathfrak{G}_n)_0 * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq k})\setminus\mathbb{Z}^n_{\geq})}$$

bijectively onto $C_c(\mathfrak{G}_n)_k$ with $\cdot * (\chi_{B_k})^*$ as the inverse. Furthermore, $\cdot * \chi_{B_k}$ is a left $C_c(\mathfrak{G}_n)_0$-module homomorphism. With χ_{B_k} being a partial isometry, $\cdot * \chi_{B_k}$ and $\cdot * (\chi_{B_k})^*$ extend continuously to provide an isomorphism between the $C^*(\mathfrak{G}_n)_0$-modules

$$C^*(\mathfrak{G}_n)_0 \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq k})\setminus\mathbb{Z}^n_{\geq})} \equiv C^*(\mathfrak{G}_n)_0 \partial_n(\widehat{I} \otimes P_{-k})$$

and $C^*(\mathfrak{G}_n)_k \equiv \overline{C_c(\mathfrak{G}_n)_k}$. So the quantum line bundle $C^*(\mathfrak{G}_n)_k$ is identified with the projection $\partial_n(\widehat{I} \otimes P_{-k})$.

For $k < 0$, we consider the direct sum decomposition as left $C_c(\mathfrak{G}_n)_0$-modules

$$\begin{aligned} C_c(\mathfrak{G}_n)_k &= \big(C_c(\mathfrak{G}_n)_k * \chi_{\{0^{(n)}\}\times(\overline{\mathbb{Z}}^{n-1}_{\geq}\setminus\mathbb{Z}^{n-1}_{\geq})\times\{0,1,\ldots,|k|-1\}}\big) \\ &\qquad\qquad \oplus \big(C_c(\mathfrak{G}_n)_k * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq|k|})\setminus\mathbb{Z}^n_{\geq})}\big) \\ &= C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}^{n-1}_{\geq}\setminus\mathbb{Z}^{n-1}_{\geq})\times\{0,1,\ldots,|k|-1\}}\big)_k \\ &\qquad\qquad \oplus \big(C_c(\mathfrak{G}_n)_k * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq|k|})\setminus\mathbb{Z}^n_{\geq})}\big). \end{aligned}$$

From

$$C_c(\mathfrak{G}_n)_0 * \chi_{B_k} * (\chi_{B_k})^* \equiv C_c(\mathfrak{G}_n)_0 * 1_{C^*(\mathfrak{G}_n)} = C_c(\mathfrak{G}_n)_0$$

and

$$C_c(\mathfrak{G}_n)_k * (\chi_{B_k})^* * \chi_{B_k} = C_c(\mathfrak{G}_n)_k * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq|k|})\setminus\mathbb{Z}^n_{\geq})}$$

we see that $\cdot * \chi_{B_{|k|}}$ is a left $C_c(\mathfrak{G}_n)_0$-module isomorphism between $C_c(\mathfrak{G}_n)_0$ and $C_c(\mathfrak{G}_n)_k * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}^{n-1}_{\geq}\times\overline{\mathbb{Z}}_{\geq|k|})\setminus\mathbb{Z}^n_{\geq})}$ with $\cdot * (\chi_{B_k})^*$ as its inverse.

On the other hand, in the $C_c(\mathfrak{G}_n)_0$-module decomposition

$$C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0,1,\ldots,|k|-1\}}\big)_k = \bigoplus_{j=0}^{|k|-1} C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{j\}}\big)_k,$$

each $C_c(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{j\}})_k$ is isomorphic to the $C_c(\mathfrak{G}_n)_0$-module

$$C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\}}\big)_{k+j}$$

with $k+j<0$ via the homeomorphism

$$(m,l,p,j)\in\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{j\}}\big)_k \subset \mathbb{Z}^{n-1}\times\mathbb{Z}\times\overline{\mathbb{Z}}_{\geq}^{n-1}\times\mathbb{Z}_{\geq}$$
$$\mapsto (m,l+j,p,0)\in\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\}}\big)_{k+j},$$

where the implicit condition $l+j\geq 0$ is equivalent to $l\geq -j$. So we focus on analyzing $C_c(\mathfrak{G}_n)_0$-modules of the form

$$C_c\big(\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\}}\big)_{-r}\big) = C_c(\mathfrak{G}_n)_{-r} * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\})}$$
$$= C_c(\mathfrak{G}_n)_{-r}\partial_n(\hat{I}\otimes P_1)$$

with $r\geq 0$. Note that the $C^*(\mathfrak{G}_n)_0$-module

$$\overline{C_c(\mathfrak{G}_n)_0\partial_n(\hat{I}\otimes P_1)} = C^*(\mathfrak{G}_n)_0\partial_n(\hat{I}\otimes P_1)$$

is identified with the projection $\partial_n(\hat{I}\otimes P_1)\equiv Q_{n-1,\mathrm{id},1}$.

For $r>0$, similar to the argument used above, it can be checked that the compact open subset

$$B'_{-r} := \big\{(0,-r,0,p,q,0)\in\mathfrak{G}_n\subset\mathbb{Z}^{n-2}\times\mathbb{Z}\times\mathbb{Z}\times\overline{\mathbb{Z}}_{\geq}^{n-2}\times\overline{\mathbb{Z}}_{\geq}\times\overline{\mathbb{Z}}_{\geq} : q\geq r\big\}\subset\mathfrak{G}_n$$

defines a partial isometry $\chi_{B'_{-r}}\in C_c(\mathfrak{G}_n)_{-r}$ with $(\chi_{B'_{-r}})^*\in C_c(\mathfrak{G}_n)_r$ such that

$$(\chi_{B'_{-r}})^* * \chi_{B'_{-r}} = \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{n-2}\times\overline{\mathbb{Z}}_{\geq r}\times\{0\})\setminus\mathbb{Z}_{\geq}^{n})} = \partial_n(I^{\otimes n-2}\otimes P_{-r}\otimes I)$$

and

$$\chi_{B'_{-r}} * (\chi_{B'_{-r}})^* = \chi_{\{0^{(n)}\}\times(\overline{\mathbb{Z}}_{\geq}^{n-1}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\}}.$$

In the decomposition

$$C_c(\mathfrak{G}_n)_{-r} * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}_{\geq}^{n-1}}\setminus\mathbb{Z}_{\geq}^{n-1})\times\{0\})}$$
$$= \big(C_c(\mathfrak{G}_n)_{-r} * (\chi_{\{0^{(n)}\}\times(\overline{\mathbb{Z}}_{\geq}^{n-2}\setminus\mathbb{Z}_{\geq}^{n-2})\times\{0,1,\ldots,r-1\}\times\{0\}})\big)$$
$$\oplus\big(C_c(\mathfrak{G}_n)_{-r} * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{n-2}\times\overline{\mathbb{Z}}_{\geq r}\times\{0\})\setminus\mathbb{Z}_{\geq}^{n})}\big)$$
$$= C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-2}\setminus\mathbb{Z}_{\geq}^{n-2})\times\{0,1,\ldots,r-1\}\times\{0\}}\big)_{-r}$$
$$\oplus\big(C_c(\mathfrak{G}_n)_{-r} * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{n-2}\times\overline{\mathbb{Z}}_{\geq r}\times\{0\})\setminus\mathbb{Z}_{\geq}^{n})}\big),$$

the second summand is isomorphic, via the right convolution $\cdot * (\chi_{B'_{-r}})^*$ by the partial isometry $(\chi_{B'_{-r}})^*$, to the $C_c(\mathfrak{G}_n)_0$-module

$$C_c(\mathfrak{G}_n)_0 * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{n-1}\times\{0\})\setminus\mathbb{Z}_{\geq}^{n})} = C_c(\mathfrak{G}_n)_0 \partial_n(\widehat{I} \otimes P_1).$$

Now we introduce the notation of a $C_c(\mathfrak{G}_n)_0$-module

$$A_{r,l} := C_c\big((\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{l}\setminus\mathbb{Z}_{\geq}^{l})\times\{0^{(n-l)}\}})_{-r}\big) \subset C_c((\mathfrak{G}_n)_{-r}) \subset C_c(\mathfrak{G}_n)$$

for $r \geq 0$ and $1 \leq l \leq n-1$. We note that the $C_c(\mathfrak{G}_n)_0$-module

$$A_{r,1} = C_c\big((\mathfrak{G}_n \restriction_{\{\infty\}\times\{0^{(n-1)}\}})_{-r}\big)$$

is isomorphic to

$$\begin{aligned}C_c\big((\mathfrak{G}_n \restriction_{\{\infty\}\times\{0^{(n-1)}\}})_0\big) &= C_c\big((\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}\times\{0^{(n-1)}\})\setminus\mathbb{Z}_{\geq}^{n}})_0\big)\\ &\equiv C_c(\mathfrak{G}_n)_0 * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}\times\{0^{(n-1)}\})\setminus\mathbb{Z}_{\geq}^{n})} = C_c(\mathfrak{G}_n)_0\partial_n(I \otimes P_1^{\otimes n-1})\end{aligned}$$

via the homeomorphism

$$\begin{aligned}\big(s, t, \infty, 0^{(n-1)}\big) \in \big(\mathfrak{G}_n \restriction_{\{\infty\}\times\{0^{(n-1)}\}}\big)_{-r} &\subset \mathbb{Z}\times\mathbb{Z}^{n-1}\times\{\infty\}\times\overline{\mathbb{Z}}_{\geq}^{n-1}\\ &\mapsto \big(s+r, t, \infty, 0^{(n-1)}\big) \in \big(\mathfrak{G}_n \restriction_{\{\infty\}\times\{0^{(n-1)}\}}\big)_0.\end{aligned}$$

Applying the same kind of arguments as shown above, we get the isomorphism of $C_c(\mathfrak{G}_n)_0$-modules

$$\begin{aligned}A_{r,l} &\cong C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{l-1}\setminus\mathbb{Z}_{\geq}^{l-1})\times\{0,1,\ldots,r-1\}\times\{0^{(n-l)}\}}\big)_{-r}\\ &\qquad\oplus \big(C_c(\mathfrak{G}_n)_0 * \chi_{\{0^{(n)}\}\times((\overline{\mathbb{Z}}_{\geq}^{l}\setminus\mathbb{Z}_{\geq}^{l})\times\{0^{(n-l)}\})}\big)\\ &\cong \bigoplus_{j=0}^{r-1} C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{l-1}\setminus\mathbb{Z}_{\geq}^{l-1})\times\{j\}\times\{0^{(n-l)}\}}\big)_{-r} \oplus \big(C_c(\mathfrak{G}_n)_0\partial_n\big(I^{\otimes l}\otimes P_1^{\otimes n-l}\big)\big)\\ &\cong \bigoplus_{j=0}^{r-1} C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{l-1}\setminus\mathbb{Z}_{\geq}^{l-1})\times\{0\}\times\{0^{(n-l)}\}}\big)_{-r+j} \oplus (C_c(\mathfrak{G}_n)_0\partial_n\big(I^{\otimes l}\otimes P_1^{\otimes n-l}\big))\\ &= \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \big(C_c(\mathfrak{G}_n)_0\partial_n\big(I^{\otimes l}\otimes P_1^{\otimes n-l}\big)\big)\end{aligned}$$

for $2 \leq l \leq n-1$. This provides a recursive formula to reduce the index l of the module $A_{r,l}$.

For $n > 2$, we define a combinatorial number $\nu_n(m,l)$ recursively by

$$\nu_n(m,l) := \sum_{s=0}^{m} \nu_n(s, l-1)$$

and $\nu_n(m,1) := 1$, for $m \geq 0$ and $2 \leq l \leq n-1$, to be used in the following theorem.

Thanks to Thomas Timmermann, as he pointed out to the author, $\nu_n(m,l)$ can be identified with a familiar combinatorial number, namely,

$$\nu_n(m,l) = C_m^{m+l-1}$$

for all $m \geq 0$ and $l \geq 1$. Indeed, if either $l = 1$ or $m = 0$ (e.g. when $m + l \leq 2$), we get easily from the definition, $\nu_n(m,l) = 1 = C_m^{m+l-1}$. On the other hand, for $l \geq 2$ and $m \geq 1$, since

$$\nu_n(m,l) = \sum_{s=0}^{m-1} \nu_n(s, l-1) + \nu_n(m, l-1) = \nu_n(m-1, l) + \nu_n(m, l-1),$$

the identification can be proved by an induction on $m + l \geq 3$ as shown in

$$\nu_n(m-1,l) + \nu_n(m,l-1) = C_{m-1}^{m+l-2} + C_m^{m+l-2} = C_m^{m+l-1},$$

which is valid due to either the induction hypothesis for $m + l - 1$ (in the case of $m + l - 1 > 2$) or the already established identification (for the case of $m + l - 1 = 2$).

Theorem 6. *For $n > 2$, the quantum line bundle $L_k \equiv C(\mathbb{S}_H^{2n-1})_k$ of degree $k \in \mathbb{Z}$ over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ is isomorphic to the finitely generated projective left module over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ determined by the projection $\partial_n(\otimes^{n-1} I \otimes P_{-k})$ if $k \geq 0$, and the projection*

$$\Big(\boxplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(m,n-2)} \partial_n\big(I \otimes P_1^{\otimes n-1}\big)\Big) \\ \boxplus \Big(\boxplus_{l=1}^{n-1} \boxplus^{\nu_n(|k|-1,l)} \partial_n\big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\big)\Big)$$

if $k < 0$.

Proof. Only the case of $k < 0$ remains to be proved as follows.

For $k < 0$, starting with the established isomorphism

$$\begin{aligned} C_c(\mathfrak{G}_n)_k &= C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1} \setminus \mathbb{Z}_{\geq}^{n-1}) \times \{0,1,\dots,|k|-1\}}\big)_k \oplus C_c(\mathfrak{G}_n)_0 \\ &= \bigoplus_{m=0}^{|k|-1} C_c\big(\mathfrak{G}_n \restriction_{(\overline{\mathbb{Z}}_{\geq}^{n-1} \setminus \mathbb{Z}_{\geq}^{n-1}) \times \{m\}}\big)_k \oplus C_c(\mathfrak{G}_n)_0 \\ &\cong \bigoplus_{m=0}^{|k|-1} A_{|k|-m,n-1} \oplus C_c(\mathfrak{G}_n)_0, \end{aligned}$$

we apply repeatedly the recursive formula

$$A_{r,l} = \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \big(C_c(\mathfrak{G}_n)_0 \partial_n\big(I^{\otimes l} \otimes P_1^{\otimes n-l}\big)\big)$$

reducing l for $A_{r,l}$ with $2 \le l \le n$ until l reaches 2 with

$$A_{r,2} \cong \Big(\bigoplus_{j=0}^{r-1} \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I \otimes P_1^{\otimes n-1} \big) \big) \Big) \oplus \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes 2} \otimes P_1^{\otimes n-2} \big) \big),$$

in order to convert all terms to $C_c(\mathfrak{G}_n)_0$-modules of the form

$$C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes j} \otimes P_1^{\otimes n-j} \big)$$

for some $0 < j \le n$.

In fact, we check inductively on $1 \le j \le n-2$ that

$$\begin{aligned} C_c(\mathfrak{G}_n)_k \cong \bigoplus_{m=0}^{|k|-1} \big(\oplus^{\nu_n(m,j)} A_{|k|-m,n-j} \big) \\ \oplus \bigoplus_{l=1}^{j} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big) \big). \quad (*) \end{aligned}$$

The case of $j = 1$ is our starting point already proved. Now assuming that it holds for j, we get by the above recursive formula

$$\begin{aligned} C_c(\mathfrak{G}_n)_k &\cong \bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(m,j)} \Big(\Big(\bigoplus_{s=0}^{|k|-m-1} A_{|k|-m-s,n-j-1} \Big) \\ &\qquad \oplus \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-j} \otimes P_1^{\otimes j} \big) \big) \Big) \\ &\qquad \oplus \Big(\bigoplus_{l=1}^{j} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big) \big) \Big) \\ &\cong \Big(\bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(m,j)} \bigoplus_{s=0}^{|k|-m-1} A_{|k|-m-s,n-j-1} \Big) \\ &\qquad \oplus \big(\oplus^{\sum_{m=0}^{|k|-1} \nu_n(m,j)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-j} \otimes P_1^{\otimes j} \big) \big) \\ &\qquad \oplus \Big(\bigoplus_{l=1}^{j} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big) \big) \Big) \\ &\cong \Big(\bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(m,j)} \bigoplus_{s=0}^{|k|-m-1} A_{|k|-m-s,n-j-1} \Big) \\ &\qquad \oplus \bigoplus_{l=1}^{j+1} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big) \big) \end{aligned}$$

$$= \Big(\bigoplus_{m=0}^{|k|-1} \bigoplus_{s=0}^{m} \oplus^{\nu_n(s,j)} A_{|k|-m,n-j-1} \Big)$$
$$\oplus \bigoplus_{l=1}^{j+1} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big)$$
$$= \Big(\bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(m,j+1)} A_{|k|-m,n-j-1} \Big)$$
$$\oplus \bigoplus_{l=1}^{j+1} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big),$$

which verifies (∗) for $j+1$.

For $j = n-2$, (∗) says

$$C_c(\mathfrak{G}_n)_k \cong \bigoplus_{m=0}^{|k|-1} \big(\oplus^{\nu_n(m,n-2)} A_{|k|-m,2} \big)$$
$$\oplus \bigoplus_{l=1}^{n-2} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big)$$
$$\cong \bigoplus_{m=0}^{|k|-1} \oplus^{\nu_n(m,n-2)} \bigg(\Big(\bigoplus_{j=0}^{|k|-m-1} \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I \otimes P_1^{\otimes n-1} \big)\big) \Big)$$
$$\oplus \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes 2} \otimes P_1^{\otimes n-2} \big)\big) \bigg)$$
$$\oplus \bigoplus_{l=1}^{n-2} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big)$$
$$\cong \big(\oplus^{\sum_{m=0}^{|k|-1} (|k|-m)\nu_n(m,n-2)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I \otimes P_1^{\otimes n-1} \big)\big)$$
$$\oplus \big(\oplus^{\sum_{m=0}^{|k|-1} \nu_n(m,n-2)} \big(C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes 2} \otimes P_1^{\otimes n-2} \big)\big)\big)$$
$$\oplus \bigoplus_{l=1}^{n-2} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big)$$
$$\cong \big(\oplus^{\sum_{m=0}^{|k|-1} (|k|-m)\nu_n(m,n-2)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I \otimes P_1^{\otimes n-1} \big)\big)$$
$$\oplus \bigoplus_{l=1}^{n-1} \big(\oplus^{\nu_n(|k|-1,l)} C_c(\mathfrak{G}_n)_0 \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big).$$

After completion, we get the $C^*((\mathfrak{G}_n)_0)$-module L_k isomorphic to

$$\big(\oplus^{\sum_{m=0}^{|k|-1} (|k|-m)\nu_n(m,n-2)} C^*((\mathfrak{G}_n)_0) \partial_n \big(I \otimes P_1^{\otimes n-1} \big)\big)$$
$$\oplus \bigoplus_{l=1}^{n-1} \big(\oplus^{\nu_n(|k|-1,l)} C^*((\mathfrak{G}_n)_0) \partial_n \big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1} \big)\big),$$

which corresponds to the projection

$$\Big(\boxplus^{\sum_{m=0}^{|k|-1}(|k|-m)\nu_n(m,n-2)}\, \partial_n\big(I \otimes P_1^{\otimes n-1}\big)\Big) \boxplus \Big(\boxplus_{l=1}^{n-1} \Big(\boxplus^{\nu_n(|k|-1,l)}\, \partial_n\big(I^{\otimes n-l+1} \otimes P_1^{\otimes l-1}\big)\Big)\Big). \qquad \square$$

Little is known about the cancellation problem and hence the classification problem for finitely generated projective modules over $C(\mathbb{P}^{n-1}(\mathfrak{T}))$. We expect that these problems will be far more complicated than those for over $C(\mathbb{S}_H^{2n-1})$ and $C(\mathfrak{T}^{\otimes n})$.

The recent work of Farsi, Hajac, Maszczyk, and Zieliński [7] identifies one of three free generators of $K_0(C(\mathbb{P}^2(\mathfrak{T})))$ as $[L_1] + [L_{-1}] - 2[I]$ (in addition to $[L_1] - [I]$ and $[I]$) constructed from a Milnor module and then expresses it in terms of elementary projections, showing a perfect consistency with our result.

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