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Quasi-homogeneity of potentials

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Abstract. In noncommutative differential calculus, Jacobi algebra (or potential algebra) plays the role of Milnor algebra in the commutative case. The study of Jacobi algebras is of broad interest to researchers in cluster algebra, representation theory and singularity theory. In this article, we study the quasi-homogeneity of a potential in a complete free algebra over an algebraic closed field of characteristic zero. We prove that a potential with finite dimensional Jacobi algebra is right equivalent to a weighted homogeneous potential if and only if the corresponding class in the 0th Hochschlid homology group of the Jacobi algebra is zero. This result can be viewed as a noncommutative version of the famous theorem of Kyoji Saito on isolated hypersurface singularities.

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1. Introduction

This is our second paper studying the Jacobi-finite potentials, after [7]. Let $F = k\langle\langle x_1, \dots, x_n \rangle\rangle$ be a complete free algebra over a field k. A *potential* Φ refers to an element in the vector space $F_{\overline{\text{cyc}}}$ consisting of elements of F modulo cyclic permutations. The cyclic derivative $D_i\Phi$ of Φ with respect to x_i is an element in F. We can associate to every Φ an associative algebra $\Lambda(F, \Phi)$, defined to be the quotient of F by the closed two sided ideal generated by $D_i\Phi$ for $i = 1, \dots, n$. We call $\Lambda(F, \Phi)$ the *Jacobi algebra* (or potential algebra) associated to F and Φ. The Jacobi algebra is an invariant of the potential. It is natural to ask to what extent the potential is determined by its Jacobi algebra.

The natural projection from F to $\Lambda(F, \Phi)$ induces a natural map from $F_{\overline{\text{cyc}}}$ to $\Lambda(F, \Phi)_{\overline{\text{cyc}}}$. We denote the image of Φ under this map by $[\Phi]$. In [7], we proved that:

Theorem 1.1 ([7, Theorem A]). *Assume that $k = \mathbb{C}$. Let $\Phi, \Psi \in F_{\overline{\text{cyc}}}$ be two potentials of order ≥ 3. Suppose that the potential algebras $\Lambda(F, \Phi)$ and $\Lambda(F, \Psi)$ are both finite dimensional. Then the following two statements are equivalent:*

(1) *There is an algebra isomorphism* $\gamma\colon \Lambda(F,\Phi) \cong \Lambda(F,\Psi)$ *so that* $\gamma_*\big([\Phi]\big) = [\Psi]$.

(2) Φ *and* Ψ *are right equivalent, that is* $\Phi = H(\Psi)$ *for some automorphism* H *of* F.

We call a potential Φ *quasi-homogeneous* if $[\Phi] = 0$. It is easy to check that if Φ is weighted homogeneous (see Definition 2.8) then $[\Phi] = 0$, i.e. weighted homogeneous $\Rightarrow$ quasi-homogeneous. By the above theorem, weighted homogeneous potentials with finite dimensional Jacobi algebras are completely classified by their Jacobi algebras. The next question maybe is given an arbitrary potential, how to determine whether it is right equivalent to a weighted homogeneous potential or not? For a potential with finite dimensional Jacobi algebra, we show that it is right equivalent to a weighted homogeneous one if and only if it is quasi-homogeneous.

Theorem 1.2 (Theorem 4.1). *Assume that* k *is an algebraically closed field of zero characteristic. Let* $\Phi \in F_{\overline{\text{cyc}}}$ *be a potential of order* ≥ 3 *such that the Jacobi algebra associated to* Φ *is finite dimensional. Then* Φ *is quasi-homogeneous if and only if* Φ *is right equivalent to a weighted-homogeneous potential of type* $(r_1,\ldots,r_n)$ *for some rational numbers* $r_1,\ldots,r_n$ *lie strictly between* 0 *and* $1/2$. *Moreover, in this case, all such types* $(r_1,\ldots,r_n)$ *agree with each other up to permutations on the indexes* $1,\ldots,n$.

One can make a formal analogue between the study of potentials with the study of hypersurface singularities. If we view the complete free algebra as the ring of formal functions on noncommutative affine space, then Theorem 1.1 is a noncommutative version of Mather–Yau theorem ([9]) and Theorem 4.1 is a noncommutative version of Saito's theorem ([11]). In fact, the proofs of Theorem 1.1 and 4.1 are to some extent inspired by the proofs of these two classical theorems, although certain conceptional gap needs to be filled in the noncommutative case.

Jacobi algebras have appeared in many mathematical areas including representation theory, topology and algebraic geometry. The finite dimensional condition should be understood as an analogue of isolated hypersurface singularity. Finite dimensional Jacobi algebras can appear at least from two sources. The first is the theory of (generalized) cluster categories ([1]). The cluster category is defined from a Ginzburg dg-algebra of dimension 3 with finite dimensional zero-th homology. The zero-th homology of a Ginzburg dg-algebra is a Jacobi algebra. It also appears in noncommutative deformation theory. Given a 3-Calabi Yau dg-category with appropriate assumptions, the noncommutative deformation functor of a noncommutative rigid object in this category is represented by a finite dimensional Jacobi algebra. For example, if $C \subset Y$ is a contractible rational curve in a smooth CY 3-fold Y then the corresponding Jacobi algebra is precisely the contraction algebra considered by Donovan and Wemyss [5]. It is natural to propose a conjectured correspondence between three dimensional quasi-homogeneous hypersurface singularities that admits small resolutions and quasi-homogeneous potentials associated to the noncommutative crepant resolutions of these singularities.

It is well known that weighted-homogeneous hypersurface singularities admit a lot of good properties. For instance, the monodromy of a weighted homogeneous hypersurface singularity is semi-simple. Weighted homogeneous potentials also have some nice properties. For example, the Ginzburg algebra of a weighted homogeneous potential carries an extra grading. The calculation of Hochschild cohomology can be greatly simplified using this grading. However, compared with the commutative case the understanding of the properties of weighted homogeneous potentials is still quite limited. Theorem 4.1 is one attempt along this line. Note that the vanishing of the class $[\Phi]$ is fairly easy to check. At least at this point, commutative and noncommutative cases have marginal differences. Remember that whether an isolated hypersurface singularity is weighted homogeneous can be checked by comparing the Milnor number and the Tyurina number.

The paper is organized as follows. In Section 2, we recall several basic facts on noncommutative calculus and Jacobi algebras. These facts are well known to experts and have been reviewed in Section 2 of [7]. We repeat it simply to make the paper as self-contained as possible. The readers who are familiar with these can skip Section 2. In Section 3, we prove a Jordan–Chevalley type theorem for decomposition of derivations on complete free algebras. This result is of independent interest. It enables us to link quasi-homogeneous potentials to weighted-homogeneous one. In Section 4, we present the proof of the main theorem.

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2. Preliminaries

In this section, we collect basic notations and terminologies that are of concern. Throughout, we fix a base commutative ring k with unit. All algebras are k-algebras, and we denote $\otimes = \otimes_k$ for the tensor product of k-modules unless specified otherwise.

Fix an integer $n \geq 1$. Let F be the complete free algebra $k\langle\langle x_1, \ldots, x_n \rangle\rangle$. Elements of F are formal series

$$\sum_w a_w w,$$

where w runs over all words in $x_1, \ldots, x_n$ and $a_w \in k$. Let $\mathfrak{m} \subseteq F$ be the ideal generated by $x_1, \ldots, x_n$. For any subspace U of F, let U^{cl} be the closure of U with respect to the $\mathfrak{m}$-adic topology on F. Note that

$$U^{\mathrm{cl}} = \cap_{r \geq 0}(U + \mathfrak{m}^r).$$

Recall that *$(k$-$)$derivation of F in a F-bimodule M* is defined to be a $(k$-$)$linear map $\delta: F \to M$ satisfies the Leibniz rule, that is

$$\delta(ab) = a\delta(b) + a\delta(b)$$

for all $a, b \in F$. We denote by $\mathrm{Der}_k(F, M)$ the set of all k-derivations of F in M, which carries a natural k-module structure. We write

$$\mathrm{Der}_k(F) := \mathrm{Der}_k(F, F)$$

and call its elements *k-derivations of F*. Clearly, derivations of F are uniquely determined by their value at generators x_j. Note that $\mathrm{Der}_k(F)$ admits neither left nor right F-module structure.

Let $F \mathbin{\hat{\otimes}} F$ be the k-module whose elements are formal series of the form

$$\sum\nolimits_{u,v} a_{u,v}\, u \otimes v,$$

where u, v runs over all words in $x_1, \ldots, x_n$ and $a_{u,v} \in k$. This is nothing but the adic completion of $F \otimes F$ with respect to the ideal $\mathfrak{m} \otimes F + F \otimes \mathfrak{m}$. It contains $F \otimes F$ as a subspace under the identification

$$\Big(\sum_u a'_u\, u\Big) \otimes \Big(\sum_v a''_v\, v\Big) \mapsto \sum_{u,v} a'_u a''_v\, u \otimes v.$$

There are two obvious F-bimodule structures on $F \mathbin{\hat{\otimes}} F$, which we call the outer and the inner bimodule structures respectively, extends those on the subspace $F \otimes F$ defined respectively by

$$a(b' \otimes b'')c := ab' \otimes b''c \quad \text{and} \quad a * (b' \otimes b'') * c := b'c \otimes ab''.$$

Unless otherwise stated, we view $F \mathbin{\hat{\otimes}} F$ as a F-bimodule with respect to the outer bimodule structure.

We call derivations of F in the F-bimodule $F \mathbin{\hat{\otimes}} F$ *double derivations* of F. The inner bimodule structure on $F \mathbin{\hat{\otimes}} F$ naturally yields a bimodule structure on the space of double derivations

$$\mathbb{D}\mathrm{er}_k(F) := \mathrm{Der}_k(F, F \mathbin{\hat{\otimes}} F).$$

For any $\delta \in \mathbb{D}\mathrm{er}_k(F)$ and any $f \in F$, we also write $\delta(f)$ in Sweedler's notation as

$$\delta(f) = \delta(f)' \otimes \delta(f)''. \tag{2.1}$$

One shall bear in mind that this notation is an infinite sum. Clearly, double derivations of F are uniquely determined by their values on generators x_j. Thus, we have double derivations

$$\frac{\partial}{\partial x_i}\colon F \to F \mathbin{\hat{\otimes}} F, \quad x_j \mapsto \delta_{i,j}\, 1 \otimes 1.$$

Moreover, every double derivation of F has a unique representation of the form

$$\sum_{i=1}^{n} \sum_{u,v} a_{u,v}^{(i)}\, u * \frac{\partial}{\partial x_i} * v, \quad a_{u,v}^{(i)} \in k, \tag{2.2}$$

where u, v run over all words on $x_1, \ldots, x_n$, and $*$ denotes the multiplication of the bimodule structure of $\mathbb{D}\mathrm{er}_k(F)$. The infinite sum (2.2) makes sense in the obvious way.

There are two obvious linear maps $\mu\colon F \mathbin{\hat{\otimes}} F \to F$ and $\tau\colon F \mathbin{\hat{\otimes}} F \to F \mathbin{\hat{\otimes}} F$ given respectively by

$$\mu\Big(\sum_{u,v} a_{u,v} u \otimes v\Big) = \sum_{w} \Big(\sum_{w=uv} a_{u,v}\Big)\, w \quad \text{and} \quad \tau\Big(\sum_{u,v} a_{u,v} u \otimes v\Big) = \sum_{u,v} a_{v,u} u \otimes v.$$

Also, putting on $\mathrm{Hom}_k(F, F)$ the F-bimodule structure defined by

$$a_1 \cdot f \cdot a_2\colon b \mapsto a_1 f(b) a_2, \quad f \in \mathrm{Hom}_k(F, F),\ a_1, a_2, b \in F.$$

Though the map $\mathbb{D}\mathrm{er}_k(F) \xrightarrow{\mu\circ -} \mathrm{Hom}_k(F, F)$ doesn't preserves bimodule structures, the map

$$\mu \circ \tau \circ -\colon \mathbb{D}\mathrm{er}_k(F) \to \mathrm{Hom}_k(F, F)$$

is clearly a homomorphism of F-bimodules. We write

$$\mathrm{cDer}_k(F) := \mathrm{im}(\mu \circ \tau \circ -)$$

and call its elements *cyclic derivations* of F. Note that by definition $\mathrm{cDer}_k(F)$ is an F-sub-bimodule of $\mathrm{Hom}_k(F, F)$, and hence is itself an F-bimodule. For each $1 \le i \le n$, let

$$D_{x_i} := \mu \circ \tau \circ \frac{\partial}{\partial x_i} \in \mathrm{cDer}_k(F).$$

These cyclic derivations were first studied by Rota, Sagan and Stein [10]. By (2.2), every cyclic derivation of F has a decomposition (not necessary unique) of the form

$$\sum_{i=1}^{n} \sum_{u,v} a_{u,v}^{(i)}\, u \cdot D_{x_i} \cdot v, \quad a_{u,v}^{(i)} \in k. \tag{2.3}$$

In the sequel, if there is no risk of confusion, we always simply write D_i for D_{x_i}.

Elements of $F_{\overline{\text{cyc}}} := F/[F,F]^{\text{cl}}$ are called *potentials* of F. Let $\pi\colon F \to F_{\overline{\text{cyc}}}$ be the canonical projection. Given a potential $\Phi \in F_{\overline{\text{cyc}}}$, there are two linear maps

$$\begin{aligned} \Phi_{\#}\colon \operatorname{Der}_k(F) &\to F_{\overline{\text{cyc}}}, & \xi &\mapsto \pi(\xi(\phi)),\\ \Phi_{*}\colon \operatorname{cDer}_k(F) &\to F, & D &\mapsto D(\phi), \end{aligned}$$

where ϕ is any representative of Φ. Note that all derivations and cyclic derivations of F are continuous with respect to the $\mathfrak{m}$-adic topology on F. Consequently,

$$\xi\big([F,F]^{\text{cl}}\big) \subseteq [F,F]^{\text{cl}}$$

for each derivation $\xi \in \operatorname{Der}_k(F)$, and

$$D\big([F,F]^{\text{cl}}\big) = 0$$

for each cyclic derivation $D \in \operatorname{cDer}_k(F)$. It follows immediately that the resulting maps $\Phi_{\#}$ and Φ_{*} are independent of the choice of ϕ.

Lemma 2.1. *For any potential $\Phi \in F_{\overline{\text{cyc}}}$, there is a commutative diagram as follows:*

$$\begin{array}{ccccc} \mathbb{D}\mathrm{er}_k(F) & \xrightarrow{\mu\circ\tau\circ-} & \operatorname{cDer}_k(F) & \xrightarrow{\Phi_*} & F \\ \downarrow{\scriptstyle \mu\circ-} & & & & \downarrow{\scriptstyle \pi} \\ \operatorname{Der}_k(F) & & \xrightarrow{\Phi_{\#}} & & F_{\overline{\text{cyc}}}. \end{array}$$

Moreover, Φ_* is a homomorphism of F-bimodules and hence $\operatorname{im}(\Phi_*)$ *is a two-sided ideal of F.*

Proof. Let $\phi \in F$ be an arbitrary representative of Φ. Note that

$$\mu(\delta(\phi)) - \mu(\tau(\delta(\phi))) \in [F,F]^{\text{cl}}$$

for all double derivations $\delta \in \mathbb{D}\mathrm{er}_k(F)$ and all formal series $\phi \in F$, the diagram commutes. The surjection of the maps π, $\mu\circ-$ and $\mu\circ\tau\circ-$ is clear. Also, we have

$$\Phi_*(a\cdot D\cdot b) = (a\cdot D\cdot b)(\phi) = aD(\phi)b = a\Phi_*(D)b$$

for all $a, b \in F$ and $D \in \operatorname{cDer}_k(F)$, so Φ_* is a homomorphism of F-bimodules. □

Recall that two words u and v on $x_1,\dots,x_n$ are *conjugate* if there are words w_1, w_2 such that $u = w_1w_2$ and $v = w_2w_1$. Equivalent classes under this equivalence relation are called *necklaces* or *conjugacy classes*. Also recall that a word u is *lexicographically smaller* than another word v if there exist factorizations $u = wx_iw'$ and $v = wx_jw''$ with $i < j$. This order relation restricts to a total order on each necklace. Let us call a word *standard* if it is maximal in its necklace.

Remark 2.2. Every potential of F has a unique representative, called the *canonical representative*, which is a formal linear combination of standard words. Given a potential $\Phi \in F_{\overline{\text{cyc}}}$, the smallest integer r such that $\Phi \in \pi(\mathfrak{m}^r)$ is called the *order of* Φ. Note that the order of a potential coincides with the order of its canonical representative.

Definition 2.3. Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential. The *Jacobi algebra* or the *potential algebra* associated to Φ is defined to be the associative algebra

$$\Lambda(F, \Phi) := F/\mathrm{J}(F, \Phi),$$

where $\mathrm{J}(F, \Phi) := \mathrm{im}(\Phi_*)$ is called the *Jacobi ideal* of F associated to Φ. Note that if k is artinian then

$$\mathrm{J}(F, \Phi) = (\Phi_*(D_{x_1}), \dots, \Phi_*(D_{x_n}))^{cl}$$

by [7, Lemma 2.6].

We denote by $\mathcal{G} := \mathrm{Aut}_k(F, \mathfrak{m})$ the group of k-algebra automorphisms of F that preserve $\mathfrak{m}$. It is a subgroup of $\mathrm{Aut}_k(F)$, the group of all k-algebra automorphisms of F. In the case when k is a field, $\mathcal{G} = \mathrm{Aut}_k(F)$. Note that $\mathcal{G}$ acts on F and $F_{\overline{\text{cyc}}}$ in the obvious way.

Definition 2.4. For potentials $\Phi, \Psi \in F_{\overline{\text{cyc}}}$, we say Φ is *(formally) right equivalent* to Ψ and write $\Phi \sim \Psi$, if Φ and Ψ lie in the same $\mathcal{G}$-orbit.

Proposition 2.5 ([4, Proposition 3.7], [7, Proposition 3.3]). *Let $\Phi \in F_{\overline{\text{cyc}}}$ and $H \in \mathcal{G}$. Then,*

$$H(\mathrm{J}(F, \Phi)) = \mathrm{J}(F, H(\Phi)).$$

Consequently, H induces an isomorphism of algebras $\Lambda(F, \Phi) \cong \Lambda(F, H(\Phi))$.

Given a potential $\Phi \in F_{\overline{\text{cyc}}}$, let $\mathfrak{m}_\Phi := \mathfrak{m}/J(F, \Phi)$, which is an ideal of $\Lambda(F, \Phi)$. By [7, Lemma 2.8], the $\mathfrak{m}_\Phi$-adic topology of $\Lambda(F, \Phi)$ is complete. Let

$$\Lambda(F, \Phi)_{\overline{\text{cyc}}} := \Lambda(F, \Phi)/[\Lambda(F, \Phi), \Lambda(F, \Phi)]^{\text{cl}}.$$

Note that if $\Lambda(F, \Phi)$ is finitely generated as a k-module then

$$\Lambda(F, \Phi)_{\overline{\text{cyc}}} = \Lambda(F, \Phi)/[\Lambda(F, \Phi), \Lambda(F, \Phi)] = HH_0(\Lambda(F, \Phi)).$$

The projection map $F \to \Lambda(F, \Phi)$ induces a natural map

$$p_\Phi\colon F_{\overline{\text{cyc}}} \to \Lambda(F, \Phi)_{\overline{\text{cyc}}}$$

with kernel $\pi(\mathrm{J}(F, \Phi))$. For any $\Theta \in F_{\overline{\text{cyc}}}$, we write $[\Theta]$ for the class $p_\Phi(\Theta)$ in $\Lambda(F, \Phi)_{\overline{\text{cyc}}}$.

Definition 2.6. A potential $\Phi \in F_{\overline{\text{cyc}}}$ is said to be *quasi-homogeneous* if the class $[\Phi]$ is zero in $\Lambda(F, \Phi)_{\overline{\text{cyc}}}$, or equivalently Φ is contained in $\pi(\mathrm{J}(F, \Phi))$.

The following result on quasi-homogeneous potentials is of interest. It is an immediate consequence of Theorem 1.1.

Theorem 2.7 ([7, Corollary 3.8]). *Let k be the complex number field. Let $\Phi, \Psi \in F_{\overline{\mathrm{cyc}}}$ be two quasi-homogeneous potentials of order ≥ 3 such that the Jacobi algebras $\Lambda(F, \Phi)$ and $\Lambda(F, \Psi)$ are both finite dimensional. Then Φ is right equivalent to Ψ if and only if $\Lambda(F, \Phi) \cong \Lambda(F, \Psi)$ as algebras.*

Definition 2.8. Let $(r_1, \ldots, r_n)$ be a tuple of rational numbers with $0 < r_1, \ldots, r_n \leq 1/2$. A potential $\Phi \in F_{\overline{\mathrm{cyc}}}$ is said to be *weighted-homogeneous of type* $(r_1, \ldots, r_n)$ if it has a representative which is a linear combination of monomials $x_{i_1} x_{i_2} \cdots x_{i_p}$ such that $r_{i_1} + r_{i_2} + \cdots + r_{i_p} = 1$.

Lemma 2.9. *Let $\Phi \in F_{\overline{\mathrm{cyc}}}$ be a potential that is right equivalent to a weighted-homogeneous potential of a certain type. Then Φ is quasi-homogeneous.*

Proof. By Proposition 2.5, quasi-homogeneous potentials are closed under the action of $\mathscr{G}$. So we may assume Φ is itself weighted-homogeneous of type $(r_1, \ldots, r_n)$ for some rational numbers $0 < r_1, \ldots, r_n \leq 1/2$. It is not hard to see that

$$\Phi = \pi\Big(\sum_{i=1}^{n} r_i x_i \cdot \Phi_*(D_{x_i})\Big).$$

The result follows. □

The aim of this paper is to study the converse of Lemma 2.9. To this end, we employ the following geometric point of view. Let $\mathrm{Der}_k^+(F)$ be the space of derivations of F that send $\mathfrak{m}$ to $\mathfrak{m}$. Intuitively, $\mathrm{Der}_k^+(F)$ can be seen as the "tangent space" of the "infinite dimensional Lie group" $\mathscr{G}$ at the identity map Id. For every potential $\Phi \in F_{\overline{\mathrm{cyc}}}$, the action of $\mathscr{G}$ on $F_{\overline{\mathrm{cyc}}}$ yields a "smooth" map

$$\lambda_\Phi : \mathscr{G} \to F_{\overline{\mathrm{cyc}}}, \quad H \mapsto H(\Phi).$$

The map $\Phi_\# = \Phi_\#|_{\mathrm{Der}_k^+(F)} : \mathrm{Der}_k^+(F) \to F_{\overline{\mathrm{cyc}}}$ can be seen as the "differential" of λ_Φ at Id.

It is clear that a potential $\Phi \in F_{\overline{\mathrm{cyc}}}$ is weighted-homogeneous of type $(r_1, \ldots, r_n)$ if and only if $\Phi_\#(\xi) = \Phi$, where $\xi \in \mathrm{Der}_k^+(F)$ is the derivation given by $\xi(x_i) = r_i x_i$. We have the following characterization of quasi-homogeneous potentials in this perspective.

Lemma 2.10. *Suppose that k is a field. Let $\Phi \in F_{\overline{\mathrm{cyc}}}$ be a potential of order ≥ 2 such that the Jacobi algebra associated to Φ is finite dimensional. Then Φ is quasi-homogeneous if and only if $\Phi_\#(\xi) = \Phi$ for some derivation $\xi \in \mathrm{Der}_k^+(F)$.*

Proof. The if part is clear by the commutative diagram in Lemma 2.1. Next we show the only if part. Assume that Φ is quasi-homogeneous. By [7, Proposition 3.14 (1)],

$$\Phi = \pi\Big(\sum_{i=1}^{n} g_i \cdot \Phi_*(D_{x_i})\Big) = (\pi \circ \Phi_*)\Big(\sum_{i=1}^{n} g_i \cdot D_{x_i}\Big)$$

for some formal series $g_1, \dots, g_n \in \mathfrak{m}$. Let $\mathbb{D}\mathrm{er}_k^+(F)$ be the space of double derivations that map $\mathfrak{m}$ to $\mathfrak{m} \mathbin{\widehat{\otimes}} F + F \mathbin{\widehat{\otimes}} \mathfrak{m}$, and let $\mathrm{cDer}_k^+(F)$ be the space of cyclic derivations that map $\mathfrak{m}$ to $\mathfrak{m}$. Then the commutative diagram in Lemma 2.1 restricts to a commutative diagram

$$\begin{array}{ccccc} \mathbb{D}\mathrm{er}_k^+(F) & \xrightarrow{\mu\circ\tau\circ-} & \mathrm{cDer}_k^+(F) & \xrightarrow{\Phi_*} & F \\ \Big\downarrow{\scriptstyle \mu\circ-} & & & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{Der}_k^+(F) & & \xrightarrow{\qquad\Phi_\#\qquad} & & F_{\overline{\mathrm{cyc}}}. \end{array}$$

Since $\sum_{i=1}^{n} g_i \cdot D_{x_i} \in \mathrm{cDer}_k^+(F)$, the above commutative diagram shows that

$$\Phi_\#(\xi) = (\pi \circ \Phi_*)\Big(\sum_{i=1}^{n} g_i \cdot D_{x_i}\Big) = \Phi$$

for some derivation $\xi \in \mathrm{Der}_k^+(F)$. This completes the proof. □

Remark 2.11. Let $\iota\colon F \to k[\![x_1, \dots, x_n]\!]$ be the algebra homomorphism given by $x_i \mapsto x_i$ for $i = 1, \dots, n$. It induces a map

$$\tilde{\iota}\colon F_{\overline{\mathrm{cyc}}} \to k[\![x_1, \dots, x_n]\!].$$

We call $\tilde{\iota}(\Phi)$ the *abelianization of* Φ for any potential $\Phi \in F_{\overline{\mathrm{cyc}}}$. It is easy to check the following statements:

(1) The abelianizations of right equivalent potentials are right equivalent as power series;

(2) The abelianization of a weighted-homogeneous potential is weighted-homogeneous of the same type as a power series;

(3) The abelianization of a quasi-homogeneous potential is quasi-homogeneous as a power series.

Here, the term "right equivalence" and "weighted-homogeneous" for power series are defined in the obvious way, and a power series is called quasi-homogeneous if it is contained in the ideal generated by its partial derivatives. Note that these terminologies are not quite the same as that of [11].

The next example is due to Brown and Wemyss [3, Example 2.1]. It indicates that the converse of the third statement above is not true, that is the abelianization of a non-quasi-homogeneous potential may be quasi-homogeneous.

Example 2.12. Let k be a field of zero characteristic and $F = k\langle\langle x, y\rangle\rangle$. Consider the potential

$$\Phi = x^2 y - \sum_{r\geq 4}(-1)^r \frac{y^r}{r}.$$

It is not hard to check that the abelianization of Φ is quasi-homogeneous. We proceed to show that Φ itself is not quasi-homogeneous. By a direct computation,

$$\Phi_*(D_x) = xy + yx \quad \text{and} \quad \Phi_*(D_y) = (x^2 + x^2 y - y^3)(1+y)^{-1}.$$

So,

$$\Lambda(F, \Phi) = \frac{k\langle\langle x, y\rangle\rangle}{(xy + yx, x^2 + x^2 y - y^3)^{\mathrm{cl}}}.$$

Consider the algebra

$$S = \frac{k\langle x, y\rangle}{(xy + yx, x^2 - y^3 + x^2 y)}.$$

A direct computation shows that $x^3 = 0$ in S and all ambiguities of the rewriting system

$$\{yx \mapsto -xy,\ y^3 \mapsto x^2 + x^2 y,\ x^3 \mapsto 0\}$$

are resolvable. By the Diamond Lemma (see [2, Theorem 1.2]), S is nine dimensional with basis

$$1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2 y,\ xy^2,\ x^2 y^2.$$

Moreover, $xy^3 = y^6 = 0$ in S. In particular, S is a local algebra. By a similar argument of the proof of [7, Lemma 2.8], the canonical morphism $S \to \Lambda(F, \Phi)$ is an isomorphism. By the division algorithm with respect to the above rewriting system,

$$\Phi = \frac{3}{4}x^2 y - \frac{1}{20}x^2 y^2 \neq 0 \quad \text{in } \Lambda(F, \Phi).$$

Note that the commutator space $[\Lambda(F, \Phi), \Lambda(F, \Phi)]$ is spanned by xy, $x^2 y$ and xy^2. So,

$$[\Phi] = -\frac{1}{20}x^2 y^2 \neq 0 \quad \text{in } \Lambda(F, \Phi)_{\mathrm{cyc}}.$$

Thus, by definition, Φ is not a quasi-homogeneous potential.

3. Jordan–Chevalley decomposition of derivations

Throughout, let F be a fixed complete free algebra $k\langle\langle x_1, \dots, x_n\rangle\rangle$ over a field k, and let $\mathfrak{m}$ be the ideal generated by $x_1, \dots, x_n$. We assume that k is algebraically closed. This section devotes to establish a Jordan–Chevalley type decomposition for derivations of F that send $\mathfrak{m}$ to $\mathfrak{m}$.

The space of all derivations of F that send $\mathfrak{m}$ to $\mathfrak{m}$ is denoted by $\mathrm{Der}_k^+(F)$. There is a natural group action of $\mathscr{G} := \mathrm{Aut}_k(F, \mathfrak{m}) = \mathrm{Aut}_k(F)$ on $\mathrm{Der}_k^+(F)$ given by

$$\mathrm{Ad}_H \xi := H \circ \xi \circ H^{-1}.$$

This action respects the Lie bracket on $\mathrm{Der}_k^+(F)$. In addition, one has $\xi(f) = bf$ if and only if $(\mathrm{Ad}_H \xi)(H(f)) = b\, H(f)$ for any $\xi \in \mathrm{Der}_k^+(F)$, any $H \in \mathscr{G}$, any $f \in F$ and any $b \in k$.

Definition 3.1. We say that a derivation $\xi \in \mathrm{Der}_k^+(F)$

(1) is *nilpotent* if it induces a nilpotent endomorphism on $\mathfrak{m}/\mathfrak{m}^2$;

(2) is *semisimple* if it has n eigenvectors in $\mathfrak{m}$ which form a basis in $\mathfrak{m}/\mathfrak{m}^2$, or equivalently there is an automorphism $H \in \mathrm{Aut}_k(F)$ such that $\mathrm{Ad}_H \xi$ has eigenvectors $x_1, \dots, x_n$.

Proposition 3.2. *Let $\xi \in \mathrm{Der}_k^+(F)$ be a semisimple derivation.*

(1) *A scalar $a \in k$ is an eigenvalue of ξ if and only if $a \in \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$, where $a_1, \dots, a_n \in k$ are the eigenvalues of the induced map of ξ on $\mathfrak{m}/\mathfrak{m}^2$.*

(2) *Every formal series $f \in F$ can be uniquely decomposed into a formal sum*

$$f = \sum_a f_a,$$

where a runs over eigenvalues of ξ and f_a is an eigenvector of ξ with eigenvalue a.

Proof. We may assume ξ has $x_1, \dots, x_n$ as eigenvectors with eigenvalue $a_1, \dots, a_n$, respectively. Then every word $w = x_{i_1} \cdots x_{i_p}$ is an eigenvector of ξ with eigenvalue $a_{i_1} + \cdots + a_{i_n}$. The result follows □

Proposition 3.3. *Let $\zeta_1, \dots, \zeta_m \in \mathrm{Der}_k^+(F)$ be semisimple derivations that commute with each other, that is $[\zeta_i, \zeta_j] = 0$ for all $i, j = 1, \dots, n$. Then there exists an automorphism $H \in \mathrm{Aut}_k(F)$ such that $\mathrm{Ad}_H \zeta_1, \dots, \mathrm{Ad}_H \zeta_m$ all have $x_1, \dots, x_n$ as eigenvectors.*

Proof. We prove it by induction on m. For $m = 1$ there is nothing to prove. Suppose that the result is true for $m = p$ and we proceed to justify the case that $m = p+1$. By the induction hypothesis, we may assume *a priori* that x_i is an eigenvector of ζ_j with eigenvalue r_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, p$. For any p-tuple $a = (a_1, \dots, a_p)$ of scalars, let F_a be the space of formal series which are eigenvectors of ζ_j with eigenvalue a_j for $j = 1\dots, p$. Since every word is a simultaneously eigenvector of $\zeta_1, \dots, \zeta_p$, every formal series $f \in F$ can be uniquely expressed as

$$f = \sum_a f_a, \quad f_a \in F_a,$$

where $a = (a_1, \dots, a_p)$ runs over all p-tuples of scalars with a_j an eigenvalue of ζ_j for $j = 1, \dots, p$. Since ζ_{p+1} commutes with $\zeta_1, \dots, \zeta_p$, it follows that if f is an eigenvector of ζ_{p+1} then f_a is also an eigenvector of ζ_{p+1} with the same eigenvalue as that of f. Indeed, one has

$$\zeta_j(\zeta_{p+1}(f_a)) = \zeta_{p+1}(\zeta_j(f_a)) = a\zeta_{p+1}(f_a), \quad j = 1, \dots, p.$$

So, if $\zeta_{p+1}(f) = bf$ then $\zeta_{p+1}(f)$ has two decompositions into simultaneous eigenvectors of $\zeta_1, \dots, \zeta_p$ as

$$\zeta_{p+1}(f) = \sum_a \zeta_{p+1}(f_a) = \sum_a bf_a.$$

It follows immediately that $\zeta_{p+1}(f_a) = bf_a$.

Let $w(x_j) := (r_{j1}, \dots, r_{jp})$ for $j = 1, \dots, n$. Let $X_1, \dots, X_s$ be the partition of $X = \{x_1, \dots, x_n\}$ by the relation that $x_i \sim x_j$ if and only if $w(x_i) = w(x_j)$. By permutation, we may assume that

$$X_1 = \{x_1, \dots, x_{l_1}\}, \quad X_2 = \{x_{l_1+1}, \dots, x_{l_2}\}, \quad \dots, \quad X_s = \{x_{l_{s-1}+1}, \dots, x_n\}$$

for some integers $0 = l_0 < l_1 < l_2 < \cdots < l_s = n$. Since ζ_{p+1} is semisimple, it has eigenvectors $f_1, \dots, f_n \in \mathfrak{m}$ that form a basis of $\mathfrak{m}/\mathfrak{m}^2$. By the above discussion, the set

$$Y_i := \left\{(f_1)_{w(x_{l_i})}, (f_2)_{w(x_{l_i})}, \dots, (f_n)_{w(x_{l_i})}\right\}$$

consists of simultaneous eigenvectors of $\zeta_1, \dots, \zeta_{p+1}$. Moreover, Y_i induces a spanning set of the subspace $V_i \subseteq \mathfrak{m}/\mathfrak{m}^2$ spanned by X_i, so we may choose

$$h_{l_{i-1}+1}, \dots, h_{l_i} \in Y_i,$$

which form a basis of V_i for $i = 1, \dots, s$. By the inverse function theorem (cf. [7, Lemma 2.13]), the algebra homomorphism $T: F \to F$ given by $x_i \mapsto h_i$ is an automorphism. We have

$$\begin{aligned}(\mathrm{Ad}_{T^{-1}}\zeta_j)(x_i) &= T^{-1}(\zeta_j(h_i)) \\ &= T^{-1}(r_{ij}h_i) = r_{ij}x_i, \quad i = 1, \dots, n, \ j = 1, \dots, p.\end{aligned}$$

So, $\mathrm{Ad}_{T^{-1}}\zeta_j = \zeta_j$ for $j = 1, \dots, p$. In addition, h_i is an eigenvector of ζ_{p+1} by the construction, so $\mathrm{Ad}_{T^{-1}}\zeta_{p+1}$ has $x_1, \dots, x_n$ as eigenvectors. Take $H = T^{-1}$, the result follows. □

A derivation $\xi \in \mathrm{Der}_k^+(F)$ is called *principal* if $\xi(x_1), \dots, \xi(x_n)$ are all homogeneous of degree 1, that is they are all linear combinations of the generators $x_1, \dots, x_n$.

Lemma 3.4. *Let $\xi \in \mathrm{Der}_k^+(F)$ be a principal derivation with a decomposition $\xi = \xi' + \xi''$ such that $\xi' \in \mathrm{Der}_k^+(F)$ is principle semisimple derivation, $\xi'' \in \mathrm{Der}_k^+(F)$ is principle nilpotent derivation and $[\xi', \xi''] = 0$. Then for any homogeneous formal series $f \in F$ and any scalar $b \in k$, there exists a homogeneous formal series $h \in F$ of the same degree as f, such that*

$$(\xi - b)h - f$$

is an eigenvector of ξ' with eigenvalue b (eigenvectors always include the zero vector).

Proof. Suppose f is of degree p. Let $F_{(p)}$ be the space of homogeneous formal series of degree p. For any scalar c, let $F_{(p;c)}$ be the space of formal series in $F_{(p)}$ which are eigenvectors of ξ' with eigenvalue c. From the property that ξ'' commutes with ξ', we have that ξ'' acts nilpotently on $F_{(p;c)}$. Since the restriction map of $\xi - b \cdot \mathrm{Id}$ on $F_{(p;c)}$ is equal to the restriction map of $(c-b)\cdot \mathrm{Id} - \xi''$ on $F_{(p;c)}$, it is invertible when $c \neq b$. Note that there exist scalars $c_1, \dots, c_q \in k$ such that

$$F_{(p)} = F_{(p;c_1)} \oplus \cdots \oplus F_{(p;c_q)}.$$

So f has a decomposition $f = f_1 + \cdots + f_q$ with $f_i \in F_{(p;c_i)}$ for $i = 1, \dots, q$. If $c_i \neq b$ then define $h_i \in F_{(p;c_i)}$ to be the preimage of f_i under the restriction map of $\xi' + \xi'' - b \cdot \mathrm{Id}$ on $F_{(p;c_i)}$, which is invertible by the above discussion; and if $c_i = b$ then define $h_i = 0$. Now consider the formal series

$$h = h_1 + \cdots + h_q \in F_{(p)}.$$

Clearly, if $b \notin \{c_1, \dots, c_q\}$ then

$$\xi'(h) + \xi''(h) - b \cdot h - f = 0 \in F_{(p;b)};$$

and if $b = c_i$ for some i then $\xi'(h) + \xi''(h) - b \cdot h - f = -f_i \in F_{(p;b)}$. □

Theorem 3.5 (Jordan–Chevalley decomposition). *For every derivation $\xi \in \mathrm{Der}_k^+(F)$, there exists a unique pair of derivations $\xi_S, \xi_N \in \mathrm{Der}_k^+(F)$ such that*

$$\xi = \xi_S + \xi_N,$$

ξ_S is semisimple, ξ_N is nilpotent and $[\xi_S, \xi_N] = 0$. Moreover, any derivation in $\mathrm{Der}_k^+(F)$ commutes with ξ if and only if it commutes with ξ_S and ξ_N.

The above decomposition of a derivation analogs to the Jordan–Chevalley decomposition of linear endomorphisms of finite dimensional vector spaces over an algebraically closed field (see [8, Proposition 4.2]). We refer to ξ_S (resp. ξ_N) the *semisimple part* (resp. *nilpotent part*) of ξ.

We will use the following notation in the argument given below. Let $r \geq 0$ be an integer. For any derivation $\eta \in \mathrm{Der}_k^+(F)$, we write $\eta_{[r]}$ to be the induced endomorphism of η on $F/\mathfrak{m}^{r+1}$. Note that if η is semisimple (resp. nilpotent) as a derivation then $\eta_{[r]}$ is semisimple (resp. nilpotent) as a linear endomorphism. For any formal series $f \in F$, we write $f_{(r)}$ (resp. $f_{(\leq r)}$) the sum of terms of degree r (resp. $\leq r$) that occurs in f. In addition, for any derivation $\eta \in \mathrm{Der}_k^+(F)$, we write $\eta_{(r)} \in \mathrm{Der}_k^+(F)$ to be the derivation given by $x_i \mapsto \eta(x_i)_{(r)}$ for $i = 1, \dots, n$.

Proof. First we show the uniqueness of the decomposition. Suppose that $\xi = \xi'_S + \xi'_N$ and $\xi = \xi''_S + \xi''_N$ are two such decompositions. Then,

$$\xi_{[s]} = (\xi'_S)_{[s]} + (\xi'_N)_{[s]} \quad \text{and} \quad \xi_{[s]} = (\xi''_S)_{[s]} + (\xi''_N)_{[s]}.$$

Since $(\xi'_S)_{[s]}, (\xi''_S)_{[s]}$ are semisimple and $(\xi'_N)_{[s]}, (\xi'_N)_{[s]}$ are nilpotent, one gets

$$(\xi'_S)_{[s]} = (\xi''_S)_{[s]} \quad \text{and} \quad (\xi'_N)_{[s]} = (\xi''_N)_{[s]}$$

by [8, Proposition 4.2 (a)], for every integer $s \geq 0$. Therefore, $\xi'_S = \xi''_S$ and $\xi'_N = \xi''_N$. This prove the uniqueness of the decomposition.

Next we show the last statement. The converse implication is clear. To see the forward implication, assume $\eta \in \mathrm{Der}_k^+(F)$ is a derivation commutes with ξ. By [8, Proposition 4.2 (b)],

$$[\eta, \xi_S]_{[s]} = [\eta_{[s]}, (\xi_S)_{[s]}] = 0, \quad s \geq 0.$$

Therefore, $[\eta, \xi_S] = 0$ and hence $[\eta, \xi_N] = [\eta, \xi] - [\eta, \xi_S] = 0$. This proves the last statement.

Finally, we show the existence of the decomposition. Note that the action of $\mathscr{G}$ on $\mathrm{Der}_k^+(F)$ respects the Lie bracket, preserves semisimpleness and nilpotentness of derivations. So we may assume *a priori* that the restriction of $\xi_{(1)}$ on $F_{(1)}$ is of the Jordan normal form with respect to the ordered basis $x_1, \dots, x_n$, that is there exists positive integers $l_1, \dots, l_r$ with

$$l_1 + \cdots + l_r = n$$

and scalars

$$a_1 = \cdots = a_{l_1}, \quad a_{l_1+1} = \cdots = a_{l_1+l_2}, \quad \dots, \quad a_{l_1+\cdots+l_{r-1}+1} = \cdots = a_n$$

such that $\xi_{(1)}(x_i) = a_i x_i$ for $i = 1, l_1+1, l_1+l_2+1, \dots,$ and $\xi_{(1)}(x_i) = a_i x_i + x_{i-1}$, otherwise. Let $\xi'_{(1)}$ be the derivation given by $x_i \mapsto a_i x_i$ for $i = 1, \dots, n$, and let $\xi''_{(1)} := \xi_{(1)} - \xi''_{(1)}$. Clearly, $\xi'_{(1)}$ is principle semisimple, $\xi''_{(1)}$ is principle nilpotent, and $[\xi'_{(1)}, \xi''_{(1)}] = 0$.

We proceed to recursively construct an infinite sequence of n-tuples of

$$\big(h_1^{(s)}, \ldots, h_n^{(s)}\big)$$

of formal series in F for $s \geq 1$ such that

(1) $h_i^{(s)}$ is homogeneous of degree s for $i = 1, \ldots, n$;

(2) $\big(\mathrm{Ad}_{(H^{(s)}\circ\cdots\circ H^{(1)})}\xi\big)(x_i)_{(\leq s)}$ is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i for $i = 1, \ldots, n$, where $H^{(r)} \in \mathscr{G}$ is the automorphism given by $x_j \mapsto x_j + h_j^{(r)}$ for $j = 1, \ldots, n$.

Take $h_1^{(1)} = \cdots = h_n^{(1)} = 0$, then the case that $s = 1$ is fulfilled. Suppose that the required tuple $(h_1^{(s)}, \ldots, h_n^{(s)})$ has been constructed for $s = 1, \ldots p$. To simplify the notation, let

$$\xi^{(p)} := \mathrm{Ad}_{(H^{(p)}\circ\cdots\circ H^{(1)})}\xi.$$

By construction,

$$(\xi^{(p)})_{(1)} = \xi_{(1)}$$

and $\xi^{(p)}(x_i)_{(\leq p)}$ is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i for $i = 1, \ldots n$. By Lemma 3.4, we may choose a homogeneous formal series $h_i^{(p+1)}$ of degree $p+1$ for $i = 1, l_1 + 1, \ldots$ such that

$$\varphi_i^{(p+1)} := (\xi_{(1)} - a_i)\big(h_i^{(p+1)}\big) - \xi^{(p)}(x_i)_{(p+1)}$$

is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i; and then apply Lemma 3.4 again, we may also choose inductively on other i a homogeneous formal series $h_i^{(p+1)}$ of degree $p+1$ such that

$$\varphi_i^{(p+1)} := (\xi_{(1)} - a_i)\big(h_i^{(p+1)}\big) - \big(\xi^{(p)}(x_i)_{(p+1)} + h_{i-1}^{(p+1)}\big)$$

is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i. It is easy to check that

$$\big(H^{(p+1)}\big)^{-1}\colon x_i \mapsto x_i - h_i^{(p+1)} + \text{H.O.T.}, \quad i = 1, \ldots, n.$$

Here, H.O.T. is an abbreviation for "higher order terms". So, for $i = 1, \ldots, n$ one has

$$\begin{aligned}
&\big(\mathrm{Ad}_{(H^{(p+1)}\circ\cdots\circ H^{(1)})}\xi\big)(x_i)\\
&\quad = \big(\mathrm{Ad}_{H^{(p+1)}}\xi^{(p)}\big)(x_i)\\
&\quad = H^{(p+1)}\big(\xi^{(p)}\big(x_i - h_i^{(p+1)}\big)\big) + \text{H.O.T.}\\
&\quad = \xi^{(p)}\big(x_i - h_i^{(p+1)}\big)_{(\leq p+1)} + H^{(p+1)}\big(\xi^{(p)}\big(x_i - h_i^{(p+1)}\big)_{(1)}\big)_{(p+1)} + \text{H.O.T.}\\
&\quad = \xi^{(p)}(x_i)_{(\leq p+1)} - \xi_{(1)}\big(h_i^{(p+1)}\big) + H^{(p+1)}\big(\xi_{(1)}(x_i)\big)_{(p+1)} + \text{H.O.T.}\\
&\quad = \xi^{(p)}(x_i)_{(\leq p)} - \varphi_i^{(p+1)} + \text{H.O.T.}
\end{aligned}$$

Here, the third equality holds because

$$H^{(p+1)}(f) = f_{(\le p+1)} + H^{(p+1)}(f_{(1)})_{(p+1)}$$

modulo $\mathfrak{m}^{p+2}$ for any formal series $f \in F$; the fourth equality holds because

$$\xi^{(p)}\big(h_i^{(p+1)}\big)_{(\le p+1)} = \xi_{(1)}\big(h_i^{(p+1)}\big) \quad \text{and} \quad \xi^{(p)}(f)_{(1)} = \xi_{(1)}(f_{(1)})$$

for any formal series $f \in F$; and the last equality holds because $\xi_{(1)}(x_i)$ is either $a_i x_i$ or $a_i x_i + x_{i-1}$ depending on i. Consequently,

$$\big(\mathrm{Ad}_{(H^{(p+1)}\circ\cdots\circ H^{(1)})}\xi\big)(x_i)_{(\le p+1)}$$

is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i for $i = 1, \ldots, n$.

Now let

$$g_i^{(s)} := \big(H^{(s)} \circ \cdots \circ H^{(1)}\big)(x_i)$$

for $i = 1, \ldots, n$ and $s \ge 1$. Since $g_i^{(s+1)} - g_i^{(s)} \in \mathfrak{m}^{s+1}$ for $s \ge 1$, the infinite sequence

$$\big(g_i^{(1)}, g_i^{(2)}, g_i^{(3)}, \ldots\big)$$

converges to a formal series g_i. Clearly,

$$(g_i)_{(\le s)} = \big(g_i^{(s)}\big)_{(\le s)}, \quad s \ge 1.$$

Let $H \in \mathcal{G}$ be the automorphism given by $H(x_i) = g_i$ for $i = 1, \ldots, n$. It is easy to check that

$$(\mathrm{Ad}_H \xi)(x_i)_{(\le s)} = \big(\mathrm{Ad}_{(H^{(s)}\circ\cdots\circ H^{(1)})}\xi\big)(x_i)_{(\le s)}, \quad s \ge 1,$$

so $(\mathrm{Ad}_H \xi)(x_i)$ is an eigenvector of $\xi'_{(1)}$ with eigenvalue a_i. In addition, one has

$$(\mathrm{Ad}_H \xi)_{(1)} = \xi_{(1)},$$

so $\mathrm{Ad}_H \xi - \xi'_{(1)}$ is a nilpotent derivation. Let

$$\xi_S := \mathrm{Ad}_{H^{-1}} \xi'_{(1)} \quad \text{and} \quad \xi_N := \mathrm{Ad}_{H^{-1}}\big(\mathrm{Ad}_H \xi - \xi'_{(1)}\big).$$

Then ξ_S is semisimple, ξ_N is nilpotent and $\xi = \xi_S + \xi_N$. Moreover,

$$\mathrm{Ad}_H[\xi_S, \xi_N] = \big[\xi'_{(1)}, \mathrm{Ad}_H \xi - \xi'_{(1)}\big] = \big[\xi'_{(1)}, \mathrm{Ad}_H \xi\big] = 0.$$

Thus, $[\xi_S, \xi_N] = 0$ and this completes the proof. □

4. Noncommutative Saito theorem

This section is devoted to establish a noncommutative analogue of the well known Saito's theorem on hypersurfaces of isolated singularity. Throughout, let F be a fixed complete free algebra $k\langle\langle x_1, \dots, x_n \rangle\rangle$ over a field k. We assume that k is algebraically closed and of characteristic 0, and we consider the rational number field $\mathbb{Q}$ as a subfield of k in the natural way.

Theorem 4.1 (NC Saito Theorem). *Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential of order ≥ 3 such that the Jacobi algebra associated to Φ is finite dimensional. Then Φ is quasi-homogeneous if and only if Φ is right equivalent to a weighted-homogeneous potential of type $(r_1, \dots, r_n)$ for some rational numbers $r_1, \dots, r_n$ lie strictly between 0 and $1/2$. Moreover, in this case, all such types $(r_1, \dots, r_n)$ agree with each other up to permutations on the indexes $1, \dots, n$.*

We address the proof of the above theorem after several lemmas.

Lemma 4.2. *Develop a formal series $f \in F$ in eigenvectors of a semisimple derivation $\xi \in \mathrm{Der}_k^+(F)$ as $f = \sum_a f_a$. Then $f \in [F, F]^{\mathrm{cl}}$ if and only if $f_a \in [F, F]^{\mathrm{cl}}$ for each eigenvalue a of ξ.*

Proof. Since any automorphism of F preserves $[F, F]^{\mathrm{cl}}$, we may assume that ξ has $x_1, \dots, x_n$ as eigenvectors. The result follows from the facts that the commutator of any two words is an eigenvector of ξ and every formal series in $[F, F]^{\mathrm{cl}}$ is a formal sum of such commutators. $\square$

Lemma 4.3. *Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential such that $\Phi_\#(\xi) = b \cdot \Phi$ for some scalar $b \in k$ and some nilpotent derivation $\xi \in \mathrm{Der}_k^+(F)$. Then either $\Phi = 0$ or $b = 0$.*

Proof. Suppose $\Phi \neq 0$. Let f be the canonical representative of Φ. Develop f as

$$f = \sum_{i \geq p} f_{(i)}$$

with $f_{(i)}$ homogeneous of degree i and $f_{(p)} \neq 0$. Since ξ is nilpotent,

$$\xi^r(\mathfrak{m}^p) \subseteq \mathfrak{m}^{p+1}$$

for some $r \gg 0$. So $\xi^r(f)$ has a decomposition

$$\xi^r(f) = \sum_{i \geq p+1} \xi^r(f)_{(i)}$$

with $\xi^r(f)_{(i)}$ homogeneous of degree i. Then

$$b^r f_{(p)} + \sum_{i \geq p+1} \left(b^r f_{(i)} - \xi^r(f)_{(i)}\right) = b^r f - \xi^r(f) \in [F, F]^{\mathrm{cl}}.$$

Consequently, $b^r f_{(p)} \in [F, F]^{\mathrm{cl}}$. Since $f_{(p)}$ is in the canonical form, $b^r f_{(p)} = 0$ and hence $b = 0$. $\square$

Lemma 4.4. *Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential such that $\Phi_\#(\xi) = b \cdot \Phi$ for some scalar $b \in k$ and some derivation $\xi \in \mathrm{Der}_k^+(F)$. Then $\Phi_\#(\xi_S) = b \cdot \Phi$ and $\Phi_\#(\xi_N) = 0$, where ξ_S and ξ_N are the semisimple part and the nilpotent part of ξ, respectively.*

Proof. Let f be the canonical representative of Φ. Develop f in eigenvectors of ξ_S as $f = \sum_a f_a$. Since $\pi(\xi(f) - bf) = \Phi_\#(\xi) - b \cdot \Phi = 0$, where $\pi : F \to F_{\overline{\text{cyc}}}$ is the projection map, we have

$$\sum_a \xi_N(f_a) + (a-b) f_a \in [F, F]^{\mathrm{cl}}.$$

Since $\xi_S(\xi_N(f_a)) = \xi_N(\xi_S f_a) = a\xi_N(f_a)$, it follows that $(a-b)f_a + \xi_N(f_a)$ is an eigenvector of ξ_S with eigenvalue a. Then Lemma 4.2 tells us that

$$\pi(f_a)_\#(\xi_N) - (b-a) \cdot \pi(f_a) = \pi\big(\xi_N(f_a) + (a-b) f_a\big) = 0$$

for every eigenvalue a of ξ_S. So by Lemma 4.3, either $a = b$ or $f_a \in [F, F]^{\mathrm{cl}}$ for every eigenvalue a of ξ_S. Now we have two cases. If b is not an eigenvalue of ξ_S then $f \in [F, F]^{\mathrm{cl}}$ and hence $\xi_S(f) - bf \in [F, F]^{\mathrm{cl}}$; if b is an eigenvalue of ξ_S then

$$f - f_b = \sum_{a \neq b} f_a \in [F, F]^{\mathrm{cl}},$$

and hence

$$\xi_S(f) - bf = \xi_S(f - f_b) - b(f - f_b) \in [F, F]^{\mathrm{cl}}.$$

In both cases,

$$\Phi_\#(\xi_S) - b \cdot \Phi = \pi(\xi_S(f) - bf) = 0.$$

Finally, $\Phi_\#(\xi_N) = \Phi_\#(\xi) - \Phi_\#(\xi_S) = 0$. □

Lemma 4.5. *Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential with finite dimensional Jacobi algebra. Suppose*

$$\Phi = \pi(g_{l+1}x_{l+1}) + \cdots + \pi(g_n x_n) + \pi(h),$$

where $l < n$, $g_{l+1}, \ldots, g_n \in k\langle\langle x_1, \ldots, x_l \rangle\rangle$ and all monomials in $h \in F$ are of total degree ≥ 2 in $x_{l+1}, \ldots, x_n$. Then $l \leq n/2$ and there are at least l nonzero formal series among $g_{l+1}, \ldots, g_n$.

Proof. Let $k[[x_1, \ldots, x_l]]$ be the commutative algebra of power series in l indeterminates. Let $\mathfrak{a}$ be the image of the Jacobi ideal $\mathrm{J}(F, \Phi)$ under the algebra homomorphism

$$\tau \colon F \to k[[x_1, \ldots, x_l]]$$

given by $x_i \mapsto x_i$ for $i = 1, \ldots, l$ and $x_i \mapsto 0$ for $i = l+1, \ldots, n$. Clearly, $\mathfrak{a}$ is a finite codimensional proper ideal of $k[[x_1, \ldots, x_l]]$ generated by $\tau(g_{l+1}), \ldots, \tau(g_n)$. By the well known Krull's height theorem, $\mathfrak{a}$ has at least l generators as a two-sided ideal of $k[[x_1, \ldots, x_l]]$, so there are at least l nonzero power series among $\tau(g_{l+1}), \ldots, \tau(g_n)$. The result follows immediately. □

Lemma 4.6. *Let $\Phi \in F_{\overline{\text{cyc}}}$ be a potential of order ≥ 3 such that the Jacobi algebra associated to Φ is finite dimensional. Suppose that $\Phi_{\#}(\xi) = b \cdot \Phi$ for some nonzero $b \neq 0$ and some semisimple derivation $\xi \in \mathrm{Der}_k^+(F)$ that has $x_1, \dots, x_n$ as eigenvectors. Then Φ is weighted-homogeneous of type $(r_1, \dots, r_n)$ for some rational numbers $r_1, \dots, r_n$ lie strictly between 0 and $1/2$.*

Proof. By assumption, $\xi(x_i) = a_i x_i$ for $i = 1, \dots, n$, where $a_i \in k$. Let $c_1, \dots, c_p$ be a basis of the vector space $\mathbb{Q}a_1 + \cdots + \mathbb{Q}a_n + \mathbb{Q}b$ over $\mathbb{Q}$. Then

$$(a_1, \dots, a_n, b)^T = D \cdot (c_1, \dots, c_p)^T$$

for some matrix $D = (d_{ij})$ of type $(n+1) \times p$ with rational number entries. Since $b \neq 0$, the last row of D is nonzero. Without lost of generality, we may assume $d_{n+1,1} \neq 0$. Define

$$(r_1, \dots, r_n) := (d_{1,1}/d_{n+1,1}, \dots, d_{n1}/d_{n+1,1}).$$

Clearly, for any integers $m_1, \dots, m_n$, if $m_1 a_1 + \cdots + m_n a_n = b$, then

$$(m_1, \dots, m_n, -1) \cdot D = 0,$$

and hence $m_1 r_1 + \cdots + m_n r_n = 1$. Let f be the canonical representative of Φ. One has

$$\xi(f) = bf$$

because $\xi(f)$ and bf are both canonical representative of $b \cdot \Phi$. It follows that for any word $w = x_{i_1} \cdots x_{i_s}$ that occurs in f, one has

$$m_1 a_1 + \cdots + m_n a_n = a_{i_1} + \cdots + a_{i_s} = b,$$

where m_i is the occurrences of x_i in the word w, and therefore

$$r_{i_1} + \cdots + r_{i_s} = m_1 r_1 + \cdots + m_n r_n = 1.$$

It remains to show $0 < r_1, \dots, r_n < 1/2$.

Now for any real number $\varepsilon \geq 0$, let P_ε (resp. Q_ε) be the number of indexes i among $1, \dots, n$ such that $r_i \leq -\varepsilon$ (resp. $r_i \geq 1/2 + \varepsilon$). We claim that for every real number $\varepsilon \geq 0$,

$$P_\varepsilon \leq Q_{2\varepsilon+1/2} \quad \text{and} \quad Q_\varepsilon \leq P_{2\varepsilon}.$$

To see the first inequality, we may assume $r_1, \dots, r_{P_\varepsilon} \leq -\varepsilon$, up to permutation on indeterminates. Then f contains no word constitutes with letters $x_1, \dots, x_{P_\varepsilon}$. By Lemma 4.5 and the assumption that all terms of f has degree ≥ 3, there are at least P_ε indexes i among $P_\varepsilon + 1, \dots, n$ such that $r_i \geq 1 + 2\varepsilon$, and so $P_\varepsilon \leq Q_{2\varepsilon+1/2}$. The second inequality can be proved similarly.

From the above two inequalities, one has $P_\varepsilon \leq P_{4\varepsilon+1}$ and $Q_\varepsilon \leq Q_{2\varepsilon+1/2}$ for every real number $\varepsilon \geq 0$. It follows that $P_0 = Q_0 = 0$, or otherwise the finite set $\{r_1, \dots, r_n\}$ is not bounded, which is absurd. Consequently, all rational numbers $r_1, \dots, r_n$ lie strictly between 0 and $1/2$. □

Proof of the equivalence statement of Theorem 4.1. The "if" part is Lemma 2.9. Next we proceed to show the "only if" part.

Assume that Φ is quasi-homogeneous. By Lemma 2.10,

$$\Phi_{\#}(\xi) = \Phi$$

for some derivation $\xi \in \mathrm{Der}_k^+(F)$. Then by Lemma 4.4,

$$\Phi_{\#}(\xi_S) = \Phi.$$

Choose an automorphism $H \in \mathrm{Aut}_k(F)$ such that the derivation

$$\mathrm{Ad}_H \xi_S = H \circ \xi_S \circ H^{-1}$$

has $x_1, \ldots, x_n$ as eigenvectors. Note that

$$H(\Phi)_{\#}(\mathrm{Ad}_H \xi_S) = H(\Phi).$$

Then by Lemma 4.6, $H(\Phi)$ is weighted-homogeneous of type $(r_1, \ldots, r_n)$ for some rational numbers $r_1, \ldots, r_n$ lie strictly between 0 and $1/2$. The result follows. □

To see the uniqueness statement of Theorem 4.1, we need the following lemma.

Lemma 4.7. *Let $\Phi \in F_{\overline{\mathrm{cyc}}}$ be a potential of order ≥ 3 such that the Jacobi algebra associated to Φ is finite dimensional. Given two semisimple derivations $\xi, \eta \in \mathrm{Der}_k^+(F)$ that commute with each other, if $\Phi_{\#}(\xi) = \Phi_{\#}(\eta)$ then $\xi = \eta$.*

Proof. By Proposition 3.3, we may assume ξ and η both have $x_1 \ldots, x_n$ as eigenvectors with eigenvalue $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ respectively.

Let f be the canonical representative of Φ. We claim that for each $1 \leq i \leq n$, the formal series f either has a monomial of the form x_i^a for some $a \geq 3$ or has a monomial with exactly one occurrence of letters other than x_i. Indeed, if the first case doesn't happen, then

$$\Phi = \pi(f) = \sum_{p \neq i} \pi(g_p \cdot x_p) + \pi(h),$$

with $g_p \in k\langle\langle x_i \rangle\rangle$ and with all monomials in h has at least two occurrences in letters other than x_i. By Lemma 4.5, there is at least one p such that $g_p \neq 0$, so the claim follows.

Construct an $n \times n$ matrix $A = (a_{ij})$ with entries in $\mathbb{N}$ as follows. For each $1 \leq i \leq n$, choose a monomial in f either of the form x_i^a for some $a \geq 3$ or of the form $x_i^b x_p x_i^c$ with $b + c \geq 2$ and $p \neq i$. Such a choose is assured by the above argument. Define the i-th row of A to be ae_i or $(b+c)e_i + e_p$, according to the choice of the monomial, where e_i, e_p denote the canonical coordinate. Since $\xi(f) = \eta(f)$, it follows that

$$A \cdot (r_1, \ldots, r_n)^T = A \cdot (s_1, \ldots, s_n)^T.$$

Moreover, since

$$a_{ii} > \sum_{p \neq i} a_{ip}, \quad i = 1, \ldots, n,$$

it follows that A is an invertible matrix. Therefore, $r_i = s_i$ for $i = 1, \ldots, n$, and hence $\xi = \eta$. □

Proof of the uniqueness statement of Theorem 4.1. Replacing Φ by an appropriate potential in its orbit, we may assume Φ is itself weighted-homogeneous of type $r = (r_1, \ldots, r_n)$ with $r_1 \leq \cdots \leq r_n$. Suppose that $H(\Phi)$ is weighted-homogeneous of type $s = (s_1, \ldots, s_n)$ for some automorphisms H of F. To see the result we must show that $r = s$ up to permutations.

Let ξ be the semisimple derivation of F given by $\xi(x_i) = r_i x_i$, and let $\zeta := \mathrm{Ad}_{H^{-1}} \eta$, where η is the semisimple derivation given by $\eta(x_i) = s_i x_i$. Develop $\zeta(x_i)$ in eigenvectors of ξ as

$$\zeta(x_i) = \sum_a \zeta(x_i)_a, \quad i = 1, \ldots, n.$$

Then define for each eigenvalue u of ξ a derivation $\zeta_u \in \mathrm{Der}_k^+(F)$ by

$$\zeta_u(x_i) = \zeta(x_i)_{r_i+u}.$$

Let f be the canonical representative of Φ. Then,

$$\xi(f) = f \quad \text{and} \quad f = \zeta(f) = \sum_u \zeta_u(f) \mod [F, F]^c$$

where u runs over all eigenvalues of ξ. It is easy to check that $\zeta_u(f)$ is an eigenvector of ξ with eigenvalue $1 + u$. Then by Lemma 4.2, one gets

$$\zeta_0(f) = f \mod [F, F]^{\mathrm{cl}}, \tag{4.1}$$

and

$$\zeta_u(f) = 0 \mod [F, F]^{\mathrm{cl}}, \quad u \neq 0. \tag{4.2}$$

It is easy to check that $[\xi, \zeta_0] = 0$. So

$$[\xi, (\zeta_0)_S] = 0$$

by Theorem 3.5. One has $\Phi_\#(\zeta_0) = \Phi$ by Equation (4.1), and hence

$$\Phi_\#((\zeta_0)_S) = \Phi$$

by Lemma 4.4. In addition, $\Phi_\#(\xi) = \Phi$. Therefore,

$$\xi = (\zeta_0)_S$$

by Lemma 4.7. Thus, the characteristic polynomial of the induced endomorphism of ζ_0 on $\mathfrak{m}/\mathfrak{m}^2$ is

$$(t - r_1)(t - r_2)\cdots(t - r_n).$$

Note that the characteristic polynomial of the induced endomorphism of ζ on $\mathfrak{m}/\mathfrak{m}^2$, which equals to that of the induced endomorphism of η on $\mathfrak{m}/\mathfrak{m}^2$, is

$$(t - s_1)(t - s_2)\cdots(t - s_n).$$

It remains to show that the induced linear endomorphisms of ζ and ζ_0 on $\mathfrak{m}/\mathfrak{m}^2$, denoted by $\widetilde{\zeta}$ and $\widetilde{\zeta}_0$ respectively, have the same characteristic polynomial.

We first claim that the linear part of $\zeta(x_i)_{r_i+u} = \zeta_u(x_i)$ is zero for $u < 0$. Indeed, since

$$\sum_{i=1}^{n} \zeta_u(x_i) \cdot D_{x_i}(f) = \zeta_u(f) \mod [F, F]^{\mathrm{cl}},$$

it follows from equation (4.2) that

$$\sum_{i=1}^{n} \zeta_u(x_i) \cdot D_{x_i}(f) = 0 \mod [F, F]^{\mathrm{cl}}, \quad u \neq 0.$$

Let $\iota\colon F \to k[\![x_1, \dots, x_n]\!]$ be the algebra map given by $x_i \mapsto x_i$. Then

$$\iota(D_{x_1}(f)), \dots, \iota(D_{x_n}(f))$$

generates a finite codimensional ideal of $k[\![x_1, \dots, x_n]\!]$ and so they form a parameter system. By [6, Theorem 8.21A (a,c)], any permutation of the sequence $\iota(D_{x_1}(f)), \dots,$ $\iota(D_{x_n}(f))$ is regular. Since

$$\sum_{i=1}^{n} \iota(\zeta_u(x_i)) \cdot \iota(D_{x_i}(f)) = 0, \quad u \neq 0,$$

it follows that for each $1 \leq i \leq n$ one has

$$\iota(\zeta_u(x_i)) \in \big(\iota(D_{x_1}(f)), \dots, \widehat{\iota(D_{x_i}(f))}, \dots, \iota(D_{x_n}(f))\big), \quad u \neq 0.$$

Since $D_{x_1}(f), \dots, D_{x_n}(f)$ are all eigenvectors of ξ of eigenvalue $\geq 1/2$ but $\zeta_u(x_i)$ is an eigenvector of ξ of eigenvalue $r_i + u < 1/2$ for $u < 0$, it follows that

$$\iota(\zeta_u(x_j)) = 0, \quad u < 0.$$

Since the linear part of $\zeta_u(x_i)$ coincide with the linear part of $\iota(\zeta_u(x_i))$, the claim follows.

Now note that

$$r_1 = \cdots = r_{l_1} < r_{l_1+1} = \cdots = r_{l_2} < \cdots < r_{l_{p-1}+1} = \cdots = r_n$$

for some integers $0 = l_0 < \cdots < l_p = n$. By the above claim, for $l_q + 1 \leq i \leq l_{q+1}$ one has

$$\widetilde{\zeta}(x_i) = \sum_{j=l_q+1}^{l_{q+1}} a_{ji} \cdot x_j + \sum_{j>l_{q+1}} a_{ji} \cdot x_j,$$

$$\widetilde{\zeta}_0(x_i) = \sum_{j=l_q+1}^{l_{q+1}} a_{ji} \cdot x_j.$$

Compare the matrices of $\widetilde{\zeta}$ and $\widetilde{\zeta}_0$ with respect to the basis $x_1, \ldots, x_n$, one gets that the characteristic polynomial of $\widetilde{\zeta}$ and $\widetilde{\zeta}_0$ are equal. This completes the proof. □

Remark 4.8. By the statements displayed in Remark 2.11, the uniqueness part of Theorem 4.1 follows readily from [11, Lemma 4.3]. However, we give a direct demonstration as above for completeness and reader's convenience. Our argument is essential the same as that of Saito's, but with more details. Of course, some tricks are employed to deal with the noncommutativity. In addition, our argument used Lemma 4.7 (and hence Proposition 3.3), which has an interest in its own right.

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