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Odd characteristic classes in entire cyclic homology and equivariant loop space homology

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Abstract. Given a compact manifold M and a smooth map $g: M \to U(l \times l; \mathbb{C})$ from M to the Lie group of unitary $l \times l$ matrices with entries in $\mathbb{C}$, we construct a Chern character $\mathrm{Ch}^{-}(g)$ which lives in the odd part of the equivariant (entire) cyclic Chen-normalized cyclic complex $\mathcal{N}_{\epsilon}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ of M, and which is mapped to the odd Bismut–Chern character under the equivariant Chen integral map. It is also shown that the assignment $g \mapsto \mathrm{Ch}^{-}(g)$ induces a well-defined group homomorphism from the K^{-1} theory of M to the odd homology group of $\mathcal{N}_{\epsilon}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$.

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Introduction

Let M be a closed Riemannian spin manifold with its Clifford multiplication

$$c: \Omega(M) \longrightarrow \mathrm{End}(S)$$

and its Dirac operator D acting in $L^2(M, S)$, and given $g \in C^{\infty}(M, U(l \times l; \mathbb{C}))$ let D_g denote the twisted Dirac operator

$$D_g := g^{-1} D g = D + c(g^{-1} dg),$$

considered to be acting on $L^2(M, S \otimes \mathbb{C}^l)$. Then with

$$D_{g,s} := (1-s) D + s D_g, \quad s \in [0,1],$$

the odd dimensional variant of Atiyah–Singer's 'index' theorem states that if M is odd dimensional, then [9]

$$\frac{1}{2\pi} \int_0^1 \mathrm{Tr}\big[\dot{D}_{g,s} \exp\big(-D_{g,s}^2\big)\big]\, ds = \int_M \widehat{A}(M) \wedge \mathrm{ch}^{-}(g), \tag{1}$$

where $\mathrm{ch}^-(g) \in \Omega^-(M)$ denotes the odd Chern character. The l.h.s. of (1) is precisely the spectral flow $\mathrm{sf}(D, D_g)$ [9]. Furthermore, on the r.h.s. of this formula, the odd Chern character can be obtained integration along the fiber of $M \times I \to M$ of the even Chern character of an appropriately chosen connection on $M \times I$ [9]. In fact, this formula can be proved by noting the l.h.s. admits an infinite dimensional version of such an even/odd periodicity [4,5] in terms of the eta form.

Being motivated by the considerations of Atiyah and Bismut [1,2] for the even-dimensional case, one finds that another very elegant and geometric, however purely formal, way to prove (1) is to assume the existence of a Duistermaat–Heckmann localization formula for the smooth loop space LM: indeed, the spin structure on M induces an orientation on LM [1] and the path integral formalism entails the elegant, however mathematically ill-defined, formula (the even-dimensional variant of this formula is well known [2] and the odd-dimensional case can be proved similarly [13])

$$\frac{1}{2\pi}\int_0^1 \mathrm{Tr}\big[\dot{D}_{g,s}\exp\big(-D_{g,s}^2\big)\big]\,ds = \int_{LM}\exp(-\beta)\wedge \mathrm{Bch}^-(g), \tag{2}$$

where $\beta \in \Omega^+(LM)$ denotes the even differential form on LM given by $\beta = E + \omega$ with E the energy functional on LM considered as a 0-form on LM and with $\omega \in \Omega^2(LM)$ the (presymplectic) 2-form given on smooth vector fields X, Y on LM by

$$\omega(X,Y) := \int_0^1 (\nabla X_t/\nabla t, Y_t)\,dt,$$

and where $\mathrm{Bch}^-(g) \in \Omega^-(M)$ denotes the odd Bismut–Chern character [3,16]. Now both differential forms $\exp(-\beta)$ and $\mathrm{Bch}^-(g)$ are equivariantly closed (cf. Section 4 for the definition of the degree -1 differential P),

$$(d + P)\exp(-\beta) = 0 = (d + P)\mathrm{Bch}^-(g)$$

and so is their product. As the fixed point set of the $\mathbb{T}$-action on LM given by rotating every loop is precisely $M \subset LM$, a hypothetical Duistermaat–Heckmann localization formula immediately gives

$$\int_{LM}\exp(-\beta)\wedge \mathrm{Bch}^-(g) = \int_M \widehat{A}(M)\wedge \exp(-\beta)|_M \wedge \mathrm{Bch}^-(g)|_M,$$

as $\widehat{A}(M)$ is the inverse of the (appropriately renormalized) Euler class of the normal bundle of $M \subset LM$. This proves (1), as clearly $\exp(-\beta)|_M = 1$ and by construction $\mathrm{Bch}^-(g)|_M = \mathrm{ch}^-(g)$.

A direct implementation of the above arguments is not possible, as the right hand side of formula (2) is not well-defined for various reasons. For example, there exists no volume measure on LM, while smooth loops have Wiener measure zero, and, on the other hand, it is notoriously difficult to produce a variant of the super complex $(\Omega(LM), d + P)$ if one replaces LM with the smooth Banach manifold of *continuous loops*. Nevertheless and strikingly, the above formal manipulations lead to the powerful machinery of hypoelliptic Dirac and Laplace operators, as is explained in [3] and the references therein.

However, a possible way out of these problems has been proposed by Getzler, Jones and Petrack (GJP) [8, 10]. In this approach, the idea is to take as model for $\Omega(LM)$ the space of equivariant Chen integrals: these are given by the image of a morphism of super complexes (cf. Section 4 below for the relevant definitions)

$$\rho\colon \big(\mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T})), b + B\big) \longrightarrow \big(\widehat{\Omega}(LM), d + P\big).$$

Above, $\mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ denotes the Chen-normalized entire cyclic (or Connes) complex of the locally convex unital DGA $\Omega_{\mathbb{T}}(M \times \mathbb{T})$, and $\widehat{\Omega}(LM)$ denotes a completed space of smooth differential forms on LM. Now the GJP-program for infinite dimensional localization is as follows: here it is conjectured that the composition

$$\int_{LM} \exp(-\beta) \wedge \rho(\cdot)\colon \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T})) \longrightarrow \mathbb{C}$$

is a mathematically well-defined continuous functional, and that

- $\int_{LM} \exp(-\beta) \wedge \rho(\cdot)$ is odd (as LM is formally odd-dimensional if M is so [3]) and co-closed, meaning that it vanishes on the exact elements of $\mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$,
- if $w \in \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ is closed, then one has the 'Duistermaat–Heckmann localization formula'

$$\int_{LM} \exp(-\beta) \wedge \rho(w) = \int_M \widehat{A}(TM) \wedge \rho(w)|_M. \tag{3}$$

If in addition one could canonically construct an element

$$\mathrm{Ch}^-(g) \in \mathcal{N}_\epsilon^-(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

such that

(i) $\mathrm{Ch}^-(g)$ is closed;

(ii) $\rho(\mathrm{Ch}^-(g)) = \mathrm{Bch}^-(g)$;

(iii) $\rho(\mathrm{Ch}^-(g))|_M = \mathrm{ch}^-(g)$,

then from the above observations we would immediately obtain a proof of (1) within the GJP-program for infinite dimensional localization. Note that in the even dimensional case such a Chern character has been constructed as an even cycle in $\mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ in [10].

The aim of this paper is precisely to construct a canonically given element

$$\mathrm{Ch}^-(g) \in \mathcal{N}_\epsilon^-(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

satisfying the above properties (i)–(iii). In fact, our main results Theorem 5.1 and Theorem 5.4 below construct $\mathrm{Ch}^-(g)$ for M a compact manifold (possibly with boundary), which satisfies (i) and (iii) and in addition (ii) if M is closed (so that LM is a well-defined smooth Fréchet manifold). We also show in Theorem 5.1 that the assignment $g \mapsto \mathrm{Ch}^-(g)$ induces a well-defined group homomorphism

$$\mathsf{K}^{-1}(M) \longrightarrow \mathcal{N}(\Omega_{\mathbb{T}}(M \times \mathbb{T})).$$

Finally, taking for granted that the even variant of $\mathrm{Ch}^-(g)$ and $\mathrm{BCh}^-(g)$ have been previously defined [2, 10], we establish an even/odd periodicity, relating these constructions to ours, showing another analogy to (1).

Note added in proof. Recently, a mathematically rigorous version of the Duistermaat–Heckmann localization formula (3) on the loop space of an *even dimensional* dimensional spin manifold has been established in [12].

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1. Cyclic bar complex of a differential graded algebra (DGA)

In the sequel, we understand all our linear spaces to be over $\mathbb{C}$. Assume we are given a unital DGA Ω, that is,

- Ω is a unital algebra;
- $\Omega = \bigoplus_{j=-\infty}^{\infty} \Omega^j$ is graded into subspaces $\Omega^j \subset \Omega$ such that $\Omega^i \Omega^j \subset \Omega^{i+j}$ for all $i, j \in \mathbb{Z}$, there is a degree $+1$ differential $d\colon \Omega \to \Omega$ which satisfies the graded Leibniz rule.

Note that the space $\underline{\Omega} := \Omega/(\mathbb{C} \cdot \mathbf{1})$ is a graded linear space (but not canonically an algebra), and the space of cyclic chains $\mathcal{C}(\Omega)$ is defined as

$$\mathcal{C}(\Omega) := \bigoplus_{n=0}^{\infty} \Omega \otimes \underline{\Omega}^{\otimes n}.$$

We give $\Omega \otimes \underline{\Omega}^{\otimes n}$ the grading

$$\Omega \otimes \underline{\Omega}^{\otimes n} = \bigoplus_{j=0}^{\infty} \bigoplus_{j_0+\dots+j_n=j-n} \Omega^{j_0} \otimes \underline{\Omega}^{j_1} \otimes \dots \otimes \underline{\Omega}^{j_n},$$

which induces a linear map

$$\Gamma\colon \mathcal{C}(\Omega) \longrightarrow \mathcal{C}(\Omega), \quad \Gamma(w_0, w_1, \dots) := \big((-1)^{\deg(w_0)} w_0, (-1)^{\deg(w_1)} w_1, \dots\big).$$

Since we have $\Gamma^2 = 1$, we can define a superstructure $\mathcal{C}(\Omega) = \mathcal{C}^+(\Omega) \oplus \mathcal{C}^-(\Omega)$ by setting

$$\mathcal{C}^{\pm}(\Omega) := \{w \in \mathcal{C}(\Omega) : \Gamma w = \pm w\}.$$

The following notation will be useful in the sequel:

Notation 1.1. Given $a \in \Omega \otimes \underline{\Omega}^{\otimes n}$ we define

$$\langle a \rangle := (\dots, a, \dots) \in \mathcal{C}(\Omega)$$

to be the cochain which has a in its n-th slot and 0 anywhere else.

We have the Hochschild map of the DGA-category

$$b\colon \mathcal{C}(\Omega) \longrightarrow \mathcal{C}(\Omega)$$

defined on $\Omega^{j_0} \otimes \underline{\Omega}^{j_1} \otimes \cdots \otimes \underline{\Omega}^{j_n}$ by

$$\begin{aligned} b\,\langle \omega_0 \otimes \cdots \otimes \omega_n \rangle = {} & \langle d\omega_0 \otimes \cdots \otimes \omega_i \otimes \cdots \otimes \omega_n \rangle \\ & - \sum_{i=1}^{n} (-1)^{j_0+\ldots+j_{i-1}-i+1} \langle \omega_0 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_n \rangle \\ & - \sum_{i=0}^{n-1} (-1)^{j_0+\ldots+j_i-i} \langle \omega_0 \otimes \cdots \otimes \omega_i \omega_{i+1} \otimes \cdots \otimes \omega_n \rangle \\ & + (-1)^{(j_n-1)(j_0+\ldots+j_{n-1}-n+1)} \langle (\omega_n \omega_0) \otimes \omega_1 \otimes \cdots \otimes \omega_{n-1} \rangle , \end{aligned}$$

and Connes' operator

$$B\colon \mathcal{C}(\Omega) \longrightarrow \mathcal{C}(\Omega),$$

which is defined on $\Omega^{j_0} \otimes \underline{\Omega}^{j_1} \otimes \cdots \otimes \underline{\Omega}^{j_n}$ by

$$\begin{aligned} & B\,\langle \omega_0 \otimes \cdots \otimes \omega_n \rangle \\ & \quad = \sum_{i=0}^{n} (-1)^{(r_{i-1}+1)(r_n - r_{i-1})} \langle 1 \otimes \omega_i \otimes \cdots \otimes \omega_n \otimes \omega_0 \otimes \cdots \otimes \omega_{i-1} \rangle , \end{aligned}$$

with $r_l = j_0 + \cdots + j_l - l$. It is a well known fact that one has

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0, \quad \Gamma b = -b\Gamma, \quad \Gamma B = -B\Gamma.$$

We get the super complex

$$\mathcal{C}^+(\Omega) \xrightarrow{b+B} \mathcal{C}^-(\Omega) \xrightarrow{b+B} \mathcal{C}^+(\Omega). \tag{4}$$

The subspace $\mathcal{D}(\Omega) \subset \mathcal{C}(\Omega)$ is defined to be the linear span of all $w \in \mathcal{C}(\Omega)$ that satisfy one of the following relations:

- For all $n \in \mathbb{N}$ there exist $1 \le r \le n$, $f \in \Omega^0$, $\omega_0 \in \Omega$, $\omega_s \in \underline{\Omega}$, $s \neq r$, with

$$\langle w_n \rangle = \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes f \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle . \tag{5}$$

- for all $n \in \mathbb{N}$ there exist $1 \le r \le n$, $f \in \Omega^0$, $\omega_0 \in \Omega$, $\omega_s \in \underline{\Omega}$, $s \neq r$, with

$$\begin{aligned} &\langle \omega_0 \otimes \cdots \otimes \omega_{r-1} f \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle \\ &\qquad + \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes df \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle \\ &\qquad - \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes f\omega_{r+1} \otimes \cdots \otimes \omega_n \rangle . \end{aligned} \tag{6}$$

The maps Γ, b, B map $\mathcal{D}(\Omega)$ to itself, so that with

$$\mathcal{D}^{\pm}(\Omega) := \{w \in \mathcal{D}(\Omega) : \Gamma w = \pm w\},$$

there is a super complex

$$\mathcal{D}^{+}(\Omega) \xrightarrow{b+B} \mathcal{D}^{-}(\Omega) \xrightarrow{b+B} \mathcal{D}^{+}(\Omega).$$

With $\mathcal{N}^{\pm}(\Omega) := \mathcal{C}^{\pm}(\Omega)/\mathcal{D}^{\pm}(\Omega)$, the induced quotient complex

$$\mathcal{N}^{+}(\Omega) \xrightarrow{b+B} \mathcal{N}^{-}(\Omega) \xrightarrow{b+B} \mathcal{N}^{+}(\Omega).$$

Whenever there is no danger of confusion, the equivalence class of $w \in \mathcal{C}(\Omega)$ in $\mathcal{N}(\Omega)$ is denoted by the same symbol again.

2. Entire cyclic homology of a locally convex unital DGA

We recall that a topological vector space is called locally convex, if the topology is induced by a family of seminorms, noting that then the topology is equivalent to the topology induced by all continuous seminorms.

Definition 2.1. By a locally convex unital DGA we understand a unital DGA Ω which is also a locally convex Hausdorff space, such that

- the differential is continuous, e.g., for every continuous seminorm ε on Ω there exists a continuous seminorm ε' on Ω such that

$$\varepsilon(d\omega) \le \varepsilon'(\omega) \quad \text{for all } \omega \in \Omega; \tag{7}$$

- the multiplication is jointly continuous, e.g., for every continuous seminorm ε on Ω there exists a continuous seminorm ε' on Ω such that

$$\varepsilon(\omega_1\omega_2) \le \varepsilon'(\omega_1)\varepsilon'(\omega_2) \quad \text{for all } \omega_1, \omega_2 \in \Omega. \tag{8}$$

The space $\underline{\Omega}$ becomes a graded locally convex Hausdorff space, and we equip the algebraic tensor product $\Omega \otimes \underline{\Omega}^{\otimes n}$ with the induced family of π-tensor seminorms, that is,

$$\varepsilon_n(\omega) = \inf\left\{ \sum_\alpha \varepsilon(\omega_0^{(1)}) \cdots \varepsilon(\omega_n^{(\alpha)}) : \omega = \sum_\alpha \omega_0^{(\alpha)} \otimes \cdots \otimes \omega_n^{(\alpha)} \right\},$$

where the sum runs through all representations of ω as a finite sum of elementary tensors, and where ϵ is a continuous seminorm on Ω.

Definition 2.2. The space of *entire cyclic chains* $\mathcal{C}_\epsilon(\Omega)$ is defined to be the closure of $\mathcal{C}(\Omega)$ with respect to the seminorms

$$\kappa_\varepsilon(w) := \sum_{n=0}^{\infty} \frac{\varepsilon_n(w_n)}{\sqrt{n!}},$$

where ε is an arbitrary continuous seminorm on Ω.

The space $\mathcal{C}_\epsilon(\Omega)$ is a complete locally convex Hausdorff space. Note that the above family of seminorms is equivalent to the family of seminorms

$$\kappa_{\varepsilon,l}(w) := \sum_{n=0}^{\infty} \frac{\varepsilon_n(w_n) l^n}{\sqrt{n!}} < \infty,$$

where ε is an arbitrary continuous seminorm on Ω and $l \in \mathbb{N}$, as $l\varepsilon$ is again a continuous seminorm and the ε_n's are cross seminorms. Thus, our growth conditions are modelled on the entire growth conditions for ungraded Banach algebras by Getzler–Szenes from [11]. We refer the reader also to Connes' original variant [7] for ungraded Banach algebras.

Before stating the next auxiliary result, we recall that a continuous linear map from a locally convex Hausdorff space $\mathcal{X}$ to a complete locally convex Hausdorff space $\mathcal{Y}$ can be uniquely extended to a continuous linear map $\widehat{\mathcal{X}} \to \mathcal{Y}$, noting that the completion $\widehat{\mathcal{X}}$ is Hausdorff again. This can be proved precisely as for normed spaces.

Lemma 2.3. *The operators Γ, b, B map $\mathcal{C}(\Omega)$ continuously to itself, in particular, with*

$$\mathcal{C}_\epsilon^{\pm}(\Omega) := \{w \in \mathcal{C}_\epsilon(\Omega) : \Gamma w = \pm w\},$$

there is a well-defined super complex

$$\mathcal{C}_\epsilon^{+}(\Omega) \xrightarrow{b+B} \mathcal{C}_\epsilon^{-}(\Omega) \xrightarrow{b+B} \mathcal{C}_\epsilon^{+}(\Omega). \tag{9}$$

Proof. Let ε be an arbitrary continuous seminorm on Ω. Clearly, one has

$$\kappa_\varepsilon(\Gamma w) \le \kappa_\varepsilon(w)$$

for all $w \in \mathcal{C}(\Omega)$.

Pick continuous seminorms $\varepsilon', \varepsilon''$ on Ω such that for all $\omega \in \Omega$ one has

$$\varepsilon(d\omega) \leq \varepsilon''(\omega)$$

and such that for all $\omega_1, \omega_2 \in \Omega$ one has

$$\varepsilon(\omega_1\omega_2) \leq \varepsilon'(\omega_1)\varepsilon'(\omega_2).$$

Using $n+1 \leq 2^n$ it is then easily checked that

$$\kappa_\varepsilon(bw) \leq C \max(\kappa_{\varepsilon'}, \kappa_{\varepsilon''})(w) \quad \text{for all } w \in \mathcal{C}(\Omega).$$

Likewise, it follows immediately that $\kappa_\varepsilon(Bw) \leq C\kappa_\varepsilon(w)$ for all $w \in \mathcal{C}(\Omega)$. □

Defining the subspace $\mathcal{D}_\epsilon(\Omega) \subset \mathcal{C}_\epsilon(\Omega)$ as the closure of $\mathcal{D}(\Omega)$, it follows automatically that the maps Γ, b, B map $\mathcal{D}(\Omega)$ continuously to itself, too, producing with

$$\mathcal{N}^\pm_\epsilon(\Omega) := \mathcal{C}^\pm_\epsilon(\Omega)/\mathcal{D}^\pm_\epsilon(\Omega)$$

the quotient complex

$$\mathcal{N}^+_\epsilon(\Omega) \xrightarrow{b+B} \mathcal{N}^-_\epsilon(\Omega) \xrightarrow{b+B} \mathcal{N}^+_\epsilon(\Omega). \tag{10}$$

Finally, we can give:

Definition 2.4. The complex (9) is called the (reduced) *entire cyclic complex* of Ω and its homology groups are denoted with $\mathsf{HC}^\pm_\epsilon(\Omega)$. Likewise, the complex (10) is called the (reduced) *Chen-normalized entire cyclic complex* of Ω and its homology groups are denoted with $\mathsf{HN}^\pm_\epsilon(\Omega)$.

Above, 'reduced' refers to the fact that we work with $\Omega \otimes \underline{\Omega}^{\otimes n}$ rather than $\Omega^{\otimes(n+1)}$, which leads to a simpler formula for the Connes differential B.

3. The unital locally convex DGA $\Omega_{\mathbb{T}}(N \times \mathbb{T})$

Assume N is a manifold (possibly with boundary) and denote with $\mathbb{T}$ the 1-sphere. We denote by $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ the smooth $\mathbb{T}$-invariant differential forms on $N \times \mathbb{T}$, where $\mathbb{T}$ acts trivially on N and by rotation on itself. Every element of $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ can be uniquely written in the form $\alpha + \vartheta_{\mathbb{T}} \wedge \beta$ for some $\alpha, \beta \in \Omega(N)$, where $\vartheta_{\mathbb{T}}$ denotes the canonical 1-form on $\mathbb{T}$. We turn $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ into a unital algebra by means of $\Omega_{\mathbb{T}}(N \times \mathbb{T}) \subset \Omega(N \times \mathbb{T})$, and give $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ the grading

$$\alpha + \vartheta_{\mathbb{T}} \wedge \beta \in \Omega^j_{\mathbb{T}}(N \times \mathbb{T}) \iff \alpha \in \Omega^j(N), \beta \in \Omega^{j+1}(N).$$

With $\partial_{\mathbb{T}}$ the canonical vector field on $\mathbb{T}$, we have the differential $d_{\mathbb{T}} = d + \iota_{\partial_{\mathbb{T}}}$ defined by

$$d_{\mathbb{T}}(\alpha + \vartheta_{\mathbb{T}} \wedge \beta) = d\alpha + \beta - \vartheta_{\mathbb{T}} \wedge d\beta, \quad \text{if } \alpha + \vartheta_{\mathbb{T}} \wedge \beta \text{ is homogeneous},$$

finally turning $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ into a unital DGA.

Remark 3.1. Given a manifold X (possibly with boundary), the wedge product and the de Rham differential is continuous with respect to the canonical locally convex structure on $\Omega(X)$ [15]. In addition, if B is a vector field on X then the contraction

$$\iota_B: \Omega(X) \longrightarrow \Omega(X)$$

is continuous, and if Y is another manifold (possibly with boundary) and if $\Psi: X \to Y$ is a smooth map, then the pullback map

$$\Psi^*: \Omega(Y) \longrightarrow \Omega(X)$$

is continuous [15].

For every continuous seminorm ε on $\Omega(N)$ we get a seminorm $\varepsilon^{\mathbb{T}}$ on $\Omega_{\mathbb{T}}(N\times\mathbb{T})$ by setting

$$\varepsilon^{\mathbb{T}}(\alpha + \vartheta_{\mathbb{T}} \wedge \beta) := \varepsilon(\alpha) + \varepsilon(\beta).$$

In view of the formula $d_{\mathbb{T}}$, the space $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ then becomes a locally convex unital DGA (by Remark 3.1) in terms of the $\varepsilon^{\mathbb{T}}$'s. As a consequence, we get the super complexes

$$\mathcal{C}^{+}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{C}^{-}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{C}^{+}(\Omega_{\mathbb{T}}(N\times\mathbb{T})), \tag{11}$$

$$\mathcal{N}^{+}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{N}^{-}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{N}^{+}(\Omega_{\mathbb{T}}(N\times\mathbb{T})), \tag{12}$$

$$\mathcal{C}^{+}_{\epsilon}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{C}^{-}_{\epsilon}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{C}^{+}(\Omega_{\mathbb{T}}(N\times\mathbb{T})), \tag{13}$$

$$\mathcal{N}^{+}_{\epsilon}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{N}^{-}_{\epsilon}(\Omega_{\mathbb{T}}(N\times\mathbb{T})) \xrightarrow{b+B} \mathcal{N}^{+}_{\epsilon}(\Omega_{\mathbb{T}}(N\times\mathbb{T})). \tag{14}$$

4. Equivariant Chen integrals

Let us consider a compact manifold N without boundary, and the space LN of smooth loops $\gamma: \mathbb{T} \to N$, where in the sequel we read $\mathbb{T}$ as $\mathbb{T} = [0,1]/\sim$. This becomes an infinite dimensional Fréchet manifold which is locally modelled on the Fréchet space $L\mathbb{R}^{\dim N}$ of smooth loops $\mathbb{T} \to \mathbb{R}^{\dim N}$. Then LN carries a natural smooth $\mathbb{T}$-action, given by rotating each loop, and the fixed point set of this action is precisely $N \subset LN$, embedded as constant loops. Given $\gamma \in LN$ the tangent space $T_\gamma LN$ is given by linear space of smooth vector fields on N along γ, that is,

$$T_\gamma(LN) = \{X \in C^\infty(\mathbb{T}, N) : X(t) \in T_{\gamma(t)}N \text{ for all } t \in \mathbb{T}\},$$

and the generator of the $\mathbb{T}$-action on LN is the vector field $\gamma \mapsto \dot{\gamma}$ on LN. Let ι denote the contraction with respect to the latter vector field. In the sequel, we understand

$$\Omega(LN) := \bigoplus_{k=0}^{\infty} \Omega^k(LM).$$

For fixed $s \in \mathbb{T}$ one has the diffeomorphism

$$\phi_s \colon LN \longrightarrow LN, \quad \gamma \longmapsto \gamma(s + \cdot)$$

induced by the $\mathbb{T}$-action, and one gets an induced operator

$$P \colon \Omega(LN) \longrightarrow \Omega(LN), \quad \text{defined on } \Omega^k(LN) \text{ by } P\alpha := \int_0^1 \phi_s^* \iota\alpha \, ds.$$

Then P becomes a degree -1 derivation. In addition, there is the usual exterior derivative

$$d \colon \Omega(LN) \longrightarrow \Omega(LN),$$

a degree $+1$ derivation. Taking only odd/even degree forms, one gets the superstructure $\Omega = \Omega^+(LN) \oplus \Omega^-(LN)$, and we get the super complex

$$\Omega^+(LN) \xrightarrow{d+P} \Omega^-(LN) \xrightarrow{d+P} \Omega^+(LN), \tag{15}$$

called the *equivariant de Rham complex of* LN. This complex does not carry much information, as the differential forms of interest, like the Bismut–Chern character below, are actually elements of

$$\prod_{k=0}^{\infty} \Omega^k(LN), \quad \text{rather than} \quad \Omega(LN) = \bigoplus_{k=0}^{\infty} \Omega^k(LN).$$

Thus, we are going to 'complete' $\Omega(LN)$ in some way. To this end, following Chen's approach [6] of constructing a smooth structure on LN in terms of plots, we consider smooth maps $f \colon X \to LN$, where X is a finite dimensional manifold (without boundary). Given a continuous seminorm ε on $\Omega(X)$ we get an induced seminorm

$$\varepsilon_f(\omega) := \varepsilon(f^*\omega) \quad \text{on } \Omega(LN).$$

The locally convex topology induced by the ϵ_f's is Hausdorff and we define $\widehat{\Omega}(LN)$ to be the completion of $\Omega(LN)$ with respect to this locally convex topology. The maps d, P and the grading operator become continuous maps $\Omega(LN) \to \Omega(LN)$: indeed, the continuity of the grading map is trivial. The continuity of d follows from

$$\varepsilon_f(d\omega) = \varepsilon\big(d[f^*\omega]\big) \le \varepsilon'(f^*\omega) = \varepsilon'_f(\omega)$$

for some continuous seminorm ε' on $\Omega(X)$, where we have used the continuity of $d \colon \Omega(X) \to \Omega(X)$. Finally, the continuity of P follows easily from the continuity of ι, which in turn follows from writing

$$\varepsilon_f(\iota\omega) = \varepsilon\big(f^*[\iota\omega]\big) = \varepsilon\big(r^*\iota_{\partial_\mathbb{T}} \widehat{f}^* j^*[\omega]\big) \le \varepsilon'_{j \circ \widehat{f}}(\omega)$$

for some continuous seminorm ε' on $\Omega(X \times \mathbb{T})$, where

$$r\colon X \longrightarrow X \times \mathbb{T}, \quad j\colon N \longrightarrow LN$$

are the canonical embeddings, and

$$\hat{f}\colon X \times \mathbb{T} \longrightarrow N$$

the map induced by $f\colon X \to LN$, and where we have used Remark 3.1 (the continuity of $r^* \iota_{\partial_{\mathbb{T}}}$, which implies the existence of ε').

We end up with the super complex

$$\widehat{\Omega}^+(LN) \xrightarrow{d+P} \widehat{\Omega}^-(LN) \xrightarrow{d+P} \widehat{\Omega}^+(LN), \tag{16}$$

called the *completed equivariant de Rham complex of* LN. The corresponding homology groups are denoted by $\widehat{\mathsf{H}}^{\pm}_{\mathbb{T}}(LN)$.

Given $t \in \mathbb{T}$ and $\alpha \in \Omega^k(N)$ one denotes with $\alpha(t) \in \Omega^k(LN)$ the form obtained by pulling α back with respect to the evaluation map $\gamma \mapsto \gamma(t)$. With this notation at hand, one has the *equivariant Chen integral* map

$$\rho\colon \mathcal{C}(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \Omega(LN),$$

which is defined by

$$\begin{aligned}\rho\,&\langle(\alpha_0 + \vartheta_{\mathbb{T}} \wedge \beta_0) \otimes \cdots \otimes (\alpha_n + \vartheta_{\mathbb{T}} \wedge \beta_n)\rangle \\ &:= \int_0^1 ds\, \phi_s^* \int_{\Delta_n} \alpha_0(0) \wedge (\iota\alpha_1(t_1) - \beta_1(t_1)) \wedge \cdots \wedge (\iota\alpha_n(t_n) - \beta_n(t_n))\, dt_1 \cdots dt_n,\end{aligned}$$

where

$$\Delta_n = \{0 \le t_1 \le \cdots \le t_n \le 1\} \subset \mathbb{R}^n$$

denotes the standard n-simplex. We will also write

$$\begin{aligned}\rho\,&\langle(\alpha_0 + \vartheta_{\mathbb{T}} \wedge \beta_0) \otimes \cdots \otimes (\alpha_n + \vartheta_{\mathbb{T}} \wedge \beta_n)\rangle \\ &\qquad = \int_0^1 ds \phi_s^* \widetilde{\rho}\,\langle(\alpha_0 + \vartheta_{\mathbb{T}} \wedge \beta_0) \otimes \cdots \otimes (\alpha_n + \vartheta_{\mathbb{T}} \wedge \beta_n)\rangle\,.\end{aligned}$$

We collect the essential properties of ρ in the following proposition:

Proposition 4.1. *The map ρ is a continuous morphism of super complexes*

$$\rho\colon \mathcal{C}(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \Omega(LN), \tag{17}$$

which in turn descends to a continuous map of super complexes

$$\rho\colon \mathcal{N}(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \Omega(LN). \tag{18}$$

In particular, by density, we obtain the continuous maps of super complexes

$$\rho\colon \mathcal{C}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \widehat{\Omega}(LN), \quad \rho\colon \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \widehat{\Omega}(LN).$$

Proof. (i) The fact that (17) is a map of superspaces follows easily from observing that

$$\mathcal{C}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T})) = \bigoplus_{j=0}^{\infty}\mathcal{C}^{2j}(\Omega_{\mathbb{T}}(N\times\mathbb{T})),$$
$$\mathcal{C}^-(\Omega_{\mathbb{T}}(N\times\mathbb{T})) = \bigoplus_{j=0}^{\infty}\mathcal{C}^{2j+1}(\Omega_{\mathbb{T}}(N\times\mathbb{T})),$$

where

$$\mathcal{C}^k(\Omega_{\mathbb{T}}(N\times\mathbb{T})) = \bigoplus_{r=0}^{\infty}\bigoplus_{l_0+\cdots+l_r=k+r}\Omega^{l_0}_{\mathbb{T}}(N\times\mathbb{T}))\otimes\underline{\Omega^{l_1}_{\mathbb{T}}(N\times\mathbb{T}))}\otimes\cdots\otimes\underline{\Omega^{l_r}_{\mathbb{T}}(N\times\mathbb{T}))},$$

and that ρ maps $\mathcal{C}^k(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\to\Omega^k(LN)$.

(ii) Next we show that $\rho(b+B)=(d+P)\rho$. Setting $\omega_j=\alpha_j+\vartheta_{\mathbb{T}}\wedge\beta_j$, we first notice

$$\begin{aligned}\widetilde{\rho}b\langle\omega_0\otimes\cdots\otimes\omega_n\rangle &= \widetilde{\rho}\,\langle d_{\mathbb{T}}\omega_0\otimes\cdots\otimes\omega_{j-1}\otimes\omega_j\otimes\omega_{j+1}\otimes\cdots\otimes\omega_n\rangle\\ &\quad-\widetilde{\rho}\Big\langle\sum_{j=1}^{n}(-1)^{r_{j-1}}\omega_0\otimes\cdots\otimes\omega_{j-1}\otimes d_{\mathbb{T}}\omega_j\otimes\omega_{j+1}\otimes\cdots\otimes\omega_n\Big\rangle\\ &\quad-\widetilde{\rho}\Big\langle\sum_{j=0}^{n-1}(-1)^{r_j}\omega_0\otimes\cdots\otimes\omega_{j-1}\otimes\omega_j\wedge\omega_{j+1}\otimes\omega_{j+2}\otimes\cdots\otimes\omega_n\Big\rangle\\ &\quad+(-1)^{(j_n-1)r_{n-1}}\widetilde{\rho}\,\langle\omega_n\wedge\omega_0\otimes\omega_1\otimes\cdots\otimes\omega_{n-1}\rangle\,. \end{aligned}\tag{19}$$

The first two lines give

$$\begin{aligned}&\int_{\Delta_n}\big(d\alpha_0(0)+\beta_0(0)\big)\wedge\big(\iota\alpha_1(t_1)-\beta_1(t_1)\big)\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)\,d^nt\\ &-\sum_{j=1}^{n}(-1)^{r_{j-1}}\int_{\Delta_n}\alpha_0(0)\big(\iota\alpha_1(t_1)-\beta_1(t_1)\big)\wedge\cdots\wedge\big(\iota\alpha_{j-1}(t_{j-1})-\beta_{j-1}(t_{j-1})\big)\\ &\qquad\wedge\big(\iota d\alpha_j(t_j)+\iota\beta_j(t_{j-1})+d\beta_j(t_j)\big)\wedge\big(\iota\alpha_{j+1}(t_{j+1})-\beta_{j+1}(t_{j+1})\big)\\ &\qquad\qquad\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)\,d^nt,\end{aligned}$$

where $d^nt=dt_1\cdots dt_n$. Using that

$$\Delta_n=\{(t_1,t_2,\ldots,t_n):0\le t_1\le\cdots\le t_{j-1}\le t_j\le t_{j+1}\le\cdots\le t_n\},$$

and that

$$\iota d\alpha_j(t_j) = \frac{d}{dt_j}\alpha_j(t_j) - d\iota\alpha_j(t_j),$$

it can be rewritten as

$$\begin{aligned}
&\int_{\Delta_n} \big(d\alpha_0(0) + \beta_0(0)\big) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&+ \sum_{j=1}^{n} (-1)^{r_{j-1}} \int_{\Delta_n} \alpha_0(0)\big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge d\big(\iota\alpha_j(t_j) - \beta_j(t_j)\big) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&- \sum_{j=1}^{n} (-1)^{r_{j-1}} \int_{\Delta_n} \alpha_0(0)\big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \frac{d}{dt_j}\alpha_j(t_j) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&- \sum_{j=1}^{n} (-1)^{r_{j-1}} \int_{\Delta_n} \alpha_0(0)\big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \iota\beta_j(t_j) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t.
\end{aligned}$$

The first two (three) lines give

$$d\widetilde{\rho}\langle\omega_0 \otimes \cdots \otimes \omega_n\rangle + \int_{\Delta_n} \beta_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t, \tag{20}$$

while the third (fourth and fifth) line can be integrated in t_j from t_{j-1} to t_{j+1} thus getting

$$\begin{aligned}
&d\widetilde{\rho}\langle\omega_0 \otimes \cdots \otimes \omega_n\rangle + \int_{\Delta_n} \beta_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&- \sum_{j=1}^{n-1} (-1)^{r_{j-1}} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \alpha_j(t_{j+1}) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_j \\
&- (-1)^{r_{n-1}} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{n-1}(t_{n-1}) - \beta_{n-1}(t_{n-1})\big) \\
&\qquad\qquad \wedge \alpha_n(1)\, d^n t_n
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=2}^{n}(-1)^{r_{j-1}} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \alpha_j(t_{j-1}) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_j \\
&+ (-1)^{r_0} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \alpha_1(0) \wedge \big(\iota\alpha_2(t_2) - \beta_2(t_2)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_1 \\
&- \sum_{j=1}^{n}(-1)^{r_{j-1}} \int_{\Delta_n} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \iota\beta_j(t_j) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t, \quad (21)
\end{aligned}
$$

where $d^n t_j = dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_n$. If in the fourth sum of integrals we change the summation variable from j to $j+1$, then make the change of variable $t_j \to t_{j+1}$, and put it together with the second sum of integrals, after noting that

$$
(-1)^{r_{j-1}}(-1)^{j_j} = -(-1)^{r_j},
$$

then summing the fourth and the second integrals, we get

$$
\begin{aligned}
&- \sum_{j=1}^{n-1}(-1)^{r_{j-1}} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \big[\alpha_j(t_{j+1}) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big)\big] \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_j \\
&+ \sum_{j=1}^{n-1}(-1)^{r_j} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \big[\big(\iota\alpha_j(t_{j+1}) - \beta_j(t_{j+1})\big) \wedge \alpha_{j+1}(t_{j+1})\big] \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_j \\
&= \sum_{j=1}^{n-1}(-1)^{r_j} \int_{\Delta_{n-1}} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \big[\big(\iota\alpha_j(t_{j+1}) - \beta_j(t_{j+1})\big) \wedge \alpha_{j+1}(t_{j+1}) + (-1)^{j_j-1}\alpha_j(t_{j+1}) \\
&\qquad\qquad \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big)\big] \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t_j \\
&= \sum_{j=1}^{n-1}(-1)^{r_j} \widetilde{\rho}\langle \omega_0 \otimes \cdots \otimes \omega_{j-1} \otimes \omega_j \wedge \omega_{j+1} \otimes \omega_{j+2} \otimes \cdots \otimes \omega_n \rangle,
\end{aligned}
$$

which including the fifth integral in (21) becomes

$$
\widetilde{\rho}\left\langle \sum_{j=0}^{n-1}(-1)^{r_j} \omega_0 \otimes \cdots \otimes \omega_{j-1} \otimes \omega_j \wedge \omega_{j+1} \otimes \omega_{j+2} \otimes \cdots \otimes \omega_n \right\rangle.
$$

This cancels the second line of (19). After noting that $\alpha_n(1) = \alpha_n(0)$, we see that the third integral in (21) is just

$$-(-1)^{(j_n-1)r_{n-1}}\widetilde{\rho}\,\langle \omega_n \wedge \omega_0 \otimes \omega_1 \otimes \cdots \otimes \omega_{n-1}\rangle\,,$$

which cancels the third line of (19). Thus, we get

$$\begin{aligned}
&\widetilde{\rho}b\,\langle \omega_0 \otimes \cdots \otimes \omega_n\rangle = d\,\widetilde{\rho}\,\langle \omega_0 \otimes \cdots \otimes \omega_n\rangle \\
&\quad + \int_{\Delta_n} \beta_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&\quad - \sum_{j=1}^{n} (-1)^{r_{j-1}} \int_{\Delta_n} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\qquad \wedge \iota\beta_j(t_j) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t.
\end{aligned} \tag{22}$$

Now, let us consider

$$\begin{aligned}
&P\,\widetilde{\rho}\,\langle \omega_0 \otimes \cdots \otimes \omega_n\rangle \\
&= \int_I ds \phi_s^* \iota \int_{\Delta_n} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t \\
&= \int_{I\times\Delta_n} \iota\alpha_0(s) \wedge \big(\iota\alpha_1(t_1+s) - \beta_1(t_1+s)\big) \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s) - \beta_n(t_n+s)\big)\, d^n t\, ds \\
&\quad - \sum_{j=1}^{n} (-1)^{r_{j-1}} \int_I ds \phi_s^* \int_{\Delta_n} \alpha_0(0) \wedge \big(\iota\alpha_1(t_1) - \beta_1(t_1)\big) \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_{j-1}(t_{j-1}) - \beta_{j-1}(t_{j-1})\big) \\
&\quad \wedge \iota\beta_j(t_j) \wedge \big(\iota\alpha_{j+1}(t_{j+1}) - \beta_{j+1}(t_{j+1})\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n) - \beta_n(t_n)\big)\, d^n t,
\end{aligned} \tag{23}$$

where now I must be identified with the circle $\mathbb{T}$, and where we used that

$$\iota(\iota\alpha_k(t_k) - \beta_k(t_k)) = -\iota\beta_k(t_k).$$

Now, for any given choice of $\overline{t} = (t_1, \ldots, t_n)$ such that $0 \leq t_1 \leq \cdots \leq t_n \leq 1$, we can understand $\mathbb{T}$ as the union of almost everywhere $n+1$ disjoint intervals defined by

$$I_j(\overline{t}) = \{s \in \mathbb{T} \,|\, t_{j-1} + s \leq 1,\ t_j + s - 1 \geq 0\}, \quad j = 1, \ldots, n+1.$$

We see that

$$D_j = \{I_j(\overline{t}) \times \overline{t} \mid \overline{t} \in \Delta_n\}$$

is an $(n+1)$-simplex for any given j, and

$$\bigcup_{j=1}^{n+1} D_j = I \times \Delta_n,$$

while $D_j \cap D_k$ has zero measure if $j \neq k$. Therefore,

$$\begin{aligned}
&\int_{I\times\Delta_n} \iota\alpha_0(s) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big)\, d^nt\, ds \\
&= \int_{I\times\Delta_n} \beta_0(s) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big)\, d^nt\, ds \\
&+ \int_{I\times\Delta_n} \big(\iota\alpha_0(s)-\beta_0\big) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big) d^nt\, ds \\
&= \int_I ds\phi_s^* \int_{\Delta_n} \beta_0(0) \wedge \big(\iota\alpha_1(t_1)-\beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n)-\beta_n(t_n)\big) d^nt\, ds \\
&+ \sum_{j=1}^{n+1} \int_{D_j} \big(\iota\alpha_0(s)-\beta_0(s)\big) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \\
&\qquad\qquad \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big)\, d^nt\, ds.
\end{aligned}$$

Now, for any given j we introduce the variables

$$\begin{aligned}
\tau_k &= t_{j+k-1}+s-1, \quad k = 1,\dots,n+1-j, \\
\tau_{n+2-j} &= s, \\
\tau_k &= t_{k+j-n-2}+s, \qquad k = n+3-j,\dots,n+1 \quad (\text{if } j \geq 2).
\end{aligned}$$

In these coordinates we have

$$D_j = \{(\tau_1,\dots,\tau_{n+1})|0 \leq \tau_1 \leq \cdots \leq \tau_{n+1} \leq 1\} \equiv \Delta_{n+1}, \quad d^nt\, ds = d^{n+1}\tau,$$

and

$$\begin{aligned}
&\big(\iota\alpha_0(s)-\beta_0(s)\big) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big) \\
&\quad = (-1)^{r_{j-1}(r_n-r_j)} 1 \wedge \big(\iota\alpha_j(\tau_1)-\beta_j(\tau_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(\tau_{n-j+1})-\beta_n(\tau_{n-j+1})\big) \\
&\qquad\quad \wedge \big(\iota\alpha_0(\tau_{n-j+2})-\beta_0(\tau_{n-j+2})\big) \wedge \cdots \wedge \big(\iota\alpha_{j-1}(\tau_{n+1})-\beta_{j-1}(\tau_{n+1})\big).
\end{aligned}$$

Integrating over $D_j = \Delta_{n+1}$ it becomes

$$\begin{aligned}
&\int_{D_j} \big(\iota\alpha_0(s)-\beta_0(s)\big) \wedge \big(\iota\alpha_1(t_1+s)-\beta_1(t_1+s)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n+s)-\beta_n(t_n+s)\big) \\
&\qquad\qquad = \rho\big\langle (-1)^{r_{j-1}(r_n-r_j)} 1 \otimes \omega_j \otimes \cdots \otimes \omega_n \otimes \omega_0 \otimes \cdots \otimes \omega_{j-1} \big\rangle,
\end{aligned}$$

and after summation over j we finally get

$$\begin{aligned}
&P\widetilde{\rho}\,\langle \omega_0 \otimes \cdots \otimes \omega_n \rangle = \widetilde{\rho}B\,\langle \omega_0 \otimes \cdots \otimes \omega_n \rangle \\
&+ \int_I ds\phi_s^* \int_{\Delta_n} \beta_0(0) \wedge \big(\iota\alpha_1(t_1)-\beta_1(t_1)\big) \wedge \cdots \wedge \big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)\, d^nt\, ds
\end{aligned}$$

$$-\sum_{j=1}^{n}(-1)^{r_{j-1}}\int_I ds\phi_s^*\int_{\Delta_n}\alpha_0(0)\wedge\big(\iota\alpha_1(t_1)-\beta_1(t_1)\big)\wedge\cdots\wedge\big(\iota\alpha_{j-1}(t_{j-1})-\beta_{j-1}(t_{j-1})\big)$$
$$\wedge\,\iota\beta_j(t_j)\wedge\big(\iota\alpha_{j+1}(t_{j+1})-\beta_{j+1}(t_{j+1})\big)\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)\,d^nt.$$

Notice that the second and third lines here are the means over $\mathbb{T}$ of the corresponding terms in (22). After taking the mean of both expressions and subtracting each other, we finally get $\rho(b+B)=(d+P)\rho$ as desired.

(iii) We now prove that $\widetilde{\rho}$ vanishes on $\mathcal{D}(\Omega_{\mathbb{T}}(N\times\mathbb{T}))$. This implies that ρ vanishes on $\mathcal{D}(\Omega_{\mathbb{T}}(N\times\mathbb{T}))$, too. For elements of the form (5) the assertion immediately follows from the fact that $\iota f(t)=0$, as $f(t)$ is a zero form. So, let us consider an element of the form (6). Since (recall that f is constant over $\mathbb{T}$)

$$\iota df(t)=\frac{d}{dt}f(t),$$

and $df=d_{\mathbb{T}}f$, we can write

$$\begin{aligned}
&\widetilde{\rho}\big(\langle\omega_0\otimes\cdots\otimes\omega_{r-1}f\otimes\omega_{r+1}\otimes\cdots\otimes\omega_n\rangle\\
&\qquad+\langle\omega_0\otimes\cdots\otimes\omega_{r-1}\otimes df\otimes\omega_{r+1}\otimes\cdots\otimes\omega_n\rangle\\
&\qquad-\langle\omega_0\otimes\cdots\otimes\omega_{r-1}\otimes f\omega_{r+1}\otimes\cdots\otimes\omega_n\rangle\big)\\
&=\int_{\Delta_{n-1}}\alpha_0(0)\wedge\cdots\wedge\big(\iota\alpha_{r-1}(t_{r-1})f(t_{r-1})-\beta_{r-1}(t_{r-1})f(t_{r-1})\big)\\
&\qquad\wedge\big(\iota\alpha_{r+1}(t_{r+1})-\beta_{r+1}(t_{r+1})\big)\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)\,d^nt_r\\
&-\int_{\Delta_{n-1}}\alpha_0(0)\wedge\cdots\wedge\big(\iota\alpha_{r-1}(t_{r-1})-\beta_{r-1}(t_{r-1})\big)\\
&\qquad\wedge\big(f(t_{r+1})\iota\alpha_{r+1}(t_{r+1})-f(t_{r+1})\beta_{r+1}(t_{r+1})\big)\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)d^nt_r\\
&+\int_{\Delta_n}\alpha_0(0)\wedge\cdots\wedge\big(\iota\alpha_{r-1}(t_{r-1})-\beta_{r-1}(t_{r-1})\big)\wedge\frac{d}{dt_r}f(t_r)\wedge\big(\iota\alpha_r(t_r)-\beta_r(t_r)\big)\\
&\qquad\wedge\cdots\wedge\big(\iota\alpha_n(t_n)-\beta_n(t_n)\big)d^nt.
\end{aligned}$$

After integrating t_r from t_{r-1} to t_{r+1} in the last term, we get exactly zero.

(iv) It remains to check the continuity of (17), which easily follow from the continuity of $\widetilde{\rho}$. To see the latter, let X be a smooth manifold (without boundary), let ε be a continuous seminorm on $\Omega(X)$, and let $f\colon X\to LN$ be smooth. For $s\in\mathbb{T}$ let r_s denote the embedding

$$X\longrightarrow X\times\mathbb{T},\ x\longmapsto(x,s).$$

Then we have

$$\begin{aligned}
&\varepsilon_f\big(\widetilde{\rho}\,\langle(\alpha_0+\vartheta_{\mathbb{T}}\wedge\beta_0)\otimes\cdots\otimes(\alpha_n+\vartheta_{\mathbb{T}}\wedge\beta_n)\rangle\,\big)\\
&\quad\leq\int_{\Delta_n}\varepsilon\big(f^*[\alpha_0(0)]\big)\prod_{i=1}^n\varepsilon\big(f^*[\iota\alpha_i(t_i)-\beta_i(t_i)]\big)\,dt_1\cdots dt_n\\
&\quad=\int_{\Delta_n}\varepsilon\big(r_0^*\widehat{f}^*\alpha_0\big)\prod_{i=1}^n\varepsilon\big(r_{t_i}^*\iota_{\partial_{\mathbb{T}}}\widehat{f}^*\alpha_i-r_{t_i}^*\widehat{f}^*\beta_i\big)\,dt_1\cdots dt_n\\
&\quad\leq\int_{\Delta_n}\varepsilon\big(r_0^*\widehat{f}^*\alpha_0\big)\prod_{i=1}^n\big(\varepsilon\big(r_{t_i}^*\iota_{\partial_{\mathbb{T}}}\widehat{f}^*\alpha_i\big)+\varepsilon\big(r_{t_i}^*\widehat{f}^*\beta_i\big)\big)\,dt_1\cdots dt_n\\
&\quad\leq\int_{\Delta_n}\widetilde{\varepsilon}(\alpha_0)\prod_{i=1}^n\big(\widetilde{\varepsilon}(\alpha_i)+\widetilde{\varepsilon}(\beta_i)\big)\,dt_1\cdots dt_n\\
&\quad\leq\frac{1}{n!}\prod_{i=0}^n\big(\widetilde{\varepsilon}(\alpha_i)+\widetilde{\varepsilon}(\beta_i)\big)=\frac{1}{n!}\widetilde{\varepsilon}_n^{\mathbb{T}}\big((\alpha_0+\vartheta_{\mathbb{T}}\wedge\beta_0)\otimes\cdots\otimes(\alpha_n+\vartheta_{\mathbb{T}}\wedge\beta_n)\big),
\end{aligned}$$

for some continuous seminorm $\widetilde{\varepsilon}$ on $\Omega(N)$. This estimate shows the continuity of $\widetilde{\rho}$ and completes the proof. □

5. Construction of cycles in $\mathcal{N}_\epsilon^-(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$ and the induced cycles in $\widehat{\Omega}^-(LM)$

Let M be a compact manifold (possibly with boundary). Given $g\in C^\infty(M,U(l\times l;\mathbb{C}))$ our aim is to construct a canonically given element

$$\mathrm{Ch}^-(g)\in\mathcal{C}_\epsilon^-(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$$

with $(b+B)\mathrm{Ch}^-(g)=0$ in the Chen normalized complex. To this end, let $I:=[0,1]$ and denote the canonical vector field on I with ∂_I. We denote the canonical Maurer–Cartan form on $U(l\times l;\mathbb{C})$ by

$$\omega\in\Omega^1\big(U(l\times l;\mathbb{C}),\mathrm{Mat}(l\times l;\mathbb{C})\big).$$

Then for all $s\in I$ we can form the covariant derivative $d+s\omega$ on the trivial vector bundle $U(l\times l;\mathbb{C})\times\mathbb{C}^l\to U(l\times l;\mathbb{C})$. Let

$$A^s\in\Omega^1\big(U(l\times l;\mathbb{C}),\mathrm{Mat}(l\times l;\mathbb{C})\big),\quad R^s\in\Omega^2\big(U(l\times l;\mathbb{C}),\mathrm{Mat}(l\times l;\mathbb{C})\big)$$

denote the connection 1-form of $d+s\omega$ and the curvature of $d+s\omega$, respectively, and

$$\mathcal{A}^s:=A^s-\vartheta_{\mathbb{T}}\wedge R^s\in\Omega_{\mathbb{T}}\big(U(l\times l;\mathbb{C})\times\mathbb{T},\mathrm{Mat}(l\times l;\mathbb{C})\big).$$

We set

$$A^s(g):=g^*A^s,\quad R_g^s:=g^*R^s,\quad \omega_g:=g^*\omega,$$

so that $A^s(g) = s\omega_g$ and by the Maurer–Cartan equation $R^s_g = (s/2)\omega_g^2$. Then we can define

$$\mathcal{A}^s(g) := A^s_g - \vartheta_{\mathbb{T}} \wedge R^s_g \in \Omega_{\mathbb{T}}(M \times \mathbb{T}, \mathrm{Mat}(l \times l; \mathbb{C})).$$

By varying s, the forms $\mathcal{A}^s(g)$ induce a form

$$\mathcal{A}(g) \in \Omega_{\mathbb{T}}(M \times I \times \mathbb{T}, \mathrm{Mat}(l \times l; \mathbb{C}))$$

and we set

$$\mathcal{B}(g) := \iota_{\partial_I} \mathcal{A}(g) \in \Omega_{\mathbb{T}}(M \times I \times \mathbb{T}, \mathrm{Mat}(l \times l; \mathbb{C})).$$

Then we can define

$$\mathcal{B}^s(g) \in \Omega_{\mathbb{T}}(M \times \mathbb{T}, \mathrm{Mat}(l \times l; \mathbb{C}))$$

to be the pullback of $\mathcal{B}(g)$ with respect to the embedding

$$M \times \mathbb{T} \longrightarrow M \times I \times \mathbb{T}, \quad (x,t) \longmapsto (x,s,t).$$

In fact, by a simple calculation one finds

$$\mathcal{A}^s(g) = s\omega_g + s(1-s)\vartheta_{\mathbb{T}} \wedge \omega_g^2, \quad \mathcal{B}^s(g) = -\vartheta_{\mathbb{T}} \wedge \omega_g, \tag{24}$$

so that $\mathcal{B}^s(g)$ actually does not depend on s. With these preparations, we can define an element

$$\mathrm{Ch}^-(g) = (\mathrm{Ch}^-_0(g), \mathrm{Ch}^-_1(g), \dots) \in \mathcal{C}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

by setting

$$\mathrm{Ch}^-_n(g) := \mathrm{Tr}_n\Big[\int_0^1 1 \otimes \sum_{k=1}^{n} \mathcal{A}^s(g)^{\otimes(k-1)} \otimes \mathcal{B}^s(g) \otimes \mathcal{A}^s(g)^{\otimes(n-k)}\, ds\Big],$$

where given linear spaces $V_0, \dots, V_n$, and $v^{(j)} \in \mathrm{Mat}(l \times l; V_i)$, $j = 0, \dots, n$, the generalized trace is defined by

$$\mathrm{Tr}_n\big[v^{(0)} \otimes \cdots \otimes v^{(n)}\big] := \sum_{i_0,\dots,i_n=1,\dots l} v^{(0)}_{i_0,i_1} \otimes v^{(1)}_{i_1,i_2} \otimes \cdots \otimes v^{(n)}_{i_n,i_0}.$$

We refer the reader to the paper [14] by Simons and Sullivan, where a construction of the usual odd Chern character $\mathrm{ch}^-(g) \in \Omega^-(M)$ (cf. formula (25) below) has been given that influenced our definition of $\mathrm{Ch}^-(g)$.

Theorem 5.1. *Let M be a compact manifold, possibly with boundary.*

(a) *One has*

$$\mathrm{Ch}^-(g) \in \mathcal{C}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})), \quad \textit{and} \quad (b+B)\mathrm{Ch}^-(g) = 0 \textit{ in } \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),$$

in particular, $\mathrm{Ch}^-(g)$ *induces a homology class*

$$\left[\mathrm{Ch}^-(g)\right] \in \mathsf{HN}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

(b) *The map*

$$\mathsf{K}^{-1}(M) \longrightarrow \mathsf{HN}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})), \quad [g] \longmapsto \left[\mathrm{Ch}^-(g)\right]$$

is a well-defined group homomorphism.

Proof. (a) It is easily seen that $\Gamma\mathrm{Ch}^-(g) = -\mathrm{Ch}^-(g)$. To show that

$$\mathrm{Ch}^-(g) \in \mathcal{C}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),$$

given a continuous seminorm ε on $\Omega_{\mathbb{T}}(M\times\mathbb{T})$ set

$$C_\varepsilon := \sup_{s\in[0,1]} \max\Big(\varepsilon(1), \max_{i,j=1,\ldots,l} \varepsilon(\mathcal{A}^s(g)_{ij}), \max_{i,j=1,\ldots,l} \varepsilon(\mathcal{B}^s(g)_{ij})\Big).$$

It is then easily checked that

$$\kappa_\varepsilon(\mathrm{Ch}^-(g)) \leq \sum_{n=0}^\infty n\frac{(l^2C_\varepsilon)^n}{\sqrt{n!}} < \infty.$$

It remains to prove

$$(b+B)\mathrm{Ch}^-(g) \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

In fact,

$$B\mathrm{Ch}^-(g) \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),$$

as every $\langle \mathrm{Ch}^-_n(g)\rangle$ contains the 0-form 1 and so is of the form (5) with $f = 1$. It remains to show that

$$b\mathrm{Ch}^-(g) \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

In order to see the latter, let us first notice that

$$(b\mathrm{Ch}^-(g))_n = \big(b\,\langle \mathrm{Ch}^-_n(g)\rangle\,\big)_n + \big(b\langle \mathrm{Ch}^-_{n+1}(g)\rangle\big)_n.$$

Using (24) and the explicit definition of b, we get

$$
\begin{aligned}
&\big(b\langle \mathrm{Ch}^-_n(g)\rangle\big)_n \\
&\quad= -\mathrm{Tr}_n\bigg[\int_0^1 1\otimes \sum_{k=1}^{n}\sum_{l=0}^{k-2}\mathcal{A}^s(g)^{\otimes l}\otimes(-s^2\omega_g^2)\otimes \mathcal{A}^s(g)^{\otimes(k-l-2)} \\
&\qquad\qquad\qquad\qquad \otimes(-\vartheta_{\mathbb{T}}\wedge\omega_g)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\bigg] \\
&\qquad+\mathrm{Tr}_n\bigg[\int_0^1 1\otimes\sum_{k=1}^{n}\sum_{l=0}^{n-k-1}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-\vartheta_{\mathbb{T}}\wedge\omega_g) \\
&\qquad\qquad\qquad\qquad \otimes\mathcal{A}^s(g)^{\otimes l}\otimes(-s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(n-k-l-1)}\,ds\bigg] \\
&\qquad-\mathrm{Tr}_n\bigg[\int_0^1 1\otimes\sum_{k=1}^{n}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(\vartheta_{\mathbb{T}}\wedge\omega_g^2+\omega_g)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(b\langle \mathrm{Ch}^-_{n+1}(g)\rangle\big)_n \\
&\quad= -\mathrm{Tr}_n\bigg[\int_0^1 1\otimes \sum_{k=1}^{n}\sum_{l=0}^{k-2}\mathcal{A}^s(g)^{\otimes l}\otimes(+s^2\omega_g^2)\otimes \mathcal{A}^s(g)^{\otimes(k-l-2)} \\
&\qquad\qquad\qquad\qquad \otimes(-\vartheta_{\mathbb{T}}\wedge\omega_g)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\bigg] \\
&\qquad+\mathrm{Tr}_n\bigg[\int_0^1 1\otimes\sum_{k=1}^{n}\sum_{l=0}^{n-k-1}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-\vartheta_{\mathbb{T}}\wedge\omega_g)\otimes\mathcal{A}^s(g)^{\otimes l} \\
&\qquad\qquad\qquad\qquad \otimes(+s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(n-k-l-1)}\,ds\bigg] \\
&\qquad-\mathrm{Tr}_n\bigg[\int_0^1 1\otimes\sum_{k=1}^{n}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-2s\vartheta_{\mathbb{T}}\wedge\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\bigg],
\end{aligned}
$$

whose sum is

$$
\begin{aligned}
&\mathrm{Tr}_n\bigg[\int_0^1 1\otimes\sum_{k=1}^{n}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes\Big(\frac{d}{ds}\mathcal{A}^s(g)\Big)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\bigg] \\
&= \mathrm{Tr}_n\bigg[\int_0^1\frac{d}{ds}\big(1\otimes\mathcal{A}^s(g)^{\otimes n}\big)\,ds\bigg] = \mathrm{Tr}_n\big[1\otimes\mathcal{A}^1(g)^{\otimes n}\big]-\mathrm{Tr}_n\big[1\otimes\mathcal{A}^0(g)^{\otimes n}\big].
\end{aligned}
$$

Thus, we finally have

$$
(b\mathrm{Ch}^-(g))_n = \mathrm{Tr}_n\big[1\otimes {}_g^{\otimes n}\big],\quad n=1,2,\ldots.
$$

We now prove that

$$\left(\dots, \mathrm{Tr}_n \left[1 \otimes {}_g^{\otimes n}\right], \dots\right) \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T})).$$

To this end we have simply to employ the properties of the generalized trace. Indeed, for $n \geq 2$ we can write

$$\begin{aligned}\left\langle \mathrm{Tr}_n\left[1 \otimes {}_g^{\otimes n}\right]\right\rangle &= \left\langle \mathrm{Tr}_n\left[1 \otimes \omega_g \otimes \omega_g \otimes {}_g^{\otimes(n-2)}\right]\right\rangle \\ &= -\left\langle \mathrm{Tr}_n\left[1 \otimes dg^{-1} \otimes dg \otimes {}_g^{\otimes(n-2)}\right]\right\rangle \\ &= -\left\langle \mathrm{Tr}_n\left[1 \otimes dg^{-1} \otimes dg \otimes {}_g^{\otimes(n-2)}\right]\right\rangle \\ &\quad - \left\langle \mathrm{Tr}_{n-1}\left[g^{-1} \otimes dg \otimes {}_g^{\otimes(n-2)}\right]\right\rangle + \left\langle \mathrm{Tr}_{n-1}\left[1 \otimes g^{-1}dg \otimes {}_g^{\otimes(n-2)}\right]\right\rangle,\end{aligned}$$

where the last two terms cancel each other because of the trace property, which is precisely of the form (6) for $f = g^{-1}$. Similarly, for $n = 1$ it is sufficient to notice that

$$\left\langle \mathrm{Tr}_1\left[1 \otimes \omega_g\right]\right\rangle = \left\langle \mathrm{Tr}_1\left[g^{-1} \otimes dg\right]\right\rangle,$$

which is of the form (5) with $f = g^{-1}$, completing the proof of

$$b\mathrm{Ch}^-(g) \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T})).$$

(b) It suffices to prove the following two facts:

(i) If $g, h \in C^\infty(M, U(l \times l; \mathbb{C}))$, then one has $\mathrm{Ch}^-(g \oplus h) = \mathrm{Ch}^-(g) + \mathrm{Ch}^-(h)$.

(ii) If $g_0, g_1 \in C^\infty(M, U(l \times l; \mathbb{C}))$ are connected by a smooth homotopy

$$g_\cdot \in C^\infty(M \times I, U(l \times l; \mathbb{C})),$$

then one has

$$\mathrm{Ch}^-(g_1) - \mathrm{Ch}^-(g_0) = (b + B)w \quad \text{in } \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

for some $w \in \mathcal{C}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$.

Here, property (i) is an immediate consequence of the properties of the generalized trace Tr_n using the block diagonal form of $g \oplus h$.

To see (ii), for any $t \in I$, we define the embedding

$$j_t : M \hookrightarrow M \times I, \quad x \longmapsto (x, t),$$

and $w = (w_0, w_1, \dots) \in \mathcal{C}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ by setting

$$w_n := -\mathrm{Tr}_n\Big[\int_0^1\int_0^1 1 \otimes \sum_{k=1}^{n}\sum_{l=0}^{k-2} j_t^*\Big(\mathcal{A}^s(g.)^{\otimes l} \otimes \iota_{\partial_I}\mathcal{A}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes(k-l-2)} \otimes \mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes(n-k)}\Big)\, ds\, dt\Big]$$
$$+ \mathrm{Tr}_n\Big[\int_0^1\int_0^1 1 \otimes \sum_{k=1}^{n}\sum_{l=0}^{n-k-1} j_t^*\Big(\mathcal{A}^s(g.)^{\otimes(k-1)} \otimes \mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes l} \otimes \iota_{\partial_I}\mathcal{A}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes(n-k-l-1)}\Big)\, ds\, dt\Big]$$
$$- \mathrm{Tr}_n\Big[\int_0^1\int_0^1 1 \otimes \sum_{k=1}^{n} j_t^*\Big(\mathcal{A}^s(g.)^{\otimes(k-1)} \otimes \iota_{\partial_I}\mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes(n-k)}\Big)\, ds\, dt\Big].$$

The $\mathcal{C}_\epsilon$ growth conditions are easily checked for w. Then again it is clear that $Bw \in \mathcal{D}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$. On the other hand, by using the identity

$$dj_t^*\iota_{\partial_I}\mathcal{A}^s(g.) = -j_t^*\iota_{\partial_I} d\mathcal{A}^s(g.) + \frac{\partial}{\partial t} j_t^*\mathcal{A}^s(g.),$$

and similarly for $\mathcal{B}^s$, and the same computations as in part a) we get, as elements in the Chen normalized complex,

$$(bw + Bw)_n = (bw)_n = \big(b\langle w_n\rangle\big)_n + \big(b\langle w_{n+1}\rangle\big)_n = \Big(\Big\langle \int_0^1 \frac{d}{dt} j_t^*\mathrm{Ch}^-(g.)\Big\rangle\Big)_n$$
$$= \mathrm{Ch}_n^-(g_1) - \mathrm{Ch}_n^-(g_0).$$

This completes the proof. □

If M has no boundary (so that LM is a well-defined Fréchet manifold), in view of $(d + P)\rho = \rho(b + B)$, we immediately get:

Corollary 5.2. *Assume M is a compact manifold without boundary. Then for all $g \in C^\infty(M, U(l \times l; \mathbb{C}))$ one has $(d + P)\rho(\mathrm{Ch}^-(g)) = 0$ in $\mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$, in particular, $\rho(\mathrm{Ch}^-(g))$ induces a homology class in $\widehat{\mathsf{H}}_{\mathbb{T}}^-(LM)$.*

Remark 5.3. There is an even version of $\mathrm{Ch}^-(g)$ given as follows: If N is a manifold and $d + C$ is a connection on a trivial vector bundle over N, then with R_C the curvature of the connection 1-form C one defines

$$\mathrm{Ch}^+(C) = (\mathrm{Ch}_0^+(C), \mathrm{Ch}_1^+(C), \dots) \in \mathcal{C}_\epsilon^+(\Omega_{\mathbb{T}}(N \times \mathbb{T}))$$

by

$$\mathrm{Ch}_n^+(C) := \mathrm{Tr}_n\big[1 \otimes (C - \vartheta_{\mathbb{T}} \wedge R_C)^{\otimes n}\big],$$

which by an analogous calculation as in the proof of Theorem 5.1 is seen to satisfy

$$(b + B)\mathrm{Ch}^+(C) = 0 \quad \text{in } \mathcal{N}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T})).$$

Then, there holds an even/odd periodicity, that is, one can obtain $\mathrm{Ch}^-(g)$ from its even variant by a fiber integration: indeed, by varying $s \in I$ in

$$A^s(g) \in \Omega_{\mathbb{T}}(M, \mathrm{Mat}(l \times l; \mathbb{C}))$$

we get a form

$$A(g) \in \Omega_{\mathbb{T}}(M \times I, \mathrm{Mat}(l \times l; \mathbb{C}))$$

and can consider the fibration

$$\pi \colon M \times I \longrightarrow M.$$

Then, for the connection $d + \widetilde{A}_g$ on the trivial vector bundle over $M \times I$, where $\widetilde{A}_g := \pi^* A_g$, one has, using the definitions of $\mathcal{A}^s(g)$ and $\mathcal{B}^s(g)$ that

$$\mathrm{Ch}^-(g) = \int_I \iota_{\partial_I} \mathrm{Ch}^+(\widetilde{A}_g) = \pi_* \mathrm{Ch}^+(\widetilde{A}_g),$$

the integration along the fibers of π.

The *odd Chern character* $\mathrm{ch}^-(g) \in \Omega^-(M)$ is the closed odd differential form defined by

$$\mathrm{ch}^-(g) := \mathrm{Tr}\Big[\sum_{j=0}^{\infty} \frac{(-1)^j j!}{(2j+1)!} (g^{-1} dg)^{\wedge(2j+1)} \Big], \tag{25}$$

and the *odd Bismut–Chern character* is the differential form

$$\mathrm{Bch}^-(g) = (\mathrm{Bch}^-_1(g), \mathrm{Bch}^-_3(g), \dots) \in \widehat{\Omega}^-(LM)$$

defined by

$$\mathrm{Bch}^-_{2n-1}(g) = \mathrm{Tr}\Big[\int_0^1 \int_{\{0 \le t_1 \le \dots t_n \le 1\}} \sum_{j=1}^{n} \bigwedge_{i=1}^{j-1} /\!\!/^s_{t_i}(g) R^s_g(t_i) \bigwedge /\!\!/^s_{t_j}(g) \dot{A}^s_g(t_j) \bigwedge_{l=j+1}^{n} /\!\!/^s_{t_l}(g) R^s_g(t_l) \, /\!\!/^s_1(g) \, dt_1 \cdots dt_n ds \Big],$$

where

$$\dot{A}^s_g = \frac{d}{ds} A^s_g = \omega_g \in \Omega^1(M, \mathrm{Mat}(l \times l; \mathbb{C})),$$

and where $/\!\!/^s_\cdot(g)$ denotes the parallel transport with respect to the connection $d + s\omega_g$ on the trivial vector bundle over M.

Theorem 5.4. *Assume M is a compact Riemannian manifold, possibly with boundary, and let $g \in C^\infty(M, U(l \times l; \mathbb{C}))$. Then one has $\rho(\mathrm{Ch}^-(g))|_M = \mathrm{ch}^-(g)$, and if M has no boundary then $\mathrm{Bch}^-(g) = \rho(\mathrm{Ch}^-(g))$.*

Note that in view of Corollary 5.2, Theorem 5.4 provides a new proof of

$$(d + P)\mathrm{Bch}^{-}(g) = 0$$

We refer the reader to [16] for a variant of this result.

Proof of Theorem 5.4. The formula $\rho(\mathrm{Ch}^{-}(g))|_M = \mathrm{ch}^{-}(g)$ is a simple consequence of the definitions, once one has noticed the formula

$$\rho\left\langle (\alpha_0 + \vartheta_{\mathbb{T}} \wedge \beta_0) \otimes \cdots \otimes (\alpha_n + \vartheta_{\mathbb{T}} \wedge \beta_n)\right\rangle|_M = \alpha_0 \wedge \cdots \wedge \alpha_n.$$

In order to see $\mathrm{Bch}^{-}(g) = \rho(g)$, given $t, s \in I$ define

$$V^s(g,t) \in \widehat{\Omega}^{-}(LM, \mathrm{Mat}(l \times l; \mathbb{C}))$$

by

$$V^s_{2n+1}(g,t) = \int_{\{0 \le t_1 \le \ldots t_{n+1} \le t\}} \sum_{j=1}^{n+1} \bigwedge_{i=1}^{j-1} /\!/^s_{t_i}(g) R^s_g(t_i) \bigwedge /\!/^s_{t_j}(g) \dot{A}^s_g(t_j)$$
$$\times \bigwedge_{l=j+1}^{n+1} /\!/^s_{t_l}(g) R^s_g(t_l)\, /\!/^s_1\,(g) dt_1 \cdots dt_{n+1},$$

and the differential form

$$W^s(g,t) \in \widehat{\Omega}^{-}(LM, \mathrm{Mat}(l \times l; \mathbb{C}))$$

by

$$W^s_{2n+1}(g,t) = \sum_{k=n+1}^{\infty} \sum_{r, j_1, \ldots, j_n = 1,\ \text{pairwise distinct}}^{k}$$
$$\times \int_{\{0 \le t_1 \le \ldots t_k \le t\}} \iota A^s_g(t_1) \cdots R^s_g(t_{j_1}) \cdots \dot{A}^s_g(t_r) \cdots R^s_g(t_{j_n}) \cdots \iota A^s_g(t_k) dt_1 \cdots dt_k.$$

Then obviously one has

$$\mathrm{Bch}^{-}(g) = \mathrm{Tr}\left[\int_0^1 V^s(g,t)|_{t=1}\, ds\right]$$

and it is easily checked from the definitions that

$$\rho(\mathrm{Ch}^{-}(g)) = \mathrm{Tr}\left[\int_0^1 W^s(g,t)|_{t=1}\, ds\right].$$

Thus, it suffices to show that $W^s(g,t) = V^s(g,t)$ for all $t, s \in I$. To see this, the essential idea is to consider for every $t, s \in I$ the even form

$$X^s(g,t) = (X^s_0(g,t), X^s_2(g,t), \dots) \in \widehat{\Omega}^+(LM, \mathrm{Mat}(l \times l; \mathbb{C})),$$

which is defined by

$$\begin{aligned} X^s_0(g,t) &= /\!/^s_t(g), \\ \frac{d}{dt} X^s_{2n}(g,t) &= X^s_{2n}(g,t)\iota A^s_g(t) + X^s_{2n-2}(g,t) R^s_g(t), \\ X^s_{2n}(g,t)|_{t=0} &= 0 \quad \text{for all } n \geq 1, \end{aligned}$$

and the odd form

$$Y^s(g,t) = (Y^s_1(g,t), Y^s_3(g,t), \dots) \in \Omega^-(LM, \mathrm{Mat}(l \times l; \mathbb{C}))$$

which is defined by

$$\begin{aligned} \frac{d}{dt} Y^s_1(g,t) &= Y^s_1(g,t)\iota A^s_g(t) + X^s_0(g,t)\dot{A}^s_g(t), \\ \frac{d}{dt} Y^s_{2n+1}(g,t) &= Y^s_{2n+1}(g,t)\iota A^s_g(t) + Y^s_{2n-1}(g,t) R^s_g(t) + X^s_{2n}(g,t)\dot{A}^s_g(t) \\ &\qquad \text{for all } n \geq 1, \\ Y^s_{2n+1}(g,t)|_{t=0} &= 0 \quad \text{for all } n. \end{aligned}$$

Noting that the sum that defines $W^s_{2n+1}(g,t)$ converges uniformly in t so that one can interchange d/dt with $\sum_{k=n+1}^{\infty}$, it is now easily checked that both $t \mapsto W^s(g,t)$ and $t \mapsto V^s(g,t)$ solve the IVP's which define $Y^s(g,t)$, so that

$$V^s(g,t) = W^s(g,t) = Y^s(g,t) \quad \text{for all } t, s \in I,$$

as was claimed. □

Remark 5.5. If N is a compact manifold without boundary and given a connection $d + C$ over a trivial vector bundle over N, the *even Bismut–Chern character* is the differential form

$$\mathrm{Bch}^+(C) = (\mathrm{Bch}^+_0(C), \mathrm{Bch}^+_2(C), \dots) \in \widehat{\Omega}^+(LN)$$

defined by

$$\mathrm{Bch}^+_{2n}(C) = \mathrm{Tr}\Big[\int_{\{0 \leq t_1 \leq \dots t_n \leq 1\}} \bigwedge_{i=1}^{n} /\!/^C_{t_i} R_C(t_i) /\!/^C_1 \; dt_1 \cdots dt_n \Big],$$

where R_C is again the curvature of $d + C$ and $/\!\!/^C_\cdot$ is the parallel transport with respect to $d + C$. Then one has another even/odd periodicity as in Remark 5.3: we can consider A_g^s as defining a connection 1-form $\widetilde{A}_g$ over a trivial vector bundle over $M \times I$. However, since $M \times I$ is a manifold with boundary, it is convenient to embed it in a larger manifold, say

$$\chi : M \times I \hookrightarrow M \times J$$

where $J = (-1, 2)$. Therefore, we extend A_g^s to $s \in J$, consider it as defining a connection 1-form $\widetilde{A}_g$ over a trivial vector bundle over $M \times J$.

The corresponding curvature

$$R_{\widetilde{A}_g} \in \Omega^2(M \times J, \mathrm{Mat}(l \times l; \mathbb{C}))$$

is given by varying $s \in J$ in

$$R_g^s + ds \wedge \dot{A}_g^s \in \Omega^2(M, \mathrm{Mat}(l \times l; \mathbb{C})).$$

Since $\iota_{\partial J} R_{\widetilde{A}_g} = \dot{A}_g^s$, after restricting to loops fibering over J, we immediately get that under integration along the fibers of

$$\pi : M \times I \longrightarrow M,$$

one has

$$\mathrm{Bch}^-_{2n-1}(g) = \int_I \chi^* \iota_{\partial_J} \mathrm{Bch}^+_{2n}(\widetilde{A}_g) = \pi_* \chi^* \mathrm{Bch}^+_{2n}(\widetilde{A}_g).$$

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