J. Noncommut. Geom. 15 (2021), 761–795
DOI 10.4171/JNCG/420

G-homotopy invariance of the analytic signature of proper co-compact G-manifolds and equivariant Novikov conjecture

Yoshiyasu Fukumoto

Abstract. The main result of this paper is the G-homotopy invariance of the G-index of the signature operator of proper co-compact G-manifolds. If proper co-compact G-manifolds X and Y are G-homotopy equivalent, then we prove that the images of their signature operators by the G-index map are the same in the K-theory of the C^*-algebra of the group G. Neither discreteness of the locally compact group G nor freeness of the action of G on X are required, so this is a generalization of the classical case of closed manifolds. Using this result, we can deduce the equivariant version of Novikov conjecture for proper co-compact G-manifolds from the strong Novikov conjecture for G.

1. Introduction

Before discussing our case of proper action of a locally compact group G, let us review the classical case of closed manifolds. For even-dimensional oriented closed manifold M, the ordinary Fredholm index of the signature operator ∂_M is equal to the signature of the manifold M which is defined using the cup product of the ordinary cohomology of M. In particular, it follows that $\mathrm{ind}(\partial_M)$ is invariant under orientation-preserving homotopy. We have the following classical and important result.

Theorem 1.1 ([7–9]). *Let M and N be even-dimensional oriented closed manifolds. Assume that M and N are orientation-preserving homotopy equivalent to each other. The fundamental groups of M and N are identified with each other via the isomorphism $\pi_1(M) \xrightarrow{\simeq} \pi_1(N)$ induced by the homotopy equivalent map and let Γ denote this group. Let ∂_M and ∂_N be their signature operators. Then, $\mathrm{ind}_\Gamma(\partial_M) = \mathrm{ind}_\Gamma(\partial_N) \in K_0(\Gamma)$.*

Notice that we can deduce the Novikov conjecture from the strong Novikov conjecture by using this theorem. Moreover, we also have a more generalized result.

Theorem 1.2 ([13, Proposition 3.3 and Theorem 3.6]). *Let a finite group G act on M and N. Assume that M and N are orientation-preserving G-equivariantly homotopy equivalent. The fundamental groups of M and N are identified with each other via the iso-*

2020 Mathematics Subject Classification. 19L47, 19K56, 19K35, 46L80, 46L87, 58A12.
Keywords. Novikov conjecture, higher signatures, almost flat bundles, proper actions.

morphism $\pi_1(M) \xrightarrow{\simeq} \pi_1(N)$ *induced by the homotopy equivalent map and let* Γ *denote this group. Let* ind_Γ^G *be the* G*-equivariant* Γ*-index map with value in* $K_0^G(C^*_{\mathrm{red}}(\Gamma)) \simeq K_0(C^*_{\mathrm{red}}(G^\Gamma))$*, where* G^Γ *denotes the group extension* $\{1\} \to \Gamma \to G^\Gamma \to G \to \{1\}$*. Then,* $\mathrm{ind}_\Gamma^G(\partial_M) = \mathrm{ind}_\Gamma^G(\partial_N) \in K_0(C^*_{\mathrm{red}}(G^\Gamma))$.

Our main theorem is a generalization of them. Let us fix the settings. Let X and Y be oriented even-dimensional complete Riemannian manifolds and let G be a second countable locally compact Hausdorff group acting on X and Y isometrically, properly, and co-compactly.

Theorem A. *Let* X *and* Y *be oriented even-dimensional complete Riemannian manifolds and let* G *be a second countable locally compact Hausdorff group acting on* X *and* Y *isometrically, properly, and co-compactly. Let* ∂_X *and* ∂_Y *be the signature operators. Assume that we have a* G*-equivariant orientation-preserving homotopy equivalent map* $f\colon Y \to X$*. Then,* $\mathrm{ind}_G(\partial_X) = \mathrm{ind}_G(\partial_Y) \in K_0(C^*(G))$.

This claim is also stated in [1] without proofs and here we will give a proof for it to obtain Corollary B. The method we use in this paper is based on [7], so we will construct a map that sends $\mathrm{ind}_G(\partial_X)$ to $\mathrm{ind}_G(\partial_Y)$. Our group C^*-algebras can be either a maximal or a reduced one.

Theorem 1.1 is the case when X and Y are the universal covering of closed manifolds M and N. Thus, analogously to the case of closed manifolds, the equivariant version of the Novikov conjecture can be deduced from the strong Novikov conjecture for the acting group G. In particular, by using this theorem and the result discussed in [3], we obtain the following equivariant version of Novikov conjecture for low-dimensional cohomologies.

Corollary B. *Let* X*,* Y*, and* G *be as above and let* L *be a* G*-Hermitian line bundle over* X *which is induced from a* G*-line bundle over* $\mathcal{E}G$*, or more generally,* G*-Hermitian line bundle* L *over* X *satisfying* $c_1(L) = 0 \in H^2(X;\mathbb{R})$*. Suppose, in addition, that* G *is unimodular and* $H_1(X;\mathbb{R}) = H_1(Y;\mathbb{R}) = \{0\}$*. Then,*

$$\int_X c_X(x)\mathcal{L}(TX) \wedge \mathrm{ch}(L) = \int_Y c_Y(y)\mathcal{L}(TY) \wedge \mathrm{ch}(f^*L),$$

where c_X *denotes the cut-off function, that is,* c_X *is an* $\mathbb{R}_{\geq 0}$*-valued compactly supported function on* X *satisfying* $\int_G c(\gamma^{-1}x)\,\mathrm{d}\gamma = 1$ *for any* $x \in X$*. In the case of the closed manifold, that is, when* X *is obtained as the universal covering of a closed manifold* M*, and the acting group is the fundamental group, the above value is equal to the ordinary, so called, higher signature* $\langle \mathcal{L}(TX) \cup \mathrm{ch}(L), [M]\rangle$*. The same result in this case of closed manifolds was obtained in* [5, 12].

Moreover, in Section 5, we will prove the G-homotopy invariance of the analytic signature twisted by almost flat bundles as in [7, Section 4]. However, we will use a different method from [7] to deal with general G-invariant elliptic operators. To be specific, we will prove the following Theorem C to obtain Corollary D.

Theorem C. *Let X be a complete oriented Riemannian manifold and let G be a locally compact Hausdorff group acting on X isometrically, properly, and co-compactly. Moreover, we assume that X is simply connected. Let D be a G-invariant properly supported elliptic operator of order 0 on G-Hermitian vector bundle over X.*

Then, there exists $\varepsilon > 0$ satisfying the following: for any finitely generated projective Hilbert B-module G-bundle E over X equipped with a G-invariant Hermitian connection such that $\|R^E\| < \varepsilon$, we have

$$\mathrm{ind}_G\big([E]\,\widehat{\otimes}_{C_0(X)}\,[D]\big) = 0 \in K_0\big(C^*_{\mathrm{Max}}(G) \otimes_{\mathrm{Max}} B\big)$$

if $\mathrm{ind}_G([D]) = 0 \in K_0(C^*_{\mathrm{Max}}(G))$. If we only consider commutative C^*-algebras for B, then the same conclusion is also valid for $C^*_{\mathrm{red}}(G)$.

Corollary D. *Consider the same conditions as Theorem A on X, Y, and G and assume additionally that X and Y are simply connected.*

Then, there exists $\varepsilon > 0$ satisfying the following: for any finitely generated projective Hilbert B-module G-bundle E over X equipped with a G-invariant Hermitian connection such that $\|R^E\| < \varepsilon$, we have

$$\mathrm{ind}_G\big([E]\,\widehat{\otimes}\,[\partial_X]\big) = \mathrm{ind}_G\big([f^*E]\,\widehat{\otimes}\,[\partial_Y]\big) \in K_0\big(C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}\,B\big).$$

If we only consider commutative C^*-algebras for B, then the same conclusion is also valid for $C^*_{\mathrm{red}}(G)$.

2. Preliminaries on proper actions

Definition 2.1. Let G be a second countable locally compact Hausdorff group. Let X be a complete Riemannian manifold.

- X is called a G-Riemannian manifold if G acts on X isometrically.
- The action of G on X is said to be proper or X is called a proper G-space if the following continuous map is proper: $X \times G \to X \times X$, $(x, \gamma) \mapsto (x, \gamma x)$.
- The action of G on X is said to be co-compact or X is called G-compact space if the quotient space X/G is compact.

Definition 2.2. The action of G on X induces actions on TX and T^*X given by

$$\begin{aligned} \gamma\colon T_xX &\to T_{\gamma x}X \\ v &\mapsto \gamma(v) := \gamma_* v \end{aligned} \quad \text{and} \quad \begin{aligned} \gamma\colon T^*_xX &\to T^*_{\gamma x}X \\ \xi &\mapsto \gamma(\xi) := (\gamma^{-1})^*\xi. \end{aligned}$$

The action on $\mathfrak{X}(X)$ and $\Omega^*(X)$ is given by

$$\gamma[V] := \gamma_* V \quad \text{and} \quad \gamma[\omega] := (\gamma^{-1})^*\omega$$

for $\gamma \in G$, $V \in \mathfrak{X}(X)$, and $\omega \in \Omega^*(X)$. Obviously, $\gamma[\omega \wedge \eta] = \gamma[\omega] \wedge \gamma[\eta]$ and $\mathrm{d}(\gamma[\omega]) = \gamma[\mathrm{d}\omega]$.

Proposition 2.3 (Slice theorem). *Let G be a second countable locally compact Hausdorff group that acts properly and isometrically on X. Then, for any neighborhood O of any point $x \in X$ there exists a compact subgroup $K \subset G$ including the stabilizer at x, $K \supset G_x := \{\gamma \in G \mid \gamma x = x\}$ and there exists a K-slice $\{x\} \subset S \subset O$.* ∎

Here $S \subset X$ is called K-slice if the following are satisfied:

- S is K-invariant; $K(S) = S$,
- the tubular subset $G(S) \subset X$ is open,
- there exists a G-equivariant map $\psi\colon G(S) \to G/K$ satisfying $\psi^{-1}([e]) = S$, called a slice map.

Corollary 2.4. *We additionally assume that X is G-compact. Then, for any open covering $X = \bigcup_{x\in X} O_x$, there exists a subfamily of finitely many open subsets $\{O_{x_i}, \dots, O_{x_N}\}$ such that*

$$\bigcup_{\gamma\in G}\bigcup_{i=1}^{N} \gamma(O_{x_i}) = X.$$

In particular, X is of bounded geometry, namely, the injective radius is bounded below and the norm of Riemannian curvature is bounded. ∎

Lemma 2.5. *Let X and Y be manifolds on which G acts properly. Suppose that the action on Y is co-compact. Let $f\colon Y \to X$ be a G-equivariant continuous map. Then, f is a proper map.*

Proof. Since the action on Y is co-compact, there exists a compact subset $F \subset Y$ satisfying $G(F) = Y$. Fix a compact subset $C \subset X$ and assume that the closed set $f^{-1}C \subset Y$ is not compact. Then, there exists a sequence $\{y_j\} \subset f^{-1}C$ tending to the infinity, that is, any compact subset in Y contains only finitely many points of $\{y_j\}$. Since the action on Y is proper, there exists a sequence $\{\gamma_j\} \subset G$ tending to the infinity satisfying $y_j \in \gamma_j F$. Then, it follows that $f(y_j) \in f(\gamma_j F) = \gamma_j f(F)$. Due to the compactness of $f(F) \subset X$ and the properness of the action on X, the sequence $\{f(y_j)\} \subset X$ tends to the infinity. However, the compact subset C cannot contain such a sequence. So, $f^{-1}C$ is compact. ∎

3. Perturbation arguments

In this section, we will discuss on some technical methods introduced in [7, Sections 1 and 2]. For now, we will forget about the manifolds and group actions. Let A be a C^*-algebra, which may not be unital. Especially we will consider $A = C^*(G)$. Let $\mathcal{E}$ be a Hilbert A-module equipped with A-valued scalar product $\langle\cdot,\cdot\rangle$. Let us fix some notations:

- $\mathbb{L}(\mathcal{E}_1, \mathcal{E}_2)$ denotes a space consisting of adjointable A-linear operators, and we also use $\mathbb{L}(\mathcal{E}) := \mathbb{L}(\mathcal{E}, \mathcal{E})$;

- $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ denotes a subspace of $\mathbb{L}(\mathcal{E}_1, \mathcal{E}_2)$ consisting of compact A-linear operators, namely, the norm closure of the space of operators whose A-rank are finite. We also use $\mathbb{K}(\mathcal{E}) := \mathbb{K}(\mathcal{E}, \mathcal{E})$.

3.1. Quadratic forms and graded modules

Definition 3.1 (Regular quadratic forms). $Q: \mathcal{E} \times \mathcal{E} \to A$ is called a quadratic form on $\mathcal{E}$ if it satisfies

$$Q(\xi, \nu) = Q(\nu, \xi)^* \quad \text{and} \quad Q(\nu, \xi a) = Q(\nu, \xi)a \quad \text{for } \nu, \xi \in \mathcal{E},\ a \in A. \tag{3.1}$$

A quadratic form Q is said to be regular if there exists an invertible operator $B \in \mathbb{L}(\mathcal{E})$ satisfying that $Q(\xi, B\nu) = \langle \xi, \nu \rangle$.

For an operator $T \in \mathbb{L}(\mathcal{E})$, let T' denote the adjoint with respect to Q, that is, an operator satisfying that $Q(T\xi, \nu) = Q(\xi, T'\nu)$. Using B, it is written as $T' = BT^*B^{-1}$.

Definition 3.2 (Compatible scalar product). Another scalar product $\langle \cdot, \cdot \rangle_1: \mathcal{E} \times \mathcal{E} \to A$ is called compatible with $\langle \cdot, \cdot \rangle$ if there exists a linear bijection $P: \mathcal{E} \to \mathcal{E}$ satisfying that $\langle \nu, \xi \rangle_1 = \langle \nu, P\xi \rangle$.

Note that P is a positive operator with respect to both scalar products, and $\sqrt{P}: (\mathcal{E}, \langle \cdot, \cdot \rangle_1) \to (\mathcal{E}, \langle \cdot, \cdot \rangle)$ is a unitary isomorphism. In particular, neither the space $\mathbb{L}(\mathcal{E})$ nor $\mathbb{K}(\mathcal{E})$ depends on the choice of a compatible scalar product.

Lemma 3.3. *Let Q be a regular quadratic form on $\mathcal{E}$. Then, there exists a compatible scalar product $\langle \cdot, \cdot \rangle_Q$ with the initial scalar product of $\mathcal{E}$ and $U \in \mathbb{L}(\mathcal{E})$ satisfying that $Q(\xi, U\nu) = \langle \xi, \nu \rangle_Q$ and $U^2 = 1$. Moreover, they are unique.*

Proof. With respect to the initial scalar product $\langle \cdot, \cdot \rangle$, we have that

$$\langle \nu, B^{-1}\xi \rangle = Q(\nu, \xi) = Q(\xi, \nu)^* = \langle \xi, B^{-1}\nu \rangle^* = \langle B^{-1}\nu, \xi \rangle = \big\langle \nu, (B^{-1})^*\xi \big\rangle,$$

which implies that B^{-1} is an invertible self-adjoint operator. Thus, it has the polar decomposition $B^{-1} = UP$ in which B^{-1}, U, and P commute one another; here U is unitary and P is positive. To be specific, U and P are given by the continuous functional calculus. Let f and g be continuous functions given by $f(x) := \frac{x}{|x|}$ and $g(x) := |x|$ on the spectrum of B^{-1}, which is contained in $\mathbb{R} \setminus \{0\}$, and set $U := f(B^{-1})$ and $P := g(B^{-1})$. Note that

$$U = P^{-1}B^{-1} = P^{-1}(B^{-1})^* = P^{-1}PU^* = U^*,$$

so it follows that $U^2 = U^*U = 1$. Let us set $\langle \nu, \xi \rangle_Q := \langle \nu, P\xi \rangle$. Then,

$$Q(\nu, U\xi) = Q(\nu, U^{-1}\xi) = Q(\nu, BP\xi) = \langle \nu, P\xi \rangle = \langle \nu, \xi \rangle_Q.$$

If there is another such operator U_1 satisfying that $U_1^2 = 1$ and that $Q(\nu, U_1\xi)$ is another scalar product, then $U_1^{-1}U$ is a positive unitary operator, which implies that $U_1^{-1}U = 1$. Thus we obtained the uniqueness. ∎

Remark 3.4. A regular quadratic form Q on a Hilbert A-module $\mathcal{E}$ determines the renewed compatible scalar product $\langle \cdot, \cdot \rangle_Q$ associated to Q and the $(\mathbb{Z}/2\mathbb{Z})$-grading given by the ± 1-eigen spaces of U. Conversely, if a Hilbert A-module $\mathcal{E}$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-grading, then it determines a regular quadratic form Q given by $Q(\nu, \xi) = \langle \nu, (-1)^{\deg(\xi)} \xi \rangle$ for homogeneous elements.

Definition 3.5. Let A be a C^*-algebra. $\mathbb{J}(A)$ denotes the space consisting of unitary equivalent classes of triples $(\mathcal{E}, Q, \delta)$, where $\mathcal{E}$ is a Hilbert A-module, Q is a regular quadratic form on $\mathcal{E}$, and $\delta\colon \mathrm{dom}(\delta) \to \mathcal{E}$ is a densely defined closed operator satisfying the following conditions:

(1) $\delta' = -\delta$, namely, $Q(-\delta(\nu), \xi) = Q(\nu, \delta(\xi))$ for $\nu, \xi \in \mathrm{dom}(\delta)$;

(2) $\mathrm{Im}(\delta) \subset \mathrm{dom}(\delta)$ and $\delta^2 = 0$;

(3) there exists $\sigma, \tau \in \mathbb{K}(\mathcal{E})$ satisfying $\sigma\delta + \delta\tau - 1 \in \mathbb{K}(\mathcal{E})$.

The typical example, which we will use for dealing with the signature, is given by Definition 4.7. Roughly speaking, $\mathcal{E}$ is a completion of the space of compactly supported differential forms Ω_c^*, Q is given by the Hodge $*$-operation, and δ is the exterior derivative.

Remark 3.6. This definition is slightly different from $\mathbf{L}_{\mathrm{nb}}(A)$ in [7, Définition 1.5] and our $\mathbb{J}(A)$ is smaller. However, it is sufficient for our purpose.

Lemma 3.7. *If a closed operator δ satisfies the condition* (3)*, then both operators $(\delta + \delta^* \pm i)^{-1}$ can be defined and they belong to $\mathbb{K}(\mathcal{E})$. Here, δ^* denotes the adjoint of δ with respect to a certain scalar product on $\mathcal{E}$.*

Proof. Since δ is a closed operator, $\delta + \delta^*$ is self-adjoint. Thus $\mathrm{Im}(\delta + \delta^* \pm i)$ are equal to $\mathcal{E}$ and both operators $\delta + \delta^* \pm i$ are invertible. We now claim that both $(\delta + \delta^* \pm i)^{-1} \in \mathbb{L}(\mathcal{E})$ are compact operators. Since $\mathrm{Im}((\delta + \delta^* \pm i)^{-1}) = \mathrm{dom}(\delta + \delta^* \pm i) = \mathrm{dom}(\delta) \cap \mathrm{dom}(\delta^*)$ and δ and δ^* are closed operators, the operators

$$\alpha_{\pm} := \delta(\delta + \delta^* \pm i)^{-1} \quad \text{and} \quad \beta_{\pm} := \delta^*(\delta + \delta^* \pm i)^{-1}$$

are closed operator defined on entire $\mathcal{E}$, which implies that they are bounded; $\alpha, \beta \in \mathbb{L}(\mathcal{E})$.

On the other hand, note that $(\sigma\delta)^2 = (\sigma\delta)(1 - \delta\tau) = \sigma\delta$ and $(\delta\tau)^2 = \delta\tau$ modulo $\mathbb{K}(\mathcal{E})$. Let p be the orthogonal projection onto $\mathrm{Im}(\delta\tau)$ and let $q = 1 - p$. Then, we have that $p(\delta\tau) = \delta\tau$ and $(\delta\tau)p = p$ modulo $\mathbb{K}(\mathcal{E})$. Moreover,

$$\begin{aligned}
(\sigma\delta)q &= (1 - \delta\tau)(1 - p) = 1 - \delta\tau - p + (\delta\tau)p = 1 - \delta\tau = \sigma\delta, \\
q(\sigma\delta) &= (1 - p)(1 - \delta\tau) = 1 - p - \delta\tau + p(\delta\tau) = 1 - p = q, \\
1 - (\delta^*\sigma^*)q - (\delta\tau)p &= 1 - (q\sigma\delta)^* - p = 1 - q^* - p = 1 - q - p = 0 \text{ modulo } \mathbb{K}(\mathcal{E}).
\end{aligned}$$

Then, set $\ell := 1 - (\delta^*\sigma^*)q - (\delta\tau)p \in \mathbb{K}(\mathcal{E})$. Now we conclude that

$$\begin{aligned}
1 &= \ell + (\delta^*\sigma^* q - \delta\tau p), \\
(\delta + \delta^* \pm i)^{-1} &= (\delta + \delta^* \pm i)^{-1}\ell + (\alpha_{\mp}^*\sigma^* q - \beta_{\mp}^*\tau p) \in \mathbb{K}(\mathcal{E})
\end{aligned}$$

because ℓ, σ, and τ belong to $\mathbb{K}(\mathcal{E})$; and $\alpha_{\pm}$ and $\beta_{\pm}$ belong to $\mathbb{L}(\mathcal{E})$. ∎

Definition 3.8. For $(\mathcal{E}, Q, \delta) \in \mathbb{J}(A)$, we define the K-theory class $\Psi(\mathcal{E}, Q, \delta) \in K_0(A)$ as follows. As in Lemma 3.3, let $\mathcal{E}$ be equipped with the compatible scalar product $\langle\cdot,\cdot\rangle_Q$ and $(\mathbb{Z}/2\mathbb{Z})$-grading associated to Q. Next, put

$$F_\delta := (\delta + \delta^*)\big(1 + (\delta + \delta^*)^2\big)^{-\frac{1}{2}} \in \mathbb{L}(\mathcal{E}),$$

where δ^* is the adjoint of δ with respect to the scalar product $\langle\cdot,\cdot\rangle_Q$. Obviously F_δ is self-adjoint and F_δ is an odd operator since $U\delta U = \delta' = -\delta$. Moreover, it follows that

$$1 - F_\delta^2 = \big(1 + (\delta + \delta^*)^2\big)^{-1} \in \mathbb{K}(\mathcal{E})$$

by the previous lemma. Then, we define $\Psi(\mathcal{E}, Q, \delta) := (\mathcal{E}, F_\delta) \in KK(\mathbb{C}, A) \cong K_0(A)$. The action of $\mathbb{C}$ on $\mathcal{E}$ is the natural multiplication.

Lemma 3.9. *For* $(\mathcal{E}, Q, \delta) \in \mathbb{J}(A)$ *satisfying* $\mathrm{Im}(\delta) = \mathrm{Ker}(\delta)$, $\Psi(\mathcal{E}, Q, \delta) = 0 \in K_0(A)$.

Proof. First, remark that $\mathrm{Im}(\delta)$ and $\mathrm{Ker}(\delta^*)$ are orthogonal to each other, and hence, $\mathrm{Im}(\delta) \cap \mathrm{Ker}(\delta^*) = \{0\}$. Indeed, for $\delta(\eta) \in \mathrm{Im}(\delta)$ and $\nu \in \mathrm{Ker}(\delta^*)$, it follows that $\langle\delta(\eta), \nu\rangle = \langle\eta, \delta^*(\nu)\rangle = 0$. Now let $\xi \in \mathrm{Ker}(\delta + \delta^*)$. Then,

$$\begin{aligned} 0 &= \big\langle\xi, (\delta + \delta^*)^2(\xi)\big\rangle = \big\langle\xi, \delta^*\delta(\xi) + \delta\delta^*(\xi)\big\rangle \\ &= \big\langle\delta(\xi), \delta(\xi)\big\rangle + \big\langle\delta^*(\xi), \delta^*(\xi)\big\rangle, \end{aligned}$$

which implies that $\xi \in \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\delta^*) = \mathrm{Im}(\delta) \cap \mathrm{Ker}(\delta^*) = \{0\}$. Therefore, $\mathrm{Ker}(F_\delta) = \{0\}$. Since F_δ is a bounded self-adjoint operator, it is invertible. To conclude, $(\mathcal{E}, F_\delta) = 0 \in KK(\mathbb{C}, A)$. ∎

3.2. Perturbation arguments

Lemma 3.10 ([7, Lemme 2.1]). *Let* $(\mathcal{E}_X, Q_X, \delta_X), (\mathcal{E}_Y, Q_Y, \delta_Y) \in \mathbb{J}(A)$. *Suppose that we have*

(1) $T \in \mathbb{L}(\mathcal{E}_X, \mathcal{E}_Y)$ *satisfying* $T(\mathrm{dom}(\delta_X)) \subset \mathrm{dom}(\delta_Y)$, $T\delta_X = \delta_Y T$ *and* T *induces an isomorphism* $[T]\colon \mathrm{Ker}(\delta_X)/\mathrm{Im}(\delta_X) \to \mathrm{Ker}(\delta_Y)/\mathrm{Im}(\delta_Y)$*;*

(2) $\phi \in \mathbb{L}(\mathcal{E}_X)$ *satisfying* $\phi(\mathrm{dom}(\delta_X)) \subset \mathrm{dom}(\delta_X)$ *and* $1 - T'T = \delta_X\phi + \phi\delta_X$*;*

(3) $\varepsilon \in \mathbb{L}(\mathcal{E}_X)$ *satisfying* $\varepsilon^2 = 1$, $\varepsilon' = \varepsilon$, $\varepsilon\delta_X = -\delta_X\varepsilon$, *and* $\varepsilon(1 - T'T) = (1 - T'T)\varepsilon$.

Then, $\Psi(\mathcal{E}_X, Q_X, \delta_X) = \Psi(\mathcal{E}_Y, Q_Y, \delta_Y) \in K_0(A)$.

Proof. First, we may assume that $\phi' = -\phi$. Indeed, since $1 - T'T = (1 - T'T)' = (\delta_X\phi + \phi\delta_X)' = -(\delta_X\phi' + \phi'\delta_X)$, we may replace ϕ by $\frac{1}{2}(\phi - \phi')$ which satisfies the same assumption.

Set $\mathcal{E} := \mathcal{E}_X \oplus \mathcal{E}_Y$, $Q := Q_X \oplus (-Q_Y)$, and $\nabla := \left[\begin{smallmatrix} \delta_X & 0 \\ 0 & -\delta_Y \end{smallmatrix}\right]$. Note that the replacing of Q_Y by $-Q_Y$ means the reversing of the grading of $\mathcal{E}_Y$. Then, it is easy to see that $\Psi(\mathcal{E}, Q, \nabla) = \Psi(\mathcal{E}_X, Q_X, \delta_X) - \Psi(\mathcal{E}_Y, Q_Y, \delta_Y)$. Therefore, it is sufficient to verify that $\Psi(\mathcal{E}, Q, \nabla) = 0$.

Let us introduce invertible operators $R_t \in \mathbb{L}(\mathcal{E})$ and a quadratic form B_t on $\mathcal{E}$ given by

$$R_t := \begin{bmatrix} 1 & 0 \\ itT\varepsilon & 1 \end{bmatrix} \quad \text{and} \quad B_t(\nu, \xi) := Q(R_t\nu, R_t\xi) = Q(R_t' R_t \nu, \xi)$$

for $t \in [0, 1]$. We claim that $(\mathcal{E}, B_t, \nabla) \in \mathbb{J}(A)$.

It is easy to see that $\nabla R_t = R_t \nabla$, and hence, $B_t(\nu, \nabla\xi) = B_t(-\nabla\nu, \xi)$. Clearly the scalar products associated to B_t and Q are compatible with each other, also the conditions (2) and (3) in the definition of $\mathbb{J}(A)$ are satisfied. Therefore, $(\mathcal{E}, B_t, \nabla) \in \mathbb{J}(A)$ and $\Psi(\mathcal{E}, B_t, \nabla) = \Psi(\mathcal{E}, Q, \nabla)$.

Next let us introduce

$$L_t := \begin{bmatrix} 1 - T'T & (i\varepsilon + t\phi)T' \\ T(i\varepsilon + t\phi) & 1 \end{bmatrix} \quad \text{and} \quad C_t(\nu, \xi) := Q(L_t\nu, \xi).$$

Let T' denote the adjoint of T with respect to Q_X and Q_Y. Notice that since $Q = Q_X \oplus (-Q_Y)$, the adjoint of the matrix $\left[\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right]$ with respect to Q is equal to $\left[\begin{smallmatrix} 0 & -T' \\ 0 & 0 \end{smallmatrix}\right]$. Thus we have that $R_t' = \left[\begin{smallmatrix} 1 & it\varepsilon T' \\ 0 & 1 \end{smallmatrix}\right]$ and that

$$R_1' R_1 = \begin{bmatrix} \varepsilon(1 - T'T)\varepsilon & i\varepsilon T' \\ iT\varepsilon & 1 \end{bmatrix} = \begin{bmatrix} (1 - T'T)\varepsilon^2 & i\varepsilon T' \\ iT\varepsilon & 1 \end{bmatrix} = L_0.$$

In particular, $B_1 = C_0$. Since L_t is invertible at $t = 0$, there exists $t_0 > 0$ such that L_t is invertible for $t \in [0, t_0]$. Besides, it is clear that $L_t' = L_t$, so C_t is a regular quadratic form for $t \in [0, t_0]$.

Moreover, consider the operator $\nabla_t := \left[\begin{smallmatrix} \delta_X & tT' \\ 0 & -\delta_Y \end{smallmatrix}\right]$. We claim that $(\mathcal{E}, C_t, \nabla_t) \in \mathbb{J}(A)$, for $t \in [0, t_0]$. The adjoint of ∇_t with respect to the quadratic form C_t is equal to $L_t^{-1}\nabla_t' L_t$, so in order to check that it is equal to $-\nabla_t$, we should check that $L_t\nabla_t = -\nabla_t' L_t$. We have

$$L_t\nabla_t = \begin{bmatrix} (1 - T'T)\delta_X & t(1 - T'T)T' - (i\varepsilon + t\phi)T'\delta_Y \\ T(i\varepsilon + t\phi)\delta_X & tT(i\varepsilon + t\phi)T' - \delta_Y \end{bmatrix},$$

$$\nabla_t' L_t = \begin{bmatrix} -\delta_X(1 - T'T) & -\delta_X(i\varepsilon + t\phi)T' \\ -tT(1 - T'T) - \delta_Y T(i\varepsilon + t\phi) & -tT(i\varepsilon + t\phi)T' + \delta_Y \end{bmatrix}.$$

Obviously, the $(1, 1)$- and $(2, 2)$-entries are the negative of each other. Besides, we can see that

$$\begin{aligned} \big[(1,2)\text{-entry of } L_t\nabla_t\big] &= t(\delta_X\phi + \phi\delta_X)T' - (i\varepsilon + t\phi)\delta_X T' \\ &= t\delta_X\phi T' - i\varepsilon\delta_X T' = \delta_X(i\varepsilon + t\phi)T' \\ &= -\big[(1,2)\text{-entry of } \nabla_t' L_t\big]. \end{aligned}$$

Since $(L_t\nabla_t)' = \nabla_t' L_t$, it automatically follows that $[(2, 1)\text{-entry of } L_t\nabla_t] = -[(2, 1)\text{-entry of } \nabla_t' L_t]$ as well, and now we obtained that $L_t\nabla_t = -\nabla_t' L_t$. It is easy to see that

$(\nabla_t)^2 = 0$. If $\sigma_X, \tau_X \in \mathbb{K}(\mathcal{E}_X)$ and $\sigma_Y, \tau_Y \in \mathbb{K}(\mathcal{E}_Y)$ satisfy $\sigma_X\delta_X + \delta_X\tau_X - 1 \in \mathbb{K}(\mathcal{E}_X)$ and $\sigma_Y\delta_Y + \delta_Y\tau_Y - 1 \in \mathbb{K}(\mathcal{E}_Y)$, then it follows that $\left[\begin{smallmatrix}\sigma_X & 0\\ 0 & -\sigma_Y\end{smallmatrix}\right]\nabla_t + \nabla_t\left[\begin{smallmatrix}\tau_X & 0\\ 0 & -\tau_Y\end{smallmatrix}\right] - 1 \in \mathbb{K}(\mathcal{E})$, since $T \in \mathbb{L}(\mathcal{E}_X, \mathcal{E}_Y)$. Thus we obtained that $(\mathcal{E}, C_t, \nabla_t) \in \mathbb{J}(\mathcal{E})$ and $\Psi(\mathcal{E}, C_t, \nabla_t) = \Psi(\mathcal{E}, B_1, \nabla) = \Psi(\mathcal{E}, Q, \nabla)$.

Finally, check that $\mathrm{Ker}(\nabla_t) = \mathrm{Im}(\nabla_t)$ for any $t \in (0, t_0]$. $\mathrm{Ker}(\nabla_t) \supset \mathrm{Im}(\nabla_t)$ is implied by $(\nabla_t)^2 = 0$, so let $\left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right] \in \mathrm{Ker}(\nabla_t)$. Then, $\theta_2 \in \mathrm{Ker}(\delta_Y)$ and $tT'\theta_2 = -\delta_X\theta_1 \in \mathrm{Im}(\delta_X)$. Since T' induces an isomorphism $[T']\colon \mathrm{Ker}(\delta_Y)/\mathrm{Im}(\delta_Y) \to \mathrm{Ker}(\delta_X)/\mathrm{Im}(\delta_X)$, it follows from the injectivity that $\theta_2 \in \mathrm{Im}(\delta_Y)$. There exists $\eta \in \mathcal{E}_2$ such that $\delta_Y\eta = \theta_2$. On the other hand, $\theta_1 + tT'\eta \in \mathrm{Ker}(\delta_X)$ and the surjectivity of $[T']$ imply that there exists $\zeta \in \mathrm{Ker}(\delta_Y)$ such that $T'\zeta = \frac{1}{t}(\theta_1 + tT'\eta)$. Therefore, $\mathrm{Im}(\nabla_t) \ni \nabla_t\left[\begin{smallmatrix}0\\ \zeta-\eta\end{smallmatrix}\right] = \left[\begin{smallmatrix}tT'(\zeta-\eta)\\ -\delta_Y(\eta)\end{smallmatrix}\right] = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$, which concludes that $\mathrm{Ker}(\nabla_t) \subset \mathrm{Im}(\nabla_t)$.

Due to Lemma 3.9, it follows that $\Psi(\mathcal{E}, C_t, \nabla_t) = 0 \in KK(\mathbb{C}, A)$ and we conclude that $\Psi(\mathcal{E}_X, Q_X, \delta_X) - \Psi(\mathcal{E}_Y, Q_Y, \delta_Y) = \Psi(\mathcal{E}, Q, \nabla) = 0$. ■

4. G-signature

4.1. Description of the analytic G-index

Let G be a second countable locally compact Hausdorff group. Let X be a G-compact proper complete G-Riemannian manifold. And let $\mathbb{V}$ be a G-Hermitian vector bundle over X. In this section, we will define and investigate a $C^*(G)$-module denoted by $\mathcal{E}(\mathbb{V})$ obtained by completing $C_c(X;\mathbb{V})$. This will be used for the definition of the index of G-invariant elliptic operators, in particular, the signature operator.

Definition 4.1 ([11, Section 5]). First, we define, on $C_c(X;\mathbb{V})$, the structure of a pre-Hilbert module over $C_c(G)$ using the action of G on $C_c(X;\mathbb{V})$ given by $\gamma[s](x) = \gamma(s(\gamma^{-1}x))$ for $\gamma \in G$.

- The action of $C_c(G)$ on $C_c(X;\mathbb{V})$ from the right is given by

$$s \cdot b = \int_G \gamma[s] \cdot b(\gamma^{-1})\Delta(\gamma)^{-\frac{1}{2}}\mathrm{d}\gamma \in C_c(X;\mathbb{V}) \tag{4.1}$$

  for $s \in C_c(X;\mathbb{V})$ and $b \in C_c(G)$. Here, Δ denotes the modular function.
- The scalar product valued in $C_c(G)$ is given by

$$\langle s_1, s_2\rangle_{\mathcal{E}}(\gamma) = \Delta(\gamma)^{-\frac{1}{2}}\langle s_1, \gamma[s_2]\rangle_{L^2(\mathbb{V})} \tag{4.2}$$

  for $s_i \in C_c(\mathbb{V})$.

Define $\mathcal{E}(\mathbb{V})$ as the completion of $C_c(\mathbb{V})$ in the norm $\|\langle s, s\rangle\|_{C^*(G)}^{\frac{1}{2}}$.

Theorem 4.2 ([11, Theorem 5.8]). *Let G be a second countable locally compact Hausdorff group. Let X be a G-compact proper complete G-Riemannian manifold. Let*

$$D\colon C_c^\infty(X;\mathbb{V}) \to C_c^\infty(X;\mathbb{V})$$

be a formally self-adjoint G-invariant first-order elliptic operator on a G-Hermitian vector bundle $\mathbb{V}$. Then, both operators $D \pm i$ have dense range as operators on $\mathcal{E}(\mathbb{V})$ and

$(D \pm i)^{-1}$ *belong to* $\mathbb{K}(\mathcal{E}(\mathbb{V}))$. *The operator* $D(1+D^2)^{-1/2} \in \mathbb{L}(\mathcal{E}(\mathbb{V}))$ *is a Fredholm operator and determines an element* $\mathrm{ind}_G(D) \in K_0(C^*(G))$. ∎

In this paper, mainly we consider $\mathbb{V}$ as $\bigwedge^* T^*X$ equipped with the $\mathbb{Z}/2\mathbb{Z}$-grading given by the Hodge $*$-operation and D as a signature operator.

Definition 4.3. Let X and Y be proper and co-compact Riemannian G-manifolds and let $\mathbb{V}$ and $\mathbb{W}$ be G-Hermitian vector bundles over X and Y, respectively. Let $T\colon C_c^\infty(X;\mathbb{V}) \to C(Y;\mathbb{W})$ be a linear operator. The support of the distributional kernel of T is given by the closure of the complement of the following union of all subsets $K_X \times K_Y \subset X \times Y$:

$$\bigcup_{\substack{\langle Ts_1, s_2\rangle = 0 \text{ for any sections} \\ s_1 \in C_c(X;\mathbb{V}) \text{ and } s_2 \in C_c(Y;\mathbb{W}) \text{ satisfying} \\ \mathrm{supp}(s_1) \subset K_X,\ \mathrm{supp}(s_2) \subset K_Y}} K_X \times K_Y.$$

T is said to be properly supported if both

$$\mathrm{supp}(k_T) \cup (K_X \times Y) \quad \text{and} \quad \mathrm{supp}(k_T) \cup (X \times K_Y) \subset X \times Y$$

are compact for any compact subset $K_X \subset X$ and $K_Y \subset Y$.

T is said to be compactly supported if $\mathrm{supp}(k_T) \subset X \times Y$ is compact.

The following proposition is used for the construction of the bounded operators on $\mathcal{E}(\mathbb{V})$.

Proposition 4.4 ([11, Proposition 5.4]). *Let* G, X, Y, $\mathbb{V}$, *and* $\mathbb{W}$ *be as above. Let*

$$T\colon C_c(X;\mathbb{V}) \to C_c(Y;\mathbb{W})$$

be a properly supported G*-invariant operator which is* L^2*-bounded. Then,* T *defines an element of* $\mathbb{L}(\mathcal{E}(\mathbb{V}), \mathcal{E}(\mathbb{W}))$.

For the proof, we will use the following two lemmas.

Lemma 4.5. *Let* $P \in \mathbb{L}(L^2(X;\mathbb{V}), L^2(Y;\mathbb{W}))$ *be a compactly supported bounded operator. Then, the operator*

$$\widetilde{P} := \int_G \gamma[P]\,\mathrm{d}\gamma$$

is well defined as a bounded operator in $\mathbb{L}(L^2(X;\mathbb{V}), L^2(Y;\mathbb{W}))$ *and the inequation* $\|\widetilde{P}\|_{\mathrm{op}} \le C\|P\|_{\mathrm{op}}$ *holds, where* C *is a constant depending on its support.*

Proof. Assume that the support of the distributional kernel of P is contained in $K_X \times K_Y$ for some compact subsets $K_X \subset X$ and $K_Y \subset Y$. We will follow the proof of [2, Lemmas 1.4 and 1.5]. Fix an arbitrary smooth section with compact support $s \in C_c^\infty(X;\mathbb{V})$ and let us consider $F_s \in L^2(G; L^2(Y;\mathbb{W}))$ given by

$$F_s(\gamma) := \gamma[P]s.$$

Note that for any $\gamma \in G$ the support of the distributional kernel of $\gamma[P]$ is contained in $\gamma(K_X) \times \gamma(K_Y)$. This is because for any $s \in C_c^\infty(X; \mathbb{V})$, it follows that $\operatorname{supp}(\gamma[P]s) \subset \gamma(K_Y)$ and $\gamma[P]s = 0$ whenever $\operatorname{supp}(s) \cap \gamma(K_X) = \emptyset$. In particular, since the actions are proper, F_s has compact support in G. In addition, again since the actions are proper, $\gamma(K_Y) \cap \eta(K_Y) = \gamma(K_Y \cap \gamma^{-1}\eta(K_Y)) = \emptyset$ if $\gamma^{-1}\eta \in G$ is outside some compact neighborhood $Z \subset G$ in particular,

$$\left\| F_s(\gamma) \right\|_{L^2(Y;\mathbb{W})} \cdot \| F_s(\eta) \|_{L^2(Y;\mathbb{W})} = 0$$

for such γ and $\eta \in G$. Recall that Z is determined only by K_Y being so independent of s. Then,

$$\begin{aligned}
\left\| \int_G F_s(\gamma)\,\mathrm{d}\gamma \right\|^2_{L^2(Y;\mathbb{W})} &= \left\| \int_G F_s(\gamma)\,\mathrm{d}\gamma \right\|_{L^2(Y;\mathbb{W})} \left\| \int_G F_s(\eta)\,\mathrm{d}\eta \right\|_{L^2(Y;\mathbb{W})} \\
&\leq \int_G \int_G \| F_s(\gamma) \|_{L^2(Y;\mathbb{W})} \| F_s(\eta) \|_{L^2(Y;\mathbb{W})} \,\mathrm{d}\gamma\,\mathrm{d}\eta \\
&\leq \int_G \| F_s(\gamma) \|_{L^2(Y;\mathbb{W})} \left(\int_G \chi_Z(\gamma^{-1}\eta) \| F_s(\eta) \|_{L^2(Y;\mathbb{W})} \,\mathrm{d}\eta \right) \mathrm{d}\gamma \\
&\leq \| F_s \|_{L^2(G)} \| \chi_Z \|_{L^2(G)} \| F_s \|_{L^2(G)} \\
&\leq |Z| \| F_s \|^2_{L^2(G)},
\end{aligned}$$

where $\chi_Z : G \to [0,1]$ is the characteristic function of C, that is $\chi_Z(\gamma) = 1$ for $\gamma \in Z$ and $\chi_Z(\gamma) = 0$ for $\gamma \notin Z$.

Next, take a compactly supported smooth function $c_1 \in C_c^\infty(X; [0,1])$ such that $c_1 = 1$ on K_X. Noting that $P = Pc_1$, we obtain

$$\begin{aligned}
\| F_s \|^2_{L^2(G)} &= \int_G \left\| F_s(\gamma) \right\|^2_{L^2(Y;\mathbb{W})} \mathrm{d}\gamma = \int_G \| \gamma P c_1 \gamma^{-1} s \|^2_{L^2(Y;\mathbb{W})} \mathrm{d}\gamma \\
&\leq \int_G \| P \|^2_{\mathrm{op}} \| c_1 \gamma^{-1} s \|^2_{L^2(X;\mathbb{V})} \mathrm{d}\gamma \\
&\leq \| P \|^2_{\mathrm{op}} \int_G \int_X \left| c_1(x) \right|^2 \left\| \gamma^{-1} s(x) \right\|^2_{\mathbb{V}} \mathrm{d}x\,\mathrm{d}\gamma \\
&\leq \| P \|^2_{\mathrm{op}} \int_G \int_X \left| c_1(\gamma^{-1}x) \right|^2 \left\| s(x) \right\|^2_{\mathbb{V}} \mathrm{d}x\,\mathrm{d}\gamma \\
&\leq \| P \|^2_{\mathrm{op}} \sup_{x \in X} \left(\int_G \left| c_1(\gamma^{-1}x) \right|^2 \mathrm{d}\gamma \right) \| s \|^2_{L^2(X;\mathbb{V})}.
\end{aligned}$$

Since the action of G is proper, $\{\gamma \in G \mid \gamma^{-1}x \in \operatorname{supp}(c_1)\} \subset G$ is compact, so the value $\int_G |c_1(\gamma^{-1}x)|^2\,\mathrm{d}\gamma$ is always finite for any fixed $x \in X$. Besides, since X/G is compact, this value is uniformly bounded:

$$C := \sup_{x \in X} \left(\int_G \left| c_1(\gamma^{-1}x) \right|^2 \mathrm{d}\gamma \right) = \sup_{[x] \in X/G} \left(\int_G \left| c_1(\gamma^{-1}x) \right|^2 \mathrm{d}\gamma \right) < \infty.$$

Recall that C depends only on K_X, not on s. We conclude that

$$\left\| \int_G \gamma[P]s \, \mathrm{d}\gamma \right\|^2_{L^2(Y;\mathbb{W})} = \left\| \int_G F_s(\gamma) \, \mathrm{d}\gamma \right\|^2_{L^2(Y;\mathbb{W})} \leq |Z| \|F_s\|^2_{L^2(G)} \leq |Z| C \cdot \|P\|^2_{\mathrm{op}} \|s\|^2_{L^2(X;\mathbb{V})}. \qquad \blacksquare$$

Lemma 4.6 ([11, Lemma 5.3]). *Let P be a bounded positive operator on $L^2(X;\mathbb{V})$ with a compactly supported distributional kernel. Then, the scalar product*

$$(s_1, s_2) \mapsto \left\langle s_1, \left(\int_G \gamma[P] \, \mathrm{d}\gamma \right) s_2 \right\rangle_{\mathcal{E}(\mathbb{V})} \in C^*(G)$$

is well defined and positive for any $s_1 = s_2 \in C_c(X;\mathbb{V})$.

Proof. Note that

$$\langle \gamma[s], P(\gamma[s]) \rangle_{L^2(X;\mathbb{V})} = \langle \sqrt{P}(\gamma[s]), \sqrt{P}(\gamma[s]) \rangle_{L^2(X;\mathbb{V})}$$

for $\gamma \in G$ and $s \in C_c(X;\mathbb{V})$. Regarding each side of the above equation as a function in $\gamma \in G$, it is clear that the left-hand side vanishes outside some compact subset in G depending on the support of s and P. This implies that $\sqrt{P}(\gamma[s])$ has a compact support in G. Take any unitary representation space $\mathcal{H}$ of G and $h \in \mathcal{H}$. By the above observation of the compact support,

$$v := \int_G \Delta(\gamma)^{-\frac{1}{2}} \sqrt{P}(\gamma[s]) \otimes \gamma[h] \, \mathrm{d}\gamma \in L^2(X;\mathbb{V}) \otimes \mathcal{H}$$

is well defined. Then, we obtain that

$$\begin{aligned}
0 &\leq \|v\|^2 \\
&= \int_G \int_G \Delta(\gamma)^{-\frac{1}{2}} \Delta(\eta)^{-\frac{1}{2}} \langle \sqrt{P}(\gamma[s]), \sqrt{P}(\eta[s]) \rangle_{L^2(X;\mathbb{V})} \langle \gamma[h], \eta[h] \rangle_{\mathcal{H}} \, \mathrm{d}\gamma \, \mathrm{d}\eta \\
&= \int_G \int_G \Delta(\gamma)^{-\frac{1}{2}} \Delta(\eta)^{-\frac{1}{2}} \langle s, \gamma^{-1}[P(\eta[s])] \rangle_{L^2(X;\mathbb{V})} \langle h, \gamma^{-1}\eta[h] \rangle_{\mathcal{H}} \, \mathrm{d}\gamma \, \mathrm{d}\eta \\
&= \int_G \int_G \Delta(\gamma)^{-1} \Delta(\gamma^{-1}\eta)^{-\frac{1}{2}} \langle s, \gamma^{-1}[P](\gamma^{-1}\eta[s]) \rangle_{L^2(X;\mathbb{V})} \langle h, \gamma^{-1}\eta[h] \rangle_{\mathcal{H}} \, \mathrm{d}\gamma \, \mathrm{d}(\gamma^{-1}\eta) \\
&= \int_G \int_G \Delta(\zeta)^{-\frac{1}{2}} \langle s, \gamma^{-1}[P](\zeta[s]) \rangle_{L^2(X;\mathbb{V})} \langle h, \zeta[h] \rangle_{\mathcal{H}} \, \mathrm{d}(\gamma^{-1}) \, \mathrm{d}\zeta \\
&= \int_G \Delta(\zeta)^{-\frac{1}{2}} \left\langle s, \left(\int_G \gamma[P] \, \mathrm{d}\gamma \right) (\zeta[s]) \right\rangle_{L^2(X;\mathbb{V})} \langle h, \zeta[h] \rangle_{\mathcal{H}} \, \mathrm{d}\zeta \\
&= \int_G \left\langle s, \left(\int_G \gamma[P] \, \mathrm{d}\gamma \right) (s) \right\rangle_{\mathcal{E}(\mathbb{V})} (\zeta) \cdot \langle h, \zeta[h] \rangle_{\mathcal{H}} \, \mathrm{d}\zeta.
\end{aligned}$$

Recall that the action of $f := \langle s, (\int_G \gamma[P] \, \mathrm{d}\gamma)(s) \rangle_{\mathcal{E}(\mathbb{V})} \in C_c(G)$ on $\mathcal{H}$ is given by $f[h] = \int_G f(\zeta)\zeta[h] \, \mathrm{d}\zeta$ for $h \in \mathcal{H}$. Thus, by rewriting the above inequality, we have $\langle h, f[h] \rangle_{\mathcal{H}} \geq 0$ for any h, which means that this f is a positive operator on any unitary representation space $\mathcal{H}$. To conclude, f is positive in $C^*(G)$ for any $s \in C_c(\mathcal{E}(\mathbb{V}))$. ∎

Proof of Proposition 4.4. Let $T_1 := \frac{1}{2}(cT^*T + T^*Tc)$, which is a bounded self-adjoint operator $L^2(X;\mathbb{V}) \to L^2(X;\mathbb{V})$. Moreover, the distributional kernel of T_1 is contained in $K \times K$ for some compact subset $K \subset X$. By Lemma 4.5, $\int_G \gamma[T_1]$ is well defined in $\mathbb{L}(L^2(X;\mathbb{V}))$ and

$$\int_G \gamma[T_1] = \int_G \frac{1}{2}\big(\gamma[c]T^*T + T^*T\gamma[c]\big) = T^*T.$$

Consider a compactly supported continuous function $f \in C_c(X;[0,1])$ satisfying that $c_1 = 1$ on K so that $c_1 T_1 c_1 = T_1$ holds. Consider the following self-adjoint operator:

$$P := c_1\big(\|T\|^2\|c\| - T_1\big)c_1 = c_1^2\|T\|^2\|c\| - T_1 \in \mathbb{L}\big(L^2(X;\mathbb{V})\big).$$

Obviously P is compactly supported and since $T_1 \leq \|T_1\| \leq \|T\|^2\|c\|$, P is positive. Using Lemma 4.6, for any $s \in C_c(\mathbb{V})$, the following value is positive:

$$\begin{aligned} 0 &\leq \left\langle s, \left(\int_G \gamma[P]\,\mathrm{d}\gamma\right)s\right\rangle_{\mathcal{E}(\mathbb{V})} \\ &\leq C\|T\|^2\|c\|\langle s,s\rangle_{\mathcal{E}(\mathbb{V})} - \left\langle s, \left(\int_G \gamma[T_1]\right)s\right\rangle_{\mathcal{E}(\mathbb{V})} \in C^*(G), \end{aligned}$$

where C is the maximum of a G-invariant bounded function $\int_G \gamma[c_1^2]$, which is independent of s. To conclude,

$$\begin{aligned} \big\langle T(s), T(s)\big\rangle_{\mathcal{E}(\mathbb{W})} = \big\langle s, T^*T(s)\big\rangle_{\mathcal{E}(\mathbb{V})} &= \left\langle s, \left(\int_G \gamma[T_1]\right)s\right\rangle_{\mathcal{E}(\mathbb{V})} \\ &\leq C\|T\|^2\|c\|\langle s,s\rangle_{\mathcal{E}(\mathbb{V})}. \end{aligned}$$ ∎

4.2. Proof of Theorem A

The theorem we will discuss is the following.

Theorem A. *Let X and Y be oriented even-dimensional complete Riemannian manifolds and let a locally compact Hausdorff group G act on X and Y isometrically, properly, and co-compactly. Let $f: Y \to X$ be a G-equivariant orientation-preserving homotopy equivalent map. Let ∂_X and ∂_Y be the signature operators. Then,* $\mathrm{ind}_G(\partial_X) = \mathrm{ind}_G(\partial_Y) \in K_0(C^*(G))$.

From now on we will slightly change the notation for simplicity. We will only consider $\mathbb{V}$ for the cotangent bundle $\bigwedge^* T^*X \otimes \mathbb{C}$. Let us use $\mathcal{E}_X$ for $\mathcal{E}(\bigwedge^* T^*X \otimes \mathbb{C})$. Let $\Omega_c^*(X)$ be the space consisting of compactly supported smooth differential forms on X, namely, $C_c^\infty(X;\mathbb{V})$. We will prove Theorem A using Lemma 3.10.

Definition 4.7. Let us introduce the following data $(\mathcal{E}, Q, \delta)$ to present the G-index of the signature operator.

- Let $C^*(G)$-valued quadratic form Q_X be defined by the formula

$$Q_X(\nu,\xi)(\gamma) := i^{k(n-k)}\Delta(\gamma)^{-\frac{1}{2}}\int_X \bar{\nu}\wedge\gamma[\xi]$$
$$\text{for } \nu\in\Omega_c^k(X),\ \nu\in\Omega_c^{n-k}(X),\ \gamma\in G; \tag{4.3}$$

  here $\bar{\nu}$ denotes the complex conjugate. If $\deg(\nu)+\deg(\xi)\neq\dim(X)$, then $Q_X(\nu,\xi):=0$. This "deg" means the degree of the differential form.
- The grading U_X determined by Q_X is given by

$$U_X(\xi) = i^{-k(n-k)} * \xi \quad \text{for } \xi\in\Omega_c^k(X), \tag{4.4}$$

  where $*$ denotes the Hodge $*$-operation.
  Clearly, $U_X^2 = 1$ and $Q_X(\nu, U_X(\xi)) = \langle\nu,\xi\rangle_{\mathcal{E}_X}$ hold.
- $\delta_X(\xi) := i^k \mathrm{d}_X\xi$ for $\xi\in\Omega_c^k(X)$, here d_X denotes the exterior derivative on X.

We will also use similar notations for Y.

Lemma 4.8. $(\mathcal{E}_X, Q_X, \delta_X)\in\mathbb{J}(C^*(G))$ *and* $\Psi(\mathcal{E}_X, Q_X, \delta_X) = \mathrm{ind}_G(\partial_X)$*, where* ∂_X *is the signature operator of* X*.*

Proof. First, obviously $\delta^2 = 0$. Applying Theorem 4.2 to the signature operator on X, it follows that $\delta_X - U_X\delta_X U_X\colon\Omega_c^*(X)\to\mathcal{E}_X$ is closable and its closure is self-adjoint. Let us use $\delta_X - U_X\delta_X U_X$ for also its closure. Since $\mathrm{Im}(\delta_X)$ and $\mathrm{Im}(-U_X\delta_X U_X)$ are orthogonal to each other with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{E}_X}$, it follows that δ_X itself is a closed operator on $\mathcal{E}$. Moreover, set

$$\sigma = \tau := \frac{\delta_X^*}{1+(\delta_X^*+\delta_X)^2}.$$

They belong to $\mathbb{K}(\mathcal{E}_X)$ since $\frac{\delta_X^*}{\delta_X^*+\delta_X\pm i}\in\mathbb{L}(\mathcal{E}_X)$ and $\frac{1}{\delta_X^*+\delta_X\pm i}\in\mathbb{K}(\mathcal{E}_X)$. Then, from Theorem 4.2, we obtain

$$\sigma\delta_X + \delta_X\tau - 1 = \frac{-1}{1+(\delta_X^*+\delta_X)^2}\in\mathbb{K}(\mathcal{E}_X).$$

Therefore, $(\mathcal{E}_X, Q_X, \delta_X)\in\mathbb{J}(C^*(G))$ and $\Psi(\mathcal{E}_X, Q_X, \delta_X) = \mathrm{ind}_G(\partial_X)$ by the definition of Ψ. ∎

Let $f\colon Y\to X$ be a G-equivariant proper orientation-preserving homotopy equivalent map between n-dimensional proper co-compact Riemannian G-manifolds. In order to construct a map $T\in\mathbb{L}(\mathcal{E}_X,\mathcal{E}_Y)$ satisfying the hypothesis of Lemma 3.10, it is sufficient to construct an L^2-bounded G-invariant operator $T\colon\Omega_c^*(X)\to\Omega_c^*(Y)$ due to Proposition 4.4.

Remark 4.9. Note that $f^*\colon\Omega_c^*(X)\to\Omega_c^*(Y)$ may not be L^2-bounded unless $f\colon Y\to X$ is a submersion. For instance, let $Y = X = [-1,1]$ and $f(y) = y^3$. Consider an L^2-form

ω on X given by $\omega(x) = \frac{1}{|x|^{1/4}}$. Actually $\|\omega\|^2_{L^2(X)} = \int_{-1}^{1} \frac{1}{|x|^{1/2}}\mathrm{d}x = 2$, however,

$$\|f^*\omega\|^2_{L^2(Y)} = \int_{-1}^{1} \frac{1}{|y|^{3/2}}\mathrm{d}y = +\infty.$$

So we need to replace f^* by a suitable operator.

Let us construct operator T that we need and investigate its properties in a slightly more general condition.

- X and Y are Riemannian manifolds and G acts on them isometrically and properly. For a while, X and Y may have a boundary and the action may not be co-compact if not mentioned.
- Let W be an oriented G-invariant fiber bundle over Y whose typical fiber is an even-dimensional unit open disk $B^k \subset \mathbb{R}^k$. Let $q\colon W \twoheadrightarrow Y$ denote the canonical projection map and $q_I\colon \Omega_c^{*+k}(W) \to \Omega_c^*(Y)$ be the integration along the fiber.
- Let us fix $\omega \in \Omega^k(W)$ to be a G-invariant closed k-form with fiber-wisely compact support such that the integral along the fiber is always equal to 1; $q_I(\omega)(y) = \int_{W_y} \omega = 1$ for any $y \in Y$. Let e_ω denote the operator given by $e_\omega(\zeta) = \zeta \wedge \omega$ for $\zeta \in \Omega^*(W)$. We can construct a G-invariant ω as follows. Let $\tau \in \Omega^k(W)$ be a k-form inducing a Thom class of W. We may assume that $\int_{W_y} \tau = 1$ for any $y \in Y$. Then, $\omega := \int_G \gamma[c\tau]\mathrm{d}\gamma$ is a desired G-invariant form.
- Suppose that we have a G-equivariant submersion $p\colon W \to X$ whose restriction on $\mathrm{supp}(\omega) \subset W$ is proper.

Definition 4.10. For the above data, let us set $T_{p,\omega} := q_I e_\omega p^*\colon \Omega_c^*(X) \to \Omega_c^*(Y)$. We may write just T_p for simplicity.

$$\begin{array}{ccc} W & & \\ {\scriptstyle q}\downarrow & \searrow^{p} & \\ Y & & X \end{array} \qquad \Omega_c^*(X) \xrightarrow[p^*]{} \Omega^*(W) \xrightarrow[e_\omega]{} \Omega_c^{*+k}(W) \xrightarrow[q_I]{} \Omega_c^*(Y).$$

Lemma 4.11. *If the actions of G are co-compact, then $T_{p,\omega}$ determines an operator in $\mathbb{L}(\mathcal{E}_X, \mathcal{E}_Y)$.*

Proof. By Proposition 4.4, it is sufficient to check that $T_{p,\omega}$ is L^2-bounded.

Since q_I is obviously L^2-bounded, only the boundedness of $e_\omega p^*\colon \Omega_c^*(X) \to \Omega_c^*(W)$ is non-trivial. Note that our proper submersion p restricted on $\mathrm{supp}(\omega) \subset W$ is a locally trivial G-invariant fibration. Let p_I denote the integration along this fibration. Then,

$$\int_W \zeta = \int_X p_I \zeta$$

holds for any compactly supported differential form $\zeta \in \Omega_c^*(W)$ satisfying $\mathrm{supp}(\zeta) \subset \mathrm{supp}(\omega)$, in particular, $\zeta = |(p^*\xi) \wedge \omega|^2 \mathrm{vol}_W \in \Omega_c^{n+k}(W)$ for $\xi \in \Omega_c^*(X)$. Let C_ω be the maximum of the norm of bounded G-invariant form $p_I(|\omega|^2 \mathrm{vol}_W) \in \Omega^n(X)$. We have

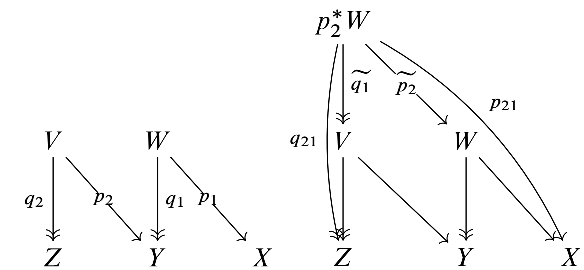
$$\begin{aligned}\left\|e_\omega p^*(\xi)\right\|_{L^2(W)}^2 &= \int_W \left|(p^*\xi) \wedge \omega\right|^2 \mathrm{vol}_W \overset{(\dagger)}{=} \int_X |\xi|^2 p_I\left(|\omega|^2 \mathrm{vol}_W\right) \\ &\leq C_\omega \int_X |\xi|^2 \mathrm{vol}_X = C_\omega \|\xi\|_{L^2(X)}^2 \quad \text{for } \xi \in \Omega_c^*(X).\end{aligned}$$

The equation (†) holds because the function $p^*|\xi|^2$ is constant along the fiber $p^{-1}(x)$. ∎

Lemma 4.12. *Let us consider proper co-compact G-manifolds X, Y, and Z and let $q_1\colon W \twoheadrightarrow Y$ and $q_2\colon V \twoheadrightarrow Z$ be G-invariant oriented disk bundles over Y and Z with typical fiber B^{k_1} and B^{k_2}. Fix G-invariant closed forms $\omega_1 \in \Omega^{k_1}(W)$ and $\omega_2 \in \Omega^{k_2}(V)$ with fiber-wisely compact support satisfying $(q_j)_I(\omega_j) = 1$. Let $p_1\colon W \to X$ $p_2\colon V \to Y$ be G-equivariant submersions whose restriction on* $\mathrm{supp}(\omega_j)$ *is proper.*

On the other hand, as in the diagram below, let us consider the pull-back bundle $p_2^*W = \{(v, w) \in V \times W \mid p_2(v) = q_1(w)\}$ over V and let us regard it as a fiber bundle over Z with projection denoted by q_{21}. Let us set $\omega_{21} := \widetilde{p_2}^*\omega_1 \wedge \widetilde{q_1}^*\omega_2 \in \Omega^*(p_2^*W)$, $p_{21} := p_1\widetilde{p_2}$, where $\widetilde{q_1}\colon p_2^*W \to V$ denotes the projection and $\widetilde{p_2}\colon p_2^*W \to W$ denotes the map induced by p_2.

Then, $T_{p_2}T_{p_1} = T_{p_{21}}\colon \mathcal{E}_X \to \mathcal{E}_Z$.



$T_{p_2}T_{p_1}\colon \mathcal{E}_X \to \mathcal{E}_Y \to \mathcal{E}_Z,$ $\qquad T_{p_{21}}\colon \mathcal{E}_X \to \mathcal{E}_Z.$

Proof. First we can see that for $\xi \in \Omega_c^*(X)$,

$$\begin{aligned}T_{p_{21}}(\xi) &= (q_{21})_I \circ e_{\omega_{21}} p_{21}^*(\xi) \\ &= (q_2)_I(\widetilde{q_1})_I\left\{\widetilde{p_2}^* p_1^*\xi \wedge (\widetilde{p_2}^*\omega_1 \wedge \widetilde{q_1}^*\omega_2)\right\} \\ &= (q_2)_I(\widetilde{q_1})_I\left\{\widetilde{p_2}^*(p_1^*\xi \wedge \omega_1) \wedge \widetilde{q_1}^*\omega_2\right\} \\ &= (q_2)_I\left\{(\widetilde{q_1})_I\left(\widetilde{p_2}^*(p_1^*\xi \wedge \omega_1)\right) \wedge \omega_2\right\} \\ &= (q_2)_I e_{\omega_2}(\widetilde{q_1})_I \widetilde{p_2}^* e_{\omega_1} p_1^*(\xi), \\ T_{p_2}T_{p_1}(\xi) &= (q_2)_I e_{\omega_2} p_2^* \circ (q_1)_I e_{\omega_1} p_1^*(\xi).\end{aligned}$$

Note that $(\widetilde{q_1})_I$ in the second bottom row is well defined because the differential form $\widetilde{p_2}^* e_{\omega_1} p_1^*(\xi)$ is compactly supported along each fiber of $\widetilde{q_1}\colon p^*W \twoheadrightarrow V$. We need to prove the commutativity of the following diagram:

$$\begin{array}{ccc} \Omega^*(p_2^* W) & & \\ \big\downarrow (\widetilde{q_1})_I & \nwarrow \widetilde{p_2}^* & \\ \Omega^*(V) & & \Omega_c^*(W) \\ & \nwarrow p_2^* & \big\downarrow (q_1)_I \\ & & \Omega^*(Y) \end{array} \tag{4.5}$$

It is easy to check this using local trivializations. Suppose that $W \twoheadrightarrow Y$ is trivialized on $U \subset Y$. Then, $p_2^* W$ is trivialized on $p_2^{-1} U \subset V$. We write these trivializations as $W|_U \simeq U \times B^k$ and $p_2^* W|_U \simeq p_2^{-1} U \times B^k$. Then, for $\zeta(y, w) = f(y, w)\,\mathrm{d}y \wedge \mathrm{d}w \in \Omega_c^*(W|_U)$,

$$\begin{aligned} \big((\widetilde{q_1})_I \widetilde{p_2}^* \zeta\big)(v) &= \int_{B^k} \big(f\big(p_2(v), w\big) p_2^*(\mathrm{d}y)\big)\mathrm{d}w \\ &= \big(p_2^*(q_1)_I \zeta\big)(v) \quad \text{for } v \in p^{-1} U \subset V. \end{aligned}$$

■

We will use the following proposition repeatedly.

Proposition 4.13. *Let W_1 and W_2 be oriented G-invariant disk bundles over Y with typical fiber B^{k_1} and B^{k_2}, and let $q_j\colon W_j \twoheadrightarrow Y$ be the projection. Let $\omega_j \in \Omega^{k_j}$ be closed forms with fiber-wisely compact support satisfying $(q_j)_I(\omega)\omega_j = 1$.*

Suppose that there exist G-equivariant submersions $p_j\colon W_j \to X$ whose restrictions on the 0-sections $p_j(\cdot, 0)\colon Y \to X$ are G-equivariant homotopic to each other.

Then, there exists a properly supported G-equivariant L^2-bounded operator ψ: $\Omega_c^*(X) \to \Omega_c^*(Y)$ satisfying that $T_{p_2,\omega_2} - T_{p_1,\omega_1} = \mathrm{d}_X \psi + \psi\, \mathrm{d}_Y$.

First, let us prove the following lemma.

Lemma 4.14. *Let $Q\colon \widetilde{W} \twoheadrightarrow Y \times [0,3]$ be a G-invariant disk bundle over $Y \times [0,3]$ and let $\omega \in \Omega^k(\widetilde{W})$ be a closed form with fiber-wisely compact support satisfying $Q_I(\omega) = 1$. Suppose that there exists a G-equivariant submersion $P\colon \widetilde{W} \to X$ whose restriction on* supp(ω) *is proper. Then, there exists a properly supported G-equivariant L^2-bounded operator $\psi\colon \Omega_c^*(X) \to \Omega_c^*(Y)$ satisfying that $T_{P(\cdot,3),\omega(\cdot,3)} - T_{P(\cdot,0),\omega(\cdot,0)} = \mathrm{d}_X \psi + \psi \mathrm{d}_Y$.*

Proof. Let $\xi \in \Omega_c^*(X)$ and $\theta := Q_I(P^*\xi \wedge \omega) \in \Omega_c^*(Y \times [0,3])$. Then, it is easy to see that

$$\int_{[0,3]} \mathrm{d}\theta = -\mathrm{d}\left(\int_{[0,3]} \theta\right) + (i_3^*\theta - i_0^*\theta),$$

where $i_t\colon Y \times \{t\} \hookrightarrow Y \times [0,3]$ denotes the inclusion map. Note that $i_t^*\theta = T_{P(\cdot,t),\omega(\cdot,t)}\xi$.

Now, set $\psi\colon \Omega_c^*(X) \to \Omega_c^*(Y)$ by the formula; $\psi(\xi) := \int_{[0,3]} Q_I(P^*\xi \wedge \omega)$ for $\xi \in \Omega_c^*(X)$. Note that the identity map $L^1([0,3]) \to L^2([0,3])$ is a continuous inclusion due to the finiteness of vol$([0,3])$; hence, the map $\int_{[0,3]}\colon \Omega_c^*(Y \times [0,3]) \to \Omega_c^*(Y)$ is L^2-bounded. Moreover, since $P^*\xi \wedge \omega$ vanishes at the boundary of each fiber of $\widetilde{W}$, the integration along the fiber commutes with taking exterior derivative, in particular,

$$\mathrm{d}\theta = \mathrm{d}Q_I(P^*\xi \wedge \omega) = Q_I \mathrm{d}(P^*\xi \wedge \omega) = Q_I\big(P^*(\mathrm{d}\xi) \wedge \omega\big).$$

To conclude, we obtain

$$\begin{aligned}\psi(\mathrm{d}\xi) &= \int_{[0,3]} \mathrm{d}Q_I(P^*\xi \wedge \omega) \\ &= -\mathrm{d}\psi(\xi) + T_{P(\cdot,3),\omega(\cdot,3)}\xi - T_{P(\cdot,0),\omega(\cdot,0)}\xi,\end{aligned}$$

■

Proof of Proposition 4.13. We need to construct $\widetilde{W}$ and P as above satisfying

$$T_{P(\cdot,0),\omega(\cdot,0)} = T_{p_1,\omega_1} \quad \text{and} \quad T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2,\omega_2}.$$

Let $h\colon Y \times [0,3] \to X$ be a re-parametrized G-homotopy between $p_1(\cdot,0)$ and $p_2(\cdot,0)$, that is, h is a G-equivariant smooth map satisfying

$$h(y,t) = p_1(y,0) \text{ for } t \in [0,1] \quad \text{and} \quad h(y,t) = p_2(y,0) \text{ for } t \in [2,3].$$

Here G acts on $[0,3]$ trivially. Moreover, consider the following fiber product $W_1 \times_Y W_2 = \{(y_1,w_1),(y_2,w_2) \in W_1 \times W_2 \mid y_0 = y_1\}$. Let us introduce a smooth map $\chi\colon [0,3] \to [0,1]$ satisfying that

$$\chi(t) = 0 \text{ for } t \in \left[0,\tfrac{1}{10}\right) \cup \left(\tfrac{29}{10},3\right] \quad \text{and} \quad \chi(t) = 1 \text{ for } t \in \left(\tfrac{9}{10},\tfrac{21}{10}\right).$$

Then,

$$\begin{aligned}\widetilde{h}\colon (W_1 \times_Y W_2) \times [0,3] &\to X \\ \big((y,t),w_1,w_2\big) &\mapsto \begin{cases} p_1\big(y,\big(1-\chi(t)\big)w_1\big) & \text{for } t \in [0,1], \\ h(y,t) & \text{for } t \in [1,2], \\ p_2\big(y,\big(1-\chi(t)\big)w_2\big) & \text{for } t \in [2,3]. \end{cases}\end{aligned}$$

This $\widetilde{h}$ is submersion as long as $\chi(t) \neq 1$ due to the submergence of p_1 and p_2. Let $BX := \{v \in TX \mid \|v\| < 1\}$ be the unit disk tangent bundle; consider the pull-back bundle $\widetilde{W} := \widetilde{h}^*BX$; let us regard it as a bundle over $Y \times [0,3]$ and set

$$\begin{aligned} P\colon \widetilde{W} &\to X \\ \big((y,t),w_1,w_2,v\big) &\mapsto \exp_{\widetilde{h}((y,t),w_1,w_2)}\big(\chi(t)v\big).\end{aligned}$$

Due to the $(\chi(t)v)$-component, P is a submersion also when $\chi(t) \neq 0$ not only when $\chi(t) \neq 1$.

Moreover, define $\omega \in \Omega^*(W)$ as $\omega := \pi_1^*\omega_1 \wedge \pi_2^*\omega_2 \wedge \widetilde{h}^*\omega_{BX}$, where $\pi_j\colon \widetilde{W} \twoheadrightarrow W_j$ for $j = 1,2$ and $\omega_{BX} \in \Omega^*(BX)$ is a G-invariant differential with fiber-wisely compact support satisfying $\int_{BX_x} \omega_{BX} = 1$. These $\widetilde{W}$, P, and ω satisfy the assumption of Lemma 4.14.

It is easy to see that $T_{P(\cdot,0),\omega(\cdot,0)} = T_{p_1,\omega_1}$ and $T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2,\omega_2}$ as follows. For the simplicity, let $\pi\colon \widetilde{W}_{Y\times\{0\}} \twoheadrightarrow W_1$ denote the projection. Note that $P(y,0) = p_1\pi$ and we can write $\omega(\cdot,0) = \pi^*\omega_1 \wedge \widetilde{\omega}$, using some $\widetilde{\omega} \in \Omega^*(\widetilde{W}_{Y\times\{0\}})$ satisfying $\pi_I\widetilde{\omega} = 1$. Then, we obtain that

$$\begin{aligned} T_{P(\cdot,0),\omega(\cdot,0)}(\xi) &= (q_1)_I\pi_I(\pi^* p_1^*\xi \wedge \pi^*\omega_1 \wedge \widetilde{\omega}) \\ &= (q_1)_I\pi_I\big(\pi^*(p_1^*\xi \wedge \omega_1) \wedge \widetilde{\omega}\big) \\ &= (q_1)_I\big((p_1^*\xi \wedge \omega_1) \wedge \pi_I\widetilde{\omega}\big) \\ &= (q_1)_I(p_1^*\xi \wedge \omega_1) = T_{p_1,\omega_1}(\xi), \end{aligned}$$

and similarly, $T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2,\omega_2}$. ■

Now let us define a map $T \in \mathbb{L}(\mathcal{E}_X, \mathcal{E}_Y)$ which satisfies the assumption of Lemma 3.10. First, remark that our map $f\colon Y \to X$ is a proper map by Lemma 2.5.

Definition 4.15. Let $BX := \{v \in TX \mid \|v\| < 1\}$ be the unit disk tangent bundle and let $W := f^*BX$ be the pull-back on Y, that is, $W = \{(y,v) \in Y \times BX \mid v \in BX|_{f(y)}\}$. Let $\widetilde{f}\colon W \to BX$ be a map given by $\widetilde{f}(x,v) := (f(x),v)$. Since the action of G on X is isometric and f is G-equivariant, G acts on BX and also on W. Consider a G-equivariant submersion given by the formula

$$\begin{aligned} p\colon W &\to X \\ (y,v) &\mapsto \exp_{f(y)}(v). \end{aligned} \tag{4.6}$$

Let us fix a G-invariant $\mathbb{R}$-valued closed n-form $\omega_0 \in \Omega^n(BX)$ with fiber-wisely compact support whose integral along the fiber is always equal to 1, and let $\omega := \widetilde{f}^*\omega_0 \in \Omega^n(W)$. For these W, p, and ω, let us set $T := T_{p,\omega}$.

Lemma 4.16. *The adjoint with respect to quadratic forms Q_X and Q_Y is given by $T' = p_I e_\omega q^*$.*

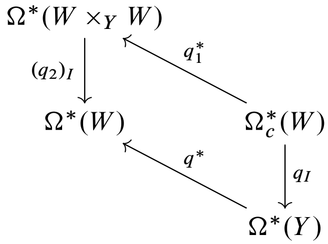
Proof. Note that $\deg(\omega) = \dim(X)$ is even; hence, ω commutes with other differential forms. For $\nu \in \Omega_c^k(Y)$ and $\xi \in \Omega_c^{n-k}(X)$,

$$\begin{aligned} \int_X p_I e_\omega q^*(\nu) \wedge \xi &= \int_X p_I(q^*\nu \wedge \omega) \wedge \xi = \int_X p_I(q^*\nu \wedge \omega \wedge p^*\xi) \\ &= \int_{BX} q^*\nu \wedge \omega \wedge p^*\xi = \int_Y q_I(q^*\nu \wedge p^*\xi \wedge \omega) \\ &= \int_Y \nu \wedge q_I(p^*\xi \wedge \omega) = \int_Y \nu \wedge T(\xi). \end{aligned}$$

Since $Q_X(\nu,\xi)(\gamma) := i^{k(n-k)}\Delta(\gamma)^{-\frac{1}{2}} \int_X \bar{\nu} \wedge \gamma[\xi]$, the proof ends by replacing ν and ξ by $\bar{\nu}$ and $\gamma[\xi]$, respectively, and using the G-invariance of T. ■

Proposition 4.17. *There exists $\phi \in \mathbb{L}(\mathcal{E}_X)$ such that $1 - T'T = \mathrm{d}_X\phi + \phi\mathrm{d}_X$.*

Proof. Consider the fiber product $W \times_Y W$ and let q_1 and $q_2 \colon W \times_Y W \to W$ denote the projections given by $q_j(y, v_1, v_2) := (y, v_j)$. Take $\zeta \in \Omega_c^*(W)$, here W is regarded as the first component of $W \times_Y W$. Using the commutativity of the diagram (4.5),



we obtain that

$$\begin{aligned} e_\omega q^* q_I(\zeta) &= e_\omega (q_2)_I q_1^*(\zeta) = (q_2)_I(q_1^*\zeta) \wedge \omega = (q_2)_I(q_1^*\zeta \wedge q_2^*\omega) \\ &= (q_2)_I e_{q_2^*\omega} q_1^*(\zeta), \end{aligned}$$

and hence,

$$T'T = p_I e_\omega q^* q_I e_\omega p^* = p_I (q_2)_I e_{q_2^*\omega} q_1^* e_\omega p^*.$$

On the other hand, since $q_1(y, 0) = q_2(y, 0)$, by Proposition 4.13, there exists a properly supported G-equivariant L^2-bounded operator $\psi_W \colon \Omega_c^*(W) \to \Omega_c^*(W)$ satisfying

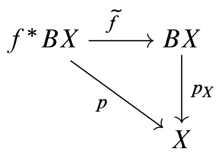
$$(q_2)_I e_{q_2^*\omega} q_2^* - (q_2)_I e_{q_2^*\omega} q_1^* = \mathrm{d}\psi_W + \psi_W \mathrm{d}.$$

Moreover, it is obvious that $(q_2)_I e_{q_2^*\omega} q_2^* = \mathrm{id}_{\Omega_c(W)}$, so we obtain

$$\begin{aligned} p_I e_\omega p^* - T'T &= p_I(\mathrm{id}_{\Omega_c(W)} - (q_2)_I e_{q_2^*\omega} q_1^*) e_\omega p^* \\ &= p_I(\mathrm{d}\psi_W + \psi_W \mathrm{d}) e_\omega p^* \\ &= \mathrm{d} \circ p_I \psi_W e_\omega p^* + p_I \psi_W e_\omega p^* \circ \mathrm{d}. \end{aligned} \tag{4.7}$$

Remark that $p_I \circ \mathrm{d} = \mathrm{d} \circ p_I$ because the act on differential forms with compact support, and $e_\omega \circ \mathrm{d} = \mathrm{d} \circ e_\omega$ because ω is a closed form.

Next let us consider submersion $p_X \colon BX \to X$ given by $(x, v) \mapsto \exp_x(v)$. Note that $p = p_X \widetilde{f}$.



Now we want to check that $p_I e_\omega p^* = (p_X)_I e_{\omega_0} p_X^*$. For any $\nu \in \Omega_c^*(X)$ and $\zeta \in \Omega_c^*(BX)$,

$$\begin{aligned} \int_X \nu \wedge p_I(\widetilde{f}^*\zeta) &= \int_W p^*\nu \wedge \widetilde{f}^*\zeta = \int_W \widetilde{f}^*(p_X^*\nu \wedge \zeta) \\ &= \deg(\widetilde{f}) \int_{BX} p_X^*\nu \wedge \zeta = \int_{BX} p_X^*\nu \wedge \zeta \\ &= \int_X \nu \wedge (p_X)_I(\zeta), \end{aligned}$$

since f is orientation-preserving proper homotopy equivalent. In particular, we obtain

$$p_I(\widetilde{f}^*\zeta) = (p_X)_I(\zeta).$$

Put $\zeta := p_X^*\xi \wedge \omega_0$ for $\xi \in \Omega_c^*(X)$ to obtain

$$\begin{aligned} p_I e_\omega p^*(\xi) &= p_I\big(\widetilde{f}^* p_X^*\xi \wedge \widetilde{f}^*\omega_0\big) = p_I\big(\widetilde{f}^*(p_X^*\xi \wedge \omega_0)\big) \\ &= (p_X)_I(p_X^*\xi \wedge \omega_0) = (p_X)_I e_{\omega_0} p_X^*(\xi). \end{aligned} \tag{4.8}$$

Let $\pi\colon BX \to X$ be the natural projection. Since $p_X(x,0) = \pi(x,0)$, by Proposition 4.13, there exists a properly supported G-equivariant L^2-bounded operator $\psi_X\colon \Omega_c^*(X) \to \Omega_c^*(X)$ satisfying

$$\pi_I e_{\omega_0}\pi^* - (p_X)_I e_{\omega_0} p_X^* = \mathrm{d}\psi_X + \psi_X\mathrm{d}. \tag{4.9}$$

On the other hand, it is obvious that $\pi_I e_{\omega_0}\pi^* = \mathrm{id}_{\Omega_c(X)}$. Therefore, combining (4.7), (4.8), and (4.9), we conclude that

$$\mathrm{id}_{\Omega_c(X)} - T'T = \mathrm{d}\phi + \phi\mathrm{d},$$

where $\phi = p_I\psi_W e_\omega p^* + \psi_X$. Since ϕ is properly supported G-invariant L^2-bounded operator, it defines an element in $\mathbb{L}(\mathcal{E}_X)$. ∎

Proof of Theorem A. First, let us check that T satisfies the assumption (1) of Lemma 3.10. Since ω is a closed form and has fiber-wisely compact support, it follows that $T\delta_X = \delta_Y T$. Let $g\colon X \to Y$ be the G-equivariant homotopy inverse of f and consider a map $S \in \mathbb{L}(\mathcal{E}_Y, \mathcal{E}_X)$ constructed in the same method as T from g instead of f in Definition 4.15. By 4.12, the composition ST is equal to the map $T_p \in \mathbb{L}(\mathcal{E}_X)$ for p satisfying that $p(\cdot,0)$ is G-equivariant homotopic to id_X. Then, by Proposition 4.13, there exists $\phi_X \in \mathbb{L}(\mathcal{E}_X)$ satisfying that $ST - (\delta_X\phi_X + \phi_X\delta_X) = T_{\mathrm{id}_X} = \mathrm{id}_{\mathcal{E}_X}$. Thus, ST induces the identity map on $\mathrm{Ker}(\delta_X)/\mathrm{Im}(\delta_X)$. Similarly TS induces the identity map on $\mathrm{Ker}(\delta_Y)/\mathrm{Im}(\delta_Y)$, and hence, T induces an isomorphism $\mathrm{Ker}(\delta_X)/\mathrm{Im}(\delta_X) \to \mathrm{Ker}(\delta_Y)/\mathrm{Im}(\delta_Y)$.

The assumption (2) of Lemma 3.10 is obtained from Proposition 4.17.

Finally, let $\varepsilon(\xi) := (-1)^k\xi$ for $\xi \in \Omega_c^k(X)$. Clearly, ε determines an operator $\varepsilon \in \mathbb{L}(\mathcal{E}_X)$, $\varepsilon^2 = 1$ and satisfies $\varepsilon' = \varepsilon$, $\varepsilon(\mathrm{dom}(\delta_X)) \subset \mathrm{dom}(\delta_X)$ and $\varepsilon\delta_X = -\delta_X\varepsilon$. Moreover, since neither T nor T' changes the order of the differential forms, ε commutes with $1 - T'T$. Thus ε satisfies the assumption (3) of Lemma 3.10. To conclude, we obtain $\mathrm{ind}_G(\partial_X) = \Psi(\mathcal{E}_X, Q_X, \delta_X) = \Psi(\mathcal{E}_Y, Q_Y, \delta_Y) = \mathrm{ind}_G(\partial_Y)$. ∎

4.3. Proof of Corollary B

To prove Corollary B, we will combine [3, Theorem A] with Theorem A. Suppose, in addition, that G is unimodular and $H_1(X;\mathbb{R}) = H_1(Y;\mathbb{R}) = \{0\}$. Let $f\colon Y \to X$ be a G-equivariant orientation-preserving homotopy invariant map and consider a G-manifold $Z := X \sqcup (-Y)$, the disjoint union of X and the orientation reversed Y. Let ∂_Z be the signature operator, then we have that $\mathrm{ind}_G(\partial_Z) = \mathrm{ind}_G(\partial_X) - \mathrm{ind}_G(\partial_Y) = 0 \in K_0(C^*(G))$.

Although the G-manifold should be connected in [3, Theorem A], however in this case, we can apply it to Z after replacing some arguments in [3] as follows.

When constructing a $U(1)$-valued cocycle $\alpha \in Z(G; U(1))$ from the given line bundle, we just use a line bundle L over X ignoring f^* over Y (see [3, Subsections 6.1 and 6.2]). When constructing a family of line bundles $\{L_t\}$ on which the central extension group G_{α^t} acts, just construct a family of line bundles $\{L_t\}$ over X in the same way and pull back on Y to obtain a family $\{f^* L_t\}$. To be specific, $f^* L_t$ is a trivial bundle $Y \times \mathbb{C}$, equipped with the connection given by $\nabla^t = \mathrm{d} + itf^*\eta$, and the action of G_{α^t} is given by

$$(\gamma, u)(y, z) = \big(\gamma y, \exp[-itf^*\psi_\gamma(x)]uz\big)$$

for $(\gamma, u) \in G_{\alpha^t}$, $y \in Y$, $z \in \mathbb{C} = (L_t)_x$. Then, consider a family of G_{α^t}-line bundles $\{L_t \sqcup f^* L_t\}$ over Z. We also need the similar replacement in [3, Definition 7.19] to obtain the global section on $L_t \sqcup f^* L_t$. Then, the remaining parts proceed similarly.

5. Index of Dirac operators twisted by almost flat bundles

Now we will discuss the Dirac operators twisted by a family of Hilbert module bundles $\{E^k\}$ whose curvature tends to zero and prove Theorem C. Such a family is called a family of almost flat bundles. In this section, it is convenient to formulate the index map using KK-theory.

5.1. G-index map in KK-theory

Lemma 5.1 ([10, Theorem 3.11]). *Let G be a second countable locally compact Hausdorff group. For any G-algebras A and B there exists a natural homomorphism*

$$j^G \colon KK^G(A, B) \to KK\big(C^*(G; A), C^*(G; B)\big).$$

Furthermore, if $x \in KK^G(A, B)$ and $y \in KK^G(B, D)$, then

$$j^G(x \mathbin{\widehat{\otimes}_B} y) = j^G(x) \mathbin{\widehat{\otimes}_{C^*(G;B)}} j^G(y). \qquad \blacksquare$$

Lemma 5.2. *Using a cut-off function $c \in C_c(X)$, one can define an idempotent $p \in C_c(G; C_0(X))$ by the formula*

$$\check{c}(\gamma)(x) = \sqrt{c(x)c(\gamma^{-1}x)\Delta(\gamma)^{-1}}.$$

In particular, it defines an element of K-theory denoted by $[c] \in K_0(C^*(G; C_0(X)))$. Moreover, the element of K-theory $[c] \in K_0(C^*(G; C_0(X)))$ does not depend on the choice of cut-off functions. $\blacksquare$

Definition 5.3 (G-index [11, Theorem 5.6]). Define

$$\mu_G \colon KK^G\big(C_0(X), \mathbb{C}\big) \to K_0\big(C^*(G)\big)$$

as the composition of

- $j^G \colon KK^G(C_0(X), \mathbb{C}) \to KK(C^*(G; C_0(X)), C^*(G))$ and

- $[c]\,\widehat{\otimes}\colon KK(C^*(G;C_0(X)),C^*(G)) \to KK(\mathbb{C},C^*(G)) \simeq K_0(C^*(G))$, i.e.,

$$\mu_G(\text{-}) := [c]\,\widehat{\otimes}_{C^*(G;C_0(X))}\, j^G(\text{-}) \in K_0\big(C^*(G)\big).$$

Remark 5.4. As in [11, Remark 4.4] or [3, Subsection 5.2], for calculating the index $\mathrm{ind}_G[D]$, it is sufficient to consider the case when the operator D is a Dirac type operator.

Let B be a unital C^*-algebra. Following Definition 5.3, we define the index maps with coefficients.

Definition 5.5. For unital C^*-algebras B, define the index map

$$\mathrm{ind}_G\colon KK^G\big(C_0(X),B\big) \to K_0\big(C^*(G;B)\big)$$

as the composition of

- $j^G\colon KK^G(C_0(X),B) \to KK(C^*(G;C_0(X)),C^*(G;B))$ and
- $[c]\,\widehat{\otimes}\colon KK(C^*(G;C_0(X)),C^*(G;B)) \to K_0(C^*(G;B))$, i.e.,

$$\mathrm{ind}_G(\text{-}) := [c]\,\widehat{\otimes}_{C^*(G;C_0(X))}\, j^G(\text{-}) \in K_0\big(C^*(G;B)\big).$$

The crossed product $C^*(G;B)$ is either maximal or reduced one. In this paper, we assume that G acts on B trivially. Then, $C^*_{\mathrm{Max}}(G;B)$ and $C^*_{\mathrm{red}}(G;B)$ will be naturally identified with $C^*_{\mathrm{Max}}(G)\otimes_{\mathrm{Max}} B$ and $C^*_{\mathrm{red}}(G)\otimes_{\min} B$, respectively. Moreover, if B is nuclear, $\otimes_{\mathrm{Max}} B$ and $\otimes_{\min} B$ are identified.

Definition 5.6. Let E be a finitely generated projective $(\mathbb{Z}/2\mathbb{Z})$-graded Hilbert B-module G-bundle. Define $C_0(X;E)$ as a space consisting of sections $s\colon X\to E$ vanishing at infinity. It is considered as a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert $C_0(X;B)$-module with the right action given by point-wise multiplications and the scalar product given by

$$\langle s_1,s_2\rangle(x) := \langle s_1(x),s_2(x)\rangle_{E_x} \in C_0(X;B).$$

Remark 5.7. The C^*-algebra $C_0(X;B)$ consisting of B-valued function vanishing at infinity is naturally identified with $C_0(X)\,\widehat{\otimes}\,B$ by [14, Theorem 6.4.17]. Similarly, if $E = X\times E_0$ is a trivial Hilbert B-module bundle over X, then $C_0(X;E)$ is naturally identified with $C_0(X)\,\widehat{\otimes}\,E_0$ as Hilbert $(C_0(X;B)\cong C_0(X)\,\widehat{\otimes}\,B)$-modules.

Definition 5.8. E defines an element in KK-theory

$$[E] = \big(C_0(X;E),0\big) \in KK^G\big(C_0(X),C_0(X)\,\widehat{\otimes}\,B\big).$$

The action of $C_0(X)$ on $C_0(X;E)$ is the point-wise multiplication.

Definition 5.9. Let E be a finitely generated Hilbert B-module bundle over X equipped with a Hermitian connection ∇^E. Let $R^E \in C^\infty(X;\mathrm{End}(E)\otimes \bigwedge^2(T^*(X)))$ denote its curvature. Then, define its norm as follows: first, define the point-wise norm as the operator norm given by

$$\|R^E\|_x := \sup\big\{\big\|R^E(u\wedge v)\big\|_{\mathbb{L}(\mathrm{E})} \mid u,v\in T_xX,\ \|u\wedge v\|=1\big\} \quad \text{for } x\in X.$$

Then, define the global norm as the supremum in $x \in X$ of the point-wise norm; $\|R^E\| := \sup_{x \in X} \|R^E\|_x$.

Theorem C. *Let X be a complete oriented Riemannian manifold and let G be a locally compact Hausdorff group acting on X isometrically, properly, and co-compactly. Moreover, assume that X is simply connected. Let D be a G-invariant properly supported elliptic operator of order 0 on G-Hermitian vector bundle over X.*

Then, there exists $\varepsilon > 0$ satisfying the following: for any finitely generated projective Hilbert B-module G-bundle E over X equipped with a G-invariant Hermitian connection such that $\|R^E\| < \varepsilon$, we have

$$\mathrm{ind}_G\big([E] \widehat{\otimes}_{C_0(X)} [D]\big) = 0 \in K_0\big(C^*_{\mathrm{Max}}(G) \otimes_{\mathrm{Max}} B\big)$$

if $\mathrm{ind}_G([D]) = 0 \in K_0(C^*_{\mathrm{Max}}(G))$. If we only consider commutative C^*-algebras for B, then the same conclusion is also valid for $C^*_{\mathrm{red}}(G)$.

5.2. Infinite product of C^*-algebras

Definition 5.10. Let B_k be a sequence of C^*-algebras.

- Define $\prod_{k \in \mathbb{N}} B_k$ as the C^*-algebra consisting of norm-bounded sequences

$$\prod_{k \in \mathbb{N}} B_k := \big\{\{b_1, b_2, \dots\} \mid b_k \in B_k,\ \sup_k \big\{\|b_k\|_{B_k}\big\} < \infty\big\}.$$

  The norm of B_k is given by $\|\{b_1, b_2, \dots\}\|_{\prod B_k} := \sup_k\{\|b_k\|_{B_k}\}$.

- Let $\bigoplus_{k \in \mathbb{N}} B_k$ be a closed two-sided ideal in $\prod_{k \in \mathbb{N}} B_k$ consisting of sequences vanishing at infinity

$$\bigoplus_{k \in \mathbb{N}} B_k := \big\{\{b_1, b_2, \dots\} \mid b_k \in B_k,\ \lim_{k \to \infty} \|b_k\| = 0\big\}.$$

  In other words, $\bigoplus_{k \in \mathbb{N}} B_k$ is a closure of the subspace in $\prod_{k \in \mathbb{N}} B_k$ consisting of sequences $\{b_1, b_2, \dots, 0, 0, \dots\}$ whose entries are zero except for finitely many of them.

- Define $\mathcal{Q}_{k \in \mathbb{N}} B_k$ as the quotient algebra given by

$$\mathcal{Q}_{k \in \mathbb{N}} B_k := \big(\prod B_k\big) / \big(\bigoplus B_k\big).$$

  The norm of $\mathcal{Q}\, B_k$ is given by $\|\{b_1, b_2, \dots\}\|_{\mathcal{Q}\, B_k} := \limsup_{k \to \infty} \|b_k\|_{B_k}$.

- If $\mathcal{E}_k$ are Hilbert B_k-modules, one can similarly define $\prod \mathcal{E}_k$ as a Hilbert $\prod B_k$-module consisting of bounded sequences

$$\prod_{k \in \mathbb{N}} \mathcal{E}_k := \big\{\{s_1, s_2, \dots\} \mid s_k \in \mathcal{E}_k,\ \sup_k \big\{\|s_k\|_{\mathcal{E}_k}\big\} < \infty\big\}.$$

The action of $\prod B_k$ and $\prod B_k$-valued scalar product are defined as follows:

$$\{s_k\} \cdot \{b_k\} := \{s_k \cdot b_k\} \in \prod \mathcal{E}_k,$$
$$\big\langle \{s_k^1\}, \{s_k^2\} \big\rangle_{\prod \mathcal{E}_k} := \big\{ \langle s_k^1, s_k^2 \rangle_{\mathcal{E}_k} \big\} \in \prod B_k,$$

for $\{s_k\}, \{s_k^1\}, \{s_k^2\} \in \prod \mathcal{E}_k$, $\{b_k\} \in \prod B_k$.
One can define similarly

$$\bigoplus_{k \in \mathbb{N}} \mathcal{E}_k := \Big\{ \{s_1, s_2, \dots\} \mid s_k \in \mathcal{E}_k,\ \lim_{k \to \infty} \|s_k\|_{\mathcal{E}_k} = 0 \Big\}$$

as a Hilbert $\prod B_k$-module, and define

$$\mathop{\mathcal{Q}}_{k \in \mathbb{N}} \mathcal{E}_k := \big(\prod \mathcal{E}_k \big) \widehat{\otimes}_\pi (\mathcal{Q}\, B_k) = \big(\prod \mathcal{E}_k \big) / \big(\bigoplus \mathcal{E}_k \big)$$

as a Hilbert $\mathcal{Q}\, B_k$-module, where $\pi\colon \prod B_k \to \mathcal{Q}\, B_k$ denotes the projection.

Example 5.11. If all of B_k are $\mathbb{C}$, then $\prod \mathbb{C} = \ell^\infty(\mathbb{N})$ and $\bigoplus \mathbb{C} = C_0(\mathbb{N})$.

Following [4, Section 3], we will construct "infinite product bundle $\prod E_k$" over X which has a structure of finite generated projective $\prod B_k$-module.

Definition 5.12. Let us fix some notations about the holonomy.

- Two paths p_0 and p_1 from x to y in X are thin homotopic to each other if there exists an endpoint preserving homotopy $h\colon [0,1] \times [0,1] \to X$ with $h(\cdot, j) = p_j$ that factors through a finite tree T,
$$h\colon [0,1] \times [0,1] \to T \to X,$$
such that both restrictions of the first map $[0,1] \times \{j\} \to T$ are piecewise-linear for $j = 0, 1$.
- The path groupoid $\mathcal{P}_1(X)$ is a groupoid consisting of all the points in X as objects. The morphism from x to y is the equivalence class of piece-wise smooth paths connecting two given points
$$\mathcal{P}_1(X)[x, y] := \big\{ p\colon [0,1] \to X \mid p(0) = x,\ p(1) = y \big\} / \sim .$$
The equivalent relationship is given by re-parametrization and thin homotopy.
- If a Hilbert B-module G-bundle E over X is given, the transport groupoid $\mathcal{T}(X; E)$ is a groupoid with the same objects as $\mathcal{P}_1(X)$. The morphisms from x to y are the unitary isomorphisms between the fibers $\mathcal{T}(X; E)[x, y] := \mathrm{Iso}_B(E_x, E_y)$.

Definition 5.13. A parallel transport of E is a continuous functor $\Phi^E \colon \mathcal{P}_1(X) \to \mathcal{T}(X; E)$. Φ^E is called ε-close to the identity if for each $x \in X$ and contractible loop $p \in \mathcal{P}_1(X)[x, x]$, it follows that

$$\big\| \Phi_p^E - \mathrm{id}_{E_x} \big\| < \varepsilon \cdot \mathrm{area}(D)$$

for any two-dimensional disk $D \subset X$ spanning p. D may be degenerated partially or completely.

Remark 5.14. Let E be a Hermitian vector bundle, in other words, a finitely generated Hilbert $\mathbb{C}$-module bundle, equipped with a compatible connection ∇. Let Φ^E be the parallel transport with respect to ∇ in the usual sense. If its curvature $R^E \in C^\infty(X; \mathrm{End}(E) \otimes \bigwedge^2(T^*(X)))$ has uniformly bounded operator norm $\|R^E\| < C$, then for any loop $p \in \mathcal{P}_1(X)[x,x]$ and any two-dimensional disk $D \subset X$ spanning p, it follows that $\|\Phi_p^E - \mathrm{id}_{E_x}\| < \int_D \|R^E\| < C \cdot \mathrm{area}(D)$; so it is C-closed to identity.

Proposition 5.15. *Let $\{E^k\}$ be a sequence of Hilbert B_k-module G-bundles over X with B_k unital C^*-algebras. Assume that each parallel transport Φ^k for E^k is ε-close to the identity uniformly, that is, ε is independent of k.*

Then, there exists a finitely generated Hilbert $(\prod_k B_k)$-module G-bundle V over X with Lipschitz continuous transition functions in diagonal form and so that the kth component of this bundle is isomorphic to the original E^k.

Moreover, if the parallel transport Φ^k for each of E^k comes from the G-invariant connection ∇^k on E^k, V is equipped with a continuous G-invariant connection induced by E^k.

Proof. We will essentially follow the proof of [4, Proposition 3.12]. For each $x \in X$ take an open ball $U_x \subset X$ of radius $\ll 1$ whose center is x. Assume that each U_x is geodesically convex. Due to Corollary 2.4 of the slice theorem, there exists a subfamily of finitely many open subsets $\{U_{x_1}, \dots, U_{x_N}\}$ such that $X = \bigcup_{\gamma \in G} \bigcup_{i=1}^N \gamma(U_{x_i})$.

Fix k. In order to simplify the notation, let $U_i := U_{x_i}$ and $\Phi_{y;x}\colon E_y^k \to E_x^k$ denote the parallel transport of E^k along the minimal geodesic from y to x for x and y in the same neighborhood $\gamma(U_i)$. Trivialize E^k via $\Phi_{y;x_i}\colon E_y^k \to E_{x_i}^k$ on each U_i. Similarly, trivialize E^k on each $\gamma(U_i)$ for $\gamma \in G$ via $\Phi_{\gamma y;\gamma x_i}\colon E_{\gamma y}^k \to E_{\gamma x_i}^k$. Note that since parallel transport commute with the action of G, it follows that $\Phi_{\gamma y,\gamma x_i} = \gamma \circ \Phi_{y;x_i} \circ \gamma^{-1}$.

These provide local trivializations for E^k whose transition functions have uniformly bounded Lipschitz constants. More precisely, we have to fix unitary isomorphisms $\phi_{\gamma x_i}\colon E_{\gamma x_i}^k \to \mathcal{E}^k$ between the fiber on γx_i and the typical fiber $\mathcal{E}^k$. Our local trivialization is $\phi_{\gamma x_i} \Phi_{y;\gamma x_i}\colon E_y^k \to \mathcal{E}^k$. If $y, z \in \gamma(U_i) \cap \eta(U_j) \neq \emptyset$, we can consider the transition function

$$y \mapsto \psi_{\gamma(U_i),\eta(U_j)}(y) := (\phi_{\eta x_j} \circ \Phi_{y;\eta x_j})(\phi_{\gamma x_i} \circ \Phi_{y;\gamma x_i})^{-1} \in \mathrm{End}_{B_k}(\mathcal{E}^k).$$

Now we will estimate its Lipschitz constant as follows:

$$\begin{aligned}
&\psi_{\gamma(U_i),\eta(U_j)}(y) - \psi_{\gamma(U_i),\eta(U_j)}(z)\\
&= (\phi_{\eta x_j} \Phi_{y;\eta x_j})(\phi_{\gamma x_i} \Phi_{y;\gamma x_i})^{-1} - (\phi_{\eta x_j} \Phi_{z;\eta x_j})(\phi_{\gamma x_i} \Phi_{z;\gamma x_i})^{-1}\\
&= \phi_{\eta x_j} \big\{ (\Phi_{y;\eta x_j}) \big(\Phi_{y;\gamma x_i}^{-1}\big) - (\Phi_{z;\eta x_j} \Phi_{y;z}) \big(\Phi_{y;z}^{-1} \Phi_{z;\gamma x_i}^{-1}\big) \big\} \phi_{\gamma x_i}^{-1}\\
&= \phi_{\eta x_j} \big\{ (\Phi_{y;\eta x_j} - \Phi_{z;\eta x_j} \Phi_{y;z}) \big(\Phi_{y;\gamma x_i}^{-1}\big) + (\Phi_{z;\eta x_j} \Phi_{y;z}) \big(\Phi_{y;\gamma x_i}^{-1} - \Phi_{y;z}^{-1} \Phi_{z;\gamma x_i}^{-1}\big) \big\} \phi_{\gamma x_i}^{-1}.
\end{aligned}$$

Since ϕ's and Φ's are isometry, it follows that

$$\begin{aligned}
&\big\| \psi_{\gamma(U_i),\eta(U_j)}(y) - \psi_{\gamma(U_i),\eta(U_j)}(z) \big\|\\
&\quad \leq \| \Phi_{y;\eta x_j} - \Phi_{z;\eta x_j} \Phi_{y;z} \| + \big\| \Phi_{y;\gamma x_i}^{-1} - \Phi_{y;z}^{-1} \Phi_{z;\gamma x_i}^{-1} \big\|
\end{aligned}$$

$$= \left\| \Phi_{y;\eta x_j} \Phi_{z;y} \Phi_{\eta x_j;z} - \mathrm{id}_{E^k_{\eta x_j}} \right\| + \left\| \Phi_{z;\gamma x_i} \Phi_{y;z} \Phi_{\gamma x_i;y} - \mathrm{id}_{E^k_{\gamma x_i}} \right\|$$
$$\leq \varepsilon \cdot \big(\mathrm{area}(D_1) + \mathrm{area}(D_2) \big). \tag{5.1}$$

Here $D_1 \subset \eta(U_j)$ is a two-dimensional disk spanning the piece-wise geodesic loops connecting ηx_j, y, z, and ηx_j and $D_2 \subset \gamma(U_i)$ is a two-dimensional disk spanning the piece-wise geodesic loop connecting γx_i, y, z, and γx_i.

We claim that there exists a constant C depending only on X such that

$$\mathrm{area}(D_1), \mathrm{area}(D_2) \leq C \cdot \mathrm{dist}(y, z) \tag{5.2}$$

if we choose suitable disks D_1 and D_2.

We verify this using the geodesic coordinate $\exp^{-1}_{\eta x_j} \colon \eta(U_j) \to T_{\eta x_j} X$ centered at $\eta x_j \mapsto 0$. More precisely, let p denote the minimal geodesic from $y = p(0)$ to $z = p(\mathrm{dist}(y, z))$ with unit speed. Consider

$$D_0 := \big\{ (r \cos\theta, r \sin\theta) \in \mathbb{R}^2 \mid 0 \leq r,\ 0 \leq \theta \leq \mathrm{dist}(y, z) \big\} \subset \mathbb{R}^2$$

and $F \colon D_0 \to \eta(U_j) \subset X$ given by

$$F(r\cos\theta, r\sin\theta) := \exp_{\eta x_j} \big(r \exp^{-1}_{\eta x_j} \big(p(\theta) \big) \big).$$

Set $D_1 := F(D_0)$. F is injective if $\exp^{-1}_{\eta x_j}(y)$ and $\pm \exp^{-1}_{\eta x_j}(z)$ are on different radial directions, in which case F is a homeomorphism onto its image, and hence $F(D_0)$ is a two-dimensional disk spanning the target loop. The Lipschitz constant of F is bounded by a constant depending on the curvature on $\eta(U_j)$, so there exists a constant $C_{\eta,j}$ depending on the Riemannian curvature on $\eta(U_j)$ satisfying

$$\mathrm{area}(D_1) \leq C_{\eta,j} \cdot \mathrm{area}(D_0) \leq C_{\eta,j} \cdot \mathrm{dist}(y, z).$$

However, the constant $C_{\eta,j}$ can be taken independent of $\eta(U_j)$ due to the bounded geometry of X implied by the slice theorem (Corollary 2.4). In the case of $\exp^{-1}_{\eta x_j}(y)$ and $\pm \exp^{-1}_{\eta x_j}(z)$ being on the same radial direction, D_1 is completely degenerated and $\mathrm{area}(D_1) = 0$. We can construct D_2 in the same manner so the claim (5.2) has been verified.

Therefore combining (5.1) and (5.2), we conclude that the Lipschitz constants of the transition functions of these local trivializations are less than $2C\varepsilon$, which are independent of E^k, U_i, and $\gamma \in G$, in particular, the products of them

$$\Psi_{\gamma(U_i),\eta(U_j)} := \{ \psi^k_{\gamma(U_i),\eta(U_j)} \}_{k \in \mathbb{N}} \colon \gamma(U_i) \cap \eta(U_j) \to \mathbb{L}_{(\prod B_k)} \Big(\prod_k \mathcal{E}^k \Big)$$

are Lipschitz continuous. So it is allowed to use them to define the Hilbert $\prod_k B_k$-module bundle V as required. Precisely V can be constructed as follows:

$$V := \bigsqcup_{\gamma,i} \Big(\gamma(U_i) \times \prod_k \mathcal{E}^k \Big) \Big/ \sim .$$

Here, $(x, v) \in \gamma(U_i) \times \prod_k \mathcal{E}^k$ and $(y, w) \in \eta(U_j) \times \prod_k \mathcal{E}^k$ are equivalent if and only if $x = y \in \gamma(U_i) \cap \eta(U_j) \neq \emptyset$ and $\Psi_{\gamma(U_i),\eta(U_j)}(v) = w$. By the construction of V, if $p_n \colon \prod_k B_k \to B_n$ denotes the projection onto the nth component, $V \widehat{\otimes}_{p_n} B_n$ is isomorphic to the original nth component E^n.

In order to verify the continuity of the induced connection, let $\{\mathbf{e}_i\}$ be any orthonormal local frame on U_i for an arbitrarily fixed E^k obtained by the parallel transport along the minimal geodesic from the center $x_i \in U_i$, namely, $\mathbf{e}_i(y) = \Phi_{x_i;y}\mathbf{e}_i(x_i)$. It is sufficient to verify that $\|\nabla^k \mathbf{e}_i\| < C$. Let $v \in T_y X$ be a unit tangent vector and $p(t) := \exp_y(tv)$ be the geodesic of unit speed with direction v,

$$\begin{aligned}
\nabla^k_v \mathbf{e}_i(y) &= \lim_{t\to 0} \frac{1}{t}\big(\Phi_{p(t);p(0)}\mathbf{e}_i\big(p(t)\big) - \mathbf{e}_i\big(p(0)\big)\big) \\
&= \lim_{t\to 0} \frac{1}{t}\big(\Phi_{p(t);p(0)}\Phi_{x_i;p(t)} - \Phi_{x_i;p(0)}\big)\mathbf{e}_i(x_i), \\
\big\|\nabla^k_v \mathbf{e}_i(y)\big\| &\le \lim_{t\to 0} \frac{1}{|t|}\big\|\Phi_{p(t);p(0)}\Phi_{x_i;p(t)} - \Phi_{x_i;p(0)}\big\| \\
&\le \lim_{t\to 0} \frac{1}{|t|}\varepsilon \cdot \operatorname{area}\big(D(t)\big),
\end{aligned}$$

where $D(t)$ is a two-dimensional disk in U_i spanning the piece-wise geodesic connecting x_i, $p(0) = y$, $p(t)$, and x_i. As above, we can find a constant $C > 0$ and disks $D(t)$ satisfying

$$\operatorname{area}\big(D(t)\big) \le C \cdot \operatorname{dist}\big(p(0), p(t)\big) = C|t|$$

for $|t| \ll 1$. Hence, we obtain $\|\nabla^k_v \mathbf{e}_i(y)\| \le C\varepsilon$. ∎

Definition 5.16. Let us define a Hilbert $(\mathcal{Q}\, B_k)$-module bundle

$$W := V \widehat{\otimes}_\pi (\mathcal{Q}\, B_k),$$

where $\pi \colon \prod B_k \twoheadrightarrow \mathcal{Q}\, B_k$ denotes the projection.

The family of parallel transport of E^k induces the parallel transport Φ^W of W which commutes with the action of G.

Proposition 5.17. *If the parallel transport of E^k is C_k-close to the identity with $C_k \searrow 0$, then the G-bundle W constructed above is a flat bundle. More precisely, the parallel transport $\Phi^W(p) \in \operatorname{Hom}(W_x, W_y)$ depends only on the ends-fixing homotopy class of $p \in \mathcal{P}_1(X)[x, y]$.*

Proof. It is sufficient to prove that for any contractive loop $p \in \mathcal{P}_1(X)[x, x]$, it satisfies $\Phi^W(p) = \mathrm{id}_{W_x}$. Fix a two-dimensional disk $D \subset X$ spanning the loop p. For arbitrary $\varepsilon > 0$, there exists n_0 such that every $k \ge n_0$ satisfies that Φ^{E^k} is $\frac{\varepsilon}{1+\operatorname{area}(D)}$-close to the identity.

$$\begin{aligned}
\big\|\Phi^W(p) - \mathrm{id}_{W_x}\big\| &= \limsup_{k\to\infty} \big\|\Phi^{E^k}(p) - \mathrm{id}\big\| \\
&\le \sup_{k \ge n_0} \big\|\Phi^{E^k}(p) - \mathrm{id}\big\|
\end{aligned}$$

$$\leq \frac{\varepsilon}{1+\operatorname{area}(D)} \cdot \operatorname{area}(D)$$
$$\leq \varepsilon.$$

This implies $\Phi^W(p) = \mathrm{id}_{W_x}$. ■

5.3. Index of the product bundle

Proposition 5.18. (1) *Let $p_n\colon \prod B_k \to B_n$ denote the projection onto the nth component and consider*

$$(1 \otimes p_n)_*\colon K_0\Big(C^*(G) \mathbin{\widehat{\otimes}} \big(\prod B_k\big)\Big) \to K_0\big(C^*(G) \mathbin{\widehat{\otimes}} B_n\big).$$

Then,

$$(1 \otimes p_n)_* \mathrm{ind}_G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big) = \mathrm{ind}_G\big([E^n] \mathbin{\widehat{\otimes}} [D]\big).$$

(2) *Let $\pi\colon \prod B_k \to \mathcal{Q}\, B_k$ denote the quotient map and consider*

$$(1 \otimes \pi)_*\colon K_0\Big(C^*(G) \mathbin{\widehat{\otimes}} \big(\prod B_k\big)\Big) \to K_0\big(C^*(G) \mathbin{\widehat{\otimes}} (\mathcal{Q}\, B_k)\big).$$

Then,

$$(1 \otimes \pi)_* \mathrm{ind}_G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big) = \mathrm{ind}_G\big([W] \mathbin{\widehat{\otimes}} [D]\big).$$

Proof. As for the first part, $[E^n] = (p_n)_*[\prod E^k] \in KK^G(C_0(X),\ C_0(X) \mathbin{\widehat{\otimes}} B_n)$ by the construction of $\prod E^k$. Then, it follows that

$$\begin{aligned}
\mathrm{ind}_G\big([E^n] \mathbin{\widehat{\otimes}} [D]\big) &= \mathrm{ind}_G\Big((p_n)_*\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big)\\
&= [c] \mathbin{\widehat{\otimes}} j^G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D] \mathbin{\widehat{\otimes}} \mathbf{p_n}\Big)\\
&= [c] \mathbin{\widehat{\otimes}} j^G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big) \mathbin{\widehat{\otimes}} j^G(\mathbf{p_n})\\
&= \mathrm{ind}_G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big) \mathbin{\widehat{\otimes}} j^G(\mathbf{p_n})\\
&= (1 \otimes p_n)_* \mathrm{ind}_G\Big(\big[\prod E^k\big] \mathbin{\widehat{\otimes}} [D]\Big),
\end{aligned}$$

where $\mathbf{p_n} = (B_n, p_n, 0) \in KK(\prod_{k\in\mathbb{N}} B_k, B_n)$. Then, note that

$$j^G(\mathbf{p_n}) = \big(C^*(G) \mathbin{\widehat{\otimes}} B_n, 1 \mathbin{\widehat{\otimes}} p_n, 0\big) \in KK\left(C^*(G) \mathbin{\widehat{\otimes}} \left(\prod_{k\in\mathbb{N}} B_k\right), C^*(G) \mathbin{\widehat{\otimes}} B_n\right).$$

Since $\pi_*[\prod E^k] = [W] \in KK^G(C_0(X), C_0(X) \mathbin{\widehat{\otimes}} (\mathcal{Q}\, B_k))$ by the construction of W, the second part can be proved in the similar way. ■

Proposition 5.19. *Let $[D] \in KK^G(C_0(X), \mathbb{C})$ be the K-homology element of X determined by a Dirac operator on a G-Hermitian vector bundle $\mathbb{V}$ over X. Suppose that W is a finitely generated flat B-module G-bundle. Assume that X is simply connected. Then,* $\mathrm{ind}_G([W] \mathbin{\widehat{\otimes}} [D]) = 0 \in K_0(C^*(G) \mathbin{\widehat{\otimes}} B)$ *if* $\mathrm{ind}_G([D]) = 0$.

In order to prove this, we introduce an element $[W]_{\mathrm{rpn}} \in KK^G(\mathbb{C}, B)$ using the holonomy representation.

Definition 5.20. • Let $\Phi_{x;y}$ denote the parallel transport of W along an arbitrary path from $x \in X$ to $y \in X$. Since X is simply connected and W is flat, it depends only on the ends of the path.

- Let us fix a base point $x_0 \in X$ and let W_{x_0} be the fiber on x_0. Define $[W]_{\mathrm{rpn}}$ as

$$[W]_{\mathrm{rpn}} := (W_{x_0}, 0) \in KK^G(\mathbb{C}, B).$$

The action of G on W_{x_0} is given by the holonomy $\rho\colon G \to \mathrm{End}_Q(W_{x_0})$,

$$\rho[\gamma](w) = (\Phi_{x_0;\gamma x_0})^{-1}\gamma(w) \quad \text{for } \gamma \in G,\ w \in W_{x_0}.$$

Lemma 5.21. *One has*

$$[W]\,\widehat{\otimes}_{C_0(X)}\,[D] = [D]\,\widehat{\otimes}_{\mathbb{C}}\,[W]_{\mathrm{rpn}} \in KK^G\big(C_0(X), B\big).$$

Proof. Recall that $[D] \in KK^G(C_0(X), \mathbb{C})$ is given by $(L^2(X; \mathbb{V}), F_D)$, where F_D denotes the operator $\frac{D}{\sqrt{1+D^2}}$, and that

$$[W]\,\widehat{\otimes}_{C_0(X)}\,[D] = \big(C_0(X; W)\,\widehat{\otimes}_{C_0(X)}\,L^2(X; \mathbb{V}), F_{D^W}\big),$$

where D^W is the Dirac operator twisted by W acting on

$$L^2(X; W \otimes \mathbb{V}) \simeq C_0(X; W)\,\widehat{\otimes}_{C_0(X)}\,L^2(X; \mathbb{V}),$$

that is,

$$D^W = \sum_j \big(\mathrm{id}_W \otimes c(e_j)\big)\big(\nabla^W_{e_j} \otimes \mathrm{id}_{\mathbb{V}} + \mathrm{id}_W \otimes \nabla^{\mathbb{V}}_{e_j}\big),$$

where $\{e_j\}$ denotes an orthogonal basis for TX and $c(\cdot)$ denotes the Clifford multiplication by $\mathrm{Cliff}(TX)$ on $\mathbb{V}$. The action of $C_0(X)$ on $C_0(X; W)$ and $L^2(X; \mathbb{V})$ are the point-wise multiplications. On the other hand,

$$[D]\,\widehat{\otimes}_{\mathbb{C}}\,[W]_{\mathrm{rpn}} = \big(L^2(X; \mathbb{V})\,\widehat{\otimes}_{\mathbb{C}}\,W_{x_0},\ F_D\,\widehat{\otimes}\,1\big).$$

The action of $C_0(X)$ is the point-wise multiplications. Note that the action of G on W_{x_0} is given by the holonomy representation ρ. It is sufficient to give a G-equivariant isomorphism

$$\varphi\colon L^2(X; \mathbb{V})\,\widehat{\otimes}_{\mathbb{C}}\,W_{x_0} \to C_0(X; W)\,\widehat{\otimes}_{C_0(X)}\,L^2(X; \mathbb{V}),$$

which is compatible with D_W and $D\,\widehat{\otimes}\,1$. Set a section for W given by

$$\overline{w}\colon x \mapsto \Phi_{x_0;x} w \in W_x \tag{5.3}$$

and define φ on a dense subspace $C_c(X; \mathbb{V})\,\widehat{\otimes}\,W_{x_0}$ as

$$\varphi(s \otimes w) := \overline{w} \cdot \chi \otimes s \quad \text{for } s \in C_c(X; \mathbb{V}) \text{ and } w \in W_{x_0},$$

where $\chi \in C_0(X)$ is an arbitrary compactly supported function on X with values in [0, 1] satisfying that $\chi(x) = 1$ for all $x \in \operatorname{supp}(s)$.

φ is independent of the choice of χ and hence well defined. Indeed, Let $\chi' \in C_c(X)$ be another such function, and let $\rho \in C_c(X)$ be a compactly supported function on X with values in [0, 1] satisfying that $\rho(x) = 1$ for all $x \in \operatorname{supp}(\chi) \cup \operatorname{supp}(\chi')$. Then, in $C_0(X; W) \widehat{\otimes}_{C_0(X)} C_c(X; \mathbb{V})$,

$$\begin{aligned}\overline{w} \cdot \chi \otimes s - \overline{w} \cdot \chi' \otimes s &= \overline{w} \cdot (\chi - \chi') \otimes s \\ &= \overline{w} \cdot \rho \cdot (\chi - \chi') \otimes s \\ &= \overline{w} \cdot \rho \otimes (\chi - \chi')s \\ &= 0.\end{aligned}$$

Now we obtain that

$$D_W \circ \varphi(s \otimes w) = D_W(\overline{w} \otimes s) = \overline{w} \otimes D(s) = \varphi \circ (D \widehat{\otimes} 1)(s \otimes w)$$

for $s \in C_c(\mathbb{V})$ and $w \in W_{x_0}$. This is because $\nabla^W \overline{w} = 0$ by its construction.

Compatibility with the action of G is verified as follows:

$$\begin{aligned}\varphi\big(\gamma(s \otimes w)\big)(x) &= \Phi_{x_0;x}\big(\rho[\gamma](w)\big) \otimes \gamma\big(s(\gamma^{-1}x)\big) \\ &= \Phi_{x_0;x}(\Phi_{x_0;\gamma x_0})^{-1}\gamma(w) \otimes \gamma\big(s(\gamma^{-1}x)\big) \\ &= \Phi_{\gamma x_0;x}\big(\gamma(b)\big) \otimes \gamma\big(s(\gamma^{-1}x)\big), \\ \gamma\big(\varphi(s \otimes w)\big)(x) &= \gamma\big((\Phi_{x_0;\gamma^{-1}x})(w) \otimes s(\gamma^{-1}x)\big) \\ &= \Phi_{\gamma x_0;x}\big(\gamma(w)\big) \otimes \gamma\big(s(\gamma^{-1}x)\big).\end{aligned}$$

Let us check that φ induces an isomorphism. For $s_1 \otimes w_1, s_2 \otimes w_2 \in C_c(X; \mathbb{V}) \widehat{\otimes}_{\mathbb{C}} W_{x_0}$, it follows that

$$\begin{aligned}&\big\langle \varphi(s_1 \otimes w_1), \varphi(s_2 \otimes w_2)\big\rangle_{C_0(X;W)\widehat{\otimes}_{C_0(X;X)} L^2(X;\mathbb{V})} \\ &= \big\langle s_1, \langle \overline{w_1} \cdot \chi, \overline{w_2} \cdot \chi\rangle_{C_0(X;W)} s_2\big\rangle_{L^2(X;\mathbb{V})} \\ &= \int_X \big\langle s_1(x), \big\langle (\Phi_{x_0;x} w_1)\chi(x), (\Phi_{x_0;x} w_2)\chi(x)\big\rangle_{W_x} s_2(x)\big\rangle_{\mathbb{V}_x} \,\mathrm{d}\operatorname{vol}(x) \\ &= \int_X \langle w_1, w_2\rangle_{W_0} \chi(x)^2 \big\langle s_1(x), s_2(x)\big\rangle_{\mathbb{V}_x} \,\mathrm{d}\operatorname{vol}(x) \\ &= \langle w_1, w_2\rangle_{W_0} \langle s_1, s_2\rangle_{L^2(X;\mathbb{V})} \\ &= \langle s_1 \otimes w_1, s_2 \otimes w_2\rangle_{L^2(X;\mathbb{V})\widehat{\otimes} W_{x_0}},\end{aligned}$$

where $\chi \in C_0(X)$ is a compactly supported function on X satisfying that $\chi(x) = 1$ for all $x \in \operatorname{supp}(s_1) \cup \operatorname{supp}(s_2)$. This implies that φ is continuous and injective.

Moreover, choose arbitrary $F \in C_c(X; W)$ and $s \in C_c(X; \mathbb{V})$. Since $\Phi_{x_0;x}^{-1}$ provides a trivialization of $W \simeq X \times W_{x_0}$, we have an isomorphism $C_c(X; W) \simeq C_c(X) \widehat{\otimes}_{\mathbb{C}} W_{x_0}$.

Remark that, however, this is not a G-equivariant isomorphism, just as pre-Hilbert $(C_0(X;B) \cong C_0(X)\,\widehat{\otimes}\,B)$-modules. Then, there exist countable subsets $\{f_1, f_2, \dots\} \subset C_c(X)$ and $\{w_1, w_2, \dots\} \subset W_{x_0}$ satisfying that $\sum_{j\in\mathbb{N}} f_j \overline{w_j} = F$ in $C_0(X;W)$. Now it follows that

$$\begin{aligned}\varphi\Bigg(\sum_{j\in\mathbb{N}} f_j s \otimes w_j\Bigg) &= \sum_{j\in\mathbb{N}} (\overline{w_j}\cdot\chi \otimes f_j s) = \sum_{j\in\mathbb{N}} (\overline{w_j}\cdot\chi f_j \otimes s) \\ &= \Bigg(\sum_{j\in\mathbb{N}} \overline{w_j} f_j\Bigg)\cdot\chi \otimes s = F \otimes s,\end{aligned}$$

where $\chi \in C_0(X)$ is a compactly supported function on X satisfying that $\chi(x) = 1$ for all $x \in \mathrm{supp}(F) \cup \mathrm{supp}(s)$. This implies that the image of φ is dense in $C_0(X;W)\,\widehat{\otimes}\,L^2(X;\mathbb{V})$. Therefore, φ induces an isomorphism. ■

Proof of Proposition 5.19. Due to the previous lemma, it follows that

$$\begin{aligned}\mathrm{ind}_G\big([W]\,\widehat{\otimes}\,[D]\big) &= \mathrm{ind}_G\big([D]\,\widehat{\otimes}\,[W]_{\mathrm{rpn}}\big) = [c]\,\widehat{\otimes}\,j^G\big([D]\,\widehat{\otimes}\,[W]_{\mathrm{rpn}}\big) \\ &= [c]\,\widehat{\otimes}\,\big(j^G[D]\big)\,\widehat{\otimes}\,\big(j^G[W]_{\mathrm{rpn}}\big) = \big(\mathrm{ind}_G[D]\big)\,\widehat{\otimes}\,\big(j^G[W]_{\mathrm{rpn}}\big).\end{aligned}$$

Thus the assumption $\mathrm{ind}_G[D] = 0$ implies $\mathrm{ind}_G([W]\,\widehat{\otimes}\,[D]) = 0$. ■

5.4. Proof of Theorem C

Proof of Theorem C. As in Remark 5.4, we may assume that D is a Dirac type operator. Assume that $\mathrm{ind}_G[D] = 0$ and we assume the converse, that is, for each $k \in \mathbb{N}$ there exists a Hilbert B_k-module G-bundle E^k over X whose curvature norm is less than $\frac{1}{k}$ satisfying that

$$\mathrm{ind}_G\big([E^k]\,\widehat{\otimes}\,[D]\big) \neq 0 \in K_0\big(C^*(G)\,\widehat{\otimes}\,B_k\big).$$

To begin with, we have an exact sequence:

$$0 \to \bigoplus B_k \xrightarrow{\iota} \prod B_k \xrightarrow{\pi} \mathcal{Q}\, B_k \to 0,$$

where ι and π are natural inclusion and projection. We also have the following exact sequence [14, Theorem T.6.26]:

$$\begin{aligned}0 \to C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}\big(\textstyle\bigoplus B_k\big) &\xrightarrow{1\widehat{\otimes}\iota} C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}\big(\textstyle\prod B_k\big) \\ &\xrightarrow{1\widehat{\otimes}\pi} C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}(\mathcal{Q}\, B_k) \to 0.\end{aligned}$$

We have the exact sequence of K-groups

$$\begin{aligned}K_0\Big(C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}\big(\textstyle\bigoplus B_k\big)\Big) &\to K_0\Big(C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}\big(\textstyle\prod B_k\big)\Big) \\ &\to K_0\big(C^*_{\mathrm{Max}}(G)\,\widehat{\otimes}_{\mathrm{Max}}(\mathcal{Q}\, B_k)\big).\end{aligned}$$

If all of B_k are commutative, then $\mathcal{Q}\, B_k$ is also commutative and hence nuclear. In that case, we also have the same exact sequences in which $C^*_{\mathrm{Max}}(G)$ and $\widehat{\otimes}_{\mathrm{Max}}$ are replaced by $C^*_{\mathrm{red}}(G)$ and $\widehat{\otimes}_{\mathrm{min}}$, respectively.

Let us start with $\mathrm{ind}_G([\prod E^k]\,\widehat{\otimes}\,[D]) \in K_0(C^*(G)\,\widehat{\otimes}\,(\prod B_k))$. Due to the flatness of W (Proposition 5.17) and Propositions 5.19 and 5.18, we have

$$(1\,\widehat{\otimes}\,\pi)_*\mathrm{ind}_G\Big(\big[\prod E^k\big]\,\widehat{\otimes}\,[D]\Big) = \mathrm{ind}_G\big([W]\,\widehat{\otimes}\,[D]\big) = 0.$$

It follows from the exactness that there exists $\zeta \in K_0(C^*(G)\,\widehat{\otimes}\,(\bigoplus B_k))$ such that

$$(1\,\widehat{\otimes}\,\iota)_*(\zeta) = \mathrm{ind}_G\Big(\big[\prod E^k\big]\,\widehat{\otimes}\,[D]\Big).$$

Lemma 5.22. *$A\,\widehat{\otimes}\,(\bigoplus_{k\in\mathbb{N}} B_k)$ is naturally isomorphic to*

$$\bigoplus_{k\in\mathbb{N}}(A\,\widehat{\otimes}\,B_k) = \varinjlim_n \bigoplus_{k=1}^{n}(A\,\widehat{\otimes}\,B_k).$$

Proof. Let C denote the direct product $\varinjlim_n \bigoplus_{k=1}^n(A\,\widehat{\otimes}\,B_k)$. Note that for the finite direct product, we have the natural isomorphism $\bigoplus_{k=1}^n(A\,\widehat{\otimes}\,B_k) \cong A\,\widehat{\otimes}\,(\bigoplus_{k=1}^n B_k)$. For each $n\in\mathbb{N}$, we have the following commutative diagram:

$$\begin{array}{ccc}
A\,\widehat{\otimes}\left(\bigoplus_{k=1}^{n} B_k\right) & \xrightarrow{\mathrm{id}_A\,\widehat{\otimes}\,\iota_n^{n+1}} & A\,\widehat{\otimes}\left(\bigoplus_{k=1}^{n+1} B_k\right) \\
& \searrow^{\mathrm{id}_A\,\widehat{\otimes}\,\iota_n} & \downarrow{\scriptstyle \mathrm{id}_A\,\widehat{\otimes}\,\iota_{n+1}} \\
& & A\,\widehat{\otimes}\left(\bigoplus_{k\in\mathbb{N}} B_k\right).
\end{array}$$

Now by using the universal property of the direct limit, we obtain a map ϕ:

$$\begin{array}{ccc}
A\,\widehat{\otimes}\left(\bigoplus_{k=1}^{n} B_k\right) & \longrightarrow & \varinjlim_n A\,\widehat{\otimes}\left(\bigoplus_{k=1}^{n} B_k\right) \\
& \searrow & \downarrow{\scriptstyle \phi} \\
& & A\,\widehat{\otimes}\left(\bigoplus_{k\in\mathbb{N}} B_k\right).
\end{array}$$

Since $\mathrm{id}_A\,\widehat{\otimes}\,\iota_n$ are isometric and injective, ϕ is isometric and injective on each subspace $A\,\widehat{\otimes}\,(\bigoplus_{k=1}^n B_k) \subset \varinjlim_n A\,\widehat{\otimes}\,(\bigoplus_{k=1}^n B_k)$. Since the union of such subspaces is dense in $\varinjlim_n A\,\widehat{\otimes}\,(\bigoplus_{k=1}^n B_k)$, it follows that ϕ itself is isometric and injective.

As for the surjectivity of ϕ, take any $a\,\widehat{\otimes}\,\{b_k\} \in A\,\widehat{\otimes}\,(\bigoplus_{k\in\mathbb{N}} B_k)$. For any $\varepsilon > 0$, there exists $n\in\mathbb{N}$ such that $\|b_k\| < \frac{\varepsilon}{1+\|a\|}$ for $k \geq n$. Then, replace b_k by 0 for all $k \geq n$ to obtain an element $\beta := \{b_1, b_2, \dots, 0_n, 0_{n+1}, \dots\} \in \bigoplus_{k\in\mathbb{N}} B_k$. Now we have that

$$a \mathbin{\widehat{\otimes}} \beta = (\mathrm{id}_A \mathbin{\widehat{\otimes}} \iota_n)\big(a \mathbin{\widehat{\otimes}} \{b_1, b_2, \dots, b_{n-1}\}\big) = \phi\big(a \mathbin{\widehat{\otimes}} \{b_1, b_2, \dots, b_{n-1}\}\big) \in \mathrm{Im}(\phi)$$

and

$$\|a \mathbin{\widehat{\otimes}} \{b_k\} - a \mathbin{\widehat{\otimes}} \beta\| \le \|a\| \|\{b_k\} - \beta\| \le \varepsilon.$$

These imply that $\mathrm{Im}(\phi)$ is dense in $A \mathbin{\widehat{\otimes}} (\bigoplus_{k\in\mathbb{N}} B_k)$ and hence, ϕ is surjective since it has a closed range. ∎

By this lemma, $C^*(G) \mathbin{\widehat{\otimes}} (\bigoplus B_k)$ is naturally isomorphic to $\bigoplus (C^*(G) \mathbin{\widehat{\otimes}} B_k)$. Besides, we have the natural isomorphism $K_0(\bigoplus(C^*(G) \mathbin{\widehat{\otimes}} B_k)) \simeq \bigoplus K_0(C^*(G) \mathbin{\widehat{\otimes}} B_k)$ [6, Proposition 4.1.15 and Remark 4.2.3], with the last $\bigoplus$ meaning the algebraic direct sum. Thus we can consider the following diagram:

$$\begin{array}{ccccc}
K_0\big(C^*(G) \mathbin{\widehat{\otimes}} (\bigoplus B_k)\big) & \xrightarrow{\iota_*} & K_0\big(C^*(G) \mathbin{\widehat{\otimes}} (\prod B_k)\big) & \xrightarrow{(1\otimes\pi)_*} & K_0\big(C^*(G) \mathbin{\widehat{\otimes}} \mathcal{Q}\, B_k\big) \\
{\scriptstyle \{(1\otimes p_k)_*\}}\big\downarrow \cong & & {\scriptstyle \{(1\otimes p_k)_*\}}\big\downarrow & & \\
\bigoplus K_0\big(C^*(G) \mathbin{\widehat{\otimes}} B_k\big) & \xrightarrow[\text{inclusion}]{} & \prod K_0\big(C^*(G) \mathbin{\widehat{\otimes}} B_k\big) & &
\end{array}$$

Since $p_k = \iota p_k$, this diagram commutes. Note that both $\bigoplus$ and $\prod$ in the bottom row are in the algebraic sense. Again due to Proposition 5.18,

$$\begin{aligned}
\big\{\mathrm{ind}_G([E^k] \mathbin{\widehat{\otimes}} [D])\big\}_{k\in\mathbb{N}} &= \big\{(1 \otimes p_k)_*\big\}\big(\mathrm{ind}_G([\textstyle\prod E^k] \mathbin{\widehat{\otimes}} [D])\big) \\
&= \big\{(1 \otimes p_k)_*\big\}\big((1 \otimes \iota)_*(\zeta)\big) \\
&= \big\{(1 \otimes p_k)_*\big\}(\zeta) \in \bigoplus K_0\big(C^*(G) \mathbin{\widehat{\otimes}} B_k\big).
\end{aligned}$$

This implies that all of $\mathrm{ind}_G([E^n] \mathbin{\widehat{\otimes}} [D]) \in K_0(C^*(G) \mathbin{\widehat{\otimes}} B_n)$ are equal to zero except for finitely many $n \in \mathbb{N}$, which contradicts to our assumption. ∎

5.5. Proof of Corollary D

To prove Corollary D, we will combine Theorem C with Theorem A. Consider the same conditions as Theorem A on X, Y, and G and assume additionally that X and Y are simply connected. Let $f : Y \to X$ be a G-equivariant orientation-preserving homotopy invariant map. Assume that for each $k \in \mathbb{N}$ there exists a Hilbert B_k-module G-bundle E^k over X whose curvature norm is less than $\frac{1}{k}$ satisfying that

$$\mathrm{ind}_G\big([E^k] \mathbin{\widehat{\otimes}} [\partial_X]\big) \neq \mathrm{ind}_G\big([f^* E^k] \mathbin{\widehat{\otimes}} [\partial_Y]\big) \in K_0\big(C^*(G) \mathbin{\widehat{\otimes}} B_k\big).$$

as in the proof of Theorem C. Consider a G-manifold $Z := X \sqcup (-Y)$, the disjoint union of X, and the orientation reversed Y with the signature operator ∂_Z on it. Although Z is not connected, however, we may apply Theorem C to ∂_Z, after replacing some argument in the proof as follows. Consider a family of Hilbert B_k-module bundles $\{E^k \sqcup f^* E^k\}$ over Z and obtain a flat bundle $W \sqcup f^* W$ as in Subsection 5.2. In order to obtain a global section $\overline{w}$ as in (5.3) in the proof of Lemma 5.21, we have used the connectedness of the base space. In this case, construct a section $\overline{w}: X \to W$ on X in the same way and pull it back on Y by f to obtain a global section on Z. The other parts are the same as above.

Funding. Research supported by the Natural Science Foundation of China (NSFC) grant no. 11771143.

References

[1] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-Algebras: 1943–1993 (San Antonio, TX, 1993)*, pp. 241–291, Contemp. Math. 167, Amer. Math. Soc., Providence, RI, 1994 Zbl 0830.46061 MR 1292018

[2] A. Connes and H. Moscovici, The L^2-index theorem for homogeneous spaces of Lie groups. *Ann. of Math. (2)* **115** (1982), no. 2, 291–330 Zbl 0515.58031 MR 647808

[3] Y. Fukumoto, On the strong Novikov conjecture of locally compact groups for low degree cohomology classes. 2016, arXiv:1604.00464

[4] B. Hanke, Positive scalar curvature, K-area and essentialness. In *Global Differential Geometry*, pp. 275–302, Springer Proc. Math. 17, Springer, Heidelberg, 2012 Zbl 1252.53039 MR 3289846

[5] B. Hanke and T. Schick, The strong Novikov conjecture for low degree cohomology. *Geom. Dedicata* **135** (2008), 119–127 Zbl 1149.19006 MR 2413333

[6] N. Higson and J. Roe, *Analytic K-Homology*. Oxford Math. Monogr., Oxford University Press, Oxford, 2000 Zbl 0968.46058 MR 1817560

[7] M. Hilsum and G. Skandalis, Invariance par homotopie de la signature à coefficients dans un fibré presque plat. *J. Reine Angew. Math.* **423** (1992), 73–99 Zbl 0731.55013 MR 1142484

[8] J. Kaminker and J. G. Miller, Homotopy invariance of the analytic index of signature operators over C^*-algebras. *J. Operator Theory* **14** (1985), no. 1, 113–127 Zbl 0614.46062 MR 789380

[9] G. G. Kasparov, Topological invariants of elliptic operators. I. K-homology. *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), no. 4, 796–838 Zbl 0337.58006 MR 0488027

[10] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture. *Invent. Math.* **91** (1988), no. 1, 147–201 Zbl 0647.46053 MR 918241

[11] G. G. Kasparov, Elliptic and transversally elliptic index theory from the viewpoint of KK-theory. *J. Noncommut. Geom.* **10** (2016), no. 4, 1303–1378 Zbl 1358.19001 MR 3597146

[12] V. Mathai, The Novikov conjecture for low degree cohomology classes. *Geom. Dedicata* **99** (2003), 1–15 Zbl 1029.19006 MR 1998926

[13] J. Rosenberg and S. Weinberger, An equivariant Novikov conjecture. With an appendix by J. P. May. *K-Theory* **4** (1990), no. 1, 29–53 Zbl 0721.57018 MR 1076524

[14] N. E. Wegge-Olsen, *K-Theory and C^*-Algebras: A Friendly Approach*. Oxford Science Publications, Oxford University Press, New York, 1993 Zbl 0780.46038 MR 1222415

Received 29 September 2017; revised 25 January 2019.

Yoshiyasu Fukumoto
Research Center for Operator Algebras, Department of Mathematics, East China Normal University, 3663 North Zhongshan Road, Shanghai 200062, China; y.fkmt.314@gmail.com