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Poisson cohomology, Koszul duality, and Batalin–Vilkovisky algebras

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Abstract. We study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich's deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality. As a corollary, the Batalin–Vilkovisky algebra structures that naturally arise in these cases are all isomorphic.

1. Introduction

In this paper, we study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich's deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality.

Let $A = \mathbb{R}[x_1, \ldots, x_n]$ be the real polynomial algebra in n variables. A Poisson bivector on A, say π, is called *quadratic* if it is in the form

$$\pi = \sum_{i_i, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{i_1 i_2}^{j_1 j_2} \in \mathbb{R}. \tag{1.1}$$

Several years ago, Shoikhet [30] observed that if π is quadratic, then the Koszul dual algebra $A^!$ of A, namely, the graded symmetric algebra $\mathbf{\Lambda}(\xi_1, \ldots, \xi_n)$ generated by n elements of degree -1, has a Poisson structure (let us call it the *Koszul dual* of π), given by

$$\pi^! = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} \xi_{j_1} \xi_{j_2} \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}}. \tag{1.2}$$

He also proved that Kontsevich's deformation quantization preserves this type of Koszul duality. Shoikhet's result motivates us to study some other properties of a Poisson algebra under Koszul duality.

First, the following theorem is clear from Shoikhet's article, once we explicitly write down the corresponding complexes.

Theorem 1.1. *Let $A = \mathbb{R}[x_1, \ldots, x_n]$ be a quadratic Poisson algebra. Denote by $A^!$ the Koszul dual Poisson algebra of A. Then there are isomorphisms*

$$\mathrm{HP}_\bullet(A) \cong \mathrm{HP}^\bullet(A^!; A^{¡}) \quad \text{and} \quad \mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!), \tag{1.3}$$

where $A^{¡} := \mathrm{Hom}_{\mathbb{R}}(A^!, \mathbb{R})$ is the linear dual of $A^!$.

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In the above theorem, $\mathrm{HP}_\bullet(-)$ is the Poisson homology, $\mathrm{HP}^\bullet(-)$ is the Poisson cohomology, and $\mathrm{HP}^\bullet(A^!; A^{¡})$ is the Poisson cohomology of $A^!$ with values in its dual space.

Historically, the Poisson homology and cohomology were introduced by Koszul [20] and Lichnerowicz [24], respectively. In 1997, Weinstein [37] introduced the notion of *unimodular* Poisson manifolds, and two years later Xu [40] proved that in this case, there is a Poincaré duality between the Poisson cohomology and homology of M. A purely algebraic version of Weinstein's notion was later formulated by Dolgushev in [9] (see also [22, 27]), and in this case we also have

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}_{n-\bullet}(A), \tag{1.4}$$

for some n depending on A.

For a *finite-dimensional* algebra such as $A^!$ above, Zhu, Van Oystaeyen, and Zhang introduced in [42] the notion of *Frobenius Poisson algebra* and proved that if they are *unimodular* in some sense (to be recalled below), then there also exists a version of Poincaré duality:

$$\mathrm{HP}^\bullet(A^!) \cong \mathrm{HP}^{\bullet-n}(A^!; A^{¡}). \tag{1.5}$$

Combining the above two versions of Poincaré duality, (1.4) and (1.5), as well as Theorem 1.1, we have the following theorem.

Theorem 1.2. *Let $A = \mathbb{R}[x_1, \ldots, x_n]$ be a quadratic Poisson algebra. Then (A, π) is unimodular if and only if its Koszul dual $(A^!, \pi^!)$ is unimodular Frobenius. In this case, one has the following commutative diagram:*

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(A) & \xrightarrow[\mathrm{PD}]{\cong} & \mathrm{HP}_{n-\bullet}(A) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\mathrm{HP}^\bullet(A^!) & \xrightarrow[\mathrm{PD}]{\cong} & \mathrm{HP}^{\bullet-n}(A^!; A^{¡}).
\end{array}$$

The main technique to prove the above theorem is the so-called "differential calculus", a notion introduced by Tamarkin and Tsygan in [31]. Later, Lambre [21] used the terminology "differential calculus with duality" to study the "noncommutative Poincaré duality" in these cases.

In the above-mentioned two references [40, 42], the authors also proved that the Poisson cohomology of a unimodular Poisson algebra (in both cases) has a Batalin–Vilkovisky algebra structure. The Batalin–Vilkovisky structure is a very important algebraic structure that has appeared in, for example, mathematical physics, Calabi–Yau geometry, and string topology. For unimodular quadratic Poisson algebras, we have the following theorem.

Theorem 1.3. *Suppose $A = \mathbb{R}[x_1, \ldots, x_n]$ is a unimodular quadratic Poisson algebra. Denote by $A^!$ its Koszul dual. Then*

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!)$$

is an isomorphism of Batalin–Vilkovisky algebras.

The above three theorems have some analogy to the case of Calabi–Yau algebras which was introduced by Ginzburg [15] in 2006. Suppose a Calabi–Yau algebra, say A, is Koszul, then its Koszul dual, denoted by $A^!$, is a symmetric Frobenius algebra. For these two algebras, we also have a version of Poincaré duality, due to van den Bergh [36] and Tradler [33], respectively (compare with (1.4) and (1.5)):

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A) \quad \text{and} \quad \mathrm{HH}^\bullet(A^!) \cong \mathrm{HH}^{\bullet-n}(A^!; A^¡).$$

In [15, §5.4] Ginzburg stated a conjecture, which he attributed to R. Rouquier, saying that for a Koszul Calabi–Yau algebra, say A, its Hochschild cohomology is isomorphic to the Hochschild cohomology of its Koszul dual $A^!$:

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!), \tag{1.6}$$

as Batalin–Vilkovisky algebras. This conjecture has recently been proved by two authors of the current paper together with G. Zhou in [6]. In fact, Theorem 1.3 may be viewed as a generalization of Rouquier's conjecture in Poisson geometry, which has been a folklore for several years.

More than just being an analogy, in [9, Theorem 3], Dolgushev proved that for the coordinate ring A of an affine Calabi–Yau Poisson variety, its deformation quantization in the sense of Kontsevich, say $A_\hbar$, is Calabi–Yau if and only if A is unimodular. Similarly, Felder and Shoikhet [12] and later Willwacher and Calaque [39] proved that, for a Frobenius Poisson algebra, its deformation quantization is again symmetric Frobenius if and only if it is unimodular. Based on these results, Dolgushev asked two questions in [9, §7] (see also [10]). The first question is whether there exists a relationship between the Poincaré duality of the Poisson (co)homology of A and the Poincaré duality of the Hochschild (co)homology of $A_\hbar$. The following theorem answers this question in the case of polynomials (the second half also includes the case of Frobenius algebras).

Theorem 1.4. (1) *Suppose $A = \mathbb{R}[x_1, \ldots, x_n]$ is a unimodular Poisson algebra. Let $A_\hbar$ be its deformation quantization. Then the diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[\![\hbar]\!]) & \xrightarrow[\mathrm{PD}]{\cong} & \mathrm{HP}_{n-\bullet}(A[\![\hbar]\!]) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow[\mathrm{PD}]{\cong} & \mathrm{HH}_{n-\bullet}(A_\hbar) \end{array}$$

commutes.

(2) *Similarly, suppose $A^! = \mathbf{\Lambda}(\xi_1, \ldots, \xi_n)$ is a unimodular Frobenius Poisson algebra, and let $A^!_\hbar$ be its deformation quantization. Then the diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A^![\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^![\![\hbar]\!]; A^¡[\![\hbar]\!]) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{HH}^\bullet(A^!_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^{\bullet-n}(A^!_\hbar; A^¡_\hbar) \end{array}$$

commutes.

In other words, the two versions of Poincaré duality, one between the Poisson cohomology and homology and the other between the Hochschild cohomology and homology, are preserved under Kontsevich's deformation quantization.

The second question that Dolgushev asked is whether there is any relationship between the roles that the unimodularity plays in the above two types of deformation quantization. The following theorem partially answers this question, although both cases that Dolgushev and Felder–Shoikhet/Willwacher–Calaque considered are more general (i.e., not necessarily Koszul).

Theorem 1.5. *Suppose $A = \mathbb{R}[x_1, \ldots, x_n]$ is a quadratic Poisson algebra. Denote by $A^!$ the Koszul dual algebra of A, and by $A_\hbar$ and $A^!_\hbar$ the Kontsevich deformation quantization of A and $A^!$, respectively. If A is unimodular (and by Theorem 1.2 $A^!$ is unimodular Frobenius), then $A_\hbar$ is Calabi–Yau, $A^!_\hbar$ is symmetric Frobenius, and the diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet\big(A[\![\hbar]\!]\big) & \xrightarrow{\cong} & \mathrm{HP}^\bullet\big(A^![\![\hbar]\!]\big) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^\bullet\big(A^!_\hbar\big) \end{array} \tag{1.7}$$

is commutative as Batalin–Vilkovisky algebra isomorphisms, where $A[\![\hbar]\!]$ and $A^![\![\hbar]\!]$ are equipped with the Poisson bivectors $\hbar\pi$ and $\hbar\pi^!$, respectively.

In other words, the theorem says that the unimodularity that appears in the deformation quantization of Calabi–Yau Poisson algebras and Frobenius Poisson algebras is related by Koszul duality. Note that in the theorem, $A_\hbar$ and $A^!_\hbar$ are Koszul dual to each other by Shoikhet [30].

Thus as a corollary, one obtains that if $A = \mathbb{R}[x_1, \ldots, x_n]$ is a unimodular quadratic Poisson algebra, then the homology and cohomology groups (Poisson and Hochschild) in Theorems 1.4 and 1.5 are all isomorphic. That is, we have the following commutative diagram of isomorphisms:

$$\begin{array}{ccccccc} & & \mathrm{HP}^\bullet\big(A^![\![\hbar]\!]\big) & & \longrightarrow & & \mathrm{HP}^{\bullet-n}\big(A^![\![\hbar]\!]; A^{¡}[\![\hbar]\!]\big) \\ & \nearrow & & & & \nearrow & \\ \mathrm{HP}^\bullet\big(A[\![\hbar]\!]\big) & & \longrightarrow & & \mathrm{HP}_{n-\bullet}\big(A[\![\hbar]\!]\big) & & \downarrow \\ & & \downarrow & & & & \\ & & \mathrm{HH}^\bullet\big(A^!_\hbar\big) & & \longrightarrow & & \mathrm{HH}^{\bullet-n}\big(A^!_\hbar; A^{¡}_\hbar\big) \\ \downarrow & \nearrow & & & \downarrow & \nearrow & \\ \mathrm{HH}^\bullet(A_\hbar) & & \longrightarrow & & \mathrm{HH}_{n-\bullet}(A_\hbar), & & \end{array}$$

where the horizontal arrows are the Poincaré duality, the vertical arrows are given by deformation quantization, and the slanted arrows are given by Koszul duality.

The rest of the paper is devoted to the proof of the above theorems. It is organized as follows: in Section 2 we collect several facts on Koszul algebras, and their application to quadratic Poisson polynomials; in Section 3 we first recall the definition of Poisson homology and cohomology, and then prove Theorem 1.1; in Section 4 we study unimodular quadratic Poisson algebras and their Koszul dual, and prove Theorem 1.2; in Section 5 we prove Theorem 1.3 by means of the so-called "differential calculus with duality;" in Section 6 we discuss Calabi–Yau algebras, their Koszul duality and the Batalin–Vilkovisky algebras associated to them; and at last, in Section 7 we discuss the deformation quantization of Poisson algebras and prove Theorems 1.4 and 1.5.

This paper has a sequel [5], where the cyclic homology groups of the Poisson mixed complexes discussed in this paper were studied and the gravity algebra structure on them was discussed.

Convention. Throughout the paper, k is a field of a characteristic zero, which we may assume to be $\mathbb{R}$ as in Section 1. All tensors and morphisms are graded over k unless otherwise specified. All complexes are graded such that the differential has degree -1; for a cochain complex, it is viewed as a chain complex by negating the original grading, and it is cohomology $\mathrm{H}^\bullet(-) := \mathrm{H}_{-\bullet}(-)$.

2. Preliminaries on Koszul algebras

In this section, we collect some necessary facts about Koszul algebras. The interested reader may refer to Loday–Vallette [26, Chapter 3] for some more details.

Let V be a finite-dimensional vector space over k. Denote by TV the free (tensor) algebra generated by V over k. Suppose R is a subspace of $V \otimes V$, and let (R) be the two-sided ideal generated by R in TV, then the quotient algebra $A := TV/(R)$ is called a *quadratic algebra*.

Consider the subspace

$$U = \bigoplus_{n=0}^{\infty} U_n := \bigoplus_{n=0}^{\infty} \bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}$$

of TV, then U is a coalgebra whose coproduct is induced from the de-concatenation of the tensor products. The *Koszul dual coalgebra* of A, denoted by $A^{¡}$, is

$$A^{¡} = \bigoplus_{n=0}^{\infty} \Sigma^{\otimes n}(U_n),$$

where Σ is the degree shifting-up (suspension) functor. $A^{¡}$ has a graded coalgebra structure induced from that of U with

$$(A^{¡})_0 = k, \quad (A^{¡})_1 = \Sigma V, \quad (A^{¡})_2 = (\Sigma \otimes \Sigma)(R), \quad \dots.$$

The *Koszul dual algebra* of A, denoted by $A^!$, is just the linear dual space of $A^{¡}$, which is then a graded algebra. More precisely, let $V^* = \mathrm{Hom}(V, k)$ be the linear dual space of

V, and let $R^{\perp}$ denote the space of annihilators of R in $V^* \otimes V^*$. Shift the grading of V^* down by one, denoted by $\Sigma^{-1}V^*$, then[1]

$$A^! = T(\Sigma^{-1}V^*)/\big(\Sigma^{-1} \otimes \Sigma^{-1} \circ R^{\perp}\big).$$

Choose a set of basis $\{e_i\}$ for V and let $\{e_i^*\}$ be their duals in V^*. There is a chain complex associated to A, called the *Koszul complex*:

$$\cdots \xrightarrow{\delta} A \otimes A^{¡}_{i+1} \xrightarrow{\delta} A \otimes A^{¡}_{i} \xrightarrow{\delta} \cdots \longrightarrow A \otimes A^{¡}_{0} \xrightarrow{\delta} k, \tag{2.1}$$

where for any $r \otimes f \in A \otimes A^{¡}$, $\delta(r \otimes f) = \sum_i e_i r \otimes \Sigma^{-1} e_i^* f$.

Definition 2.1 (Koszul algebra). A quadratic algebra $A = TV/(R)$ is called *Koszul* if the Koszul chain complex (2.1) is acyclic.

Example 2.2 (Polynomials). Let $A = k[x_1, x_2, \ldots, x_n]$ be the space of polynomials (the symmetric tensor algebra) with n generators. Then A is a Koszul algebra and its Koszul dual algebra $A^!$ is the graded symmetric algebra $\mathbf{\Lambda}(\xi_1, \xi_2, \ldots, \xi_n)$, with grading $|\xi_i| = -1$.

Lemma 2.3 (Shoikhet [30]). *Let $A = k[x_1, \ldots, x_n]$ with a bivector π in the form* (1.1). *Then (A, π) is quadratic Poisson if and only if $(A^!, \pi^!)$ is quadratic Poisson, where $\pi^!$ is given by* (1.2).

So far, we have assumed that V is a k-linear space. In Section 7, we will study the deformed algebras, which are algebras over $k[\![\hbar]\!]$. In [30], Shoikhet proved that the definitions and results in the above subsections remain to hold for algebras over a discrete evaluation ring, such as $k[\![\hbar]\!]$. For example, $k[x_1, \ldots, x_n][\![\hbar]\!]$ is Koszul dual to $\mathbf{\Lambda}(\xi_1, \ldots, \xi_n)[\![\hbar]\!]$ as graded algebras over $k[\![\hbar]\!]$ (see [30, Theorem 0.3]).

3. Poisson homology and cohomology

The notions of Poisson homology and cohomology were introduced by Koszul [20] and Lichnerowicz [24], respectively. Later, Huebschmann [16] studied both of them from a purely algebraic perspective.

For a commutative algebra A, in the following we denote by $\Omega^p(A)$ the set of p-th Kähler differential forms of A, and by $\mathfrak{X}^p(A; M)$ the space of skew-symmetric multilinear maps $A^{\otimes p} \to M$ that are derivations in each argument (note that, by our convention, elements in $\Omega^p(A)$ and in $\mathfrak{X}^p(A; M)$ have gradings p and $-p$, respectively). In the following, if $M = A$, we write $\mathfrak{X}^p(A; M)$ simply by $\mathfrak{X}^p(A)$. Note that from the universal property of Kähler differentials, there is an identity of A-modules:

$$\mathfrak{X}^p(A; M) = \mathrm{Hom}_A\big(\Omega^p(A), M\big). \tag{3.1}$$

[1] In the literature such as [26], $A^!$ is defined to be $T(V^*)/R^{\perp}$, or equivalently, $(A^!)_i \cong \Sigma^i\,\mathrm{Hom}((A^{¡})_i, k)$, but not $\mathrm{Hom}((A^{¡})_i, k)$. This will cause some issues in our later calculations, so in this paper, we take $A^!$ as given above, or equivalently $A^! = \mathrm{Hom}(A^{¡}, k)$.

Definition 3.1 (Koszul [20]). Suppose (A, π) is a Poisson algebra. Then the *Poisson chain complex* of A, denoted by $\mathrm{CP}_\bullet(A)$, is

$$\cdots \longrightarrow \Omega^{p+1}(A) \xrightarrow{\partial} \Omega^{p}(A) \xrightarrow{\partial} \Omega^{p-1}(A) \xrightarrow{\partial} \cdots \longrightarrow \Omega^{0}(A) = A, \tag{3.2}$$

where ∂ is given by

$$\begin{aligned}\partial(f_0 df_1 \wedge \cdots \wedge df_p) = &\sum_{i=1}^{p}(-1)^{i-1}\{f_0, f_i\} df_1 \wedge \cdots \widehat{df_i} \cdots \wedge df_p \\ &+ \sum_{1\le i<j\le p} (-1)^{j-i} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \widehat{df_i} \cdots \widehat{df_j} \cdots \wedge df_p.\end{aligned}$$

The associated homology is called the *Poisson homology* of A and is denoted by $\mathrm{HP}_\bullet(A)$.

Definition 3.2 (Lichnerowicz [24]). Suppose (A, π) is a Poisson algebra and M is a left Poisson A-module. The *Poisson cochain complex* of A with values in M, denoted by $\mathrm{CP}^\bullet(A; M)$, is the cochain complex

$$M = \mathfrak{X}^0(A; M) \xrightarrow{\delta} \cdots \longrightarrow \mathfrak{X}^p(A; M) \xrightarrow{\delta} \mathfrak{X}^{p+1}(A; M) \xrightarrow{\delta} \cdots,$$

where δ is given by

$$\begin{aligned}\delta(P)(f_0, f_1, \ldots, f_p) := &\sum_{0\le i\le p} (-1)^i \{f_i, P(f_0, \ldots, \widehat{f_i}, \ldots, f_p)\} \\ &+ \sum_{0\le i<j\le p} (-1)^{i+j} P\big(\{f_i, f_j\}, f_1, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_p\big).\end{aligned}$$

The associated cohomology is called the *Poisson cohomology* of A with values in M and is denoted by $\mathrm{HP}^\bullet(A; M)$. In particular, if $M = A$, then the cochain complex is denoted by $\mathrm{CP}^\bullet(A)$, and the cohomology is called the *Poisson cohomology* of A and is denoted by $\mathrm{HP}^\bullet(A)$.

Note that in the above definition, the Poisson cochain complex, viewed as a chain complex, is negatively graded, and the coboundary δ has degree -1. However, by our convention, the Poisson cohomology is positively graded.

Remark 3.3 (The graded case). The Poisson homology and cohomology can be defined for *graded* Poisson algebras as well. In this case,

$$\Omega^p(A) = \bigoplus_{n\in\mathbb{Z}} \big\{f_0 df_1 \wedge \cdots \wedge df_n \mid f_i \in A,\ |f_0| + |f_1| + \cdots + |f_n| + n = p\big\}$$

and $\mathfrak{X}^p(A; M)$ is again given by $\mathrm{Hom}_A(\Omega^p(A), M)$. The boundary maps are completely analogous to those of Poisson chain and cochain complexes (with Koszul's sign convention taken into account).

Proof of Theorem 1.1. (1) We first show the first isomorphism in (1.3). Since $A = k[x_1, \ldots, x_n]$, we have an explicit expression for $\Omega^\bullet(A)$, which is

$$\Omega^\bullet(A) = \mathbf{\Lambda}(x_1, \ldots, x_n, dx_1, \ldots, dx_n), \tag{3.3}$$

where $\mathbf{\Lambda}$ means the graded symmetric tensor product, $|x_i| = 0$, and $|dx_i| = 1$, for $i = 1, \ldots, n$. Similarly,

$$\Omega^\bullet(A^!) = \mathbf{\Lambda}(\xi_1, \ldots, \xi_n, d\xi_1, \ldots, d\xi_n),$$

where $|\xi_i| = -1$ and $|d\xi_i| = 0$, for $i = 1, \ldots, n$, and therefore

$$\begin{aligned}
&\mathfrak{X}^\bullet(A^!; A^¡)\\
&= \mathrm{Hom}_{A^!}\left(\Omega^\bullet(A^!), A^¡\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\ldots,\xi_n)}\left(\mathbf{\Lambda}(\xi_1, \ldots, \xi_n, d\xi_1, \ldots, d\xi_n), \mathrm{Hom}\left(\mathbf{\Lambda}(\xi_1, \ldots, \xi_n), k\right)\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\ldots,\xi_n)}\left(\mathbf{\Lambda}(\xi_1, \ldots, \xi_n) \otimes \mathbf{\Lambda}(d\xi_1, \ldots, d\xi_n), \mathrm{Hom}\left(\mathbf{\Lambda}(\xi_1, \ldots, \xi_n), k\right)\right)\\
&= \mathrm{Hom}\left(\mathbf{\Lambda}(d\xi_1, \ldots, d\xi_n), \mathrm{Hom}\left(\mathbf{\Lambda}(\xi_1, \ldots, \xi_n), k\right)\right)\\
&= \mathrm{Hom}\left(\mathbf{\Lambda}(d\xi_1, \ldots, d\xi_n) \otimes \mathbf{\Lambda}(\xi_1, \ldots, \xi_n), k\right)\\
&= \mathrm{Hom}\left(\mathbf{\Lambda}(d\xi_1, \ldots, d\xi_n, \xi_1, \ldots, \xi_n), k\right)\\
&= \boldsymbol{\lambda}\left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}, \xi_1^*, \ldots, \xi_n^*\right).
\end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4), there is a canonical grading preserving an isomorphism of vector spaces:

$$\begin{aligned}
\Phi : \Omega^\bullet(A) &\to \mathfrak{X}^\bullet(A^!; A^¡),\\
x_i &\mapsto \frac{\partial}{\partial \xi_i},\\
dx_i &\mapsto \xi_i^*, \quad i = 1, \ldots, n.
\end{aligned} \tag{3.5}$$

It is a direct check that Φ is a chain map, and thus we obtain an isomorphism of Poisson complexes:

$$\Phi : \mathrm{CP}_\bullet(A) \cong \mathrm{CP}^{-\bullet}(A^!; A^¡), \tag{3.6}$$

which then induces an isomorphism on the homology.

(2) We now show the second isomorphism in (1.3). Similar to the above argument, we have

$$\begin{aligned}
\mathrm{CP}^\bullet(A) &= \mathrm{Hom}_A\left(\Omega^\bullet(A), A\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(x_1,\ldots,x_n)}\left(\mathbf{\Lambda}(x_1, \ldots, x_n, dx_1, \ldots, dx_n), \mathbf{\Lambda}(x_1, \ldots, x_n)\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(x_1,\ldots,x_n)}\left(\mathbf{\Lambda}(x_1, \ldots, x_n) \otimes \mathbf{\Lambda}(dx_1, \ldots, dx_n), \mathbf{\Lambda}(x_1, \ldots, x_n)\right)\\
&= \mathrm{Hom}\left(\mathbf{\Lambda}(dx_1, \ldots, dx_n), \mathbf{\Lambda}(x_1, \ldots, x_n)\right)\\
&= \mathbf{\Lambda}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \otimes \mathbf{\Lambda}(x_1, \ldots, x_n)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\mathrm{CP}^\bullet(A^!) &= \mathrm{Hom}_{A^!}\left(\Omega^\bullet(A^!), A^!\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\dots,\xi_n)}\left(\mathbf{\Lambda}(\xi_1,\dots,\xi_n, d\xi_1,\dots,d\xi_n), \mathbf{\Lambda}(\xi_1,\dots,\xi_n)\right)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\dots,\xi_n)}\left(\mathbf{\Lambda}(\xi_1,\dots,\xi_n)\otimes\mathbf{\Lambda}(d\xi_1,\dots,d\xi_n), \mathbf{\Lambda}(\xi_1,\dots,\xi_n)\right)\\
&= \mathrm{Hom}\left(\mathbf{\Lambda}(d\xi_1,\dots,d\xi_n), \mathbf{\Lambda}(\xi_1,\dots,\xi_n)\right)\\
&= \boldsymbol{\lambda}\left(\frac{\partial}{\partial\xi_1},\dots,\frac{\partial}{\partial\xi_n}\right)\otimes\mathbf{\Lambda}(\xi_1,\dots,\xi_n).
\end{aligned}\tag{3.8}$$

Under the identity

$$x_i \mapsto \frac{\partial}{\partial\xi_i}, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i, \tag{3.9}$$

we again obtain an isomorphism of chain complexes:

$$\Psi : \mathrm{CP}^\bullet(A) \cong \mathrm{CP}^\bullet(A^!).$$

This completes the proof. ∎

4. Unimodular Poisson algebras and Koszul duality

In this section, we study *unimodular* Poisson algebras. We are particularly interested in the algebraic structures on their Poisson cohomology and homology groups, which are summarized by a *differential calculus*, a notion introduced by Tamarkin and Tsygan in [31].

Definition 4.1 (Differential calculus; Tamarkin–Tsygan [31]). Let $\mathrm{H}^\bullet$ and $\mathrm{H}_\bullet$ be graded vector spaces. A *differential calculus* is the sextuple

$$\left(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d\right)$$

satisfying the following conditions:

(1) $(\mathrm{H}^\bullet, \cup, [-,-])$ is a Gerstenhaber algebra; that is, $(\mathrm{H}^\bullet, \cup)$ is a graded commutative algebra, $(\mathrm{H}^\bullet, [-,-])$ is a degree 1 or -1 graded Lie algebra, and the product and Lie bracket are compatible in the following sense:

$$[P\cup Q, R] = P\cup[Q,R] + (-1)^{pq} Q\cup[P,R],$$

for homogeneous $P, Q, R \in V$ of degree p, q, r, respectively;

(2) $\mathrm{H}_\bullet$ is a graded (left) module over $(\mathrm{H}^\bullet, \cup)$ via the map

$$\iota : \mathrm{H}^n \otimes \mathrm{H}_m \to \mathrm{H}_{m-n}, \quad f\otimes\alpha \mapsto \iota_f\alpha,$$

for any $f \in \mathrm{H}^n$ and $\alpha \in \mathrm{H}_m$;

(3) there is a map $d : \mathrm{H}_\bullet \to \mathrm{H}_{\bullet+1}$ satisfying $d^2 = 0$. Moreover, if we set $L_f := [d, \iota_f] = d\iota_f - (-1)^{|f|}\iota_f d$, then

$$(-1)^{|f|+1}\iota_{[f,g]} = [L_f, \iota_g] := L_f \iota_g - (-1)^{|g|(|f|+1)}\iota_g L_f.$$

In the following, if $\cup$, ι, $[-,-]$ and d are clear from the context, we will simply write a differential calculus by $(\mathrm{H}^\bullet, \mathrm{H}_\bullet)$ for short.

4.1. Differential calculus on Poisson (co)homology

Suppose A is a commutative algebra. Besides the de Rham differential on $\Omega^\bullet(A)$, we have the following operations on $\mathfrak{X}^\bullet(A)$ and $\Omega^\bullet(A)$.

(1) Wedge (cup) product: suppose $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, then the *wedge product* of P and Q, denoted by $P \cup Q$, is a polyvector in $\mathfrak{X}^{p+q}(A)$ defined by

$$\begin{aligned}(P \cup Q)&(f_1, f_2, \ldots, f_{p+q})\\ &:= \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}) \cdot Q(f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q)}),\end{aligned}$$

where σ runs over all (p, q)-shuffles of $(1, 2, \ldots, p+q)$.

(2) Schouten bracket: suppose $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, then their *Schouten bracket*, denoted by $[P, Q]$, is an element in $\mathfrak{X}^{p+q-1}(A)$ given by

$$\begin{aligned}[P, Q]&(f_1, f_2, \ldots, f_{p+q-1})\\ &:= \sum_{\sigma \in S_{q,p-1}} \mathrm{sgn}(\sigma) P\big(Q(f_{\sigma(1)}, \ldots, f_{\sigma(q)}), f_{\sigma(q+1)}, \ldots, f_{\sigma(q+p-1)}\big)\\ &\quad - (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p,q-1}} \mathrm{sgn}(\sigma) Q\big(P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}),\\ &\qquad\qquad f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q-1)}\big).\end{aligned}$$

(3) Contraction (inner product): suppose $P \in \mathfrak{X}^p(A)$ and $\omega = df_1 \wedge \cdots \wedge df_n \in \Omega^n(A)$, then the *contraction* of P with ω, denoted by $\iota_P(\omega)$, is an A-linear map with values in $\Omega^{n-p}(A)$ given by

$$\iota_P(\omega) = \begin{cases} \displaystyle\sum_{\sigma \in S_{p,n-p}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}) df_{\sigma(p+1)} \wedge \cdots \wedge df_{\sigma(n)}, & \text{if } n \geq p,\\ 0, & \text{otherwise.}\end{cases}$$

(4) Lie derivative: the *Lie derivative* is given by the Cartan formula, namely, for $P \in \mathfrak{X}^p(A)$ and $\omega \in \Omega^n(A)$, the Lie derivative of ω with respect to P is given by

$$L_P\omega := [\iota_P, d] = \iota_P(d\omega) - (-1)^p d(\iota_P\omega),$$

where d is the de Rham differential.

Theorem 4.2. *Suppose A is a Poisson algebra. Then*

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A), \cup, \iota, [-,-], d\big),$$

where d is the de Rham differential, is a differential calculus.

Proof. We only have to show that the operations listed above respect the Poisson boundary and coboundary. It is a direct check and can be found in [23, Chapter 3]. ∎

In the following, we will give another differential calculus structure for a Poisson algebra, which will be used later.

(1) For any $P \in \mathfrak{X}^p(A)$ and $\phi \in \mathfrak{X}^q(A; A^*)$, let $\iota_P^*(\phi) \in \mathfrak{X}^{p+q}(A; A^*)$ be given by

$$\begin{aligned}(\iota_P^*\phi)&(f_1, \dots, f_{p+q})\\ &:= \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \dots, f_{\sigma(p)}) \cdot \phi(f_{\sigma(p+1)}, \dots, f_{\sigma(p+q)}).\end{aligned} \tag{4.1}$$

It is clear that ι^* is associative, i.e., $\iota_Q^* \circ \iota_P^* = \iota_{P\cup Q}^*$. Also, ι^* respects the Poisson coboundary maps, which is completely analogous to the proof of that $\cup$ commutes with the Poisson coboundary map (cf. [23, §4.3]).

(2) Observe that

$$\begin{aligned}\mathfrak{X}^\bullet(A; A^*) &= \mathrm{Hom}_A\big(\Omega^\bullet(A), A^*\big)\\ &= \mathrm{Hom}_A\big(\Omega^\bullet(A), \mathrm{Hom}(A, k)\big)\\ &= \mathrm{Hom}_A\big(\Omega^\bullet(A) \otimes A, k\big)\\ &= \mathrm{Hom}\big(\Omega^\bullet(A), k\big).\end{aligned} \tag{4.2}$$

By dualizing the de Rham differential d on $\Omega^\bullet(A)$, we obtain a differential d^* on $\mathrm{Hom}(\Omega^\bullet(A), k)$, i.e., on $\mathfrak{X}^\bullet(A; A^*)$. It is proved in [42, Theorem 4.10] that d^* commutes with the Poisson boundary.

(3) For any $P \in \mathfrak{X}^\bullet(A)$ and $\omega \in \mathfrak{X}^\bullet(A; A^*)$, let $L_P\omega := [\iota_P^*, d^*](\omega)$; it is a direct check that

$$[L_P, \iota_Q^*] = \iota_{[P,Q]}^*.$$

By (1)–(3) listed above, we obtain the following.

Theorem 4.3. *Suppose A is a Poisson algebra and let A^* be its dual space. Then*

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}^\bullet(A; A^*), \cup, \iota^*, [-,-], d^*\big)$$

is a differential calculus.

We next introduce two DG Lie algebras associated to the above two differential calculi. Let us start with the notion of negative cyclic homology.

Definition 4.4 (Cyclic homology; cf. Jones [17] and Kassel [18]). Suppose $(\mathrm{C}_\bullet, b, B)$ is a mixed complex, with $|b| = -1$ and $|B| = 1$. Let u be a free variable of degree -2 which commutes with b and B. The *negative cyclic chain complex* of $\mathrm{C}_\bullet$ is

$$\big(\mathrm{C}_\bullet[\![u]\!], b + uB\big),$$

and is denoted by $\mathrm{CC}^-_\bullet(\mathrm{C}_\bullet)$. The associated homology is called the *negative cyclic homology* of $\mathrm{C}_\bullet$ and is denoted by $\mathrm{HC}^-_\bullet(\mathrm{C}_\bullet)$.

Remark 4.5 (Cyclic cohomology). Suppose $(\mathrm{C}^\bullet, b, B)$ is a mixed cochain complex, namely, $|b| = 1$ and $|B| = -1$. By negating the degrees of $\mathrm{C}^\bullet$, we obtain a mixed chain complex, denoted by $(\mathrm{C}_\bullet, b, B)$ with $|b| = -1$ and $|B| = 1$. By our convention, the *cyclic cohomology* of $(\mathrm{C}^\bullet, b, B)$, denoted by $\mathrm{HC}^\bullet(\mathrm{C}^\bullet)$, is the *cohomology* of the negative cyclic complex of $(\mathrm{C}_\bullet, b, B)$.

Consider the mixed complex $\Omega^\bullet(A)$ with differential $(0, d)$, where d is the de Rham differential. Equip $\mathfrak{X}^\bullet(A)$ with trivial differential. Since $\Omega^\bullet(A)$ is a Lie module over $\mathfrak{X}^\bullet(A)$ whose action commutes with d, the negative cyclic complex $(\Omega^\bullet(A)[\![u]\!], ud)$ is a DG module over $\mathfrak{X}^\bullet(A)$. Consider the semi-direct product

$$\mathfrak{P}(A)^\# := \Sigma\mathfrak{X}^\bullet(A) \ltimes \Sigma^{-1-n}\Omega^\bullet(A)[\![u]\!], \tag{4.3}$$

where n is an arbitrary integer number. It is a DG Lie algebra with differential $(0, ud)$.

Similarly, for the mixed complex $(\mathfrak{X}(A^*), 0, d^*)$, we have the DG Lie algebra

$$\mathfrak{P}^\circ(A)^\# := \Sigma\mathfrak{X}^\bullet(A) \ltimes \Sigma^{-1-n}\mathfrak{X}^\bullet(A; A^*)[\![u]\!], \tag{4.4}$$

with a differential given by $(0, ud^*)$.

4.2. Unimodular Poisson algebras

Suppose A is a commutative algebra and $\eta \in \Omega^n(A)$. We say η is a volume form if $\mathfrak{X}^\bullet(A) \xrightarrow{\iota_{(-)}\eta} \Omega^{n-\bullet}(A)$ is an isomorphism of vector spaces. Now suppose A is Poisson, then we have the diagram

$$\begin{array}{ccc} \mathfrak{X}^\bullet(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n-\bullet}(A) \\ \uparrow \delta & & \uparrow \partial \\ \mathfrak{X}^{\bullet-1}(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n-\bullet+1}(A), \end{array} \tag{4.5}$$

which may not be commutative, i.e., η may not be a Poisson cycle. We say A is *unimodular* if there exists a volume form η such that (4.5) commutes.

In terms of the DG Lie algebra (4.3), being unimodular is equivalent to the following.

Proposition 4.6. *Let A, π, and η be as above. Then the bivector π is unimodular Poisson if and only if $(\Sigma\pi, \Sigma^{-1-n}\eta)$ is a Maurer–Cartan element of the DG Lie algebra (4.3).*

The proof is a direct check, and we leave it to the interested reader. Recall that for a DG Lie algebra (L, d), any Maurer–Cartan element, say $a \in L$, gives a new DG Lie algebra structure on L with differential $\tilde{d} = d + [a, -]$. Denote this DG Lie algebra by L_a. Going back to the above proposition, in the following we write

$$\mathfrak{P}(A, \eta) := \mathfrak{P}(A)^{\#}_{(\Sigma\pi, \Sigma^{-1-n}\eta)},$$

which will be used later in Section 7.

The following is also immediate from (4.5).

Theorem 4.7 (Xu [40]). *Suppose A is a unimodular Poisson algebra with the volume form of degree n. Then* $(\mathrm{HP}^\bullet(A), \mathrm{HP}_{n-\bullet}(A))$ *forms a differential calculus with duality, and therefore there exists an isomorphism (the Poincaré duality)*

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}_{n-\bullet}(A).$$

4.3. Unimodular Frobenius Poisson algebras

Now, we go to unimodular Frobenius Poisson algebras, a notion introduced by Zhu, Van Oystaeyen, and Zhang in [42].

Suppose $A^!$ is a finite-dimensional graded not-necessarily commutative algebra. $A^!$ is called *symmetric Frobenius* if it is equipped with a bilinear, non-degenerate symmetric pairing

$$\langle -, - \rangle : A^! \otimes A^! \to k$$

of degree n which is cyclically invariant, that is, $\langle a, b \cdot c\rangle = (-1)^{(|a|+|b|)|c|}\langle c, a \cdot b\rangle$, for all homogeneous $a, b, c \in A^!$. This is equivalent to saying that there is an $A^!$-bimodule isomorphism

$$\eta^! : (A^!)^\bullet \to (A^{¡})_{n+\bullet}, \quad \text{for some } n \in \mathbb{N},$$

where $A^{¡} = (A^!)^*$. In this case, we may view $\eta^!$ as an element in $\mathrm{Hom}_{A^!}(A^!, A^{¡}) \subset \mathfrak{X}^\bullet(A^!; A^{¡})$. Now assume $A^!$ is Poisson, then we have a diagram

$$\begin{array}{ccc} \mathfrak{X}^\bullet(A^!) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^{\bullet-n}(A^!; A^{¡}) \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ \mathfrak{X}^{\bullet-1}(A^!) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^{\bullet-1-n}(A^!; A^{¡}). \end{array} \tag{4.6}$$

According to Zhu–Van Oystaeyen–Zhang [42], if there exists $\eta^! \in \mathfrak{X}^\bullet(A^!; A^{¡})$ such that $\iota^*_{(-)}\eta^!$ is an isomorphism, then $\eta^!$ is called a *volume form*, and if furthermore the diagram (4.6) commutes, then $A^!$ is called a *unimodular Frobenius Poisson algebra* of degree n (in [42] the authors call it *unimodular Frobenius Poisson*). From the definition, we immediately arrive at the following theorem.

Theorem 4.8 (Zhu–Van Oystaeyen–Zhang [42]). *Suppose $A^!$ is a unimodular Frobenius Poisson algebra with the volume form of degree n. Then* $(\mathrm{HP}^\bullet(A^!), \mathrm{HP}^{\bullet-n}(A^!; A^{¡}))$ *forms* *a differential calculus with duality and therefore there exists an isomorphism*

$$\mathrm{HP}^\bullet(A^!) \cong \mathrm{HP}^{\bullet-n}(A^!; A^¡).$$

In this paper, since we are interested in $A = k[x_1, \ldots, x_n]$ or $A^! = \mathbf{\Lambda}(\xi_1, \ldots, \xi_n)$, we always assume that the volume form is constant. The following is completely analogous to Proposition 4.6.

Proposition 4.9. *Suppose $A^! = \mathbf{\Lambda}(\xi_1, \ldots, \xi_n)$ with volume form $\eta^!$. Then a bivector $\pi^!$ is unimodular Frobenius Poisson if and only if $(\Sigma\pi^!, \Sigma^{-1-n}\eta^!)$ is a Maurer–Cartan element of the DG Lie algebra $\mathfrak{P}^\circ(A^!)^\#$ given by (4.4).*

In Section 7, we shall use the DG Lie algebra

$$\mathfrak{P}^\circ(A^!, \eta^!) := \mathfrak{P}^\circ(A^!)^\#_{(\Sigma\pi^!, \Sigma^{-1-n}\eta^!)}.$$

Proof of Theorem 1.2. First, we show that a quadratic Poisson algebra $(A = k[x_1, \ldots, x_n], \pi)$ is unimodular if and only if $(A^!, \pi^!)$ is unimodular Frobenius. In fact, recall that for $A = k[x_1, \ldots, x_n]$,

$$\begin{aligned}
\mathfrak{X}^\bullet(A) &= \mathbf{\Lambda}\left(x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),\\
\Omega^\bullet(A) &= \mathbf{\Lambda}(x_1, \ldots, x_n, dx_1, \ldots, dx_n),\\
\mathfrak{X}^\bullet(A^!) &= \mathbf{\Lambda}\left(\xi_1, \ldots, \xi_n, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right),\\
\mathfrak{X}^\bullet(A^!; A^¡) &= \mathbf{\Lambda}\left(\xi_1^*, \ldots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right).
\end{aligned}$$

Let

$$\eta = dx_1 dx_2 \cdots dx_n \quad \text{and} \quad \eta^! = \xi_1^*\xi_2^* \cdots \xi_n^*,$$

where $\eta^!$ is understood as a contraction, namely,

$$\eta^!(\xi_{i_1} \cdots \xi_{i_p}) := \sum_{\sigma \in S_{p,n-p}} \langle \xi_{i_1} \cdots \xi_{i_p}, \xi^*_{\sigma(1)} \cdots \xi^*_{\sigma(p)} \rangle \cdot \xi^*_{\sigma(p+1)} \cdots \xi^*_{\sigma(n)}.$$

Then under the identification

$$x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad dx_i \mapsto \xi_i^*, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i \tag{4.7}$$

the diagram

$$\begin{array}{ccc}
\mathfrak{X}^\bullet(A) = \mathbf{\Lambda}\left(x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) & \xrightarrow{\iota_{(-)}\eta} & \Omega^\bullet(A) = \mathbf{\Lambda}(x_1, \ldots, x_n, dx_1, \ldots, dx_n) \\
\downarrow \cong & & \downarrow \cong \\
\mathfrak{X}^\bullet(A^!) = \mathbf{\Lambda}\left(\xi_1, \ldots, \xi_n, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^\bullet(A^!; A^¡) = \mathbf{\Lambda}\left(\xi_1^*, \ldots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right)
\end{array} \tag{4.8}$$

commutes. This means η is a Poisson cycle for A if and only if $\eta^!$ is a Poisson cocycle for $A^!$, which proves the claim.

Second, for A as above, we show that the diagram

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HP}^\bullet(A^!) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^!; A^{¡}) \end{array} \tag{4.9}$$

commutes. In fact, the two vertical isomorphisms are given by Theorem 1.1, and the two horizontal isomorphisms are given by Theorems 4.7 and 4.8, respectively. The commutativity of the diagram (4.9) follows from the chain level commutative diagram (4.8). ■

Remark 4.10. By the same identification (4.7), one immediately sees that for a quadratic Poisson algebra A and its Koszul dual $A^!$, the two DG Lie algebras given by (4.3) and (4.4) are isomorphic.

5. Poisson cohomology and the Batalin–Vilkovisky algebra

The purpose of this section is to show that for unimodular quadratic Poisson polynomial algebras, the horizontal isomorphisms in (4.9) naturally induce on $\mathrm{HP}^\bullet(A)$ and $\mathrm{HP}^\bullet(A^!)$ a Batalin–Vilkovisky algebra structure, and the vertical isomorphisms in (4.9) are isomorphisms of Batalin–Vilkovisky algebras. We start with the notion of *differential calculus with duality*.

Definition 5.1 (Lambre [21]). A differential calculus $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d)$ is called a *differential calculus with duality* if there exists an integer n and an element $\eta \in \mathrm{H}_n$ such that

(a) $\iota_1\eta = \eta$, where $1 \in \mathrm{H}^0$ is the unit, $d(\eta) = 0$, and

(b) for any $i \in \mathbb{Z}$,

$$\mathrm{PD}(-) := \iota_{(-)}\eta : \mathrm{H}^i \to \mathrm{H}_{n-i} \tag{5.1}$$

is an isomorphism.

Such isomorphism PD is called the *van den Bergh duality* (also called *the noncommutative Poincaré duality*), and η is called the *volume form*.

Definition 5.2 (Batalin–Vilkovisky algebra). Suppose $(V, \bullet)$ is a graded commutative algebra. A *Batalin–Vilkovisky algebra* structure on V is the triple $(V, \bullet, \Delta)$ such that

(1) $\Delta : V^i \to V^{i-1}$ is a differential, that is, $\Delta^2 = 0$, and

(2) Δ is a second-order operator, that is,

$$\begin{aligned}\Delta(a \bullet b \bullet c) = \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{(|a|-1)|b|} b \bullet \Delta(a \bullet c) \\ - (\Delta a) \bullet b \bullet c - (-1)^{|a|} a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c).\end{aligned}$$

Equivalently, if we define the bracket

$$[a,b] := (-1)^{|a|+1}\big(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)\big),$$

then $[-,-]$ is a derivation with respect to $\bullet$ for each component. In other words, a Batalin–Vilkovisky algebra is a Gerstenhaber algebra $(V, \bullet, [-,-])$ with a differential $\Delta : V^i \to V^{i-1}$ such that

$$[a,b] = (-1)^{|a|+1}\big(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)\big), \tag{5.2}$$

for any $a, b \in V$ (cf. [14, Proposition 1.2]). Δ is also called the Batalin–Vilkovisky operator or the generator (of the Gerstenhaber bracket).

Now suppose $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d, \eta)$ is a differential calculus with duality. Let $\Delta : \mathrm{H}^\bullet \to \mathrm{H}^{\bullet-1}$ be the linear operator such that

$$\begin{array}{ccc} \mathrm{H}^\bullet & \xrightarrow{\Delta} & \mathrm{H}^{\bullet-1} \\ \downarrow{\scriptstyle \mathrm{PD}} & & \downarrow{\scriptstyle \mathrm{PD}} \\ \mathrm{H}_{n-\bullet} & \xrightarrow{d} & \mathrm{H}_{n-\bullet+1} \end{array} \tag{5.3}$$

commutes. Then we have the following theorem.

Theorem 5.3 (Lambre [21]). *Let* $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d, \eta)$ *be a differential calculus with duality. Then the triple* $(\mathrm{H}^\bullet, \cup, \Delta)$ *is a Batalin–Vilkovisky algebra.*

The proof can be found in the work of Lambre [21, Théorème 1.6]; however, since some details in loc. cit. are omitted, we give a proof here for completeness.

Proof. Since $(\mathrm{H}^\bullet, \cup, [-,-])$ is a Gerstenhaber algebra, we only need to show that the Gerstenhaber bracket is compatible with the operator Δ in (5.3); that is, equation (5.2) holds. For any homogeneous elements $f, g \in \mathrm{H}^\bullet$, by the definition of Poincaré duality PD (5.1) and the Cartan formulae (Lemma 6.3), we have

$$\begin{aligned}
&(-1)^{|f|+1}\mathrm{PD}\big([f,g]\big)\\
&= (-1)^{|f|+1}\iota_{[f,g]}(\eta) = [L_f, \iota_g](\eta) = L_f \iota_g(\eta) - (-1)^{|g|(|f|+1)}\iota_g L_f(\eta)\\
&= d\iota_f\iota_g(\eta) - (-1)^{|f|}\iota_f d\iota_g(\eta) - (-1)^{|g|(|f|+1)}\iota_g d\iota_f(\eta) + (-1)^{|g|(|f|+1)+|f|}\iota_g\iota_f d(\eta)\\
&= d \circ \mathrm{PD}(f \cup g) - (-1)^{|g|(|f|+1)}\iota_g d \circ \mathrm{PD}(f) - (-1)^{|f|}\iota_f d \circ \mathrm{PD}(g)\\
&= \mathrm{PD}\left(\Delta(f \cup g)\right) - (-1)^{|g|(|f|+1)}\iota_g\, \mathrm{PD}\left(\Delta(f)\right) - (-1)^{|f|}\iota_f\, \mathrm{PD}\left(\Delta(g)\right)\\
&= \iota_{\Delta(f\cup g)}(\eta) - (-1)^{|g|(|f|+1)}\iota_g\iota_{\Delta(f)}(\eta) - (-1)^{|f|}\iota_f\iota_{\Delta(g)}(\eta)\\
&= \big(\iota_{\Delta(f\cup g)} - (-1)^{|g|(|f|+1)}\iota_{g\cup\Delta(f)} - (-1)^{|f|}\iota_{f\cup\Delta(g)}\big)(\eta)\\
&= \mathrm{PD}\left(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)\right).
\end{aligned}$$

Since PD is an isomorphism, we thus have

$$[f,g] = (-1)^{|f|+1}\big(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)\big). \qquad \blacksquare$$

Corollary 5.4 (see also Xu [40] and Zhu–Van Oystaeyen–Zhang [42]). *Suppose A is a unimodular Poisson or a unimodular Frobenius Poisson algebra. Then* $\mathrm{HP}^\bullet(A)$ *admits a Batalin–Vilkovisky algebra structure.*

Proof. If A is unimodular Poisson, then Theorems 4.2 and 4.7 imply that the pair $(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A))$ is, in fact, a differential calculus with duality; similarly, if A is unimodular Frobenius Poisson, then Theorems 4.3 and 4.8 imply that the pair $(\mathrm{HP}^\bullet(A), \mathrm{HP}^\bullet(A; A^*))$ is a differential calculus with duality. The theorem then follows from Theorem 5.3. ∎

Proof of Theorem 1.3. Note that in Theorem 1.2, the right vertical isomorphism preserves the Kähler differential as well as the volume form, that is, the two differential calculi with duality

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A)\big) \quad \text{and} \quad \big(\mathrm{HP}^\bullet(A^!), \mathrm{HP}^\bullet(A^!; A^¡)\big)$$

are isomorphic. Combining with Corollary 5.4, the theorem follows. ∎

Remark 5.5. Not all quadratic Poisson algebras are unimodular. For example, for $A = \mathbb{R}[x_1, x_2, x_3]$, Etingof–Ginzburg [11, Lemma 4.2.3 and Corollary 4.3.2] showed that any unimodular Poisson structure is of the form

$$\{x, y\} = \frac{\partial\phi}{\partial z}, \quad \{y, z\} = \frac{\partial\phi}{\partial x}, \quad \{z, x\} = \frac{\partial\phi}{\partial y},$$

for some $\phi \in A$ (taking ϕ to be cubic then the Poisson structure is quadratic); for $A = \mathbb{C}[x_1, x_2, x_3, x_4]$, Pym [28, §3] showed that any unimodular quadratic Poisson bracket on A may be written uniquely in the form

$$\{f, g\} := \frac{df \wedge dg \wedge d\alpha}{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}, \quad f, g \in A,$$

where $\alpha = \sum_{i=1}^4 \alpha_i dx_i \in \Omega^1(A)$ such that $\alpha \wedge d\alpha = 0$, and α_i's are homogeneous cubic polynomials satisfying $\sum_{i=1}^4 x_i\alpha_i = 0$.

6. Calabi–Yau algebras

At the end of Section 1, we sketched some analogy between unimodular Poisson algebras and Calabi–Yau algebras. In the following two sections, we study their relationships in more detail.

6.1. Calabi–Yau algebras and the Batalin–Vilkovisky algebra structure

Definition 6.1 (Calabi–Yau algebra; Ginzburg [15]). Let A be an associative algebra over k. A is called a *Calabi–Yau algebra of dimension n* if

(1) A is homologically smooth, that is, A, viewed as an A^e-module, has a bounded resolution of finitely generated projective A^e-modules, and

(2) there is an isomorphism

$$\mathrm{RHom}_{A^e}(A, A\otimes A) \cong \Sigma^{-n} A \tag{6.1}$$

in the derived category $D(A^e)$ of A^e-modules.

In the above definition, A^e is the enveloping algebra of A, namely, $A^e := A\otimes A^{\mathrm{op}}$. There are a lot of examples of Calabi–Yau algebras, such as the universal enveloping algebra of semi-simple Lie algebras, the skew-product of complex polynomials with a finite subgroup of $\mathrm{SL}(n,\mathbb{R})$, the Yang–Mills algebras, etc.

We next study van den Bergh's noncommutative Poincaré duality for Calabi–Yau algebras [36]. To this end, we first recall the differential calculus structure for associative algebras.

For a unital associative algebra A, let $\bar{A} = A/k$ be its augmentation, and $A\to\bar{A} : a\mapsto\bar{a}$ be the projection. Denote by $(\bar{\mathrm{C}}^\bullet(A),\delta)$ and $(\bar{\mathrm{C}}_\bullet(A), b)$ the reduced Hochschild cochain and chain complexes of A (the reader may refer to Loday [25] for notations). Recall that the *Gerstenhaber cup product* and the *Gerstenhaber bracket* on $\bar{\mathrm{C}}^\bullet(A)$ are given as follows: for any $f\in\bar{\mathrm{C}}^n(A)$ and $g\in\bar{\mathrm{C}}^m(A)$,

$$f\cup g(\bar{a}_1,\ldots,\bar{a}_{n+m}) := (-1)^{nm} f(\bar{a}_1,\ldots,\bar{a}_n)g(\bar{a}_{n+1},\ldots,\bar{a}_{n+m})$$

and

$$\{f,g\} := f\circ g - (-1)^{(|f|+1)(|g|+1)} g\circ f,$$

where

$$\begin{aligned} &f\circ g(\bar{a}_1,\ldots,\bar{a}_{n+m-1})\\ &:= \sum_{i=0}^{n-1}(-1)^{(|g|+1)i} f\big(\bar{a}_1,\ldots,\bar{a}_i,\overline{g(\bar{a}_{i+1},\ldots,\bar{a}_{i+m})},\bar{a}_{i+m+1},\ldots,\bar{a}_{n+m-1}\big).\end{aligned}$$

Gerstenhaber proved in [13, Theorems 3–5] that $\cup$ and $\{-,-\}$ are well-defined on the cohomology level, and moreover, $\cup$ is graded commutative. Therefore, we obtain on the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ a Gerstenhaber algebra structure.

Next, we consider the action of the Hochschild cochain complex on the Hochschild chain complex. Given any homogeneous elements $f\in\bar{\mathrm{C}}^n(A)$ and $\alpha = (a_0,\bar{a}_1,\ldots,\bar{a}_m)\in\bar{\mathrm{C}}_m(A)$,

(1) the *cap product* $\cap : \bar{\mathrm{C}}^n(A)\times\bar{\mathrm{C}}_m(A)\to\bar{\mathrm{C}}_{m-n}(A)$ is given by

$$f\cap\alpha := \begin{cases}\big(a_0 f(\bar{a}_1,\ldots,\bar{a}_n),\bar{a}_{n+1},\ldots,\bar{a}_m\big), & \text{if } m\geq n,\\ 0, & \text{otherwise.}\end{cases} \tag{6.2}$$

If we denote by $\iota_f(-) := f\cap -$ the contraction operator, then

$$\iota_f\iota_g = (-1)^{|f||g|}\iota_{g\cup f} = \iota_{f\cup g};$$

(2) the *Lie derivative* $L : \bar{C}^n(A) \times \bar{C}_m(A) \to \bar{C}_{m-n}(A)$ is given as follows: for any $\alpha = (a_0, \bar{a}_1, \ldots, \bar{a}_m) \in \bar{C}_m(A)$, if $n \le m+1$, then

$$L_f(\alpha) := \sum_{i=0}^{m-n} (-1)^{(n+1)i} \big(a_0, \bar{a}_1 \ldots, \bar{a}_i, \overline{f(\bar{a}_{i+1}, \ldots, \bar{a}_{i+n})}, \ldots, \bar{a}_m\big) + \sum_{i=m-n+1}^{m} (-1)^{m(i+1)+n+1} \big(f(\bar{a}_{i+1}, \ldots, \bar{a}_m, \bar{a}_0, \ldots, \bar{a}_{n-m+i-1}), \bar{a}_{n-m+i}, \ldots, \bar{a}_i\big),$$

where the second sum is taken over all cyclic permutations such that a_0 is inside of f, and otherwise if $n > m+1$, $L_f(\alpha) = 0$;

(3) the *Connes operator* $B : \bar{C}_\bullet(A) \to \bar{C}_{\bullet+1}(A)$ is given by

$$B(\alpha) := \sum_{i=0}^{m} (-1)^{mi} (1, \bar{a}_i, \ldots, \bar{a}_m, \bar{a}_0, \ldots, \bar{a}_{i-1}).$$

The following two lemmas first appeared in the work of Daletskiĭ–Gelfand–Tsygan [7], which we learned from Tamarkin–Tsygan in [31].

Lemma 6.2. *Keep the notations as in the above definition. Then*

(1) $(\bar{C}_\bullet(A), b, \cap)$ *is a DG module over* $(\bar{C}^\bullet(A), \delta, \cup)$*, that is,*

$$\iota_{\delta f} = (-1)^{|f|+1}[b, \iota_f], \quad \iota_f \iota_g = \iota_{f \cup g},$$

for any homogeneous elements $f, g \in \bar{C}^\bullet(A)$*;*

(2) *for any homogeneous elements* $f, g \in \bar{C}^\bullet(A)$*,*

$$[L_f, L_g] = L_{\{f,g\}},$$

and in particular $(-1)^{|f|+1}[b, L_f] + L_{\delta f} = 0$.

Lemma 6.3 (Homotopy Cartan formulae). *Suppose* ι*,* L*, and* B *are given as above and* $f, g \in \bar{C}^\bullet(A)$ *are any homogeneous elements.*

(1) *Define an operation (cf.* [31, Equ. (3.5)]*)*

$$S_f(\alpha) := \sum_{i=0}^{m-n} \sum_{j=i+n}^{m} (-1)^{\eta_{ij}} \big(1, \bar{a}_{j+1}, \ldots, \bar{a}_m, \bar{a}_0, \ldots, \bar{a}_i, \overline{f(\bar{a}_{i+1}, \ldots, \bar{a}_{i+n})}, \bar{a}_{i+n+1}, \ldots, \bar{a}_j\big)$$

for any $\alpha = (a_0, \bar{a}_1, \ldots, \bar{a}_m) \in \bar{C}_m(A; A)$ *(the sum is taken over all cyclic permutations and* a_0 *always appears on the left of* f*), where* $\eta_{ij} := (n+1)m + (m-j)m + (n+1)(j-i)$*. Then one has*

$$L_f = [B, \iota_f] + [b, S_f] - S_{\delta f}. \tag{6.3}$$

(2) *Define*

$$T(f,g)(\alpha) := \sum_{i=l-n+2}^{l} \sum_{j=0}^{n+i-l-2} (-1)^{\theta_{ij}} \big(f(\bar{a}_{i+1}, \ldots, \bar{a}_l, \bar{a}_0, \ldots, \bar{a}_j, \overline{g(\bar{a}_{j+1}, \ldots, \bar{a}_{j+m})}, \ldots, \bar{a}_{n+m+i-l-2}), \ldots, \bar{a}_i \big)$$

for any $\alpha = (a_0, \bar{a}_1, \ldots, \bar{a}_l) \in \bar{\mathrm{C}}_l(A; A)$, *where* $\theta_{ij} = (m+1)(i+j+l) + l(i+1)$. *Then one has*

$$[L_f, \iota_g] - (-1)^{|f|+1} \iota_{\{f,g\}} = \big[b, T(f,g)\big] - T(\delta f, g) - T(f, \delta g). \tag{6.4}$$

The above two lemmas say that Definitions 4.1 (2) and 4.1 (3) hold up homotopy on the chain level. Together with Gerstenhaber's theorem, we have the following.

Theorem 6.4 (Daletskiĭ–Gelfand–Tsygan [7]). *Let A be an associative algebra. Then the sextuple*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, \iota, \{-,-\}, B\big)$$

is a differential calculus.

In [8, Proposition 5.5], de Thanhoffer de Völcsey and van den Bergh proved that, for a Calabi–Yau algebra A of dimension n, there exists a class $\eta \in \mathrm{HH}_n(A)$ such that the contraction

$$\mathrm{HH}^\bullet(A) \xrightarrow{-\cap\eta} \mathrm{HH}_{n-\bullet}(A) \tag{6.5}$$

is an isomorphism. This immediately implies the following.

Theorem 6.5 ([15, 21]). *Suppose A is a Calabi–Yau algebra A of dimension n. Then*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, \iota, \{-,-\}, B\big)$$

is a differential calculus with duality and, in particular, $(\mathrm{HH}^\bullet(A), \cup, \Delta)$ *is a Batalin–Vilkovisky algebra.*

6.2. Symmetric Frobenius algebras and the Batalin–Vilkovisky algebra structure

We now recall a differential calculus structure on the Hochschild complexes of symmetric Frobenius algebras.

First, for an associative algebra A, denote $A^* := \mathrm{Hom}(A, k)$, which is an A-bimodule. Denote by $\bar{\mathrm{C}}^\bullet(A; A^*)$ the reduced Hochschild cochain complex of A with values in A^*. Then under the identity

$$\bar{\mathrm{C}}^\bullet(A; A^*) = \bigoplus_{n\geq 0} \mathrm{Hom}(\bar{A}^{\otimes n}, A^*) = \bigoplus_{n\geq 0} \mathrm{Hom}(A \otimes \bar{A}^{\otimes n}, k) = \mathrm{Hom}\big(\bar{\mathrm{C}}_\bullet(A), k\big), \tag{6.6}$$

one may equip on $\bar{\mathrm{C}}^\bullet(A; A^*)$ the dual Connes differential, which is denoted by B^*, i.e., $B^*(g) := (-1)^{|g|} g \circ B$ for homogeneous $g \in \bar{\mathrm{C}}^\bullet(A; A^*)$. B^* commutes with the Hochschild coboundary map δ and thus is well-defined on the homology level.

Second, let

$$\begin{aligned}\bar{\mathrm{C}}^\bullet(A)\times\bar{\mathrm{C}}^\bullet(A;A^*) &\xrightarrow{\cap^*} \bar{\mathrm{C}}^\bullet(A;A^*)\\ (f,\alpha) &\longmapsto \iota_f^*(\alpha):=(-1)^{|f||\alpha|}\alpha\circ\iota_f,\end{aligned}\tag{6.7}$$

for any homogeneous $f\in\bar{\mathrm{C}}^\bullet(A)$ and $\alpha\in\bar{\mathrm{C}}^\bullet(A;A^*)$. We have the following.

Theorem 6.6. *Let A be an associative algebra. Then*

$$\big(\mathrm{HH}^\bullet(A),\mathrm{HH}^\bullet(A;A^*),\cup,\iota^*,\{-,-\},B^*\big)$$

is a differential calculus.

Proof. By the definition of differential calculus, we only need to show the last two equalities given in Definition 4.1.

(1) By the definition of ι^* and Lemma 6.2 (1), one has

$$\begin{aligned}\iota_f^*\iota_g^*(\alpha) &= (-1)^{|g||\alpha|}\iota_f^*(\alpha\circ\iota_g)=(-1)^{|g||\alpha|+|f|(|\alpha|+|g|)}(\alpha\circ\iota_g)\circ\iota_f\\ &=(-1)^{|g||\alpha|+|f|(|\alpha|+|g|)}\alpha\circ(\iota_{g\cup f})=(-1)^{|f||g|}\iota_{g\cup f}^*\alpha=\iota_{f\cup g}^*(\alpha),\end{aligned}$$

for any homogenous elements $f,g\in\mathrm{HH}^\bullet(A)$ and $\alpha\in\mathrm{HH}^\bullet(A;A^*)$. This means that the cap product is a left module action.

(2) Given any homogenous elements $f\in\mathrm{HH}^\bullet(A)$ and $\alpha\in\mathrm{HH}^\bullet(A;A^*)$, define

$$L_f^*(\alpha):=(-1)^{|f||\alpha|+|\alpha|+1}\alpha\circ L_f\big(=[B^*,\iota_f^*](\alpha)\big),\tag{6.8}$$

and by Lemma 6.3 one has

$$\begin{aligned}[L_f^*,\iota_g^*](\alpha) &= \big(L_f^*\iota_g^*-(-1)^{(|f|+1)|g|}\iota_g^*L_f^*\big)(\alpha)\\ &=(-1)^{(|f|+1)(|\alpha|+|g|)+|g||\alpha|+1}\alpha\circ(\iota_gL_f)-(-1)^{(|f|+|g|+1)|\alpha|+1}\alpha\circ(L_f\iota_g)\\ &=(-1)^{(|f|+|g|+1)|\alpha|}\alpha\circ\big([L_f,\iota_g]\big)\\ &=(-1)^{(|f|+|g|+1)|\alpha|}\alpha\circ\big((-1)^{|f|+1}\iota_{\{f,g\}}\big)\\ &=(-1)^{|f|+1}\iota_{\{f,g\}}^*(\alpha).\end{aligned}$$

This completes the proof. ∎

Now suppose $A^!$ is a symmetric Frobenius algebra. Recall that the existence of the degree n cyclic pairing is equivalent to an isomorphism

$$\eta: A^!\cong\Sigma^{-n}A^{¡}$$

as $A^!$-bimodules. Such η may be viewed as an element in $\bar{\mathrm{C}}^{-n}(A^!;A^{¡})$, which is a cocycle, and hence represents a cohomology class. By abuse of notation, this class is also denoted by η. The following map:

$$\begin{aligned}&-\cap^*\eta:\bar{\mathrm{C}}^\bullet(A^!)\\ &=\bigoplus_{q\ge 0}\mathrm{Hom}\big((\bar{A}^!)^{\otimes q},A^!\big)\xrightarrow{\eta\circ-}\bigoplus_{q\ge 0}\mathrm{Hom}\big((\bar{A}^!)^{\otimes q},\Sigma^{-n}A^{¡}\big)=\bar{\mathrm{C}}^{\bullet-n}(A^!;A^{¡}),\end{aligned}\tag{6.9}$$

where $\eta \circ -$ means composing with η, gives an isomorphism on the cohomology (due to Tradler [33]). Thus we have the following.

Theorem 6.7 ([21,33]). *Suppose $A^!$ is a symmetric Frobenius algebra of degree n.*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}^\bullet(A; A^*), \cup, \iota^*, \{-,-\}, B^*\big)$$

is a differential calculus with duality and, in particular, $\mathrm{HH}^\bullet(A^!)$ *is a Batalin–Vilkovisky algebra.*

Remark 6.8. Suppose $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, \{-,-\}, B)$ is a differential calculus, then ι and the Lie derivative $L = [\iota, B]$ is nothing but saying that $\mathrm{H}_\bullet$ is a Gerstenhaber module over $\mathrm{H}^\bullet$. From this point of view, the two differential calculus structures given in Theorems 4.8 and 6.7 can be understood in the following way: since $(\mathrm{HP}^\bullet(A^!), \mathrm{HP}_\bullet(A^!))$ already forms a differential calculus and $\mathrm{HP}^\bullet(A^!; A^¡)$ is the linear dual of $\mathrm{HP}_\bullet(A^!)$ (see (4.2)), the Gerstenhaber module structure on $\mathrm{HP}^\bullet(A^!; A^¡)$ is exactly the dual (or say adjoint) of Gerstenhaber module structure on $\mathrm{HP}_\bullet(A^!)$. Analogously, by (6.6), $\mathrm{HH}^\bullet(A^!; A^¡)$ is the linear dual of $\mathrm{HH}_\bullet(A^!)$, and thus the differential calculus structure on $(\mathrm{HH}^\bullet(A^!), \mathrm{HH}^\bullet(A^!; A^¡))$ can also be understood from this point of view.

6.3. Koszul Calabi–Yau algebras and Rouquier's conjecture

Analogously to the quadratic Poisson algebra case, the Koszul dual of a Koszul Calabi–Yau algebra is symmetric Frobenius (chronologically this fact is discovered first), and we have the following theorem due to van den Bergh (see [35, Theorem 9.2] or [6, Proposition 28] for a proof): *Suppose A is a Koszul algebra and let $A^!$ be its Koszul dual algebra. Then A is Calabi–Yau of dimension n if and only if $A^!$ is symmetric Frobenius of degree n.*

It has been well-known that for a Koszul algebra, say A,

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!),$$

as Gerstenhaber algebras, and Rouquier conjectured (it is stated by Ginzburg [15]) that, for a Koszul Calabi–Yau algebra, the above two Batalin–Vilkovisky algebras are isomorphic, which turns out to be true (see [6, Theorem A] for a proof).

Theorem 6.9 (Rouquier's conjecture). *Suppose A is a Koszul Calabi–Yau algebra. Denote by $A^!$ and by $A^¡$ the Koszul dual algebra and coalgebra of A, respectively. Then*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A)\big) \quad \textit{and} \quad \big(\mathrm{HH}^\bullet(A^!), \mathrm{HH}^\bullet(A^!; A^¡)\big)$$

are isomorphic as differential calculi with duality. In particular, $\mathrm{HH}^\bullet(A)$ *and* $\mathrm{HH}^\bullet(A^!)$ *are isomorphic as Batalin–Vilkovisky algebras.*

The key point of the proof is that, with the differentials properly assigned on $A \otimes A^!$ and $A \otimes A^¡$, respectively,

$$\bar{\mathrm{C}}^\bullet(A; A) \simeq A \otimes A^! \simeq \bar{\mathrm{C}}^\bullet(A^!; A^!) \quad \text{and} \quad \bar{\mathrm{C}}_\bullet(A; A) \simeq A \otimes A^¡ \simeq \bar{\mathrm{C}}^\bullet(A^!; A^¡),$$

and via these quasi-isomorphisms, the volume forms as well as the contractions given by (6.2) and (6.7) are identical on the above middle terms (compare with the proof of Theorem 1.2).

Example 6.10 (The polynomial case). Let $A = \mathbb{R}[x_1, x_2, \dots, x_n]$, which is n-Calabi–Yau. Its Koszul dual algebra $A^! = \mathbf{\Lambda}(\xi_1, \xi_2, \dots, \xi_n)$ is symmetric Frobenius. As in the Poisson case, the volume classes on $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A^!; A^¡)$ are, via the above quasi-isomorphisms, represented by $1 \otimes \xi_1^* \cdots \xi_n^*$ in $A \otimes A^¡$.

We would like to summarize some results of the previous two subsections in terms of DG Lie algebras analogous to the ones given by (4.3) and (4.4).

For an n-Calabi–Yau algebra A with volume form η, $(0, \Sigma^{-1-n}\eta)$ is a Maurer–Cartan element of the following DG Lie algebra of semi-direct product:

$$\mathfrak{D}(A)^{\#} := \Sigma\bar{\mathrm{C}}^\bullet(A) \ltimes \Sigma^{-1-n}\overline{\mathrm{CC}_\bullet^-}(A). \tag{6.10}$$

Let

$$\mathfrak{D}(A, \eta) := \mathfrak{D}(A)^{\#}_{(0, \Sigma^{-1-n}\eta)},$$

then it is a DG Lie algebra, and it will be studied in the next section.

For a symmetric Frobenius algebra $A^!$ with volume form $\eta^!$, we similarly have the DG Lie algebra

$$\bar{\mathfrak{D}}^\circ(A^!)^{\#} := \Sigma\bar{\mathrm{C}}^\bullet(A^!) \ltimes \Sigma^{-1-n}\overline{\mathrm{CC}}^\bullet(A^!), \tag{6.11}$$

and $(0, \Sigma^{-1-n}\eta^!)$ is a Maurer–Cartan element. However, this is not exactly the DG Lie algebra that we will discuss in the next section. In fact, let us first consider the Connes cyclic cochain complex $\mathrm{CC}_\lambda^\bullet(A^!)$, which is a cyclically invariant subcomplex of $\mathrm{C}^\bullet(A^!)$, the linear dual of the Hochschild chain complex of A (recall that it is identified with $\mathrm{C}^\bullet(A; A^*)$). It is then a direct check that $\mathrm{CC}_\lambda^\bullet(A^!)$ is closed under the Lie derivative of $\bar{\mathrm{C}}^\bullet(A^!)$, and hence

$$\mathfrak{D}^\circ(A^!)^{\#} := \Sigma\bar{\mathrm{C}}^\bullet(A^!) \ltimes \Sigma^{-1-n}\mathrm{CC}_\lambda^\bullet(A^!) \tag{6.12}$$

is a DG Lie algebra. Since $\eta^!$ is a cyclically invariant inner product of $A^!$, then $(0, \Sigma^{-1-n}\eta^!)$ is a Maurer–Cartan element of this DG Lie algebra. Observing that $\overline{\mathrm{CC}}^\bullet(A^!)$ is quasi-isomorphic to the Connes cyclic cochain complex $\mathrm{CC}_\lambda^\bullet(A^!)$ (see Loday [25, §2.4] for more details), which is compatible with the Lie derivative actions, we thus have a quasi-isomorphism of DG Lie algebras:

$$\bar{\mathfrak{D}}^\circ(A^!)^{\#} \simeq \mathfrak{D}^\circ(A^!)^{\#}.$$

In the following, we write

$$\mathfrak{D}(A, \eta) := \mathfrak{D}(A)^{\#}_{(0, \Sigma^{-1-n}\eta)} \quad \text{and} \quad \mathfrak{D}^\circ(A^!, \eta) := \mathfrak{D}^\circ(A^!)^{\#}_{(0, \Sigma^{-1-n}\eta^!)}. \tag{6.13}$$

7. Deformation quantization

In this section, we take k to be a field containing $\mathbb{R}$. Dolgushev [9, Theorem 3] proved that for a Calabi–Yau algebra, if it is unimodular Poisson, then its deformation quantization is again Calabi–Yau. Analogously, Felder–Shoikhet in [12, Corollary 1] and Willwacher–Calaque in [39, Theorem 37] proved that for a symmetric Frobenius algebra, if it is unimodular Frobenius Poisson, then its deformation quantization is again symmetric Frobenius. We use their results to prove Theorems 1.4 and 1.5.

The following proposition is a rephrase of the results of Section 4 for $A[\![\hbar]\!]$ (see Propositions 4.6 and 4.9).

Proposition 7.1. (1) *Let $A = k[x_1, \ldots, x_n]$ and $\hbar$ be a formal variable. For the algebra $A[\![\hbar]\!]$ over $k[\![\hbar]\!]$ together with a bivector*

$$\pi_\hbar := \hbar \cdot \pi_0 + \hbar^2 \cdot \pi_1 + \cdots \in \hbar \cdot \mathfrak{X}^2\big(A[\![\hbar]\!]\big)$$

and an n-form

$$\eta_\hbar := \hbar \cdot \eta_1 + \hbar^2 \cdot \eta_2 + \cdots \in \hbar \cdot \Omega^n\big(A[\![\hbar]\!]\big),$$

the pair $(\pi_\hbar, \eta_0 + \eta_\hbar)$ gives on $A[\![\hbar]\!]$ a unimodular Poisson structure if and only if $(\Sigma\pi_\hbar, \Sigma^{-1-n}\eta_\hbar)$ is a Maurer–Cartan element of the DG Lie algebra $\mathfrak{P}(A[\![\hbar]\!], \eta_0)$.

(2) *Suppose $A^! = \mathbf{\Lambda}(\xi_1, \ldots, \xi_n)$ with a volume form $\eta_0^!$. Then for a bivector $\pi_\hbar^! \in \hbar \cdot \mathfrak{X}^{-2}(A^![\![\hbar]\!])$ and an n-form $\eta_\hbar^! \in \hbar \cdot \mathfrak{X}^\bullet(A^![\![\hbar]\!]; A^{¡}[\![\hbar]\!])$, the pair $(\pi_\hbar^!, \eta_0^! + \eta_\hbar^!)$ gives a unimodular Frobenius Poisson structure on $A^![\![\hbar]\!]$ if and only if $(\Sigma\pi_\hbar^!, \Sigma^{-1-n}\eta_\hbar^!)$ is a Maurer–Cartan element of the DG Lie algebra $\mathfrak{P}^\circ(A^![\![\hbar]\!], \eta_0^!)$.*

For Calabi–Yau algebras and symmetric Frobenius algebras, we have similar results (see (6.10)–(6.13)), due to de Thanhoffer de Völcsey–van den Bergh [8] and Terilla–Tradler [32], respectively (the interested reader may refer to these two works for proofs).

Proposition 7.2. (1) ([8, Theorem 8.1]) *Suppose A is an n-Calabi–Yau algebra with multiplication μ_0 and volume form η_0. Then an element $\mu_\hbar \in \hbar \cdot \bar{\mathrm{C}}^{-2}(A[\![\hbar]\!])$ and an n-form $\eta_\hbar \in \hbar \cdot \overline{\mathrm{CC}}_n^-(A[\![\hbar]\!])$ such that $(\mu_0 + \mu_\hbar, \eta_0 + \eta_\hbar)$ gives a Calabi–Yau structure on $A[\![\hbar]\!]$ if and only if $(\Sigma\mu_\hbar, \Sigma^{-1-n}\eta_\hbar)$ is a Maurer–Cartan element of the DG Lie algebra $\mathfrak{D}(A[\![\hbar]\!], \eta_0)$.*

(2) ([32, Theorem 3.7]) *Suppose $A^!$ is a symmetric Frobenius algebra with volume n-form $\eta_0^!$. Then an element $\mu_\hbar^! \in \hbar \cdot \bar{\mathrm{C}}^2(A^![\![\hbar]\!])$ and an n-form $\eta_\hbar^! \in \hbar \cdot \mathrm{CC}_\lambda^n(A^![\![\hbar]\!])$ such that $(\mu_0^! + \mu_\hbar^!, \eta_0^! + \eta_\hbar^!)$ gives a symmetric Frobenius algebra structure on $A^![\![\hbar]\!]$ if and only if $(\Sigma\mu_\hbar^!, \Sigma^{-1-n}\eta_\hbar^!)$ is a Maurer–Cartan element of the DG Lie algebra $\mathfrak{D}^\circ(A^![\![\hbar]\!], \eta_0^!)$.*

In fact, in both works, the authors also showed that the DG Lie algebras appeared in the above proposition are quasi-isomorphic, up to a degree shift, to the negative cyclic chain complex and the cyclic cochain complex, respectively.

7.1. Deformation quantization of Calabi–Yau Poisson algebras

In this subsection, we prove Theorem 1.4 (1).

Recall that for a Poisson algebra A with bracket $\{-,-\}$, its *deformation quantization*, denoted by $A_\hbar$, is a $k[\![\hbar]\!]$-linear associative product (called the *star-product*) on $A[\![\hbar]\!]$

$$a * b = a \cdot b + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots,$$

where $\hbar$ is the formal parameter and μ_i are bilinear operators, satisfying

$$\lim_{\hbar\to 0}\frac{1}{\hbar}(a*b-b*a) = \{a,b\}, \quad \text{for all } a,b \in A.$$

In [19], Kontsevich constructed, for A being the algebra of smooth functions on a Poisson manifold, an explicit L_∞-quasi-isomorphism from the space of polyvector fields to the Hochschild cochain complex of A, and therefore there is a one-to-one correspondence between the equivalence classes of star-products and the equivalence classes of Poisson algebra structures on $A[\![\hbar]\!]$. Thus via Kontsevich's map, the Poisson bivector $\hbar\pi$ on $A[\![\hbar]\!]$ gives a star-product on $A[\![\hbar]\!]$, which is called *Kontsevich's deformation quantization*.

Note that $\Omega^\bullet(A)$ and $\bar{\mathrm{C}}_\bullet(A)$ are modules over $\mathfrak{X}^\bullet(A)$ and over $\bar{\mathrm{C}}^\bullet(A)$, respectively, and in [34, Conjecture 5.3.2], Tsygan conjectured that Kontsevich's deformation quantization also gives an L_∞-quasi-isomorphism of L_∞-modules between $\bar{\mathrm{C}}_\bullet(A)$ and $\Omega^\bullet(A)$. This is known as Tsygan's Formality Conjecture for chains, and is proved by Shoikhet in [29, Theorem 1.3.1]. Shoikhet also conjectured that such an L_∞-morphism is also compatible with the cap product, which was later proved by Calaque and Rossi in [3, Theorem A].

Recall that on $\Omega^\bullet(A)$ and $\bar{\mathrm{C}}_\bullet(A)$, we have the de Rham differential operator and the Connes boundary operator, respectively. One naturally expects that the L_∞-quasi-isomorphism constructed above respects these two operators. This is known as the Cyclic Formality Conjecture for chains, and is proved by Willwacher in [38, Theorem 1.3 and Corollary 1.4].

With the above results, one obtains the following theorem, due to Dolgushev [9, Theorem 3] (see also [8, equation (1.3)]), whose proof is therefore only sketched.

Theorem 7.3. *Let $A = k[x_1,\ldots,x_n]$ be a Poisson algebra. Then the deformation quantization of A is Calabi–Yau if and only if A is unimodular.*

Sketch of proof. Denote by $\mathfrak{U}$ and $\mathfrak{S}$ the L_∞-quasi-isomorphisms of Kontsevich and Willwacher, respectively. Then the works [19,38] are equivalent to saying that there exists a roof of L_∞-quasi-isomorphisms

$$\begin{array}{ccc} & \Sigma\mathfrak{X}^\bullet(A[\![\hbar]\!]) \ltimes \Sigma^{-1-n}\overline{\mathrm{CC}}_\bullet^-(A[\![\hbar]\!]) & \\ {\scriptstyle id\times\mathfrak{S}}\swarrow & & \searrow{\scriptstyle \mathfrak{U}\times id} \\ \Sigma\mathfrak{X}^\bullet(A[\![\hbar]\!]) \ltimes \Sigma^{-1-n}\Omega^\bullet(A[\![\hbar]\!])[\![u]\!] & & \Sigma\bar{\mathrm{C}}^\bullet(A[\![\hbar]\!]) \ltimes \Sigma^{-1-n}\overline{\mathrm{CC}}_\bullet^-(A[\![\hbar]\!]) \end{array}$$

of DG Lie algebras (see [8, §11.3] for a proof).

Recall that from Example 6.10 the volume forms in the three DG Lie modules are the same on the homology level. Twisting the differentials with the corresponding volume forms in each of the DG Lie algebra in the above roof we get a new roof of L_∞-quasi-isomorphisms. This then implies that we have an L_∞-quasi-isomorphism of DG Lie algebras:

$$\mathfrak{P}(A[\![\hbar]\!], \eta_0) \overset{\simeq}{\dashrightarrow} \mathfrak{D}(A[\![\hbar]\!], \eta_0),$$

where the dotted arrow means that the quasi-isomorphism is given by a sequence of (roofs of) L_∞-morphisms.

As a corollary, the Maurer–Cartan elements of $\mathfrak{P}(A[\hbar], \eta_0)$ (up to gauge equivalence) are in one-to-one correspondence, via the above L_∞-quasi-isomorphisms, with the Maurer–Cartan elements of $\mathfrak{D}(A[\![\hbar]\!], \eta_0)$. In particular, by Propositions 7.1 (1) and 7.2 (1), if A is unimodular Poisson, then $A_\hbar$ is Calabi–Yau, and vice versa. ∎

Proof of Theorem 1.4 (1). It is proved by Calaque and Rossi in [3, Theorem 6.1] that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{X}^\bullet(A[\![\hbar]\!]) & \rightsquigarrow & \Omega^\bullet(A[\![\hbar]\!]) \\ \simeq \downarrow \mathfrak{U} & & \simeq \uparrow \mathfrak{S} \\ \bar{\mathrm{C}}^\bullet(A[\![\hbar]\!]) & \rightsquigarrow & \bar{\mathrm{C}}_\bullet(A[\![\hbar]\!]), \end{array} \tag{7.1}$$

where the horizontal curved arrows mean the cap product. Since A is unimodular Poisson, $A_\hbar$ is Calabi–Yau, and $\mathfrak{S}$ maps the volume form of $A[\![\hbar]\!]$ to the volume form of $A_\hbar$ under the Hochschild–Kostant–Rosenberg map, then we thus obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A[\![\hbar]\!]) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}_{n-\bullet}(A_\hbar) \end{array}$$

by Theorem 4.7 and the noncommutative Poincaré duality (6.5). ∎

7.2. Deformation quantization of Frobenius Poisson algebras

We first rephrase Kontsevich's Cyclic Formality Conjecture for *cochains*, published by Felder–Shoihket [12, §1], in the case $k^{0|n}$. Note that, in this case, the space of functions $\mathcal{O}(k^{0|n}) \cong A^! := \Lambda^\bullet(\xi_1, \ldots, \xi_n)$.

Recall that, by Cattaneo and Felder [4, Appendix], Kontsevich's L_∞-quasi-isomorphism holds for the supermanifold case. Denote this quasi-isomorphism again by $\mathfrak{U}$. The following is stated by Felder–Shoikhet [12] and proved by Willwacher–Calaque [39, Theorem 2] (see also [12] for some partial results).

Lemma 7.4 (Formality for cochains). *For $A^! = \mathcal{O}(k^{0|n}) \cong \Lambda^\bullet(\xi_1, \ldots, \xi_n)$, there exists an L_∞-quasi-isomorphism of Lie modules:*

$$\mathfrak{V} : \left(\mathcal{X}^\bullet(A^![\![\hbar]\!]; A^{¡}[\![\hbar]\!])[\![u]\!], ud^*\right) \xrightarrow{\simeq} \left(\mathrm{CC}^\bullet_\lambda(A^![\![\hbar]\!]), \delta\right).$$

In other words, there exists an L_∞-quasi-isomorphism of Lie algebras:

$$\mathfrak{U} \times \mathfrak{V} : \Sigma\mathcal{X}^\bullet\big(A^![\![\hbar]\!]\big) \ltimes \Sigma^{-1-n}\mathcal{X}^\bullet\big(A^![\![\hbar]\!]; A^¡[\![\hbar]\!]\big)[\![u]\!] \xrightarrow{\simeq} \Sigma\bar{\mathrm{C}}^\bullet\big(A^![\![\hbar]\!]\big) \ltimes \Sigma^{-1-n}\mathrm{CC}^\bullet_\lambda\big(A^![\![\hbar]\!]\big).$$

Again we recommend [8, §11] for the formulae of the (Taylor) expansion of $\mathfrak{U} \times \mathfrak{V}$. Also, we mention that the first term of the above L_∞-quasi-isomorphism $\mathfrak{V}$ is the Hochschild–Kostant–Rosenberg map, which then preserves the volume forms on each side. Therefore, we get a quasi-isomorphism

$$\mathfrak{P}^\circ\big(A^![\![\hbar]\!], \eta_0^!\big) \simeq \mathfrak{D}^\circ\big(A^![\![\hbar]\!], \eta_0^!\big)$$

as DG Lie algebras. As a corollary, we have the following theorem, due to Felder–Shoikhet [12, Corollary 1] and Willwacher–Calaque [39, Theorem 37].

Theorem 7.5. *For $A^! = \Lambda^\bullet(\xi_1, \dots, \xi_n)$, the deformation quantization of $A^!$ is symmetric Frobenius if and only if $A^!$ is unimodular Frobenius.*

Proof of Theorem 1.4 (2). Recall that $\Omega^\bullet(A^![\hbar])$ and $\bar{\mathrm{C}}_\bullet(A^![\hbar])$ are Lie modules over $\mathcal{X}^\bullet(A^![\![\hbar]\!])$ and $\bar{\mathrm{C}}^\bullet(A^![\![\hbar]\!])$, respectively. Applying Calaque–Rossi's result (7.1) to $A^![\![\hbar]\!]$, we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{X}^\bullet(A^![\![\hbar]\!]) & \rightsquigarrow & \Omega^\bullet(A^![\![\hbar]\!]) \\
\simeq \big\downarrow \mathfrak{U} & & \simeq \big\uparrow \mathfrak{S} \\
\bar{\mathrm{C}}^\bullet(A^![\![\hbar]\!]) & \rightsquigarrow & \bar{\mathrm{C}}_\bullet(A^![\![\hbar]\!]).
\end{array}$$

Now consider the adjoint actions of the Lie algebras to the linear dual spaces of the Lie modules (see Remark 6.8), then we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{X}^\bullet(A^![\![\hbar]\!]) & \rightsquigarrow & \mathcal{X}^\bullet(A^![\![\hbar]\!]; A^¡[\![\hbar]\!]) \\
\simeq \big\downarrow \mathfrak{U} & & \simeq \big\downarrow \mathfrak{S}^* \\
\bar{\mathrm{C}}^\bullet(A^![\![\hbar]\!]) & \rightsquigarrow & \bar{\mathrm{C}}^\bullet(A^![\![\hbar]\!]; A^¡[\![\hbar]\!]).
\end{array}$$

Taking the homology in the above commutative diagram and applying the Poincaré duality, whose existence is guaranteed by Theorem 7.5, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(A^![\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^![\![\hbar]\!]; A^¡[\![\hbar]\!]) \\
\big\downarrow \cong & & \big\downarrow \cong \\
\mathrm{HH}^\bullet(A^!_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^{\bullet-n}(A^!_\hbar; A^¡_\hbar).
\end{array}$$

This completes the proof. ∎

Proof of Theorem 1.5. By Shoikhet [30, Theorem 0.3] (see also [2, Theorem 8.6]), $A_\hbar$ and $A^!_\hbar$ are Koszul dual algebras over $k[\![\hbar]\!]$, and hence the theorem follows from a combination of Theorems 1.3, 1.4, and 6.9. ∎

7.3. Twisted Poincaré duality for Poisson algebras

For a general associative algebra, say A, it may not be Calabi–Yau, and therefore there may not exist any Poincaré duality between $\mathrm{HH}^\bullet(A)$ and $\mathrm{HH}_\bullet(A)$. In [1], Brown and Zhang introduced the so-called "twisted Poincaré duality" for associative algebras. That is, for such A, keeping its left A-module structure (the multiplication) as usual, the right A-module structure of A is the multiplication composed with an automorphism $\sigma : A \to A$. Denote such A-bimodule by A_σ, then Brown and Zhang showed that for a lot of algebras, there exists a twisted Poincaré duality $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A; A_\sigma)$ for some $n \in \mathbb{N}$ (cf. [1, Corollary 5.2]). In this case, A is called a *twisted Calabi–Yau* algebra of dimension n.

Such a phenomenon also occurs for Poisson algebras. Namely, not all Poisson algebras are unimodular, and hence there may not exist an isomorphism between $\mathrm{HP}^\bullet(A)$ and $\mathrm{HP}_\bullet(A)$. In [22, 27, 41, 42], the authors studied the so-called twisted Poincaré duality for Poisson algebras, similarly to that of associative algebras. They also studied some comparisons with twisted Calabi–Yau algebras. However, it would be very interesting to study the relationships between the deformation quantization of twisted unimodular Poisson algebras and twisted Calabi–Yau algebras, and obtain a theorem similar to Theorem 1.5 in this twisted case.

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