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C^* exponential length of commutators unitaries in AH-algebras

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Abstract. For each unital C^*-algebra A, we denote $\mathrm{cel}_{\mathrm{CU}}(A) = \sup\{\mathrm{cel}(u) : u \in \mathrm{CU}(A)\}$, where $\mathrm{cel}(u)$ is the exponential length of u and $\mathrm{CU}(A)$ is the closure of the commutator subgroup of $U_0(A)$. In this paper, we prove that $\mathrm{cel}_{\mathrm{CU}}(A) \geq 2\pi$ provided that A is an AH-algebra with slow dimension growth whose real rank is not zero. On the other hand, we prove that $\mathrm{cel}_{\mathrm{CU}}(A) \leq 2\pi$ when A is an AH-algebra with ideal property and of no dimension growth (if we further assume that A is not of real rank zero, we have $\mathrm{cel}_{\mathrm{CU}}(A) = 2\pi$).

1. Introduction

Let A be a unital C^*-algebra and $U(A)$ its unitary group. We denote by $U_0(A)$ the connected component of $U(A)$ containing the identity. A unitary element $u \in U(A)$ belongs to $U_0(A)$ if and only if u has the form

$$u = \prod_{j=1}^{n} \exp(i h_j),$$

where n is a positive integer and h_j is self-adjoint for every $1 \leq j \leq n$. For $u \in U_0(A)$, the exponential rank of u was defined by Phillips and Ringrose [45] and the exponential length of u was defined by Ringrose [46]. Recall the definition of C^* exponential length as follows.

Definition 1.1. For $u \in U_0(A)$, the C^* exponential length of u, denoted by $\mathrm{cel}(u)$, is defined as

$$\mathrm{cel}(u) = \inf\left\{ \sum_{j=1}^{k} \|h_j\| : u = \prod_{j=1}^{k} \exp(i h_j), h_j = h_j^* \right\}.$$

Define

$$\mathrm{cel}(A) = \sup\left\{\mathrm{cel}(u) : u \in U_0(A)\right\}.$$

From [46], $\mathrm{cel}(u)$ is exactly the infimum of the lengths of rectifiable paths from u to $\mathbf{1}_A$ in $U(A)$. Equivalently, $\mathrm{cel}(u)$ is also the infimum of the lengths of smooth paths from u to $\mathbf{1}_A$.

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Exponential rank and exponential length have been studied extensively (see [33,41–43, 46,50,51]) and they have played important roles in the classification of C^*-algebras (see [11,17,32,34,35]).

Phillips [41] proved that the exponential rank of a unital purely infinite simple C^*-algebra is $1 + \varepsilon$ and its exponential length is π. Lin [30] proved that for any unital C^*-algebra A of real rank zero, $u \in U_0(A)$, and $\varepsilon > 0$, there exists a self-adjoint element $h \in A$ with $\|h\| = \pi$ such that

$$\|u - \exp(ih)\| < \varepsilon.$$

This means that $\mathrm{cel}(u) \leq \pi$. Phillips [43] showed that when A does not have a real rank zero, even for $A = M_n(C([0,1]))$, $\mathrm{cel}(A)$ can be ∞.

Definition 1.2. We denote by $\mathrm{CU}(A)$ the closure of the commutator subgroup of $U_0(A)$ and define the C^* exponential length of $\mathrm{CU}(A)$ to be

$$\mathrm{cel}_{\mathrm{CU}}(A) = \sup\{\mathrm{cel}(u) : u \in \mathrm{CU}(A)\}.$$

In the study of the classification of simple amenable C^*-algebras, one often has to calculate the exponential length of unitaries in $\mathrm{CU}(A)$. Pan and Wang [39] constructed a simple AI algebra (inductive limit of $M_n(C([0,1]))$) A such that $\mathrm{cel}_{\mathrm{CU}}(A) \geq 2\pi$. Applying Lin's Lemma 4.5 in [36], one has that $\mathrm{cel}_{\mathrm{CU}}(A)$ is exactly 2π.

Definition 1.3. An AH-algebra A is the C^*-algebra inductive limit of a sequence $A = \lim(A_n, \phi_{n,n+1})$ with $A_n = \bigoplus_{j=1}^{t_n} P_{n,j} M_{[n,j]}(C(X_{n,j})) P_{n,j}$, where $[n,j]$ and t_n are positive integers, $X_{n,j}$ are compact metrizable spaces, and $P_{n,j} \in M_{[n,j]}(C(X_{n,j}))$ are projections. In this paper, we will only consider unital AH-algebras, and hence we will always assume that all the maps $\phi_{n,n+1}$ are unital.

At first look, the class of AH-algebras is a quite special class of C^*-algebras, but remarkably, many important C^*-algebras arising from the study of foliation manifolds and dynamical systems have been proved to be in the class of AH-algebras. These C^*-algebras include the foliation algebra of Kronecker foliation, two-dimensional and higher-dimensional non-commutative tori (see [6,31]), and the cross product C^*-algebras of a minimal dynamical system over a finite-dimensional space provided that the image of the K_0-group in the affine space of tracial state space is dense (see [38]). In fact, it is a conjecture that all stably finite simple separable nuclear C^*-algebras are the inductive limits of sub-algebras of the above A_n's (see [13,37]). Let us point out that the class of AH-algebras plays an important role in the classification programs (see [7–12,14–17, 26–29,31]).

In [36], Lin has obtained the following two main theorems (we rephrase the theorems in the language of AH-algebras).

Theorem A ([36, Theorem 4.6]). *Suppose that A is a $\mathcal{Z}$-stable simple C^*-algebra such that $A \otimes UHF$ is an AH-algebra of slow dimension growth (this class includes all simple AH-algebras of no dimension growth and the Jiang–Su algebra $\mathcal{Z}$). Then* $\mathrm{cel}_{\mathrm{CU}}(A) \leq 2\pi$.

Theorem B ([36, Theorem 5.11 and Corollary 5.12]). *For any unital non-elementary simple AH-algebra B (i.e., B is not isomorphic to $M_n(\mathbb{C})$) of slow dimension growth, there exists a unital simple AH-algebra A of no dimension growth such that*

$$\big(K_0(A), K_0(A)_+, K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, K_1(B)\big) \quad \textit{and} \quad \mathrm{cel}_{\mathrm{CU}}(A) > \pi.$$

It is proved in [49] that the class of simple unital non-elementary AH-algebras with no dimension growth and the class of simple unital non-elementary AH-algebras with slow dimension growth are the same (see also [11, 17, 34]).

Our main theorem in this article is that, for all (not necessarily simple) AH-algebras A with slow dimension growth, if A is not of real rank zero, then $\mathrm{cel}_{\mathrm{CU}}(A) \geq 2\pi$. This theorem greatly generalizes and strengthens Lin's Theorem B above. If we further assume A to be simple, combining with Lin's Theorem A above, then $\mathrm{cel}_{\mathrm{CU}}(A) = 2\pi$. This gives the complete calculation of $\mathrm{cel}_{\mathrm{CU}}(A)$ for simple AH-algebras A of slow dimension growth (note that for the real rank zero case, it is already known by [30] that $\mathrm{cel}(A) = \pi$). We will extend such a calculation of $\mathrm{cel}_{\mathrm{CU}}(A)$ of simple AH-algebras A to the AH-algebras of no dimension growth with ideal property. We will also prove that $\mathrm{cel}_{\mathrm{CU}}(M_n(\mathcal{Z})) \geq 2\pi$ for the Jiang–Su algebra $\mathcal{Z}$. Combining with Lin's Theorem A, we have $\mathrm{cel}_{\mathrm{CU}}(M_n(\mathcal{Z})) = 2\pi$.

In Section 2, we will introduce some notation and some known results for preparation. In Section 3, we will prove our main theorem. In Section 4, we will deal with AH-algebras with ideal property. In Section 5, we will calculate $\mathrm{cel}_{\mathrm{CU}}(M_n(\mathcal{Z}))$.

2. Notation and some known results

First, we give some useful lemmas.

Proposition 2.1 ([39, Lemma 2.5]). *Let $u \in C([0,1])$ be defined by $u(t) = \exp(i\alpha(t))$. Then*

$$\mathrm{cel}(u) = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha(t) - 2k\pi|.$$

Proposition 2.2 ([39, Corollary 3.5]). *Let H_s be a rectifiable path in $U(M_k(C([0,1])))$. For any $\varepsilon > 0$, there exists a piecewise smooth path F_s in $U(M_k(C([0,1])))$ such that*

(1) $\|H_s - F_s\|_\infty < \varepsilon$ *for all* $s \in [0,1]$*;*

(2) $|\,\mathrm{length}_s(H_s) - \mathrm{length}_s(F_s)| < \varepsilon$*;*

(3) $F_s(t)$ *has no repeated eigenvalues for any* $(s,t) \in [0,1] \times [0,1]$.

Moreover, if for any $t \in [0,1]$, $H_1(t)$ has no repeated eigenvalues, then F can be chosen to satisfy that $F_1(t) = H_1(t)$ for all $t \in [0,1]$.

Remark 2.3. In Proposition 2.2, if $H_s(0)$ and $H_s(1)$ have no repeated eigenvalues, respectively, then F can be chosen to satisfy that $F_s(0) = H_s(0)$ and $F_s(1) = H_s(1)$ for all $s \in [0,1]$.

Let Y be a compact metric space. Let $P \in M_{k_1}(C(Y))$ be a projection with $\operatorname{rank}(P) = k \leq k_1$. For each y, there exists a unitary $u_y \in M_{k_1}(\mathbb{C})$ (depending on y) such that

$$P(y) = u_y^* \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} u_y,$$

where there are k 1's on the diagonal. If the unitary u_y can be chosen to be continuous in y, the projection P is called a trivial projection. It is well known that every projection $Q \in M_{k_1}(C(Y))$ is locally trivial. That is, for each $y_0 \in Y$, there exists an open set U_{y_0} containing y_0 such that one can choose the above u_y to be continuous on U_{y_0}. If P is a trivial projection in $M_{k_1}(C(Y))$, we have

$$PM_{k_1}\big(C(Y)\big)P \cong M_k\big(C(Y)\big).$$

Following the notations in [39], we give the following definitions.

Definition 2.4. Given a metric space (Y, d), we write

$$Y^k = \underbrace{Y \times Y \times \cdots \times Y}_{k}.$$

We define an equivalent relation on Y^k as follows: two elements $(x_1, x_1, \ldots, x_k)$, $(y_1, y_2, \ldots, y_k) \in Y^k$ are equivalent if there exists a permutation $\sigma \in S_k$ such that $x_{\sigma(i)} = y_i$ for each $1 \leq i \leq k$. Define

$$P^k Y = Y^k / \sim$$

with the metric

$$d\big([x_1, x_2, \ldots, x_k], [y_1, y_2, \ldots, y_k]\big) = \min_{\sigma \in S_k} \max_{1 \leq j \leq k} |x_j - y_{\sigma(j)}|.$$

Let us further assume that Y is compact. Let $F^k Y = \operatorname{Hom}(C(Y), M_k(\mathbb{C}))_1$, the space of all unital homomorphisms from $C(Y)$ to $M_k(\mathbb{C})$. Then for any $\phi \in F^k Y$, there are k-points $y_1, y_2, \ldots, y_k$ (with multiplicity) and a unitary $u \in M_k(\mathbb{C})$ such that

$$\phi(f) = u \begin{bmatrix} f(y_1) & & & \\ & f(y_2) & & \\ & & \ddots & \\ & & & f(y_k) \end{bmatrix} u^* \quad \text{for all } f \in C(Y).$$

Define $\operatorname{Sp}(\phi)$ to be the set $\{y_1, y_2, \ldots, y_k\}$ (counting multiplicity, see [17]). Considering $\operatorname{Sp}(\phi)$ as a k-tuple, $(y_1, y_2, \ldots, y_k)$, it is not uniquely determined since the order of k-tuple is up to a choice; but as an element in $P^k Y$, it is unique. Therefore, we write $\operatorname{Sp}(\phi) \in$

P^kY. Then $F^kY \ni \phi \mapsto \mathrm{Sp}(\phi) \in P^kY$ gives a continuous map $\Pi : F^kY \to P^kY$. (Note that F^kY is endowed with the standard topology so that for any $\phi_1, \phi_2, \ldots, \phi_n, \ldots \in F^kY$ and $\phi \in F^kY$, $\phi_n \to \phi$ if and only if, for any $f \in C(Y)$, $\phi_n(f) \to \phi(f)$.)

Proposition 2.5 ([39, Remark 3.9]). *Let F_s be a path in $U(M_k(C([0,1])))$ such that $F_s(t)$ has no repeated eigenvalues for any $(s,t) \in [0,1] \times [0,1]$. Let $\Lambda : [0,1] \times [0,1] \to P^kS^1$ be the eigenvalue map of $F_s(t)$, that is, $\Lambda(s,t) = [x_1(s,t), x_2(s,t), \ldots, x_k(s,t)]$, where $\{x_i(s,t)\}_{i=1}^k$ are eigenvalues of the matrix $F_s(t)$. Then there are continuous functions $f_1, f_2, \ldots, f_k : [0,1] \times [0,1] \to S^1$ such that*

$$\Lambda(s,t) = \left[f_1(s,t), f_2(s,t), \ldots, f_k(s,t)\right].$$

For each $(s,t) \in [0,1] \times [0,1]$, there exists a unitary $U_s(t)$ such that

$$F_s(t) = U_s(t)\,\mathrm{diag}\left[f_1(s,t), f_2(s,t), \ldots, f_k(s,t)\right]U_s(t)^*.$$

Fix $1 \le i \le n$. For each $(s,t) \in [0,1] \times [0,1]$, let $p_i(s,t)$ be the spectral projection of $F_s(t) \in M_n(\mathbb{C})$ with respect to the eigenvalue $f_i(s,t)$ (of $F_s(t)$); this is a well-defined rank one projection continuously depending on (s,t), since the continuous matrix-valued function $F_s(t)$ has distinct eigenvalues. Hence $F_s(t) = \sum_{i=1}^k f_i(s,t)p_i(s,t)$. Since all projections in $M_n(C([0,1] \times [0,1]))$ are trivial, it is straightforward to prove that the unitary $U_s(t)$ above can be chosen to depend on s and t continuously.

Proposition 2.6 ([39, Lemma 3.11]). *Let F_s be a path in $U(M_n(C([0,1])))$ and $f_s^1(t)$, $f_s^2(t), \ldots, f_s^n(t)$ be continuous functions such that*

$$F_s(t) = U_s(t)\,\mathrm{diag}\left[f_s^1(t), f_s^2(t), \ldots, f_s^n(t)\right]U_s(t)^*,$$

where $U_s(t)$ are unitaries. Suppose that for any $(s,t) \in [0,1] \times (0,1)$, $f_s^i(t) \neq f_s^j(t)$ if $i \neq j$. Then

$$\mathrm{length}_s(F_s) \ge \max_{1 \le i \le n} \left\{\mathrm{length}_s(f_s^i)\right\},$$

where f_s^i is regarded as a path in $U(C([0,1]))$.

Proof. If the unitary F_s satisfies the stronger condition that for any $(s,t) \in [0,1] \times [0,1]$ and $i \neq j$, $f_s^i(t) \neq f_s^j(t)$, then this is Lemma 3.11 of [39]. To prove the general case one can apply Lemma 3.11 of [39] to $F_s|_{[0,1]\times[\delta,1-\delta]}$ (whose length is at most $\mathrm{length}_s(F_s)$) and note that $\mathrm{length}_s(f_s^i) = \sup_\delta\{\mathrm{length}_s(f_s^i|_{[0,1]\times[\delta,1-\delta]})\}$. ■

Definition 2.7 ([3, Definition 1.1]). Let $a = a^* \in PM_n(C(X))P$, where X is a connected compact metric space. For each $x \in X$, the eigenvalues of $a(x) \in P(x)M_n(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$ form a set of $\mathrm{rank}(P)$ real numbers (with multiplicity), which could be regarded as an element of $P^k\mathbb{R}$, where $k = \mathrm{rank}(P)$, and we will denote this element by $\mathrm{Eg}(a)(x)$. On the other hand, the topology on the space $\mathbb{R}$ is given by the linear order on $\mathbb{R}$ which induces a natural continuous map from $P^k\mathbb{R}$ to $\mathbb{R}^k$, by labeling the k-tuple in the increasing order. Namely, for any $[x_1, x_2, \ldots, x_k] \in P^k\mathbb{R}$ with $x_i \in \mathbb{R}$ (for $1 \le i \le k$), define

$$\alpha\big([x_1, x_2, \ldots, x_k]\big) = \big(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\big) \in \mathbb{R}^k,$$

where σ is a permutation of $\{1, 2, \ldots, k\}$ and $x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(k)}$. This α identifies $P^k\mathbb{R}$ as a subset of $\mathbb{R}^k$. The map $x \mapsto \mathrm{Eg}(a)(x) \in P^k\mathbb{R} \subset \mathbb{R}^k$ gives k continuous maps from X to $\mathbb{R}$. We will call these k continuous maps the eigenvalue list $E(a)$ of a. Namely, the *eigenvalue list* of a is defined as

$$E(a)(x) = \big\{h_1(x), h_2(x), \ldots, h_k(x)\big\},$$

where $h_i(x)$ is the ith lowest eigenvalue of $a(x)$, counted with multiplicity.

The *variation of the eigenvalues* of a is denoted by $\mathrm{EV}(a)$ and is defined as

$$\mathrm{EV}(a) = \max_{1 \le i \le n} \left\{ \max_{t,s \in X} |h_i(t) - h_i(s)| \right\}.$$

Here, when we use $\mathrm{Eg}(a) : X \to P^k\mathbb{R}$ and $E(a) : X \to \mathbb{R}^k$, we have $E(a) = \iota \circ \mathrm{Eg}(a)$, where $\iota : P^k\mathbb{R} \to \mathbb{R}^k$ is the natural inclusion.

Remark 2.8. (1) In this paper, we will often consider $a \in A_+$ with $\|a\| \le 1$. Then $\mathrm{Sp}(a) \subset [0, 1]$. This element a naturally defines a homomorphism $\phi : C([0, 1]) \to A$ by $\phi(h) = a$, where $h : [0, 1] \to [0, 1]$ is the identity function: $h(t) = t$. Let $A = PM_n(C(X))P$ as in Definition 2.7. Then $E(a)$ is a map from X to $[0, 1]^k$ (where $k = \mathrm{rank}(P)$) and $\mathrm{Eg}(a)$ is a map from X to $P^k[0, 1]$.

(2) Let $P, Q \in M_n(C(X))$ be projections which satisfy $P < Q$. An element $a \in (PM_n(C(X))P)_+$ can also be regarded as an element in $QM_n(C(X))Q$. The eigenvalue list $E_{PM_n(C(X))P}(a)$ of a as an element in $PM_n(C(X))P$ and the eigenvalue list $E_{QM_n(C(X))Q}(a)$ of a as an element in $QM_n(C(X))Q$ are related in the following way. Suppose $\mathrm{rank}(P) = k$ and $\mathrm{rank}(Q) = l$. If

$$E_{PM_n(C(X))P}(a) = \big\{h_1(x), h_2(x), \ldots, h_k(x)\big\},$$

then

$$E_{QM_n(C(X))Q}(a) = \big\{\underbrace{0, \ldots, 0}_{l-k}, h_1(x), h_2(x), \ldots, h_k(x)\big\}.$$

In particular, the eigenvalue variation of a positive element $a \in PM_n(C(X))P \subset QM_n(C(X))Q$ is independent of the choice of $PM_n(C(X))P$ or $QM_n(C(X))Q$. (This is not true for general self-adjoint elements.) So when we discuss eigenvalue list or eigenvalue variation of a positive element a in an upper left corner sub-algebra $PM_n(C(X))P$ of $QM_n(C(X))Q$, we do not need to specify in which algebra the calculations are made; that is, we will omit those $l - k$ constant 0 functions from our eigenvalue list.

(3) Let $P, Q \in M_n(C(X))$ be projections which satisfy $P < Q$. Suppose that $a \in PM_n(C(X))P \subset QM_n(C(X))Q$ is a (not necessarily positive) self-adjoint element such that none of the functions in the eigenvalue list $E(a)$ of a (considered as in $PM_n(C(X))P$)

is crossing over point 0; that is, they are either non-positive functions or non-negative functions. Then we can also ignore in which algebra (in the corner sub-algebra $PM_n(C(X))P$ or in the algebra $QM_n(C(X))Q$) the calculations are made when we calculate the eigenvalue list and the eigenvalue variation.

More precisely, if

$$E_{PM_n(C(X))P}(a) = \{h_1(x), \ldots, h_i(x), h_{i+1}(x), \ldots, h_k(x)\}$$

with $h_i(x) \le 0 \le h_{i+1}(x)$ for all $x \in X$, then

$$E_{QM_n(C(X))Q}(a) = \{h_1(x), \ldots, h_i(x), \underbrace{0, \ldots, 0}_{l-k}, h_{i+1}(x), h_k(x)\}.$$

In this case, we will also omit those $l-k$ constant 0 functions from our eigenvalue list for $E_{QM_n(C(X))Q}(a)$.

In general, for a self-adjoint element $a \in PM_n(C(X))P \subset QM_n(C(X))Q$, one has

$$\mathrm{EV}_{PM_n(C(X))P}(a) \ge \mathrm{EV}_{QM_n(C(X))Q}(a).$$

Definition 2.9. If $A = \lim(A_n = \bigoplus_{j=1}^{k_n} P_{n,j} M_{[n,j]}(C(X_{n,j}))P_{n,j}, \phi_{n,m})$ is a (non-zero) unital inductive limit system with simple limit, then the following slow dimension growth condition was introduced by Blackadar et al. [4]:

$$\lim_{n\to\infty} \max_j \left\{ \frac{\dim(X_{n,j}) + 1}{\mathrm{rank}(P_{n,j})} \right\} = 0.$$

For a general AH inductive limit system, we will use the following slow dimension growth condition: for any summand $A_n^i = P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$ of a fixed A_n,

$$\lim_{m\to\infty} \max_{i,j} \left\{ \frac{\dim(X_{m,j}) + 1}{\mathrm{rank}\left(\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})\right)} \;\Big|\; \phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i}) \neq 0 \right\} = 0,$$

where $\phi_{n,m}^{i,j}$ is the partial map of $\phi_{n,m}$ from A_n^i to A_m^j. This notion of slow dimension growth condition is used in most literatures (see [5]). In particular, in this definition, it is automatically true that $\lim_{m\to\infty} \mathrm{rank}(P_{m,j}) = \infty$.

An inductive limit system $A = \lim(\bigoplus_{j=1}^{k_n} P_{n,j} M_{[n,j]}(C(X_{n,j}))P_{n,j}, \phi_{n,m})$ is called of no dimension growth if $\sup_{n,j} \dim(X_{n,j}) < +\infty$. For a general non-simple inductive limit system, no dimension growth does not imply slow dimension growth, as it does not automatically imply that $\lim_{m\to+\infty} \mathrm{rank}(P_{m,j}) = \infty$.

We avoid to use the more general concept of slow dimension growth introduced by Gong [15] which does not imply that $\lim_{m\to\infty} \mathrm{rank}(P_{m,j}) = \infty$, since, in this case, our main theorem is not true (see Proposition 3.11).

Notation 2.10. For inductive limit

$$A = \lim\left(A_n = \bigoplus_{i=1}^{k_n} A_n^i, \phi_{n,m}\right),$$

where $A_n^i = P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}$, we will use $\phi_{n,m}^{i,j}\colon A_n^i \to A_m^j$ to denote the partial map $\pi_j \circ \phi_{n,m}|_{A_n^i}$, where $\pi_j\colon A_m \to A_m^j$ is the projection map from A_m to its jth block.

Let us also denote $\pi_j \circ \phi_{n,m}$ by $\phi_{n,m}^{-,j}$ which is the homomorphism from A_n to A_m^j.

Proposition 2.11 ([5, Corollaries 1.3 and 1.4]). *Let $A = \varinjlim(A_n, \phi_{n,m})$ be a C^*-algebra inductive limit system. Assume that each A_n is of the form*

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}\big(C(X_{n,i})\big) P_{n,i},$$

where k_n and $[n,i]$ are positive integers, $X_{n,i}$ are connected compact Hausdorff spaces, and $M_{[n,i]}$ are $[n,i] \times [n,i]$ matrices. If A has slow dimension growth (see [5, Corollary 1.4]*) or has no dimension growth (see* [5, Corollary 1.3]*), then the following are equivalent:*

(1) *A has a real rank zero;*

(2) *for any $a \in (A_n)_+$ with $\|a\| = 1$ and $\varepsilon > 0$, there exists an $m > n$ such that for any block*

$$A_m^j = P_j M_{[m,j]}\big(C(X_{m,j})\big) P_j, \quad 1 \le j \le k_m,$$

one has

$$\mathrm{EV}\left(\phi_{n,m}^{-,j}(a)\right) < \varepsilon.$$

In general, (1) implies (2) is always true. Predated [5], it was proved in [3] that if $\dim X_{n,k} \le 2$ for all n and k, then (2) implies (1).

The following proposition and remark are to discuss how the eigenvalue functions behave under a homomorphism from a single block to a single block.

Proposition 2.12 ([8, Section 1.4]). *Let $\phi : QM_{l_1}(C(X))Q \to PM_{k_1}(C(Y))P$ be a unital homomorphism, where X, Y are connected compact metric spaces, and P, Q are projections in $M_{l_1}(C(X))$ and $M_{k_1}(C(Y))$, respectively. Assume that* $\mathrm{rank}(P) = k$*, which is a multiple of* $\mathrm{rank}(Q) = l$*. Then for each $y \in Y$, $\phi(f)(y)$ only depends on the value of $f \in QM_{l_1}(C(X))Q$ at finite many points $x_1(y), x_2(y), \ldots, x_{k/l}(y)$, where $x_i(y)$ may repeat. In fact, if one identifies $Q(x_i(y))M_{l_1}(\mathbb{C})Q(x_i(y))$ with $M_l(\mathbb{C})$, and still denotes the image of $f(x_i(y))$ in $M_l(\mathbb{C})$ by $f(x_i(y))$, then there is a unitary $U_y \in M_{k_1}(C(Y))$ such that*

$$\phi(f)(y) = P(y)U_y \begin{bmatrix} f(x_1(y))_{l\times l} & & & & & & \\ & f(x_2(y))_{l\times l} & & & & & \\ & & \ddots & & & & \\ & & & f(x_{k/l}(y))_{l\times l} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} U_y^* P(y).$$

Obviously, U_y depends on the identification of $Q(x_i(y))M_{l_1}(\mathbb{C})Q(x_i(y))$ and $M_l(\mathbb{C})$.

We denote the set (possibly with multiplicity) $\{x_1(y), x_2(y), \ldots, x_{k/l}(y)\}$ by $\mathrm{Sp}(\phi|_y)$.

Remark 2.13. One can regard $\mathrm{Sp}(\phi|_y) := [x_1(y), x_2(y), \ldots, x_{k/l}(y)]$ as an element in $P^{k/l}X$. Then $Y \ni y \mapsto \mathrm{Sp}(\phi|_y) \in P^{k/l}X$ defines a map $\phi^* : Y \to P^{k/l}X$.

Let X, Y, Z be connected compact metric spaces, and let $\alpha : Y \to P^k X$ and $\beta : Z \to P^l Y$ be two maps. Then α naturally induces a map $\tilde{\alpha} : P^l Y \to P^{kl} X$. We will call the map $\tilde{\alpha} \circ \beta : Z \to P^{kl} X$ the composition of α and β and denote it by $\alpha \circ \beta$. Namely, if $\alpha(y) = [\alpha_1(y), \alpha_2(y), \ldots, \alpha_k(y)] \in P^k X$, for all $y \in Y$, and $\beta(z) = [\beta_1(z), \beta_2(z), \ldots, \beta_l(z)] \in P^l Y$, for all $z \in Z$, then $\alpha \circ \beta$ is defined as follows:

$$\alpha \circ \beta(z) = \big[\alpha_i\big(\beta_j(z)\big) : 1 \leq i \leq k,\ 1 \leq j \leq l\big] \in P^{kl} X \quad \text{for all } z \in Z.$$

We have the following facts.

(a) Let $\phi : QM_{l_1}(C(X))Q \to PM_{k_1}(C(Y))P$, $\psi : PM_{k_1}(C(Y))P \to RM_{m_1}(C(Z))R$ be two unital homomorphisms. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^* : Z \to P^{st} X$, with $\phi^* : Y \to P^s X$ and $\psi^* : Z \to P^t Y$, where $s = \mathrm{rank}(P)/\mathrm{rank}(Q)$ and $t = \mathrm{rank}(R)/\mathrm{rank}(P)$.

(b) Let $\phi : QM_n(C(X))Q \to PM_m(C(Y))P$ $(\mathrm{rank}(Q) = k, \mathrm{rank}(P) = kl)$ be a unital homomorphism and let $f \in (QM_n(C(X))Q)_{s.a}$. Using the above notation, we have

$$\mathrm{Eg}\,\big(\phi(f)\big) = \mathrm{Eg}(f) \circ \phi^* : Y \to P^{kl}\mathbb{R},$$

see Definition 2.7. Let us write the eigenvalue list $E(f) : X \to \mathbb{R}^k$ (of f) as

$$E(f)(x) = \big\{h_1(x) \leq h_2(x) \leq \cdots \leq h_k(x)\big\}$$

with $h_i : X \to \mathbb{R}$ being continuous functions for all i. It follows that

$$\mathrm{Eg}\,\big(\phi(f)\big)(y) = \big[(h_1 \circ \phi^*)(y), (h_2 \circ \phi^*)(y), \ldots, (h_k \circ \phi^*)(y)\big] \in P^{kl}\mathbb{R}.$$

For each $1 \leq i \leq k$, we write the element $(h_i \circ \phi^*)(y) \in P^l\mathbb{R}$ as element $(g_{i,1}(y), g_{i,2}(y), \ldots, g_{i,l}(y)) \in \mathbb{R}^l$ in increasing order $(g_{i,j}(y) \leq g_{i,j+1}(y))$. Then $g_{i,j} : Y \to [0, 1]$ are continuous functions with $\mathrm{rang}(g_{i,j}) \subset \mathrm{rang}(h_i)$. Also we have $\mathrm{Eg}(\phi(f))(y) = [g_{i,j}(y);\ 1 \leq i \leq k,\ 1 \leq j \leq l]$. (Note that, in this calculation, we did not get a precise order of all the eigenfunctions $g_{i,j}$, so we use $\mathrm{Eg}(\phi(f))(y)$ instead of $E(\phi(f))(y)$.)

3. Main theorem

First we give some useful results.

Lemma 3.1. *If a unitary u satisfies $\|u - \mathbf{1}_A\| < \varepsilon < 2$, then* $\mathrm{cel}(u) \leq \frac{\pi}{2}\varepsilon$.

Proof. For $\alpha \in (0, \pi)$, a direct calculation shows that $|\exp(i\alpha) - 1| = 2\sin(\frac{\alpha}{2})$ and $\{\exp(i\theta) : \theta \in [-\alpha, \alpha]\} = \{z \in S^1 : |z - 1| \leq |\exp(i\alpha) - 1|\}$. By $\|u - \mathbf{1}_A\| < \varepsilon < 2$, we have $u = \exp(ih)$ with $h = h^*$ and $\|h\| < 2\arcsin(\frac{\varepsilon}{2})$. Since $\arcsin(x) < \frac{\pi}{2}x$ for $0 < x < \frac{1}{2}$, it follows that $\mathrm{cel}(u) < 2\arcsin(\frac{\varepsilon}{2}) < \frac{\pi}{2}\varepsilon$. ∎

Corollary 3.2. *If $u, v \in U_0(A)$ and $\|u - v\| < \varepsilon < 2$, then $|\operatorname{cel}(u) - \operatorname{cel}(v)| \le \frac{\pi}{2}\varepsilon$.*

Proof. It is easy to see that $|\operatorname{cel}(u) - \operatorname{cel}(v)| \le \operatorname{cel}(u^*v)$ and $\|u^*v - \mathbf{1}_A\| = \|u - v\| < \varepsilon < 2$. Then the conclusion follows from Lemma 3.1. ■

Theorem 3.3. *Suppose that $u \in U_0(M_n(C([0,1])))$ has distinct eigenvalues $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n : [0,1] \to S^1$ are continuous and regarded as elements in $U_0(C([0,1]))$. Then*

$$\operatorname{cel}(u) \ge \max_{1 \le j \le n} \operatorname{cel}(\alpha_j).$$

Proof. Let $H_s(\cdot)$ be a unitary path from u to $\mathbf{1}$. Applying Proposition 2.2, for any $\varepsilon > 0$, we may assume that there exists a piecewise smooth path $F_s(\cdot)$ such that

(1) $\|H_s - F_s\|_\infty < \varepsilon$ for all $s \in [0,1]$;

(2) $|\operatorname{length}_s(H_s) - \operatorname{length}_s(F_s)| < \varepsilon$;

(3) $F_s(t)$ has no repeated eigenvalues for any $(s,t) \in [0,1] \times [0,1]$;

(4) $\Lambda(F_1(t)) = [\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)]$, where $\Lambda(\cdot)$ denotes the eigenvalue list of an $n \times n$ matrix.

By Proposition 2.5, there exist continuous functions $\beta_1(\cdot,\cdot), \beta_2(\cdot,\cdot), \ldots, \beta_n(\cdot,\cdot)$ such that

$$\Lambda\big(F_s(t)\big) = \big[\beta_1(s,t), \beta_2(s,t), \ldots, \beta_n(s,t)\big].$$

Then

$$\Lambda\big(F_1(t)\big) = \big[\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)\big] = \big[\beta_1(1,t), \beta_2(1,t), \ldots, \beta_n(1,t)\big]$$

and

$$\Lambda(F_0(t)) = \big[\beta_1(0,t), \beta_2(0,t), \ldots, \beta_n(0,t)\big].$$

For each $1 \le j \le n$, we have

$$\big|\beta_j(0,t) - 1\big| \le \max_{1 \le j \le n} \big|\beta_j(0,t) - 1\big| \le \big\|F_s(\cdot) - H_s(\cdot)\big\| < \varepsilon.$$

By Lemma 3.1, we have

$$\operatorname{cel}\big(\beta_j(0,\cdot)\big) \le \frac{\pi}{2}\varepsilon, \quad 1 \le j \le n.$$

Hence,

$$\operatorname{cel}\big(\beta_j(1,\cdot)\big) \le \operatorname{cel}\big(\beta_j(0,\cdot)\big) + \operatorname{length}_s\big(\beta_j(s,\cdot)\big), \quad 1 \le j \le n.$$

By Proposition 2.6, we have

$$\operatorname{length}_s(F_s) \ge \max_{1 \le j \le n} \big\{\operatorname{length}_s\big(\beta_j(s,\cdot)\big)\big\}.$$

It follows that

$$\operatorname{length}_s(F_s) \ge \max_{1 \le j \le n} \big\{\operatorname{cel}\big(\beta_j(1,\cdot)\big)\big\} - \frac{\pi}{2}\varepsilon = \max_{1 \le j \le n} \operatorname{cel}(\alpha_j) - \frac{\pi}{2}\varepsilon.$$ ■

Applying the above theorem, we will prove the following result.

Theorem 3.4. *Let $u \in M_n(C([0,1]))$ with $u(t) = \exp(iH(t))$, where the eigenvalue list of H,*

$$E(H)(t) = \{h_1(t), h_2(t), \dots, h_n(t)\},$$

satisfies that

$$\alpha \le h_1(t) \le h_2(t) \le \cdots \le h_n(t) \le \alpha + 2\pi$$

for some $\alpha \in \mathbb{R}$. Then

$$\operatorname{cel}(u) \ge \max_{1\le j\le n} \operatorname{cel}\left(\exp\left(ih_j(\cdot)\right)\right) = \max_{1\le j\le n} \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} \left|h_j(t) - 2k\pi\right|.$$

Proof. By [47, Corollary 1.3], without loss of generality, we may assume that

$$H(t) = \operatorname{diag}\left[h_1(t), h_2(t), \dots, h_n(t)\right].$$

Denote $a := \min_{t\in[0,1]} h_1(t)$.

Case 1. $\alpha < a$. For any $0 < \varepsilon < \min\{a-\alpha, 1\}$, choose ε_i, $i = 1, 2, \dots, n$, satisfying that

$$-\varepsilon < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n < 0.$$

Then

$$\alpha < a - \varepsilon < h_1(t) + \varepsilon_1 < \cdots < h_n(t) + \varepsilon_n \le 2\pi + \alpha.$$

Let $g_j(t) = h_j(t) + \varepsilon_j$, $G(t) = \operatorname{diag}[g_1(t), g_2(t), \dots, g_n(t)]$, and $v(t) = \exp(iG(t))$. It is obvious that

$$\begin{aligned}
\|v(t) - u(t)\| &= \left\| \operatorname{diag}\left[\exp\left(ig_1(t)\right) - \exp\left(ih_1(t)\right), \dots, \exp\left(ig_n(t)\right) - \exp\left(ih_n(t)\right)\right]\right\| \\
&= \max_{1\le j\le n} \left\{\left\| \exp\left(ig_j(t)\right) - \exp\left(ih_j(t)\right)\right\|\right\} \\
&= \max_{1\le j\le n} \left\{\left| \exp(i\varepsilon_j) - 1\right|\right\} \\
&= \max_{1\le j\le n} \left\{2\left| \sin\left(\frac{\varepsilon_j}{2}\right)\right|\right\} \\
&\le \max_{1\le j\le n} \left\{|\varepsilon_j|\right\} < \varepsilon < 1.
\end{aligned}$$

By Corollary 3.2, we have

$$\left|\operatorname{cel}\left(v(\cdot)\right) - \operatorname{cel}\left(u(\cdot)\right)\right| \le \frac{\pi}{2}\varepsilon.$$

Notice that

$$\begin{aligned}
\left|h_j(t) + \varepsilon_j - 2\pi k\right| &\ge \left|h_j(t) - 2\pi k\right| - |\varepsilon_j| \\
&> \left|h_j(t) - 2\pi k\right| - \varepsilon \quad \text{for all } 1 \le j \le n,\ k \in \mathbb{Z}.
\end{aligned}$$

It follows from Theorem 3.3 and Lemma 2.1 that

$$\begin{aligned}\operatorname{cel}\big(v(\cdot)\big) &\ge \max_{1\le j\le n} \operatorname{cel}\big(\exp\big(ig_j(t)\big)\big) = \max_{1\le j\le n}\min_{k\in\mathbb{Z}}\max_{t\in[0,1]} \big|h_j(t)+\varepsilon_j-2\pi k\big| \\ &\ge \max_{1\le j\le n}\min_{k\in\mathbb{Z}}\max_{t\in[0,1]} \big|h_j(t)-2\pi k\big|-\varepsilon.\end{aligned}$$

Hence we have

$$\operatorname{cel}\big(u(\cdot)\big) \ge \max_{1\le j\le n}\min_{k\in\mathbb{Z}}\max_{t\in[0,1]} \big|h_j(t)-2\pi k\big|-\varepsilon-\frac{\pi}{2}\varepsilon.$$

Case 2. $\boldsymbol{\alpha = a}$. Fix $\varepsilon \in (0,1)$. For any $1\le j\le n$, set $g_j(t)=\max\{h_j(t),\alpha+\varepsilon\}$. Define $G(t)=\operatorname{diag}[g_1(t),\dots,g_n(t)]$ and $v(t)=\exp(iG(t))$. It follows that

$$\alpha < \alpha+\varepsilon \le g_1(t)\le\cdots\le g_n(t)\le 2\pi+\alpha \quad \text{for all } t\in[0,1].$$

By the proof of Case 1, we have

$$\operatorname{cel}\big(v(\cdot)\big) \ge \max_{1\le j\le n} \operatorname{cel}\big(\exp\big(ig_j(\cdot)\big)\big).$$

Since $|g_j(t)-h_j(t)|<\varepsilon<1$ for all $t\in[0,1]$, we also have $\|v(\cdot)-u(\cdot)\|<\varepsilon<1$. Applying Corollary 3.2, we have

$$\big|\operatorname{cel}\big(\exp\big(ig_j(\cdot)\big)\big)-\operatorname{cel}\big(\exp\big(ih_j(\cdot)\big)\big)\big| \le \frac{\pi}{2}\varepsilon, \quad \text{for all } 1\le j\le n,$$

and

$$\big|\operatorname{cel}\big(v(\cdot)\big)-\operatorname{cel}\big(u(\cdot)\big)\big| \le \frac{\pi}{2}\varepsilon.$$

This means that

$$\operatorname{cel}\big(\exp\big(ig_j(\cdot)\big)\big) \ge \operatorname{cel}\big(\exp\big(ih_j(\cdot)\big)\big)-\frac{\pi}{2}\varepsilon, \quad \text{for all } 1\le j\le n,$$

and

$$\operatorname{cel}\big(u(\cdot)\big) \ge \operatorname{cel}\big(v(\cdot)\big)-\frac{\pi}{2}\varepsilon.$$

Hence

$$\begin{aligned}\operatorname{cel}\big(u(\cdot)\big) &\ge \max_{1\le j\le n} \operatorname{cel}\big(\exp\big(ih_j(\cdot)\big)\big)-\pi\varepsilon \\ &= \max_{1\le j\le n}\min_{k\in\mathbb{Z}}\max_{t\in[0,1]} \big|h_j(t)-2\pi k\big|-\pi\varepsilon.\end{aligned}$$

■

Corollary 3.5. *Let X be a connected compact metric space. Let $P\in M_m(C(X))$ be a projection with $\operatorname{rank}(P)=n$ and let $u\in PM_m(C(X))P$ be with $u(x)=\exp(iH(x))$, where the eigenvalue list of H,*

$$E(H)(x)=\big\{h_1(x),h_2(x),\dots,h_n(x)\big\},$$

satisfies

$$\alpha\le h_1(x)\le h_2(x)\le\cdots\le h_n(x)\le\alpha+2\pi$$

for some $\alpha\in\mathbb{R}$. Then

$$\operatorname{cel}(u)\ge \max_{1\le j\le n}\min_{k\in\mathbb{Z}}\max_{x\in X}\big|h_j(x)-2k\pi\big|.$$

Proof. Fix $j \in \{1, 2, \dots, n\}$, and let $x_0 \in X$ and $x_1 \in X$ be a minimum point and a maximum point of $\{h_j(x)\}_{x\in X}$, respectively. Choose an embedding $\iota : [0, 1] \to X$ satisfying that $\iota(0) = x_0$ and $\iota(1) = x_1$. Then

$$\min_{k\in\mathbb{Z}} \max_{x\in X} \left|h_j(x) - 2k\pi\right| = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} \left|h_j\big(\iota(t)\big) - 2k\pi\right|.$$

Note that $\mathrm{cel}(u) \geq \mathrm{cel}(\iota^*(u))$, where $\iota^* : PM_m(C(X))P \to P|_{[0,1]}M_m(C([0,1]))P|_{[0,1]} \cong M_n(C([0,1]))$ is given by $\iota^*(f)(t) = f(\iota(t))$. (Note that any projection in $M_m(C([0,1]))$ is trivial, so $P|_{[0,1]}M_m(C([0,1]))P|_{[0,1]} \cong M_n(C([0,1]))$.) Applying Theorem 3.4, we get the corollary. ∎

We shall use the following lemma and its corollary.

Lemma 3.6. *Let $f_1, f_2, \dots, f_n$ be a set of continuous functions from X to $[0,1]$, where X is a connected compact metric space. Let $[c,d] \subset [0,1]$ be a non-degenerated subinterval. Suppose that there exists no $1 \leq j \leq n$ such that $[c,d] \subset \mathrm{rang}(f_j)$. Let $h_k(x)$ be the kth lowest value of $\{f_1(x), f_2(x), \dots, f_n(x)\}$ for any $1 \leq k \leq n$ and any $x \in X$. Then there does not exist $1 \leq k \leq n$ such that $[c,d] \subset \mathrm{rang}(h_k)$.*

Proof. If there exists some $1 \leq k \leq n$ such that $[c,d] \subset \mathrm{rang}(h_k)$, we can choose $x, y \in X$ such that $h_k(x) = c$ and $h_k(y) = d$. Let $A = \{j : f_j(x) \leq c\}$, $B = \{i : f_i(y) \geq d\}$. Since $h_k(x) = c$, we have $|A| \geq k$. Similarly, from $h_k(y) = d$, we have $|B| \geq n - k + 1$. But $|A \cup B| \leq n$. There exists a $p \in A \cap B$. That is, $f_p(x) \leq c$ and $f_p(y) \geq d$. Since f_p is continuous, we have $[c,d] \subset \mathrm{rang}(f_p)$, a contradiction. ∎

Corollary 3.7. (a) *Let $\phi : PM_n(C(X))P \to QM_m(C(Y))Q$ be a unital homomorphism, where X, Y are connected compact metric spaces, and let $a \in PM_n(C(X))P$ be a self-adjoint element such that $E(a) = (h_1, h_2, \dots, h_{\mathrm{rank}(P)})$ and $E(\phi(a)) = (f_1, f_2, \dots, f_{\mathrm{rank}(Q)})$ with $h_i : X \to \mathbb{R}$ and $f_k : Y \to [0,1]$ being continuous functions. Let $[c,d] \subset \mathbb{R}$ be an interval. Then if there is a k such that $[c,d] \subset \mathrm{rang}(f_k)$, then there is an i, such that $[c,d] \subset \mathrm{rang}(h_i)$. Consequently,* $\mathrm{EV}(\phi(a)) \leq \mathrm{EV}(a)$.

(b) *Let $p_1, p_2 \in PM_n(C(X))P$ be two orthogonal projections and let $a_1 \in p_1M_n(C(X))p_1$, $a_2 \in p_2M_n(C(X))p_2$ be two self-adjoint elements, where X is a connected compact metric space. Then* $\mathrm{EV}(a_1 + a_2) \leq \max\{\mathrm{EV}(a_1), \mathrm{EV}(a_2)\}$.

Proof. (a) By Remark 2.13 (b), there are continuous functions $\{g_{i,j} : 1 \leq i \leq \mathrm{rank}(P),\ 1 \leq j \leq \mathrm{rank}(Q)/\mathrm{rank}(P)\}$, with $g_{i,j} : Y \to \mathbb{R}$, such that for each $y \in Y$, as elements in $P^{\mathrm{rank}(Q)}\mathbb{R}$,

$$[f_1, f_2, \dots, f_{\mathrm{rank}(Q)}] = \big[g_{i,j} : 1 \leq i \leq \mathrm{rank}(P),\ 1 \leq j \leq \mathrm{rank}(Q)/\mathrm{rank}(P)\big]$$

and such that $\mathrm{rang}(g_{i,j}) \subset \mathrm{rang}(h_i)$. Then part (a) follows from Lemma 3.6.

(b) Part (b) also follows from Lemma 3.6. ∎

Lemma 3.8. *Let $A = \varinjlim(A_n, \phi_{n,m})$ be a C^*-algebra inductive limit system. Assume that each A_n is of the form*

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}\big(C(X_{n,i})\big) P_{n,i},$$

where k_n and $[n,i]$ are positive integers, $X_{n,i}$ are connected compact metric spaces, and $M_{[n,i]}$ are $[n,i] \times [n,i]$ matrices. Suppose condition (2) of Proposition 2.11 does not hold for the inductive limit system (in the case of slow dimension growth or no dimension growth, this is equivalent to the condition that A is not of real rank zero). There exists an interval $[c,d] \subset [0,1]$, a positive integer n, and $x \in (A_n)_+$ with $\|x\| = 1$ such that for any $m \geq n$, $\phi_{n,m}(x)$ admits the following representation:

$$\phi_{n,m}(x) = \{y_k^m\}_{k=1}^{k_m} \in A_m = \bigoplus_{k=1}^{k_m} P_{m,k} M_{[m,k]}\big(C(X_{m,k})\big) P_{m,k}, \tag{3.1}$$

and there exist $1 \leq k(m) \leq k_m$ and $1 \leq i(m) \leq \operatorname{rank}(P_{m,k(m)})$ such that

$$[c,d] \subset \operatorname{rang}\big(h_{i(m)}^{k(m)}\big),$$

where $h_j^{k(m)}(t)$ is the jth lowest eigenvalue of $y_{k(m)}^m(t)$, for $1 \leq j \leq \operatorname{rank}(P_{m,k(m)})$, and $\operatorname{rang}(h)$ is the range of function h.

Proof. Since condition (2) of Proposition 2.11 does not hold, there exist $\varepsilon > 0$, a positive integer n, and $x \in (A_n)_+$ with $\|x\| = 1$ such that for any $m \geq n$ and $\phi_{n,m}(x)$ admitting representation (3.1), there exist $1 \leq k(m) \leq k_m$, $1 \leq i(m) \leq \operatorname{rank}(P_{m,k(m)})$, and $t_{i(m)}, s_{i(m)} \in X_{m,k}$ such that

$$\big|h_{i(m)}^{k(m)}(t_{i(m)}) - h_{i(m)}^{k(m)}(s_{i(m)})\big| \geq \varepsilon,$$

where $h_i^{k(m)}(t)$ is the ith lowest eigenvalue of $y_{k(m)}^m(t)$ for $1 \leq i \leq \operatorname{rank}(P_{m,k(m)})$. For $m \geq n$, we denote by $I_{i(m)}^{k(m)}$ the closed interval with end points $h_{i(m)}^{k(m)}(s_{i(m)})$ and $h_{i(m)}^{k(m)}(t_{i(m)})$. We also denote by $J_{i(m)}^{k(m)}$ the closed interval satisfying

$$\text{middle point of } J_{i(m)}^{k(m)} = \text{middle point of } I_{i(m)}^{k(m)} \quad \text{and} \quad \big|J_{i(m)}^{k(m)}\big| = \frac{1}{2}\big|I_{i(m)}^{k(m)}\big| \geq \frac{\varepsilon}{2}.$$

Choose a positive integer N such that $\frac{2}{N} < \varepsilon$. We denote $a_p = \frac{p}{N}$ for $0 \leq p \leq N$. Since $|J_{i(m)}^{k(m)}| \geq \frac{1}{2}\varepsilon$ and $J_{i(m)}^{k(m)} \subset [0,1]$ for all $m \geq n$, then there exist a $0 \leq p \leq N$ and a subsequence m_j such that

$$a_p \in J_{i(m_j)}^{k(m_j)} \quad \text{for all } j \geq 1.$$

Denote $I = [a_p - \frac{\varepsilon}{4}, a_p]$ and $J = [a_p, a_p + \frac{\varepsilon}{4}]$, then $I \subset I_{i(m_j)}^{k(m_j)}$ or $J \subset I_{i(m_j)}^{k(m_j)}$ for each $j \geq 1$. Without loss of generality, we assume that $I \subset I_{i(m_j)}^{k(m_j)}$ for each $j \geq 1$. Otherwise, we shall choose a subsequence of $\{m_j\}_{j=1}^{\infty}$.

We have proved that the conclusion holds for m_j for each $j \geq 1$. For $m \geq n$, there exists $j \geq 1$ such that $m_{j-1} < m \leq m_j (m_0 = n)$. We consider

$$\phi_{m,m_j}^{l,k(m_j)} : P_{m,l} M_{[m,l]}\big(C(X_{m,l})\big) P_{m,l} \to P_{m_j,k(m_j)} M_{[m_j,k(m_j)]}\big(C(X_{m_j,k(m_j)})\big) P_{m_j,k(m_j)}$$

the homomorphism which is the composition of the restriction of ϕ_{m,m_j} on the lth block of A_m and the quotient map from A_{m_j} to the $k(m_j)$th block of A_{m_j}.

We claim that there exist $1 \leq k(m) \leq k_m$ and $1 \leq i(m) \leq \operatorname{rank}(P_{m,k(m)})$ such that

$$I \subset \operatorname{rang}\big(h_{i(m)}^{k(m)}\big),$$

where $h_i^{k(m)}(t)$ is the ith lowest eigenvalue of $y_{k(m)}^m(t)$. Otherwise, for each $1 \leq k \leq k_m$ and $1 \leq i \leq \operatorname{rank}(P_{m,k})$, $\operatorname{rang}(h_i^k)$ does not contain the interval I. By Corollary 3.7, we conclude that there exists no $1 \leq k \leq k_{m_j}$ and $1 \leq i \leq \operatorname{rank}(P_{m_j,k})$ such that

$$I \subset \operatorname{rang}(g_i^k),$$

where $g_i^k(t)$ is the ith lowest eigenvalue of $y_k^{m_j}(t)$, a contradiction. ■

In the proof of the following theorem, we will use, from Notation 2.10, $\phi_{n,m}^{-,j} = \pi_j \circ \phi_{n,m}$, which is the homomorphism from A_n to A_m^j, where $\pi_j : A_m \to A_m^j$ is the projection map to the jth block.

Theorem 3.9. *Let A be a unital AH-algebra with slow dimension growth condition which is not of real rank zero. Then*

$$\operatorname{cel}_{\mathrm{CU}}(A) \geq 2\pi.$$

Proof. Let $A = \lim(A_n, \phi_{n,n+1})$ be an AH-algebra with

$$A_n = \bigoplus_{j=1}^{t_n} P_{n,j} M_{[n,j]}\big(C(X_{n,j})\big) P_{n,j}.$$

By [2], without loss of generality, one may assume that $X_{n,j}$ are finite simplicial complexes (see also [10, Theorem 2.1]). Furthermore, we can assume that each $X_{n,j}$ is connected.

Since A is unital, there is a k_0 such that for all $k \geq k_0$, $\phi_{k,k+1}(1_{A_k}) = 1_{A_{k+1}}$ and $\phi_{k,\infty}(1_{A_k}) = 1_A$. Without loss of generality, we assume $k_0 = 1$. For any $\varepsilon > 0$, choose an integer L such that $\frac{2\pi}{L} < \varepsilon$. Since A has slow dimension growth, it follows from a standard argument by using the stability property of vector bundles (see [23, Chapter 9, Theorem 1.2]) that there exist a positive integer n and a full projection $p \in A_n$ such that

$$L[p] < [1_{A_n}] < L_1[p] \tag{3.2}$$

for some positive integer L_1. Note that A is not of real rank zero. We know that $\phi_{n,\infty}(p) \times A\phi_{n,\infty}(p)$ is stably isomorphic to A, and hence

$$\phi_{n,\infty}(p)A\phi_{n,\infty}(p) = \lim\big(\phi_{n,m}(p)A_m\phi_{n,m}(p), \widetilde{\phi}_{m,m'}\big)$$

is also not of real rank zero, where $\widetilde{\phi}_{m,m'}$ denotes the restriction map

$$\phi_{m,m'}|_{\phi_{n,m}(p)A_m\phi_{n,m}(p)} : \phi_{n,m}(p)A_m\phi_{n,m}(p) \to \phi_{n,m'}(p)A_{m'}\phi_{n,m'}(p),$$

for $m' \geq m \geq n$. That is, condition (1) of Proposition 2.11 – the limit algebra to be of real rank zero – does not hold for the inductive system $(\phi_{n,m}(p)A_m\phi_{n,m}(p), \widetilde{\phi}_{m,m'})$. By the equivalence in Proposition 2.11, condition (2) of the proposition also does not hold. By Lemma 3.8, there exist an interval $[c,d] \subset [0,1]$, an integer $n_1 \geq n$, and a positive element $x \in (\phi_{n,n_1}(p)A_{n_1}\phi_{n,n_1}(p))_+$ with $\|x\| = 1$ such that for every $m \geq n_1$, $\widetilde{\phi}_{n_1,m}(x)$ has the following representation:

$$\widetilde{\phi}_{n_1,m}(x) = \big(y_1^m, y_2^m, \ldots, y_{k_m}^m\big) \in \bigoplus_{i=1}^{k_m} \phi_{n,m}^{-,i}(p)A_m^i\phi_{n,m}^{-,i}(p).$$

There exist $1 \leq k(m) \leq k_m$ and $1 \leq i(m) \leq \operatorname{rank}(\phi_{n,m}^{-,k(m)}(p))$ such that

$$[c,d] \subset \operatorname{rang}\big(h_{i(m)}^{k(m)}\big),$$

where $h_i^{k(m)}(t)$ is the ith lowest eigenvalue of $y_{k(m)}^m(t)$ for $1 \leq i \leq \operatorname{rank}(\phi_{n,m}^{-,k(m)}(p))$.

Since p is a full projection in A_n and $L[p] < [1_{A_n}]$, there exists a set of mutually orthogonal rank one projections $p_1, p_2, \ldots, p_L \in A_n$ such that $p_1 = p$, $p_i \sim p_j \sim p$ and $\sum_{i=1}^L p_i < \mathbf{1}_{A_n}$. Let

$$q = \sum_{i=1}^L p_i.$$

It is easy to see that qA_nq and $M_L(pA_np)$ are isomorphic. So we can identify $M_L(pA_np)$ with $qA_nq \subset A_n$ and identify $M_L(\phi_{n,n_1}(p)A_{n_1}\phi_{n,n_1}(p))$ with $\phi_{n,n_1}(q)A_{n_1}\phi_{n,n_1}(q) \subset A_{n_1}$.

We define the continuous functions

$$\chi : [0,1] \to [0,1], \quad \chi(t) = \begin{cases} 0, & t \in [0,c], \\ \frac{1}{d-c}(t-c), & t \in [c,d], \\ 1, & t \in [d,1], \end{cases}$$

$$\chi_1 : [0,1] \to \Big[0, \frac{1}{L}\Big], \quad \chi_1(t) = \frac{1}{L}t,$$

and

$$\chi_2 : [0,1] \to \Big[-1 + \frac{1}{L}, 0\Big], \quad \chi_2(t) = \Big(-1 + \frac{1}{L}\Big)t.$$

Set

$$h = \begin{bmatrix} e^{2\pi i \chi_2 \circ \chi(x)} & 0 & \cdots & 0 \\ 0 & e^{2\pi i \chi_1 \circ \chi(x)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{2\pi i \chi_1 \circ \chi(x)} \end{bmatrix}_{L \times L} \in \phi_{n,n_1}(q) A_{n_1} \phi_{n,n_1}(q),$$

where $\chi(x) \in \phi_{n,n_1}(p) A_{n_1} \phi_{n,n_1}(p)$ and $\chi_i \circ \chi(x) \in \phi_{n,n_1}(p) A_{n_1} \phi_{n,n_1}(p)$, $i = 1, 2$, are functional calculus of positive element x. We also identify

$$\phi_{n,n_1}(q) A_{n_1} \phi_{n,n_1}(q) \cong M_L\big(\phi_{n,n_1}(p) A_{n_1} \phi_{n,n_1}(p)\big).$$

Let $u = h \oplus (\mathbf{1}_{A_{n_1}} - \phi_{n,n_1}(q))$. It is easy to check that $\det(u(z)) = 1$ for all $z \in \mathrm{Sp}(A_{n_1})$ and $u \in U_0(A_{n_1})$. It follows from [44] that $u \in \mathrm{CU}(A_{n_1})$.

We shall show that $\mathrm{cel}(\phi_{n_1,m}(u)) \geq 2\pi - \varepsilon$ for all $m \geq n_1$. For a fixed $m \geq n_1$, we have $\widetilde{\phi}_{n_1,m}(h) = \exp(2\pi i H)$, where

$$H = \begin{bmatrix} \chi_2 \circ \chi\big(\widetilde{\phi}_{n_1,m}(x)\big) & 0 & \cdots & 0 \\ 0 & \chi_1 \circ \chi\big(\widetilde{\phi}_{n_1,m}(x)\big) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \chi_1 \circ \chi\big(\widetilde{\phi}_{n_1,m}(x)\big) \end{bmatrix}_{L \times L}.$$

It follows that

$$\widetilde{\phi}_{n_1,m}(x) \in \phi_{n,m}(p) A_m \phi_{n,m}(p)$$

and hence $H \in M_L(\phi_{n,m}(p) A_m \phi_{n,m}(p)) = \phi_{n,m}(q) A_m \phi_{n,m}(q) \subset A_m$.

Note that $\widetilde{\phi}_{n_1,m}(x) = (y_1^m, y_2^m, \ldots, y_{k_m}^m)$ with each $y_j^m \in \phi_{n,m}^{-,j}(p) A_m^j \phi_{n,m}^{-,j}(p)$. There exist $1 \leq k(m) \leq k_m$ and $1 \leq i(m) \leq \mathrm{rank}(\phi_{n,m}^{-,k(m)}(p))$ such that

$$[c, d] \subset \mathrm{rang}\,\big(h_{i(m)}^{k(m)}\big),$$

where $h_i^{k(m)}(t)$ is the ith lowest eigenvalue of $y_{k(m)}^m(t)$ for $1 \leq i \leq \mathrm{rank}(\phi_{n,m}^{-,k(m)}(p))$. Again notice that

$$\begin{aligned} &\chi_2 \circ \chi\big(\widetilde{\phi}_{n_1,m}(x)\big) \\ &\quad = \big(\chi_2 \circ \chi(y_1^m), \chi_2 \circ \chi(y_2^m), \ldots, \chi_2 \circ \chi(y_{k_m}^m)\big) \in \bigoplus_{i=1}^{k_m} \phi_{n,m}^{-,i}(p) A_m^i \phi_{n,m}^{-,i}(p) \end{aligned}$$

and

$$\begin{aligned} &\chi_1 \circ \chi\big(\widetilde{\phi}_{n_1,m}(x)\big) \\ &\quad = \big(\chi_1 \circ \chi(y_1^m), \chi_1 \circ \chi(y_2^m), \ldots, \chi_1 \circ \chi(y_{k_m}^m)\big) \in \bigoplus_{i=1}^{k_m} \phi_{n,m}^{-,i}(p) A_m^i \phi_{n,m}^{-,i}(p). \end{aligned}$$

Write $H = (H_1, H_2, \ldots, H_{k_m}) \in \bigoplus_{i=1}^{k_m} \phi_{n,m}^{-,i}(q) A_m^i \phi_{n,m}^{-,i}(q)$. It follows that

$$H_i = \begin{bmatrix} \chi_2 \circ \chi(y_i^m) & 0 & \cdots & 0 \\ 0 & \chi_1 \circ \chi(y_i^m) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \chi_1 \circ \chi(y_i^m) \end{bmatrix}_{L\times L}.$$

This means that

$$\widetilde{\phi}_{n_1,m}(h) = \exp(2\pi i H) = \big(\exp(2\pi i H_1), \exp(2\pi i H_2), \ldots, \exp(2\pi i H_{k_m})\big),$$

and hence

$$\mathrm{cel}\left(\widetilde{\phi}_{n_1,m}(h)\right) \geq \mathrm{cel}\left(\exp(2\pi i H_k)\right) \quad \text{for all } 1 \leq k \leq k_m.$$

In particular, we have $\mathrm{cel}(\widetilde{\phi}_{n_1,m}(h)) \geq \mathrm{cel}(\exp(2\pi i H_{k(m)}))$ and

$$\mathrm{cel}\left(\phi_{n_1,m}(u)\right) \geq \mathrm{cel}\left(\exp(2\pi i H_{k_m}) \oplus \left(\mathbf{1}_{A_m^{k(m)}} - \phi_{n,m}^{-,k(m)}(q)\right)\right).$$

Let $\mathrm{rank}(\phi_{n,m}^{-,k(m)}(p)) = K$. The eigenvalue list of $H_{k(m)}$ satisfies that

$$\begin{aligned} -1 + \frac{1}{L} &\leq \chi_2 \circ \chi \circ h_K^{k(m)} \leq \chi_2 \circ \chi \circ h_{K-1}^{k(m)} \leq \cdots \leq \chi_2 \circ \chi \circ h_1^{k(m)} \\ &\leq \underbrace{\chi_1 \circ \chi \circ h_1^{k(m)} \leq \chi_1 \circ \chi \circ h_1^{k(m)} \leq \cdots \leq \chi_1 \circ \chi \circ h_1^{k(m)}}_{L-1} \\ &\leq \underbrace{\chi_1 \circ \chi \circ h_2^{k(m)} \leq \chi_1 \circ \chi \circ h_2^{k(m)} \leq \cdots \leq \chi_1 \circ \chi \circ h_2^{k(m)}}_{L-1} \\ &\leq \cdots \\ &\leq \underbrace{\chi_1 \circ \chi \circ h_K^{k(m)} \leq \chi_1 \circ \chi \circ h_K^{k(m)} \leq \cdots \leq \chi_1 \circ \chi \circ h_K^{k(m)}}_{L-1} \\ &\leq \frac{1}{L}. \end{aligned}$$

That is, $\widetilde{\phi}_{n_1,m}^{-,k(m)}(h) = \exp(2\pi i H_{k(m)})$ satisfies the condition of Corollary 3.5. Note that each function in the above list is either non-negative (with range $[0, \frac{1}{L}]$) or non-positive (with range $[-1 + \frac{1}{L}, 0]$) – none of the functions is crossing over point zero. Applying (3) of Remark 2.8, we know that $\phi_{n_1,m}^{-,k(m)}(u) = \exp(2\pi i H_{k(m)}) \oplus (\mathbf{1}_{A_m^{k(m)}} - \phi_{n,m}^{-,k(m)}(q))$ also satisfies the condition of Corollary 3.5. By the corollary, we have

$$\mathrm{cel}\left(\phi_{n_1,m}^{-,k(m)}(u)\right) \geq \min_{j \in \mathbb{Z}} \max_{y \in X_{m,k(m)}} 2\pi \left|\chi_2 \circ \chi \circ h_{i(m)}^{k(m)}(y) - j\right|.$$

Noting that $[c, d] \subset \mathrm{rang}(h_{i(m)}^{k(m)}) \subset [0, 1]$, by the definitions of χ and χ_2, we have

$$\mathrm{rang}\left(\chi_2 \circ \chi \circ h_{i(m)}^{k(m)}\right) = \left[-1 + \frac{1}{L}, 0\right]$$

and hence

$$\min_{j\in\mathbb{Z}} \max_{y\in X_{m,k(m)}} 2\pi\left|\chi_2\circ\chi\circ h_{i(m)}^{k(m)}(y)-j\right| = \left(1-\frac{1}{L}\right)2\pi \geq 2\pi-\varepsilon. \qquad \blacksquare$$

Remark 3.10. Evidently, our proof also works for the case of no dimension growth provided that $\lim_{n\to\infty}\min_i\{\mathrm{rank}(\mathbf{1}_{A_n^i})\}=\infty$. Note that we use the slow dimension growth in two places: one is to get (3.2); the other is to get the implication from that the algebra is not of real rank zero to that condition (2) of Proposition 2.11 does not hold. For the above case, both can be done without the above-mentioned slow dimension growth condition but with slow dimension condition in [15] (which includes all non-dimension growth inductive limits) and $\lim_{n\to\infty}\min_i\{\mathrm{rank}(\mathbf{1}_{A_n^i})\}=\infty$.

For all $k\geq 1$ and $u\in \mathrm{CU}(M_k(C([0,1])))$, Lin [36] proved that $\mathrm{cel}(u)\leq 2\pi$. In fact, it will be proved that $\mathrm{cel}(u)\leq \frac{k-1}{k}2\pi$ in the following proposition. This proposition shows that the slow dimension growth condition used in this article cannot be replaced by Gong's slow dimension growth condition in [15], which does not imply that

$$\lim_{n\to\infty}\min_i\left\{\mathrm{rank}(\mathbf{1}_{A_n^i})\right\}=\infty.$$

Proposition 3.11. $\mathrm{cel}_{\mathrm{CU}}(M_k(C([0,1])))=\frac{k-1}{k}2\pi$.

Proof. From the construction in [39], we know that $\mathrm{cel}_{\mathrm{CU}}(M_k(C([0,1])))\geq\frac{k-1}{k}2\pi$. The following proof of $\mathrm{cel}_{\mathrm{CU}}(M_k(C([0,1])))\leq\frac{k-1}{k}2\pi$ is inspired by [21, Section 3] (see also [36, proof of Lemma 4.2]). Let $u\in\mathrm{CU}(M_k(C([0,1])))$ and $\varepsilon>0$. Using the proof of Lemma 4.2 in [36], one can find $v\in\mathrm{CU}(M_k(C([0,1])))$ satisfying the following conditions:

(1) $v(t)=\sum_{j=1}^k\exp(2\pi i h_j(t))p_j(t)$, where $h_j(t)\in C([0,1])_{s.a}$ and $\{p_1,p_2,\ldots,p_k\}$ is a set of mutually orthogonal rank one projections;

(2) $\sum_{j=1}^k h_j(t)=0$ for all $t\in[0,1]$, which implies that $\det(v(t))=0$ for all $t\in[0,1]$ and hence $v\in\mathrm{CU}(M_k(C([0,1])))$;

(3) $h_j(t)-h_l(t)\notin\mathbb{Z}$ for any $t\in[0,1]$ when $j\neq l$. This implies that $v(t)$ has distinct eigenvalues. Furthermore, one can require that

$$\max_{1\leq j\leq k}h_j(0)-\min_{1\leq j\leq k}h_j(0)<1,$$

which implies $0<\max_{1\leq j\leq k}h_j(t)-\min_{1\leq j\leq k}h_j(t)<1$ for all $t\in[0,1]$, by continuity of the functions h_j;

(4) $|h_j(t)|<1$ for all $t\in[0,1]$ and $1\leq j\leq k$;

(5) $\|u-v\|<\varepsilon$.

We shall show that

$$\|h_j\|<\frac{k-1}{k}\qquad\text{for all } 1\leq j\leq k.$$

By condition (3), without loss of generality, we can assume that

$$h_1(t) > h_2(t) > \cdots > h_k(t) \quad \text{and} \quad h_1(t) - h_k(t) < 1 \quad \text{for all } t \in [0,1].$$

For fixed $1 \le k_0 \le k$ and $t \in [0,1]$, we have

$$\begin{aligned} 0 &= h_1(t) + h_2(t) + \cdots + h_{k_0}(t) + \cdots + h_k(t) \\ &> k_0 h_{k_0}(t) + (k-k_0) h_k(t) \\ &> k_0 h_{k_0}(t) + (k-k_0)\big(h_1(t) - 1\big) \\ &> k_0 h_{k_0}(t) + (k-k_0)\big(h_{k_0}(t) - 1\big) \\ &= k h_{k_0}(t) - k + k_0. \end{aligned}$$

Hence

$$h_{k_0}(t) < \frac{k-k_0}{k}.$$

On the other hand,

$$\begin{aligned} 0 &= h_1(t) + h_2(t) + \cdots + h_{k_0}(t) + \cdots + h_k(t) \\ &< (k_0-1) h_1(t) + (k-k_0+1) h_{k_0}(t) \\ &< (k_0-1)\big(1 + h_k(t)\big) + (k-k_0+1) h_{k_0}(t) \\ &< (k_0-1)\big(1 + h_{k_0}(t)\big) + (k-k_0+1) h_{k_0}(t) \\ &< k h_{k_0}(t) + k_0 - 1. \end{aligned}$$

Hence

$$h_{k_0}(t) > -\frac{k_0-1}{k}.$$

It follows that

$$\|h_{k_0}\| < \frac{k-1}{k}.$$

Let

$$v_s(t) = \sum_{j=1}^{k} \exp\big(2\pi i s h_j(t)\big) p_j(t) \quad \text{for all } s \in [0,1], t \in [0,1].$$

Then v_s is a path with $v_0(t) = v(t)$ and $v_1(t) = \mathbf{1}$. Furthermore,

$$\begin{aligned} \text{length}_s(v_s) &= \int_0^1 \left\| \frac{dv_s}{ds} \right\| ds \\ &= \int_0^1 \left\| \sum_{j=1}^{k} 2\pi i h_j(t) \exp\big(2\pi i s h_j(t)\big) p_j(t) \right\| ds \\ &= 2\pi \int_0^1 \max_{1 \le j \le k} \|h_j\| ds \\ &< 2\pi \frac{k-1}{k}. \end{aligned}$$

By condition (5) and Corollary 3.2, it follows that

$$\mathrm{cel}(u) \leq \mathrm{cel}(v) + \frac{\pi}{2}\varepsilon \leq \mathrm{length}_s(v_s) + \frac{\pi}{2}\varepsilon < 2\pi\frac{k-1}{k} + \frac{\pi}{2}\varepsilon.$$

Since ε can be arbitrarily small, we have $\mathrm{cel}(u) \leq 2\pi\frac{k-1}{k}$. ■

4. Exponential length in AH-algebras with ideal property

Definition 4.1. We say that a C^*-algebra A has the ideal property if every closed two-sided ideal of A is generated as ideals by the projections inside the ideal.

It is easy to see that all simple AH-algebras and all real rank zero C^*-algebras have the ideal property. In this part, we shall show that $\mathrm{cel}_{\mathrm{CU}}(A) \leq 2\pi$ for all AH-algebras with the ideal property of no dimension growth.

As in [8], we denote by $T_{II,k}$ the two-dimensional connected simplicial complex with $H^1(T_{II,k}) = 0$ and $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$, and also, we denote by I_k the subalgebra of $M_k(C([0,1])) = C([0,1], M_k(\mathbb{C}))$, consisting of functions f satisfying $f(0) \in \mathbb{C}1_k$ and $f(1) \in \mathbb{C}1_k$. I_k is called Elliott dimension drop interval algebra. As in [18], we denote by $\mathcal{HD}$ the class of algebras of direct sums of building blocks of forms $M_l(I_k)$ and $PM_n(C(X))P$, with X being one of the spaces $\{pt\}$, $[0,1]$, S^1, and $T_{II,k}$, and with $P \in M_n(C(X))$ being a projection. A C^*-algebra is called an $\mathcal{AHD}$ algebra if it is an inductive limit of algebras in $\mathcal{HD}$. In [19, 20, 24], it is proved that all AH-algebras with ideal property of no dimension growth are $\mathcal{AHD}$ algebras.

Lemma 4.2 ([39, Corollary 3.2]). *Let $Z = \{u \in U(M_n(\mathbb{C})) : u$ has repeated eigenvalues$\}$. Then Z is the union of finitely many submanifolds of $U(M_n(\mathbb{C}))$ of codimensions at least three.*

Since $\dim(T_{II,k}) = 2$, the following lemma follows from Lemma 4.2 and a standard transversal argument.

Lemma 4.3. *Let $u \in U(PM_n(C(T_{II,k}))P)$, where P is a projection in $M_n(C(T_{II,k}))$. For any $\varepsilon > 0$, there exists $v \in U(PM_n(C(T_{II,k}))P)$ such that*

(1) $\|u - v\| \leq \varepsilon$;

(2) $\mathrm{Sp}(v(y)) = \{\beta_1(y), \beta_2(y), \ldots, \beta_k(y)\}$, *where* $k = \mathrm{rank}(P)$ *and* $\beta_i(y) \neq \beta_j(y)$ *for all* $i \neq j$ *and* $y \in T_{II,k}$.

Let $F^kS^1 = \mathrm{Hom}(C(S^1), M_k(\mathbb{C}))_1$ and $\Pi : F^kS^1 \to P^kS^1$ be defined as in Definition 2.4. Let $\mathring{F}^kS^1$ be the set of homomorphism with distinct spectrum and $\mathring{P}^kS^1 = \Pi(\mathring{F}^kS^1)$.

Lemma 4.4. *$\pi_1(\mathring{P}^kS^1) = \mathbb{Z}$ is torsion free.*

Proof. Note that F^kS^1 is homeomorphic to $U_k(\mathbb{C}) = U(k)$ and $\mathring{F}^kS^1$ corresponds to the set of all unitaries $u \in U(k)$ with distinct spectrum, which is a union of finitely many sub-manifolds of $U(k)$ of codimensions at least three. Hence $\pi_1(\mathring{F}^kS^1) = \pi_1(U(k)) = \mathbb{Z}$.

Consider the fibration map $\Pi|_{\mathring{F}^kS^1} : \mathring{F}^kS^1 \to \mathring{P}^kS^1$ whose fiber is the simply connected flag manifold $U(k)/\underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{k}$; hence we get the desired result. ∎

Lemma 4.5. *Let $F : T_{II,k} \to P^kS^1$ be a continuous function. Suppose*

$$F(t) = \big[x_1(t), x_2(t), \ldots, x_k(t)\big],$$

and for all $t \in T_{II,k}$, $x_i(t) \neq x_j(t)$ for $i \neq j$. Then there are continuous functions $f_1, f_2, \ldots, f_k : T_{II,k} \to S^1$ such that

$$F(t) = \big[f_1(t), f_2(t), \ldots, f_k(t)\big] \quad \textit{for all } t \in T_{II,k}.$$

Proof. Note that the restriction of the map $\pi : (S^1)^k \to P^kS^1$ on $\pi^{-1}(\mathring{P}^kS^1)$ is a covering map and $\pi_1(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$ is a torsion group. The lemma follows from Lemma 4.4 and the lifting Proposition 1.33 in [22]. ∎

Theorem 4.6. *Let $u \in \mathrm{CU}(PM_n(C(T_{II,k}))P)$, then, for any $\varepsilon > 0$, there exists a self-adjoint element $h \in PM_n(C(T_{II,k}))P$ with $\|h\| \leq 1$ such that $\|u - \exp(2\pi i h)\| < \varepsilon$. In particular,* $\mathrm{cel}(u) \leq 2\pi$.

Proof. The proof is inspired by the proof of Lemma 4.2 of [36] (see also Remark 3.13 of [21]). By Lemma 4.3, for any $\varepsilon > 0$, there exists $v \in \mathrm{CU}(PM_n(C(T_{II,k}))P)$ such that $v(y)$ has distinct eigenvalues for all $y \in T_{II,k}$ and $\|u - v\| \leq \varepsilon$. By Lemma 4.5, one can write

$$\mathrm{Sp}\big(v(y)\big) = \big\{\beta_1(y), \beta_2(y), \ldots, \beta_q(y)\big\}$$

for continuous functions $\beta_j : T_{II,k} \to S^1$, $j = 1, 2, \ldots, q$, where $q = \mathrm{rank}(P)$ and $\beta_l(y) \neq \beta_j(y)$ for all $l \neq j$ and $y \in T_{II,k}$.

Fix a base point $y_0 \in T_{II,k}$. We can choose some $b_j \in C(T_{II,k})_{\mathrm{s.a.}}$ such that $\beta_j(y) = \exp(2\pi i b_j(y))$, where $b_j(y_0) \in (-\frac{1}{2}, \frac{1}{2}]$, $j = 1, 2, \ldots, q$. Since $v \in \mathrm{CU}(PM_n(C(T_{II,k}))P)$, one obtains $\det(v(y)) = 1$ for all $y \in T_{II,k}$. Then $\sum_{j=1}^{q} b_j(y_0) = m$ for some integer m. Since $b_j(y_0) \in (0, 1]$, we have $-q < m < q$.

If $m \geq 1$, without loss of generality, we can assume that $b_1(y_0) > b_2(y_0) > \cdots > b_q(y_0)$. It follows that $b_m(y_0) > 0$. Define $a_j(y) = b_j(y) - 1$, for $j = 1, 2, \ldots, m$, $y \in T_{II,k}$, and $a_j(y) = b_j(y)$, for $j > m$, $y \in T_{II,k}$.

Then

$$\sum_{j=1}^{q} a_j(y_0) = 0 \quad \text{and} \quad \big|a_j(y_0)\big| < 1 \quad \text{for all } j = 1, 2, \ldots, q. \tag{4.1}$$

Also, since $b_j(y_0) > -\frac{1}{2}$, we have $\min_j a_j(y_0) = b_m(y_0) - 1$. Note that $\max_j a_j(y_0) < b_m(y_0)$, we have

$$\max_j a_j(y_0) - \min_j a_j(y_0) < 1. \tag{4.2}$$

If $m \le -1$, we directly assume that $b_1(y_0) < b_2(y_0) < \cdots < b_q(y_0)$. It follows that $b_m(y_0) < 0$. Define $a_j(y) = b_j(y) + 1$, for $j = 1, 2, \ldots, m$, $y \in T_{II,k}$, and $a_j(y) = b_j(y)$, for $j > m$, $y \in T_{II,k}$. Then (4.1) and (4.2) also hold.

Hence, $\beta_j(t) = \exp(2\pi i b_j(y)) = \exp(2\pi i a_j(y))$ for each $1 \le j \le q$. Since

$$\det\big(v(y)\big) = 1 \quad \text{for all } y \in T_{II,k},$$

we have

$$\sum_{j=1}^{q} a_j(y_0) \in \mathbb{Z} \quad \text{for all } y \in T_{II,k}.$$

By the continuity of the functions a_j and the connectedness of $T_{II,k}$, we know that $\sum_{j=1}^{q} a_j(y)$ is a constant function of $y \in T_{II,k}$. By (4.1), we have

$$\sum_{j=1}^{q} a_j(y) = 0 \quad \text{for all } y \in T_{II,k}. \tag{4.3}$$

Since $\beta_l(y) \ne \beta_j(y)$ for any $l \ne j$ and $y \in T_{II,k}$, we have

$$a_l(y) - a_j(y) \notin \mathbb{Z} \quad \text{for all } y \in T_{II,k},\ l \ne j.$$

Again, by the continuity of the functions a_j and the connectedness of $T_{II,k}$, and also by (4.2), we have

$$0 < \max_j a_j(y) - \min_j a_j(y) < 1 \quad \text{for all } y \in T_{II,k}. \tag{4.4}$$

For each fixed y, by (4.3), either $a_j(y) = 0$ for all $1 \le j \le q$, which is impossible since $a_j(y) \ne a_l(y)$ when $j \ne l$, or $a_j(y) < 0$ for some j and $a_l(y) > 0$ for some other l. By (4.4), we have

$$\big|a_j(y)\big| < 1 \quad \text{for all } y \in T_{II,k}.$$

Fix $j \in \{1, 2, \ldots, q\}$. For any $y \in T_{II,k}$, let $p_j(y)$ be the spectrum projection of $v(y)$ with respect to the spectrum $\exp(2\pi i a_j(y))$, which is rank one projection continuously depending on y. Then $v(y) = \sum_{j=1}^{q} \exp(2\pi i a_j(y)) p_j(y)$.

Let $h \in (PM_n(C(T_{II,k}))P)_{s.a}$ be defined by $h(y) = \sum_{j=1}^{q} a_j(y) p_j(y)$. Then $v = \exp(2\pi i h)$. Furthermore, $\|h\| = \max_{j,y} |a_j(y)| \le 1$. Consequently, $\|u - \exp(2\pi i h)\| = \|u - v\| < \varepsilon$. ■

Using a similar method, we can get the following result.

Theorem 4.7. *Let $u \in \mathrm{CU}(PM_n(C(X))P)$, where X is one of the spaces $\{pt\}$, $[0,1]$, and S^1; and P is a projection in $M_n(C(X))$. Then for any $\varepsilon > 0$, there exists a self-adjoint element $h \in PM_n(C(X))P$ with $\|h\| \le 1$ such that $\|u - \exp(2\pi i h)\| < \varepsilon$. In particular,* $\mathrm{cel}(u) \le 2\pi$.

Proof. The theorem for the cases of $\{pt\}$ and $[0,1]$ is trivial and can be proved in a way similar to, but simpler than, the proof of the theorem for the case $T_{II,k}$. For the case of S^1, we can do it as follows. First one perturbs $u \in M_k(C(S^1))$ to an element $v \in M_k(C(S^1))$ such that $v(z)$ has a distinct spectrum for any $z \in S^1$. Then $z \mapsto \mathrm{Sp}(v(z))$ defines a map, $\mathrm{Sp}(v) : S^1 \to \mathring{P}^k S^1$. Note that this map defines the zero element in $\pi_1(\mathring{P}^k S^1) = \mathbb{Z}$, since $[v] = 0 \in K_1(C(S^1))$. Hence the map $\mathrm{Sp}(v)$ can be lifted to a map from S^1 to $(S^1)^k$ as in Lemma 4.5. Then the other part of the proof of the theorem for the case of $T_{II,k}$ (see Lemma 4.3 and Theorem 4.6) can be applied here. ∎

Now we are going to prove the following result. Its proof is similar to that of Lemma 3.14 in [21] (see also [21, Lemma 3.12]).

Theorem 4.8. *Let $u \in \mathrm{CU}(M_l(I_k))$. Then for any $\varepsilon > 0$, there exists a self-adjoint element $h \in M_l(I_k)$ with $\|h\| \le 1$ such that $\|u - \exp(2\pi i h)\| < \varepsilon$. In particular,* $\mathrm{cel}(u) \le 2\pi$.

Proof. By [21, Lemma 3.10], a unitary $w \in M_l(I_k)$ is in $\mathrm{CU}(M_l(I_k))$ if and only if for any irreducible representation $\psi : M_l(I_k) \to M_\bullet(\mathbb{C})$ (where $\bullet = l$ or lk), $\psi(w) \in \mathrm{CU}(M_\bullet(\mathbb{C}))$, which is equivalent to $\det(\psi(w)) = 1$.

By [21, Lemma 3.10], one can write $u(0) = a_0 \otimes 1_k \in M_l(\mathbb{C}) \otimes 1_k$ and $u(1) = a_1 \otimes 1_k \in M_l(\mathbb{C}) \otimes 1_k$, where $a_i \in \mathrm{CU}(M_l(\mathbb{C}))$ for $i = 1, 2$. After a small perturbation of the unitary u inside $\mathrm{CU}(M_l(I_k))$, one can assume that both $a_0 \in M_l(\mathbb{C})$ and $a_1 \in M_l(\mathbb{C})$ have l distinct spectra. This can be done as follows. Let $\eta > 0$ and $\delta > 0$ be fixed. Choose $a_0' \in \mathrm{CU}(M_l(\mathbb{C}))$ and $a_1' \in \mathrm{CU}(M_l(\mathbb{C}))$, both with distinct spectra, such that $\|a_0' - a_0\| < \eta$ and $\|a_1' - a_1\| < \eta$. Define two paths $\xi_0 : [-\delta, 0] \to \mathrm{CU}(M_l(\mathbb{C}))$ and $\xi_1 : [1, 1+\delta] \to \mathrm{CU}(M_l(\mathbb{C}))$ such that $\xi_0(-\delta) = a_0'$, $\xi_0(0) = a_0$, $\xi_1(1) = a_1$, $\xi_1(1+\delta) = a_1'$, $\|\xi_0(t) - a_0\| < \eta$ for $t \in [-\delta, 0]$, and $\|\xi_1(t) - a_1\| < \eta$ for $t \in [1, 1+\delta]$. Define $\tilde{u} : [-\delta, 1+\delta] \to \mathrm{CU}(M_{lk}(\mathbb{C}))$ by

$$\tilde{u}(t) = \begin{cases} \xi_0(t) \otimes 1_k, & t \in [-\delta, 0], \\ u(t), & t \in [0,1], \\ \xi_1(t) \otimes 1_k, & t \in [1, 1+\delta]. \end{cases}$$

Reparametrising $\tilde{u}$ so as to shrink the interval of definition of $\tilde{u}$ from $[-\delta, 1+\delta]$ to $[0,1]$ proportionally, we obtain $u' \in \mathrm{CU}(M_l(I_k))$ with $u'(0) = a_0' \otimes 1_k$ and $u'(1) = a_1' \otimes 1_k$. Evidently, $\|u' - u\|$ can be made arbitrarily small provided that η and δ are small enough. By [21, Lemma 3.10], $u' \in \mathrm{CU}(M_l(I_k))$. Without loss of generality, we simply assume that a_0 (for $u(0) = a_0 \otimes 1_k$) and a_1 (for $u(1) = a_1 \otimes 1_k$) have distinct spectra.

It is easy to prove (see (4.1) and (4.2) and the corresponding paragraphs in the proof of Theorem 4.6) that there exists $-1 < \lambda_1 < \lambda_2 < \cdots < \lambda_l < 1$ with $\sum_{j=1}^{l} \lambda_j = 0$ (note that by [21, Lemma 3.10], $\det(a_0) = 1$) and $\lambda_l - \lambda_1 < 1$, such that

$$\mathrm{Sp}(a_0) = \{\exp(2\pi i\lambda_1), \exp(2\pi i\lambda_2), \dots, \exp(2\pi i\lambda_l)\}.$$

Similarly, there exist $-1 < \lambda_1' < \lambda_2' < \cdots < \lambda_l' < 1$ with $\sum_{j=1}^{l} \lambda_j' = 0$ and $\lambda_l' - \lambda_1' < 1$, such that

$$\mathrm{Sp}(a_1) = \{\exp(2\pi i\lambda_1'), \exp(2\pi i\lambda_2'), \dots, \exp(2\pi i\lambda_l')\}.$$

One can find two self-adjoint elements $b_0, b_1 \in M_l(\mathbb{C})$ with

$$\mathrm{Sp}(b_0) = \{\lambda_1, \lambda_2, \dots, \lambda_l\} \quad \text{and} \quad \mathrm{Sp}(b_1) = \{\lambda_1', \lambda_2', \dots, \lambda_l'\}$$

such that $a_0 = \exp(2\pi i b_0)$ and $a_1 = \exp(2\pi i b_1)$. In fact, since $\max_j \lambda_j - \min_j \lambda_j = \lambda_l - \lambda_1 < 1$, one can choose a branch of logarithm function log which takes $\exp(2\pi i\lambda_j)$ to $2\pi i\lambda_j$ and define $b_0 = \frac{1}{2\pi i}\log(a_0)$ (the element b_1 can be defined similarly).

By [21, Lemma 3.12], there exist mutually orthogonal rank one projections, $\{p_j\}_{j=1}^{kl} \subset M_{kl}(C[0,1])$, and continuous functions, $h_j : [0,1] \to \mathbb{R}$, such that the unitary

$$v := \sum_{j=1}^{lk} \exp(2\pi i h_j) p_j \in M_{lk}\big(C[0,1]\big)$$

satisfies the following:

(1) $v(0) = u(0)$, $v(1) = u(1)$, and, consequently, $v \in M_l(I_k)$;

(2) $v(t)$ has distinct spectra for any $t \in (0,1)$;

(3) $v \in \mathrm{CU}(M_{kl}(C[0,1]))$ (combining with (1), $v \in \mathrm{CU}(M_l(I_k))$);

(4) $\|v - u\| < \varepsilon$.

Since $\sum_{j=1}^{lk} \exp(2\pi i h_j(0)) p_j(0) = v(0) = u(0) = \exp(2\pi i b_0) \otimes 1_k$, there exist integers $n_j \in \mathbb{Z}$ such that

$$\{h_1(0) - n_1, h_2(0) - n_2, \dots, h_{kl}(0) - n_{kl}\} = \{\underbrace{\lambda_1, \dots, \lambda_1}_{k}, \underbrace{\lambda_2, \dots, \lambda_2}_{k}, \dots, \underbrace{\lambda_l, \dots, \lambda_l}_{k}\}$$

as sets with multiplicity. Replacing each function h_j by the function $h_j - n_j$ for any $j \in \{1, 2, \dots, kl\}$, we can assume that $\sum_{j=0}^{kl} h_j(0) p_j(0) = b_0 \otimes 1_k$ (in fact, both of them are the logarithm functions (of the same branch) of the unitary $u(0) = v(0)$, multiplying by $\frac{1}{2\pi i}$).

Let $h = \sum_{j=0}^{lk} h_j p_j \in M_{kl}(C[0,1])$. Then we have $v = \exp(2\pi i h)$. We need to prove that $h \in M_l(I_k)$ and $\|h\| \le 1$.

By [21, Lemma 3.10], $\det(v(t)) = 0$ for all $t \in [0,1]$. And note that $v(t)$ has distinct spectra for any $t \in (0,1)$. The following argument is used in Remark 3.13 and the proof of Lemma 3.14 in [21] (we refer to [21] for details). From $\sum_{j=1}^{lk} h_j(0) = k \cdot (\sum_{j=1}^{l} \lambda_j) = 0$, we get $\sum_{j=1}^{lk} h_j(t) = 0$ for all $t \in [0,1)$; and from $\max_{1 \le j \le kl} h_j(0) - \min_{1 \le j \le kl} h_j(0) = \lambda_l - \lambda_1 < 1$, we get $\max_{1 \le j \le kl} h_j(t) - \min_{1 \le j \le kl} h_j(t) < 1$ for all $t \in [0,1)$. Furthermore, one gets $-1 \le h_j(t) \le 1$ for all $t \in [0,1]$ and hence $\|h\| \le 1$.

We still need to prove that $h \in M_l(I_k)$. Since $h(0) = b_0 \otimes 1_k \in M_l(\mathbb{C}) \otimes 1_k$, it suffices to prove that $h(1) \in M_l(\mathbb{C}) \otimes 1_k$. Let us prove that $h(1) = b_1 \otimes 1_k$. From $\exp(2\pi i h(1)) = v(1) = u(1) = \exp(2\pi i b_1) \otimes 1_k$, one gets

$$\{h_1(1), h_2(1), \dots, h_{kl}(1)\} = \{\underbrace{\lambda'_1, \dots, \lambda'_1}_{k}, \underbrace{\lambda'_2, \dots, \lambda'_2}_{k}, \dots, \underbrace{\lambda'_l, \dots, \lambda'_l}_{k}\} \mod \mathbb{Z}.$$

From

$$\sum_{j=1}^{k} \lambda'_j = 0, \quad \sum_{j=1}^{kl} h_j(1) = 0,$$
$$\max_{1 \le j \le l} \lambda_j - \min_{1 \le j \le l} \lambda_j < 1, \quad \max_{1 \le j \le kl} h_j(1) - \min_{1 \le j \le kl} h_j(1) \le 1,$$

it is easy to prove that $h_j(1) \in \{\lambda'_1, \lambda'_2, \dots, \lambda'_l\}$ for all $j \in \{1, 2, \dots, kl\}$. In fact, for each j, there is a unique $m_j \in \mathbb{Z}$ and $i(j) \in \{1, 2, \dots, l\}$ such that $h_j(1) = \lambda'_{i(j)} + m_j$. We claim that all $m_j = 0$. If one of them is positive, say $m_{j_1} > 0$, then there is j_2 such that $m_{j_2} < 0$. Therefore,

$$\begin{aligned} h_{j_1}(1) - h_{j_2}(1) &= (\lambda'_{i(j_1)} + m_{j_1}) - (\lambda'_{i(j_2)} + m_{j_2}) \\ &= (m_{j_1} - m_{j_2}) + (\lambda'_{i(j_1)} - \lambda'_{i(j_2)}) > 2 + (-1) = 1, \end{aligned}$$

which is a contradiction. Hence $h(1) = b_1 \otimes 1_k$ (again both of them are the logarithm functions (of the same branch) of the unitary $u(1) = v(1)$, multiplying by $\frac{1}{2\pi i}$). Thus we get that $h \in M_l(I_k)$, as desired. ■

Now we get the following result.

Theorem 4.9. *Let A be an AH-algebra with the ideal property and with no dimension growth. Then for any $\varepsilon > 0$ and any $u \in \mathrm{CU}(A)$, there exists a self-adjoint element h in A with $\|h\| \le 1$ such that $\|u - \exp(2\pi i h)\| < \varepsilon$. In particular,* $\mathrm{cel}_{\mathrm{CU}}(A) \le 2\pi$

Proof. We assume that $A = \lim(A_n, \phi_{n,n+1})$, where $A_n \in \mathcal{HD}$ for each $n \ge 1$. Using Theorems 4.6, 4.7, and 4.8, for any $u \in \mathrm{CU}(A_n)$ we have $\mathrm{cel}(\phi_{n,m}(u)) \le 2\pi$ for all $m \ge n$. Noting that $\mathrm{cel}(\phi_{n,\infty}(u)) = \inf_{m \ge n} \mathrm{cel}(\phi_{n,m}(u)) \le 2\pi$, hence $\mathrm{cel}(\phi_{n,\infty}(u)) \le 2\pi$. ■

The above theorem generalizes Theorem 4.6 of [36] (see Theorem A in the introduction) for the case of simple AH-algebra.

The following theorem is the main theorem of this section. This theorem is not quite a consequence of Theorem 3.9 and Theorem 4.9 since it does not assume that

$$\lim_{n \to \infty} \mathrm{rank}(P_{n,i}) = \infty.$$

But we assume that A has the ideal property.

Theorem 4.10. *Let A be an AH-algebra with the ideal property and with no dimension growth. If we further assume that A is not of real rank zero, then* $\mathrm{cel}_{\mathrm{CU}}(A) = 2\pi$.

To prove the above result, we need the following Pasinicu dichotomy lemma.

Proposition 4.11 ([40, Lemma 2.11]). *Let*

$$A = \varinjlim \left(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}\big(C(X_{n,i})\big) P_{n,i}, \phi_{n,m} \right)$$

be an AH-algebra with the ideal property and with no dimension growth condition. Then for any n, any finite subset $F_n^i \subset P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i} \subset A_n$, any $\varepsilon > 0$, and any positive integer N, there exists $m_0 > n$ such that each partial map $\phi_{n,m}^{i,j}$ with $m \geq m_0$ satisfies either:

(1) $\operatorname{rank}(\phi_{n,m}^{i,j}(P_{n,i})) \geq N(\dim X_{m,j} + 1)$ *or*

(2) *there is a homomorphism*

$$\psi_{n,m}^{i,j} : A_n^i \to \phi_{n,m}^{i,j}(P_{n,i}) A_m^j \phi_{n,m}^{i,j}(P_{n,i})$$

with a finite-dimensional image such that $\|\phi_{n,m}^{i,j}(f) - \psi_{n,m}^{i,j}(f)\| < \varepsilon$ *for all* $f \in F_n^i$.

Remark 4.12. Let X, Y be connected finite simplicial complexes. Let $f \in PM_n(C(X))P$ be a self-adjoint element and $\phi, \psi : PM_n(C(X))P \to QM_m(C(Y))Q$ be two unital homomorphisms with ψ factoring through a finite-dimensional algebra such that $\|\phi(f) - \psi(f)\| < \varepsilon$. Then all functions in the eigenvalue list of $\psi(f)$ are constant functions and, consequently, $\operatorname{EV}(\psi(f)) = 0$. Also by Weyl's inequality in [48], $\operatorname{EV}(\phi(f)) < \varepsilon$.

Proof of Theorem 4.10. Since A is not of real rank zero, by Proposition 2.11 (for the case of no dimension growth), there exist $\delta_0 > 0$, N, $x \in (A_N)_+$, with $\|x\| = 1$, and a subsequence $\{A_{n_k}\}_{k=2}^{\infty}$ with $n_2 > N$ such that for any $k \geq 2$, there is a block $A_{n_k}^j$ with

$$\operatorname{EV}\left(\phi_{N,n_k}^{-,j}(x)\right) \geq \delta_0. \tag{4.5}$$

To save notations, we can directly assume that $N = 1$ and $n_k = k$ for every $k \geq 2$.

For any $\varepsilon > 0$ and $L > \frac{2\pi}{\varepsilon}$, by Proposition 4.11 and Remark 4.12, there exists $m_0 > 1$ such that for any $m \geq m_0$ and $A_m^j = P_{m,j} M_{[m,j]}(C(X_{m,j})) P_{m,j}$, either

$$\operatorname{rank}(P_{m,j}) \geq L(\dim X_{m,j} + 1)$$

or

$$\operatorname{EV}\left(\phi_{1,m}^{-,j}(x)\right) < \delta_0.$$

We denote

$$\Lambda = \left\{1 \leq j \leq t_{m_0} : \operatorname{rank}(P_{m_0,j}) \geq L(\dim X_{m_0,j} + 1)\right\}.$$

Let $P = \bigoplus_{j \in \Lambda} P_{m_0,j}$, $R = \bigoplus_{j \notin \Lambda} P_{m_0,j}$, and $x^j = \phi_{1,m_0}^{-,j}(x)$. Set

$$x_1 = \bigoplus_{j \in \Lambda} x^j = P\phi_{1,m_0}(x)P \quad \text{and} \quad x_2 = \bigoplus_{j \notin \Lambda} x^j = R\phi_{1,m_0}(x)R.$$

Then $\mathrm{EV}(x^j) < \delta_0$ for $j \notin \Lambda$. By Corollary 3.7, for any $m > m_0$ and any $j \in \{1, 2, \ldots, t_m\}$, we have $\phi^{-,j}_{m_0,m}(x_2) < \delta_0$. By (4.5), for any $m > m_0$, there is a $j \in \{1, 2, \ldots, t_m\}$ such that $\phi^{-,j}_{1,m}(x) \geq \delta_0$. Note that $\phi^{-,j}_{1,m}(x) = \phi^{-,j}_{m_0,m}(x_1) + \phi^{-,j}_{m_0,m}(x_2)$. By Corollary 3.7 (b), $\mathrm{EV}(\phi^{-,j}_{m_0,m}(x_1)) \geq \delta_0$.

Hence by Proposition 2.11,

$$\phi_{m_0,\infty}(P)A\phi_{m_0,\infty}(P) = \lim\left(\phi_{m_0,m}(P)A_m\phi_{m_0,m}(P), \phi_{m,m'}\right)$$

is not of real rank zero.

By [23, Theorem 1.2, p. 112], for each $j \in \Lambda$, there exists a set of mutually orthogonal rank one projections $p_1^{(j)}, p_2^{(j)}, \ldots, p_L^{(j)}$ with $p_l^{(j)} < P_{m_0,j}$ and $p_l^{(j)} \sim p_1^{(j)}$ for each $1 \leq l \leq L$. Let $q = \bigoplus_{j\in\Lambda} p_1^{(j)}$. There exists an integer $W \in \mathbb{N}$ such that

$$P < W[q].$$

It follows that $\phi_{m_0,\infty}(q)A\phi_{m_0,\infty}(q)$ is stably isomorphic to $\phi_{m_0,\infty}(P)A\phi_{m_0,\infty}(P)$ and hence $\phi_{m_0,\infty}(q)A\phi_{m_0,\infty}(q) = \lim(\phi_{m_0,m}(q)A_m\phi_{m_0,m}(q), \phi_{m,m'})$ is not of real rank zero. By Lemma 3.8, there exist an interval $[c, d] \subset [0, 1]$, an integer $m_1 \geq m_0$, and $y \in (\phi_{m_0,m_1}(q)A_{m_1}\phi_{m_0,m_1}(q))_+$ with $\|y\| = 1$ with the following property. For any $m \geq m_1$, writing $\widetilde{\phi}_{m_1,m} := \phi_{m_1,m}|_{\phi_{m_0,m_1}(q)A_{m_1}\phi_{m_0,m_1}(q)}$ and writing $\widetilde{\phi}_{m_1,m}(y)$ as

$$\widetilde{\phi}_{m_1,m}(y) = \left(z_1^m, z_2^m, \ldots, z_{k_m}^m\right) \in \bigoplus_{j=1}^{k_m} \phi^{-,j}_{m_0,m}(q)A_m^j\phi^{-,j}_{m_0,m}(q) = \phi_{m_0,m}(q)A_m\phi_{m_0,m}(q),$$

then there exist $1 \leq k(m) \leq k_m$ and $1 \leq i(m) \leq \mathrm{rank}(\phi^{-,k(m)}_{m_0,m}(q))$ such that

$$[c, d] \subset \mathrm{rang}\left(h^{k(m)}_{i(m)}\right),$$

where $h_i^{k(m)}$ is the ith lowest eigenvalue of $y^m_{k(m)}$ for $1 \leq i \leq \mathrm{rank}(\phi^{-,k(m)}_{m_0,m}(q))$.

Let $Q = \sum_{l=1}^L \bigoplus_{j\in\Lambda} p_l^{(j)}$. Then $QA_{m_0}Q \cong M_L(qA_{m_0}q)$. Hence one can identify $M_L(qA_{m_0}q)$ as a subalgebra of A_{m_0} and $M_L(\phi_{m_0,m}(q)A_m\phi_{m_0,m}(q))$ as a subalgebra of A_m.

Repeating the proof in Theorem 3.9, one can prove that there is a $u \in \mathrm{CU}(A)$ such that

$$\mathrm{cel}(u) \geq 2\pi\left(1 - \frac{1}{L}\right) \geq 2\pi - \varepsilon.$$

Consequently, $\mathrm{cel}_{\mathrm{CU}}(A) \geq 2\pi$. ∎

5. Exponential length in the Jiang–Su algebra

We shall show that there exists a unitary $u \in \mathrm{CU}(\mathcal{Z})$ such that $\mathrm{cel}(u) \geq 2\pi$. First, we review the construction of the Jiang–Su algebra $\mathcal{Z}$. We refer the readers to [25] for details. Denote by $I[m_0, m, m_1]$ the dimension drop algebra

$$\left\{f \in C\left([0, 1], M_m\right) : f(0) \in M_{m_0} \otimes \mathbf{1}_{m/m_0}, f(1) \in \mathbf{1}_{m/m_1} \otimes M_{m_1}\right\},$$

where m_0, m_1, and m are positive integers with m divisible by both m_0 and m_1. If m_0 and m_1 are relatively prime and $m = m_0 m_1$, then $I[m_0, m, m_1]$ is called a prime dimension drop algebra.

The Jiang–Su algebra is constructed as below. Let $A_1 = I[2, 6, 3]$. Suppose that a prime dimension drop algebra $A_m = I[p_m, d_m, q_m]$ is chosen for some $m \geq 1$. We construct A_{m+1} and $\phi_{m,m+1} : A_m \to A_{m+1}$ as follows.

Choose $k_0^{(m)}$ and $k_1^{(m)}$ to be the first two prime numbers that are greater than $2d_m$. Then

$$k_0^{(m)} > 2p_m, \quad k_1^{(m)} > 2q_m, \quad (k_0^{(m)} p_m, k_1^{(m)} q_m) = 1.$$

Let

$$p_{m+1} = k_0^{(m)} p_m, \quad q_{m+1} = k_1^{(m)} q_m, \quad d_{m+1} = p_{m+1} q_{m+1},$$

and

$$A_{m+1} = I[p_{m+1}, d_{m+1}, q_{m+1}].$$

Obviously, A_{m+1} is a prime dimension drop algebra. Denote $k^{(m)} = k_0^{(m)} k_1^{(m)}$. Choose $r_0^{(m)}$ such that

$$0 < r_0^{(m)} \leq q_{m+1} \quad \text{and} \quad q_{m+1} | (k^{(m)} - r_0^{(m)}).$$

Choose $r_1^{(m)}$ such that

$$0 < r_1^{(m)} \leq p_{m+1} \quad \text{and} \quad p_{m+1} | (k^{(m)} - r_1^{(m)}).$$

Define

$$\xi_j^{(m)}(x) = \begin{cases} \frac{x}{2}, & 1 \leq j \leq r_0^{(m)}, \\ \frac{1}{2}, & r_0^{(m)} < j \leq k^{(m)} - r_1^{(m)}, \\ \frac{x+1}{2}, & k^{(m)} - r_1^{(m)} < j \leq k^{(m)}. \end{cases}$$

It follows that

$$\xi_j^{(m)}(0) = \begin{cases} 0, & 1 \leq j \leq r_0^{(m)}, \\ \frac{1}{2}, & r_0^{(m)} < j \leq k^{(m)}, \end{cases}$$

and

$$\xi_j^{(m)}(1) = \begin{cases} \frac{1}{2}, & 1 \leq j \leq k^{(m)} - r_1^{(m)}, \\ 1, & k^{(m)} - r_1^{(m)} < j \leq k^{(m)}. \end{cases}$$

Obviously, we have

$$r_0^{(m)} q_m \equiv k^{(m)} q_m = k_0^{(m)} q_{m+1} \equiv 0 \pmod{q_{m+1}}.$$

It follows that $q_{m+1} | r_0^{(m)} q_m$. Notice that $q_{m+1} | (k^{(m)} - r_0^{(m)})$. There exists a unitary $u_0 \in M_{d_{m+1}}$ such that

$$u_0^* \begin{bmatrix} f\left(\xi_1^{(m)}(0)\right) & 0 & \cdots & 0 \\ 0 & f\left(\xi_2^{(m)}(0)\right) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f\left(\xi_{k^{(m)}}^{(m)}(0)\right) \end{bmatrix} u_0 \in M_{p_{m+1}} \otimes \mathbf{1}_{q_{m+1}} \subset M_{p_{m+1}} \otimes M_{q_{m+1}}$$

for all $f \in A_m$. Define the morphism $\rho_0 : A_m \to M_{p_{m+1}} \otimes \mathbf{1}_{q_{m+1}} \subset M_{p_{m+1}} \otimes M_{q_{m+1}}$ by

$$\rho_0(f) := u_0^* \operatorname{diag}\big[f\big(\xi_1^{(m)}(0)\big), f\big(\xi_2^{(m)}(0)\big), \ldots, f\big(\xi_{k^{(m)}}^{(m)}(0)\big)\big]u_0, \quad \text{for all } f \in A_m.$$

On the other hand, we have

$$p_{m+1} | r_1^{(m)} p_m, \quad p_{m+1} | (k^{(m)} - r_1^{(m)}).$$

There exists a unitary $u_1 \in M_{d_{m+1}}$ such that

$$u_1^* \begin{bmatrix} f\big(\xi_1^{(m)}(1)\big) & 0 & \cdots & 0 \\ 0 & f\big(\xi_2^{(m)}(1)\big) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f\big(\xi_{k^{(m)}}^{(m)}(1)\big) \end{bmatrix} u_1 \in \mathbf{1}_{p_{m+1}} \otimes M_{q_{m+1}} \subset M_{p_{m+1}} \otimes M_{q_{m+1}}$$

for all $f \in A_m$. Define the morphism $\rho_1 : A_m \to \mathbf{1}_{p_{m+1}} \otimes M_{q_{m+1}} \subset M_{p_{m+1}} \otimes M_{q_{m+1}}$ by

$$\rho_1(f) := u_1^* \operatorname{diag}\big[f\big(\xi_1^{(m)}(1)\big), f\big(\xi_2^{(m)}(1)\big), \ldots, f\big(\xi_{k^{(m)}}^{(m)}(1)\big)\big]u_1 \quad \text{for all } f \in A_m.$$

Let u be any continuous path of unitaries in $M_{d_{m+1}}$ connecting u_0 and u_1 and let $\phi_{m,m+1}$ be given as follows:

$$\phi_{m,m+1}(f) = u^* \begin{bmatrix} f \circ \xi_1^{(m)} & 0 & \cdots & 0 \\ 0 & f \circ \xi_2^{(m)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f \circ \xi_{k^{(m)}}^{(m)} \end{bmatrix} u \quad \text{for all } f \in A_m.$$

Theorem 5.1 ([25, Proposition 2.5]). *The Jiang–Su algebra $\mathcal{Z}$ can be written as the limit $\mathcal{Z} = \lim_n(A_n, \phi_{n,n+1})$, such that each connecting map $\phi_{m,n} = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_{m+1} \circ \phi_m$ has the form*

$$\phi_{m,n}(f) = U^* \begin{bmatrix} f \circ \xi_1 & 0 & \cdots & 0 \\ 0 & f \circ \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \circ \xi_k \end{bmatrix} U,$$

where U is a continuous path in $U(M_{d_n})$, $k = k^{(m)}k^{(m+1)} \cdots k^{(n-1)}$ and

$$\xi_1 \le \xi_2 \le \cdots \le \xi_k.$$

In fact, each ξ_j can be chosen from the following list:

$$\xi(t) = \frac{l}{2^{n-m}}, \quad \text{where } l \in \mathbb{Z},\ 0 < l < 2^{n-m},$$

or

$$\xi(t) = \frac{t+l}{2^{n-m}}, \quad \text{where } l \in \mathbb{Z},\ 0 \le l < 2^{n-m}.$$

In particular, among the above functions ξ_i, the smallest function $\xi_1(t) = \frac{t}{2^{n-m}}$ and the largest function $\xi_\bullet(t) = \frac{t+(2^{n-m}-1)}{2^{n-m}}$, where $\bullet = k$. (For convenience, when we quote this theorem, we will use $\xi_\bullet$ for ξ_k, so k can be used for another purpose.)

Remark 5.2. We shall use $x^{\sim k}$ to denote $x, x, \ldots, x$ (k times) for the notation of set with multiplicity. For example, $\{x^{\sim 2}, y^{\sim 3}\} = \{x, x, y, y, y\}$. As in the construction of $\mathcal{Z}$, we have

$$\{\xi_1(0), \xi_2(0), \ldots, \xi_k(0)\} = \left\{\frac{l}{2^{n-m}}^{\sim j_l}\right\}_{l=0}^{2^{n-m}-1},$$

where $q_n | j_l$ for all $0 \le l \le 2^{n-m} - 1$, and

$$\{\xi_1(1), \xi_2(1), \ldots, \xi_k(1)\} = \left\{\frac{l}{2^{n-m}}^{\sim s_l}\right\}_{l=1}^{2^{n-m}},$$

where $p_n | s_l$ for all $1 \le l \le 2^{n-m}$.

Lemma 5.3. *Let $\mathcal{Z} = \lim(A_n, \phi_{n,n+1})$ be the Jiang–Su algebra defined above. If $v \in A_n$ is a unitary and u_s is a smooth path of unitaries connecting v and $\mathbf{1}_{A_n}$, then for any $\varepsilon > 0$, there exists another smooth path $v_s \in A_n$ of unitaries such that*

1. $\|v_s - u_s\| < \varepsilon$;
2. $|\text{length}(v_s) - \text{length}(u_s)| < \varepsilon$;
3. $v_s(0) = \exp(2\pi i \sum_{j=1}^{p_n} h_j a_j) \otimes \mathbf{1}_{q_n}$, *where $\{a_j\}_{j=1}^{p_n}$ is a set of mutually orthogonal rank one projections in $C([0,1], M_{p_n})$ and $h_j \in C([0,1])_{\text{s.a.}}$, with*
$$\exp\left(2\pi i h_j(s)\right) \ne \exp\left(2\pi i h_k(s)\right) \quad \textit{for } j \ne k \textit{ and } s \in [0,1];$$
4. $v_s(1) = \exp(2\pi i \sum_{j=1}^{q_n} g_j b_j) \otimes \mathbf{1}_{p_n}$, *where $\{b_j\}_{j=1}^{q_n}$ is a set of mutually orthogonal rank one projections in $C([0,1], M_{q_n})$ and $g_j \in C([0,1])_{\text{s.a.}}$, with*
$$\exp\left(2\pi i g_j(s)\right) \ne \exp\left(2\pi i g_k(s)\right) \quad \textit{for } j \ne k \textit{ and } s \in [0,1].$$

Proof. For any $0 < \varepsilon < 1$, there is a number $\delta > 0$ such that $\|u_s(t_1) - u_s(t_2)\| < \frac{\varepsilon}{2}$ and $\|\frac{du_s}{ds}(t_1) - \frac{du_s}{ds}(t_2)\| < \frac{\varepsilon}{3}$ for any $s \in [0,1]$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$.

Since $u_s \in A_n$, one can write $u_s(0)$ as $u_s(0) = \gamma^{(0)}(s) \otimes \mathbf{1}_{q_n}$, where $\gamma^{(0)}$ is a unitary in $C([0,1], M_{p_n})$. By [36, Lemma 4.1], there exist a set of mutually orthogonal rank one projections $\{a_j\}_{j=1}^{p_n}$ in $C([0,1], M_{p_n})$ and self-adjoint elements $h_j \in C([0,1])_{\text{s.a.}}$ with $\exp(2\pi i h_j(s)) \ne \exp(2\pi i h_k(s))$ for $j \ne k$ and $s \in [0,1]$ such that

$$\|\gamma^{(0)}(s) - \overline{\gamma}^{(0)}(s)\| < \frac{\varepsilon}{6\left(1 + \max_{s\in[0,1]} \left\|\frac{du_s(0)}{ds}\right\|\right)} \quad \text{and} \quad \left\|\frac{d\gamma^{(0)}(s)}{ds} - \frac{d\overline{\gamma}^{(0)}(s)}{ds}\right\| < \frac{\varepsilon}{3},$$

for all $s \in [0,1]$, where $\overline{\gamma}^{(0)}(s) = \exp(2\pi i \sum_{j=1}^{p_n} h_j a_j)$.

On the other hand, $u_s(1)$ can be written as $u_s(1) = \gamma^{(1)}(s) \otimes \mathbf{1}_{p_n}$, where $\gamma^{(1)}$ is a unitary in $C([0,1], M_{q_n})$. By [36, Lemma 4.1], there exist a set of mutually orthogonal rank one projections $\{b_j\}_{j=1}^{q_n}$ in $C([0,1], M_{q_n})$ and $g_j \in C([0,1])_{\text{s.a.}}$ with $\exp(2\pi i g_j(s)) \neq \exp(2\pi i g_k(s))$ for $j \neq k$ and $s \in [0,1]$ such that

$$\|\gamma^{(1)}(s) - \overline{\gamma}^{(1)}(s)\| < \frac{\varepsilon}{6\big(1 + \max_{s\in[0,1]} \big\|\frac{du_s(1)}{ds}\big\|\big)} \quad \text{and} \quad \left\|\frac{d\gamma^{(1)}(s)}{ds} - \frac{d\overline{\gamma}^{(1)}(s)}{ds}\right\| < \frac{\varepsilon}{3}$$

for all $s \in [0,1]$, where $\overline{\gamma}^{(1)}(s) = \exp(2\pi i \sum_{j=1}^{q_n} g_j b_j)$.

Write $v^{(0)}(s) = \overline{\gamma}^{(0)}(s) \otimes \mathbf{1}_{q_n}$ and $v^{(1)}(s) = \overline{\gamma}^{(1)}(s) \otimes \mathbf{1}_{p_n}$. Then

$$\big\|v^{(0)}(s) - u_s(0)\big\| = \big\|\big(\gamma^{(0)}(s) - \overline{\gamma}^{(0)}(s)\big) \otimes \mathbf{1}_{q_n}\big\| < \frac{\varepsilon}{6\big(1 + \max_{s\in[0,1]} \big\|\frac{du_s(0)}{ds}\big\|\big)}$$

and

$$\big\|v^{(1)}(s) - u_s(1)\big\| = \big\|(\gamma^{(1)}(s) - \overline{\gamma}^{(1)}(s)) \otimes \mathbf{1}_{p_n}\big\| < \frac{\varepsilon}{6\big(1 + \max_{s\in[0,1]} \big\|\frac{du_s(1)}{ds}\big\|\big)}$$

for all $s \in [0,1]$. Furthermore,

$$\left\|\frac{dv^{(0)}(s)}{ds} - \frac{du_s(0)}{ds}\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\|\frac{dv^{(1)}(s)}{ds} - \frac{du_s(1)}{ds}\right\| < \frac{\varepsilon}{3}.$$

Since

$$\big\|u_s^*(0)v^{(0)}(s) - \mathbf{1}_{A_n}\big\| = \big\|v^{(0)}(s) - u_s(0)\big\| < \frac{\varepsilon}{6} < \frac{1}{6},$$

there exists $H \in M_{d_n}(C([0,1]))_{\text{s.a.}}$ with $\|H\| < 1$ such that $u_s^*(0)v^{(0)}(s) = \exp(2\pi i H(s))$. Also, there exists $G \in M_{d_n}(C([0,1]))_{\text{s.a.}}$ with $\|G\| < 1$ such that

$$u_s^*(1)v^{(1)}(s) = \exp\big(2\pi i G(s)\big).$$

In fact, $H(s) = \frac{1}{2\pi i}\log(u_s^*(0)v^{(0)}(s))$ and $G(s) = \frac{1}{2\pi i}\log(u_s^*(1)v^{(1)}(s))$.

We denote

$$w(s,t) = \begin{cases} u_s(0)\exp\big(2\pi i \frac{-t}{\delta} H(s)\big), & -\delta \le t < 0, \\ u_s(t), & 0 \le t \le 1, \\ u_s(1)\exp\big(2\pi i \frac{t-1}{\delta} G(s)\big), & 1 < t \le 1+\delta. \end{cases}$$

Let $v_s(t) = w(s, (1+2\delta)t - \delta)$ for $(s,t) \in [0,1]\times[0,1]$. Then v_s is a path in A_n and it satisfies conditions (3) and (4).

For $t \in [0, \frac{\delta}{1+2\delta})$, by the choice of δ, we have

$$\begin{aligned} \big\|v_s(t) - u_s(t)\big\| &= \left\|u_s(0)\exp\Big(2\pi i \frac{\delta - (1+2\delta)t}{\delta} H(s)\Big) - u_s(t)\right\| \\ &\le \left\|u_s(0)\exp\Big(2\pi i \frac{\delta - (1+2\delta)t}{\delta} H(s)\Big) - u_s(0)\| + \|u_s(0) - u_s(t)\right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) - \mathbf{1}_{A_n} \right\| + \left\| u_s(0) - u_s(t) \right\| \\
&\leq \left\| \exp\left(2\pi i H(s)\right) - \mathbf{1}_{A_n} \right\| + \left\| u_s(0) - u_s(t) \right\| \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{2} \\
&= \frac{2\varepsilon}{3}.
\end{aligned}
$$

For $t \in [\frac{\delta}{1+2\delta}, \frac{1+\delta}{1+2\delta}]$, we have $|(1+2\delta)t - \delta - t| < \delta$ and hence

$$
\left\| v_s(t) - u_s(t) \right\| = \left\| u_s\left((1+2\delta)t - \delta\right) - u_s(t) \right\| < \frac{\varepsilon}{2}.
$$

For $t \in (\frac{1+\delta}{1+2\delta}, 1]$, by the choice of δ, we have

$$
\begin{aligned}
\left\| v_s(t) - u_s(t) \right\| &= \left\| u_s(1) \exp\left(2\pi i \frac{(1+2\delta)t-\delta-1}{\delta} G(s)\right) - u_s(t) \right\| \\
&\leq \left\| u_s(1) \exp\left(2\pi i \frac{(1+2\delta)t-\delta-1}{\delta} G(s)\right) - u_s(1) \right\| + \left\| u_s(1) - u_s(t) \right\| \\
&= \left\| \exp\left(2\pi i \frac{(1+2\delta)t-\delta-1}{\delta} G(s)\right) - \mathbf{1}_{A_n} \right\| - \left\| u_s(1) - u_s(t) \right\| \\
&= \left\| \exp(2\pi i G(s)) - \mathbf{1}_{A_n} \right\| - \left\| u_s(1) - u_s(t) \right\| \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{2} \\
&= \frac{2\varepsilon}{3}.
\end{aligned}
$$

It follows that $\|v_s - u_s\| < \varepsilon$.

For $t \in [0, \frac{\delta}{1+2\delta})$, a direct calculation shows that

$$
\begin{aligned}
&\left| \left\| \frac{dv_s}{ds} \right\| - \left\| \frac{du_s}{ds} \right\| \right| \\
&\quad = \left| \sup_{t\in[0,1]} \left\| u_s(0) \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) 2\pi i \frac{\delta-(1+2\delta)t}{\delta} \frac{dH(s)}{ds} \right.\right. \\
&\qquad \left.\left. + \frac{du_s(0)}{ds} \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) \right\| - \sup_{t\in[0,1]} \left\| \frac{du_s(t)}{ds} \right\| \right| \\
&\quad \leq \sup_{t\in[0,1]} \left\| u_s(0) \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) 2\pi i \frac{\delta-(1+2\delta)t}{\delta} \frac{dH(s)}{ds} \right\| \\
&\qquad + \sup_{t\in[0,1]} \left\| \frac{du_s(0)}{ds} \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) - \frac{du_s(t)}{ds} \right\| \\
&\quad \leq 2\pi \left\| \frac{dH(s)}{ds} \right\| + \sup_{s\in[0,1]} \left\| \frac{du_s(0)}{ds} - \frac{du_s(t)}{ds} \right\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\| \frac{du_s(0)}{ds} \right\| \sup_{t\in[0,1]} \left\| \mathbf{1}_{A_n} - \exp\left(2\pi i \frac{\delta-(1+2\delta)t}{\delta} H(s)\right) \right\| \\
&\le \left\| -u_s^*(0)\frac{du_s(0)}{ds} u_s^*(0)v^{(0)}(s) + u_s^*(0)\frac{dv^{(0)}(s)}{ds} \right\| + \frac{\varepsilon}{3} \\
&\quad + \left\| \frac{du_s(0)}{ds} \right\| \left\| \mathbf{1}_{A_n} - \exp\left(2\pi i H(s)\right) \right\| \\
&\le \left\| \frac{du_s(0)}{ds} - \frac{dv^{(0)}(s)}{ds} \right\| + 2 \left\| \frac{du_s(0)}{ds} \right\| \left\| u_s^*(0)v^{(0)}(s) - \mathbf{1}_{A_n} \right\| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

Similarly,

$$
\left| \left\| \frac{dv_s}{ds} \right\| - \left\| \frac{du_s}{ds} \right\| \right| < \varepsilon \quad \text{for all } t \in \left[\frac{1+\delta}{1+2\delta}, 1 \right].
$$

For $t \in [\frac{\delta}{1+2\delta}, \frac{1+\delta}{1+2\delta}]$, we have $|(1+2\delta)t - \delta - t| < \delta$ and hence

$$
\left| \left\| \frac{dv_s(t)}{ds} \right\| - \left\| \frac{du_s(t)}{ds} \right\| \right| = \left\| \frac{du_s((1+2\delta)t-\delta)}{ds} - \frac{du_s(t)}{ds} \right\| < \frac{\varepsilon}{3}.
$$

It follows that $|\|\frac{dv_s}{ds}\| - \|\frac{du_s}{ds}\|| < \varepsilon$ for all $s \in [0,1]$. Therefore,

$$
\left| \mathrm{length}_s(v_s) - \mathrm{length}_s(u_s) \right| = \left| \int_0^1 \left\| \frac{dv_s}{ds} \right\| ds - \int_0^1 \left\| \frac{du_s}{ds} \right\| ds \right| < \varepsilon. \qquad \blacksquare
$$

Remark 5.4. Notice that

$$
\exp\left(2\pi i h_j(s)\right) \ne \exp\left(2\pi h_k(s)\right), \quad \exp\left(2\pi i g_j(s)\right) \ne \exp(2\pi i g_k(s))
$$

for any $j \ne k$ and $s \in [0,1]$. In Lemma 5.3, one can choose the initial values $h_j(0)$ of h_j to satisfy

$$
h_1(0) < h_2(0) < \cdots < h_{p_n}(0) \quad \text{and} \quad h_{p_n}(0) - h_1(0) < 1.
$$

Using the fact that $h_j(s) - h_k(s) \notin \mathbb{Z}$ (i.e., $\exp(2\pi i h_j(s)) \ne \exp(2\pi h_k(s))$), it is easy to prove that (see [21, proof of Lemma 3.14])

$$
h_1(s) < h_2(s) < \cdots < h_{p_n}(s) \quad \text{and} \quad h_{p_n}(s) - h_1(s) < 1
$$

for all $s \in [0,1]$. Similarly, in Lemma 5.3, we can also assume that

$$
g_1(s) < g_2(s) < \cdots < g_{q_n}(s) \quad \text{and} \quad g_{q_n}(s) - g_1(s) < 1
$$

for all $s \in [0,1]$.

Lemma 5.5. *Let $\mathcal{Z} = \lim(A_n, \phi_{n,n+1})$ be the Jiang–Su algebra defined above. If $v \in A_n$ is a unitary and u_s is a path of unitaries connecting v and $\mathbf{1}_{A_n}$, then for any $\varepsilon > 0$, there exists another path $v_s \in A_n$ of unitaries such that*

(1) $\|v_s - u_s\| < \varepsilon$;

(2) $|\operatorname{length}(v_s) - \operatorname{length}(u_s)| < \varepsilon$;

(3) $v_s(t) = \exp(2\pi i H_s(t))$, $H_s(t) = \sum_{j=1}^{d_n} \lambda_j(s,t) p_j(s,t)$, *where* $\{p_j\}_{j=1}^{d_n}$ *is a set of mutually orthogonal rank one projections in* $C([0,1]\times[0,1], M_{d_n})$ *and* $\lambda_j \in C([0,1]\times[0,1])_{\text{s.a.}}$, *with* $\lambda_1(s,t) < \lambda_2(s,t) < \cdots < \lambda_{d_n}(s,t)$ *for all* $(s,t) \in [0,1]\times(0,1)$ *and* $\lambda_{d_n}(s,t) - \lambda_1(s,t) < 1$ *for all* $(s,t) \in [0,1]\times[0,1]$.

Proof. For any $0 < \eta < 1$, since $u_s(t)$ is uniformly continuous on $[0,1]\times[0,1]$, there exists $\delta_1 > 0$ such that

$$\|u_s(t_1) - u_s(t_2)\| < \frac{\eta}{4}$$

for all $s \in [0,1]$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < 4\delta_1$. Since $u_s(0), u_s(1) \in C([0,1]) \otimes M_{d_n}$, there are continuous functions $f_i, g_i : [0,1] \to \mathbb{C}$, $i = 1, 2, \ldots, d_n$, such that $\{f_1(s), f_2(s), \ldots, f_{d_n}(s)\}$ are the eigenvalues for $u_s(0)$ and $\{g_1(s), g_2(s), \ldots, g_{d_n}(s)\}$ are the eigenvalues for $u_s(1)$, respectively. By Lemma 5.3, without loss of generality, we may assume that $u_s(0)$ and $u_s(1)$ can be written in the following forms:

$$u_s(0) = U^{(0)}(s) \begin{bmatrix} \exp(2\pi i f_1(s)) & & & \\ & \exp(2\pi i f_2(s)) & & \\ & & \ddots & \\ & & & \exp(2\pi i f_{d_n}(s)) \end{bmatrix} \left(U^{(0)}(s)\right)^* \tag{5.1}$$

for all $s \in [0,1]$ and

$$u_s(1) = U^{(1)}(s) \begin{bmatrix} \exp(2\pi i g_1(s)) & & & \\ & \exp(2\pi i g_2(s)) & & \\ & & \ddots & \\ & & & \exp(2\pi i g_{d_n}(s)) \end{bmatrix} \left(U^{(1)}(s)\right)^* \tag{5.2}$$

for all $s \in [0,1]$, where $U^{(0)}$, $U^{(1)}$ are unitaries in $C([0,1]) \otimes M_{d_n}$. By Remark 5.4, we can assume that

$$f_1(s) \le f_2(s) \le \cdots \le f_{d_n}(s) \quad \text{and} \quad f_{d_n}(s) - f_1(s) < 1 \tag{5.3}$$

for all $s \in [0,1]$ (note that $d_n = p_n q_n$ and each function h_j for $j = 1, 2, \ldots, p_n$, in Remark 5.4 (also see Lemma 5.3), repeats q_n times in the list of f_j above). Similarly, we also have

$$g_1(s) \le g_2(s) \le \cdots \le g_{d_n}(s) \quad \text{and} \quad g_{d_n}(s) - g_1(s) < 1 \tag{5.4}$$

for all $s \in [0,1]$.

Let $0 < \delta < \delta_1$ be such that $\frac{2\delta}{1-4\delta} < \delta_1$. One can choose $h_1, h_2, \ldots, h_{d_n} \in C([0,1])_{\text{s.a.}}$ such that

$$h_1(s) < h_2(s) < \cdots < h_{d_n}(s) \quad \text{and} \quad h_{d_n}(s) - h_1(s) < 1, \tag{5.5}$$

for $s \in [0,1]$ and such that $\| \exp(2\pi i h_j(s)) - \exp(2\pi i f_j(s))\| < \frac{\eta}{4}$ for $1 \le j \le d_n$ and $s \in [0,1]$.

Also we can choose $k_1, k_2, \ldots, k_{d_n} \in C([0,1])_{\text{s.a.}}$ such that

$$k_1(s) < k_2(s) < \cdots < k_{d_n}(s) \quad \text{and} \quad k_{d_n}(s) - k_1(s) < 1 \tag{5.6}$$

for $s \in [0,1]$ and such that

$$\| \exp\left(2\pi i k_j(s)\right) - \exp\left(2\pi i g_j(s)\right)\| < \frac{\eta}{4} \quad \text{for } 1 \le j \le d_n \text{ and } s \in [0,1].$$

We define a new path $\widetilde{u}_s$ as follows:

$$\widetilde{u}_s(t) = \begin{cases} U^{(0)}(s)\,\mathrm{diag}\left[\exp\left(2\pi i\left(\frac{\delta - t}{\delta} f_j(s) + \frac{t}{\delta} h_j(s)\right)\right)\right]_{j=1}^{d_n} \left(U^{(0)}(s)\right)^*, \\ \hfill t \in [0,\delta], \\ U^{(0)}(s)\,\mathrm{diag}\left[\exp\left(2\pi i\left(\frac{t-\delta}{\delta} f_j(s) + \frac{2\delta - t}{\delta} h_j(s)\right)\right)\right]_{j=1}^{d_n} \left(U^{(0)}(s)\right)^*, \\ \hfill t \in (\delta, 2\delta], \\ u_s\left(\frac{t-2\delta}{1-4\delta}\right), \hfill t \in (2\delta, 1-2\delta], \\ U^{(1)}(s)\,\mathrm{diag}\left[\exp\left(2\pi i\left(\frac{t-1+2\delta}{\delta} k_j(s) + \frac{1-\delta-t}{\delta} g_j(s)\right)\right)\right]_{j=1}^{d_n} \left(U^{(1)}(s)\right)^*, \\ \hfill t \in (1-2\delta, 1-\delta], \\ U^{(1)}(s)\,\mathrm{diag}\left[\exp\left(2\pi i\left(\frac{t-1+\delta}{\delta} g_j(s) + \frac{1-t}{\delta} k_j(s)\right)\right)\right]_{j=1}^{d_n} \left(U^{(1)}(s)\right)^*, \\ \hfill t \in (1-\delta, 1]. \end{cases}$$

As in the construction, it is easy to see that $\widetilde{u}_s$ is a path of unitaries in A_n.

For $t \in (0,\delta]$, it follows from (5.3) and (5.5) that

$$\frac{\delta - t}{\delta} f_1(s) + \frac{t}{\delta} h_1(s) < \frac{\delta - t}{\delta} f_2(s) + \frac{t}{\delta} h_2(s) < \cdots < \frac{\delta - t}{\delta} f_{d_n}(s) + \frac{t}{\delta} h_{d_n}(s)$$

and that

$$\left(\frac{\delta - t}{\delta} f_{d_n}(s) + \frac{t}{\delta} h_{d_n}(s)\right) - \left(\frac{\delta - t}{\delta} f_1(s) + \frac{t}{\delta} h_1(s)\right) < 1.$$

Hence, $\widetilde{u}_s(t)$ has no repeated eigenvalues for any $(s,t) \in [0,1] \times (0,\delta]$. Similarly, by (5.4) and (5.6), $\widetilde{u}_s(t)$ has no repeated eigenvalues for any $(s,t) \in [0,1] \times [1-\delta, 1)$. Moreover, when $t \in [0,\delta]$, we have

$$\begin{aligned} \left|\widetilde{u}_s(t) - u_s(t)\right| &\le \left|\widetilde{u}_s(t) - \widetilde{u}_s(0)\right| + \left|\widetilde{u}_s(0) - u_s(t)\right| \\ &\le \max_{1 \le j \le d_n} \left| \exp\left(2\pi i \frac{t}{\delta}\left(h_j(s) - f_j(s)\right)\right) - 1 \right| + \left|u_s(0) - u_s(t)\right| \end{aligned}$$

$$\leq \max_{1\leq j\leq d_n} \left|\exp\left(2\pi i\left(h_j(s)-f_j(s)\right)\right)-1\right| + \frac{\eta}{4}$$
$$= \max_{1\leq j\leq dn} \left|\exp\left(2\pi i f_j(s)\right)-\exp\left(2\pi i h_j(s)\right)\right| + \frac{\eta}{4}$$
$$\leq \frac{\eta}{2}.$$

For $t \in (\delta, 2\delta]$, we have

$$\begin{aligned}
\left|\widetilde{u}_s(t)-u_s(t)\right| &\leq \left|\widetilde{u}_s(t)-\widetilde{u}_s(\delta)\right| + \left|\widetilde{u}_s(\delta)-\widetilde{u}_s(2\delta)\right| + \left|\widetilde{u}_s(2\delta)-u_s(t)\right| \\
&\leq \max_{1\leq j\leq d_n} \left|\exp\left(2\pi i \frac{t-\delta}{\delta}\left(f_j(s)-h_j(s)\right)\right)-1\right| + \frac{\eta}{4} + \left|u_s(0)-u_s(t)\right| \\
&\leq \max_{1\leq j\leq d_n} \left|\exp\left(2\pi i\left(f_j(s)-h_j(s)\right)\right)-1\right| + \frac{\eta}{4} + \frac{\eta}{4} \\
&= \max_{1\leq j\leq d_n} \left|\exp\left(2\pi i f_j(s)\right)-\exp\left(2\pi i h_j(s)\right)\right| + \frac{\eta}{2} \\
&\leq \frac{3\eta}{4}.
\end{aligned}$$

In the same way, we have

$$\left|\widetilde{u}_s(t)-u_s(t)\right| \leq \frac{3\eta}{4} \quad \text{for all } t \in (1-2\delta, 1].$$

Furthermore, for $t \in [2\delta, 1-2\delta]$, it is easy to see that

$$\left|\frac{t-2\delta}{1-4\delta}-t\right| = \left|\frac{4t\delta-2\delta}{1-4\delta}\right| < \frac{2\delta}{1-4\delta} < \delta_1.$$

Hence

$$\left|\widetilde{u}_s(t)-u_s(t)\right| = \left|u_s\left(\frac{t-2\delta}{1-4\delta}\right)-u_s(t)\right| < \frac{\eta}{4}.$$

It follows that

$$\|\widetilde{u}_s-u_s\| < \frac{3\eta}{4}, \quad \text{for all } s \in [0,1].$$

In the construction of $\widetilde{u}_s$, it is easy to see that the lengths of $\widetilde{u}_s$ and u_s are close if η is small enough.

As we mentioned before, the unitaries $\widetilde{u}_s|_{\{\delta\}} \in M_{d_n}(C([0,1]\times\{\delta\}))$ and $\widetilde{u}_s|_{\{1-\delta\}} \in M_{d_n}(C([0,1]\times\{1-\delta\}))$, as the boundaries of $\widetilde{u}_s|_{[\delta,1-\delta]} \in M_{d_n}(C([0,1]\times[\delta,1-\delta]))$, both have distinct eigenvalues. By Proposition 2.2 and Remark 2.3, there exists another unitary, $\widetilde{\widetilde{u}}_s \in U(M_{d_n}(C([0,1]\times[\delta,1-\delta])))$, such that

$$\left\|\widetilde{\widetilde{u}}_s-\widetilde{u}_s|_{[\delta,1-\delta]}\right\| < \frac{\eta}{2},$$
$$\left|\text{length}_s\left(\widetilde{\widetilde{u}}_s\right)-\text{length}_s\left(\widetilde{u}_s|_{[\delta,1-\delta]}\right)\right| < \eta,$$

$\widetilde{\widetilde{u}}_s$ has distinct eigenvalues for any $(s,t) \in [0,1]\times[\delta,1-\delta]$, and

$$\widetilde{\widetilde{u}}_s(\delta) = \widetilde{u}_s(\delta), \quad \widetilde{\widetilde{u}}_s(1-\delta) = \widetilde{u}_s(1-\delta). \tag{5.7}$$

Define the path $v_s \in C([0,1], A_n)$ by

$$v_s(t) = \begin{cases} \widetilde{u}_s(t), & 0 \le t \le \delta, \\ \widetilde{\widetilde{u}}_s(t), & \delta < t \le 1-\delta, \\ \widetilde{u}_s(t), & 1-\delta < t \le 1. \end{cases}$$

Note that the initial value $v_s(0) = \widetilde{u}_s(0) = u_s(0)$ is in the form of (5.1) with the function f_j described in (5.3). One can choose the functions $\lambda_j : [0,1] \times [0,1] \to \mathbb{R}$ with $\lambda_j(s,0) = f_j(s)$ and $\lambda_1(s,t) \le \lambda_2(s,t) \le \cdots \le \lambda_{d_n}(s,t)$ such that

$$\mathrm{Sp}\left(v_s(t)\right) = \left\{\exp\left(2\pi i \lambda_j(s,t)\right)\right\}_{j=1}^{d_n}.$$

Since $v_s(t)$ has distinct eigenvalues for any $(s,t) \in [0,1] \times (0,1)$, by (5.3), we can prove that

$$\lambda_1(s,t) < \lambda_2(s,t) < \cdots < \lambda_{d_n}(s,t), \quad \lambda_{d_n}(s,t) - \lambda_1(s,t) < 1 \quad \text{for any } (s,t) \in [0,1] \times [0,1).$$

To see this, by (5.3)

$$\lambda_{d_n}(s,0) - \lambda_1(s,0) < 1.$$

Since $\exp(2\pi i \lambda_{d_n}(s,t)) \ne \exp(2\pi i \lambda_1(s,t))$ for any $(s,t) \in [0,1] \times (0,1)$, we have

$$\lambda_{d_n}(s,t) - \lambda_1(s,t) \notin \mathbb{Z}.$$

Notice that $\lambda_{d_n}(\cdot,\cdot)$ and $\lambda_1(\cdot,\cdot)$ are continuous. We have

$$\lambda_{d_n}(s,t) - \lambda_1(s,t) < 1 \quad \text{for all } (s,t) \in [0,1] \times [0,1).$$

Furthermore, there exist a permutation $\sigma \in S_{d_n}$ and d_n integers $\{m_j\}_{j=1}^{d_n}$ such that

$$\lambda_j(s,1) = g_{\sigma(j)}(s) + m_j \quad \text{for all } 1 \le j \le d_n.$$

It is easy to check that

$$\lambda_{d_n}(s,t) - \lambda_1(s,t) < 1 \quad \text{for any } (s,t) \in [0,1] \times [0,1].$$

For $(s,t) \in [0,1] \times (0,1)$, let $p_j(s,t)$ be the spectral projection of the unitary $v_s(t)$ corresponding to the eigenvalue $\exp(2\pi i \lambda_j(s,t))$, which is a rank one projection and continuously depends on $(s,t) \in [0,1] \times (0,1)$. From the definition of $v_s(t)$ (see the definition of $\widetilde{u}_s(t)$) for $t \in [0,\delta]$, we know that $\lambda_j(s,t) = \frac{\delta-t}{\delta} f_1(s) + \frac{t}{\delta} h_1(s)$ and that

$$p_j(s,t) = U^{(0)}(s) \,\mathrm{diag}\big[\underbrace{0,\ldots,0}_{j-1}, 1, \underbrace{0,\ldots,0}_{d_n-j}\big] U^{(0)}(s) \quad \text{for } (s,t) \in [0,1] \times (0,\delta],$$

which are constant projection valued functions with respect to $t \in (0,\delta]$. Similarly,

$$p_j(s,t) = U^{(1)}(s) \,\mathrm{diag}\big[\underbrace{0,\ldots,0}_{j-1}, 1, \underbrace{0,\ldots,0}_{d_n-j}\big] U^{(1)}(s) \quad \text{for } (s,t) \in [0,1] \times [1-\delta,1),$$

which are also constant projection valued functions with respect to $t \in [1-\delta, 1)$. Hence, the projection valued function p_j can be continuously extended to $[0,1] \times [0,1]$. Let $H_s(t) = \sum_{j=1}^{d_n} \lambda_j(s,t) p_j(s,t)$. Then $v_s(t) = \exp(2\pi i H_s(t))$ holds for all $(s,t) \in [0,1] \times (0,1)$, and, therefore, it holds for all $(s,t) \in [0,1] \times [0,1]$ by continuity. ■

Theorem 5.6. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then* $\mathrm{cel}_{\mathrm{CU}}(\mathcal{Z}) \geq 2\pi$.

Proof. Let $\mathcal{Z} = \lim_m A_m$ be the Jiang–Su algebra. Fix $\alpha \in (0,1)$ such that $1-\alpha(>0)$ is very small (it will be specified later about how small it should be). For each $m \geq 1$, we define a unitary $u \in A_m$ as follows:

$$u(t) = \begin{bmatrix} \exp\left(2\pi i h_1(t)\right) & & \\ & \ddots & \\ & & \exp\left(2\pi i h_{d_m}(t)\right) \end{bmatrix}_{d_m \times d_m},$$

where $h_i(t) = \frac{q_m - 1}{q_m}\alpha t$ for each $1 \leq i \leq p_m$, $h_i(t) = -\frac{1}{q_m}\alpha t$ for each $p_m + 1 \leq i \leq d_m$. (Here we identify $\mathbf{1}_{p_m} \otimes M_{q_m} \ni \mathbf{1} \otimes (a_{ij})_{q_m \times q_m}$ with $(a_{ij}\mathbf{1}_{p_m}) \in M_{p_m q_m}$.) It follows from [21, Lemma 3.10] that $u \in \mathrm{CU}(A_m)$. It is easy to calculate that

$$\mathrm{Sp}(u) \subset \left\{ \exp(2\pi i \lambda) : -\frac{1}{q_m}\alpha \leq \lambda \leq \frac{q_m - 1}{q_m}\alpha \right\}. \tag{5.8}$$

For any fixed $n \geq m$, denote $v = \phi_{m,n}(u)$. Let $u_s(t)$ be a unitary path in A_n with $u_0(t) = v(t)$ and $u_1(t) = \mathbf{1}_{A_n}$.

For any $0 < \varepsilon < \frac{1}{2^{n-m}} \cdot \frac{1-\alpha}{4q_m}$, by Lemma 5.5, there exists another piecewise smooth unitary path $v_s(t)$ such that

(1) $\|v_s - u_s\| < \frac{\varepsilon}{2}$;

(2) $|\operatorname{length}_s(v_s) - \operatorname{length}_s(u_s)| < \frac{\varepsilon}{2}$;

(3)

$$v_s(t) = U_s(t) \begin{bmatrix} \exp\left(2\pi i f_1(s,t)\right) & & & \\ & \exp\left(2\pi i f_2(s,t)\right) & & \\ & & \ddots & \\ & & & \exp\left(2\pi i f_{d_n}(s,t)\right) \end{bmatrix} U_s(t)^*,$$

where each $f_j(s,t) : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and $f_1(s,t) < f_2(s,t) < \cdots < f_{d_n}(s,t)$ for all $(s,t) \in [0,1] \times (0,1)$ and $f_{d_n}(s,t) - f_1(s,t) < 1$ for all $(s,t) \in [0,1] \times [0,1]$.

In particular, we have

$$\exp\left(2\pi i f_j(s,t)\right) \neq \exp\left(2\pi i f_k(s,t)\right) \quad \text{for } j \neq k \text{ and } (s,t) \in [0,1] \times (0,1).$$

By the construction of the Jiang–Su algebra, we have

$$u_0(t) = v(t) = \phi_{m,n}(u)(t)$$

$$= b^* \begin{bmatrix} \exp\big(2\pi i \mu_1(t)\big) & 0 & \cdots & 0 \\ 0 & \exp\big(2\pi i \mu_2(t)\big) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \exp\big(2\pi i \mu_{d_n}(t)\big) \end{bmatrix} b,$$

where

$$-\frac{1}{q_m}\alpha \leq \mu_1(t) \leq \mu_2(t) \leq \cdots \leq \mu_{d_n}(t) \leq \frac{q_m - 1}{q_m}\alpha, \tag{5.9}$$

for all $t \in [0,1]$, and b is a unitary element in $M_{d_n}(C([0,1]))$.

Let

$$X_\alpha = \Big\{ r \cdot \exp(2\pi i \lambda) \in \mathbb{C} : r \in \mathbb{R}_+, \Big(\frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}\Big) < \lambda < 1 + \Big(-\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi}\Big) \Big\} \subset \mathbb{C}$$

and

$$Y_\alpha = \Big\{ x + yi \in \mathbb{C} : x \in \mathbb{R}, \Big(-\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi}\Big)2\pi \leq y \leq \Big(\frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}\Big)2\pi \Big\} \subset \mathbb{C}.$$

Then $\mathrm{Sp}(v_0), \mathrm{Sp}(u_0) \subset \mathbb{C} \setminus X_\alpha$. Consider the logarithm function $\log : \mathbb{C} \setminus X_\alpha \to Y_\alpha$. Then $\|\frac{1}{2\pi}\log(v_0) - \frac{1}{2\pi}\log(u_0)\| < \frac{\varepsilon}{2\pi}$, since X_α is a wedge shaped region with angle at $0 \in \mathbb{C}$ of angle size

$$1 + \Big(-\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi}\Big) - \Big(\frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}\Big) = (1 - \alpha) - \frac{\varepsilon}{\pi} > 2\varepsilon$$

and $\|v_0 - u_0\| < \varepsilon/2$. By Weyl's inequality in [48],

$$\|f_j(0,t) - \mu_j(t)\| < \frac{\varepsilon}{2\pi} \quad \text{for all } j \in \{1, 2, \ldots, d_n\}. \tag{5.10}$$

From $u_1 = 1_{A_n}$ and $\|u_1 - v_1\| < \frac{\varepsilon}{2}$, by the version of Weyl's inequality for unitaries (see [1]), one gets

$$\big\| \exp\big(2\pi i f_j(1,t)\big) - 1 \big\| < \frac{\varepsilon}{2}.$$

Hence, for each $j \in \{1, 2, \ldots, d_n\}$, there is an integer l_j such that

$$\big\| f_j(1,t) - l_j \big\| < \frac{\varepsilon}{2\pi}. \tag{5.11}$$

By Proposition 2.6, we have $\mathrm{length}_s(v_s(\cdot)) \geq \mathrm{length}_s(\exp(2\pi i f_j(s,\cdot)))$ for each j. Hence, by (5.10) and (5.11),

$$\mathrm{length}_s\,\big(\exp\big(2\pi i f_j(s,\cdot)\big)\big) \geq 2\pi \max_{t\in[0,1]} \big|\mu_j(t) - l_j\big| - 2\varepsilon.$$

From the construction of Jiang–Su algebra (see Theorem 5.1),

$$\mu_1(t) = -\frac{\alpha}{q_m}\big(\xi_\bullet(t)\big) = \frac{t + 2^{n-m} - 1}{2^{n-m}} \cdot \frac{-\alpha}{q_m} \leq 0$$

and

$$\mu_{d_n}(t) = \frac{q_m - 1}{q_m}\big(\xi_{\bullet}(t)\big) = \frac{t + 2^{n-m} - 1}{2^{n-m}} \cdot \frac{(q_m - 1)\alpha}{q_m} \geq \frac{2^{n-m} - 1}{2^{n-m}} \cdot \frac{(q_m - 1)\alpha}{q_m}. \tag{5.12}$$

We divide the discussion into the following cases.

Case 1. $l_{j_0} \geq 2$ for some $1 \leq j_0 \leq d_n$. Then

$$\text{length}_s\left(e^{2\pi i f_{j_0}(s,\cdot)}\right) \geq 2\pi \max_{t\in[0,1]} \left|\mu_{j_0}(t) - l_{j_0}\right| - 2\varepsilon \geq 2\pi - 2\varepsilon.$$

Case 2. $l_{j_1} \leq -1$ for some $1 \leq j_1 \leq d_n$. Then

$$\text{length}_s\left(e^{2\pi i f_{j_1}(s,\cdot)}\right) \geq 2\pi \max_{t\in[0,1]} \left|\mu_{j_1}(t) - l_{j_1}\right| - 2\varepsilon \geq 2\pi - 2\varepsilon.$$

Case 3. $l_j = 0$ for all $1 \leq j \leq d_n$. Then

$$\text{length}_s\left(e^{2\pi i f_{d_n}(s,\cdot)}\right) \geq 2\pi \max_{t\in[0,1]} \left|\mu_{d_n}(t) - 0\right| - 2\varepsilon.$$

It follows that

$$\text{length}_s\left(e^{2\pi i f_{d_n}(s,\cdot)}\right) \geq 2\pi\left(\frac{2^{n-m} - 1}{2^{n-m}} \cdot \frac{(q_m - 1)\alpha}{q_m}\right) - 2\varepsilon. \tag{5.13}$$

Case 4. $l_j = 1$ for all $1 \leq j \leq d_n$. Then

$$\text{length}_s\left(e^{2\pi i f_1(s,\cdot)}\right) \geq 2\pi \max_{t\in[0,1]} \left|\mu_1(t) - 1\right| - 2\varepsilon \geq 2\pi - 2\varepsilon.$$

Case 5. All l_j are either 0 or 1 (i.e., $\{1, 2, \ldots, d_n\} = \{j : l_j = 0\} \cup \{j : l_j = 1\}$) and $\{j : l_j = 0\} \neq \emptyset, \{j : l_j = 1\} \neq \emptyset$.

There exists $1 \leq K < d_n$ such that $l_j = 0$ for all $1 \leq j \leq K$ and $l_j = 1$ for all $K + 1 \leq j \leq d_n$. Note that $f_{d_n}(s,t) - f_1(s,t) < 1$ for all $(s,t) \in [0,1] \times [0,1]$. By the boundary condition of $v_s \in A_n$, we have $q_n \mid K$ and $p_n \mid (d_n - K)$. Since $d_n = p_n q_n$, we have $p_n \mid K$. It follows that $K = 0$ or $K = d_n$, a contradiction. This means that Case 5 will not occur. ∎

Furthermore, we can obtain the following more general result with almost the same proof.

Theorem 5.7. *Let $\mathcal{Z}$ be the Jiang–Su algebra and k a positive integer. Then*

$$\text{cel}_{\text{CU}}\left(M_k(\mathcal{Z})\right) \geq 2\pi.$$

(Combining with Lin's result, one gets $\text{cel}_{\text{CU}}(M_k(\mathcal{Z})) = 2\pi$.)

Proof (Sketch). Let $\mathcal{Z} = \lim_m A_m$ be the Jiang–Su algebra. Write

$$M_k(\mathcal{Z}) = \lim_m \big(M_k(A_m), \phi_{m,n}\big).$$

We shall also use $\phi_{m,n}$ to denote $\phi_{m,n}|_{A_m}$ when we identify A_m as the upper left corner of $M_k(A_m)$. Fix $m \geq 1$. Let $0 < \beta < \alpha < 1$ be such that $\beta < 1 - \alpha$ and that $1 - \alpha$ is very small. Define a unitary $u^1 \in A_m$ as follows:

$$u^1(t) = \begin{bmatrix} e^{2\pi i h_1(t)} & & \\ & \ddots & \\ & & e^{2\pi i h_{d_m}(t)} \end{bmatrix}_{d_m \times d_m},$$

where $h_i(t) = \frac{q_m - 1}{q_m}(\beta + (\alpha - \beta)t)$ for each $1 \leq i \leq p_m$ and $h_i(t) = -\frac{1}{q_m}(\beta + (\alpha - \beta)t)$ for each $p_m + 1 \leq i \leq d_m$. We denote

$$u = \operatorname{diag}[u^1, u^2, \ldots, u^k] \in M_k(A_m),$$

where $u^i(t) = \mathbf{1}_{A_m}$ for each $2 \leq i \leq k$. It follows that $u \in \mathrm{CU}(M_k(A_m))$.

For any $n > m + 1$, suppose that u_s is a path connecting $\phi_{m,n}(u)$ and $\mathbf{1}_{M_k(A_n)}$. Fix a positive number $\varepsilon < \frac{1}{2^{n-m}} \cdot \frac{\beta}{4q_m}$. Let v_s be described in the proof of Theorem 5.6 with properties (1), (2), and (3) there. In particular, we have

$$v_s(t) = U_s(t) \operatorname{diag}\big[\exp\big(2\pi i f_1(s,t)\big), \exp\big(2\pi i f_2(s,t)\big), \ldots, \exp\big(2\pi i f_{d_n k}(s,t)\big)\big] U_s(t)^*,$$

where each $f_j(s,t) : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and

$$f_1(s,t) < f_2(s,t) < \cdots < f_{d_n k}(s,t) \tag{5.14}$$

for all $(s,t) \in [0,1] \times (0,1)$ and

$$f_{d_n k}(s,t) - f_1(s,t) < 1 \tag{5.15}$$

for all $(s,t) \in [0,1] \times [0,1]$.

With our modified construction of u^1, one gets

$$\mathrm{Sp}\big(\phi_{m,n}(u^1)\big) \subset \mathrm{Sp}(u^1) \subset \left\{\exp(2\pi i \lambda) : \lambda \in \left[-\frac{1}{q_m}\alpha, -\frac{1}{q_m}\beta\right] \cup \left[\frac{q_m - 1}{q_m}\beta, \frac{q_m - 1}{q_m}\alpha\right]\right\}$$

which is stronger than (5.8) in the proof of Theorem 5.6. Denote $w_1 = \phi_{m,n}(u^1)$ and $w = \phi_{m,n}(u)$. Then for any $t \in [0,1]$, the eigenvalues of $w_1(t)$ can be written as

$$\big\{\exp\big(2\pi i \mu_j(t)\big)\big\}_{j=1}^{d_n}$$

such that

$$\begin{aligned} -\frac{1}{q_m}\alpha \leq \mu_1(t) \leq \mu_2(t) \leq \cdots \leq \mu_L(t) \\ \leq -\frac{1}{q_m}\beta < 0 < \frac{q_m - 1}{q_m}\beta \leq \mu_{L+1}(t) \end{aligned}$$

$$\leq \mu_{L+2}(t) \leq \cdots \leq \mu_{d_n}(t) \leq \frac{q_m - 1}{q_m}\alpha$$

for some positive integer $L < d_n$ (compare with (5.9)). And the eigenvalues of $w(t)$ can be listed as

$$\left\{\left\{\exp\left(2\pi i \mu_j(t)\right)\right\}_{j=1}^{L}, \exp(2\pi i 0)^{\sim(k-1)d_n}, \left\{\exp\left(2\pi i \mu_j(t)\right)\right\}_{j=L+1}^{d_n}\right\}.$$

Note that $\|v_0 - w\| < \varepsilon/2$. One has

$$\mathrm{Sp}(v_0) \subset \left\{\exp(2\pi i \lambda) : -\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi} \leq \lambda \leq \frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}\right\}.$$

Hence the above f_j can be chosen to satisfy

$$-\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi} \leq f_1(0,t) \leq f_2(0,t) \leq \cdots \leq f_{d_n k}(0,t) \leq \frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}.$$

(Compare with (5.9).) Then similar to (5.10), one can get

$$\begin{aligned} \left\| f_j(0,t) - \mu_j(t) \right\| &< \frac{\varepsilon}{2\pi} \quad \text{for all } j \in \{1, 2, \ldots, L\}, \\ \left\| f_j(0,t) - 0 \right\| &< \frac{\varepsilon}{2\pi} \quad \text{for all } j \in \left\{L+1, L+2, \ldots, d_n(k-1) + L\right\}, \\ \left\| f_{d_n(k-1)+j}(0,t) - \mu_j(t) \right\| &< \frac{\varepsilon}{2\pi} \quad \text{for all } j \in \{L+1, L+2, \ldots, d_n\}. \end{aligned}$$

The ranges of functions $f_j(0,\cdot)$ are in three mutually disjoint open intervals:

$$\begin{aligned} &\text{for } 1 \leq j \leq L, && f_j(0,t) \in \left(-\frac{1}{q_m}\alpha - \frac{\varepsilon}{2\pi}, -\frac{1}{q_m}\beta + \frac{\varepsilon}{2\pi}\right); \\ &\text{for } L+1 \leq j \leq d_n(k-1) + L, && f_j(0,t) \in \left(-\frac{\varepsilon}{2\pi}, +\frac{\varepsilon}{2\pi}\right); \\ &\text{for } d_n(k-1) + L + 1 \leq j \leq d_n k, && f_j(0,t) \in \left(\frac{q_m - 1}{q_m}\beta - \frac{\varepsilon}{2\pi}, \frac{q_m - 1}{q_m}\alpha + \frac{\varepsilon}{2\pi}\right). \end{aligned}$$

Repeating the proof of Theorem 5.6, we can prove that

$$\mathrm{length}_s(v_s) \geq 2\pi\left(\frac{2^{n-m} - 1}{2^{n-m}} \cdot \frac{(q_m - 1)\boldsymbol{\alpha}}{q_m}\right) - 2\varepsilon. \tag{5.16}$$

Note that

$$\begin{aligned} \mu_{d_n}(t) &= \frac{q_m - 1}{q_m}\left(\beta + (\alpha - \beta)\xi_\bullet(t)\right) \\ &= \frac{q_m - 1}{q_m}\left(\beta + (\alpha - \beta)\frac{t + 2^{n-m} - 1}{2^{n-m}}\right) \geq \left(\frac{2^{n-m} - 1}{2^{n-m}} \cdot \frac{(q_m - 1)\boldsymbol{\alpha}}{q_m}\right). \end{aligned}$$

(Compare with (5.12).)

We are not going to give all the details. Instead, we will briefly describe some steps of the proof of the corresponding part of Theorem 5.6 (see (5.13) there) and point out only the new issue involved in our new case which needs a slightly different treatment.

As in the proof of Theorem 5.6 (see (5.11) there), by $\|v_s - u_s\| < \varepsilon/2$ and $u_1 = \mathbf{1}_{M_k(A_n)}$, for each $j \in \{1, 2, \ldots, d_n k\}$, there is an integer l_j such that

$$\|f_j(1,t) - l_j\| < \frac{\varepsilon}{2\pi}.$$

We will divide the proof of (5.16) into five cases according to the possible values of l_j, the same as what we did in the proof of (5.13) (see the proof of Theorem 5.6).

For Cases 1, 2, 3, and 4, the proofs are exactly the same. So we only need to deal with Case 5: all l_j are either 0 or 1, with $\{j : l_j = 0\} \neq \emptyset$ and $\{j : l_j = 1\} \neq \emptyset$.

So we have $\{1, 2, \ldots, d_n k\} = \{j : l_j = 0\} \cup \{j : l_j = 1\}$. By (5.14) and (5.15), there exists $1 \le K < d_n k$ such that $l_j = 0$ for all $1 \le j \le K$ and $l_j = 1$ for all $K + 1 \le j \le d_n k$.

If $K \le d_n(k-1)$, then, for $j = d_n(k-1) + 1$, we have $\|f_j(0,t)\| < \frac{\varepsilon}{2\pi}$ and $\|f_j(1,t) - 1\| < \frac{\varepsilon}{2\pi}$. Then $\text{length}_s(\exp(2\pi i f_j(s,\cdot))) \ge 2\pi - 2\varepsilon$. Consequently, (5.16) holds for this case. If $K > d_n(k-1)$, we shall show that $K = d_n k$. Otherwise, we have $d_n(k-1) < K < d_n k$. By (5.14) and (5.15), we have $q_n | K$ and $p_n | (d_n k - K)$. Since $q_n | d_n$, we have $q_n | [K - (k-1)d_n]$. We denote $s = \frac{K-(k-1)d_n}{q_n}$ and $l = \frac{d_n k - K}{p_n}$. Then $s, l \in \mathbb{N}$. Then we have

$$p_n l + q_n s = (d_n k - K) + \big(K - d_n(k-1)\big) = d_n = p_n q_n.$$

Hence, we have $p_n \mid s$ and $q_n \mid l$. We denote $s' = \frac{s}{p_n}$ and $l' = \frac{l}{q_n}$. Then $l' + s' = 1$. This leads to a contradiction. So we have $K = k d_n$. This means that $l_j = 0$ for each $1 \le j \le d_n k$. This is contracted to the fact that $\{j : l_j = 1\} \neq \emptyset$. So "$K > d_n(k-1)$" does not occur. It follows that (5.16) holds for any case.

Consequently, $\text{cel}_{\text{CU}}(M_k(\mathcal{Z})) \ge 2\pi$. ■

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