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On localized signature and higher rho invariant of fibered manifolds

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Abstract. The higher index of the signature operator is a far-reaching generalization of the signature of a closed oriented manifold. When two closed oriented manifolds are homotopy equivalent, one can define a secondary invariant of the relative signature operator, called higher rho invariant. The higher rho invariant detects the topological nonrigidity of a manifold. In this paper, we prove product formulas for the higher index and the higher rho invariant of the signature operator on a fibered manifold. Our result implies the classical product formula for the numerical signature of a fibered manifold obtained by Chern, Hirzebruch, and Serre (1957). We also give a new proof of the product formula for the higher rho invariant of the signature operator on a product manifold, which is parallel to the product formula for the higher rho invariant of Dirac operator on a product manifold obtained by Xie and Yu (2014) and Zeidler (2016).

1. Introduction

The signature of a $4k$-dimensional manifold is defined to be the signature of the cup product as a nondegenerate symmetric bilinear form on the vector space of $2k$-cohomology classes. In [1], Chern, Hirzebruch, and Serre established a product formula for the signature of a fibered manifold. More precisely, let $F \to E \to B$ be a fibered manifold with base manifold B and fiber manifold F. If $\pi_1(B)$ acts trivially on $H^*_{dR}(F)$, the de Rham cohomology of F, we have the following product formula:

$$\mathrm{sgn}(B) \times \mathrm{sgn}(F) = \mathrm{sgn}(E). \tag{1.1}$$

The signature of a manifold is also equal to the Fredholm index of the signature operator. By taking into account the fundamental group of the manifold, one can introduce higher invariants of the signature operator, which lie in the K-theory of certain geometric C^*-algebras. Let M be an m-dimensional closed manifold with fundamental group $\pi_1(M) = G$ and universal covering space $\widetilde{M}$. Let D_M^{sgn} be the signature operator on M. The higher index of D_M^{sgn}, $\mathrm{Ind}(D_M^{\mathrm{sgn}})$, is a generalization of the Fredholm index and is defined to be an element in $K_m(C^*(\widetilde{M})^G)$, where $C^*(\widetilde{M})^G$ is the equivariant Roe algebra of $\widetilde{M}$ and is Morita equivalent to the reduced group C^*-algebra $C_r^*(G)$. The higher index of the signature operator is invariant under homotopy equivalence and oriented cobordism and plays a fundamental role in the study of classification of manifolds. On the other hand,

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D_M^{sgn} defines a K-homology class $[D_M^{\mathrm{sgn}}]$ in $K_m(C_L^*(\widetilde{M})^G)$, the K-theory of the equivariant localization algebra. See Sections 2 and 3 for the explicit definitions of the equivariant geometric C^*-algebras, the higher index, and the K-homology class of the signature operator.

Furthermore, if $f : M' \to M$ is an orientation-preserving homotopy equivalence of closed manifolds, then there exists a concrete homotopy that realizes the equality

$$\operatorname{Ind}\left(D_{M'}^{\mathrm{sgn}}\right) = \operatorname{Ind}\left(D_M^{\mathrm{sgn}}\right) \in K_m\left(C^*(\widetilde{M})^G\right),$$

where m is the dimension of M and M'. This homotopy allows one to define a secondary invariant of the signature operator associated to the homotopy equivalence f, called higher rho invariant, in the K-theory of the equivariant obstruction algebra $C_{L,0}^*(\widetilde{M})^G$. The higher rho invariant of the signature operator associated to a homotopy equivalence plays a central role in estimating the topological nonrigidity of a manifold (cf. [5, 8, 9, 15, 21]).

Inspired by Chern, Hirzebruch, and Serre's product formula, we prove a product formula for the higher index and the higher rho invariant of the signature operator on a fibered manifold. More precisely, consider a closed fibered manifold $F \to E \to B$ with base space B and fiber F. Denote the fundamental group of E by G and the fundamental group of B by H. Let $\widetilde{E}$ and $\widetilde{B}$ be the universal covering spaces of E and B, respectively. Set $n = \dim F$ and $m = \dim B$. We first define a family localization algebra $C_L^*(\widetilde{E}; \widetilde{B})^G$ and a family obstruction algebra $C_{L,0}^*(\widetilde{E}; \widetilde{B})^G$. We show that there are naturally defined product maps:

$$\phi : K_m\left(C_L^*(\widetilde{B})^H\right) \otimes K_n\left(C_L^*(\widetilde{E}, \widetilde{B})^G\right) \to K_{m+n}\left(C_L^*(\widetilde{E})^G\right),$$
$$\phi_0 : K_m\left(C_L^*(\widetilde{B})^H\right) \otimes K_n\left(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\right) \to K_{m+n}\left(C_{L,0}^*(\widetilde{E})^G\right).$$

Taking advantage of the fiberwise signature operator, we introduce a family K-homology class of the family signature operator along F in $K_n(C_L^*(\widetilde{E}; \widetilde{B})^G)$, denoted by $[D_{E,B}^{\mathrm{sgn}}]$, and a family higher rho invariant $\rho(f; B)$ in $K_n(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G)$, associated to a fiberwise homotopy equivalence $f : E' \to E$. The following theorem is a product formula for the K-homology class of the signature operator on a fibered manifold, which implies the product formula for the higher index of the signature operator.

Theorem 1.1. *Let $F \to E \to B$ be a fibered manifold with base space B and fiber F. Denote the fundamental group of E by G and the fundamental group of B by H. Let $\widetilde{E}$ and $\widetilde{B}$ be the universal covering spaces of E and B, respectively. Write* $\dim F = n$ *and* $\dim B = m$. *Let $[D_{E,B}^{\mathrm{sgn}}]$ be the family K-homology class of the family signature operator in $K_n(C_L^*(\widetilde{E}, \widetilde{B})^G)$. One has the following product formula for the family K-homology class of the family signature operator*

$$k_{mn} \cdot \phi\left([D_B^{\mathrm{sgn}}] \otimes [D_{E,B}^{\mathrm{sgn}}]\right) = [D_E^{\mathrm{sgn}}],$$

where $k_{mn} = 1$ when mn is even and $k_{mn} = 2$ otherwise; and ϕ is the product map

$$\phi : K_m\left(C_L^*(\widetilde{B})^H\right) \otimes K_n\left(C_L^*(\widetilde{E}, \widetilde{B})^G\right) \to K_{m+n}\left(C_L^*(\widetilde{E})^G\right).$$

We also obtain the following product formula for the higher rho invariant of the signature operator on a fibered manifold.

Theorem 1.2. *Let $F \to E \to B$ and $F' \to E' \to B$ be two fibered manifolds with base space B and fiber F and F', respectively. Let $f : E' \to E$ be a fiberwise homotopy equivalence. Denote the fundamental group of E and E' by G and the fundamental group of B by H. Let $\widetilde{E}$, $\widetilde{E}'$, and $\widetilde{B}$ be the universal covering spaces of E, E', and B, respectively. Write* $\dim F = n$ *and* $\dim B = m$. *Let $\rho(f; B)$ be the family higher rho invariant associated to the fiberwise homotopy equivalence f defined in $K_n(C^*_{L,0}(\widetilde{E}, \widetilde{B})^G)$. One has the following product formula for the higher rho invariant:*

$$k_{mn} \cdot \phi_0\big([D_B^{\mathrm{sgn}}] \otimes \rho(f; B)\big) = \rho(f),$$

where $k_{mn} = 1$ when mn is even and $k_{mn} = 2$ otherwise, and ϕ_0 is the product map

$$\phi_0 : K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C^*_{L,0}(\widetilde{E}, \widetilde{B})^G\big) \to K_{m+n}\big(C^*_{L,0}(\widetilde{E})^G\big).$$

As an application of Theorem 1.1, we give an alternative proof of the product formula of Chern, Hirzebruch, and Serre (cf. [1]). Also, the product formula for the higher rho invariant stated in Theorem 1.2 can be applied to study the behavior of the higher rho map in [15] under fibration and thus can be applied to study the topological nonrigidity of a fibered manifold.

We mention that the product formula for the higher index of the signature operator (Proposition 4.1) has been obtained by Wahl in [13, Theorem 5.7]. In this paper, we give a new proof of Wahl's product formula. On the other hand, the product formula for the higher rho invariant of a positive scalar curvature metric on a product manifold has been proved by Siegel in his thesis [11], by Xie and Yu in [17], and by Zeidler in [20]. Their results and Theorems 6.8 and 6.9 in [15] inspire us to study the product formula for the higher rho invariant of the signature operator.

The paper is organized as follows. In Section 2, we briefly recall some definitions of geometric C^*-algebras that we use throughout the paper. In Section 3, we revisit the construction of several higher invariants associated to the signature operator. Next, in Section 4, we prove the product formulas for the higher index and the higher rho invariant of the signature operator on a product manifold. In Section 5, we generalize the product formulas to fibered manifolds and prove Theorems 1.1 and 1.2. We shall define an auxiliary C^*-algebra consisting of operators that can be localized along the base manifold and use Mayer–Vietoris arguments to prove the product formula on a fibered manifold.

2. Preliminary

The aim of this section is to briefly recall some basic definitions of geometric C^*-algebras used throughout the paper. For more details, we refer the readers to [15].

Let X be a proper metric space and G a finitely presented discrete group. Suppose that G acts on X properly by isometries. For simplicity, we assume that the G-action is free. Let $C_0(X)$ be the C^*-algebra consisting of all complex-valued continuous functions on X that vanish at infinity. An X-module is a separable Hilbert space H_X equipped with a $*$-representation of $C_0(X)$. It is called "nondegenerate" if the $*$-representation is nondegenerate and "standard" if no nonzero function in $C_0(X)$ acts as a compact operator. Additionally, we assume that H_X is equipped with a unitary representation of G which is compatible with the $C_0(X)$-representation; that is,

$$\forall f \in C_0(X),\ g \in G, \quad \pi(g)\phi(f) = \phi(g \cdot f)\pi(g),$$

where ϕ (resp. π) is the $C_0(X)$ (resp. G)-representation on H_X and $g \cdot f(x) = f(g^{-1}x)$.

Now, let us recall the definitions of propagation and local compactness of operators.

Definition 2.1. Under the above assumptions, let T be a bounded linear operator acting on H_X.

(1) The *propagation* of T is defined by

$$\operatorname{prop}(T) = \sup\left\{d(x, y) \mid (x, y) \in \operatorname{Supp}(T)\right\}, \tag{2.1}$$

where $\operatorname{Supp}(T)$ is the complement (in $X \times X$) of the set of points $(x, y) \in X \times X$ such that there exist $f_1, f_2 \in C_0(X)$ such that $f_1 T f_2 = 0$ and $f_1(x) f_2(y) \neq 0$.

(2) T is said to be *locally compact* if both fT and Tf are compact for all $f \in C_0(X)$.

In the following, we recall the definitions of the equivariant Roe algebra, localization algebra, and obstruction algebra.

Definition 2.2. Let H_X be a standard nondegenerate X-module and $B(H_X)$ the set of all bounded linear operators on H_X.

(1) The *G-equivariant Roe algebra* of X, denoted by $C^*(X)^G$, is the C^*-algebra generated by all G-equivariant locally compact operators with finite propagation in $B(H_X)$.

(2) The *G-equivariant localization algebra* $C_L^*(X)^G$ is the C^*-algebra generated by all uniformly norm-bounded and uniformly norm-continuous functions $f : [1, \infty) \to C^*(X)^G$ such that

$$\forall t \in [1, \infty),\ \operatorname{prop}\left(f(t)\right) < \infty \text{ and } \operatorname{prop}\left(f(t)\right) \to 0 \text{ as } t \to \infty.$$

(3) The *G-equivariant obstruction algebra* $C_{L,0}^*(X)^G$ is defined to be the kernel of the following evaluation map:

$$\begin{aligned} \operatorname{ev} : C_L^*(X)^G &\to C^*(X)^G \\ f &\mapsto f(1). \end{aligned}$$

In particular, $C_{L,0}^*(X)^G$ is an ideal of $C_L^*(X)^G$.

(4) Let $\mathcal{U}^*(X)^G$ be the C^*-algebra generated by all G-equivariant operators with finite propagation in $B(H_X)$. Similarly, we define $\mathcal{U}_L^*(X)^G$ and $\mathcal{U}_{L,0}^*(X)^G$. It is easy to see that $C^*(X)^G$, $C_L^*(X)^G$, and $C_{L,0}^*(X)^G$ are two-sided closed ideals of $\mathcal{U}^*(X)^G$, $\mathcal{U}_L^*(X)^G$, and $\mathcal{U}_{L,0}^*(X)^G$, respectively.

Remark 2.3. Up to isomorphism, $C^*(X)^G$ does not depend on the choice of the standard nondegenerate X-module. The same holds for $C_L^*(X)^G$ and $C_{L,0}^*(X)^G$ (cf. [16, Part II], [17, Remark 2.7]).

When X is a Galois G-covering of a closed Riemannian manifold, $L^2(X)$ is a standard nondegenerate X-module. In this case, there is an equivalent definition of the equivariant Roe algebra.

Definition 2.4. Let X be a Galois G-covering of a closed Riemannian manifold. Set $\mathbb{C}[X]^G$ as a $*$-algebra consisting of integral operators given by

$$f \mapsto \int_X k(x,y) f(y)\, dy, \quad \forall f \in L^2(X),$$

where $k : X \times X \to \mathbb{C}$ is uniformly continuous, uniformly bounded on $X \times X$, has finite propagation, i.e.,

$$\exists \delta > 0, \text{ s.t. } k(x,y) = 0, \text{ if } d(x,y) > \delta,$$

and is G-equivariant, i.e., $k(gx, gy) = k(x,y)$ for any $g \in G$. The G-equivariant Roe algebra is the operator norm completion of $\mathbb{C}[X]^G$.

Remark 2.5. As we have assumed that the base manifold is closed, the G-action on X is cocompact. If we remove the cocompactness, the C^*-algebra given in Definition 2.4 will not coincide with the equivariant Roe algebra given in Definition 2.2.

Suppose that $T \in \mathbb{C}[X]^G$ has a corresponding Schwartz kernel $k(x,y)$. The support of T defined in Definition 2.2 is simply given by

$$\operatorname{Supp}(T) = \overline{\{(x,y) \in X \times X : k(x,y) \neq 0\}}. \tag{2.2}$$

Similarly, we define the G-equivariant localization algebra $C_L^*(X)^G$ to be the completion of all paths on $t \in [1, +\infty)$ with value in $\mathbb{C}[X]^G$, which are uniformly continuous and uniformly bounded with respect to the operator norm and have propagation going to zero as t goes to infinity.

Definition 2.4 can be easily generalized to the case where $L^2(X)$ is replaced by L^2-section of a Hermitian vector bundle over X on which G acts by isometries. The above definitions coincide with Definition 2.2, as those C^*-algebras are independent of the choice of the standard nondegenerate X-module.

Now, we consider the product of two manifolds. Let M, N be two closed Riemannian manifolds and $\widetilde{M}, \widetilde{N}$ their Galois G,H-covering spaces, respectively. For any pair of integral operators in $\mathbb{C}[\widetilde{M}]^G$ and $\mathbb{C}[\widetilde{N}]^H$, their tensor product is well-defined as an operator in $\mathbb{C}[\widetilde{M} \times \widetilde{N}]^{G\times H}$. Thus, the following homomorphisms are naturally defined:

$$\begin{aligned}
\psi &: K_m\big(C^*(\widetilde{M})^G\big) \otimes K_n\big(C^*(\widetilde{N})^H\big) \to K_{m+n}\big(C^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big),\\
\psi_L &: K_m\big(C_L^*(\widetilde{M})^G\big) \otimes K_n\big(C_L^*(\widetilde{N})^H\big) \to K_{m+n}\big(C_L^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big),\\
\psi_{L,0} &: K_m\big(C_L^*(\widetilde{M})^G\big) \otimes K_n\big(C_{L,0}^*(\widetilde{N})^H\big) \to K_{m+n}\big(C_{L,0}^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big).
\end{aligned} \tag{2.3}$$

More precisely,

$$\psi : K_m\big(C^*(\widetilde{M})^G\big) \otimes K_n\big(C^*(\widetilde{N})^H\big) \to K_{m+n}\big(C^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big)$$

is defined to be the following composition of maps:

$$\begin{aligned} K_m\big(C^*(\widetilde{M})^G\big) \otimes K_n\big(C^*(\widetilde{N})^H\big) &\xrightarrow{\otimes_K} K_{m+n}\big(C^*(\widetilde{M})^G \otimes C^*(\widetilde{N})^H\big) \\ &\longrightarrow K_{m+n}\big(C^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big). \end{aligned}$$

The maps ψ_L and $\psi_{L,0}$ are defined similarly.

3. Higher invariants associated to signature operator

In this section, we recall a formula of the higher index of the signature operator, which was obtained by Higson and Roe in [3, 4]. After that, we will give the construction of K-homology class of the signature operator, which was originally introduced by Weinberger, Xie, and Yu in [15]. Finally, we give the definition of the higher rho invariant associated to a controlled homotopy equivalence of manifolds.

In this section, all manifolds mentioned are not assumed to be compact or connected unless otherwise noted.

3.1. Higher index of signature operator

Let M be a Riemannian manifold of dimension m. Let $\widetilde{M}$ be a Galois G-covering space of M, where G is a finitely presented discrete group. Denote by $\Lambda^p(\widetilde{M})$ the L^2-completion of compactly supported smooth differential p-forms on $\widetilde{M}$. Let $d_{\widetilde{M}}$ be the de Rham differential on $\widetilde{M}$, which is an unbounded operator from $\Lambda^p(\widetilde{M})$ to $\Lambda^{p+1}(\widetilde{M})$. We will write $\Lambda^{\text{even}}(\widetilde{M}) = \bigoplus_k \Lambda^{2k}(\widetilde{M})$, $\Lambda^{\text{odd}}(\widetilde{M}) = \bigoplus_k \Lambda^{2k+1}(\widetilde{M})$, and $\Lambda(\widetilde{M}) = \Lambda^{\text{even}}(\widetilde{M}) \oplus \Lambda^{\text{odd}}(\widetilde{M})$.

Let $D_{\widetilde{M}}$ be the Hodge–de Rham operator; i.e., $D_{\widetilde{M}} = d_{\widetilde{M}} + d^*_{\widetilde{M}}$. Let $*$ be the Hodge $*$-operator of $\Lambda^*(\widetilde{M})$. Define the Poincaré duality operator $S_{\widetilde{M}}$ by

$$\begin{aligned} S_{\widetilde{M}} : \Lambda^p(\widetilde{M}) &\to \Lambda^{n-p}(\widetilde{M}), \\ \omega &\mapsto i^{p(p-1)+[\frac{n}{2}]} * \omega. \end{aligned}$$

Remark 3.1. Here, $S_{\widetilde{M}}$ is a special case of the Poincaré duality operator defined in [3, Definition 3.1]. We see that the square of $S_{\widetilde{M}}$ is equal to 1, but, in general, we do not need this fact to define the higher signature. We will pay special attention to this distinction in Section 4 when proving the product formulas, especially for the product formula for the higher rho invariants.

We now recall the representative of the higher index of the signature operator according to the parity of m.

Odd case. When m is odd, the signature operator D_M^{sgn} is given by

$$iD_{\widetilde{M}} S_{\widetilde{M}} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{even}}(\widetilde{M}).$$

The following is shown in [3, 4].

(1) $D_{\widetilde{M}} \pm S_{\widetilde{M}}$ are both invertible.

(2) The invertible operator

$$(D_{\widetilde{M}} + S_{\widetilde{M}})(D_{\widetilde{M}} - S_{\widetilde{M}})^{-1} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{even}}(\widetilde{M})$$

belongs to $(C^*(\widetilde{M})^G)^+$ and thus defines a class in $K_1(C^*(\widetilde{M})^G)$ denoted by $\operatorname{Ind}(D_M^{\text{sgn}})$.

(3) The higher index of the signature operator (cf. [4, Section 5.2.2]) is equal to $\operatorname{Ind}(D_M^{\text{sgn}})$.

Even case. The even case is parallel. In this case, the signature operator D_M^{sgn} is the Hodge–de Rham operator $D_{\widetilde{M}}$ under the grading given by the Poincaré duality $S_{\widetilde{M}}$.

The following is shown in [3, 4].

(1) $D_{\widetilde{M}} \pm S_{\widetilde{M}}$ are both invertible.

(2) Let $P_+(D_{\widetilde{M}} \pm S_{\widetilde{M}})$ be the positive spectral projections of invertible operators $D_{\widetilde{M}} \pm S_{\widetilde{M}}$, respectively. Then, we have $P_+(D_{\widetilde{M}} \pm S_{\widetilde{M}}) \in \mathcal{U}^*(\widetilde{M})^G$ and

$$P_+(D_{\widetilde{M}} + S_{\widetilde{M}}) - P_+(D_{\widetilde{M}} - S_{\widetilde{M}}) \in C^*(\widetilde{M})^G.$$

Thus, the formal difference

$$[P_+(D_{\widetilde{M}} + S_{\widetilde{M}})] - [P_+(D_{\widetilde{M}} - S_{\widetilde{M}})]$$

determines a K-theory class in $K_0(C^*(\widetilde{M})^G)$ denoted by $\operatorname{Ind}(D_M^{\text{sgn}})$.

(3) The higher index of the signature operator (for definition, see [10, Section 4.3] or [4, Section 5.2.1]) is equal to $\operatorname{Ind}(D_M^{\text{sgn}})$.

3.2. K-homology class of signature operator

In this subsection, we recall the definition of the K-homology class of the signature operator.

Let M be an m-dimensional closed oriented Riemannian manifold. We use the rescaling trick to define the K-homology class of the signature operator, which lies in the K-theory of the localization algebra $C_L^*(\widetilde{M})^G$.

Let M_t be the Riemannian manifold equipped with the metric $g^{M_t} = t^2 g^M$, where g^M is the Riemannian metric of M. Let $D_{\widetilde{M}_t}$ and $S_{\widetilde{M}_t}$ be the Hodge–de Rham operator and the Poincaré duality operator on $\widetilde{M}_t$, respectively. As in the previous subsection, we construct the higher index of the signature operator on $\widetilde{M}_t$ using the specific representatives

$$(D_{\widetilde{M}_t} + S_{\widetilde{M}_t})(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})^{-1} : \Lambda^{\text{even}}(\widetilde{M}_t) \to \Lambda^{\text{even}}(\widetilde{M}_t)$$

and

$$P_+(D_{\widetilde{M}_t} + S_{\widetilde{M}_t}) - P_+(D_{\widetilde{M}_t} - S_{\widetilde{M}_t}) : \Lambda(\widetilde{M}_t) \to \Lambda(\widetilde{M}_t)$$

when m is odd and even, respectively.

The Hilbert space $\Lambda(\widetilde{M}_t)$ is naturally isomorphic to $\Lambda(\widetilde{M})$. Thus, the above operators are also viewed as acting on $\Lambda^{\text{even}}(\widetilde{M})$ or $\Lambda(\widetilde{M})$ by conjugation. We will still use the same notations to denote the corresponding operator acting on $\widetilde{M}$.

Lemma 3.2. (a) *When m is odd, the path*

$$t \mapsto \left((D_{\widetilde{M}_t} + S_{\widetilde{M}_t})(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})^{-1} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{even}}(\widetilde{M})\right), \quad t \in [1, +\infty),$$

lies in the localization algebra $C_L^*(\widetilde{M})^G$.

(b) *When m is even, the paths*

$$t \mapsto \left(P_+(D_{\widetilde{M}_t} \pm S_{\widetilde{M}_t}) : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})\right), \quad t \in [1, +\infty),$$

lie in $\mathcal{U}_L^*(\widetilde{M})^G$. Moreover, the difference

$$t \mapsto \left(P_+(D_{\widetilde{M}_t} + S_{\widetilde{M}_t}) - P_+(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})\right)$$

lies in the localization algebra $C_L^*(\widetilde{M})^G$.

Assuming Lemma 3.2 for a while, we are able to define the K-homology class of the signature operator as follows.

Definition 3.3. The K-homology class of the signature operator on M, which will be denoted by $[D_M^{\text{sgn}}]$, is defined, respectively, by

- the K-theory class in $K_1(C_L^*(\widetilde{M})^G)$ represented by the invertible element

$$t \mapsto \left((D_{\widetilde{M}_t} + S_{\widetilde{M}_t})(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})^{-1} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{even}}(\widetilde{M})\right), \quad t \in [1, +\infty),$$

  when m is odd;
- the K-theory class in $K_0(C_L^*(\widetilde{M})^G)$ represented by the formal difference of the projections

$$\left[t \mapsto P_+(D_{\widetilde{M}_t} + S_{\widetilde{M}_t})\right] - \left[t \mapsto P_+(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})\right],$$

  when m is even.

Now, we are going to prove Lemma 3.2. We will only give the proof of part (a). The proof of the other part is parallel.

We first give a proof of Lemma 3.2 using the fact $S_{\widetilde{M}}^2 = 1$, and then give an alternative proof without this fact for further generalization. Denote by T_t the restriction of $(D_{\widetilde{M}_t} + S_{\widetilde{M}_t})(D_{\widetilde{M}_t} - S_{\widetilde{M}_t})^{-1}$ to $\Lambda^{\text{even}}(\widetilde{M})$. It is easy to give a concrete formula of T_t by

$$\begin{aligned} T_t &= (D_{\widetilde{M}_t} + S_{\widetilde{M}_t})(D_{\widetilde{M}_t} - S_{\widetilde{M}_t}) \\ &= \left(\frac{D_{\widetilde{M}}}{t} + S_{\widetilde{M}}\right)\left(\frac{D_{\widetilde{M}}}{t} - S_{\widetilde{M}}\right) = \frac{\frac{iD_{\widetilde{M}}S_{\widetilde{M}}}{t} + i}{\frac{iD_{\widetilde{M}}S_{\widetilde{M}}}{t} - i}. \end{aligned} \tag{3.1}$$

The second equality follows from direct computations, and the last equality follows from the fact that $S_{\widetilde{M}}^2 = 1$. Note that

$$\frac{\frac{iD_{\widetilde{M}}S_{\widetilde{M}}}{t} + i}{\frac{iD_{\widetilde{M}}S_{\widetilde{M}}}{t} - i}$$

is the Cayley transform of $iD_{\widetilde{M}}S_{\widetilde{M}}$. Now, the standard propagation speed argument shows that $T_t \in C_L^*(\widetilde{M})^G$.

Remark 3.4. Recall that $iD_{\widetilde{M}}S_{\widetilde{M}} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{odd}}(\widetilde{M})$ is the usual signature operator on the odd dimensional manifold $\widetilde{M}$ (cf. [4, Section 5.2]). Equation (3.1) actually shows that T_t agrees with the standard construction of the representative element of the K-homology class of the elliptic differential operator $iD_{\widetilde{M}}S_{\widetilde{M}} : \Lambda^{\text{even}}(\widetilde{M}) \to \Lambda^{\text{odd}}(\widetilde{M})$ in [16, Chapter 8].

However, for further generalization, we give an alternative proof of Lemma 3.2 based only on the followings:

(i) $D_{\widetilde{M}_t}$ is a first-order self-adjoint elliptic differential operator on $\widetilde{M}_t$;

(ii) the norm of the principal symbol of $D_{\widetilde{M}_t}$ is uniformly bounded in $t \in [1, \infty)$;

(iii) $D_{\widetilde{M}_t} + S_{\widetilde{M}_t}$ and $D_{\widetilde{M}_t} - S_{\widetilde{M}_t}$ are invertible, and their quotient

$$\big(D_{\widetilde{M}_t} + S_{\widetilde{M}_t}\big)\big(D_{\widetilde{M}_t} - S_{\widetilde{M}_t}\big)^{-1}$$

is uniformly bounded in t with respect to the operator norm;

(iv) $S_{\widetilde{M}_t}$ is self-adjoint and uniformly bounded and anti-commutes with $D_{\widetilde{M}_t}$;

(v) $S_{\widetilde{M}_t}$ has zero propagation.

Proof of Lemma 3.2. This proof is in spirit from [15, Appendix A]. We prove part (a) in detail only. The proof of part (b) is similar.

It is sufficient to prove that T_t can be approximated by a path of operators of which the propagation goes to zero as t goes to infinity. Here, the approximation should be uniformly in $t \in [1, \infty)$ with respect to the operator norm.

Let CM be $M \times \mathbb{R}^+$ equipped with a conic metric $dt^2 + t^2(g^M)^2$. When restricted to a slice $M \times \{t\}$, the Riemannian metric tensor agrees with the one on M_t constructed above.

We first prove the case when $t \in \mathbb{N}^+$. Let $\coprod_n M_n$ be the disjoint union of M_n, $n = 1, 2, 3, \ldots$. This is a Riemannian manifold, of which the Riemannian metric tensor is defined piecewise. We may also equip $\coprod_n M_n$ with the metric induced by the inclusion

$$\coprod_n M_n \cong M \times \{1, 2, 3, \ldots\} \subset CM,$$

which agrees with the metric induced by the Riemannian metric tensor on each piece. See Figure 1.

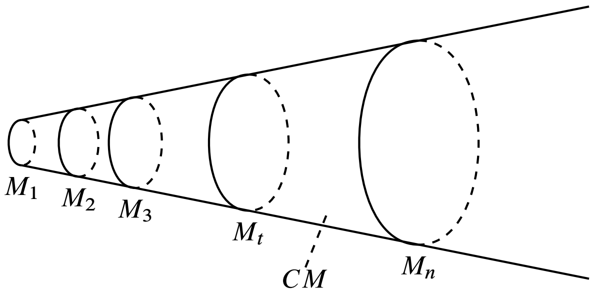

Figure 1. $\coprod_n M_n$ in CM.

Similarly, we define $\coprod_n \widetilde{M}_n$ and the following invertible operator:

$$T = \big(D_{\coprod_n \widetilde{M}_n} + S_{\coprod_n \widetilde{M}_n}\big)\big(D_{\coprod_n \widetilde{M}_n} - S_{\coprod_n \widetilde{M}_n}\big)^{-1} : \Lambda^{\text{even}}\Big(\coprod_n \widetilde{M}_n\Big) \to \Lambda^{\text{even}}\Big(\coprod_n \widetilde{M}_n\Big).$$

Since every component of $\coprod_n \widetilde{M}_n$ does not interact with others, T actually agrees with T_n when restricted to $\widetilde{M}_n$.

We claim that, for every $\varepsilon > 0$, there exists an operator T' acting on $\Lambda^{\text{even}}(\coprod_n \widetilde{M}_n)$ such that $\|T - T'\| < \varepsilon$ with respect to the operator norm, and T' has finite propagation with respect to the metric on $\coprod_n \widetilde{M}_n$. Let T_n' be the restriction of T' to $\widetilde{M}_n$. If our claim holds, then the propagation of T_n' goes to zero as $n \to \infty$ with respect to the metric on $\widetilde{M}$.

The local compactness of operators on $\widetilde{M}_n$ only concerns finitely many components. Thus, it is equivalent to prove our claim by showing that $T - 1$ lies in $C^*(\coprod_n \widetilde{M}_n)^G$, the equivariant Roe algebra defined in Definition 2.2.

Now, we prove our claim above. Since $D_{\coprod_n \widetilde{M}_n}$ is self-adjoint, $(D_{\coprod_n \widetilde{M}_n} + i)$ is invertible. Observe that

$$\begin{aligned} T &= \big(D_{\coprod_n \widetilde{M}_n} + S_{\coprod_n \widetilde{M}_n}\big)\big(D_{\coprod_n \widetilde{M}_n} - S_{\coprod_n \widetilde{M}_n}\big)^{-1} \\ &= 1 + 2S_{\coprod_n \widetilde{M}_n}\big(D_{\coprod_n \widetilde{M}_n} - S_{\coprod_n \widetilde{M}_n}\big)^{-1} \\ &= 1 + 2S_{\coprod_n \widetilde{M}_n}\big(D_{\coprod_n \widetilde{M}_n} + i\big)^{-1}\big(1 - \big(S_{\coprod_n \widetilde{M}_n} + i\big)\big(D_{\coprod_n \widetilde{M}_n} + i\big)^{-1}\big)^{-1}. \end{aligned}$$

Furthermore, by the formula of the Fourier transformation, we have that

$$\frac{1}{D_{\coprod_n \widetilde{M}_n} + i} = -i \int_{-\infty}^{0} e^{-|x|} e^{-ixD_{\coprod_n \widetilde{M}_n}}\, dx.$$

Although $\coprod_n \widetilde{M}_n$ is noncompact, the principal symbol of $D_{\coprod_n \widetilde{M}_n}$ is uniformly bounded on $\coprod_n \widetilde{M}_n$. Therefore, the wave operator $e^{-ixD_{\coprod_n \widetilde{M}_n}}$ has finite propagation $C|x|$ for some constant $C > 0$ on $\coprod_n \widetilde{M}_n$ (cf. [16, Chapter 8] and [2, Chapter 10]). Define

$$X_N = -i \int_{-N}^{0} e^{-|x|} e^{-ixD_{\coprod_n \widetilde{M}_n}}\, dx.$$

Then, $X_N \to (D_{\coprod_n \widetilde{M}_n} + i)^{-1}$ with respect to the operator norm as $N \to \infty$, and X_N has finite propagation on $\coprod_n \widetilde{M}_n$. This shows that

$$(D_{\coprod_n \widetilde{M}_n} + i)^{-1} \in C^*\Big(\coprod_n \widetilde{M}_n\Big)^G.$$

Besides, the Poincaré duality operator $S_{\coprod_n \widetilde{M}_n}$ is defined pointwise so that it has zero propagation on $\coprod_n \widetilde{M}_n$.

Now, we apply the above argument verbatim to $\coprod_n \widetilde{M}_{n+s}$ for all $s \in [0,1]$. This completes the proof. ∎

3.3. Controlled homotopy equivalence and higher rho invariant

In this subsection, we recall the construction of the higher rho invariant of the signature operator associated to a homotopy equivalence.

The higher rho invariant associated to a smooth homotopy equivalence was first introduced by Higson and Roe in [3–5]. Later, in [9], Piazza and Schick gave an index theoretic definition of the higher rho invariant of the signature operator. In [21], Zenobi extended these definitions to a notion of higher rho invariant associated to a topological homotopy equivalence.

In [15], Weinberger, Xie, and Yu constructed the higher rho invariant of the signature operator associated to a homotopy equivalence by a piecewise-linear approach. In this paper, we adapt their construction to give a differential geometric approach to the definition of the higher rho invariant. It is not hard to see that our construction here is equivalent to the one given in [18, Section 8].

Definition 3.5. Let M' and M be two Riemannian manifolds. Let $f : M' \to M$ be a smooth homotopy equivalence with smooth inverse $g : M \to M'$. Denote by h'_t, $t \in [0,1]$, (resp. h_t, $t \in [0,1]$) the smooth homotopy between fg and $\mathrm{id}_{M'}$ (resp. gf and id_M). We say that f is a controlled homotopy equivalence if there exists a positive constant C such that

(1) the diameter of the set $\{h'_t(a) : 0 \le t \le 1\} \subset M'$ is bounded by C uniformly for all $a \in M'$;

(2) the diameter of the set $\{h_t(b) : 0 \le t \le 1\} \subset M$ is bounded by C uniformly for all $b \in M$.

Remark 3.6. If M' and M are closed manifolds, then any homotopy equivalence $f : M' \to M$ is automatically controlled. Furthermore, the lift of f to their Galois covering is also controlled.

Let $f : M' \to M$ be a controlled homotopy equivalence. Suppose that $\widetilde{M}, \widetilde{M}'$ are Galois G-covering spaces of M, M', respectively. Write

$$D = D_{\widetilde{M}'} \oplus D_{\widetilde{M}} \quad \text{and} \quad S = \begin{pmatrix} S_{\widetilde{M}'} & \\ & -S_{\widetilde{M}} \end{pmatrix}$$

acting on $\Lambda^*(\widetilde{M}') \otimes \Lambda^*(\widetilde{M})$. The controlled homotopy equivalence $f: M' \to M$ induces a map from $\Lambda^p(\widetilde{M})$ to $\Lambda^p(\widetilde{M}')$, which we will still denote by f. In general, the induced map is not a bounded operator. However, we may apply the Hilsum–Skandalis submersion (cf. [7, p. 74], [14, p. 157], and [18, p. 34]) to construct a bounded operator $\mathcal{T}_f$ out of f. Without loss of generality, we might as well assume that f is a bounded operator.

Now, let us recall the definition of the higher rho invariant associated to the controlled homotopy equivalence f according to the parity of $\dim M$. We mention here that the construction is due to Higson and Roe (cf. [3,5]) and Weinberger, Xie, and Yu (cf. [15]).

Odd case. We first assume that both M' and M are odd dimensional. Via conjugating by f on the first summand $\Lambda^*(\widetilde{M}')$, we may identify D and S with their corresponding operators acting on $\Lambda^*(\widetilde{M}) \oplus \Lambda^*(\widetilde{M})$. Under this identification, the invertible element defined by

$$(D+S)(D-S)^{-1}|_{\Lambda^{\text{even}}(\widetilde{M}')\oplus\Lambda^{\text{even}}(\widetilde{M})} \in M_2\big(C_L^*(\widetilde{M})^G\big)^+$$

represents the K-theory element $f_* \operatorname{Ind}(D^{\text{sgn}}_{M'}) - \operatorname{Ind}(D^{\text{sgn}}_{M}) \in K_1(C^*(\widetilde{M})^G)$. The construction in Definition 3.3 gives rise to an invertible element

$$(D_t+S_t)(D_t-S_t)^{-1}|_{\Lambda^{\text{even}}(\widetilde{M}')\oplus\Lambda^{\text{even}}(\widetilde{M})} \in M_2\big(C_L^*(\widetilde{M})^{G,+}\big).$$

In particular, we have $D_1 = D$ and $S_1 = S$. Since f is a controlled homotopy equivalence, it gives rise to a canonical path that connects $(D+S)(D-S)^{-1}$ to the identity operator as shown by Higson and Roe in [3, Theorem 4.3]. The path is constructed out of the following path $S_f(t)$ connecting S with $-S$:

$$S_f(t) = \begin{cases} \begin{pmatrix} (1-t)S_{\widetilde{M}'} + tf^*S_{\widetilde{M}}f & 0 \\ 0 & -S_{\widetilde{M}} \end{pmatrix}, & t\in[0,1], \\ \begin{pmatrix} \cos\left(\frac{\pi}{2}(t-1)\right)f^*S_{\widetilde{M}}f & \sin\left(\frac{\pi}{2}(t-1)\right)f^*S_{\widetilde{M}} \\ \sin\left(\frac{\pi}{2}(t-1)\right)S_{\widetilde{M}}f & -\cos\left(\frac{\pi}{2}(t-1)\right)S_{\widetilde{M}} \end{pmatrix}, & t\in[1,2], \\ \begin{pmatrix} 0 & e^{i\pi(t-2)}S_{\widetilde{M}}f \\ e^{-i\pi(t-2)}f^*S_{\widetilde{M}} & 0 \end{pmatrix}, & t\in[2,3], \\ -\begin{pmatrix} 0 & e^{i\pi(4-t)}S_{\widetilde{M}}f \\ e^{-i\pi(4-t)}f^*S_{\widetilde{M}} & 0 \end{pmatrix}, & t\in[3,4], \\ -\begin{pmatrix} \cos\left(\frac{\pi}{2}(5-t)\right)f^*S_{\widetilde{M}}f & \sin\left(\frac{\pi}{2}(5-t)\right)f^*S_{\widetilde{M}} \\ \sin\left(\frac{\pi}{2}(5-t)\right)S_{\widetilde{M}}f & -\cos\left(\frac{\pi}{2}(5-t)\right)S_{\widetilde{M}} \end{pmatrix}, & t\in[4,5], \\ -\begin{pmatrix} (t-5)S_{\widetilde{M}'} + (6-t)f^*S_{\widetilde{M}}f & 0 \\ 0 & -S_{\widetilde{M}} \end{pmatrix}, & t\in[5,6]. \end{cases} \tag{3.2}$$

For any $t \in [0,6]$, $S_f(t)$ satisfies the following conditions:

(1) $D \pm S_f(t)$ are both invertible;

(2) the invertible operator

$$(D+S)\big(D-S_f(t)\big)^{-1} : \Lambda^{\text{even}}(\widetilde{M}) \oplus \Lambda^{\text{even}}(\widetilde{M}') \to \Lambda^{\text{even}}(\widetilde{M}) \oplus \Lambda^{\text{even}}(\widetilde{M}')$$

belongs to $M_2(C^*(\widetilde{M})^G)^+$.

Definition 3.7. The higher rho invariant $\rho(f)$ is the class in $K_1(C^*_{L,0}(\widetilde{M})^G)$ represented by the following invertible element:

$$\begin{cases} (D+S)\big(D+S_f(t-1)\big)^{-1}|_{\Lambda^{\text{even}}(\widetilde{M}')\oplus\Lambda^{\text{even}}(\widetilde{M})}, & t \in [1,7], \\ (D_{t-6}+S_{t-6})(D_{t-6}-S_{t-6})^{-1}|_{\Lambda^{\text{even}}(\widetilde{M}')\oplus\Lambda^{\text{even}}(\widetilde{M})}, & t \geqslant 7. \end{cases} \tag{3.3}$$

Even case. The even dimensional case is parallel to the odd case above. The construction in Definition 3.3 gives rise to a path of differences of projections

$$P_+(D_t+S_t) - P_+(D_t-S_t) \in M_2\big(C^*_L(\widetilde{M})^G\big),$$

with

$$P_+(D_1+S_1) - P_+(D_1-S_1) = P_+(D+S) - P_+(D-S).$$

Let $S_f(t)$ be as above. Similarly, for any fixed t, we have that

(1) $D - S_f(t)$ is invertible,

(2) $P_+(D-S_f(t))$ belongs to $M_2(\mathcal{U}^*(\widetilde{M})^G)$, and

$$P_+(D+S) - P_+\big(D-S_f(t)\big) \in M_2\big(C^*(\widetilde{M})^G\big).$$

The higher rho invariant associated to a controlled homotopy equivalence f is defined as follows.

Definition 3.8. Write

$$\begin{aligned} \Theta_{f,+}(t) &= \begin{cases} P_+(D+S), & t \in [1,7], \\ P_+(D_{t-6}+S_{t-6}), & t \geqslant 7, \end{cases} \\ \Theta_{f,-}(t) &= \begin{cases} P_+\big(D+S_f(t-1)\big), & t \in [1,7], \\ P_+(D_{t-6}-S_{t-6}), & t \geqslant 7. \end{cases} \end{aligned} \tag{3.4}$$

Since $\Theta_{f,\pm}$ are projections in $M_2(\mathcal{U}^*_{L,0}(\widetilde{M})^G)$ and their difference lies in $M_2(C^*_{L,0}(\widetilde{M})^G)$, the formal difference

$$[\Theta_{f,+}] - [\Theta_{f,-}]$$

defines a K-theoretic class $\rho(f)$ in $K_0(C^*_{L,0}(\widetilde{M})^G)$, called higher rho invariant.

4. Product formula

In this section, we will prove the product formula for the higher rho invariant of the signature operator associated to a homotopy equivalence. We only consider the case of product manifolds for now. The general case of fibered manifolds will be discussed in the next section.

Proposition 4.1. *Let M, N be two manifolds with dimensions m, n and fundamental groups G, H, respectively. Under the product map*

$$\psi : K_m\big(C^*(\widetilde{M})^G\big) \otimes K_n\big(C^*(\widetilde{N})^H\big) \to K_{m+n}\big(C^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big),$$

one has

$$k_{mn} \cdot \psi\big(\mathrm{Ind}\,\big(D_M^{\mathrm{sgn}}\big) \otimes \mathrm{Ind}\,\big(D_N^{\mathrm{sgn}}\big)\big) = \mathrm{Ind}\,\big(D_{M\times N}^{\mathrm{sgn}}\big),$$

where

$$k_{mn} = \begin{cases} 1, & mn \text{ is even}, \\ 2, & mn \text{ is odd}. \end{cases}$$

Proof. In the following, we omit the mention of ψ for simplicity. We avoid using the fact that the Poincaré duality operator $S_{\widetilde{M}}^2$ squares 1 on $\widetilde{M}$, in order to allow the proof to be generalized later. We consider four cases according to the parity of both $\dim M$ and $\dim N$.

Even times odd. We first suppose that $\dim M$ is even and $\dim N$ is odd.

Write $B_{\widetilde{M}\pm} = D_{\widetilde{M}} \pm S_{\widetilde{M}}$ for short. On the product manifold $\widetilde{M} \times \widetilde{N}$, the de Rham differential $d_{\widetilde{M}\times\widetilde{N}}$ is given by

$$d_{\widetilde{M}\times\widetilde{N}} = d_{\widetilde{M}}\hat{\otimes}1 + 1\hat{\otimes}d_{\widetilde{N}} = d_{\widetilde{M}} \otimes 1 + E_{\widetilde{M}} \otimes d_{\widetilde{N}},$$

where $E_{\widetilde{M}}$ is the even-odd grading operator for $\Lambda(\widetilde{M})$. Therefore, the Hodge–de Rham operator on $\widetilde{M} \times \widetilde{N}$ is given by

$$D_{\widetilde{M}\times\widetilde{N}} = D_{\widetilde{M}} \otimes 1 + E_{\widetilde{M}} \otimes D_{\widetilde{N}}.$$

Now, we decompose $\Lambda^{\mathrm{even}}(\widetilde{M} \times \widetilde{N})$ into the direct sum of $\Lambda^{\mathrm{even}}(\widetilde{M}) \otimes \Lambda^{\mathrm{even}}(\widetilde{N})$ and $\Lambda^{\mathrm{odd}}(\widetilde{M}) \otimes \Lambda^{\mathrm{odd}}(\widetilde{N})$. As $\dim N$ is odd, the Hodge $*$-operator as well as the Poincaré duality operator $S_{\widetilde{N}}$ reverses the parity of $\Lambda(\widetilde{N})$. Note that $S_{\widetilde{N}}^2 : \Lambda^p(\widetilde{N}) \to \Lambda^p(\widetilde{N})$ is a multiple of identity. Therefore, we identify $\Lambda(\widetilde{M}) \otimes \Lambda^{\mathrm{odd}}(\widetilde{N})$ as $\Lambda(\widetilde{M}) \otimes \Lambda^{\mathrm{even}}(\widetilde{N})$ via $1 \otimes S_{\widetilde{N}}$. Under this identification and by [4, Section 5.2.2], the higher index of the signature operator $D_{\widetilde{M}\times\widetilde{N}}^{\mathrm{sgn}}$ is represented by the following invertible operator:

$$\big(B_{\widetilde{M}}^{+} \otimes 1 + 1 \otimes S_{\widetilde{N}} D_{\widetilde{N}}\big)\big(B_{\widetilde{M}}^{-} \otimes 1 + 1 \otimes S_{\widetilde{N}} D_{\widetilde{N}}\big)^{-1} :$$
$$\Lambda(\widetilde{M}) \otimes \Lambda^{\mathrm{even}}(\widetilde{N}) \to \Lambda(\widetilde{M}) \otimes \Lambda^{\mathrm{even}}(\widetilde{N}).$$

Since $B^+_{\widetilde{M}}$ is invertible, we define a path of bounded operators

$$W_{+,s} = \frac{B^+_{\widetilde{M}}}{|B^+_{\widetilde{M}}|^s} \otimes 1 + 1 \otimes S_{\widetilde{N}} D_{\widetilde{N}}.$$

For the invertible operator $B^{\pm}_{\widetilde{M}}$, we denote by $P^+(B^{\pm}_{\widetilde{M}})$ (resp. $P^-(B^{\pm}_{\widetilde{M}})$) the spectral projection of the positive (resp. negative) part of $B^{\pm}_{\widetilde{M}}$. We see that

$$\begin{aligned} W_{+,0} &= B^+_{\widetilde{M}} \otimes 1 + 1 \otimes S_{\widetilde{N}} D_{\widetilde{N}}, \\ W_{+,1} &= \big(P^+(B^+_{\widetilde{M}}) - P^-(B^+_{\widetilde{M}})\big) \otimes 1 + 1 \otimes S_{\widetilde{N}} D_{\widetilde{N}}. \end{aligned}$$

Since $D_{\widetilde{N}}$ anti-commutes with $S_{\widetilde{N}}$, we have

$$W^*_{+,s} W_{+,s} = \big(B^+_{\widetilde{M}}\big)^{2(1-s)} \otimes 1 + 1 \otimes D^2_{\widetilde{N}} > 0.$$

Thus, $W_{+,s}$ is a path of invertible operators for every s in $[0,1]$.

Similarly, we define a path of invertible operator $W_{-,s}$. Thus, via the homotopy $W_{+,s}(W_{-,s})^{-1}$, the higher index of the signature operator on $M \times N$ is also represented by the invertible operator

$$W_{+,1}(W_{-,1})^{-1} : \Lambda(\widetilde{M}) \otimes \Lambda^{\text{even}}(\widetilde{N}) \to \Lambda(\widetilde{M}) \otimes \Lambda^{\text{even}}(\widetilde{N}).$$

We rewrite the expression above using $1 = P^+(B^{\pm}_{\widetilde{M}}) + P^-(B^{\pm}_{\widetilde{M}})$.

$$\begin{aligned} \big[W_{+,1}(W_{-,1})^{-1}\big] &= \big[P^+\big(B^+_{\widetilde{M}}\big) \otimes (S_{\widetilde{N}} D_{\widetilde{N}} + 1)(S_{\widetilde{N}} D_{\widetilde{N}} - 1)^{-1} + P^-\big(B^+_{\widetilde{M}}\big) \otimes 1\big] \\ &\quad - \big[P^+\big(B^-_{\widetilde{M}}\big) \otimes (S_{\widetilde{N}} D_{\widetilde{N}} + 1)\big(S_{\widetilde{N}} D_{\widetilde{N}} - 1\big)^{-1} + P^-\big(B^-_{\widetilde{M}}\big) \otimes 1\big] \\ &= \big(\big[P^+\big(B^+_{\widetilde{M}}\big)\big] - \big[P^+\big(B^-_{\widetilde{M}}\big)\big]\big) \times \big[(D_{\widetilde{N}} + S_{\widetilde{N}})(D_{\widetilde{N}} - S_{\widetilde{N}})^{-1}\big] \\ &= \operatorname{Ind}\big(D^{\text{sgn}}_M\big) \times \operatorname{Ind}\big(D^{\text{sgn}}_N\big). \end{aligned}$$

The last two equalities follow from the definition of product of K-groups and the formula for the higher index of the signature operator in Section 3.1.

Odd times even. Suppose that M is odd dimensional and N is even dimensional. Straightforward computation shows that

- $S_{\widetilde{M} \times \widetilde{N}} = S_{\widetilde{M}} \otimes S_{\widetilde{N}}$ on $\Lambda(\widetilde{M}) \otimes \Lambda^{\text{even}}(\widetilde{N})$,
- $S_{\widetilde{M} \times \widetilde{N}} = -S_{\widetilde{M}} \otimes S_{\widetilde{N}}$ on $\Lambda(\widetilde{M}) \otimes \Lambda^{\text{odd}}(\widetilde{N})$,

and

- $d_{\widetilde{M} \times \widetilde{N}} = d_{\widetilde{M}} \otimes 1 + 1 \otimes d_{\widetilde{N}}$ on $\Lambda(\widetilde{M})^{\text{odd}} \otimes \Lambda(\widetilde{N})$,
- $d_{\widetilde{M} \times \widetilde{N}} = d_{\widetilde{M}} \otimes 1 - 1 \otimes d_{\widetilde{N}}$ on $\Lambda(\widetilde{M})^{\text{even}} \otimes \Lambda(\widetilde{N})$.

Let $\Lambda_{\pm}(\widetilde{N})$ be the ± 1 eigenspace of $S_{\widetilde{N}}$. We make the following identifications.

- Under the decompositions

$$\Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}) = \begin{matrix} \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{even}}(\widetilde{N}) \end{matrix}$$

and

$$\Lambda^{\text{odd}}(\widetilde{M} \times \widetilde{N}) = \begin{matrix} \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{even}}(\widetilde{N}), \end{matrix}$$

we identify $\Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda(\widetilde{N})$ with $\Lambda^{\text{odd}}(\widetilde{M} \times \widetilde{N})$ by

$$\begin{pmatrix} -B_{\widetilde{M}+} \otimes 1 & & & \\ & B_{\widetilde{M}-} \otimes 1 & & \\ & & 1 \otimes 1 & \\ & & & 1 \otimes 1 \end{pmatrix} : \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}) \to \Lambda^{\text{odd}}(\widetilde{M} \times \widetilde{N}).$$

- Under the decompositions

$$\Lambda^{\text{even}}(\widetilde{M} \times \widetilde{N}) = \begin{matrix} \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{+}^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_{-}^{\text{odd}}(\widetilde{N}) \end{matrix}$$

and

$$\Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}) = \begin{matrix} \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_+^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_-^{\text{even}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_+^{\text{odd}}(\widetilde{N}) \\ \oplus \\ \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda_-^{\text{odd}}(\widetilde{N}), \end{matrix}$$

we identify $\Lambda^{\text{even}}(\widetilde{M} \times \widetilde{N})$ with $\Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda(\widetilde{N})$ by

$$\begin{pmatrix} 1 \otimes 1 & & & \\ & -1 \otimes 1 & & \\ & & -B_{\widetilde{M}-} \otimes 1 & \\ & & & -B_{\widetilde{M}+} \otimes 1 \end{pmatrix} : \Lambda^{\text{even}}(\widetilde{M} \times \widetilde{N}) \to \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}).$$

With these identifications, we have

$$\begin{aligned} & d_{\widetilde{M}\times\widetilde{N}} + d^*_{\widetilde{M}\times\widetilde{N}} + S_{\widetilde{M}\times\widetilde{N}} \\ &= \begin{cases} B_{\widetilde{M}+} \otimes 1 + B_{\widetilde{M}+} \otimes D_{\widetilde{N}} & \text{on } \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_+^{\text{even}}(\widetilde{N}), \\ B^2_{\widetilde{M}-} B_{\widetilde{M}+} \otimes 1 + B_{\widetilde{M}+} \otimes D_{\widetilde{N}} & \text{on } \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_+^{\text{odd}}(\widetilde{N}), \\ -B_{\widetilde{M}-} \otimes 1 + B_{\widetilde{M}-} \otimes D_{\widetilde{N}} & \text{on } \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_-^{\text{even}}(\widetilde{N}), \\ -B^2_{\widetilde{M}+} B_{\widetilde{M}-} \otimes 1 + B_{\widetilde{M}-} \otimes D_{\widetilde{N}} & \text{on } \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda_+^{\text{odd}}(\widetilde{N}). \end{cases} \end{aligned}$$

Note that $B^2_{\widetilde{M}\pm}$ are positive invertible operators. It follows that $d_{\widetilde{M}\times\widetilde{N}} + d^*_{\widetilde{M}\times\widetilde{N}} + S_{\widetilde{M}\times\widetilde{N}}$ is homotopic to

$$\begin{pmatrix} B_{\widetilde{M}+} & 0 \\ 0 & B_{\widetilde{M}-} \end{pmatrix} S_{\widetilde{N}} + \begin{pmatrix} B_{\widetilde{M}-} & 0 \\ 0 & B_{\widetilde{M}+} \end{pmatrix} D_{\widetilde{N}} : \Lambda^{\text{odd}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}) \to \Lambda^{\text{even}}(\widetilde{M}) \otimes \Lambda(\widetilde{N}),$$

where the matrix form is written with respect to the decomposition

$$\Lambda(\widetilde{N}) = \Lambda_+(\widetilde{N}) \oplus \Lambda_-(\widetilde{N}).$$

Since $D_{\widetilde{N}}$ is off-diagonal, $d_{\widetilde{M}\times\widetilde{N}} + d^*_{\widetilde{M}\times\widetilde{N}} + S_{\widetilde{M}\times\widetilde{N}}$ is in turn homotopic to

$$V = \begin{pmatrix} B_{\widetilde{M}+} & 0 \\ 0 & B_{\widetilde{M}-} \end{pmatrix} S_{\widetilde{N}} f(D_{\widetilde{N}}) + \begin{pmatrix} B_{\widetilde{M}-} & 0 \\ 0 & B_{\widetilde{M}+} \end{pmatrix} g(D_{\widetilde{N}}),$$

where $g(x) = \frac{x}{\sqrt{1+x^2}}$ and $f(x) = \frac{1}{\sqrt{1+x^2}}$. In the meantime, $d_{\widetilde{M}\times\widetilde{N}} + d^*_{\widetilde{M}\times\widetilde{N}} - S_{\widetilde{M}\times\widetilde{N}}$ is homotopic to

$$U = \begin{pmatrix} B_{\widetilde{M}-} & 0 \\ 0 & B_{\widetilde{M}+} \end{pmatrix} S_{\widetilde{N}} f(D_{\widetilde{N}}) + \begin{pmatrix} B_{\widetilde{M}-} & 0 \\ 0 & B_{\widetilde{M}+} \end{pmatrix} g(D_{\widetilde{N}}).$$

It follows that

$$VU^{-1} : \Lambda^{\text{even}}(\widetilde{M} \times \widetilde{N}) \to \Lambda^{\text{even}}(\widetilde{M} \times \widetilde{N})$$
$$VU^{-1} = \big((d_{\widetilde{M}} + d^*_{\widetilde{M}}) \otimes 1 + S_{\widetilde{M}} \otimes S_2 S_1 S_2\big) \begin{pmatrix} B^{-1}_{\widetilde{M}-} & \\ & B^{-1}_{\widetilde{M}+} \end{pmatrix},$$

where

$$S_1 = S_{\widetilde{N}} \quad \text{and} \quad S_2 = g(D_{\widetilde{N}}) + S_{\widetilde{N}} f(D_{\widetilde{N}}).$$

Note that $S_2 S_1 S_2$ is a symmetry; i.e., $S_2 S_1 S_2$ can be approximated by finite propagation operators, and $(S_2 S_1 S_2)^2 - 1$ belongs to $C^*(\widetilde{N})^H$. We define

$$P = \frac{S_2 S_1 S_2 + 1}{2}.$$

Now, one can see that the higher index of $D^{\text{sgn}}_{M \times N}$ is actually represented by

$$\begin{aligned}
&\big((d_{\widetilde{M}} + d^*_{\widetilde{M}}) \otimes 1 + S_{\widetilde{M}} \otimes S_2 S_1 S_2\big) \begin{pmatrix} B^{-1}_{\widetilde{M}-} & 0 \\ 0 & B^{-1}_{\widetilde{M}+} \end{pmatrix} \\
&= \big(B_{\widetilde{M}+} \otimes P + B_{\widetilde{M}-} \otimes (1-P)\big)\left(\begin{pmatrix} B^{-1}_{\widetilde{M}-} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B^{-1}_{\widetilde{M}+} \end{pmatrix} \right) \\
&= \begin{pmatrix} B_{\widetilde{M}+} B^{-1}_{\widetilde{M}-} & 0 \\ 0 & 1 \end{pmatrix} \otimes P + \begin{pmatrix} 1 & 0 \\ 0 & B_{\widetilde{M}-} B^{-1}_{\widetilde{M}+} \end{pmatrix} \otimes (1-P) \\
&= \left(\begin{pmatrix} B_{\widetilde{M}+} B^{-1}_{\widetilde{M}-} & 0 \\ 0 & B_{\widetilde{M}+} B^{-1}_{\widetilde{M}-} \end{pmatrix} \otimes P + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1-P) \right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & B_{\widetilde{M}-} B^{-1}_{\widetilde{M}+} \end{pmatrix} \otimes 1.
\end{aligned}$$

This represents the K-theoretic class

$$[B_{\widetilde{M}+} B^{-1}_{\widetilde{M}-}] \otimes \left([P] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \right)$$

in $K_1(C^*(\widetilde{M} \times \widetilde{N})^{G \times H})$. As shown in [4, Section 5.2.1], we have that

$$[P] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = [P_+(D_{\widetilde{N}} + S_{\widetilde{N}})] - [P_+(D_{\widetilde{N}} - S_{\widetilde{N}})].$$

This completes the proof for the case of odd times even.

Even times even. Suppose that both M and N are even dimensional. In [15], it is shown that

$$\operatorname{Ind}\big(D^{\text{sgn}}_N\big) \otimes \operatorname{Ind}\big(D^{\text{sgn}}_{\mathbb{R}}\big) = \operatorname{Ind}\big(D^{\text{sgn}}_{N \times \mathbb{R}}\big),$$
$$\operatorname{Ind}\big(D^{\text{sgn}}_{M \times N}\big) \otimes \operatorname{Ind}\big(D^{\text{sgn}}_{\mathbb{R}}\big) = \operatorname{Ind}\big(D^{\text{sgn}}_{M \times N \times \mathbb{R}}\big).$$

Note that $\mathrm{Ind}(D^{\mathrm{sgn}}_{\mathbb{R}})$ is the generator of

$$K_1\big(C^*(\mathbb{R})\big) \cong K_1\big(C^*_L(\mathbb{R})\big) \cong \mathbb{Z}.$$

Now, we have, by the even times odd case,

$$\begin{aligned}
&\mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_N\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{N\times\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N\times\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N}\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right).
\end{aligned}$$

It follows that

$$\mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_N\right) = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N}\right).$$

Odd times odd. Let M and N be both odd dimensional manifolds. In this case, as shown in [15], we have

$$\begin{aligned}
2\,\mathrm{Ind}\left(D^{\mathrm{sgn}}_N\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right) &= \mathrm{Ind}\left(D^{\mathrm{sgn}}_{N\times\mathbb{R}}\right),\\
\mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N}\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right) &= \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N\times\mathbb{R}}\right).
\end{aligned}$$

Now, we have, by the odd times even case,

$$\begin{aligned}
&2\,\mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_N\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{N\times\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N\times\mathbb{R}}\right)\\
&\quad = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N}\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_{\mathbb{R}}\right).
\end{aligned}$$

It follows that

$$2\,\mathrm{Ind}\left(D^{\mathrm{sgn}}_M\right) \otimes \mathrm{Ind}\left(D^{\mathrm{sgn}}_N\right) = \mathrm{Ind}\left(D^{\mathrm{sgn}}_{M\times N}\right). \qquad \blacksquare$$

Note that, in the proof of Proposition of 4.1, we actually do not require that $S^2_{\widetilde{M}} = 1$. Therefore, the argument above can be easily generalized to show the following proposition and theorem.

Proposition 4.2. *With the same notations, under the product map*

$$\psi_L : K_m\big(C^*_L(\widetilde{M})^G\big) \otimes K_n\big(C^*_L(\widetilde{N})^H\big) \to K_{m+n}\big(C^*_L(\widetilde{M}\times\widetilde{N})^{G\times H}\big),$$

one has

$$k_{mn}\cdot\psi_L\big([D^{\mathrm{sgn}}_M]\otimes[D^{\mathrm{sgn}}_N]\big) = [D^{\mathrm{sgn}}_{M\times N}],$$

where

$$k_{mn} = \begin{cases} 1, & mn \text{ is even},\\ 2, & mn \text{ is odd}.\end{cases}$$

Theorem 4.3. *Suppose that M', M are two closed oriented Riemannian manifolds and $f : M' \to M$ is a homotopy equivalence. Write $m = \dim M' = \dim M$. Let $\widetilde{M}'$, $\widetilde{M}$ be their Galois G-covering spaces, respectively. Under the product map*

$$\psi_{L,0} : K_m\big(C_L^*(\widetilde{M})^G\big) \otimes K_n\big(C_{L,0}^*(\widetilde{N})^H\big) \to K_{m+n}\big(C_{L,0}^*(\widetilde{M} \times \widetilde{N})^{G\times H}\big),$$

we have

$$k_{mn} \cdot \psi_{L,0}\big(\rho(f) \otimes \big[D_N^{\mathrm{sgn}}\big]\big) = \rho(f \times I_N),$$

where $I_N : N \to N$ is the identity map and

$$k_{mn} = \begin{cases} 1, & mn \text{ is even,} \\ 2, & mn \text{ is odd.} \end{cases}$$

5. Product formula for fibered manifolds

In this section, we generalize the product formula given in the previous section to fibered manifolds. We will first introduce a series of family geometric C^*-algebras with respect to the fibration. Next, we define a family version of K-homology class and higher rho invariant of the fiberwise signature operator in the K-theory of these C^*-algebras. Finally, we prove Theorems 1.1 and 1.2.

5.1. Family algebras

In this subsection, we introduce family geometric C^*-algebras associated to a fibered manifold.

Let $\pi : E \to B$ be a fibration with fiber F and base space B. Assume that E, F, and B are closed connected oriented Riemannian manifolds. The fibration induces a long exact sequence of homotopy groups

$$\cdots \to \pi_2(B) \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \xrightarrow{\pi_*} \pi_1(B) \to 0.$$

Denote by $\widetilde{E}$ and $\widetilde{B}$ the universal covering of E and B. From the exactness of the above sequence, we see that $\partial(\pi_2(B))$ is a normal subgroup of $\pi_1(F)$. Write

$$\Gamma = \pi_1(F)/\partial\big(\pi_2(B)\big).$$

The above exact sequence shows that $\widetilde{E}$ is also a fibration on $\widetilde{B}$ with fiber projection $\widetilde{\pi} : \widetilde{E} \to \widetilde{B}$ and fiber $\widetilde{F}$, the Galois Γ-covering of F.

From now on, we will write $G = \pi_1(E)$ and $H = \pi_1(B)$ for short. Recall that the equivariant Roe algebra $C^*(\widetilde{E})^G$ is defined to be the completion of G-equivariant, locally compact operators with finite propagation as in Definition 2.2. Now, let us define the equivariant family Roe algebra.

First, we construct an equivariant Roe algebra bundle over B. View the fiber bundle E over B as gluing many pieces of local trivialization by a series of diffeomorphisms of F. More precisely, there exists an open cover $\{V_\alpha\}$ of B and continuous maps $\varphi_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{Diff}(F)$ such that the fiber bundle E is equivalent to the tuple $(V_\alpha \times F, \varphi_{\alpha\beta})$; that is, every continuous section s of E is equivalent to a series of continuous maps $s_\alpha : V_\alpha \to F$ satisfying the cocycle condition, $\varphi_{\alpha\beta}(x)s_\alpha(x) = s_\beta(x)$ for any $x \in V_\alpha \cap V_\beta$.

By our previous arguments, when turning to the universal covering space, $\widetilde{E}$ is also an $\widetilde{F}$-bundle over $\widetilde{B}$. Assume that the open set V_α is small enough so that its lifting to $\widetilde{B}$ can be written as the disjoint union of open sets $\coprod_\gamma U_{\alpha,\gamma}$ with each $U_{\alpha,\gamma}$ diffeomorphic to V_α. We renumber the open cover $\{U_{\alpha,\gamma}\}$ of $\widetilde{B}$ by $\{U_j\}$. Therefore, we obtain a $\pi_1(B)$-equivariant open cover $\{U_j\}$ of $\widetilde{B}$, each open set of which is diffeomorphic to the Euclidean space and trivializes $\widetilde{E}$. Also, the transition maps lift to $\{\psi_{ij}\}$ with $\psi_{ij} : U_i \cap U_j \to \mathrm{Diff}(\widetilde{F})$. Recall the subspace $\mathbb{C}[\widetilde{F}]^\Gamma$ of the equivariant Roe algebra $C^*(\widetilde{F})^\Gamma$ as in Definition 2.4. For any $x \in U_i \cap U_j$, $\psi_{ij}(x)$ induces an automorphism $\psi_{ij,*}(x)$ of $C^*(\widetilde{F})^\Gamma$ by conjugation (i.e., for any $T \in C^*(\widetilde{F})^\Gamma$, $\psi_{ij,*}(x)T = \psi_{ij}(x)T\psi_{ij}^{-1}(x)$), which maps $\mathbb{C}[\widetilde{F}]^\Gamma$ to itself. This induces the following fiber bundle.

Definition 5.1 (Equivariant family Roe algebra). Recall that $G = \pi_1(E)$ and $\Gamma = \pi_1(F)/\partial(\pi_2(B))$. A continuous section of the fiber bundle given by $(\{U_i\}, \{\psi_{ij,*}\})$ is defined by a series of norm-continuous maps $s_i : U_i \to C^*(\widetilde{F})^\Gamma$ satisfying the cocycle condition, $\psi_{ij,*}(x)s_i(x) = s_j(x)$ for any $x \in U_i \cap U_j$. Let $\mathbb{C}[\widetilde{E}, \widetilde{B}]^G$ be the collection of uniformly norm-bounded and uniformly norm-continuous sections that are invariant under $\pi_1(B)$-action and have uniformly finite propagation on $\widetilde{B}$. The norm of such a section $\{s_j\}$ is defined to be $\sup_j \sup_{x\in U_j} \|s_j(x)\|$. Denote the completion of $\mathbb{C}[\widetilde{E}, \widetilde{B}]^G$ by $C^*(\widetilde{E}, \widetilde{B})^G$.

We mention that Definition 5.1 is related to the "Groupoid Roe algebra" given in [12, Definition 3.6] by Tang, Willett, and Yao.

It is easy to verify that the above definition is independent of the local trivialization. Similarly, we define the corresponding equivariant family localization algebra.

Definition 5.2 (Equivariant family localization algebra and obstruction algebra). The equivariant family localization algebra $C_L^*(\widetilde{E}, \widetilde{B})^G$ is the completion of uniformly norm-bounded and uniformly norm-continuous paths $s : [1, +\infty) \to C^*(\widetilde{E}, \widetilde{B})^G$ such that the propagation of $s(t)$ goes to zero as t goes to infinity uniformly on $\widetilde{B}$, where the norm of $s(t)$ is defined to be $\sup_{t\in[1,+\infty)} ||s(t)||$. The equivariant family obstruction algebra $C_{L,0}^*(\widetilde{E}, \widetilde{B})^G$ is then defined to be the kernel of the family assembly map:

$$\begin{aligned} \mathrm{ev} : C_L^*(\widetilde{E}, \widetilde{B})^G &\to C^*(\widetilde{E}, \widetilde{B})^G \\ s &\mapsto s(1). \end{aligned}$$

5.2. Product map of K-theory

In this subsection, we construct the product map on the family C^*-algebras.

Theorem 5.3. *Recall that $G = \pi_1(E)$ and $H = \pi_1(B)$. There are product maps*

$$\phi : K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C_L^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{m+n}\big(C_L^*(\widetilde{E})^G\big),$$
$$\phi_0 : K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{m+n}\big(C_{L,0}^*(\widetilde{E})^G\big)$$

that generalize the maps defined in (2.3).

Proof. Without loss of generality, we assume that both m and n are even. We will only give in detail the construction of

$$\phi_0 : K_0\big(C_L^*(\widetilde{B})^H\big) \otimes K_0\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) \to K_0\big(C_{L,0}^*(\widetilde{E})^G\big).$$

Suppose that $f_t \in (C_L^*(\widetilde{B})^H)^+$ represents a K_0-class, which has finite propagation that goes to zero as t goes to infinity and is a 1/10-projection; i.e., $f_t^* = f_t$ and $\|f_t^2 - f_t\| < 1/10$. Similarly, we suppose that $g_t \in (C_{L,0}^*(\widetilde{E}, \widetilde{B})^G)^+$ is a 1/10-projection, has finite propagation that goes to zero uniformly as t goes to infinity, and satisfies that $g_1 = 1$. Furthermore, we assume that $f_t - 1$ and $g_t - 1$ are given by kernel operators acting on L^2-sections as in Definition 2.4.

Choose $r > 0$ small enough such that, for any $x \in B$, the restriction of the fiber bundle E to the r-ball near x is trivial. Since f_t and f_{t+M} are homotopic for any $M > 0$, we may assume that the propagation of f_t is smaller than r. By the local triviality, we define

$$\phi\big([f_t] \otimes [g_t]\big) = \big[(f_t - 1) \otimes (g_t - 1) + 1\big],$$

where $(f_t - 1) \otimes (g_t - 1) + 1 \in (C_{L,0}^*(\widetilde{E})^G)^+$ is given by

$$\begin{aligned}\big(\big((f_t - 1) \otimes (g_t - 1)\big)h\big)(x, y) \\ = \int_{\widetilde{B}} \int_{\widetilde{F}} (f_t - 1)(x, x') \otimes (g_t - 1)_{x'}(y, y')h(x', y')\, dy'\, dx',\end{aligned} \tag{5.1}$$

with $h \in L^2(\widetilde{E})$. The above expression makes sense as the propagation of f_t is small enough. It is easy to verify that $(f_t - 1) \otimes (g_t - 1) + 1$ is at most a 3/10-projection, which gives rise to a K_0-class.

Now, passing to the matrix algebra and the Grothendieck group, we obtain the product map. ∎

5.3. Family higher invariants

In this subsection, we introduce the family version of higher invariants of the signature operator on a fibered manifold and prove Theorems 1.1 and 1.2.

On a fibered manifold E, the vertical differentials and Poincaré duality are well-defined as they are compatible with the transition maps. Thus, the family K-homology class of the vertical signature operator $[D_{E,B}^{\mathrm{sgn}}] \in K_{\dim F}(C_L^*(\widetilde{E}, \widetilde{B})^G)$ is defined similarly to Definition 3.3.

Theorem 5.4 (Theorem 1.1). *One has the following product formula for the family K-homology class of the family signature operator along F:*

$$k_{B,F} \cdot \phi\big([D_B^{\text{sgn}}] \otimes [D_{E,B}^{\text{sgn}}]\big) = [D_E^{\text{sgn}}],$$

where $k_{B,F} = 1$ when $\dim B \cdot \dim F$ is even and $k_{B,F} = 2$ otherwise, and ϕ is the product map

$$\phi : K_{\dim B}\big(C_L^*(\widetilde{B})^H\big) \otimes K_{\dim F}\big(C_L^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{\dim E}\big(C_L^*(\widetilde{E})^G\big).$$

We shall also define a family higher rho invariant associated to a fiberwise homotopy equivalence. Suppose we have two fibrations over the same base space B,

$$F' \to E' \xrightarrow{\pi'} B \quad \text{and} \quad F \to E \xrightarrow{\pi} B.$$

Let $f : E' \to E$ be a fiberwise homotopy equivalence; that is, the following diagram commutes:

$$\begin{array}{ccc} E' & \xrightarrow{\quad f \quad} & E \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & B & \end{array}$$

as well as replacing f with its homotopy inverse and the corresponding homotopy. Using the vertical differential and the Poincaré duality operator, we obtain a family higher rho invariant $\rho(f; B) \in K_{\dim F}(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G)$ as in Definition 3.7.

Theorem 5.5 (Theorem 1.2). *With the same notation as above, one has the following product formula for the family higher rho invariant associated to a fiberwise homotopy equivalence:*

$$k_{B,F}\phi\big([D_B^{\text{sgn}}] \otimes \rho(f; B)\big) = \rho(f),$$

where $k_{B,F} = 1$ when $\dim B \cdot \dim F$ is even and $k_{B,F} = 2$ otherwise, and ϕ_0 is the product map

$$\phi_0 : K_{\dim B}\big(C_L^*(\widetilde{B})_r^H\big) \otimes K_{\dim F}\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{\dim E}\big(C_{L,0}^*(\widetilde{E})^G\big).$$

In the following, we only prove Theorem 5.5 in detail. The proof for Theorem 5.4 is similar.

We need some definitions to prepare for the proof of Theorem 5.5.

Definition 5.6. For any element $T \in C^*(\widetilde{E})^G$, we define the propagation of T along the base space $\widetilde{B}$ by

$$\text{prop}_{\widetilde{B}}(T) = \sup\big\{d\big(\widetilde{\pi}(x), \widetilde{\pi}(y)\big) : (x, y) \in \text{Supp}(T)\big\},$$

where $\widetilde{\pi}$ is the lift of the fiber projection $\pi : E \to B$ and d is distance on $\widetilde{B}$.

This is an analogue of the first part of Definition 2.1. It is obvious that every operator in $C^*(\widetilde{E})^G$ with finite propagation has finite propagation along $\widetilde{B}$.

We need the following C^*-algebra generated by elements in $C^*_{L,0}(\widetilde{E})^G$ that can be localized horizontally.

Definition 5.7. Define $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$ to be the C^*-algebra generated by paths

$$\begin{aligned} f : [1,+\infty) &\to C^*_{L,0}(\widetilde{E})^G \\ s &\mapsto f(s) \end{aligned}$$

such that

(1) f is uniformly norm-continuous and uniformly norm-bounded;

(2) for any $s \in [1,+\infty)$ and $t \in [1,+\infty)$, $f(s)(t)$ has finite propagation along $\widetilde{B}$ as an element in $C^*(\widetilde{E})^G$;

(3) for any $s \in [1,+\infty)$, $\sup_{t\in[1,+\infty)} \mathrm{prop}_{\widetilde{B}}(f(s)(t)) < \infty$ and

$$\lim_{s\to+\infty} \sup_{t\in[1,+\infty)} \mathrm{prop}_{\widetilde{B}}\big(f(s)(t)\big) = 0.$$

The norm of $f \in C^*_{\widetilde{B},L,0}(\widetilde{E})^G$ is given by the supremum over s of the norm of $f(s)$ in $C^*_{L,0}(\widetilde{E})^G$; that is, $\|f\| = \sup_{s\geq 1} \|f(s)\|$.

There is an evaluation map

$$\mathrm{ev} : C^*_{\widetilde{B},L,0}(\widetilde{E})^G \to C^*_{L,0}(\widetilde{E})^G,$$

which induces a K-theoretic map denoted by ev_*.

If X is a closed Riemannian manifold, the equivariant localization algebra $C^*_L(\widetilde{X})^{\pi_1 X}$ admits a Mayer–Vietoris sequence for a partition of X. More precisely, if U_1, U_2 are two open sets on X and $\widetilde{U}_1, \widetilde{U}_2$ are their lifts to $\widetilde{X}$, then we have the following six-term exact sequence (cf. [19, Proposition 3.11]):

$$\begin{array}{ccccc}
K_0\big(C^*_L(\widetilde{U}_1 \cap \widetilde{U}_2)^{\pi_1 X}\big) & \longrightarrow & \begin{array}{c} K_0\big(C^*_L(\widetilde{U}_1)^{\pi_1 X}\big) \\ \oplus \\ K_0\big(C^*_L(\widetilde{U}_2)^{\pi_1 X}\big) \end{array} & \longrightarrow & K_0\big(C^*_L(\widetilde{U}_1 \cup \widetilde{U}_2)^{\pi_1 X}\big) \\
\uparrow & & & & \downarrow \\
K_1\big(C^*_L(\widetilde{U}_1 \cup \widetilde{U}_2)^{\pi_1 X}\big) & \longleftarrow & \begin{array}{c} K_1\big(C^*_L(\widetilde{U}_1)^{\pi_1 X}\big) \\ \oplus \\ K_1\big(C^*_L(\widetilde{U}_2)^{\pi_1 X}\big) \end{array} & \longleftarrow & K_1\big(C^*_L(\widetilde{U}_1 \cap \widetilde{U}_2)^{\pi_1 X}\big).
\end{array}$$

As the C^*-algebra $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$ is generated by elements that can be localized along B, it also admits a Mayer–Vietoris sequence as above for two open sets on the base space.

Proposition 5.8. *Let U_1, U_2 be two open sets on B and $\widetilde{U}_1, \widetilde{U}_2$ their lifts to $\widetilde{B}$. Let $\widetilde{E}_{\widetilde{U}_1}$ and $\widetilde{E}_{\widetilde{U}_2}$ be the restrictions of $\widetilde{E}$ to $\widetilde{U}_1$ and $\widetilde{U}_2$, respectively. One has the following six-term exact sequence:*

$$\begin{array}{ccccc} K_0\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1}\cap \widetilde{E}_{\widetilde{U}_2})^G\big) & \longrightarrow & \begin{array}{c} K_0\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1})^G\big) \\ \oplus \\ K_0\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_2})^G\big)\end{array} & \longrightarrow & K_0\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1}\cup \widetilde{E}_{\widetilde{U}_2})^G\big) \\ \uparrow & & & & \downarrow \\ K_1\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1}\cup \widetilde{E}_{\widetilde{U}_2})^G\big) & \longleftarrow & \begin{array}{c} K_1\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1})^G\big) \\ \oplus \\ K_1\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_2})^G\big)\end{array} & \longleftarrow & K_1\big(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{U}_1}\cap \widetilde{E}_{\widetilde{U}_2})^G\big). \end{array}$$

Proof. The proof of Proposition 5.8 is essentially the same as the proof of Proposition 3.11 in [19]. For any open subset Y of B, we define $(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_Y$ to be the C^*-subalgebra of $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$ generated by all paths $f : [1,\infty) \to C^*_{L,0}(\widetilde{E})^G$ such that, for any $s, t \in [1,\infty)$, $\operatorname{prop}(f(s,t)) < \infty$ as an operator in $\mathbb{C}[\widetilde{E}]^G$ and, for any $\varepsilon > 0$, there exists $S > 0$ such that, for any $s > S$ and $t \in [1,\infty)$, $\operatorname{Supp}(f(s,t))$ lies in the ε-neighborhood of $\widetilde{E}_{\widetilde{Y}} \times \widetilde{E}_{\widetilde{Y}}$, where $\widetilde{E}_{\widetilde{Y}}$ is the restriction of $\widetilde{E}$ to Y.

There is a natural inclusion

$$i : C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{Y}})^G \to (C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_Y.$$

For any $f \in (C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_Y$, we have a homotopy

$$\begin{aligned} H : [0,1] &\to \big(C^*_{\widetilde{B},L,0}(\widetilde{E})^G\big)_Y \\ \lambda &\mapsto \big(s \mapsto f(s+\lambda s_0)\big), \end{aligned}$$

connecting f and $f_{s_0} : s \mapsto f(s+s_0)$, whose support is closed to $\widetilde{E}_{\widetilde{Y}} \times \widetilde{E}_{\widetilde{Y}}$. This shows that, for any $\delta > 0$, any K-theory element of $(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_Y$ admits a representative whose support lies in $\widetilde{E}_{\widetilde{Y}_\delta} \times \widetilde{E}_{\widetilde{Y}_\delta}$, where Y_δ is the δ-neighborhood of Y. As an analogue of [19, Proposition 3.7], we see that $C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{Y}})^G$ and $C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{Y}_\delta})^G$ are isomorphic on K-theoretic level for small δ. This shows that the K-theoretic map i_* is surjective. The injectivity of i_* goes similarly.

Note that $(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_{U_1}$ and $(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)_{U_2}$ are closed ideals of $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$. And we also have that

$$\big(C^*_{\widetilde{B},L,0}(\widetilde{E})^G\big)_{U_1} + \big(C^*_{\widetilde{B},L,0}(\widetilde{E})^G\big)_{U_2} = C^*_{\widetilde{B},L,0}(\widetilde{E})^G.$$

Now, the proposition follows from the K-theoretic six-term exact sequence (cf. [6, Lemma 3.1]). ■

In the following Lemma, we show that there exists a natural map

$$\phi_{L,0} : K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{m+n}\big(C_{\widetilde{B},L,0}^*(\widetilde{E})^G\big)$$

compatible with ϕ_0 defined in Theorem 5.3.

Lemma 5.9. *The product map ϕ_0 defined in Theorem 5.3 factors through the evaluation map*

$$\mathrm{ev}_* : K_*\big(C_{\widetilde{B},L,0}^*(\widetilde{E})^G\big) \to K_*\big(C_{L,0}^*(\widetilde{E})^G\big).$$

That is, for any $m, n \in \{0, 1\}$, there exists a map

$$\phi_{L,0} : K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) \to K_{m+n}\big(C_{\widetilde{B},L,0}^*(\widetilde{E})^G\big)$$

such that the following diagram commutes:

$$\begin{array}{ccc} K_m\big(C_L^*(\widetilde{B})^H\big) \otimes K_n\big(C_{L,0}^*(\widetilde{E}, \widetilde{B})^G\big) & \xrightarrow{\ \phi_{L,0}\ } & K_{m+n}\big(C_{\widetilde{B},L,0}^*(\widetilde{E})^G\big) \\ & \searrow^{\phi_0} & \Big\downarrow \mathrm{ev}_* \\ & & K_{m+n}\big(C_{L,0}^*(\widetilde{E})^G\big). \end{array}$$

Proof. Without loss of generality, we assume that both m and n are zero. With the same notations as in the proof of Theorem 5.3, we define $\phi_{L,0}$ by

$$\phi\big([f_t] \otimes [g_t]\big) = \big[(f_{t+s-1} - 1) \otimes (g_t - 1) + 1\big],$$

where $(f_{t+s-1} - 1) \otimes (g_t - 1) + 1 \in (C_{L,0}^*(\widetilde{E})^G)^+$ is given by

$$\begin{aligned} &\big(\big((f_{t+s-1} - 1) \otimes (g_t - 1)\big)h\big)(x, y) \\ &\quad = \int_{\widetilde{B}} \int_{\widetilde{F}} (f_{t+s-1} - 1)(x, x') \otimes \big(g_t - 1\big)_{x'}(y, y')h(x', y')\, dy'\, dx', \end{aligned} \tag{5.2}$$

with $h \in L^2(\widetilde{E})$. Here, $t \in [1, +\infty)$ is the parameter in $C_{L,0}^*(\widetilde{E})^G$ and $s \in [1, +\infty)$ is the extra parameter in $C_{\widetilde{B},L,0}^*(\widetilde{E})^G$. The expression makes sense as we may assume that the propagation of f_t is small enough. After passing to the matrix algebra and the Grothendieck group, we obtain the map $\phi_{L,0}$. The commuting diagram follows directly from the definition. ■

Lemma 5.10. *With the same notations, for the fiberwise homotopy equivalence f, there exists a K-theory class $\rho_L(f) \in K_{\dim E}(C_{\widetilde{B},L,0}^*(\widetilde{E})^G)$ such that* $\mathrm{ev}_*(\rho_L(f)) = \rho(f) \in K_{\dim E}(C_{L,0}^*(\widetilde{E})^G)$.

Proof. The construction of $\rho_L(f)$ is an analogy of Definition 3.3. Denote by g^B, $g^{E'}$, and g^E the metric on B, E', and E, respectively. For $s \in [1,\infty)$, let B_s, E'_s, and E_s be the Riemannian manifolds equipped with metric tensor $g^{B_s} = s^2 g^B$, $g^{E'_s} = g^{E'} + s^2(\pi')^* g^B$, and $g^{E_s} = g^E + s^2\pi^* g^B$. Now, we have the following fibrations:

$$F' \to E'_s \xrightarrow{\pi'} B_s \quad \text{and} \quad F \to E_s \xrightarrow{\pi} B_s,$$

as well as the homotopy equivalence $f : E'_s \to E_s$. These data define a family of elements in $C^*_{L,0}(\widetilde{E}_s)^G$ parameterized by $s \in [1,\infty)$ which represents the higher rho invariant in $K_{\dim E}(C^*_{L,0}(\widetilde{E}_s)^G)$ as in Definitions 3.7 and 3.8. As $C^*_{L,0}(\widetilde{E}_s)^G$ is naturally isomorphic to $C^*_{L,0}(\widetilde{E})^G$, the above construction gives rise to a path in $C^*_{L,0}(\widetilde{E})^G$ parameterized by $s \in [1,\infty)$.

It remains to show that the above path is an element in $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$. We will prove this using the proof of Lemma 3.2. Note that the Hodge–de Rham operator on E_s and E'_s and the Poincaré duality operator S_f defined by (3.2) satisfy all conditions but the last one listed before the proof of Lemma 3.2. For the last condition (v), although in general S_f does not have zero propagation on the total space of the fibration, it has zero propagation along $\widetilde{B}_s$ as f is a fiberwise map. Therefore, a similar proof shows that the path lies in $C^*_{\widetilde{B},L,0}(\widetilde{E})^G$ and defines a K-theoretic element $\rho_L(f)$ in $K_{\dim E}(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)$ according to the parity of $\dim E$.

When $s = 1$, the above definition gives exactly the higher rho invariant

$$\rho(f) \in K_{\dim E}\big(C^*_{L,0}(\widetilde{E})^G\big).$$

Therefore, we have

$$\mathrm{ev}_*(\rho_L(f)) = \rho(f) \in K_{\dim E}\big(C^*_{L,0}(\widetilde{E})^G\big). \qquad \blacksquare$$

Now, we are ready to prove Theorem 5.5. We will go through the proof in detail only for the case where the dimensions of B and F are both even. The other cases are totally similar.

Proof of Theorem 5.5. Let $\rho_L(f) \in K_{\dim E}(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)$ be the K-theoretic class constructed in the proof of Lemma 5.10. We shall show that

$$\phi_{L,0}\big([D_B^{\mathrm{sgn}}] \otimes \rho(f;B)\big) = \rho_L(f) \in K_0\big(C^*_{\widetilde{B},L,0}(\widetilde{E})^G\big) \tag{5.3}$$

by Mayer–Vietoris arguments. And the theorem follows from Lemmas 5.9 and 5.10.

We first assume a special case where $E = F \times B$, a trivial fiber bundle over B. In this case, the family algebra $C^*_{L,0}(\widetilde{E},\widetilde{B})^G$ is isomorphic to $C(B) \otimes C^*_{L,0}(\widetilde{F})^\Gamma$. The product map $\phi_{L,0}$ and the localized higher rho invariant $\rho_L(f)$ are constructed in Lemmas 5.9 and 5.10, respectively. Using the same construction as in Section 4, we will obtain line (5.3) for this trivial case.

Now, we turn to the general situation. For simplicity, we assume that the base space B admits a triangulation that makes it a simplicial complex. Assume that the diameter of every simplex on B is small enough so that the restriction of E on every simplex is trivial. Let $B^{(k)}$ be a small open neighborhood of the k-skeleton of B, which contains the k-skeleton of B as a deformation retraction. In particular, $B^{(k)} = B$ when k is $\dim B$. Denote the lift of $B^{(k)}$ to $\widetilde{B}$ by $\widetilde{B}^{(k)}$ and the restriction of $\widetilde{E}$ to $\widetilde{B}^{(k)}$ by $\widetilde{E}_{\widetilde{B}^{(k)}}$.

For any K-theory element in $K_*(C^*_{\widetilde{B},L,0}(\widetilde{E})^G)$, its restriction to $\widetilde{E}_{\widetilde{B}^{(k)}}$ is well defined by multiplying the element by the characteristic function of $\widetilde{E}_{\widetilde{B}^{(k)}}$ on both sides. Similarly, for $K_*(C^*_L(\widetilde{B})^H)$ and $K_*(C^*_L(\widetilde{E},\widetilde{B})^G)$, we will prove that line (5.3) holds when restricted to $\widetilde{E}_{\widetilde{B}^{(k)}}$ by induction on k.

When k is zero, $B^{(0)}$ is a disjoint union of small balls in B, to which the restriction of E is trivial. Therefore, line (5.3) holds on $\widetilde{E}_{\widetilde{B}^{(0)}}$. Now, we assume that line (5.3) holds on $\widetilde{E}_{\widetilde{B}^{(k)}}$. Let Δ be the disjoint union of the interior of every $(k+1)$-simplex in $B^{(k+1)}$. Denote the lift of Δ to $\widetilde{B}$ by $\widetilde{\Delta}$ and the restriction of $\widetilde{E}$ to $\widetilde{\Delta}$ by $\widetilde{E}_{\widetilde{\Delta}}$. Note that $B^{(k+1)} = \Delta \cup B^{(k)}$. By Proposition 5.8, we have the following six-term exact sequence:

$$\begin{array}{ccccc}
K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k)}} \bigcap \widetilde{E}_{\widetilde{\Delta}}\big)^G\big) & \to & \begin{array}{c} K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k)}}\big)^G\big) \\ \oplus \\ K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{\Delta}}\big)^G\big) \end{array} & \longrightarrow & K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k+1)}}\big)^G\big) \\
\uparrow & & & & \downarrow \\
K_1\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k+1)}}\big)^G\big) & \longleftarrow & \begin{array}{c} K_1\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k)}}\big)^G\big) \\ \oplus \\ K_1\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{\Delta}}\big)^G\big) \end{array} & \leftarrow & K_1\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k)}} \bigcap \widetilde{E}_{\widetilde{\Delta}}\big)^G\big).
\end{array}$$

From the assumption that the diameter of each simplex of B is small, the restriction of E to Δ or $\Delta \cap B^{(k)}$ is a disjoint union of trivial bundles. Direct computations show that

$$\partial\big(\phi_{L,0}\big(\big[D_B^{\mathrm{sgn}}\big] \otimes \rho(f;B)\big) - \rho_L(f)\big)$$

is trivial in $K_1(C^*_{\widetilde{B},L,0}(\widetilde{E}_{\widetilde{B}^{(k)}} \bigcap \widetilde{E}_{\widetilde{\Delta}})^G)$; thus, it lies in the image of the map

$$\begin{array}{c} K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k)}}\big)^G\big) \\ \oplus \\ K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{\Delta}}\big)^G\big) \end{array} \to K_0\big(C^*_{\widetilde{B},L,0}\big(\widetilde{E}_{\widetilde{B}^{(k+1)}}\big)^G\big).$$

Then, along with the inductive hypothesis, the K-theory classes represented by

$$\phi_{L,0}\big(\big[D_B^{\mathrm{sgn}}\big] \otimes \rho(f;B)\big) - \rho_L(f)$$

restricted to $\widetilde{E}_{\widetilde{\Delta}}$, $\widetilde{E}_{\widetilde{B}^{(k)}}$ vanishes, which shows that

$$\phi_{L,0}\big(\big[D_B^{\mathrm{sgn}}\big] \otimes \rho(f;B)\big) - \rho_L(f)$$

is the image of trivial class. Now, line (5.3) follows when $k = \dim B$. ∎

6. For a special fiber bundle

In this section, we show that Theorem 5.4 implies the product formula for the numerical signature on a fibered manifold given by Chern, Hirzebruch, and Serre in [1].

Consider the fiber bundle $\pi : E \to B$ with fiber F, with all those spaces being $4k$-dimensional oriented closed Riemannian manifolds. Assume that $\pi_1(B)$ acts trivially on $H^*_{dR}(F)$, the de Rham cohomology of F. We would like to use our product formula to prove the original formula introduced by Chern, Hirzebruch, and Serre in [1]; namely,

$$\mathrm{sgn}(B) \times \mathrm{sgn}(F) = \mathrm{sgn}(E).$$

Consider the K-theoretic index map

$$\mathrm{Ind}_E : K_0\big(C^*_L(\widetilde{E})^{\pi_1(E)}\big) \to K_0(C^*_L(pt)) \cong \mathbb{Z}$$

induced by the map that crushes the whole space to a point and forgets the group action. Under this, the localized index of the signature operator will be mapped to its graded Fredholm index, i.e., $\mathrm{sgn}(E)$. Besides, we replace E by the base space B and obtain

$$\mathrm{Ind}_B : K_0\big(C^*_L(\widetilde{B})^{\pi_1(B)}\big) \to K_0(C^*_L(pt)) \cong \mathbb{Z}.$$

Recall that the equivariant family localization algebra $C^*_L(\widetilde{E}, \widetilde{B})^{\pi_1(E)}$ is the collection of some sections of a C*-bundle over B. Any element $s(t) \in C^*_L(\widetilde{E}, \widetilde{B})^{\pi_1(E)}$ is viewed as a family of operators $s(t)_x \in C^*_L(\widetilde{F})^{\Gamma}$ for $x \in \widetilde{B}$. Thus, we define a family index map

$$\mathrm{Ind}_{E,B} : K_0\big(C^*_L(\widetilde{E}, \widetilde{B})^{\pi_1(E)}\big) \to K_0\big(C(\widetilde{B})^{\pi_1(B)}\big) \cong K_0\big(C(B)\big)$$

by taking indices along the fiber.

Moreover, we have the following classical pairing of K-homology and K-theory:

$$\langle \cdot, \cdot \rangle : K_0\big(C^*_L(\widetilde{B})^{\pi_1(B)}\big) \times K_0\big(C(B)\big) \to \mathbb{Z}.$$

From the construction above, we see that the following diagram commutes:

$$\begin{array}{ccc}
K_0\big(C^*_L(\widetilde{B})^{\pi_1(B)}\big) \otimes K_0\big(C^*_L(\widetilde{E}, \widetilde{B})^{\pi_1(E)}\big) & \xrightarrow{\ \phi\ } & K_0\big(C^*_L(\widetilde{E})^{\pi_1(E)}\big) \\
\downarrow{\scriptstyle 1\otimes \mathrm{Ind}_{E,B}} & & \downarrow{\scriptstyle \mathrm{Ind}_E} \\
K_0\big(C^*_L(\widetilde{B})^{\pi_1(B)}\big) \otimes K_0(C(B)) & \xrightarrow{\ \langle\cdot,\cdot\rangle\ } & \mathbb{Z}
\end{array}$$

Therefore, we have the identity

$$\mathrm{sgn}(E) = \big\langle [D_B], \mathrm{Ind}_{E,B}\,\big([D_{E,B}]\big)\big\rangle.$$

Since F is even dimensional, the family index $\mathrm{Ind}_{E,B}([D_{E,B}])$ living in $K_0(C(B))$ can be viewed as a virtual vector bundle over B. The local picture of such a vector bundle is $[\ker(D_F)] - [\operatorname{coker}(D_F)]$. As we have assumed that $\pi_1(B)$ acts on $H^*_{dR}(E)$ trivially, the virtual bundle is indeed a trivial bundle; i.e., it comes from the inclusion $\mathbb{Z} \cong K_0(C(pt)) \to K_0(C(B))$. Moreover, the preimage of $\operatorname{Ind}_{E,B}([D_{E,B}])$ under the inclusion is actually

$$\dim\ker(D_F) - \dim\operatorname{coker}(D_F) = \operatorname{sgn}(F).$$

Thus, the pairing map $\langle\cdot,\cdot\rangle$ is simplified as follows:

$$\big\langle [D_B], \operatorname{Ind}_{E,B}\big([D_{E,B}]\big)\big\rangle = \big\langle [D_B], \operatorname{sgn}(F)\big\rangle = \operatorname{Ind}_B\big([D_B]\big) \times \operatorname{sgn}(F) = \operatorname{sgn}(B) \times \operatorname{sgn}(F).$$

From this, we obtain the classical product formula for the numerical signature given by Chern, Hirzebruch, and Serre.

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