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Equivalences of (co)module algebra structures over Hopf algebras

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Abstract. We introduce the notion of *support equivalence* for (co)module algebras (over Hopf algebras), which generalizes in a natural way (weak) an equivalence of gradings. We show that for each equivalence class of (co)module algebra structures on a given algebra A, there exists a unique universal Hopf algebra H together with an H-(co)module structure on A such that any other equivalent (co)module algebra structure on A factors through the action of H. We study support equivalence and the universal Hopf algebras mentioned above for group gradings, Hopf–Galois extensions, actions of algebraic groups, and cocommutative Hopf algebras. We show how the notion of support equivalence can be used to reduce the classification problem of Hopf algebra (co)actions. We apply support equivalence in the study of the asymptotic behavior of codimensions of H-identities and, in particular, to the analogue (formulated by Yu. A. Bahturin) of Amitsur's conjecture, which was originally concerned with ordinary polynomial identities. As an example, we prove this analogue for all unital H-module structures on the algebra $F[x]/(x^2)$ of dual numbers.

1. Introduction

Module and comodule algebras over Hopf algebras (see the definitions in Sections 4.1 and 5.1) appear in many areas of mathematics and physics. These notions allow a unified approach to algebras with various kinds of an additional structure: group gradings, group actions by automorphisms, (skew) derivations, etc. Another important class of examples arises from (affine) algebraic geometry: if G is an affine algebraic group G acting morphically on an affine algebraic variety X, then the algebra A of regular functions on X is an H-comodule algebra, where H is the algebra of regular functions on G. At the same time, A is a $U(\mathfrak{g})$-module algebra, where $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of the algebraic group G (see, e.g., [1] and Section 6.1). Taking this into account, one may view a (not necessarily commutative) (co)module algebra as an action of a quantum group on a non-commutative space.

The above point of view is also advocated in [29], where the notion of a universal coacting Hopf algebra $\mathsf{aut}(A)$ on an algebra A was introduced (see also [39]), which plays the role of a symmetry group in non-commutative geometry. In order to classify all (co)module algebra structures on a given algebra A, one therefore should understand the Manin–Hopf algebra $\mathsf{aut}(A)$, as well as its quotients. For particular cases, a description

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of $\mathsf{aut}(A)$ has been obtained in, for example, [33,34]. However, finding an explicit description of $\mathsf{aut}(A)$ seems to be a very wild problem. Furthermore, for this universal coacting Hopf algebra of an algebra A to exist, one needs to impose a finiteness condition on the algebra A: it needs to be finite dimensional (see [31, Proposition 1.3.8, Remark 2.6.4]) or at least a rigid object in a suitable monoidal category. An example of the latter is a graded algebra for which each homogeneous component is finite dimensional, which was the setting of [29]. Without such a finiteness condition, one can indeed see that $\mathsf{aut}(A)$ does not always exist (see [3] for an explicit example). Therefore, we propose here a refinement of Manin's construction by studying comodule algebras up to *support equivalence* and show that a Hopf algebra coacting universally up to support equivalence exists for a given comodule algebra A without any finiteness assumption on A.

Inspiration for our approach comes from the theory of group graded algebras. In this context, two gradings are called *equivalent* if there exists an algebra isomorphism between the graded algebras that maps each homogeneous component onto a homogeneous component (see Section 3.1). Remark that no group (iso)morphism between the grading groups is required, but only a bijection between their supports (the support of a grading group is the set of all group elements for which the corresponding homogeneous component is nonzero). It turns out that when one studies the structure of a graded algebra (e.g., graded ideals, graded subspaces, radicals, etc.) or graded polynomial identities of graded algebras, the grading group itself does not play an important role, but can be replaced by any other group that realizes the same decomposition of the algebra into graded components. Using this approach, fine gradings on exceptional simple Lie algebras have been classified up to equivalence [13, Chapter 6]. In [16], the authors studied the possibility of regrading finite dimensional algebras by finite groups.

In Theorem 3.7, we give a criterion for the equivalence of gradings in terms of operators from the dual of the corresponding group algebra. As group graded algebras are exactly comodule algebras over a group algebra, we propose a generalization of the above notions of equivalence for arbitrary comodule algebras, which we call *support equivalence* for comodule algebras. Using the correspondence between H^*-actions and H-coactions for finite dimensional Hopf algebras H, we also define support equivalence for module algebra structures and, in particular, for group actions. Despite the formal duality, in Proposition 6.6 we show that equivalent actions of infinite dimensional Hopf algebras are in general not as close to each other as equivalent coactions.

Among all groups that realize a given grading there is a distinguished one called the universal group of the grading (see [13, Definition 1.17], [32] and Definition 3.4 below). It is easy to see that a similar universal group exists for group actions too (Remark 3.12). Generalizing these constructions, we show in Theorems 4.8 and 5.5 that for a given (co)module algebra A not necessarily finite dimensional, there exists a unique Hopf algebra H with a (co)action on A, which is universal among all Hopf algebras that admit a support equivalent (co)action on A. As mentioned before, our universal Hopf algebra provides a refinement of the Manin–Hopf algebra which is universal among *all* coactions on A (not only those that are support equivalent).

We study in particular equivalences of actions of algebraic groups and their associated Lie algebras. More precisely, for a connected affine algebraic group G over an algebraically closed field F of a characteristic 0 and its associated Lie algebra $\mathfrak{g}$, we show that the $U(\mathfrak{g})$- and the FG-module structures on a finite dimensional algebra with a rational action are support equivalent (see Theorem 6.2).

Although the universal Manin–Hopf algebra of an algebra is in general very difficult to calculate, there are several interesting classes of actions for which the universal Hopf algebra defined here can be computed explicitly. Firstly, in case the (co)action defines a Hopf–Galois extension, the universal Hopf algebra is precisely the original Hopf algebra (see Theorems 4.16 and 5.9). Similarly, the universal Hopf algebra for the standard action of a Hopf algebra H on the algebra H^* is precisely H itself (Theorem 5.8). Furthermore, it turns out that the universal Hopf algebra of a comodule structure corresponding to a grading is just the group algebra of the universal group of this grading (Theorem 4.11). Quite surprisingly, the same result does no longer hold in the case of group actions. Indeed, in Proposition 6.6, we show that the universal Hopf algebra of a group action can contain non-trivial primitive elements. More generally, we prove that the universal Hopf algebra of an action of a cocommutative Hopf algebra is not necessarily cocommutative (see Example 6.5). Therefore, we also consider the universal cocommutative Hopf algebras of actions. We prove their existence and describe them explicitly in the case of an algebraically closed base field of a characteristic 0, thanks to the Milnor–Moore decomposition (see Theorem 6.3). In some cases, the universal Hopf algebra coincides with the cocommutative one and, hence, can be described completely. This is the case, for instance, if one can prove that the universal Hopf algebra is cocommutative too; see Proposition 6.6.

Finally, we apply our results to polynomial identities and show that the codimensions of polynomial H_1- and H_2-identities for an algebra with support equivalent H_1- and H_2-module structures coincide and their polynomial H_1- and H_2-non-identities can be identified in a natural way (Lemma 6.11). This allows us to prove that the analog of Amitsur's conjecture holds for polynomial H-identities of the algebra $F[x]/(x^2)$ (see Theorem 6.12).

The paper is organized as follows.

In Section 3.1, we recall the definitions of an isomorphism and an equivalence of gradings as well as of the universal group of a grading and give a criterion for two gradings to be equivalent in terms of the algebras of linear endomorphisms of the corresponding graded algebras (Theorem 3.7). In Section 3.2, we introduce the corresponding notion of equivalence of group actions and calculate the universal group of an action.

In Section 4.1, we give a definition of support equivalence of comodule structures as well as a criterion for such an equivalence in terms of comodule maps. In Section 4.2, we introduce universal Hopf algebras of comodule structures and prove their existence. In addition, we show that in the case of a group grading the universal Hopf algebra of the corresponding comodule structure is just the group algebra of the universal group of the grading (Theorem 4.11) and any coaction which is support equivalent to a grading can be always reduced to a grading (see the precise statement in Theorem 4.12). Also we show that the universal Hopf algebra can be viewed as a functor from the preorder of all comodule structures on a given algebra (with the respect to the relation "finer/coarser") to the category of Hopf algebras. In Section 4.3, we consider comodule Hopf–Galois extensions and show that the corresponding universal Hopf algebra is the original coacting Hopf algebra.

In Section 5.1, we give a definition of support equivalence of module structures. We introduce universal Hopf algebras of module structures and prove their existence in Section 5.2. In Section 5.3, we consider module Hopf–Galois extensions and show that the corresponding universal Hopf algebra is the original acting Hopf algebra.

In Section 6.1, we consider the classical correspondence between connected affine algebraic groups G over an algebraically closed field of a characteristic 0 and their Lie algebras $\mathfrak{g}$. Namely, we prove that FG-actions and $U(\mathfrak{g})$-actions are support equivalent (Theorem 6.2). Section 6.2 deals with universal cocommutative Hopf algebras. The notion of a universal cocommutative Hopf algebra is then used to calculate the universal Hopf algebra (Proposition 6.6). In Section 6.4, we show how support equivalences of module structures can be applied to classify module structures on a given algebra and to polynomial H-identities.

2. Preliminaries

Throughout this paper, F denotes a field and, unless specified otherwise, all vector spaces, tensor products, homomorphisms, (co)algebras, and Hopf algebras are over F. For a coalgebra C, we use the Sweedler Σ-notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for all $c \in C$, with a suppressed summation sign.

Our notation for the standard categories is as follows: Alg_F (algebras over F), Coalg_F (coalgebras over F), and Hopf_F (Hopf algebras over F). Recall that there exists a left adjoint functor $L\colon \mathrm{Coalg}_F \to \mathrm{Hopf}_F$ for the forgetful functor $U\colon \mathrm{Hopf}_F \to \mathrm{Coalg}_F$ (see [38]) and we denote by $\eta\colon \mathrm{id}_{\mathrm{Coalg}_F} \Rightarrow UL$ the unit of this adjunction. Moreover, there exists a right adjoint functor $R\colon \mathrm{Alg}_F \to \mathrm{Hopf}_F$ for the forgetful functor $U\colon \mathrm{Hopf}_F \to \mathrm{Alg}_F$ (see [2, 11]). The counit of this adjunction will be denoted by $\mu\colon UR \Rightarrow \mathrm{id}_{\mathrm{Alg}_F}$. Remark that although we use the same character U for both forgetful functors, it will be clear from the context which forgetful functor is meant.

Given a bialgebra (or a Hopf algebra) H, a (not necessarily associative) algebra A is called an *H-comodule algebra* if it admits a right H-comodule structure $\rho\colon A \to A \otimes H$ which is an algebra homomorphism ($A \otimes H$ has the usual tensor product algebra structure). Furthermore, A is called a *unital* H-comodule algebra if there exists an identity element $1_A \in A$ such that $\rho(1_A) = 1_A \otimes 1_H$. The map ρ is called a *comodule algebra structure* on A and will be written in Sweedler notation as $\rho(a) = a_{(0)} \otimes a_{(1)}$, for all $a \in A$, again with a suppressed summation sign. Explicitly, the fact that ρ is an algebra homomorphism reads, for all $a, b \in A$,

$$(ab)_{(0)} \otimes (ab)_{(1)} = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}.$$

Note that any H-comodule algebra map $\rho\colon A \to A \otimes H$ gives rise to an algebra homomorphism $\zeta\colon H^* \to \operatorname{End}_F(A)$ defined by $\zeta(h^*)a = h^*(a_{(1)})a_{(0)}$ for all $a \in A$ and $h^* \in H^*$. Here H^* is the vector space dual to H, endowed with the structure of the algebra dual to the coalgebra H. Recall that if H is finite dimensional, then H^* is a Hopf algebra too.

A (not necessarily associative) algebra A is called an *H-module algebra* if it admits a left H-module structure such that

$$h(ab) = (h_{(1)}a)(h_{(2)}b) \quad \text{for all } a, b \in A,\ h \in H. \tag{2.1}$$

We denote by ζ the homomorphism of algebras $H \to \operatorname{End}_F(A)$ defined by $\zeta(h)a = ha$, for all $h \in H$ and $a \in A$, and we call it a *module algebra structure* on A. An H-module algebra A will be called *unital* if there exists an identity element $1_A \in A$ such that $h1_A = \varepsilon(h)1_A$ for all $h \in H$.

3. Equivalences of group gradings and group actions

3.1. Group gradings

When studying graded algebras, one has to determine when two graded algebras can be considered “the same” or equivalent.

Recall that $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$ is a *grading* on a (not necessarily associative) algebra A by a group G if $A^{(g)}A^{(h)} \subseteq A^{(gh)}$ for all $g, h \in G$. Then G is called the *grading group* of Γ. The algebra A is called *graded* by G.

Let

$$\Gamma_1\colon A_1 = \bigoplus_{g\in G_1} A_1^{(g)} \quad \text{and} \quad \Gamma_2\colon A_2 = \bigoplus_{g\in G_2} A_2^{(g)} \tag{3.1}$$

be two gradings, where G_1 and G_2 are groups and A_1 and A_2 are algebras.

The most restrictive case is when we require that both grading groups coincide.

Definition 3.1 (e.g., [13, Definition 1.15]). Gradings (3.1) are *isomorphic* if $G_1 = G_2$ and there exists an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras such that $\varphi(A_1^{(g)}) = A_2^{(g)}$ for all $g \in G_1$.

In this case, we say that A_1 and A_2 are *graded isomorphic*.

If one studies the graded structure of a graded algebra or its graded polynomial identities [5, 6, 9, 14, 21], then it is not really important by elements of which group the graded components are indexed. A replacement of the grading group leaves both graded subspaces and graded ideals graded. In the case of graded polynomial identities, reindexing the graded components leads only to renaming the variables. Here we come naturally to the notion of (weak) equivalence of gradings.

Definition 3.2. We say that gradings (3.1) are *(weakly) equivalent* if there exists an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras such that for every $g_1 \in G_1$ with $A_1^{(g_1)} \neq 0$ there exists $g_2 \in G_2$ such that $\varphi(A_1^{(g_1)}) = A_2^{(g_2)}$.

Obviously, if gradings are isomorphic, then they are equivalent. It is important to notice that the converse is not true.

However, if gradings (3.1) are equivalent and $\varphi\colon A_1 \xrightarrow{\sim} A_2$ is the corresponding isomorphism of algebras, then $\Gamma_3\colon A_1 = \bigoplus_{g\in G_2} \varphi^{-1}(A_2^{(g)})$ is a G_2-grading on A_1 isomorphic to Γ_2 and the grading Γ_3 is obtained from Γ_1 just by reindexing the homogeneous components. Therefore, when gradings (3.1) are equivalent, we say that Γ_1 *can be regraded* by G_2.

If $A_1 = A_2$ and φ in Definition 3.2 is the identity map, we say that Γ_1 and Γ_2 are *realizations of the same grading* on A as, respectively, G_1- and G_2-gradings.

For a grading $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$, we denote by $\operatorname{supp}\Gamma := \{g \in G \mid A^{(g)} \neq 0\}$ its support.

Remark 3.3. Each equivalence between gradings Γ_1, Γ_2 induces a bijection $\operatorname{supp}\Gamma_1 \xrightarrow{\sim} \operatorname{supp}\Gamma_2$.

Each group grading on an algebra can be realized as a G-grading for many different groups G, however it turns out that there is one distinguished group among them; see [13, Definition 1.17], [32].

Definition 3.4. Let Γ be a group grading on an algebra A. Suppose that Γ admits a realization as a G_Γ-grading for some group G_Γ. Denote by $\varkappa_\Gamma$ the corresponding embedding $\operatorname{supp}\Gamma \hookrightarrow G_\Gamma$. We say that $(G_\Gamma, \varkappa_\Gamma)$ is the *universal group of the grading* Γ if for any realization of Γ as a grading by a group G with $\psi\colon \operatorname{supp}\Gamma \hookrightarrow G$ there exists a unique homomorphism $\varphi\colon G_\Gamma \to G$ such that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{supp}\Gamma & \xrightarrow{\varkappa_\Gamma} & G_\Gamma \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\varphi} \\ & & G \end{array}$$

Remarks 3.5. (a) For each grading Γ, one can define a category $\mathcal{C}_\Gamma$, where the objects are all pairs (G, ψ) such that G is a group and Γ can be realized as a G-grading with $\psi\colon \operatorname{supp}\Gamma \hookrightarrow G$ being the embedding of the support. In this category, the set of morphisms between (G_1, ψ_1) and (G_2, ψ_2) consists of all group homomorphisms $f\colon G_1 \to G_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{supp}\Gamma & \xrightarrow{\psi_1} & G_1 \\ & {\scriptstyle\psi_2}\searrow & \downarrow{\scriptstyle f} \\ & & G_2 \end{array}$$

Then $(G_\Gamma, \varkappa_\Gamma)$ is the initial object of $\mathcal{C}_\Gamma$.

(b) It is easy to see that if $\Gamma\colon A = \bigoplus_{g\in\operatorname{supp}\Gamma} A^{(g)}$ is a group grading, then the universal group G_Γ of the grading Γ is isomorphic to $\mathcal{F}_{[\operatorname{supp}\Gamma]}/N$, where $\mathcal{F}_{[\operatorname{supp}\Gamma]}$ is the free group on the set $[\operatorname{supp}\Gamma] := \{[g] \mid g \in \operatorname{supp}\Gamma\}$ and N is the normal closure of the words $[g][h][t]^{-1}$ for pairs $g, h \in \operatorname{supp}\Gamma$ such that $A^{(g)}A^{(h)} \neq 0$, where $t \in \operatorname{supp}\Gamma$ is defined by $A^{(g)}A^{(h)} \subseteq A^{(t)}$.

(c) Of course, from the linguistic point of view, it would be more logic to say in our definition "finer or equivalent" instead of just "finer" and "coarser or equivalent" instead of just "coarser"; however throughout this paper we drop the words "or equivalent" for brevity.

Let $\Gamma_1 \colon A = \bigoplus_{g\in G} A^{(g)}$ and $\Gamma_2 \colon A = \bigoplus_{h\in H} A^{(h)}$ be two gradings on the same algebra A. If for every $g \in G$ there exists $h \in H$ such that $A^{(g)} \subseteq A^{(h)}$, then we say that Γ_1 is *finer* than Γ_2 and Γ_2 is *coarser* than Γ_1. It is easy to see that this relation is a preorder and Γ_1 is both finer and coarser than Γ_2 if and only if id_A is an equivalence of Γ_1 and Γ_2. Moreover, the universal group of the grading is the functor from this preorder to the category of groups: if Γ_1 is finer than Γ_2, the functor assigns the homomorphism $G_{\Gamma_1} \to G_{\Gamma_2}$ defined by $[g] \mapsto [h]$ for $A^{(g)} \subseteq A^{(h)}$.

Now, in order to make it possible to transfer the relation of support equivalence and the relation "coarser/finer" to (co)module algebra structures, we translate these relations into the language of linear operators.

If an algebra $A = \bigoplus_{g\in G} A^{(g)}$ is graded by a group G, then we have an $(FG)^*$-action on A, where $(FG)^*$ is the algebra dual to the group coalgebra FG, i.e., $(FG)^*$ is the algebra of all functions $G \to F$ with pointwise operations: $ha = h(g)a$ if $g \in G$, $a \in A^{(g)}$, and $h \in (FG)^*$.

Lemma 3.6. *Let* (3.1) *be two group gradings and let* $\varphi\colon A_1 \xrightarrow{\sim} A_2$ *be an isomorphism of algebras. Denote by* $\zeta_i \colon (FG_i)^* \to \operatorname{End}_F(A_i)$ *the homomorphism from* $(FG_i)^*$ *to the algebra* $\operatorname{End}_F(A_i)$ *of* F*-linear operators on* A_i *induced by the* $(FG_i)^*$*-action,* $i = 1, 2$*, and denote by* $\tilde\varphi$ *the isomorphism* $\operatorname{End}_F(A_1) \xrightarrow{\sim} \operatorname{End}_F(A_2)$ *defined by* $\tilde\varphi(\psi)(a) = \varphi(\psi(\varphi^{-1}(a)))$ *for* $\psi \in \operatorname{End}_F(A_1)$ *and* $a \in A_2$*. Then the inclusion*

$$\tilde\varphi\big(\zeta_1\big((FG_1)^*\big)\big) \supseteq \zeta_2\big((FG_2)^*\big) \tag{3.2}$$

holds if and only if for every $g_1 \in G_1$ *there exists* $g_2 \in G_2$ *such that* $\varphi(A_1^{(g_1)}) \subseteq A_2^{(g_2)}$.

Proof. If for every $g_1 \in G_1$ there exists $g_2 \in G_2$ such that $\varphi(A_1^{(g_1)}) \subseteq A_2^{(g_2)}$, then each $A_2^{(g_2)}$ is a direct sum of some of $\varphi(A_1^{(g_1)})$ since

$$\bigoplus_{g_1\in G_1} \varphi\big(A_1^{(g_1)}\big) = \varphi(A_1) = A_2 = \bigoplus_{g_2\in G_2} A_2^{(g_2)}.$$

Note that the set $\zeta_i((FG_i)^*)$ consists of all the linear operators that act by a scalar operator on each homogeneous component $A_i^{(g_i)}$, $g_i \in G_i$. Hence $\tilde\varphi(\zeta_1((FG_1)^*))$ consists of all the linear operators on A_2 that act on each $\varphi(\zeta_1(A_1^{(g_1)}))$ by a scalar operator. Since each $A_2^{(g_2)}$ is a direct sum of some of $\varphi(A_1^{(g_1)})$, all the operators from $\zeta_2((FG_2)^*)$ act by a scalar operator on each of $\varphi(A_1^{(g_1)})$ too. Therefore, (3.2) holds.

Conversely, suppose (3.2) holds. Denote by p_{g_2}, where $g_2 \in \operatorname{supp}\Gamma_2$, the projection on $A_2^{(g_2)}$ along $\bigoplus_{h\in\operatorname{supp}\Gamma_2,\, h\neq g_2} A_2^{(h)}$, i.e.,

$$p_{g_2}a := \begin{cases} a & \text{if } a \in A_2^{(g_2)}, \\ 0 & \text{if } a \in \bigoplus_{h\in\operatorname{supp}\Gamma_2,\, h\neq g_2} A_2^{(h)}. \end{cases}$$

Then (3.2) implies that $p_{g_2} \in \tilde{\varphi}(\zeta_1((FG_1)^*))$ for all $g_2 \in \operatorname{supp}\Gamma_2$. In particular, p_{g_2} is acting as a scalar operator on all the components $\varphi(A_1^{(g)})$, where $g \in \operatorname{supp}\Gamma_1$. Since $p_{g_2}^2 = p_{g_2}$, for every $g \in \operatorname{supp}\Gamma_1$ either $p_{g_2}\varphi(A_1^{(g)}) = 0$ or $p_{g_2}a = a$ for all $a \in \varphi(A_1^{(g)})$. Hence $A_2^{(g_2)} = \operatorname{im}(p_{g_2})$ is a direct sum of some of $\varphi(A_1^{(g)})$, where $g \in \operatorname{supp}\Gamma_1$. Since

$$\bigoplus_{g_1\in\operatorname{supp}\Gamma_1} \varphi(A_1^{(g_1)}) = \bigoplus_{g_2\in\operatorname{supp}\Gamma_2} A_2^{(g_2)},$$

for every $g_1 \in G_1$ there exists $g_2 \in G_2$ such that $\varphi(A_1^{(g_1)}) \subseteq A_2^{(g_2)}$. ∎

From Lemma 3.6, we immediately deduce the criteria that are crucial to transfer the notion of equivalence and the relation "finer/coarser" from gradings to (co)module structures.

Theorem 3.7. *Let (3.1) be two group gradings. Then an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras defines an equivalence of gradings if and only if*

$$\tilde{\varphi}\big(\zeta_1\big((FG_1)^*\big)\big) = \zeta_2\big((FG_2)^*\big),$$

where $\zeta_i\colon (FG_i)^* \to \operatorname{End}_F(A_i)$ is the homomorphism from $(FG_i)^*$ to the algebra $\operatorname{End}_F(A_i)$ of F-linear operators on A_i induced by the $(FG_i)^*$-action, $i = 1, 2$, and the isomorphism $\tilde{\varphi}\colon \operatorname{End}_F(A_1) \xrightarrow{\sim} \operatorname{End}_F(A_2)$ is defined by $\tilde{\varphi}(\psi)(a) = \varphi(\psi(\varphi^{-1}(a)))$ for $\psi \in \operatorname{End}_F(A_1)$ and $a \in A_2$.

Proof. We apply Lemma 3.6 to φ and φ^{-1}. ∎

Theorem 3.8. *Let $\Gamma_1\colon A = \bigoplus_{g\in G_1} A^{(g)}$ and $\Gamma_2\colon A = \bigoplus_{g\in G_2} A^{(g)}$ be two gradings on the same algebra A and let $\zeta_i\colon (FG_i)^* \to \operatorname{End}_F(A)$ be the corresponding homomorphisms. Then Γ_1 is finer than Γ_2 if and only if $\zeta_1((FG_1)^*) \supseteq \zeta_2((FG_2)^*)$.*

Proof. We apply Lemma 3.6 to $A = A_1 = A_2$ and $\varphi = \mathrm{id}_A$. ∎

3.2. Group actions

Suppose we have a dual situation: for $i = 1, 2$ a group G_i is acting on an algebra A_i by automorphisms, where $\zeta_i\colon G_i \to \operatorname{Aut}(A_i)$ are the corresponding group homomorphisms. Inspired by Theorem 3.7, we introduce the following definition.

Definition 3.9. We say that actions ζ_1 and ζ_2 are *equivalent via an isomorphism* φ if $\varphi\colon A_1 \xrightarrow{\sim} A_2$ is an isomorphism of algebras such that

$$\tilde{\varphi}\big(\langle\zeta_1(G_1)\rangle_F\big) = \langle\zeta_2(G_2)\rangle_F,$$

where the F-linear span $\langle\cdot\rangle_F$ is taken in the corresponding $\mathrm{End}(A_i)$ and the isomorphism $\tilde{\varphi}\colon \mathrm{End}_F(A_1) \xrightarrow{\sim} \mathrm{End}_F(A_2)$ is defined by $\tilde{\varphi}(\psi)(a) = \varphi(\psi(\varphi^{-1}(a)))$ for $\psi \in \mathrm{End}_F(A_1)$ and $a \in A_2$.

If $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$ is a grading on an algebra A by a group G, then there exists a standard $\mathrm{Hom}(G, F^\times)$-action on A by automorphisms:

$$\chi a := \chi(g)a \quad \text{for } \chi \in \mathrm{Hom}(G, F^\times),\ a \in A^{(g)},\ g \in G. \tag{3.3}$$

Extend each $\chi \in \mathrm{Hom}(G, F^\times)$ by linearity to a map $FG \to F$. Then this $\mathrm{Hom}(G, F^\times)$-action becomes just the restriction of the map $\zeta\colon (FG)^* \to \mathrm{End}_F(A)$ corresponding to Γ.

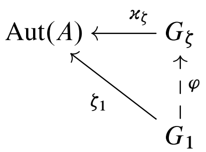
In the case when F is algebraically closed of a characteristic 0 and G is finite abelian, the group $\mathrm{Hom}(G, F^\times)$ is usually denoted by $\widehat{G}$ and called the *Pontryagin dual group*. The classical theorem on the structure of finitely generated abelian groups implies then $\widehat{G} \cong G$. Moreover, (3.3) defines a one-to-one correspondence between G-gradings and $\widehat{G}$-actions. (See the details, e.g., in [15, Section 3.2].) The following proposition shows that in this case equivalent G-gradings correspond to equivalent $\widehat{G}$-actions.

Proposition 3.10. *Let* (3.1) *be two group gradings by finite abelian groups G_1 and G_2 and let ζ_i be the corresponding $(FG_i)^*$-actions, $i = 1, 2$. Suppose that the base field F is algebraically closed of a characteristic* 0. *Then Γ_1 and Γ_2 are equivalent as group gradings if and only if $\zeta_1|_{\widehat{G}_1}$ and $\zeta_2|_{\widehat{G}_2}$ are equivalent as group actions.*

Proof. The orthogonality relations for characters imply that the elements of $\widehat{G}_i$ form a basis in $(FG_i)^*$. Hence $\zeta_i((FG_i)^*) = \langle \zeta_i(\widehat{G}_i)\rangle$, for $i = 1, 2$, and the proposition follows from Theorem 3.7. ∎

Return now to the case of arbitrary groups G_1 and G_2 and an arbitrary field F. As in the case of gradings, we can identify A_1 and A_2 via φ. Then the equivalence of ζ_1 and ζ_2 means that the images of G_1 and G_2 generate the same subalgebra in the algebra of F-linear operators on $A_1 = A_2$.

Definition 3.11. Let $\zeta\colon G \to \mathrm{Aut}(A)$ be an action of a group G on an algebra A and let $\varkappa_\zeta\colon G_\zeta \to \mathrm{Aut}(A)$ be an action of a group G_ζ equivalent to ζ via the identity isomorphism id_A. We say that the pair $(G_\zeta, \varkappa_\zeta)$ is a *universal group of the action* ζ if for any other action $\zeta_1\colon G_1 \to \mathrm{Aut}(A)$ equivalent to ζ via id_A there exists a unique group homomorphism $\varphi\colon G_1 \to G_\zeta$ such that the following diagram is commutative:



Remark 3.12. Consider the group $\mathcal{U}(\langle\zeta(G)\rangle_F) \cap \mathrm{Aut}(A)$, where $\mathcal{U}(\langle\zeta(G)\rangle_F)$ is the group of invertible elements of the algebra $\langle\zeta(G)\rangle_F \subseteq \mathrm{End}_F(A)$. Since for every G_1, as

above, the image of G_1 in $\mathrm{Aut}(A)$ belongs to $\mathcal{U}(\langle\zeta_1(G_1)\rangle_F) \cap \mathrm{Aut}(A) = \mathcal{U}(\langle\zeta(G)\rangle_F) \cap \mathrm{Aut}(A)$, the universal group of the action ζ is (up to an isomorphism) the couple $(G_\zeta, \varkappa_\zeta)$, where

$$G_\zeta := \mathcal{U}\big(\langle\zeta(G)\rangle_F\big) \cap \mathrm{Aut}(A)$$

and $\varkappa_\zeta$ is the natural embedding $G_\zeta \subseteq \mathrm{Aut}(A)$.

Let $\zeta_i\colon G_i \to \mathrm{Aut}(A)$, $i = 1, 2$, be two group actions on A by automorphisms. We say that ζ_1 is *finer* than ζ_2 and ζ_2 is *coarser* than ζ_1 if $\langle\zeta_2(G)\rangle_F \subseteq \langle\zeta_1(G)\rangle_F$. Again, it is easy to see that this relation is a preorder and ζ_1 is both finer and coarser than ζ_2 if and only if id_A is an equivalence of ζ_1 and ζ_2. Moreover, the universal group of the action is the functor from this preorder to the category of groups: if ζ_1 is finer than ζ_2, the functor assigns the embedding $\mathcal{U}(\langle\zeta(G_2)\rangle_F) \cap \mathrm{Aut}(A) \subseteq \mathcal{U}(\langle\zeta(G_1)\rangle_F) \cap \mathrm{Aut}(A)$.

Proposition 3.13. *Let* (3.1) *be two group gradings by finite abelian groups* G_1 *and* G_2 *and let* ζ_i *be the corresponding* $(FG_i)^*$*-actions,* $i = 1, 2$*. Suppose that the base field* F *is algebraically closed of a characteristic* 0*. Then* Γ_1 *is finer than* Γ_2 *if and only if* $\zeta_1|_{\widehat{G}_1}$ *is finer than* $\zeta_2|_{\widehat{G}_2}$.

Proof. Again, the orthogonality relations for characters imply that the elements of $\widehat{G}_i$ form a basis in $(FG_i)^*$. Hence $\zeta_i((FG_i)^*) = \langle\zeta_i(\widehat{G}_i)\rangle$, for $i = 1, 2$, and the proposition follows from Theorem 3.8. ∎

4. Comodule algebras

4.1. Support equivalence of comodule structures on algebras

Inspired by Theorem 3.7, we give the following definition.

Definition 4.1. Let A_i be (not necessarily associative) H_i-comodule algebras for Hopf algebras H_i, $i = 1, 2$. We say that comodule structures on A_1 and A_2 are *support equivalent* via the algebra isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ if

$$\tilde{\varphi}\big(\zeta_1(H_1^*)\big) = \zeta_2(H_2^*), \tag{4.1}$$

where ζ_i is the algebra homomorphism $H_i^* \to \mathrm{End}_F(A_i)$ induced by the comodule algebra structure on A_i and the isomorphism $\tilde{\varphi}\colon \mathrm{End}_F(A_1) \xrightarrow{\sim} \mathrm{End}_F(A_2)$ is defined by

$$\tilde{\varphi}(\psi)(a) = \varphi\big(\psi\big(\varphi^{-1}(a)\big)\big) \quad \text{for } \psi \in \mathrm{End}_F(A_1) \text{ and } a \in A_2.$$

As the only equivalences we will consider in the sequel are support equivalences, we will just use the term *equivalence*.

It is easy to see that each support equivalence of comodule algebras maps H_1-subcomodules to H_2-subcomodules.

As in the case of gradings, we can restrict our consideration to the case when $A_1 = A_2$ and φ is an identity map.

Let A be an H-comodule algebra with a comodule map $\rho\colon A \to A \otimes H$ and the corresponding homomorphism of algebras $\zeta\colon H^* \to \operatorname{End}_F(A)$. Choose a basis $(a_\alpha)_\alpha$ in A and let $\rho(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$, where $h_{\beta\alpha} \in H$. Denote by $C(\rho)$ the F-linear span of all such $h_{\alpha\beta}$. Now since $(\rho \otimes \mathrm{id}_H)\rho = (\mathrm{id}_A \otimes \Delta_H)\rho$ and $(\mathrm{id}_A \otimes \varepsilon)\rho = \mathrm{id}_A$, our definition of $h_{\alpha\beta}$ implies

$$\Delta h_{\alpha\beta} = \sum_\gamma h_{\alpha\gamma} \otimes h_{\gamma\beta}, \quad \varepsilon(h_{\alpha\beta}) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta, \end{cases} \quad \text{for all } \alpha, \beta. \tag{4.2}$$

In particular, $C(\rho)$ is a subcoalgebra of H. It is easy to see that

$$\ker \zeta = C(\rho)^\perp := \{\lambda \in H^* \mid \lambda(C(\rho)) = 0\}$$

and $\zeta(H^*) \cong C(\rho)^*$. In other words, $C(\rho)$ is the minimal subcoalgebra $C \subseteq H$ such that $\rho(A) \subseteq A \otimes C$.

Definition 4.2. Given an H-comodule algebra A with coaction $\rho\colon A \to A \otimes H$, call the coalgebra $C(\rho)$ constructed above the *support coalgebra* of the coaction ρ.

Remark 4.3. The support coalgebra was introduced by J. A. Green in [25] under the name "coefficient space". We prefer to use here the name "support coalgebra" as in the case of a grading it is exactly the linear span of the support of the grading.

Proposition 4.4. *Let A_i be H_i-comodule algebras for Hopf algebras H_i, $i = 1, 2$. Then an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras is an equivalence of comodule algebra structures $\rho_i\colon A_i \to A_i \otimes H_i$, $i = 1, 2$, if and only if there exists an isomorphism $\tau\colon C(\rho_1) \xrightarrow{\sim} C(\rho_2)$ of coalgebras such that the following diagram is commutative:*

$$\begin{array}{ccc} A_1 & \xrightarrow{\rho_1} & A_1 \otimes C(\rho_1) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi\otimes\tau} \\ A_2 & \xrightarrow{\rho_2} & A_2 \otimes C(\rho_2), \end{array} \tag{4.3}$$

i.e., two comodule algebras are support equivalent if and only if they have isomorphic support coalgebras and they are isomorphic as comodules over their support coalgebra.

Proof. Suppose φ is a support equivalence of ρ_1 and ρ_2. Choose as above some basis $(a_\alpha)_\alpha$ in A_1. Let $a'_\alpha := \varphi(a_\alpha)$, $\rho_1(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$, and $\rho_2(a'_\alpha) = \sum_\beta a'_\beta \otimes h'_{\beta\alpha}$, where $h_{\beta\alpha} \in H_1$, $h'_{\beta\alpha} \in H_2$.

Assume that $\sum_{\alpha,\beta} t_{\beta\alpha} h_{\beta\alpha} = 0$ for some $t_{\beta\alpha} \in F$, where only a finite number of $t_{\beta\alpha}$ are nonzero. Define linear functions $\tau_\alpha\colon A \to F$ by $\tau_\alpha(a_\beta) = t_{\beta\alpha}$. Then for any $\lambda \in H_1^*$ we have

$$\sum_\alpha \tau_\alpha(\zeta_1(\lambda) a_\alpha) = \sum_{\alpha,\beta} t_{\beta\alpha} \lambda(h_{\beta\alpha}) = 0.$$

Hence

$$\sum_\alpha \tau_\alpha\big(\varphi^{-1}\big(\tilde\varphi(\zeta_1(\lambda)) a'_\alpha\big)\big) = 0 \quad \text{for all } \lambda \in H_1^*.$$

Now (4.1) implies

$$\sum_{\alpha} \tau_\alpha\big(\varphi^{-1}\big(\zeta_2(\lambda')a'_\alpha\big)\big) = 0$$

and

$$\sum_{\alpha,\beta} t_{\beta\alpha}\lambda'(h'_{\beta\alpha}) = 0 \quad \text{for all } \lambda' \in H_2^*.$$

As a consequence, $\sum_{\alpha,\beta} t_{\beta\alpha}h'_{\beta\alpha} = 0$. Applying the same argument for φ^{-1}, we obtain that if $\sum_{\alpha,\beta} t_{\beta\alpha}h'_{\beta\alpha} = 0$ for some $t_{\beta\alpha} \in F$, then $\sum_{\alpha,\beta} t_{\beta\alpha}h_{\beta\alpha} = 0$. Hence there are the same linear dependencies among $h_{\alpha\beta}$ and among $h'_{\alpha\beta}$. Taking into account that $C(\rho_1) = \langle h_{\alpha\beta} \mid \alpha,\beta\rangle_F$ and $C(\rho_2) = \langle h'_{\alpha\beta} \mid \alpha,\beta\rangle_F$, we can correctly define a linear map $\tau: C(\rho_1) \to C(\rho_2)$ by $\tau(h_{\alpha\beta}) = h'_{\alpha\beta}$ for all α,β and the reverse map $\tau^{-1}: C(\rho_2) \to C(\rho_1)$ by $\tau^{-1}(h'_{\alpha\beta}) = h_{\alpha\beta}$ for all α,β. In particular, the map τ is a bijection. By (4.2) the map τ is a homomorphism of coalgebras. Finally, (4.3) holds.

Conversely, suppose (4.3) holds. Then $\zeta_i(\lambda)$ for $\lambda \in H_i^*$ is determined by the values of λ on $C(\rho_i)$. Now (4.3) implies (4.1). ■

Remark 4.5. The proof of Proposition 4.4 makes in fact no use of the algebra structures of A_i and H_i, nor of the coalgebra structures of H_i. In fact, it is possible to define a notion of support equivalence for arbitrary linear maps $A \to B \otimes Q$, where A, B, and Q are just vector spaces, and in such a setting Proposition 4.4 remains valid mutatis mutandis (see [4, Proposition 2.6]).

4.2. Universal Hopf algebra of a comodule algebra structure

Analogously to the universal group of a grading, we would like to introduce the universal Hopf algebra of a given comodule algebra structure. It will be the initial object of the category $\mathcal{C}_A^H$ defined below.

Let A be an H-comodule algebra for a Hopf algebra H and let $\zeta: H^* \to \mathrm{End}_F(A)$ be the corresponding algebra homomorphism. Consider the category $\mathcal{C}_A^H$, where

(1) the objects are H_1-comodule algebra structures on the algebra A for arbitrary Hopf algebras H_1 over F such that $\zeta_1(H_1^*) = \zeta(H^*)$, where $\zeta_1: H_1^* \to \mathrm{End}_F(A)$ is the algebra homomorphism corresponding to the H_1-comodule algebra structure on A;

(2) the morphisms from an H_1-comodule algebra structure on A with the corresponding homomorphism ζ_1 to an H_2-comodule algebra structure with the corresponding homomorphism ζ_2 are all Hopf algebra homomorphisms $\tau: H_1 \to H_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{End}_F(A) & \xleftarrow{\ \zeta_2\ } & H_2^* \\ & \nwarrow^{\zeta_1} & \downarrow{\scriptstyle \tau^*} \\ & & H_1^* \end{array} \tag{4.4}$$

Remark 4.6. The commutative diagram (4.4) means that

$$\zeta_2(h^*)(a) = \zeta_1\big(\tau^*(h^*)\big)(a),$$
$$h^*(a_{[1]})a_{[0]} = h^*\big(\tau(a_{(1)})\big)a_{(0)}$$

for all $h^* \in H_2$ and $a \in A$, where $\rho_1(a) = a_{(0)} \otimes a_{(1)}$ and $\rho_2(a) = a_{[0]} \otimes a_{[1]}$ are the H_1- and H_2-comodule maps on A, respectively. Hence

$$a_{[0]} \otimes a_{[1]} = a_{(0)} \otimes \tau(a_{(1)})$$

for all $a \in A$ and (4.4) is equivalent to the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \rho_1\ } & A \otimes H_1 \\ & \searrow^{\rho_2} & \big\downarrow{\scriptstyle \mathrm{id}_A \otimes \tau} \\ & & A \otimes H_2 \end{array}$$

Now we are going to show that $\mathcal{C}_A^H$ possesses an initial object (which is always unique up to an isomorphism).

Recall that by L we denote the left adjoint functor to the forgetful functor $U\colon \mathrm{Hopf}_F \to \mathrm{Coalg}_F$. Let $\rho\colon A \to A \otimes H$ be the map defining a right H-comodule structure on A. Since $C(\rho) \subseteq H$ is an embedding, $\eta_{C(\rho)}\colon C(\rho) \to L(C(\rho))$ is an embedding too. Hence for our choice of $C(\rho)$ the algebra A is a right $C(\rho)$-comodule and therefore a right $L(C(\rho))$-comodule. Now we will factor $L(C(\rho))$ by a specific Hopf ideal I to turn A into an $L(C(\rho))/I$-comodule algebra.

Choose a basis $(a_\alpha)_\alpha$ in A and $h_{\alpha\beta}$ as in the previous section.

Lemma 4.7. *Let $a_\alpha a_\beta = \sum_v k^v_{\alpha\beta} a_v$ for some structure constants $k^v_{\alpha\beta} \in F$ and denote by I_0 the ideal of $L(C(\rho))$ generated by*

$$\sum_{r,q} k^\gamma_{rq}\eta_{C(\rho)}(h_{r\alpha})\eta_{C(\rho)}(h_{q\beta}) - \sum_u k^u_{\alpha\beta}\eta_{C(\rho)}(h_{\gamma u}) \tag{4.5}$$

for all possible choices of indices α, β, γ. Then I_0 is a coideal.

Proof. First of all, note that for every α, β, γ we have

$$\varepsilon\bigg(\sum_{r,q} k^\gamma_{rq}\eta_{C(\rho)}(h_{r\alpha})\eta_{C(\rho)}(h_{q\beta}) - \sum_u k^u_{\alpha\beta}\eta_{C(\rho)}(h_{\gamma u})\bigg) = k^\gamma_{\alpha\beta} - k^\gamma_{\alpha\beta} = 0$$

and therefore $\varepsilon(I_0) = 0$. Moreover, a direct computation using (4.2) gives

$$\Delta\bigg(\sum_{r,q} k^\gamma_{rq}\eta_{C(\rho)}(h_{r\alpha})\eta_{C(\rho)}(h_{q\beta}) - \sum_u k^u_{\alpha\beta}\eta_{C(\rho)}(h_{\gamma u})\bigg)$$
$$\overset{(4.2)}{=} \sum_{r,q,a,b} k^\gamma_{rq}\, \eta_{C(\rho)}(h_{ra})\eta_{C(\rho)}(h_{qb}) \otimes \eta_{C(\rho)}(h_{a\alpha})\eta_{C(\rho)}(h_{b\beta})$$
$$- \sum_{u,v} k^u_{\alpha\beta}\, \eta_{C(\rho)}(h_{\gamma v}) \otimes \eta_{C(\rho)}(h_{vu})$$

$$= \sum_{a,b}\Bigg(\underline{\sum_{r,q} k^{\gamma}_{rq}\eta_{C(\rho)}(h_{ra})\eta_{C(\rho)}(h_{qb}) - \sum_{v} k^{v}_{ab}\eta_{C(\rho)}(h_{\gamma v})}\Bigg) \otimes \eta_{C(\rho)}(h_{a\alpha})\eta_{C(\rho)}(h_{b\beta})$$

$$+ \sum_{v}\Bigg(\eta_{C(\rho)}(h_{\gamma v}) \otimes \Bigg(\underline{\sum_{a,b} k^{v}_{ab}\eta_{C(\rho)}(h_{a\alpha})\eta_{C(\rho)}(h_{b\beta}) - \sum_{u} k^{u}_{\alpha\beta}\eta_{C(\rho)}(h_{vu})}\Bigg)\Bigg)$$

and obviously the underlined terms belong to I_0, as desired. ∎

Now define I to be the ideal generated by spaces $S^n I_0$ for all $n \in \mathbb{N}$, where S is the antipode of $L(C(\rho))$. Obviously, I is a Hopf ideal and $H^{\rho} := L(C(\rho))/I$ is a Hopf algebra.

Denote by $\bar{\eta}\colon C(\rho) \to L(C(\rho))/I$ the map induced by $\eta_{C(\rho)}$ and define an H^{ρ}-comodule algebra structure $\varkappa^{\rho}$ on A by $\varkappa^{\rho}(a_{\alpha}) := \sum_{\beta} a_{\beta} \otimes \bar{\eta}(h_{\beta\alpha})$. The relations (4.5) ensure that A indeed becomes an H^{ρ}-comodule algebra.

Theorem 4.8. *The pair $(H^{\rho}, \varkappa^{\rho})$ is the initial object of the category $\mathcal{C}^H_A$.*

Proof. We first notice that the embedding $C(\rho) \hookrightarrow H$ induces a homomorphism of Hopf algebras $\varphi\colon L(C(\rho)) \to H$ such that the diagram

$$\begin{array}{ccc} C(\rho) & \xrightarrow{\eta_{C(\rho)}} & L\big(C(\rho)\big) \\ & \searrow & \downarrow{\scriptstyle \varphi} \\ & & H \end{array}$$

is commutative. Since A is an H-comodule algebra, the generators (4.5) of I belong to the kernel of φ and there exists a homomorphism of Hopf algebras $\bar{\varphi}\colon L(C(\rho))/I \to H$ such that the diagram

$$\begin{array}{ccc} C(\rho) & \xrightarrow{\bar{\eta}} & L\big(C(\rho)\big)/I \\ & \searrow & \downarrow{\scriptstyle \bar{\varphi}} \\ & & H \end{array}$$

is commutative. Hence the map $\bar{\eta}$ is injective and therefore $\bar{\eta}\colon C(\rho) \to \bar{\eta}(C(\rho))$ is a coalgebra isomorphism. Since $\bar{\eta}(C(\rho))$ is the support coalgebra of $\varkappa^{\rho}$, Proposition 4.4 implies that the structure of an H^{ρ}-comodule algebra on A defined by $\varkappa^{\rho}$ belongs to $\mathcal{C}^H_A$.

Suppose now that A is an H_1-comodule algebra for some other Hopf algebra H_1 and the corresponding comodule structure $\rho_1\colon A_1 \to A_1 \otimes H_1$ is equivalent to ρ. Then by Proposition 4.4 there exists an isomorphism $\tau_0\colon C(\rho) \xrightarrow{\sim} C(\rho_1)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & A \otimes C(\rho) \\ & {\scriptstyle \rho_1}\searrow & \downarrow{\scriptstyle \mathrm{id}_A \otimes \tau_0} \\ & & A \otimes C(\rho_1) \end{array}$$

Let i_1 be the embedding $C(\rho_1) \subseteq H_1$. Then $i_1\tau_0 = \tau_1\eta_{C(\rho)}$ for a unique homomorphism of Hopf algebras $\tau_1\colon L(C(\rho)) \to H_1$. Note that since A is an H_1-comodule algebra, again all generators (4.5) of the ideal I are in the kernel. Hence we get a homomorphism of Hopf algebras $\tau\colon L(C(\rho))/I \to H_1$ providing the desired arrow in $\mathcal{C}_A^H$. This arrow is unique since, by Takeuchi's construction of the functor L (see [38]), $L(C(\rho))/I$ is generated as an algebra by $\bar{\eta}(h_{\alpha\beta})$ and their images under S. ∎

We call the pair $(H^\rho, \varkappa^\rho)$ the *universal Hopf algebra* of ρ.

Remark 4.9. If $\rho\colon A \to A \otimes H$ is a right H-comodule algebra structure on A such that $C(\rho)$ is a pointed coalgebra, then the corresponding universal Hopf algebra of ρ is pointed as well. Indeed, it follows from [40, Proposition 3.3] that $L(C(\rho))$ is pointed and its coradical is given by $L(C(\rho))_0 = FT$, where $T = \mathcal{F}_{G(C(\rho))}$ is the free group generated by the set $G(C(\rho))$ of group-like elements of $C(\rho)$. Now since the canonical projection $\pi\colon L(C(\rho)) \to L(C(\rho))/I$ is a surjective coalgebra homomorphism, it follows from [12, Exercise 5.5.2] or [30, Corollary 5.3.5] that $L(C(\rho))/I$ is pointed and its coradical is $\pi(FT)$.

Theorem 4.10. *Suppose A is a unital H-comodule algebra. Then A is a unital H^ρ-comodule coalgebra too. As a consequence, unital comodule structures can be equivalent only to unital comodule structures.*

Proof. We can include 1_A into a basis, say, $a_1 := 1_A$. We have

$$h_{\alpha 1} = \begin{cases} 0 & \text{if } \alpha \neq 1, \\ 1_H & \text{if } \alpha = 1 \end{cases}$$

and

$$k^\beta_{\alpha 1} = k^\beta_{1\alpha} = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1_F & \text{if } \alpha = \beta \end{cases}$$

for all α, β. Moreover, $\varepsilon(h_{11}) = 1$ and $\Delta h_{11} = h_{11} \otimes h_{11}$. Now (4.5) for $\alpha = 1$ implies $\bar{\eta}(h_{11})\bar{\eta}(h_{\gamma\beta}) = \bar{\eta}(h_{\gamma\beta})$ and for $\beta = 1$ implies $\bar{\eta}(h_{\gamma\alpha})\bar{\eta}(h_{11}) = \bar{\eta}(h_{\gamma\alpha})$.

Note that

$$S\big(\bar{\eta}(h_{11})\big)\bar{\eta}(h_{11}) = \bar{\eta}(h_{11})S\big(\bar{\eta}(h_{11})\big) = 1_{H^\rho}$$

since $\Delta\bar{\eta}(h_{11}) = \bar{\eta}(h_{11}) \otimes \bar{\eta}(h_{11})$.

Using induction on k (the base $k = 0$ has been already proven), we get

$$\begin{aligned} S^k\big(\bar{\eta}(h_{\gamma\beta})\big)\bar{\eta}(h_{11}) &= S\big(\bar{\eta}(h_{11})S^{k-1}\big(\bar{\eta}(h_{\gamma\beta})\big)\big)\bar{\eta}(h_{11}) \\ &= S^k\big(\bar{\eta}(h_{\gamma\beta})\big)S\big(\bar{\eta}(h_{11})\big)\bar{\eta}(h_{11}) = S^k\big(\bar{\eta}(h_{\gamma\beta})\big). \end{aligned}$$

The equality $\bar{\eta}(h_{11})S^k(\bar{\eta}(h_{\gamma\beta})) = S^k(\bar{\eta}(h_{\gamma\beta}))$ is proven analogously. In other words, $\bar{\eta}(h_{11})$ is the identity element of H^ρ and A is a unital H^ρ-comodule algebra. ∎

Theorem 4.11 below shows that in the case of gradings the construction above yields the group algebra of the universal group of the corresponding grading.

Theorem 4.11. *Let $\Gamma\colon A = \bigoplus_{g\in G} A^{(g)}$ be a grading on an algebra A by a group G. Denote by $\rho\colon A \to A \otimes FG$ the corresponding comodule map. Let G_Γ be the universal group of Γ and let $\rho_\Gamma\colon A \to A \otimes FG_\Gamma$ be the corresponding comodule map. Then (FG_Γ, ρ_Γ) is the universal Hopf algebra of the comodule structure ρ.*

Proof. If H is the group algebra FG for some group G, i.e., A is G-graded, then we can choose $(a_\alpha)_\alpha$ to be a homogeneous basis in A. In this case, $h_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and each $h_{\alpha\alpha}$ is the group element corresponding to the homogeneous component of a_α. The generators (4.5) defining I now correspond to the relations defining the universal group of the grading. From Takeuchi's construction [38] it follows that $L(C(\rho))$ is the group algebra of the free group generated by the support of the grading and $H^\rho = L(C(\rho))/I$ is the group algebra of the universal group of the grading. ∎

Theorem 4.12. *If $\rho\colon A \to A \otimes H$ is an H-comodule structure equivalent to a group grading, then there exists a Hopf subalgebra $H_1 \subseteq H$, isomorphic to a group Hopf algebra, such that $\rho(A) \subseteq A \otimes H_1$.*

Proof. In order to deduce this from Theorem 4.11, it is sufficient to consider the homomorphic image H_1 of $L(C(\rho))/I$ and use the fact that any homomorphism of Hopf algebras maps group-like elements to group-like elements. ∎

Next we prove that taking the universal Hopf algebra of a comodule algebra yields a functor. To start with, given an algebra A, we define the category $\mathcal{C}_A$ as follows:

(1) the objects are pairs (H, ρ), where H is a Hopf algebra and $\rho\colon A \to A \otimes H$ is a right H-comodule algebra structure on A;

(2) the morphisms between two objects (H_1, ρ_1) and (H_2, ρ_2) are coalgebra homomorphisms $\tau\colon C(\rho_1) \to C(\rho_2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\ \rho_1\ } & A \otimes C(\rho_1) \\ & {\scriptstyle \rho_2}\searrow & \downarrow {\scriptstyle \mathrm{id}_A \otimes \tau} \\ & & A \otimes C(\rho_2) \end{array} \tag{4.6}$$

Theorem 4.13. *If $\rho_i\colon A \to A \otimes H_i$, $i = 1, 2$, are comodule structures on A and $\zeta_i\colon H_i^* \to \mathrm{End}_F(A)$ are the corresponding H_i^*-actions, then in $\mathcal{C}_A$ there exists at most one morphism from (H_1, ρ_1) to (H_2, ρ_2). Furthermore, this morphism exists if and only if $\zeta_2(H_2^*) \subseteq \zeta_1(H_1^*)$. In particular, $\mathcal{C}_A$ is a preorder (in the sense of* [28, Chapter I, Section 2]*)*[1].

Proof. As before, choose a basis $(a_\alpha)_\alpha$ in A and let $\rho_1(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$, where $h_{\beta\alpha} \in H_1$ and $\rho_2(a_\alpha) = \sum_\beta a_\beta \otimes h'_{\beta\alpha}$, where $h'_{\beta\alpha} \in H_2$.

The proof of Proposition 4.4 implies that if $\zeta_2(H_2^*) \subseteq \zeta_1(H_1^*)$, then if among $h_{\beta\alpha}$ there exists some linear dependence, the same linear dependence holds among $h'_{\beta\alpha}$. Therefore,

[1]Or, in another terminology, a thin category.

there exists a linear map $\tau: C(\rho_1) \to C(\rho_2)$ such that $\tau(h_{\beta\alpha}) = h'_{\beta\alpha}$, making (4.6) commutative. By (4.2) τ is a coalgebra homomorphism.

Conversely, suppose (4.6) is commutative for some coalgebra homomorphism

$$\tau: C(\rho_1) \to C(\rho_2).$$

Then $\tau(h_{\beta\alpha}) = h'_{\beta\alpha}$ for all α and β. Since $C(\rho_1)$ is the F-linear span of all $h_{\beta\alpha}$, such τ is unique and there exists at most one morphism from (H_1, ρ_1) to (H_2, ρ_2), which is always surjective. For every $f \in \zeta_2(H_2^*)$, there exists $\alpha \in C(\rho_2)^*$ such that we have

$$fa = (\mathrm{id}_A \otimes \alpha)\rho_2(a) \quad \text{for all } a \in A.$$

The commutativity of (4.6) implies $fa = (\mathrm{id}_A \otimes \tau^*(\alpha))\rho_1(a) = \tau^*(\alpha)a$, where $\tau^*(\alpha) \in C(\rho_1)^*$. In other words, $f \in \zeta_1(H_1^*)$ and $\zeta_2(H_2^*) \subseteq \zeta_1(H_1^*)$. ∎

If $\zeta_2(H_2^*) \subseteq \zeta_1(H_1^*)$, we say that ρ_1 is *finer* than ρ_2 and ρ_2 is *coarser* than ρ_1. Note that Theorem 3.8 implies that this definition agrees with the one for gradings. Again, id_A is an equivalence of ρ_1 and ρ_2 if and only if ρ_1 is both finer and coarser than ρ_2. Furthermore, the proof of Theorem 4.8 implies that the universal Hopf algebra of a comodule structure ρ is universal not only among the structures equivalent to ρ, but also among all the structures that are coarser than ρ.

We claim that any morphism $\tau: (H_1, \rho_1) \to (H_2, \rho_2)$ in $\mathcal{C}_A$ induces a Hopf algebra homomorphism between the corresponding universal Hopf algebras $L(C(\rho_1))/I_1$ and $L(C(\rho_2))/I_2$, respectively.

To this end, consider (a_α) to be a basis of A over F and let $C(\rho_1)$, $C(\rho_2)$ be the F-spans of all $h_{\alpha\beta}$, respectively $t_{\alpha\beta}$, where $\rho_1(a_\alpha) = \sum_\beta a_\beta \otimes h_{\beta\alpha}$ and $\rho_2(a_\alpha) = \sum_\beta a_\beta \otimes t_{\beta\alpha}$. Then, the coalgebra homomorphism $\tau: C(\rho_1) \to C(\rho_2)$ determines a unique Hopf algebra homomorphism $\varphi: L(C(\rho_1)) \to L(C(\rho_2))$ such that the following diagram is commutative:

$$\begin{array}{ccc} C(\rho_1) & \xrightarrow{\eta_{C(\rho_1)}} & L\big(C(\rho_1)\big) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\varphi} \\ C(\rho_2) & \xrightarrow[\eta_{C(\rho_2)}]{} & L\big(C(\rho_2)\big), \end{array} \qquad \text{i.e., } \varphi\eta_{C(\rho_1)} = \eta_{C(\rho_2)}\tau.$$

Let $\pi_1: L(C(\rho_1)) \to L(C(\rho_1))/I_1$ and $\pi_2: L(C(\rho_2)) \to L(C(\rho_2))/I_2$ be the canonical projections. Now notice that by the commutativity of (4.6), we obtain $\tau(h_{\beta\alpha}) = t_{\beta\alpha}$ for all α, β which together with the commutativity of the above diagram implies by a straightforward computation that all generators (4.5) of I_1 belong to the kernel of $\pi_2\varphi$. Therefore, there exists a unique Hopf algebra homomorphism

$$\bar{\tau}: L\big(C(\rho_1)\big)/I_1 \to L\big(C(\rho_2)\big)/I_2$$

such that the following diagram is commutative:

$$\begin{array}{ccc} L\big(C(\rho_1)\big) & \xrightarrow{\pi_2\varphi} & L\big(C(\rho_2)\big)/I_2, \\ {\scriptstyle \pi_1}\big\downarrow & \nearrow_{\overline{\tau}} & \\ L\big(C(\rho_1)\big)/I_1 & & \end{array} \qquad \text{i.e., } \overline{\tau}\pi_1 = \pi_2\varphi. \tag{4.7}$$

We have in fact defined a functor from $\mathcal{C}_A$ to the category of Hopf algebras in the following theorem.

Theorem 4.14. *There exists a functor* $F\colon \mathcal{C}_A \to \mathrm{Hopf}_F$ *given as follows:*

$$F(H,\rho) = L(C(\rho))/I \quad \textit{and} \quad F(\tau) = \overline{\tau}.$$

Proof. We only need to show that F respects composition of morphisms. Indeed, consider $\tau_1\colon (H,\rho) \to (H_1,\rho_1)$ and $\tau_2\colon (H_1,\rho_1) \to (H_2,\rho_2)$ two morphisms in $\mathcal{C}_A$. We obtain two unique Hopf algebra homomorphisms $\varphi_1\colon L(C(\rho)) \to L(C(\rho_1))$ and $\varphi_2\colon L(C(\rho_1)) \to L(C(\rho_2))$ such that

$$\varphi_1\eta_{C(\rho)} = \eta_{C(\rho_1)}\tau_1, \tag{4.8}$$

$$\varphi_2\eta_{C(\rho_1)} = \eta_{C(\rho_2)}\tau_2. \tag{4.9}$$

Furthermore, we have unique Hopf algebra homomorphisms

$$\overline{\tau_1}\colon L(C(\rho))/I \to L\big(C(\rho_1)\big)/I_1 \quad \text{and} \quad \overline{\tau_2}\colon L\big(C(\rho_1)\big)/I_1 \to L\big(C(\rho_2)\big)/I_2$$

such that

$$\overline{\tau_1}\pi = \pi_1\varphi_1, \tag{4.10}$$

$$\overline{\tau_2}\pi_1 = \pi_2\varphi_2. \tag{4.11}$$

Similarly, there exist two unique Hopf algebra homomorphisms $\psi\colon L(C(\rho)) \to L(C(\rho_2))$ and $\overline{\tau_2\tau_1}\colon L(C(\rho))/I \to L(C(\rho_2))/I_2$ such that

$$\psi\,\eta_{C(\rho)} = \eta_{C(\rho_2)}\tau_2\tau_1, \tag{4.12}$$

$$\overline{\tau_2\tau_1}\pi = \pi_2\psi. \tag{4.13}$$

The proof will be finished once we show that $\overline{\tau_2\tau_1} = \overline{\tau_2}\,\overline{\tau_1}$. First we prove that $\psi = \varphi_2\varphi_1$. Indeed, using (4.8) and (4.9) we obtain

$$\varphi_2\,\varphi_1\eta_{C(\rho)} = \varphi_2\eta_{C(\rho_1)}\tau_1 = \eta_{C(\rho_2)}\tau_2\tau_1.$$

As ψ is the unique Hopf algebra homomorphism for which (4.12) holds, we obtain $\psi = \varphi_2\varphi_1$. Furthermore, using (4.11) and (4.10) yields

$$\pi_2\varphi_2\varphi_1 = \overline{\tau_2}\pi_1\varphi_1 = \overline{\tau_2}\,\overline{\tau_1}\pi.$$

Finally, the uniqueness of the Hopf algebra homomorphism for which (4.13) holds implies $\overline{\tau_2\tau_1} = \overline{\tau_2}\,\overline{\tau_1}$, as desired. ■

Remark 4.15. At this point, we should recall that there is another construction in the literature related to the universal Hopf algebra of a comodule algebra, namely Manin's universal coacting Hopf algebra (see, for instance, [31, Proposition 1.3.8, Remark 2.6.4]). However, the latter construction is different from the one introduced in the present paper as, roughly speaking, it involves all possible coactions on a certain algebra not only those equivalent to a given one. More precisely, if A is a given algebra, it can be easily seen that the universal coacting Hopf algebra of A is precisely the initial object in the category whose objects are all comodule algebra structures on A and the morphisms between two such objects (H_1, ρ_1) and (H_2, ρ_2) are Hopf algebra homomorphisms $f : H_1 \to H_2$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\rho_1} & A \otimes H_1 \\ & {\scriptstyle\rho_2}\searrow & \Big\downarrow {\scriptstyle \mathrm{id}_A \otimes f} \\ & & A \otimes H_2 \end{array}$$

In other words, our construction can be considered as a refinement of Manin's as we get more information on each class of equivalent coactions. Furthermore, note that Manin's universal coacting Hopf algebra was shown to exist only for finite dimensional algebras while our construction can be performed for any arbitrary algebra.

4.3. H-comodule Hopf–Galois extensions

Let H be a Hopf algebra, let A be a nonzero unital H-comodule algebra, and let $\rho\colon A \to A \otimes H$ be its comodule map. Denote by $A^{\mathrm{co}\,H}$ the subalgebra of coinvariants, i.e., $A^{\mathrm{co}\,H} := \{a \in A \mid \rho(a) = a \otimes 1_H\}$. Recall that A is called a *Hopf–Galois extension* of $A^{\mathrm{co}\,H}$ if the linear map $\mathsf{can}\colon A \otimes_{A^{\mathrm{co}\,H}} A \to A \otimes H$ defined below is bijective:

$$\mathsf{can}(a \otimes b) := ab_{(0)} \otimes b_{(1)}.$$

Our next result computes the universal Hopf algebra of a Hopf–Galois extension.

Theorem 4.16. *Let $A/A^{\mathrm{co}\,H}$ be a Hopf–Galois extension. Then (H, ρ) is the universal Hopf algebra of ρ.*

Proof. Note that the surjectivity of can implies $C(\rho) = H$. Hence for every Hopf algebra H_1 and every H_1-comodule structure ρ_1 on A equivalent to ρ, there exists a unique coalgebra homomorphism $\tau\colon H \to H_1$ such that

$$\begin{array}{ccc} A & \xrightarrow{\rho} & A \otimes H \\ & {\scriptstyle\rho_1}\searrow & \Big\downarrow {\scriptstyle \mathrm{id}_A \otimes \tau} \\ & & A \otimes H_1 \end{array}$$

The only thing left to prove now is that τ is a Hopf algebra homomorphism.

First, since by Theorem 4.10 unital comodule structures can be equivalent only to unital comodule structures, $\rho_1(1_A) = 1_A \otimes 1_{H_1}$. At the same time, we have $\rho_1(1_A) = (\mathrm{id}_A \otimes \tau)\rho(1_A) = 1_A \otimes \tau(1_H)$. Hence $\tau(1_H) = 1_{H_1}$.

Second, for every $a, b \in A$ we have

$$a_{(0)}b_{(0)} \otimes \tau(a_{(1)}b_{(1)}) = \rho_1(ab) = \rho_1(a)\rho_1(b) = a_{(0)}b_{(0)} \otimes \tau(a_{(1)})\tau(b_{(1)}). \tag{4.14}$$

We claim that

$$ab_{(0)} \otimes \tau(hb_{(1)}) = ab_{(0)} \otimes \tau(h)\tau(b_{(1)}) \quad \text{for } a, b \in A,\ h \in H. \tag{4.15}$$

Choose a basis $(h_\alpha)_\alpha$ in H and fix some basis element h_β. The surjectivity of can implies that there exist $a_i, b_i \in A$ such that $a \otimes h_\beta = \sum_i a_i b_{i(0)} \otimes b_{i(1)}$. Note that $\rho(b_i) = \sum_\alpha b_{i\alpha} \otimes h_\alpha$ for some $b_{i\alpha} \in A$, where for each i only a finite number of $b_{i\alpha}$ is nonzero. Hence $a \otimes h_\beta = \sum_{i,\alpha} a_i b_{i\alpha} \otimes h_\alpha$ and

$$\sum_i a_i b_{i\alpha} = \begin{cases} a & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{4.16}$$

Thus

$$\begin{aligned} ab_{(0)} \otimes \tau(h_\beta)\tau(b_{(1)}) &= \sum_{i,\alpha} a_i b_{i\alpha} b_{(0)} \otimes \tau(h_\alpha)\tau(b_{(1)}) \\ &\overset{(4.14)}{=} \sum_{i,\alpha} a_i b_{i\alpha} b_{(0)} \otimes \tau(h_\alpha b_{(1)}) \overset{(4.16)}{=} ab_{(0)} \otimes \tau(h_\beta b_{(1)}). \end{aligned}$$

In other words, we have proved (4.15) in the case $h = h_\beta$. Since β was an arbitrary index and both sides of (4.15) are linear in h, (4.15) is proved for arbitrary h.

Fix now arbitrary $h \in H$ and some basis element h_γ. Again the surjectivity of can implies that there exists $c_i, d_i \in A$ such that $1_A \otimes h_\gamma = \sum_i c_i d_{i(0)} \otimes d_{i(1)}$. We can rewrite $\rho(d_i) = \sum_\alpha d_{i\alpha} \otimes h_\alpha$ for some $d_{i\alpha} \in A$. Then $1_A \otimes h_\gamma = \sum_{i,\alpha} c_i d_{i\alpha} \otimes h_\alpha$ and

$$\sum_i c_i d_{i\alpha} = \begin{cases} 1_A & \text{if } \alpha = \gamma, \\ 0 & \text{if } \alpha \neq \gamma. \end{cases} \tag{4.17}$$

Then

$$1_A \otimes \tau(h)\tau(h_\gamma) = \sum_{i,\alpha} c_i d_{i\alpha} \otimes \tau(h)\tau(h_\alpha) \overset{(4.15)}{=} \sum_{i,\alpha} c_i d_{i\alpha} \otimes \tau(hh_\alpha) \overset{(4.17)}{=} 1_A \otimes \tau(hh_\gamma).$$

Hence $\tau(hh_\gamma) = \tau(h)\tau(h_\gamma)$. Since γ was an arbitrary index, τ is a bialgebra homomorphism and therefore a Hopf algebra homomorphism. Thus (H, ρ) is the universal Hopf algebra of ρ. ■

Remark 4.17. In the proof of Theorem 4.16 we have used only the surjectivity of can.

If we consider the standard G-grading on the group algebra FG of a group G, then the universal group of this grading is isomorphic to G. In the case of a comodule structure we can obtain a similar result.

Corollary 4.18. *Let H be a Hopf algebra. Then the universal Hopf algebra of the H-comodule algebra structure on H defined by the comultiplication $\Delta\colon H \to H \otimes H$ is again (H, Δ).*

Proof. We have $H^{\operatorname{co} H} \cong F$ and H/F is a Hopf–Galois extension by [12, Examples 6.4.8 (1)]. The desired conclusion now follows from Theorem 4.16. ∎

Example 4.19. Let H be a Hopf algebra and let A be a unital H-module algebra. We denote by $A\#H$ the corresponding *smash product*, i.e., $A\#H = A \otimes H$ as a vector space with multiplication given as follows:

$$(a\#h)(b\#g) = a(h_{(1)}b)\#h_{(2)}g,$$

where we denote the element $a \otimes h \in A \otimes H$ by $a\#h$. Then, we have an H-comodule algebra structure on $A\#H$ given by

$$\rho\colon A\#H \to (A\#H) \otimes H, \quad \rho(a\#h) = a\#h_{(1)} \otimes h_{(2)}.$$

Then $(A\#H)^{\operatorname{co} H} \cong A$ and $A\#H/A$ is a Hopf–Galois extension by [12, Examples 6.4.8 (2)]. Now Theorem 4.16 implies that the universal Hopf algebra of ρ is $(H, \operatorname{id}_A \otimes \Delta)$.

5. Module structures on algebras

5.1. Support equivalence of module structures on algebras

Analogously, we can introduce the notion of equivalence of module structures on algebras.

Definition 5.1. Let A_i be H_i-module algebras for Hopf algebras H_i, $i = 1, 2$. We say that an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras is a *support equivalence of module algebra structures* on A_1 and A_2 if

$$\tilde{\varphi}\big(\zeta_1(H_1)\big) = \zeta_2(H_2), \tag{5.1}$$

where ζ_i is the module algebra structure on A_i and the isomorphism $\tilde{\varphi}\colon \operatorname{End}_F(A_1) \xrightarrow{\sim} \operatorname{End}_F(A_2)$ is defined by the conjugation by φ. In this case, we call module algebra structures on A_1 and A_2 *support equivalent via the isomorphism* φ and, as in the comodule algebra case, we will say just *equivalent* for short.

It is easy to see that each equivalence of module algebra structures maps H_1-submodules to H_2-submodules. In Lemma 6.11 below, we show that one can identify the corresponding relatively free H_1- and H_2-module algebras and that codimensions of polynomial H-identities for equivalent algebras coincide.

As in the previous sections, we can restrict our consideration to the case when $A_1 = A_2$ and φ is an identity map.

Let A be a left H-module algebra and $\zeta\colon H \to \operatorname{End}_F(A)$ the corresponding module map. Denote by $\tilde{x}$ the image of $x \in H$ in $H/\ker\zeta$.

Then the map $\hat{\zeta}\colon H/\ker\zeta \to \operatorname{End}_F(A)$, where

$$\hat{\zeta}(\tilde{x}) := \zeta(x)$$

for all $x \in H$, defines on A a structure of an $H/\ker\zeta$-module. Notice that $\hat{\zeta}$ is obviously injective and, moreover, we have $\zeta(H) = \hat{\zeta}(H/\ker\zeta)$. Proposition 5.2 is an analog of Proposition 4.4 for module algebras.

Proposition 5.2. *Let A_i be H_i-module algebras for Hopf algebras H_i, $i = 1, 2$. Then an isomorphism $\varphi\colon A_1 \xrightarrow{\sim} A_2$ of algebras is an equivalence of module algebra structures $\zeta_i\colon H_i \to \operatorname{End}_F(A_i)$, $i = 1, 2$, if and only if there exists an isomorphism $\lambda : H_1/\ker\zeta_1 \xrightarrow{\sim} H_2/\ker\zeta_2$ of algebras such that the following diagram is commutative:*

$$\begin{array}{ccc} H_1/\ker\zeta_1 & \xrightarrow{\hat{\zeta}_1} & \operatorname{End}_F(A_1) \\ \lambda\downarrow & & \downarrow\tilde{\varphi} \\ H_2/\ker\zeta_2 & \xrightarrow[\hat{\zeta}_2]{} & \operatorname{End}_F(A_2) \end{array} \tag{5.2}$$

Proof. Suppose there exists an isomorphism $\lambda : H_1/\ker\zeta_1 \xrightarrow{\sim} H_2/\ker\zeta_2$ of algebras such that diagram (5.2) is commutative. Hence, as λ is in particular surjective, we obtain

$$\tilde{\varphi}\big(\zeta_1(H_1)\big) = \tilde{\varphi}\big(\hat{\zeta}_1(H_1/\ker\zeta_1)\big) = \hat{\zeta}_2\big(\lambda(H_1/\ker\zeta_1)\big) = \hat{\zeta}_2(H_2/\ker\zeta_2) = \zeta_2(H_2),$$

as desired. Therefore, the module algebra structures on A_1 and A_2 are equivalent via the isomorphism φ.

Conversely, assume now that the isomorphism of algebras $\varphi\colon A_1 \xrightarrow{\sim} A_2$ is an equivalence of module algebra structures on A_1 and A_2. Then if we identify $H_1/\ker\zeta_1$ with $\zeta_1(H_1)$ and $H_2/\ker\zeta_2$ with $\zeta_2(H_2)$, we can take λ to be the restriction of $\tilde{\varphi}$ on $\zeta_1(H_1)$. In other words, $\lambda(\tilde{x}) = \tilde{y}_x$, where $y_x \in H_2$ such that $\tilde{\varphi}(\hat{\zeta}_1(\tilde{x})) = \hat{\zeta}_2(\tilde{y}_x)$. Then λ is a well-defined algebra isomorphism which makes diagram (5.2) commutative. ∎

Note that if an H-module algebra A is unital, then the identity element 1_A is a common eigenvector for all operators from H and it retains this property for all equivalent module algebra structures. The following proposition implies that if the original module algebra structure is unital, then all module algebra structures equivalent to it are unital too.

Proposition 5.3. *Let A be an H-module algebra for some Hopf algebra H. Suppose there exists the identity element $1_A \in A$ such that 1_A is a common eigenvector for all operators from H. Then A is a unital H-module algebra.*

Proof. Denote by $\lambda \in H^*$ the linear function such that $h1_A = \lambda(h)1_A$. Since A is an H-module algebra, we have $\lambda(h_1h_2) = \lambda(h_1)\lambda(h_2)$ and $\lambda(h) = \lambda(h_{(1)})\lambda(h_{(2)})$ for all

$h, h_1, h_2 \in H$. Moreover, $\lambda(1_H) = 1_F$. Hence

$$\begin{aligned}\lambda(h) &= \lambda(h_{(1)})\varepsilon(h_{(2)})\lambda(1_H) = \lambda(h_{(1)})\lambda(h_{(2)})\lambda(Sh_{(3)}) \\ &= \lambda(h_{(1)})\lambda(Sh_{(2)}) = \lambda\big(h_{(1)}(Sh_{(2)})\big) = \varepsilon(h)\lambda(1_H) = \varepsilon(h)\end{aligned}$$

and A is a unital H-module algebra. ■

It is well known that if H is a finite dimensional Hopf algebra, then H^* is a Hopf algebra too and the notions of an H-module and H^*-comodule algebras coincide [12, Proposition 6.2.4].

Proposition 5.4. *Let $\rho_1 : A \to A \otimes H_1$ and $\rho_2 : A \to A \otimes H_2$ be two comodule structures on an algebra A, where H_1 and H_2 are finite dimensional Hopf algebras. Let $\zeta_i : H_i^* \to$* $\mathrm{End}_F(A)$*, $i = 1, 2$, be the corresponding homomorphisms of algebras. Then ρ_1 and ρ_2 are equivalent comodule structures if and only if ζ_1 and ζ_2 are equivalent module structures.*

Proof. Follows directly from the definitions. ■

5.2. Universal Hopf algebra of a module algebra structure

Analogously to the case of comodule algebras, if A is an H-module algebra for a Hopf algebra H and $\zeta : H \to \mathrm{End}_F(A)$ is the corresponding algebra homomorphism, one can consider the category ${}_H\mathcal{C}_A$ where

(1) the objects are H_1-module algebra structures on the algebra A for arbitrary Hopf algebras H_1 over F such that $\zeta_1(H_1) = \zeta(H)$, where $\zeta_1 : H_1 \to \mathrm{End}_F(A)$, is the algebra homomorphism corresponding to the H_1-module algebra structure on A;

(2) the morphisms from an H_1-module algebra structure on A with the corresponding homomorphism ζ_1 to an H_2-module algebra structure with the corresponding homomorphism ζ_2 are all Hopf algebra homomorphisms $\tau : H_1 \to H_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{End}_F(A) & \xleftarrow{\ \zeta_1\ } & H_1 \\ & \nwarrow^{\zeta_2} & \downarrow{\scriptstyle \tau} \\ & & H_2 \end{array}$$

Recall that by $R : \mathrm{Alg}_F \to \mathrm{Hopf}_F$ we denote the right adjoint functor for the forgetful functor $U : \mathrm{Hopf}_F \to \mathrm{Alg}_F$. The counit of this adjunction is denoted by $\mu : UR \Rightarrow \mathrm{id}_{\mathrm{Alg}_F}$.

Suppose $\zeta_1 : H_1 \to \mathrm{End}_F(A)$ is a structure of an H_1-module algebra on A that is equivalent to ζ. Then $\zeta_1(H_1) = \zeta(H)$ and there exists a unique Hopf algebra homomorphism $\varphi_1 : H_1 \to R(\zeta(H))$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{\ \varphi_1\ } & R\big(\zeta(H)\big) \\ & \searrow_{\zeta_1} & \downarrow{\scriptstyle \tilde{\mu}} \\ & & \zeta(H) \end{array}$$

where we denote $\tilde{\mu} = \mu_{\zeta(H)}$.

Now consider a subalgebra H_ζ of $R(\zeta(H))$ generated by $\varphi_1(H_1)$ for all such structures ζ_1. Obviously, H_ζ is a Hopf algebra. Let $\psi_\zeta := \tilde{\mu}|_{H_\zeta}$.

Theorem 5.5. *The homomorphism ψ_ζ defines on A a structure of an H_ζ-module algebra which is the terminal object in ${}_H\mathcal{C}_A$.*

Proof. An arbitrary element h_0 of H_ζ can be presented as a linear combination of elements $\varphi_1(h_1)\cdots\varphi_n(h_n)$, where $n \in \mathbb{N}$, $h_i \in H_i$, H_i are Hopf algebras, $\varphi_i\colon H_i \to R(\zeta(H))$ are homomorphisms of Hopf algebras, and each homomorphism $\tilde{\mu}\varphi_i$ defines on A a structure of an H_i-module algebra equivalent to ζ. Hence

$$\begin{aligned} h_0(ab) &= \varphi_1(h_1)\cdots\varphi_n(h_n)(ab) = h_1\big(h_2(\cdots h_n(ab)\cdots)\big) \\ &= \big(h_{1(1)}h_{2(1)}\cdots h_{n(1)}a\big)\big(h_{1(2)}h_{2(2)}\cdots h_{n(2)}b\big) \\ &= \big(\varphi_1(h_{1(1)})\varphi_2(h_{2(1)})\cdots\varphi_n(h_{n(1)})a\big)\big(\varphi_1(h_{1(2)})\varphi_2(h_{2(2)})\cdots\varphi_n(h_{n(2)})b\big) \\ &= \big(\big(\varphi_1(h_1)\cdots\varphi_n(h_n)\big)_{(1)}a\big)\big(\big(\varphi_1(h_1)\cdots\varphi_n(h_n)\big)_{(2)}b\big) \quad \text{for all } a, b \in A. \end{aligned}$$

Note that the comultiplication in the middle is calculated each time in the corresponding Hopf algebra H_i. Also we have used that each φ_i is, in particular, a homomorphism of coalgebras. Therefore, A is an H_ζ-module algebra.

Now the choice of $H_\zeta \subseteq R(\zeta(H))$ implies that (H_ζ, ψ_ζ) is the terminal object of ${}_H\mathcal{C}_A$. ∎

We call (H_ζ, ψ_ζ) the *universal Hopf algebra* of ζ.

If ζ_1 and ζ_2 are two module structures on A and $\zeta_2(H_2) \subseteq \zeta_1(H_1)$, we say that ζ_1 is *finer* than ζ_2 and ζ_2 is *coarser* than ζ_1. Again, id_A is an equivalence of ζ_1 and ζ_2 if and only if ζ_1 is both finer and coarser than ζ_2. Note that we could define H_ζ as a subalgebra of $R(\zeta(H))$ generated by the images of all Hopf algebras whose action is coarser than ζ. Then the action of H_ζ would still be equivalent to ζ, but the proof of Theorem 5.5 would imply that H_ζ is universal not only among module structures equivalent to ζ, but also among all the module structures coarser than ζ. The uniqueness of H_ζ implies that the original H_ζ satisfies this property too, i.e., H_ζ is universal among all the module structures coarser than ζ.

As in the case of comodule algebra structures, we will see that taking the universal Hopf algebra of a module algebra yields a functor. Indeed, given an algebra A we define the category ${}_A\mathcal{C}$ as follows:

(1) the objects are pairs (H, ζ), where H is a Hopf algebra and $\zeta\colon H \to \mathrm{End}_F(A)$ is a left H-module algebra structure on A;

(2) the morphisms between two objects (H, ζ) and (H', ζ') are algebra homomorphisms $\lambda\colon \zeta(H) \to \zeta'(H')$ such that the following diagram is commutative:

$$\begin{array}{ccc} \zeta(H) & \xrightarrow{\hat{\zeta}} & \mathrm{End}_F(A) \\ {\scriptstyle\lambda}\downarrow & \nearrow_{\hat{\zeta}'} & \\ \zeta'(H') & & \end{array}$$

(Here $\hat{\zeta}$ and $\hat{\zeta}'$ are natural embeddings.)

Note that the existence of an arrow from (H,ζ) to (H',ζ') just means that ζ is coarser than ζ' and ${}_A\mathcal{C}$ is just the preorder of all module structures on A with respect to the relation "coarser/finer".

Moreover, any morphism $\lambda\colon \zeta(H)\to\zeta'(H')$ in ${}_A\mathcal{C}$ induces a unique Hopf algebra homomorphism $\overline{\lambda}\colon R(\zeta(H))\to R(\zeta'(H'))$ such that the following diagram commutes:

$$\begin{array}{ccc} R\big(\zeta(H)\big) & \xrightarrow{\mu_{\zeta(H)}} & \zeta(H) \\ {\scriptstyle\overline{\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ R\big(\zeta'(H')\big) & \xrightarrow[\mu_{\zeta'(H')}]{} & \zeta'(H') \end{array}$$

We claim that $\overline{\lambda}_{|H_\zeta}\subseteq H'_{\zeta'}$. Indeed, the existence of the arrow λ implies that ζ' is coarser than ζ. Hence, if an H_1-module structure ζ_1 is coarser than ζ, then ζ_1 is coarser than ζ' too and $\bar{\lambda}$ maps the image of H_1 in $R(\zeta(H))$ to $H'_{\zeta'}$.

We can now define the desired functor.

Theorem 5.6. *There exists a functor $G\colon {}_A\mathcal{C}\to \mathrm{Hopf}_F$ given as follows:*

$$G(H,\zeta)=H_\zeta \quad \textit{and} \quad G(\lambda)=\overline{\lambda}_{|H_\zeta}.$$

Proof. We only need to prove that G respects composition of morphisms. To this end, let (H,ζ), (H',ζ'), and (H'',ζ'') be objects in ${}_A\mathcal{C}$ and $\lambda_1:\zeta(H)\to\zeta'(H')$, $\lambda_2:\zeta'(H')\to\zeta''(H'')$ two morphisms in ${}_A\mathcal{C}$. Then $G(\lambda_1)=\overline{\lambda_1}_{|H_\zeta}$ and $G(\lambda_2)=\overline{\lambda_2}_{|H'_{\zeta'}}$, where $\overline{\lambda_1}$ and $\overline{\lambda_2}$ are the unique Hopf algebra homomorphisms such that the following diagrams commute:

$$\begin{array}{ccc} R\big(\zeta(H)\big) & \xrightarrow{\mu_{\zeta(H)}} & \zeta(H) \\ {\scriptstyle\overline{\lambda_1}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda_1} \\ R\big(\zeta'(H')\big) & \xrightarrow[\mu_{\zeta'(H')}]{} & \zeta'(H') \end{array} \qquad \begin{array}{ccc} R\big(\zeta'(H')\big) & \xrightarrow{\mu_{\zeta'(H')}} & \zeta'(H') \\ {\scriptstyle\overline{\lambda_2}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda_2} \\ R\big(\zeta''(H'')\big) & \xrightarrow[\mu_{\zeta''(H'')}]{} & \zeta''(H'') \end{array} \tag{5.3}$$

Moreover, $G(\lambda_2\lambda_1)=\overline{\lambda_2\lambda_1}_{|H_\zeta}$, where $\overline{\lambda_2\lambda_1}$ is the unique Hopf algebra homomorphism which makes the following diagram commutative:

$$\begin{array}{ccc} R\big(\zeta(H)\big) & \xrightarrow{\mu_{\zeta(H)}} & \zeta(H) \\ {\scriptstyle\overline{\lambda_2\lambda_1}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda_2\lambda_1} \\ R\big(\zeta''(H'')\big) & \xrightarrow[\mu_{\zeta''(H'')}]{} & \zeta''(H'') \end{array}$$

Using (5.3) one can easily check that $\overline{\lambda_2}\,\overline{\lambda_1}$ makes the above diagram commutative as well and therefore we obtain $G(\lambda_2\lambda_1)=G(\lambda_2)G(\lambda_1)$. This finishes the proof. ∎

Remark 5.7. Inspired by Manin's universal coacting Hopf algebra of an algebra, we can construct the universal acting Hopf algebra of an algebra. More precisely, any given algebra A can be endowed with an $M(A,A)$-module algebra structure θ, where $M(A,A)$ is the universal measuring bialgebra of A (see [37, Chapter VII]). By [11, Theorem 3.1], there exists a Hopf algebra $H_*(M(A,A))$ together with a bialgebra homomorphism

$$\beta\colon H_*\big(M(A,A)\big) \to M(A,A)$$

such that for any other bialgebra homomorphism $f\colon H \to M(A,A)$ from a Hopf algebra H to $M(A,A)$ there exists a unique Hopf algebra homomorphism $g\colon H \to H_*(M(A,A))$ which makes the following diagram commutative:

$$\begin{array}{ccc} H_*\big(M(A,A)\big) & \xrightarrow{\beta} & M(A,A) \\ {\scriptstyle g}\uparrow & \nearrow{\scriptstyle f} & \\ H & & \end{array}$$

Thus $(H_*(M(A,A)), \theta(\beta \otimes 1_A))$ is the terminal object in the category whose objects are all module algebra structures on A and the morphisms between two such objects (H_1,ψ_1) and (H_2,ψ_2) are Hopf algebra homomorphisms $f\colon H_1 \to H_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} H_1 \otimes A & \xrightarrow{\psi_1} & A \\ {\scriptstyle f\otimes 1_A}\downarrow & \nearrow{\scriptstyle \psi_2} & \\ H_2 \otimes A & & \end{array}$$

We call $(H_*(M(A,\ A)), \theta(\beta \otimes 1_A))$ the *universal acting Hopf algebra of the algebra A*. For further details as well as the construction of the (co)universal acting Hopf algebra of a coalgebra we refer to [3].

Theorem 5.8 below provides an analog of Corollary 4.18 for module structures.

Theorem 5.8. *Let H be a Hopf algebra. Denote by $\zeta\colon H \to \mathrm{End}_F(H^*)$ the homomorphism defined by $(\zeta(h)\lambda)(t) := \lambda(th)$ for all $h,t \in H$, $\lambda \in H^*$. Then ζ is a unital H-module structure on the algebra H^* and the universal Hopf algebra of ζ is again (H,ζ).*

Proof. Indeed, if $\lambda,\mu \in H^*$, we have

$$\begin{aligned} \big(\zeta(h)(\lambda\mu)\big)(t) &= (\lambda\mu)(th) = \lambda(t_{(1)}h_{(1)})\mu(t_{(2)}h_{(2)}) \\ &= (h_{(1)}\lambda)(t_{(1)})(h_{(2)}\mu)(t_{(2)}) \\ &= \big((h_{(1)}\lambda)(h_{(2)}\mu)\big)(t) \end{aligned}$$

for all $h,t \in H$ and $h\varepsilon = \varepsilon(h)\varepsilon$. Hence ζ is a unital H-module structure on the algebra H^*.

Consider an H_1-module structure $\zeta_1\colon H_1 \to \mathrm{End}_F(H^*)$ such that $\zeta_1(H_1) = \zeta(H)$. Then Proposition 5.3 implies that ζ_1 is a unital module structure. Note that ζ is an embedding. Hence if we restrict the codomain $\mathrm{End}_F(H^*)$ of ζ and ζ_1 to $\zeta_1(H_1) = \zeta(H)$, the map ζ becomes an isomorphism between H and $\zeta(H)$ and there exists exactly one unital homomorphism $\tau\colon H_1 \to H$ of algebras such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{End}_F(H^*) & \xleftarrow{\ \zeta_1\ } & H_1 \\ & {}^{\zeta}\nwarrow & \downarrow{\scriptstyle\tau} \\ & & H \end{array}$$

This map τ satisfies the following equality:

$$\big(\zeta_1(h)\lambda\big)(t) = \lambda\big(t\tau(h)\big) \tag{5.4}$$

for all $h \in H_1$, $t \in H$, and $\lambda \in H^*$. In order to show that τ is a homomorphism of Hopf algebras, it is enough to show that τ is a homomorphism of coalgebras. Substituting in (5.4) $t = 1_H$ and $\lambda = \varepsilon_H$, we get $\varepsilon_H(\tau(h)) = (\zeta_1(h)\varepsilon_H)(1_H) = \varepsilon_{H_1}(h)\varepsilon_H(1_H) = \varepsilon_{H_1}(h)$. Considering arbitrary $\lambda_1, \lambda_2 \in H^*$ and using (5.4) once again, we obtain

$$\begin{aligned} \lambda_1\big(\tau(h_{(1)})\big)\lambda_2\big(\tau(h_{(2)})\big) &= \big(\zeta_1(h_{(1)})\lambda_1\big)(1_H)\big(\zeta_1(h_{(2)})\lambda_2\big)(1_H) \\ &= \big(\zeta_1(h_{(1)})\lambda_1 \otimes \zeta_1(h_{(2)})\lambda_2\big)(1_H \otimes 1_H) \\ &= \big(\big(\zeta_1(h_{(1)})\lambda_1\big)\big(\zeta_1(h_{(2)})\lambda_2\big)\big)(1_H) \\ &= \big(\zeta_1(h)(\lambda_1\lambda_2)\big)(1_H) = (\lambda_1\lambda_2)\big(\tau(h)\big) \\ &= \lambda_1\big(\tau(h)_{(1)}\big)\lambda_2\big(\tau(h)_{(2)}\big). \end{aligned}$$

Since $\lambda_1, \lambda_2 \in H^*$ were arbitrary, we get $\tau(h_{(1)}) \otimes \tau(h_{(2)}) = \tau(h)_{(1)} \otimes \tau(h)_{(2)}$ for all $h \in H$ and τ is indeed a homomorphism of coalgebras. Therefore, (H, ζ) is the universal Hopf algebra of ζ. ■

5.3. *H*-module Hopf–Galois extensions

In the sequel, we prove a result analogous to Theorem 4.16 for H-module algebras. Let A be a nonzero unital H-module algebra for a Hopf algebra H and consider $\zeta\colon H \to \mathrm{End}_F(A)$ to be the corresponding H-action on A. Denote by A^H its subalgebra of invariants, i.e., $A^H := \{a \in A \mid ha = \varepsilon(h)a \text{ for all } h \in H\}$. The algebra A is a *Hopf–Galois extension* of A^H if the linear map $\mathsf{can}\colon A \otimes_{A^H} A \to \mathrm{Hom}_F(H, A)$, defined by

$$\mathsf{can}(a \otimes b)(h) := a(hb),$$

is injective and has a dense image in the *finite topology* (i.e., the compact-open topology on $\mathrm{Hom}_F(H, A)$ defined for the discrete topologies on H and A).

Theorem 5.9. *Let A/A^H be a Hopf–Galois extension. Then (H, ζ) is the universal Hopf algebra of ζ.*

Proof. The density of the image of can implies that for every $h \neq 0$ there exist $a, b \in A$ such that $a(hb) \neq 0$. In other words, $\ker \zeta = 0$. Hence for every Hopf algebra H_1 and every H_1-module structure ζ_1 on A equivalent to ζ via id_A, there exists a unique algebra homomorphism $\tau\colon H_1 \to H$ such that the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\ \zeta\ } & \mathrm{End}_F(A) \\ {\scriptstyle \tau}\uparrow & \nearrow_{\zeta_1} & \\ H_1 & & \end{array}$$

The only thing we need to prove now is that τ is a Hopf algebra homomorphism.

First, since A is a unital H-module algebra, by Proposition 5.3, A is a unital H_1-module algebra too. Hence $\varepsilon(h)1_A = h1_A = \tau(h)1_A = \varepsilon(\tau(h))1_A$ for all $h \in H_1$. Therefore, $\varepsilon(h) = \varepsilon(\tau(h))$.

Let $(h_\alpha)_\alpha$ be a basis in H. We claim that for every $\lambda_{\alpha\beta} \in F$ such that only a finite number of them is nonzero, the condition

$$\sum_{\alpha,\beta} \lambda_{\alpha\beta}(h_\alpha a)(h_\beta b) = 0 \quad \text{for all } a, b \in A \tag{5.5}$$

implies $\lambda_{\alpha\beta} = 0$ for all α, β.

Indeed, suppose (5.5) holds. Let Λ be a finite set of indices such that $\lambda_{\alpha\beta} = 0$ unless $\alpha, \beta \in \Lambda$. The density of the image of can implies that for every γ there exist $a_{\gamma i}, b_{\gamma i} \in A$ such that

$$\sum_i a_{\gamma i}(h_\alpha b_{\gamma i}) = \begin{cases} 1_A & \text{if } \alpha = \gamma, \\ 0 & \text{if } \alpha \neq \gamma \text{ and } \alpha \in \Lambda. \end{cases}$$

Then by (5.5) for every $b \in A$ and every γ we have

$$\sum_{\beta \in \Lambda} \lambda_{\gamma\beta} h_\beta b = \sum_{\substack{\alpha,\beta \in \Lambda, \\ i}} \lambda_{\alpha\beta} a_{\gamma i}(h_\alpha b_{\gamma i})(h_\beta b) = 0.$$

Now $\ker \zeta = 0$ implies $\sum_\beta \lambda_{\gamma\beta} h_\beta = 0$ and $\lambda_{\gamma\beta} = 0$ for all β, γ.

In virtue of (2.1), for every $h \in H_1$ and $a, b \in A$ we have

$$\begin{aligned} \big(\tau(h_{(1)})a\big)\big(\tau(h_{(2)})b\big) &= (h_{(1)}a)(h_{(2)}b) \\ &= h(ab) = \tau(h)(ab) \\ &= \big(\tau(h)_{(1)}a\big)\big(\tau(h)_{(2)}b\big). \end{aligned}$$

By (5.5) we obtain $\Delta(\tau(h)) = (\tau \otimes \tau)\Delta(h)$ for all $h \in H_1$ and τ is a Hopf algebra homomorphism. Thus (H, ζ) is indeed the universal Hopf algebra of ζ. ∎

Remark 5.10. In the proof of Theorem 5.9 we have used only the density of the image of can.

6. Applications

6.1. Rational actions of affine algebraic groups

Let G be an affine algebraic group over an algebraically closed field F and let $\mathcal{O}(G)$ be the algebra of regular functions on G. Then $\mathcal{O}(G)$ is a Hopf algebra, where the comultiplication Δ and the antipode S are induced by, respectively, the multiplication and taking the inverse element in G, and where the counit ε of $\mathcal{O}(G)$ is just the calculation of the value at 1_G. (See the details, e.g., in [1, Chapter 4].) Suppose G is acting *rationally* by automorphisms on a finite dimensional algebra A, i.e., for a given basis $a_1, \dots, a_n$ in A there exist $\omega_{ij} \in \mathcal{O}(G)$, where $1 \leqslant i, j \leqslant n$, such that $ga_j = \sum_{i=1}^{n} \omega_{ij}(g)a_i$ for all $1 \leqslant j \leqslant n$ and $g \in G$. This implies that A is an $\mathcal{O}(G)$-comodule algebra, where $\rho(a_j) := \sum_{i=1}^{n} a_i \otimes \omega_{ij}$ for $1 \leqslant j \leqslant n$. At the same time, A is an $\mathcal{O}(G)^\circ$-module algebra, where $\mathcal{O}(G)^\circ$ is the finite dual of $\mathcal{O}(G)$: $f^* a_j = \sum_{i=1}^{n} f^*(\omega_{ij})a_i$ for all $1 \leqslant j \leqslant n$ and $f^* \in \mathcal{O}(G)^\circ$. The Lie algebra $\mathfrak{g}$ of G is the subspace consisting of all *primitive* elements of $\mathcal{O}(G)^\circ$, i.e., $f^* \in \mathcal{O}(G)^\circ$ such that $\Delta(f^*) = f^* \otimes 1 + 1 \otimes f^*$, and the $\mathfrak{g}$-action on A by derivations is just the restriction of the $\mathcal{O}(G)^\circ$-action. At the same time, the group G itself can be identified with the group of *group-like* elements of $\mathcal{O}(G)^\circ$, i.e., $f^* \in \mathcal{O}(G)^\circ$ such that $\Delta(f^*) = f^* \otimes f^*$ and $f^* \neq 0$.

Hence three Hopf algebras are acting on A: $\mathcal{O}(G)^\circ$, FG, and $U(\mathfrak{g})$ (the universal enveloping algebra of the Lie algebra $\mathfrak{g}$). In Theorem 6.2 below, we prove that all three actions are equivalent in the case when G is connected. In order to show this, we need an auxiliary lemma, that is a generalization of [23, Lemma 3].

Lemma 6.1. *Let V be a comodule over a coalgebra C with a coaction map $\rho: V \to V \otimes C$. Suppose that there exists a finite dimensional subcoalgebra $D \subseteq C$ such that $\rho(V) \subseteq V \otimes D$ (e.g., V is finite dimensional itself). Let $\zeta: C^* \to \operatorname{End}_F(V)$ be the corresponding action of the algebra C^* on V defined by $c^*v := c^*(v_{(1)})v_{(0)}$ for $c^* \in C^*$ and $v \in V$. Let $A \subseteq C^*$ be a dense subalgebra, i.e., $A^\perp := \{c \in C \mid a(c) = 0$ for all $a \in A\} = 0$. Then $\zeta(A) = \zeta(C^*)$.*

Proof. It is sufficient to show that the restriction of both C^* and A to D coincides with D^*. The first one is obvious since every linear function on D can be extended to a linear function on the whole C. The second is proved as follows. The elements of A viewed as linear functions on D form a subspace W in D^*. If $W \neq D^*$, the finite dimensionality of D implies that there exists $d \in D$, $d \neq 0$, such that $w(d) = 0$ for all $w \in W$. As a consequence, $d \in A^\perp$ and we get a contradiction to the density of A in C^*. Therefore, the restriction of A to D coincides with D^*, and we have $\zeta(c^*) = \zeta(a)$ for any $a \in A$, $c^* \in C^*$ such that $c^*|_D = a|_D$ which finally implies $\zeta(A) = \zeta(C^*)$. ∎

Now we are ready to prove the theorem.

Theorem 6.2. *Let G be a connected affine algebraic group over an algebraically closed field F of a characteristic 0 acting rationally by automorphisms on a finite dimensional* *algebra A. Let $\mathfrak{g}$ be the Lie algebra of G. Then the corresponding FG-action, $U(\mathfrak{g})$-action, and $\mathcal{O}(G)^\circ$-action on A are equivalent.*

Proof. By Lemma 6.1, it is sufficient to prove that the images of FG, $U(\mathfrak{g})$, and $\mathcal{O}(G)^\circ$ are dense in $\mathcal{O}(G)^*$. For FG this follows from the definition of $\mathcal{O}(G)$. If we show that $U(\mathfrak{g})$ is dense in $\mathcal{O}(G)^*$, we will get automatically that $\mathcal{O}(G)^\circ$ is dense too, since $U(\mathfrak{g}) \subseteq \mathcal{O}(G)^\circ$.

Let $I := \ker \varepsilon$, the set of all polynomial functions from $\mathcal{O}(G)$ that take zero at 1_G. By [30, Proposition 9.2.5],

$$U(\mathfrak{g}) = \left\{a \in \mathcal{O}(G)^\circ \mid a(I^n) = 0 \text{ for some } n \in \mathbb{N}\right\} = \mathcal{O}(G)',$$

where $\mathcal{O}(G)'$ is the irreducible component of ε in $\mathcal{O}(G)^\circ$ (see [30, Definition 5.6.1]). By [26, Chapter II, Section 7.3] the connectedness of G implies that G is irreducible as a variety. Hence by a corollary of Krull's theorem [7, Corollary 10.18] we have $\bigcap_{n\geqslant 1} I^n = 0$ and [30, Proposition 9.2.10] implies that $U(\mathfrak{g}) = \mathcal{O}(G)'$ is dense in $\mathcal{O}(G)^*$. Hence $\mathcal{O}(G)^\circ$ is dense in $\mathcal{O}(G)^*$ too and the FG-action, the $U(\mathfrak{g})$-action, and the $\mathcal{O}(G)^\circ$-action on A are equivalent. ■

6.2. Actions of cocommutative Hopf algebras

If we restrict our consideration to actions of cocommutative Hopf algebras, we can define the notion of a universal cocommutative Hopf algebra of a given action, i.e., such a Hopf algebra whose action is universal among all actions of cocommutative Hopf algebras equivalent to a given one. Besides its own interest, the universal cocommutative Hopf algebra can help calculating the Hopf algebra H that is universal among equivalent actions of all Hopf algebras, not necessarily cocommutative ones, if one manages to prove that H is cocommutative too, as we do in Proposition 6.6 below.

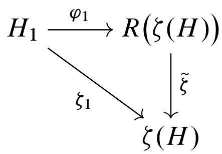
Analogously to the case of arbitrary Hopf algebras, if A is an H-module algebra for a cocommutative Hopf algebra H and $\zeta\colon H \to \mathrm{End}_F(A)$ is the corresponding algebra homomorphism, one can consider the category ${}_H\mathcal{C}_A^{\mathrm{coc}}$ where

(1) the objects are H_1-module algebra structures on the algebra A for cocommutative Hopf algebras H_1 over F such that $\zeta_1(H_1) = \zeta(H)$, where $\zeta_1\colon H_1 \to \mathrm{End}_F(A)$ is the algebra homomorphism corresponding to the H_1-module algebra structure on A;

(2) the morphisms from an H_1-module algebra structure on A with the corresponding homomorphism ζ_1 to an H_2-module algebra structure with the corresponding homomorphism ζ_2 are all Hopf algebra homomorphisms $\tau\colon H_1 \to H_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{End}_F(A) & \xleftarrow{\ \zeta_1\ } & H_1 \\ & \nwarrow^{\zeta_2} & \downarrow{\scriptstyle\tau} \\ & & H_2 \end{array}$$

Suppose $\zeta_1\colon H_1 \to \mathrm{End}_F(A)$ is a structure of an H_1-module algebra on A that is equivalent to ζ. Then $\zeta_1(H_1) = \zeta(H)$ and there exists a unique Hopf algebra homomorphism $\varphi_1\colon H_1 \to R(\zeta(H))$ such that the following diagram commutes:



where we denote $\tilde{\zeta} = \mu_{\zeta(H)}$.

Now consider a subalgebra H_ζ^{coc} of $R(\zeta(H))$ generated by $\varphi_1(H_1)$ for all such structures ζ_1, where H_1 is a cocommutative Hopf algebra. Obviously, H_ζ^{coc} is a Hopf algebra. Let $\psi_\zeta^{\mathrm{coc}} := \tilde{\zeta}|_{H_\zeta^{\mathrm{coc}}}$.

The same argument as in Theorem 5.5 shows that $(H_\zeta^{\mathrm{coc}}, \psi_\zeta^{\mathrm{coc}})$ is a terminal object in ${}_H\mathcal{C}_A^{\mathrm{coc}}$. We call this terminal object the *universal cocommutative Hopf algebra* of ζ.

In the case of an algebraically closed field of a characteristic 0, the Cartier–Gabriel–Kostant theorem (see, e.g., [30, Corollary 5.6.4 and Theorem 5.6.5]) allows us to give a concrete description of the universal cocommutative Hopf algebra, using a similar technique as in [24].

Theorem 6.3. *Let A be an H-module algebra for a cocommutative Hopf algebra H over an algebraically closed field F of a characteristic 0 and let $\zeta\colon H \to \mathrm{End}_F(A)$ be the corresponding algebra homomorphism. Let*

$$G_0 := \mathcal{U}\big(\zeta(H)\big) \cap \mathrm{Aut}(A) \quad \textit{and} \quad L_0 := \mathrm{Der}(A) \cap \zeta(H),$$

where $\mathcal{U}(\zeta(H))$ is the group of invertible elements of the algebra $\zeta(H)$ and $\mathrm{Der}(A)$ is the Lie algebra of derivations of A viewed as a subspace of $\mathrm{End}_F(A)$. Define H_0 to be the Hopf algebra which as a coalgebra is just $U(L_0) \otimes FG_0$, as an algebra is the smash product $U(L_0) \# FG_0$ (the G_0-action on L_0 is the standard one by conjugations) and the antipode is defined by

$$S(w \# g) := (1 \# g^{-1})(Sw \# 1)$$

for all $w \in U(L_0)$ and $g \in G_0$. Then (H_0, ζ_0) is the terminal object in ${}_H\mathcal{C}_A^{\mathrm{coc}}$. (Here ζ_0 is the H_0-action on A induced by the G- and L-actions on A.)

Proof. By the Cartier–Gabriel–Kostant theorem, $H \cong U(L) \# FG$, where G is the group of group-like elements of H and L is the Lie algebra of primitive elements of H. Since $\zeta(G) \subseteq G_0$, $\zeta(L) \subseteq L_0$, and G and L generate H as an algebra, G_0 and L_0 generate $\zeta(H)$ and we have $\zeta_0(H_0) = \zeta(H)$, i.e., ζ_0 is equivalent to ζ.

Let $\zeta_1\colon H_1 \to \mathrm{End}_F(A)$ be an action of another cocommutative Hopf algebra H_1 equivalent to ζ. Again, by the Cartier–Gabriel–Kostant theorem, $H_1 \cong U(L_1) \# FG_1$, where G_1 is the group of group-like elements of H_1 and L_1 is the Lie algebra of primitive elements of H_1. We have to show that there exists a unique Hopf algebra homomorphism $\tau\colon H_1 \to H_0$ such that $\zeta_0\tau = \zeta_1$.

It is not difficult to see (using, say, the techniques of [30, Proposition 5.5.3 (2)]) that the Lie algebra of primitive elements of $H_0 = U(L_0) \# FG_0$ coincides with $L_0 \otimes 1$, which we identify with L_0, while the group of group-like elements of H_0 coincides with $1 \otimes G_0$, which we identify with G_0. Since τ (if it indeed exists) maps group-like elements to group-like ones and primitive elements to primitive ones, we must have $\tau(G_1) \subseteq G_0$ and $\tau(L_1) \subseteq L_0$. Note that $\zeta_0|_{G_0} = \mathrm{id}_{G_0}$ and $\zeta_0|_{L_0} = \mathrm{id}_{L_0}$. Hence $\tau|_{G_1}$ and $\tau|_{L_1}$ are uniquely determined by $\zeta_1|_{G_1}: G_1 \to G_0$ and $\zeta_1|_{L_1}: L_1 \to L_0$. Since H_1 is generated as an algebra by G_1 and L_1, there exists at most one Hopf algebra homomorphism $\tau: H_1 \to H_0$ such that $\zeta_0 \tau = \zeta_1$. Now we define τ to be the Hopf algebra homomorphism induced by $\zeta_1|_{G_1}$ and $\zeta_1|_{L_1}$. ∎

Below we provide a criterion for universal cocommutative and universal Hopf algebras to coincide.

Theorem 6.4. *Let A be an H-module algebra for a cocommutative Hopf algebra H over an algebraically closed field F of a characteristic 0 and let $\zeta: H \to \mathrm{End}_F(A)$ be the corresponding algebra homomorphism. Then the universal cocommutative Hopf algebra $(H_\zeta^{\mathrm{coc}}, \psi_\zeta^{\mathrm{coc}})$ of ζ is its universal Hopf algebra if and only if every Hopf algebra action $\zeta_1: H_1 \to \mathrm{End}_F(A)$ equivalent to ζ factors through some cocommutative Hopf algebra H_2, i.e., there exist a Hopf algebra action $\zeta_2: H_2 \to \mathrm{End}_F(A)$ and a Hopf algebra homomorphism θ making the following diagram commutative:*

$$\begin{array}{ccc} H_1 & \xrightarrow{\zeta_1} & \mathrm{End}_F(A) \\ {\scriptstyle \theta}\downarrow & \nearrow{\scriptstyle \zeta_2} & \\ H_2 & & \end{array}$$

Proof. The "only if" part follows immediately from the definition of the universal Hopf algebra of an action.

Suppose that every Hopf algebra action equivalent to ζ factors through some cocommutative Hopf algebra H_2. Without loss of generality, we may assume that the corresponding homomorphism θ is surjective and therefore the H_2-action is equivalent to ζ. Since every such H_2 is cocommutative, this implies that every Hopf algebra action equivalent to ζ factors through $\psi_\zeta^{\mathrm{coc}}$. Consider now the universal Hopf algebra (H_ζ, ψ_ζ) of ζ. The universal property of ψ_ζ and the fact that ψ_ζ factors through $\psi_\zeta^{\mathrm{coc}}$ too imply that there exist Hopf algebra homomorphisms θ_1, θ_2 making the following diagram commutative:

$$\begin{array}{ccc} H_\zeta^{\mathrm{coc}} & \xrightarrow{\psi_\zeta^{\mathrm{coc}}} & \mathrm{End}_F(A) \\ {\scriptstyle \theta_2}\uparrow\downarrow{\scriptstyle \theta_1} & \nearrow{\scriptstyle \psi_\zeta} & \\ H_\zeta & & \end{array}$$

Now the uniqueness of the comparison maps in the definitions of both ψ_ξ and ψ_ξ^{coc} implies that $\theta_1\theta_2 = \text{id}_{H_\xi}$ and $\theta_2\theta_1 = \text{id}_{H_\xi^{\text{coc}}}$. In particular, we may identify (H_ξ, ψ_ξ) with $(H_\xi^{\text{coc}}, \psi_\xi^{\text{coc}})$. ∎

Now we give an example of an action of a cocommutative Hopf algebra with a non-cocommutative universal Hopf algebra. This will show that the universal cocommutative Hopf algebra does not always coincide with the universal Hopf algebra. In fact, it is sufficient to present an algebra A with an H-action equivalent to an action of a cocommutative Hopf algebra such that $(h_{(1)}a)(h_{(2)}b) \neq (h_{(2)}a)(h_{(1)}b)$ for some $h \in H$, $a, b \in A$. Then such an H-action can never factor through a cocommutative algebra. Let us present an example of this situation.

Example 6.5. Let F be a field, let $A = F1_A \oplus Fa \oplus Fb \oplus Fab$ be the F-algebra, where $a^2 = b^2 = ba = 0$, and let S_3 be the 3rd symmetric group. Consider the equivalent S_3- and the $\mathbb{Z}/4\mathbb{Z}$-gradings on A defined by

$$A^{(\text{id})} := A^{(\bar{0})} := F1_A, \quad A^{((12))} := A^{(\bar{1})} := Fa,$$
$$A^{((23))} := A^{(\bar{2})} := Fb, \quad A^{((123))} := A^{(\bar{3})} := Fab.$$

Since the gradings are equivalent, by Theorem 3.7 the corresponding $(FS_3)^*$- and $(F(\mathbb{Z}/4\mathbb{Z}))^*$-actions are equivalent too, while $(F(\mathbb{Z}/4\mathbb{Z}))^*$ is commutative cocommutative and $(FS_3)^*$ is commutative non-cocommutative. In view of the above, to prove that the universal Hopf algebra of these actions is not cocommutative, it suffices to show that there exists an element $h \in (FS_3)^*$ such that $(h_{(1)}a)(h_{(2)}b) \neq (h_{(2)}a)(h_{(1)}b)$. Recall that if G is a finite group and $(h_g)_{g\in G}$ is the basis of $(FG)^*$ dual to the basis $(g)_{g\in G}$ of FG, the comultiplication on $(FG)^*$ is given by $\Delta h_g = \sum_{st=g} h_s \otimes h_t$. Hence

$$\big((h_{(123)})_{(1)}a\big)\big((h_{(123)})_{(2)}b\big) = \sum_{\sigma\rho=(123)} (h_\sigma a)(h_\rho b) = ab \neq$$
$$\big((h_{(123)})_{(2)}a\big)\big((h_{(123)})_{(1)}b\big) = \sum_{\sigma\rho=(123)} (h_\rho a)(h_\sigma b) = 0.$$

Therefore, the dual base element $h_{(123)}$ in $(FS_3)^*$ satisfies the needed property and hence the universal Hopf algebra is not cocommutative.

Recall that Theorem 4.11 asserts that the universal Hopf algebra of a coaction that corresponds to a group grading is just the group algebra of the universal group of the grading. In Proposition 6.6 below, we show that the analogous result for group actions does not hold, i.e., the universal Hopf algebra of a group action is not necessarily a group algebra. In addition, here we demonstrate how the notion of the universal cocommutative Hopf algebra can be used to calculate the universal Hopf algebra.

Proposition 6.6. *Let $A = F[x]/(x^2)$, where F is an algebraically closed field,* char $F = 0$*. Consider G to be the cyclic group of order 2 with the generator c. Define a G-action on A by $c\bar{x} = -\bar{x}$. Then the universal Hopf algebra of the corresponding FG-action*

$\zeta_0 \colon FG \to \mathrm{End}_F(A)$ equals (H, ζ), where $H = F[y] \otimes FF^\times$ as an algebra and a coalgebra where the coalgebra structure on $F[y]$ is defined by $\Delta(y) = 1 \otimes y + y \otimes 1$ and $\varepsilon(y) = 0$. The antipode S of H and the action $\zeta \colon H \to \mathrm{End}_F(A)$ are defined by

$$S(y^k \otimes \lambda) = (-1)^k y^k \otimes \lambda^{-1} \quad \text{and} \quad \zeta(y^k \otimes \lambda)\bar{x} = \lambda$$

for $k \in \mathbb{Z}_+$ and $\lambda \in F^\times$. In particular, H is not a group algebra since it contains non-trivial primitive elements.

Proof. To this end, we first identify $\mathrm{End}_F(A)$ with the algebra $M_2(F)$ of 2×2 matrices by fixing the basis $\bar{1}, \bar{x}$ in A. Then $\zeta_0(1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\zeta_0(c) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since $\zeta_0(FG)$ is the linear span of $\zeta_0(1)$ and $\zeta_0(c)$, we get that $\zeta_0(FG)$ is the subalgebra of all diagonal matrices in $M_2(F)$. Moreover,

$$\mathrm{Aut}(A) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in F^\times \right\} \quad \text{and} \quad \mathrm{Der}(A) = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in F \right\}.$$

Hence $\mathcal{U}(\zeta_0(FG)) \cap \mathrm{Aut}(A) = \mathrm{Aut}(A) \cong F^\times$ and $\zeta_0(FG) \cap \mathrm{Der}(A)$ is a one-dimensional Lie algebra whose universal enveloping algebra is isomorphic to $F[y]$. Now Theorem 6.3 implies that the universal cocommutative Hopf algebra of ζ_0 is indeed (H, ζ).

By Theorem 6.4, in order to show that (H, ζ) is universal as a not necessarily cocommutative Hopf algebra, it is sufficient to show that any other Hopf algebra action $\zeta_1 \colon H_1 \to \mathrm{End}_F(A)$, equivalent to ζ_0, factors through a cocommutative Hopf algebra.

Suppose $\zeta_1 \colon H_1 \to \mathrm{End}_F(A)$ is an H_1-module structure on A equivalent to ζ_0. Then $\zeta_1(H_1)$ is the algebra of all diagonal 2×2 matrices. In particular, $\bar{x}$ is a common eigenvector of operators from H_1. Define $\varphi \in H_1^*$ by $h\bar{x} = \varphi(h)\bar{x}$ for all $h \in H_1$. Then $\varphi \colon H_1 \to F$ is a unital algebra homomorphism and therefore a group-like element of the Hopf algebra H°. At the same time Proposition 5.3 implies $h\bar{1} = \varepsilon(h)\bar{1}$ for all $h \in H_1$.

The powers φ^k, $k \in \mathbb{Z}$, of the element φ in the group $G(H^\circ)$ of the group-like elements of H° are defined by the formula

$$\varphi^k(h) = \begin{cases} \varphi(h_{(1)}) \dots \varphi(h_{(k)}) & \text{if } k \geqslant 1, \\ \varepsilon(h) & \text{if } k = 0, \\ \varphi(Sh_{(1)}) \dots \varphi(Sh_{(k)}) & \text{if } k \leqslant -1 \end{cases}$$

for every $h \in H_1$. Let

$$I := \bigcap_{k \in \mathbb{Z}} \ker(\varphi^k).$$

Since all $\varphi^k \colon H_1 \to F$ are unital algebra homomorphisms, I is an ideal. Moreover, as we include in the definition of I the negative powers of φ too, we have $SI \subseteq I$.

We claim that I is a coideal too, i.e., $\Delta(I) \subseteq H_1 \otimes I + I \otimes H_1$. We first notice that

$$H_1 \otimes I + I \otimes H_1 = \bigcap_{k, \ell \in \mathbb{Z}} \ker(\varphi^k \otimes \varphi^\ell). \tag{6.1}$$

The inclusion

$$H_1 \otimes I + I \otimes H_1 \subseteq \bigcap_{k,\ell \in \mathbb{Z}} \ker(\varphi^k \otimes \varphi^\ell)$$

is obvious. In order to prove the converse inclusion, we choose a basis $(a_\alpha)_{\alpha \in \Lambda_1 \cup \Lambda_2}$ in H_1 that contains a basis $(a_\alpha)_{\alpha \in \Lambda_1}$ of I. Suppose

$$w := \sum_{\alpha,\beta \in \Lambda_1 \cup \Lambda_2} \gamma_{\alpha\beta}\, a_\alpha \otimes a_\beta \in \bigcap_{k,\ell \in \mathbb{Z}} \ker(\varphi^k \otimes \varphi^\ell),$$

where only a finite number of coefficients $\gamma_{\alpha\beta} \in F$ are nonzero. From the definition of I we obtain

$$\sum_{\alpha,\beta \in \Lambda_1 \cup \Lambda_2} \gamma_{\alpha\beta} \varphi^k(a_\alpha) a_\beta \in I \quad \text{for every } k \in \mathbb{Z}.$$

The properties of a_α imply that for all $\beta \in \Lambda_2$

$$\sum_{\alpha \in \Lambda_1 \cup \Lambda_2} \gamma_{\alpha\beta} \varphi^k(a_\alpha) = 0,$$

i.e.,

$$\sum_{\alpha \in \Lambda_1 \cup \Lambda_2} \gamma_{\alpha\beta} a_\alpha \in I.$$

Hence $\gamma_{\alpha\beta} = 0$ for all $\alpha, \beta \in \Lambda_2$. Therefore, $w \in H_1 \otimes I + I \otimes H_1$ and (6.1) follows. Since for every $k, \ell \in \mathbb{Z}$ and $h \in I$ we have

$$(\varphi^k \otimes \varphi^\ell)(h_{(1)} \otimes h_{(2)}) = \varphi^{k+\ell}(h) = 0,$$

the S-invariant ideal I is a coideal and therefore a Hopf ideal.

Recall that $I \subseteq \ker \varepsilon \cap \ker \varphi$. Hence the homomorphism ζ_1 factors through the Hopf algebra H_1/I, i.e., the following diagram is commutative:

$$\begin{array}{ccc} H_1 & \xrightarrow{\ \pi\ } & H_1/I \\ & \searrow^{\zeta_1} & \downarrow \\ & & \mathrm{End}_F(A) \end{array}$$

where $\pi \colon H_1 \twoheadrightarrow H_1/I$ is the natural surjective homomorphism.

In order to show that H_1/I is cocommutative, it is sufficient to prove that

$$h_{(1)} \otimes h_{(2)} - h_{(2)} \otimes h_{(1)} \in H_1 \otimes I + I \otimes H_1 \quad \text{for every } h \in H_1.$$

By (6.1) it is sufficient to check that

$$(\varphi^k \otimes \varphi^\ell)(h_{(1)} \otimes h_{(2)} - h_{(2)} \otimes h_{(1)}) = 0.$$

Indeed,

$$(\varphi^k \otimes \varphi^\ell)(h_{(1)} \otimes h_{(2)} - h_{(2)} \otimes h_{(1)}) = \varphi^{k+\ell}(h) - \varphi^{k+\ell}(h) = 0.$$

Therefore, H_1/I is cocommutative, as desired. ∎

6.3. Unital module structures on $F[x]/(x^2)$

In this section, we classify all unital module structures on $F[x]/(x^2)$. Throughout, H_4 denotes the Sweedler Hopf algebra. Recall that H_4 is generated as an algebra by two elements c and v subject to the relations $c^2 = 1$, $v^2 = 0$, and $vc = -cv$, while the coalgebra structure and the antipode are given as follows:

$$\Delta(c) = c \otimes c, \quad \Delta(v) = c \otimes v + v \otimes 1, \quad S(c) = c, \quad S(v) = -cv.$$

Theorem 6.7. *Let $\zeta\colon H \to \operatorname{End}_F(A)$ be a unital H-module structure on $A = F[x]/(x^2)$, where H is a Hopf algebra,* char $F \neq 2$. *Then ζ is equivalent to one of the following module structures on A:*

(1) *the action of F on A by the multiplication by scalars;*

(2) *the FG-action, where $G = \langle c\rangle_2$, defined by $c\bar{x} = -\bar{x}$;*

(3) *the H_4-action defined by $c\bar{1} = \bar{1}$, $c\bar{x} = -\bar{x}$, $v\bar{1} = 0$, $v\bar{x} = \bar{1}$.*

Proof. Again, fix the basis $\bar{1}, \bar{x}$ and identify $\operatorname{End}_F(A)$ with $M_2(F)$. Since ζ is unital, there exist $\alpha, \beta \in H^*$ such that $\zeta(h) = \left(\begin{smallmatrix} \varepsilon(h) & \beta(h) \\ 0 & \alpha(h) \end{smallmatrix}\right)$ for every $h \in H$.

Note that (2.1) implies

$$\begin{aligned} 0 = h(\bar{x}^2) = (h_{(1)}\bar{x})(h_{(2)}\bar{x}) &= \big(\beta(h_{(1)})\bar{1} + \alpha(h_{(1)})\bar{x}\big)\big(\beta(h_{(2)})\bar{1} + \alpha(h_{(2)})\bar{x}\big) \\ &= \beta(h_{(1)})\beta(h_{(2)})\bar{1} + \big(\alpha(h_{(1)})\beta(h_{(2)}) + \beta(h_{(1)})\alpha(h_{(2)})\big)\bar{x} \end{aligned}$$

and

$$\beta(h_{(1)})\beta(h_{(2)}) = 0, \tag{6.2}$$

$$\alpha(h_{(1)})\beta(h_{(2)}) + \beta(h_{(1)})\alpha(h_{(2)}) = 0 \tag{6.3}$$

for all $h \in H$.

We have $\dim \zeta(H) = \dim\langle \alpha, \beta, \varepsilon\rangle_F$.

If $\dim \zeta(H) = 3$, then $\zeta(H)$ is the subalgebra of all upper triangular matrices and ζ is equivalent to (3).

Suppose $\dim \zeta(H) \leqslant 2$. If α and ε are linearly dependent, then $\varepsilon \neq 0$ implies $\alpha = \gamma\varepsilon$ for some $\gamma \in F$. Therefore, $1 = \alpha(1_H) = \gamma\varepsilon(1_H) = \gamma$ and $\alpha = \varepsilon$. By (6.3), $2\beta(h) = 0$ and $\beta = 0$. Then $\zeta(H)$ is the algebra of all scalar matrices and ζ is equivalent to (1).

Suppose $\dim \zeta(H) \leqslant 2$, but α and ε are linearly independent. Then $\beta = \lambda\varepsilon + \gamma\alpha$ for some $\lambda, \gamma \in F$. Applying both sides of this equality to 1_H, we get $0 = \lambda + \gamma$ since $\zeta(1_H) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, $\beta = \lambda(\varepsilon - \alpha)$ and (6.2) and (6.3) imply

$$\lambda^2\big(\varepsilon(h) - 2\alpha(h) + \alpha(h_{(1)})\alpha(h_{(2)})\big) = 0, \tag{6.4}$$

$$2\lambda\big(\alpha(h) - \alpha(h_{(1)})\alpha(h_{(2)})\big) = 0 \tag{6.5}$$

for all $h \in H$. Suppose $\lambda \neq 0$. Since char $F \neq 2$, (6.5) implies

$$\alpha(h_{(1)})\alpha(h_{(2)}) = \alpha(h).$$

Then from (6.4) we get

$$\alpha(h) = \varepsilon(h)$$

which contradicts with the linear independence of α and ε. Therefore, $\lambda = 0$ and $\beta = 0$. Thus $\zeta(H)$ is the subalgebra of all diagonal matrices and ζ is equivalent to (2). ■

6.4. Polynomial H-identities

In this section, we will show that the module algebras classified in the previous section all satisfy the analog of Amitsur's conjecture for polynomial H-identities.

We begin with a quick introduction into the theory of polynomial H-identities (see also [8, 10, 22]). All algebras in this section are associative, but not necessarily unital.

We denote by $F\langle X\rangle$ the *free non-unital associative algebra* on the countable set $X = \{x_1, x_2, x_3, \dots\}$, i.e., the algebra of all polynomials without a constant term in non-commuting variables $x_1, x_2, x_3, \dots$ with coefficients from F. Then

$$F\langle X\rangle = \bigoplus_{n=1}^{\infty} F\langle X\rangle^{(n)},$$

where $F\langle X\rangle^{(n)}$ is the linear span of monomials of total degree n.

Let H be a Hopf algebra. Then the *free non-unital associative H-module algebra* on X is

$$F\langle X|H\rangle := \bigoplus_{n=1}^{\infty} F\langle X\rangle^{(n)} \otimes H^{\otimes n},$$

where

$$h \cdot v \otimes h_1 \otimes \cdots \otimes h_n := v \otimes h_{(1)}h_1 \otimes \cdots \otimes h_{(n)}h_n \quad \text{for } v \in F\langle X\rangle^{(n)} \text{ and } h_i \in H$$

and $(v_1 \otimes w_1)(v_2 \otimes w_2) := v_1v_2 \otimes w_1 \otimes w_2$ for $v_1 \in F\langle X\rangle^{(k)}$, $v_2 \in F\langle X\rangle^{(\ell)}$, $w_1 \in H^{\otimes k}$, $w_2 \in H^{\otimes \ell}$, $k, \ell \in \mathbb{N}$. We use the notation $x_{i_1}^{h_1} \cdots x_{i_n}^{h_n} := x_{i_1} \cdots x_{i_n} \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_n$.

Let $(h_\alpha)_{\alpha\in\Lambda}$ be a basis in H. Then $F\langle X|H\rangle$ is isomorphic as an algebra to the free non-unital associative algebra on the set $\{x_n^{h_\alpha} \mid n \in \mathbb{N},\ \alpha \in \Lambda\}$.

Now we identify X with the subset $\{x_n^1 \mid n \in \mathbb{N}\} \subseteq F\langle X|H\rangle$. Denote by $\iota\colon X \to F\langle X|H\rangle$ the corresponding embedding. Then $F\langle X|H\rangle$ satisfies the following universal property: for any map $\varphi\colon X \to B$, where B is an H-module algebra, there exists a unique homomorphism of algebras and H-modules $\bar\varphi\colon F\langle X|H\rangle \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ \iota\ } & F\langle X|H\rangle \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle\bar\varphi} \\ & & B \end{array}$$

On the generators, $\bar\varphi$ is defined as follows:

$$\bar\varphi\big(x_{i_1}^{h_1} \cdots x_{i_n}^{h_n}\big) = \big(h_1\varphi(x_{i_1})\big) \cdots \big(h_n\varphi(x_{i_n})\big).$$

Remark 6.8. Note that in a similar way one can define the free H-module algebra on any set. Then $F\langle -|H\rangle$ becomes the left adjoint functor to the forgetful functor from the category of not necessarily unital associative H-module algebras to the category of sets.

The elements of $F\langle X|H\rangle$ are called *H-polynomials*. For a given H-module algebra A the intersection $\mathrm{Id}^H(A)$ of the kernels of all possible H-module algebra homomorphisms $F\langle X|H\rangle \to A$ is called the set of *polynomial H-identities*. Taking into account the universal property of $F\langle X|H\rangle$, it is easy to see that $\mathrm{Id}^H(A)$ consists of all H-polynomials that vanish under all substitutions of elements of A for their variables. In addition, $\mathrm{Id}^H(A)$ is an H-invariant ideal of $F\langle X|H\rangle$ invariant under all endomorphisms of $F\langle X|H\rangle$ as an H-module algebra.

The algebra $F\langle X|H\rangle/\mathrm{Id}^H(A)$ is called the *relatively free H-module algebra of the variety of H-module algebras generated by A*. Indeed, it is easy to see that $F\langle X|H\rangle/\mathrm{Id}^H(A)$ satisfies the same universal property as $F\langle X|H\rangle$ except that we consider only H-module algebras B satisfying $\mathrm{Id}^H(A) \subseteq \mathrm{Id}^H(B)$.

In some polynomial H-identities below we use other variables such as $x, y, \ldots$, always assuming that in fact these variables coincide with some of the variables $x_1, x_2, \ldots$.

Example 6.9. Consider the $\mathbb{Z}/2\mathbb{Z}$-grading on $M_2(F)$ defined by $M_2(F)^{(\bar{0})} = \left\{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\right\}$ and $M_2(F)^{(\bar{1})} = \left\{\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\right\}$ and the corresponding $(F(\mathbb{Z}/2\mathbb{Z}))^*$-action. Then

$$x^{h_0} y^{h_0} - y^{h_0} x^{h_0} \in \mathrm{Id}^{(F(\mathbb{Z}/2\mathbb{Z}))^*}\big(M_2(F)\big),$$

where $h_0 \in (F(\mathbb{Z}/2\mathbb{Z}))^*$ is defined by $h_0(\bar{0}) = 1$ and $h_0(\bar{1}) = 0$.

The classification of all varieties of algebras with respect to their ideals of polynomial identities seems to be a wild problem. This is the reason why numeric characteristics of polynomial identities are studied. One of the most important numeric characteristics is the codimension sequence.

Let

$$P_n^H := \left\langle x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \cdots x^{h_n}_{\sigma(n)} \mid \sigma \in S_n,\ h_i \in H \right\rangle_H \subset F\langle X|H\rangle,$$

where S_n is the nth symmetric group and $n \in \mathbb{N}$. The elements of P_n^H are called *multilinear H-polynomials* and the elements of $P_n^H \cap \mathrm{Id}^H(A)$ are called *multilinear polynomial H-identities* of A.

Using the linearization process [15, Section 1.3], it is not difficult to see that over a field of a characteristic 0 every polynomial H-identity of an H-module algebra A is equivalent to a finite set of multilinear polynomial H-identities of A. Therefore, the spaces $P_n^H \cap \mathrm{Id}^H(A)$, where $n \in \mathbb{N}$, contain all the information of polynomial H-identities of A. For a given H-module algebra A the number $c_n^H(A) := \dim\left(\frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)}\right)$, $n \in \mathbb{N}$, is called the *nth codimension of polynomial H-identities* of A.

In the case when $H = F$, we obtain ordinary polynomial identities and their codimensions. In the case when $H = (FG)^*$ for some finite group G, every H-module algebra A is just a G-graded algebra. Here one can introduce graded polynomial identities and their codimensions $c_n^{G-\mathrm{gr}}(A)$, however it turns out that $c_n^{G-\mathrm{gr}}(A) = c_n^{(FG)^*}(A)$ for every $n \in \mathbb{N}$ [17, Lemma 1].

In the 1980's, a conjecture concerning the asymptotic behavior of codimensions of ordinary polynomial identities was made by S. A. Amitsur [15, Conjecture 6.1.3]. This conjecture was proved in 1999 by A. Giambruno and M. V. Zaicev [15, Theorem 6.5.2]. For polynomial H-identities the analog of Amitsur's conjecture can be formulated in the following form, which belongs to Yu. A. Bahturin.

Conjecture 6.10. *Let A be a finite dimensional associative H-module algebra for a Hopf algebra H over a field of a characteristic* 0*. Then there exists an integer* $\mathrm{PIexp}^H(A) := \lim_{n\to\infty} \sqrt[n]{c_n^H(A)}$.

Conjecture 6.10 was proved in [17, Theorem 3] in case H is a finite dimensional semisimple Hopf algebra (this result was later generalized by Ya. Karasik [27] for the case when A is a not necessarily finite dimensional PI-algebra) and in [18, Theorem 1] for H-module algebras A such that the Jacobson radical $J(A)$ is an H-submodule (the requirement that $A/J(A)$ is a direct sum of H-simple algebras is satisfied by [35, Theorem 1.1], [36, Lemma 4.2]). In the case when $J(A)$ is not an H-submodule, the conjecture was proved only for Hopf algebras H that are iterated Ore extensions of finite dimensional semisimple Hopf algebras [22, Corollary 7.4].

The notion of equivalence of module structures reduces drastically the number of cases one has to consider in order to prove Conjecture 6.10.

Lemma 6.11. *Let $\zeta_1 \colon H_1 \to \mathrm{End}_F(A_1)$ and $\zeta_2 \colon H_2 \to \mathrm{End}_F(A_2)$ be equivalent module structures on algebras A_1 and A_2. Then there exists an algebra isomorphism*

$$F\langle X|H_1\rangle/\,\mathrm{Id}^{H_1}(A_1) \mathrel{\tilde{\to}} F\langle X|H_2\rangle/\,\mathrm{Id}^{H_2}(A_2)$$

that, for every $n \in \mathbb{N}$, restricts to an isomorphism

$$\frac{P_n^{H_1}}{P_n^{H_1} \cap \mathrm{Id}^{H_1}(A_1)} \mathrel{\tilde{\to}} \frac{P_n^{H_2}}{P_n^{H_2} \cap \mathrm{Id}^{H_2}(A_2)}.$$

In particular, $c_n^{H_1}(A_1) = c_n^{H_2}(A_2)$.

Proof. Let $\varphi \colon A_1 \mathrel{\tilde{\to}} A_2$ be an equivalence of H_1- and H_2-module structures and let $\tilde{\varphi} \colon \mathrm{End}_F(A_1) \mathrel{\tilde{\to}} \mathrm{End}_F(A_2)$ be the corresponding isomorphism of algebras of linear operators. We have $\tilde{\varphi}(\zeta_1(H_1)) = \zeta_2(H_2)$. Hence there exist F-linear maps $\xi \colon H_1 \to H_2$ and $\theta \colon H_2 \to H_1$ (which are not necessary homomorphisms) such that the following diagram commutes in both ways:

$$\begin{array}{ccc} H_1 & \underset{\theta}{\overset{\xi}{\rightleftarrows}} & H_2 \\ {\scriptstyle \zeta_1}\downarrow & & \downarrow{\scriptstyle \zeta_2} \\ \mathrm{End}_F(A_1) & \xrightarrow{\tilde{\varphi}} & \mathrm{End}_F(A_2) \end{array}$$

Then

$$\xi(h)\varphi(a) = \varphi(ha) \quad \text{for every } h \in H_1 \text{ and } a \in A_1 \tag{6.6}$$

and

$$\varphi\big(\theta(h)a\big) = h\varphi(a) \quad \text{for every } h \in H_2 \text{ and } a \in A_1. \tag{6.7}$$

Now we define the algebra homomorphisms

$$\tilde{\xi}\colon F\langle X \mid H_1\rangle \to F\langle X \mid H_2\rangle \quad \text{and} \quad \tilde{\theta}\colon F\langle X \mid H_2\rangle \to F\langle X \mid H_1\rangle$$

by

$$\tilde{\xi}(x_k^h) := x_k^{\xi(h)} \quad \text{for } h \in H_1,\ k \in \mathbb{N} \quad \text{and} \quad \tilde{\theta}(x_k^h) := x_k^{\theta(h)} \quad \text{for } h \in H_2,\ k \in \mathbb{N}.$$

Equations (6.6) and (6.7) imply that

$$\tilde{\xi}\big(\operatorname{Id}^{H_1}(A_1)\big) \subseteq \operatorname{Id}^{H_2}(A_2) \quad \text{and} \quad \tilde{\theta}\big(\operatorname{Id}^{H_2}(A_2)\big) \subseteq \operatorname{Id}^{H_1}(A_1).$$

Hence $\tilde{\xi}$ and $\tilde{\theta}$ induce homomorphisms $\bar{\xi}\colon F\langle X \mid H_1\rangle / \operatorname{Id}^{H_1}(A_1) \to F\langle X \mid H_2\rangle / \operatorname{Id}^{H_2}(A_2)$ and $\bar{\theta}\colon F\langle X \mid H_2\rangle / \operatorname{Id}^{H_2}(A_2) \to F\langle X \mid H_1\rangle / \operatorname{Id}^{H_1}(A_1)$. Note that (6.6) and (6.7) also imply $\theta(\xi(h))a = ha$ for all $a \in A_1$ and $h \in H_1$ and $\xi(\theta(h))a = ha$ for all $a \in A_2$ and $h \in H_2$. Hence $x^h - x^{\theta(\xi(h))} \in \operatorname{Id}^{H_1}(A_1)$ and $x^h - x^{\xi(\theta(h))} \in \operatorname{Id}^{H_2}(A_2)$. As a consequence,

$$\bar{\theta}\bar{\xi} = \operatorname{id}_{F\langle X \mid H_1\rangle / \operatorname{Id}^{H_1}(A_1)} \quad \text{and} \quad \bar{\xi}\bar{\theta} = \operatorname{id}_{F\langle X \mid H_2\rangle / \operatorname{Id}^{H_2}(A_2)},$$

i.e., we get the desired isomorphism.

Comparing degrees of H-polynomials, we get

$$\bar{\xi}\left(\frac{P_n^{H_1}}{P_n^{H_1} \cap \operatorname{Id}^{H_1}(A_1)}\right) \subseteq \frac{P_n^{H_2}}{P_n^{H_2} \cap \operatorname{Id}^{H_2}(A_2)}, \qquad \bar{\theta}\left(\frac{P_n^{H_2}}{P_n^{H_2} \cap \operatorname{Id}^{H_2}(A_2)}\right) \subseteq \frac{P_n^{H_1}}{P_n^{H_1} \cap \operatorname{Id}^{H_1}(A_1)}$$

for every $n \in \mathbb{N}$. ∎

Now we can prove Conjecture 6.10 for any unital Hopf algebra action on $F[x]/(x^2)$.

Theorem 6.12. *Suppose $F[x]/(x^2)$ is a unital H-module algebra for some Hopf algebra H over a field F of a characteristic 0. Denote by d the dimension of the maximal H-invariant nilpotent ideal in $F[x]/(x^2)$. Then there exist $C_1, C_2 > 0$ and $r_1, r_2 \in \mathbb{R}$ such that*

$$C_1 n^{r_1} (2-d)^n \leqslant c_n^H\big(F[x]/(x^2)\big) \leqslant C_2 n^{r_2} (2-d)^n \tag{6.8}$$

for all $n \in \mathbb{N}$.

In particular, the analog of Amitsur's conjecture holds for polynomial H-identities of $F[x]/(x^2)$.

Proof. Lemma 6.11 implies that it is sufficient to prove (6.8) for module structures described in Theorem 6.7. In the first two cases the Jacobson radical of $F[x]/(x^2)$, which equals $F\bar{x}$, is H-invariant, i.e., $d = 1$ and (6.8) follows from [18, Theorem 1]. In the last case $F[x]/(x^2)$ is an H_4-simple algebra (see [19, 20]), i.e., $d = 0$ and (6.8) follows from [22, Theorem 7.1, Corollary 7.4]. ∎

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