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Covering-monopole map and higher degree in non-commutative geometry

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Abstract. We analyze the monopole map over the universal covering space of a compact four-manifold. We induce the property of local properness of the covering-monopole map under the condition of closedness of the Atiyah–Hitchin–Singer (AHS) complex. In particular, we construct a higher degree of the covering-monopole map when the linearized equation is isomorphic. This induces a homomorphism between the K-groups of the group C^*-algebra. We apply a non-linear analysis on the covering space, which is related to L^p cohomology. We also obtain various Sobolev estimates on the covering spaces.

By applying the Singer conjecture on L^2 cohomology, we propose a conjecture of an aspherical version of the $\frac{10}{8}$-inequality. This is satisfied for a large class of four-manifolds, including some complex surfaces of general type.

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1. Introduction

There has been a significant development in gauge theory on the study of smooth structures in four dimensions. It is based on the construction of a moduli space that is given by a set of solutions to some non-linear elliptic partial differential equations modulo gauge symmetry. It has been revealed that these moduli spaces contain deep information on the topology of the underlying four-manifolds [11, 15].

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In relation to the Seiberg–Witten (SW) theory, Bauer and Furuta introduced a new invariant [4]. To explain this invariant, we use the analogy of the finite-dimensional case. Formally speaking, the SW moduli space is given by the zero set of a map between configuration spaces that are Hilbert manifolds. If the moduli space is zero-dimensional, then, by definition, its algebraic number is the SW invariant. It is fundamental in differential topology that, in a case when a map is given between finite-dimensional compact manifolds, such an algebraic number can be recovered from the degree of the map through K-theory. See [1] for the topological K-theory. One may say that Bauer–Furuta (BF) theory can be considered an infinite-dimensional version of this degree theory based on the concept of finite-dimensional approximation.

In this paper, we develop a covering version of the degree theory and study the monopole map over the universal covering space of a compact four-manifold. In particular, it is crucial to induce properness of the map in order to apply the framework of algebraic topology to the map.

Later, we explain the motivation of such a construction, but first, we state our main theorem. Let X be the universal covering space of a compact, oriented smooth four-manifold M.

Theorem 1.1. *Suppose the Atiyah–Hitchin–Singer (AHS) complex has closed range over the Sobolev spaces on X. Then, the covering-monopole map is locally strongly proper.*

We can apply a framework of algebraic topology constructed in [26].

Corollary 1.2. *Suppose the AHS complex has closed range as above. Assume, moreover, the following conditions:*

- *the Dirac operator over X is invertible;*
- *the second L^2 cohomology of the AHS complex vanishes.*

Then, the covering-monopole map gives a Γ-equivariant $*$-homomorphism

$$\tilde{\mu}^* : S\mathfrak{C}(H) \to S\mathfrak{C}_{\tilde{\mu}}(H')$$

between certain C^*-algebras, where $\Gamma := \pi_1(M)$ is the fundamental group of M.

In particular, the map induces a homomorphism on K-theory:

$$\tilde{\mu}^* : K^*\big(C^*(\Gamma)\big) \to K^*\big(S\mathfrak{C}_{\tilde{\mu}}(H') \rtimes \Gamma\big).$$

We shall present some examples of four-manifolds whose covering spaces satisfy these conditions with respect to their spin structures.

We first describe our motivation for introducing such a covering version of BF theory from some historical perspectives. Classical surgery theory has revealed that the fundamental group significantly impacts the smooth structure on a manifold. In high dimensions, a smooth structure is reduced to the algebraic topology of the group ring. Noncommutative geometry created a new framework that unifies the Atiyah–Singer index theorem with coefficients and surgery theory, passing through representation theory [7].

This topic led to the significant development of analysis over the universal covering space. Atiyah–Singer index theory has been extensively developed over non-compact manifolds. The construction by Gromov and Lawson is fundamental and revealed a deep relation to the non-existence of positive scalar curvature metrics [20].

The study of smooth structures is a core aspect of both fields of non-commutative geometry and gauge theory (see [24]). In gauge theory, the tangent space of a moduli space is given by the index bundle of the family of elliptic operators parametrized by the moduli space. Thus, the Atiyah–Singer index theorem is the fundamental object as the local model of the moduli theory. Hence, using Atiyah–Singer index theory, both fields led to important developments in differential topology.

It would be quite natural to try to combine both theories by introducing a systematic tool to analyze a smooth structure on a four-manifold from the perspective of the fundamental group, and to construct a gauge theory over non-compact four-manifolds in the framework of non-commutative geometry. This paper is the first step in tackling this project by using SW theory and BF theory. It aims to construct an infinite-dimensional degree theory in non-commutative geometry. This would also provide motivation to develop analysis of L^p cohomology theory, which appears naturally in a non-linear analysis over non-compact spaces.

For a better understanding of our construction, we describe its finite-dimensional version. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map. We denote the set of continuous functions vanishing at infinity as $C_0(X)$. Later, we will also use $C_c(X)$ to denote the set of continuous functions with compact support. It induces a K-theory map $\varphi_* : K_*(\mathbb{R}^n) \to K_*(\mathbb{R}^n)$ via the composition of functions $f \in C_0(\mathbb{R}^n)$ as $f \circ \varphi \in C_0(\mathbb{R}^n)$. This gives the degree map in a standard sense. If a discrete group Γ acts on $\mathbb{R}^n$ and φ is Γ-equivariant, then the following equivariant degree map is induced on the equivariant K-theory:

$$\varphi_* : K_*^{\Gamma}(\mathbb{R}^n) := K_*(\mathbb{R}^n \rtimes \Gamma) \to K_*(\mathbb{R}^n \rtimes \Gamma).$$

If Γ acts on $\mathbb{R}^n$ freely, then we have the induced map $\varphi_* : K_*(\mathbb{R}^n/\Gamma) \to K_*(\mathbb{R}^n/\Gamma)$ over the classifying space. The homotopy class of $\varphi : \mathbb{R}^n/\Gamma \to \mathbb{R}^n/\Gamma$ is determined by the induced group homomorphism $\varphi_* : \Gamma \to \Gamma$, where $\Gamma = \pi_1(\mathbb{R}^n/\Gamma)$. Note that a straightforward analogue of the degree in an infinite-dimensional case does not exist, because the infinite-dimensional unitary group is contractible.

Higson, Kasparov, and Trout constructed a C^*-algebra, which is a kind of an infinite-dimensional Clifford algebra, and induced an infinite-dimensional version of Bott periodicity between Hilbert spaces in K-theory [22]. In this paper, we combine the constructions of the BF degree theory with Higson–Kasparov–Trout Bott periodicity and introduce the K-theoretic degree of the covering-monopole map. Our main aim here is to construct a covering-monopole operator that is given by an equivariant $*$-homomorphism between two Clifford C^*-algebras, which we call the *higher degree* of the covering-monopole map. It induces a homomorphism between the equivariant K-groups.

To achieve this, we require some analytic conditions. The first is the closedness of the AHS complex which consists of a part of the linearized operator of the covering-monopole map. This type of property has been studied deeply with respect to L^2 cohomology theory, and we can find plenty of instances of four-manifolds whose covering spaces satisfy such a property [18]. In this paper, we construct the higher degree when the linearized map is isomorphic. We also present examples of four-manifolds satisfying this type of property. General cases will be considered in other papers. We also include some basic analysis on the covering-monopole map over general four-manifolds. Note that we do not assume isomorphism of the linearized map until Section 6.

1.1. Review of SW theory and BF theory

Let us recall the construction of the SW moduli space. Let M be an oriented closed four-manifold equipped with a spin^c structure, and let $S^\pm$ and L be the associated rank-2 Hermitian bundles and their determinant bundle, respectively. The Clifford multiplication $T^*M \times S^\pm \to S^\mp$ induces a linear map $\rho : \Lambda^2 \to \mathrm{End}_{\mathbb{C}}(S^+)$ whose kernel is the sub-bundle of anti-self-dual (ASD) 2-forms and the image is the sub-bundle of trace-free skew-Hermitian endomorphisms.

The configuration space for the SW map consists of the set of $U(1)$ connections over L and sections of positive spinors. The map associates as

$$F(A, \phi) := \big(D_A(\phi), F^+(A) - \sigma(\phi)\big),$$

where $F^+(A)$ is the self-dual part of the curvature of A, and A induces a connection over the spinor bundles, which gives the associated Dirac operator. Then, $\sigma(\phi)$ is given as the trace-free endomorphism

$$\phi \otimes \phi^* - \frac{1}{2}|\phi|^2 \mathrm{id},$$

which is regarded as a self-dual 2-form on M via ρ.

The gauge group acts on the configuration space, which is the set of automorphisms on the principal spin^c bundle that cover the identity on the frame bundle. It is given by a map from M to the center S^1 of $\mathrm{Spin}^c(4)$.

The SW map F is equivariant with respect to the $U(1)$ gauge group actions $\mathfrak{G}(L)$, and its moduli space is given by the total set of solutions divided by the gauge group action:

$$\mathfrak{M}(M) := \big\{(A, \phi) : F(A, \phi) = 0\big\} \big/ \mathfrak{G}(L).$$

Now recall a basic differential topology. Let M and N be two compact oriented manifolds, both with dimension n, and consider a smooth map $f : M \to N$. There are two ways to extract the degree of f. The first is to count the algebraic number of the inverse image of a generic point of f. The second is to use the multiplication number of the pull-back $f^* : H^n(N : \mathbb{Z}) \to H^n(M : \mathbb{Z})$. In general, both numbers coincide and the value is called the degree of f. Let us consider the case when the SW moduli space has zero dimension and apply the two different interpretations of the degree to the SW map. The SW invariant corresponds to the first way. The degree construction of the map by the algebro-topological method is the basic idea of BF theory, which corresponds to the second way.

One of the key differences from the finite-dimensional case is that the spaces are Sobolev spaces which are locally non-compact. Hence, more functional analytic ideas are required. Let us recall a part of the construction of BF theory, which is based on a rather abstract formalism of homotopy theory on infinite-dimensional spaces by Schwarz [32]. Let H', H be two separable Hilbert spaces and let $F = l + c : H' \to H$ be a Fredholm map between them such that the linearized map l is Fredholm and its non-linear part c is compact on each bounded set. More precisely, c maps a bounded set to a relatively compact subset in H. Then, the restrictions of F on "large" finite-dimensional linear subspaces $V' \subset H'$ composed with the projections to the image of l become "asymptotically proper" in some sense as follows:

$$\mathrm{pr} \circ F : V' \cap B_r \to V = l(V'),$$

where $B_r \subset H'$ is the open ball with radius r for sufficiently large $r > 1$. This gives a well-defined element in the stable cohomotopy group from F.

BF theory applies the above framework to the monopole map, which is a modified version of the SW map, since the SW map is not proper. The monopole map μ is defined for the quadruplet (A, ϕ, a, f), where A is a spin^c connection, ϕ is a positive spinor (section of S^+), and a and f are a 1-form and a locally constant function, respectively. Let Conn be the set of spin^c-connections. Then,

$$\begin{aligned} \mu : \mathrm{Conn} \times \big(\Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M)\big) \\ \to \mathrm{Conn} \times \big(\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)\big), \\ (A, \phi, a, f) \mapsto \big(A, D_{A+a}\phi, F^+_{A+a} - \sigma(\phi), d^*(a) + f, a_{\mathrm{harm}}\big), \end{aligned}$$

where a_{harm} is the harmonic projection of a. The map μ is equivariant with respect to the action by the gauge group $\mathfrak{G} = \mathrm{map}(M, \mathbf{T})$.

The subspace $A + \ker(d) \subset \mathrm{Conn}$ is invariant under the free action of the based gauge group $\mathfrak{G}_0 \subset \mathfrak{G}$, where the based gauge group consists of all automorphisms of the bundle whose values are the identity at a base point. Its quotient is isomorphic to the space of equivalent classes of flat connections $\mathrm{Pic}(M) = H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$. Let us consider the quotient spaces

$$\begin{aligned} \mathfrak{A} &:= \big(A + \ker(d)\big) \times_{\mathfrak{G}_0} \big(\Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M)\big), \\ \mathfrak{C} &:= \big(A + \ker(d)\big) \times_{\mathfrak{G}_0} \big(\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)\big). \end{aligned}$$

The monopole map descends to the fibered map

$$\mu : \mathfrak{A} \to \mathfrak{C}$$

over $\mathrm{Pic}(M)$. In general, a Hilbert bundle over a compact space admits trivialization such that the bundle isomorphisms $\mathfrak{A} \cong H' \times \mathrm{Pic}(M)$ and $\mathfrak{C} \cong H \times \mathrm{Pic}(M)$ hold by Kuiper's theorem. Let us consider the composition with the projection

$$\mathrm{pr} \circ \mu : \mathfrak{A} \to H.$$

Let $b^+(M)$ be the dimension of the space of self-dual harmonic 2-forms.

Theorem 1.3 ([4]). *Let M be a compact oriented smooth four-manifold. The monopole map over M defines an element in the stable cohomotopy group.*

If $b^+(M) \geq b^1(M) + 1$, then the group admits a natural homomorphism to the group of integers, and the image of the element coincides with the SW invariant.

In this paper, we use the Clifford C^*-algebras $S\mathfrak{C}(H)$. We now state a special case of our construction.

Proposition 1.4. *Let M be as above with $b^1(M) = 0$. Suppose the Fredholm index of l is zero. Then, μ induces a $*$-homomorphism:*

$$\mu^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H').$$

Moreover, the induced map

$$\mu^* : K\big(S\mathfrak{C}(H)\big) \cong \mathbb{Z} \to K\big(S\mathfrak{C}(H')\big) \cong \mathbb{Z}$$

is given by multiplication by the SW invariant.

Our aim is to extend the construction of the $*$-homomorphism over the universal covering space of a compact oriented smooth four-manifold equivariantly with respect to the fundamental group action.

1.2. Covering-monopole map

Let M be a compact oriented smooth Riemannian four-manifold, and let $X = \widetilde{M}$ be its universal covering space equipped with the lift of the metric. We denote the fundamental group $\pi_1(M)$ by Γ. Let us fix a spin^c structure on M. We assume that there exists a solution (A_0, ψ_0) to the SW equations over M. Then, we denote their respective lifts by $\widetilde{A}_0, \widetilde{\psi}_0$ over X. Note that both $\widetilde{A}_0$ and $\widetilde{\psi}_0$ cannot be in L^2 if they are non-zero. Furthermore, we have to choose a solution as a base point. Otherwise, any solution over the universal covering space cannot be in L^2. At this moment, it is not necessary to require (ir)reducibility of the base point.

In this paper, we shall introduce the covering-monopole map $\widetilde{\mu} = \widetilde{\mu}_{A_0,\psi_0}$ at the base (A_0, ψ_0) given by

$$\widetilde{\mu} : L^2_k(X; \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \to L^2_{k-1}\big(X; \widetilde{S}^- \oplus (\Lambda^2_+ \oplus \Lambda^0) \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X),$$

where $H^1_{(2)}(X)$ is the first L^2 cohomology group with respect to the induced metric.

In general, the de Rham differential does not have closed range between the Sobolev spaces over a non-compact manifold. This leads to two different cases of L^2 cohomology theory, reduced or unreduced ones. If the AHS complex has closed range over X, then these L^2 cohomology groups coincide and they are uniquely defined. Hereinafter, we assume the closedness of the AHS complex over X.

Let

$$\mathfrak{G}_{k+1}(\widetilde{L}) := \exp\big(L^2_{k+1}(X; i\mathbb{R})\big)$$

be the Sobolev gauge group. It is well known that this space admits a structure of a Hilbert manifold and is a group for the pointwise multiplication for $k \geq 2$ (see [30, p. 59]). Later on, we always assume $k \geq 2$ (see Section 2.3).

With respect to the gauge group action, we verify that the covering-monopole map admits the Γ-equivariant global slice

$$\tilde{\mu} : L^2_k(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^* \to L^2_{k-1}(X; \tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}) \oplus H^1_{(2)}(X).$$

The linearized operator of the covering-monopole map over X is Γ-equivariant at the base point, and Atiyah's Γ-index coincides with

$$\dim_\Gamma d\tilde{\mu} = \operatorname{ind} D - \chi_{\mathrm{AHS}}(M) - \dim_\Gamma H^1_{(2)}(X) = \operatorname{ind} D - \dim_\Gamma H^+_{(2)}(X),$$

where $\chi_{\mathrm{AHS}}(M) = b_0(M) - b_1(M) + b_2^+(M)$.

Remark 1.5. Let us consider the kernel of

$$d : L^2_{k+1}(X : \Lambda^1 \otimes i\mathbb{R}) \to L^2_k(X : \Lambda^2 \otimes i\mathbb{R})$$

as a closed linear subspace $\operatorname{Ker} d \subset L^2_{k+1}(X : \Lambda^1 \otimes i\mathbb{R})$ and set $\mathbf{A}_0 = \tilde{A}_0 + \operatorname{Ker} d$. A covering version of the BF formalism is the $\mathfrak{G}_{k+1}(L) \rtimes \Gamma$ equivariant monopole map

$$\begin{aligned}\tilde{\mu} : \mathbf{A}_0 &\times L^2_k(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \\ &\to \mathbf{A}_0 \times \left[L^2_{k-1}\big(X; \tilde{S}^- \oplus (\Lambda^0 \oplus \Lambda^2_+) \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X)\right].\end{aligned}$$

The quotient space by the gauge group is fibered over the first L^2 cohomology group:

$$\mathbf{A}_0 \times_{\mathfrak{G}_{k+1}(L)} L^2_k(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cong H^1_{(2)}(X) \times L^2_k(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R})$$

and the latter space is similar. By projecting to the fiber, we obtain the $\mathfrak{G}_{k+1}(L) \rtimes \Gamma$ equivariant monopole map

$$\tilde{\mu} : H^1_{(2)}(X) \times L^2_k(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \to L^2_{k-1}\big(X; \tilde{S}^- \oplus (\Lambda^0 \oplus \Lambda^2_+) \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X).$$

The Γ-index of the liberalized map is given by

$$\dim_\Gamma d\tilde{\mu} = \operatorname{ind} D - \chi_{\mathrm{AHS}}(M),$$

which is a topological invariant of the base manifold M. However, we encounter difficulty in analyzing this space, because it is not proper whenever $H^1_{(2)}(X)$ does not vanishes. We will not use this version of the map in the remainder of this paper.

Let $F = l + c : H' \to H$ be a smooth map between Hilbert spaces, where l is its linear part. We set $W = l(W') \subset H$ for a finite-dimensional linear subspace $W' \subset H'$. Consider $\mathrm{pr} \circ F : W' \to W$, which is the restriction of F composed with the orthogonal projection to W. Then, we obtain the induced homomorphism

$$(\mathrm{pr} \circ F)^* : C_0(W) \to C_0(W')$$

if it is proper. The basic idea is to regard this as an approximation of the original map $F : H' \to H$. When F is Fredholm, such a finite-dimensional restriction works effectively if we choose a sufficiently large dimension of W.

In our case of the covering-monopole map, we must construct an "induced map" between function spaces over infinite-dimensional linear spaces, by using a family of approximations as above. Our approach is to use the infinite-dimensional Clifford C^*-algebras $S\mathfrak{C}(H)$ [22], which are defined through a kind of limit of $C_0(W, \mathrm{Cl}(W))$ over all finite-dimensional linear subspaces $W \subset H$. To induce a $*$-homomorphism from $S\mathfrak{C}(H)$, we construct another C^*-algebra $S\mathfrak{C}_F(H')$. In the case of the monopole map over a compact four-manifold, $F^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H')$ is really constructed and $S\mathfrak{C}_F(H')$ is given by the image $F^*(S\mathfrak{C}(H))$.

Throughout this paper, a compact subset of a manifold refers to a compact submanifold of codimension zero possibly with a smooth boundary. Let $E \to X$ be a vector bundle and $H' := L^2_k(X; E)$ the Sobolev space with the open r-ball denoted by $B_r \subset H'$. For a compact subset $K \Subset X$, let $L^2_k(K; E)_0$ be the closure of $C^\infty_c(K; E)$ by the Sobolev L^2_k norm, where the latter is the set of smooth functions whose supports lie in the interior of K.

Definition 1.1. Let $F : H' \to H$ be a smooth map between Hilbert spaces. It is strongly proper if

(1) the preimage of a bounded set is contained in some bounded set, and

(2) the restriction of F on any ball B_r is proper.

Suppose the Hilbert spaces consist of Sobolev spaces over X. Then, the map F is locally strongly proper if it is strongly proper over the restriction on $L^2_k(K; E)_0$ for any compact subset $K \Subset X$.

An important case of strongly proper map is given by the monopole map between Sobolev spaces over a closed four-manifold. In our case, these two properties hold locally since the base space is non-compact.

Let

$$F = l + c : H' := L^2_k(X; E') \to H := L^2_{k-1}(X; E)$$

be a Γ-equivariant locally strongly proper map, where l is a first-order elliptic differential operator and c is pointwise and locally compact on each bounded set (see Section 5.3.1).

In variation (B) below Definition 5.1, we introduce adaptedness for some finite-dimensional approximation of a Sobolev space, which is a kind of compatibility condition with respect to an exhaustion of X by compact subsets.

In the case when Γ acts on X, we will introduce a weakly finite Γ-approximation in variation (A), which requires that the intersection of a weakly finite approximation with its Γ-translation also approximates the Sobolev space.

Proposition 1.6. *Suppose l is isomorphic. Then, there is an adapted family of finite-dimensional linear subspaces $\{W'_i\}_i$ which finitely Γ-approximates F.*

This is verified in Corollary 6.7, and it follows from Proposition 6.12 that F induces a Γ-equivariant $*$-homomorphism:

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H').$$

By applying the infinite-dimensional Bott periodicity by [22], F^* induces a homomorphism on the K-theory of the full group C^*-algebra:

$$F^* : K\big(C^*(\Gamma)\big) \to K\big(S\mathfrak{C}_F(H') \rtimes \Gamma\big).$$

This is a general framework. We obtain Corollary 1.2 by applying Proposition 1.6 to Theorem 1.1.

Remark 1.7. (1) Asymptotic morphism is a notion between C^*-algebras that is weaker than the usual $*$-homomorphism [8], but still induces a homomorphism in K-theory. One might expect that there is a way to construct a Γ-equivariant asymptotic morphism $\widetilde{\mu}^*$ from $S\mathfrak{C}(H)$ to $S\mathfrak{C}(H')$ over some classes of covering-monopole maps.

(2) So far, we have assumed the condition that the linearized operator gives an isomorphism. In general, non-zero kernel or co-kernel subspaces are both infinite-dimensional if the fundamental group is infinite. To eliminate this condition, we must use some method to stabilize these infinite-dimensional spaces, such as Kasparov's KK-theory for a general construction.

1.3. Higher $\frac{10}{8}$ conjecture

Furuta verified the following constraint on the topology of smooth four-manifolds [16]. In fact, a stronger estimate $b^2(M) \geq \frac{10}{8}|\sigma(M)| + 2$ is given there.

Theorem 1.8. *Let M be a compact smooth spin four-manifold. Then, the inequality*

$$b^2(M) \geq \frac{10}{8}\big|\sigma(M)\big|$$

holds, where $\sigma(M)$ is the signature of M.

The proof uses the type of finite-dimensional approximation described here, with representation-theoretic observation over the $\mathbb{H}$ and $\mathbb{R}$ as Pin_2 modules. More specifically, the monopole map is reduced to a $G := \mathrm{Pin}_2$-equivariant map $\mu' : V' \to V$ between finite-dimensional Pin_2 modules V' and V. Then, the induced map on the equivariant K_G-theory is computed as multiplication by the degree of the monopole map:

$$\alpha = 2^{b^+(M)+\frac{\sigma(M)}{8}-1}(1-c),$$

where c is a part of a generating set of the representation ring $R(G)$. Then, the above inequality follows from the positivity that $b^+(M) + \frac{\sigma(M)}{8} - 1 \geq 0$ with some elementary observation. Thus, computation of the induced map is a core part of the induction of the inequality.

We propose a higher version of the $\frac{10}{8}$-inequality.

Conjecture 1.9. *Suppose M is a compact aspherical smooth spin four-manifold. Then, the inequality*

$$\chi(M) \geq \frac{10}{8}\left|\sigma(M)\right|$$

holds, where $\chi(M)$ is the Euler characteristic of M.

Strategy. (1) We propose the covering version of the inequality. Let $X = \widetilde{M}$ be the universal covering space. Then, the covering $\frac{10}{8}$-inequality

$$b^2_\Gamma(X) \geq \frac{10}{8}\left|\sigma_\Gamma(X)\right| = \frac{10}{8}\left|\sigma(M)\right|$$

holds, where $b^2_\Gamma(X)$ is the second L^2 Betti number and

$$\sigma_\Gamma(X) = \dim_\Gamma H^+_{(2)}(X) - \dim_\Gamma H^-_{(2)}(X)$$

is the Γ-signature, which is equal to the signature of M by Atiyah's Γ-index theorem.

(2) The Singer conjecture states that if M is aspherical of even dimension $2m$, then the L^2 Betti numbers vanish except the middle dimension

$$b^i_\Gamma(X) = 0, \quad i \neq m.$$

The Singer conjecture is known to be true for Kähler hyperbolic manifolds according to Gromov [18]. This is a stronger version of the Hopf conjecture, which states that, under the same conditions, the non-negativity

$$(-1)^m \chi(M) \geq 0$$

holds. The Hopf conjecture is true for four-dimensional hyperbolic manifolds according to Chern [6]. This supports the Singer conjecture in four-dimensions.

Suppose the Singer conjecture is true for an aspherical four-manifold M. Then, we have the equalities

$$\chi(M) = \chi_\Gamma(X) = b^0_\Gamma(X) - b^1_\Gamma(X) + b^2_\Gamma(X) - b^3_\Gamma(X) + b^4_\Gamma(X) = b^2_\Gamma(X).$$

(3) We have the inequality in Conjecture 1.9 if we combine (1) and (2) above. To verify (1), we compute the "degree" of the covering-monopole map.

Let us check that, in the case of non-positive signature values, the above inequality $\chi(M) \geq \frac{10}{8}|\sigma(M)|$ follows from another inequality,

$$\chi_{\text{AHS}}(M) \geq -\frac{1}{8}\sigma(M),$$

where $\chi_{\text{AHS}}(M) = b^0(M) - b^1(M) + b^+(M)$. Assume that $\sigma(M) \leq 0$ is non-positive. Then, the equalities

$$\chi(M) + \frac{10}{8}\sigma(M)$$
$$= b^0(M) - b^1(M) + b^2(M) - b^3(M) + b^4(M) + b + -b^-(M) + \frac{1}{4}\sigma(M)$$

$$= 2\big(b^0(M) - b^1(M) + b^+(M)\big) + \frac{1}{4}\sigma(M)$$
$$= 2\chi_{\mathrm{AHS}}(M) + \frac{1}{4}\sigma(M)$$

hold, according to the Poincaré duality.

1.3.1. Concrete cases. So far, we have obtained various affirmative estimates that support the conjecture.

Lemma 1.10. (1) *If M is an ashperical surface bundle, then the inequality $\chi(M) \geq 2|\sigma(M)|$ holds* [28].

(2) *Suppose the intersection form of M is even, and its fundamental group is amenable or realized by π_1 of a closed hyperbolic manifold of* $\dim \geq 3$. *Then, the inequality $\chi(M) \geq \frac{10}{8}|\sigma(M)|$ holds* [5].

In (1), one can replace 2 with 3 if, moreover, M admits a complex structure. The proof is rather different from our approach. Note that M is aspherical if both the base and the fiber surfaces have their genus values ≥ 1.

Example 1.11. Atiyah constructed complex algebraic surfaces Z_g with non-zero signatures [2]. They admit the structure of fiber bundles whose base and fiber spaces are both Riemann surfaces of genus ≥ 3 and, hence, they are aspherical. It is well known that the total space of a fiber bundle is aspherical if the base and the fiber spaces are both aspherical. One can check this by using the standard homotopy exact sequence of the fibration.

Their signatures and Euler characteristics are, respectively, given by $\sigma(Z_g) = (g-1)2^{2g+1}$ and $\chi(Z_g) = (g-1)2^{2g+2}(2g+1)$. Then, surely the inequality

$$\chi(Z_g) = 2(2g+1)\big|\sigma(Z_g)\big| > \frac{10}{8}\big|\sigma(Z_g)\big|$$

holds.

For (2), Bohr developed a very interesting argument that replies on some group-theoretic properties. In Bohr's case, M is not necessarily assumed to be aspherical.

Lemma 1.12. *Suppose M is a complex surface of general type with $c_1^2 \geq 0$. Then, the inequality $\chi(M) \geq \frac{12}{8}|\sigma(M)|$ holds.*

The condition of $c_1^2 \geq 0$ holds if it is minimal, otherwise $c_1^2 = 3c_2$ holds (see [3]). The latter case is given by the unit ball in $\mathbb{C}^2$ divided by a discrete group action by Yau.

Proof. Recall the formulas $\sigma(M) = \frac{1}{3}(c_1^2 - 2c_2)$ with $\chi(M) = c_2(M)$. Moreover, positivity $c_2 > 0$ holds.

We consider two cases and suppose $\sigma > 0$ holds. Then, the strict inequality

$$\frac{10}{8}\big|\sigma(M)\big| = \frac{10}{8}\frac{1}{3}(c_1^2 - 2c_2) \leq \frac{10}{24}c_2 = \frac{10}{24}\chi(M)$$

holds according to the Miyaoka–Yau inequality $c_1^2 \leq 3c_2$.

Suppose $\sigma(M) \leq 0$ holds. Then,

$$\frac{10}{8}|\sigma(M)| = \frac{10}{8}\frac{1}{3}(2c_2 - c_1^2) \leq \frac{10}{12}c_2 = \frac{10}{12}\chi(M)$$

holds by non-negativity of the Chern number. ■

Proposition 1.13. *The covering $\frac{10}{8}$-inequality*

$$b_\Gamma^2(X) \geq \frac{10}{8}|\sigma(M)|$$

holds for spin four-manifolds with a residually finite fundamental group.

Proof. Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be the tower by normal subgroups of finite indices with $\bigcap_i \Gamma_i = 1$. Let M_i be Γ/Γ_i spin coverings of M with $\pi_1(M_i) = \Gamma_i$ and let $X = \tilde{M}$ be the universal covering of M with $\pi_1(M) = \Gamma$.

Note that any covering space of a spin manifold is also spin. By Furuta, the inequalities hold:

$$b_2(M_i) \geq \frac{10}{8}|\sigma(M_i)| + 2.$$

Denote $m_i = |\Gamma/\Gamma_i|$ and divide both sides by m_i as

$$\frac{b_2(M_i)}{m_i} \geq \frac{10}{8}\left|\frac{\sigma(M_i)}{m_i}\right| + \frac{2}{m_i}.$$

First, the equality $\frac{\sigma(M_i)}{m_i} = \sigma(M)$ holds because the signature is multiplicative under finite covering. It follows that

$$\lim_i \frac{b_2(M_i)}{m_i} = b_\Gamma^2(X)$$

converges to the L^2-Betti number (see [29, Theorem 13.49]). Then, the conclusion holds because $\frac{2}{m_i} \to 0$. (The argument was suggested by Yosuke Kubota.) ■

It is believed that most word hyperbolic groups are residually finite. For example, the fundamental groups of hyperbolic manifolds are residually finite.

Corollary 1.14. *Suppose a four-manifold M is aspherical and spin. Moreover, assume that $\pi_1(M)$ is residually finite and Kähler-hyperbolic.*

Then, the aspherical $\frac{10}{8}$ conjecture holds.

Example 1.15. (1) A Kähler manifold is Kähler hyperbolic if it is homotopy equivalent to a manifold which admits a metric of negative curvature.

A Kähler manifold M is Kähler hyperbolic if $\pi_1(M)$ is word-hyperbolic and $\pi_2(M) = 0$ holds. In particular, an aspherical Kähler manifold M is Kähler hyperbolic if $\pi_1(M)$ is word-hyperbolic [18] (see also [29]).

(2) An irreducible symmetric Hermitian space of non-compact type G/K is Kähler hyperbolic, where G is a connected non-compact simple adjoint Lie group and K is a

maximal connected and compact Lie subgroup of G with the center S^1 [34]. A complex manifold is Kähler hyperbolic if it is biholomorphic to a bounded symmetric domain in the complex plane [18].

(3) The product space of two Kähler hyperbolic manifolds is also Kähler hyperbolic.

In particular, a compact manifold is Kähler hyperbolic if its universal covering space is a symmetric Hermitian space of non-compact type. In fact, it is a product of irreducible symmetric Hermitian spaces of non-compact type.

(4) A finitely generated linear group Γ is residually finite. Let G/K be the case of (2). If $M := \Gamma\backslash G/K$ is a compact Kähler manifold, then M or product spaces of M are all Kähler hyperbolic such that $\pi_1(M)$ is residually finite, where $\Gamma \subset G$ is a co-compact discrete subgroup.

One of the approaches to attack the aspherical $\frac{10}{8}$-inequality is to seek for spin four-manifolds with two conditions of Kähler hyperbolicity and residual finiteness of the fundamental group. The aspherical $\frac{10}{8}$-inequality has been verified for such a class based on using a family of normal coverings of finite index. Our ultimate goal to is to eliminate the second condition of residual finiteness and to present a more straight method by developing the analysis on the universal covering spaces of compact spin four-manifolds.

Remark 1.16. It is known that an oriented and definite four-manifold must have a diagonal and, hence, odd-type intersection form [9, 10] (see also [4]). An aspherical four-manifold with a definite form is not expected to satisfy the inequality in Conjecture 1.9 if it exists. In fact, in that case, the inequality is given by

$$2(1-b_1)+b_2 \geq \frac{10}{8}b_2,$$

which is equivalent to $8(1-b_1) \geq b_2 \geq 0$. Then, we have a contradiction to the above inequality if $b_1 > 1$ holds. Thus, the $b_1 = 1$ (and hence $b_2 = 0$) and $b_1 = 0$ cases might survive. Both cases seem rare for aspherical four-manifolds.

2. Monopole map

In Section 2, we briefly review the SW and BF theories over compact four-manifolds. Then, we extend their constructions over universal covering spaces of compact four-manifolds. The L^2 cohomology of their fundamental groups plays an important role in their extensions.

The setting of Sobolev spaces over the covering space X is explained in Section 3.

2.1. Clifford algebras

Let V be a real four-dimensional Euclidean space and consider the $\mathbb{Z}_2$-graded Clifford algebra $\mathrm{Cl}(V) = \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$.

Let S be the unique complex four-dimensional irreducible representation of $\mathrm{Cl}(V)$. The complex involution is defined by

$$\omega_{\mathbb{C}} = -e_1 e_2 e_3 e_4,$$

where $\{e_i\}_i$ is any orthonormal basis. The involution decomposes S into its eigen bundles as $S = S^+ \oplus S^-$ and induces the eigenspace decomposition

$$\mathrm{Cl}_0(V) \otimes \mathbb{C} \cong \big(\mathrm{Cl}_0(V) \otimes \mathbb{C}\big)^+ \oplus \big(\mathrm{Cl}_0(V) \otimes \mathbb{C}\big)^- \tag{2.1}$$

via left multiplication. It turns out that the following isomorphisms hold:

$$\big(\mathrm{Cl}_0(V) \otimes \mathbb{C}\big)^{\pm} \cong \mathrm{End}_{\mathbb{C}}(S^{\pm}).$$

Passing through the vector space isomorphism $\mathrm{Cl}_0(V) \cong \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4$, the first component of the right-hand side of (2.1) corresponds as

$$\big(\mathrm{Cl}_0(V) \otimes \mathbb{C}\big)^+ \cong \mathbb{C}\left(\frac{1+\omega_{\mathbb{C}}}{2}\right) \oplus \big(\Lambda^2_+(V) \otimes \mathbb{C}\big),$$

where the self-dual form corresponds to the trace-free part. Then, for any vector $v \in S^+$, $v \otimes v^* \in \mathrm{End}(S^+)$ can be regarded as an element of a self-dual 2-form:

$$\sigma(v) := v \otimes v^* - \frac{|v|^2}{2}\mathrm{id} \in \Lambda^2_+(V) \otimes i\mathbb{R},$$

if its trace part is extracted.

2.2. Sobolev spaces

In this subsection, we include some basic materials on Sobolev spaces over a complete Riemannian manifold. Let $(X; g)$ be an n-dimensional complete Riemannian manifold of bounded geometry in the sense that the injectivity radius is uniformly positive at any point, and the C^k-norm of the curvature is uniformly bounded at any point for any $k \geq 0$. Then, $(X; g)$ admits a uniform local chart $\varphi_x : D \hookrightarrow X$ with $\varphi(0) = x$, where $D \subset \mathbb{R}^n$ is the unit disk. A family of smooth functions $\{a_i\}_i$ on D is called uniformly bounded if they admit uniformly bounded C^k-norms:

$$\sup_i \|a_i\|_{C^k(D)} < \infty$$

for any $k \geq 0$.

Let E be a Euclidean vector bundle on X and take a connection A_0 on E. One may assume that there is a trivialization

$$\psi_x : E|\varphi_x(D) \cong \varphi_x(D) \times \mathbb{R}^m$$

for each $x \in X$. One may further assume that ψ_x preserves their metrics, when we equip with the standard inner product on $\mathbb{R}^m$. Then, we can obtain $\psi_x : E|\varphi_x(D) \cong D \times \mathbb{R}^m$ via φ_x. By pulling back A_0 on S by using φ_x, it can be expressed as

$$\varphi_x^*(A_0) = d + a_x,$$

where a_x is a matrix-valued 1-form on D. One can choose A_0 such that the family $\{a_x\}_{x \in X}$ is uniformly bounded as above. We call such a connection also uniformly bounded. Let χ_i be a partition of unity subordinate to a covering $B_i := \varphi_{x_i}(D) \subset X$ for some $x_i \in X$. Then, we write

$$\nabla_{A_0}(s) := \sum_i \psi_i^{-1}\big(d\big(\psi_i(\chi_i s)\big)\big),$$

where $\psi_i := \psi_{x_i}$. Note that $d(\psi_i(\chi_i s)) \in \Omega_c^1(D, \mathbb{R}^m)$.

There are many other connections that are uniformly bounded. In fact, for any uniformly bounded element $a \in \Omega^1(X, \operatorname{End} E)$, the sum $A := A_0 + a$ is also uniformly bounded, where $\operatorname{End} E := P \times_{O(n)} \mathfrak{o}(n)$ and P is the frame bundle.

The Levi–Civita connection induces a connection ∇ on the tensor power $\otimes\Omega^1(X)$. Then, the pair (∇_{A_0}, ∇) gives a connection on $\Omega^*(X, \operatorname{End} E)$, which we also denote by ∇_{A_0}. One can operate ∇_{A_0} successively on a section $s \in \Omega^0(X, \operatorname{End} E)$ and then obtain

$$\nabla_{A_0}^l(s) \in \Omega^l(X, \operatorname{End} E).$$

Definition 2.1. The Sobolev space of E is given by completion of $\Omega_c^0(X, \operatorname{End} E)$ by the norm

$$\|s\|_{L_k^2}^2 := \sum_{i=0}^k |\nabla_{A_0}^i(s)|^2(x) \operatorname{vol}.$$

One can verify straightforwardly that the equivalent class of the norms is independent of the choice of A_0 and g in the sense that, for two choices of such pairs, the corresponding Sobolev norms $\| \ \|'$, $\| \ \|$ are equivalent as

$$c^{-1}\| \ \|' \leq \| \ \| \leq c\| \ \|'$$

for some constant $c > 0$. The equivalent class of the norm is independent if we use any uniformly bounded elliptic operators instead of a uniformly bounded connection, where the former means that the coefficients of the differential operator are uniformly bounded.

2.3. Monopole map over compact four-manifolds

Let M be an oriented compact Riemannian four-manifold equipped with a spin^c structure. Let $S^\pm$ and L be the Hermitian rank-2 bundles and the determinant bundle, respectively.

Let A_0 be a smooth $U(1)$ connection on L. With a Riemannian metric on M, A_0 induces a spin^c connection and the associated Dirac operator D_{A_0} on $S^\pm$. We set a large $k \geq 2$ and consider the configuration space

$$\mathfrak{D} = \big\{(A_0 + a, \psi) : a \in L_k^2(M; \Lambda^1 \otimes i\mathbb{R}), \psi \in L_k^2(M; S^+)\big\}.$$

Then, we have the SW map

$$F : \mathfrak{D} \to L^2_{k-1}(M; S^- \oplus \Lambda^2_+ \otimes i\mathbb{R})$$
$$(A_0 + a, \psi) \to \big(D_{A_0+a}(\psi), F^+_{A_0+a} - \sigma(\psi)\big).$$

Note that the space of connections is independent of the choice of A_0 as long as M is compact.

There is symmetry $\mathfrak{G}_* := L^2_{k+1}(M; S^1)_*$ that acts on $\mathfrak{D}$ by the group of based automorphisms with the identity at $* \in M$ on the spin^c bundle. The action of the gauge group g on the spinors is the standard one and on 1-form is given by

$$a \mapsto a + g^{-1}dg.$$

It is trivial for both 0 and self-dual 2-forms.

It follows that F is equivariant with respect to the gauge group action and, hence, the gauge group acts on the zero set

$$\tilde{\mathfrak{M}} = \big\{(A_0 + a, \psi) \in \mathfrak{D} : F(A_0 + a, \psi) = 0\big\}.$$

Moreover, the quotient space $\mathfrak{B} := \mathfrak{D}/\mathfrak{G}_*$ is Hausdorff.

Definition 2.2. The based SW moduli space is given by the quotient space

$$\mathfrak{M} = \tilde{\mathfrak{M}}/\mathfrak{G}_*.$$

Any connection $A_0 + a$ with $a \in L^2_k(M; \Lambda^1 \otimes i\mathbb{R})$ can be assumed to satisfy $\operatorname{Ker} d^*(a) = 0$ after gauge transform. Such a gauge group element is unique, since it is based. Therefore, locally constant functions cannot appear. The slice map is given by the restriction

$$SW : L^2_k(M; S^+) \oplus (A_0 + \operatorname{Ker} d^*) \to L^2_{k-1}(M; S^- \oplus \Lambda^2_+ \otimes i\mathbb{R})$$

whose zero set consists of the based moduli space equipped with a natural S^1 action,

$$\tilde{\mathfrak{M}}/\mathfrak{G}_* = SW^{-1}(0) \subset (A_0 + \operatorname{Ker} d^*) \oplus L^2_k(M; S^+).$$

We now list some of the remarkable properties of the SW moduli space.

(1) The infinitesimal model of the moduli space is given by the elliptic complex

$$0 \to L^2_{k+1}(M; i\mathbb{R}) \to L^2_k(M; \Lambda^1 \otimes i\mathbb{R} \oplus S^+)$$
$$\to L^2_{k-1}(M; \Lambda^2_+ \otimes i\mathbb{R} \oplus S^-) \to 0,$$

where the first map is given by $a \to (2da, -a\psi)$ and the second is given by

$$\begin{pmatrix} d^+ & -D\sigma(\psi) \\ \frac{1}{2}\psi & D_A \end{pmatrix}$$

at (A, ψ), where $D\sigma(\psi)$ is the differential of σ at ψ. Thus, the formal dimension is given by

$$\operatorname{ind} SW = -\chi_{\mathrm{AHS}} + \operatorname{ind} D_{A_0},$$

where the right-hand side is the sum of the negative Euler characteristic of the AHS complex and the index of the Dirac operator.

(2) The moduli space consists of only the trivial solution when M admits a positive scalar curvature.

(3) The moduli space is always compact or empty.

(4) A part of the linearized map

$$0 \to L^2_{k+1}(M) \to L^2_k(M; \Lambda^1) \to L^2_{k-1}(M; \Lambda^2_+) \to 0$$

is called the AHS complex. Let us compute the first cohomology group.

Lemma 2.1. *Let M be a closed four-manifold. The first cohomology of the AHS complex*

$$H^1(M) = \operatorname{Ker} d^+ / \operatorname{im} d$$

is isomorphic to the 1st de Rham cohomology.

Proof. There is a canonical linear map $H^1_{dR}(M) \to H^1(M)$.

Suppose $d^+(a) = 0$ holds. We verify that $d(a) = 0$ also holds. By the Stokes' theorem, we have the equalities

$$\begin{aligned} 0 = \int_M d(a) \wedge d(a) &= \int_M d^+(a) \wedge d^+(a) + \int_M d^-(a) \wedge d^-(a) \\ &= \int_M d^-(a) \wedge d^-(a) = -\|d^-(a)\|^2_{L^2}. \end{aligned}$$

Therefore, the inverse linear map $H^1(M) \to H^1_{dR}(M)$ exists. ■

The monopole map is given by

$$\begin{aligned} \mu : \mathrm{Conn} \times \big[\Gamma(S^+) \oplus \Omega^1(M) \otimes i\mathbb{R} \oplus H^0(M; i\mathbb{R})\big] \\ \to \mathrm{Conn} \times \big[\Gamma(S^-) \oplus \big(\Omega^+(M) \oplus \Omega^0(M)\big) \otimes i\mathbb{R} \oplus H^1(M; i\mathbb{R})\big] \\ (A, \phi, a, f) \mapsto \big(A, D_{A+a}\phi, F^+_{A+a} - \sigma(\phi), d^*(a) + f, a_{\mathrm{harm}}\big), \end{aligned}$$

where Conn is the space of spin^c-connections and μ is equivariant with respect to the gauge group action. The subspace $A + \ker(d) \subset \mathrm{Conn}$ is invariant under the free action of the based gauge group, and its quotient is isomorphic to the isomorphism classes of the flat bundles

$$\mathrm{Pic}(M) = H^1(M; \mathbb{R}) / H^1(M; \mathbb{Z}).$$

Now, we define the quotient spaces

$$\begin{aligned} \mathfrak{A} &:= \big(A + \ker(d)\big) \times_{\mathfrak{G}_*} \big[\Gamma(S^+) \oplus \Omega^1(M) \otimes i\mathbb{R} \oplus H^0(M; i\mathbb{R})\big], \\ \mathfrak{C} &:= \big(A + \ker(d)\big) \times_{\mathfrak{G}_*} \big[\Gamma(S^-) \oplus \big(\Omega^+(M) \oplus \Omega^0(M)\big) \otimes i\mathbb{R} \oplus H^1(M; i\mathbb{R})\big]. \end{aligned}$$

Then, the monopole map descends to the fibered map

$$\mu = \widetilde{\mu}/\mathfrak{G}_* : \mathfrak{A} \to \mathfrak{C}$$

over $\mathrm{Pic}(M)$. For a fixed k, consider the fiberwise L^2_k Sobolev completion and denote it as $\mathfrak{A}_k$. Similarly, we define $\mathfrak{C}_{k-1}$. Then, the monopole map extends to a smooth map $\mu : \mathfrak{A}_k \to \mathfrak{C}_{k-1}$ over $\mathrm{Pic}(M)$. It is a well-known fact that a Hilbert bundle over a compact space admits trivialization, which follows from a fact that the unitary group of an infinite-dimensional separable Hilbert space is contractible. Given a trivialization $\mathfrak{C}_{k-1} = \mathrm{Pic}(M) \times H_{k-1}$, one gets that the monopole map can be obtained by composition with the projection

$$\mu : \mathfrak{A}_k \to \mathfrak{C}_{k-1} = \mathrm{Pic}(M) \times H_{k-1} \to H_{k-1}.$$

There is a finite-dimensional reduction of a strongly proper map in our sense, which allows us to define *degree*. BF theory verifies that the monopole map $\mu : \mathfrak{A}_k \to H_{k-1}$ is strongly proper when the underlying four-manifold is compact. Thus, we can define a degree as an element in the S^1 equivariant stable co-homotopy group. There is a natural homomorphism from the S^1 equivariant stably co-homotopy group to integer. The image of the degree coincides with the SW invariant if $b^+ > b^1 + 1$ holds [4].

2.4. SW map on the universal covering space

Let (M, g) be a smooth and closed Riemannian four-manifold equipped with a spin^c structure. Denote its universal covering space and fundamental group by $X = \widetilde{M}$ and $\Gamma = \pi_1(M)$, respectively. We equip with the lift of the metric on X. The spinor bundle $S = S^+ \oplus S^-$ over M is also lifted as $\widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-$ over X, which are all Γ-invariant.

Later on, we will assume that a universal covering space X is non-compact (and hence its fundamental group is infinite).

Remark 2.2. Among various classes of non-compact manifolds, gauge theory has been intensively developed over cylindrical four-manifolds. For cylindrical four-manifolds, a standard analytical approach has been established, which works both for SW theory and Yang–Mills theory (for example, [33]). A striking analytic property of gauge theory over a cylindrical four-manifold is the exponential decay phenomenon for solutions under a mild condition on the slice three-manifold. It follows from this property that the L^2 norm of its curvature is integral in the case of an instanton. On the other hand, for more general classes of non-compact four-manifolds such as the hyperbolic four-plane, it is not so difficult to have an instanton whose L^2 norm of the curvature takes a non-integral value. As a result, it cannot satisfy the exponential decay estimate. It seems impossible to obtain a gauge theoretic method of analysis that can work for a general class of complete Riemannian non-compact four-manifolds.

Recall Sobolev spaces in Section 2.2. Let us fix a connection over M and lift it on X such that it is Γ-invariant. It is uniformly bounded on X in the sense defined in Section 2.2, so it can be used with the lifted metric to introduce Sobolev spaces $L^2_k(X)$. Their norms are also Γ-invariant. For a Euclidean bundle $E \to M$, we consider its lift $\widetilde{E} \to X$. Then, we obtain the Sobolev spaces $L^2_k(X, \widetilde{E})$ with the coefficient. These are also equipped with Γ-invariant metrics. Later on, we assume this property.

Let A_0 be a spin^c connection over M, and let (A_0, ψ_0) be a smooth solution to the SW equations such that the following equalities hold:

$$\begin{cases} D_{A_0}(\psi_0) = 0, \\ F^+_{A_0} - \sigma(\psi_0) = 0. \end{cases}$$

Later on, we will assume that there exists a solution (A_0, ψ_0) over M as above. This is, of course, a non-trivial condition.

Let us denote its lift by $(\widetilde{A}_0, \widetilde{\psi}_0)$ over X and put

$$\begin{cases} D_{\widetilde{A}_0, \widetilde{\psi}_0}(\psi, a) = D_{\widetilde{A}_0 + a}(\widetilde{\psi}_0 + \psi) - D_{\widetilde{A}_0}(\widetilde{\psi}_0) = a(\widetilde{\psi}_0 + \psi) + D_{\widetilde{A}_0}(\psi), \\ \sigma(\widetilde{\psi}_0, \psi) := \sigma(\widetilde{\psi}_0 + \psi) - \sigma(\widetilde{\psi}_0). \end{cases}$$

Lemma 2.3. *For $k \geq 1$, they extend to the continuous maps*

$$D_{\widetilde{A}_0, \widetilde{\psi}_0} : L^2_k\big((X, g); \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \to L^2_{k-1}\big((X, g); \widetilde{S}^-\big),$$
$$\sigma(\widetilde{\psi}_0, \ ) : L^2_k\big((X, g); \widetilde{S}^+\big) \to L^2_{k-1}\big((X, g); \Lambda^2_+ \otimes i\mathbb{R}\big).$$

If $k \geq 3$, then the second map defines the continuous map

$$\sigma(\widetilde{\psi}_0, \ ) : L^2_k\big((X, g); \widetilde{S}^+\big) \to L^2_k\big((X, g); \Lambda^2_+ \otimes i\mathbb{R}\big).$$

Proof. Note the equality

$$\sigma(\widetilde{\psi}_0, \psi) = \widetilde{\psi}_0 \otimes \psi^* + \psi \otimes \widetilde{\psi}_0^* - \langle \widetilde{\psi}_0, \psi \rangle \mathrm{id} + \sigma(\psi).$$

Since $\psi_0 \in C^\infty(M; S^+)$ is smooth, there is a constant C such that the following estimates hold:

$$\big\| \sigma(\widetilde{\psi}_0, \psi) \big\|_{L^2_{k-1}} \leq C \|\psi\|_{L^2_{k-1}} + \big\|\sigma(\psi)\big\|_{L^2_{k-1}}.$$

Let us consider the last term. Note the estimate

$$\big\|\nabla^l(\psi \otimes \psi^*)\big\|_{L^2} = \big\|\Sigma_{\alpha+\beta=l} \nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\big\|_{L^2} \leq \Sigma_{\alpha+\beta=l} \big\|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\big\|_{L^2}$$

with $l \leq k-1$. It follows from Lemma 3.2 (1) below that

$$\begin{aligned} \big\|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\big\|^2_{L^2(X)} &= \Sigma_{\gamma \in \Gamma} \big\|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\big\|^2_{L^2(\gamma(K))} \\ &\leq \Sigma_{\gamma \in \Gamma} \big\|\nabla^\alpha(\psi)\big\|^2_{L^4(\gamma(K))} \big\|\nabla^\beta(\psi^*)\big\|^2_{L^4(\gamma(K))} \\ &\leq C\, \Sigma_{\gamma \in \Gamma} \big\|\nabla^\alpha(\psi)\big\|^2_{L^2_1(\gamma(K))} \big\|\nabla^\beta(\psi^*)\big\|^2_{L^2_1(\gamma(K))} \\ &\leq C \|\psi\|^4_{L^2_k(X)}, \end{aligned}$$

where $K \subset X$ is a fundamental domain of the covering.

In particular, we obtain the estimate $\|\sigma(\psi)\|_{L^2_{k-1}} \leq C\|\psi\|^2_{L^2_k}$ and, hence, we obtain the estimate

$$\|\sigma(\tilde{\psi}_0, \psi)\|_{L^2_{k-1}} \leq C\big(\|\psi\|_{L^2_{k-1}} + \|\psi\|^2_{L^2_k}\big).$$

This implies that $\sigma(\tilde{\psi}_0, \ )$ is continuous from L^2_k to L^2_{k-1}.

The estimate for $D_{\tilde{A}_0,\tilde{\psi}_0}$ is obtained in the same way.

Now, suppose $k \geq 3$ and consider $\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)$ with $\alpha + \beta = l \leq k$. Suppose both α and β are less than or equal to $k-1$. Then, by the same argument as above, the L^2 norm is bounded by $C\|\psi\|^4_{L^2_k(X)}$. Next, suppose $\alpha = k \geq 3$ and hence $\beta = 0$. Then, there is a constant C with $\|\psi\|_{C^0(\gamma(K))} \leq C\|\psi\|_{L^2_k(\gamma(K))}$ by Lemma 3.2 (2) below. So we obtain the estimates

$$\begin{aligned}\|\nabla^k(\psi) \otimes \psi^*\|^2_{L^2(X)} &= \Sigma_{\gamma\in\Gamma}\|\nabla^k(\psi) \otimes \psi^*\|^2_{L^2(\gamma(K))} \\ &\leq \Sigma_{\gamma\in\Gamma}\|\psi\|^2_{L^2_k(\gamma(K))}\|\psi^*\|^2_{C^0(\gamma(K))} \\ &\leq C\|\psi\|^2_{L^2_k(\gamma(K))}\|\psi^*\|^2_{L^2_k(\gamma(K))} \leq C\|\psi\|^4_{L^2_k(X)}.\end{aligned}$$

∎

Later, we will choose a large $k \gg 1$, unless otherwise stated.

Definition 2.3. The covering-SW map at the base (A_0, ψ_0) is given by

$$F_{\tilde{A}_0,\tilde{\psi}_0} : L^2_k\big((X,g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \to L^2_{k-1}\big((X,g); \tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big)$$

$$(\psi, a) \mapsto \big(D_{\tilde{A}_0+a}(\tilde{\psi}_0+\psi), F^+_{\tilde{A}_0+a} - \sigma(\tilde{\psi}_0+\psi)\big) = \big(D_{\tilde{A}_0,\tilde{\psi}_0}(\psi, a), d^+(a) - \sigma(\tilde{\psi}_0, \psi)\big).$$

The fundamental group Γ acts equivariantly on the covering-SW map.

The gauge group $\mathfrak{G}_{k+1}$ over the spin^c bundle is defined by

$$\mathfrak{G}_{k+1} = \exp\big(L^2_{k+1}(X; i\mathbb{R})\big),$$

which is based at infinity, and admits the structure of a Hilbert commutative Lie group for $k \geq 3$ (see Corollary 3.3 (a) below). The action is given by

$$(A, \psi) \mapsto \big((\det g)^*(A), g^{-1}\psi\big)$$

for $g \in \mathfrak{G}_{k+1}$. Note the equality $\det\sigma = \sigma^2$ for $\sigma : X \to S^1 \subset \mathbb{C}$. These two groups are combined into $\mathfrak{G}_{k+1} \rtimes \Gamma$, and the covering-SW map is equivariant with respect to this new group. Here, the homomorphism $\Gamma \to \mathrm{Aut}(\mathfrak{G}_{k+1})$ is given by the pull-back via deck transformations.

Lemma 2.4. *The gauge group acts equivariantly on the covering-SW map at the base (A_0, ψ_0).*

Proof. We must verify that a gauge-transformed connection has to be in $\tilde{A}_0 + L^2_k$. Let $g = \exp(f)$ with $f \in L^2_{k+1}(X; i\mathbb{R})$. Then, the conclusion follows from the equality

$$(\det g)^*(\tilde{A}_0) - \tilde{A}_0 = 2df \in L^2_k(X; \Lambda^1 \otimes i\mathbb{R}).$$

∎

2.4.1. Reducible case. The covering-SW map becomes simpler if we use a reducible solution $(A_0, 0)$ as the base, in which case A_0 satisfies the ASD equation $F^+_{A_0} = 0$. Then, the map is given by

$$F_{\tilde{A}_0} : L^2_k\big((X,g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \to L^2_{k-1}\big((X,g); \tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big)$$
$$(\psi, a) \mapsto \big(D_{\tilde{A}_0+a}(\psi), d^+(a) - \sigma(\psi)\big).$$

Note that the moduli space of $U(1)$ ASD connections consists of the space of harmonic ASD 2-forms on M whose cohomology class coincides with the first Chern class of the $U(1)$ bundle.

2.4.2. Equivariant gauge fixing. Let us say that AHS complex is *closed* if the differentials

$$0 \to L^2_{k+1}(X) \to L^2_k(X;\Lambda^1) \to L^2_{k-1}(X;\Lambda^2_+) \to 0$$

have closed range (see [23]).

Lemma 2.5. *Suppose the AHS complex is closed. Then, the first cohomology group* $H^1(X) = \operatorname{Ker} d^+ / \operatorname{im} d$ *is isomorphic to the* L^2 *first de Rham cohomology.*

Proof. This follows by the same argument as Lemma 2.1. ■

Remark 2.6. Later, we see several classes of universal covering spaces whose AHS complexes are closed. In many cases, this property depends only on the large-scale analytic property of their fundamental groups.

Suppose the AHS complex is closed and consider the space of L^2 harmonic one-forms:

$$\mathfrak{H}^1 = \operatorname{Ker}\big[d^* \oplus d^+ : L^2_k\big((X,g);\Lambda^1 \otimes i\mathbb{R}\big) \to L^2_{k-1}\big((X,g);(\Lambda^0 \oplus \Lambda^2_+) \otimes i\mathbb{R}\big)\big].$$

For any $a \in L^2_k((X,g);\Lambda^1 \otimes i\mathbb{R})$, consider the orthogonal decomposition $a = a_1 \oplus a_2 \in \mathfrak{H}^1 \oplus (\mathfrak{H}^1)^\perp$. We denote a_1 by $a_{\text{harm}} \in \mathfrak{H}^1$.

Let us state the equivariant gauge fixing.

Proposition 2.7. *Suppose the AHS complex is closed. Then, there is a global and* Γ*-equivariant gauge fixing such that the covering-SW map is restricted to the slice*

$$F_{\tilde{A}_0,\tilde{\psi}_0} : L^2_k(X;\tilde{S}^+) \oplus L^2_k(X;\Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^* \to L^2_{k-1}\big((X,g);\tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big).$$

More strongly, the following holds. For any $A = \tilde{A}_0 + a$ *with* $a \in L^2_k(X;\Lambda^1 \otimes i\mathbb{R})$, *there is* $\sigma \in \mathfrak{G}_{k+1}$ *such that* $(\det \sigma)^*(A) := \tilde{A}_0 + a'$ *satisfies the equality* $d^*(a') = 0$ *with the estimate*

$$\|a'\|_{L^2_k(X)} \le C\big(\|d^+(a)\|_{L^2_{k-1}(X)} + \|a_{\text{harm}}\|\big)$$

for some constant C *independent of* A_0.

Compare this with [30, Lemma 5.3.1].

Proof. We have divided the proof into three steps.

Step 1. Take two elements $A := \widetilde{A}_0 + a$ and $A' := \widetilde{A}_0 + a'$ with $d^*a = d^*a' = 0$. Suppose $A' = (\det\sigma)^*(A)$ could hold for some $\sigma = \exp(f) \in \mathfrak{G}_{k+1}$. Then, the equality

$$a' = a + 2df$$

should hold. Applying d^* on both sides, we obtain $d^*df = 0$. Then,

$$0 = \langle d^*df, f\rangle_{L^2} = \|df\|^2_{L^2},$$

which gives $df = 0$ and the equality $A = A'$. Moreover, $f \equiv 0$ holds, because X is non-compact.

This implies that the quotient map

$$L^2_k(X; \widetilde{S}^+) \oplus L^2_k(X; \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^* \hookrightarrow L^2_k(X; \widetilde{S}^+) \oplus L^2_k(X; \Lambda^1 \otimes i\mathbb{R})/\mathfrak{G}_{k+1}$$

is injective.

Step 2. Note that when one finds such a constant for some A_0, then it holds for any choice of the base, since the SW moduli space is compact over the base compact manifold M (see Section 2.3).

It follows from the assumption that there is a bounded linear map

$$\Delta^{-1} : d^*\big(L^2_k(X; \Lambda^1 \otimes i\mathbb{R})\big) \to L^2_{k+1}(X; i\mathbb{R})$$

that inverts the Laplacian. Let us set

$$s_0 = -\frac{1}{2}\Delta^{-1}\big(d^*(a)\big) \in L^2_{k+1}(X; i\mathbb{R})$$

and $\sigma_0 = \exp(s_0) \in \mathfrak{G}_{k+1}$. For $a' = a + 2\sigma_0^{-1}d\sigma_0$, we have

$$\det(\sigma_0)^*A = \widetilde{A}_0 + a'$$

with the equality $d^*(a') = 0$.

Step 3. Let us consider

$$d^* \oplus d^+ : L^2_k(X; \Lambda^1 \otimes i\mathbb{R}) \to L^2_{k-1}(X; \Lambda^0 \oplus \Lambda^2_+) \otimes i\mathbb{R}.$$

Its kernel is the space of harmonic 1-forms. We decompose $a' = h + b$, where $h = a'_{\text{harm}}$ is the harmonic form and b lies in the orthogonal subspace. Then, it follows from closedness that there is a bound

$$\|b\|_{L^2_k} \le C\big(\big\|d^*(b)\big\|_{L^2_{k-1}} + \big\|d^+(b)\big\|_{L^2_{k-1}}\big) = C\big\|d^+(b)\big\|_{L^2_{k-1}}.$$

Moreover, $d^+(b) = F^+_A - F^+_{\widetilde{A}_0} = d^+(a')$ holds. Thus, we obtain

$$\|a'\|^2_{L^2_k} \le C\big(\big\|d^+(a')\big\|_{L^2_{k-1}} + \|a'_{\text{harm}}\|\big).$$

Since both the equalities $d^+(a') = d^+(a)$ and $a'_{\text{harm}} = a_{\text{harm}}$ hold, this concludes the proof. ∎

2.4.3. Covering-SW moduli space. Let us consider the closed subset

$$\mathfrak{M}(A_0, \psi_0) := F^{-1}_{\tilde{A}_0,\tilde{\psi}_0}(0) \subset L^2_k(X; \tilde{S}^+) \oplus L^2_k(X; \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^*.$$

It is non-empty since $[(0,0)]$ is an element in it. If $F_{\tilde{A}_0,\tilde{\psi}_0}$ has a regular value at 0 such that its differential is surjective on $\mathfrak{M}(A_0, \psi_0)$, then it is a regular manifold equipped with the induced Γ action. Its Γ-dimension is equal to

$$\operatorname{ind} D_{A_0} - \chi_{\mathrm{AHS}},$$

where $\operatorname{ind} D_{A_0}$ is the index of D_{A_0} and χ_{AHS} is the AHS-Euler characteristic on M.

Note that if an element $g \in \Gamma$ is infinite cyclic, then the g-action is free except at the origin $[(0,0)]$.

Choose any $x, y \in F^{-1}_{\tilde{A}_0,\tilde{\psi}_0}(0)$ and consider its differential

$$d(F_{\tilde{A}_0,\tilde{\psi}_0})_x : L^2_k(X; \tilde{S}^+) \oplus L^2_k(X; \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^* \to L^2_{k-1}\big((X,g); \tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big).$$

We denote $d(F_{\tilde{A}_0,\tilde{\psi}_0})_x$ by dF_x for brevity.

Lemma 2.8. (1) *For $x = (\psi, a)$, the following formula holds:*

$$dF_x(c,\xi) = \big(D_{\tilde{A}_0+a}(\xi) + c(\tilde{\psi}_0 + \psi), d^+(c) - (\tilde{\psi}_0 + \psi) \otimes \xi^* - \xi \otimes (\tilde{\psi}_0 + \psi)^*\big).$$

(2) *Let $k \geq 3$. The difference $dF_x - dF_y$ is compact.*

Proof. We have

$$\begin{aligned} dF_x(c,\xi) &= \frac{d}{dt}\big(D_{\tilde{A}_0+a+tc}(\tilde{\psi}_0 + \psi + t\xi), d^+(a+tc) - \sigma(\tilde{\psi}_0, \psi + t\xi)\big)_{t=0} \\ &= \big(D_{\tilde{A}_0+a}(\xi) + c(\tilde{\psi}_0 + \psi), d^+(c) - (\tilde{\psi}_0 + \psi) \otimes \xi^* - \xi \otimes (\tilde{\psi}_0 + \psi)^*\big). \end{aligned}$$

Let $x = (\psi, a)$ and $y = (b, \phi)$. Their difference is given by

$$(dF_x - dF_y)(c,\xi) = \big((a-b)\xi + c(\psi - \phi), -(\psi - \phi) \otimes \xi^* - \xi \otimes (\psi - \phi)^*\big).$$

Since all $a, b, c, \phi, \psi, \xi \in L^2_k$, their products all lie in L^2_k by Corollary 3.3. If a, b, ψ, ϕ all have compact support, then compactness follows from the Sobolev multiplication with Rellich's lemma. In general, they can be approximated by compactly supported smooth functions as the space of compact operators is a closed set in the space of bounded operators. Hence, the difference is still compact. ∎

2.5. Covering-monopole map

Let M be a compact oriented four-manifold equipped with a spin^c structure, and let $X = \tilde{M}$ be its universal covering space with $\pi_1(M) = \Gamma$. Let $H^1_{(2)}(X)(\bar{H}^1_{(2)}(X))$ be the first (reduced) L^2 cohomology group. The L^2 cohomology groups coincide with each other, that is, $\bar{H}^*_{(2)}(X) = H^*_{(2)}(X)$ when the AHS complex is closed.

Let (A_0, ψ_0) be a solution to the SW equations over M and denote its lift by $(\tilde{A}_0, \tilde{\psi}_0)$ over X.

Definition 2.4. The covering-monopole map at the base (A_0, ψ_0) is the $\mathfrak{G}_{k+1} \rtimes \Gamma$ equivariant map given by

$$\widetilde{\mu} : L^2_k\big((X, g); \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \to L^2_{k-1}\big((X, g); \widetilde{S}^- \oplus (\Lambda^2_+ \oplus \Lambda^0) \otimes i\mathbb{R}\big) \oplus \bar{H}^1_{(2)}(X)$$
$$(\phi, a) \mapsto \big(F_{\widetilde{A}_0, \widetilde{\psi}_0}, (\phi, a), d^*(a), [a]\big),$$

where [ ] is the orthogonal projection to the reduced cohomology group.

Remark 2.9. Even if the first de Rham cohomology group $H^1_{dR}(X; \mathbb{R}) = 0$ vanishes, the first reduced cohomology group may survive. For an element in the latter cohomology, there associates a "gauge-group action" that can eliminate it. Clearly such an action does not lie in the L^2 Sobolev space. Its behavior at infinity appears complicated such that they will "move" quite "slowly" at infinity.

Suppose the AHS complex is closed. Then, the covering-monopole map restricts on the slice

$$\widetilde{\mu} : L^2_k\big((X, g); \widetilde{S}^+\big) \oplus L^2_k\big((X, g); \Lambda^1 \otimes i\mathbb{R}\big) \cap \operatorname{Ker} d^*$$
$$\to L^2_{k-1}\big((X, g); \widetilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X)$$
$$(\phi, a) \mapsto \big(F_{\widetilde{A}_0, \widetilde{\psi}_0}(\phi, a), [a]\big)$$

which is a Γ-equivariant map (see Proposition 2.7).

Lemma 2.10. *The Γ-index of the linearized map is given by*

$$\begin{aligned} \dim_\Gamma d\widetilde{\mu} &= \operatorname{ind} D - \big(b_0(M) - b_1(M) + b_2^+(M)\big) - \dim_\Gamma H^1_{(2)}(X) \\ &= \operatorname{ind} D - \dim_\Gamma H^+_{(2)}(X), \end{aligned}$$

where ind D *is the index of the Dirac operator over* M.

Proof. This follows from Atiyah's Γ-index theorem. ∎

Remark 2.11. The Γ-dimension is a topological invariant of the base manifold M when one of $H^1_{(2)}(X) = 0$ or $H^+_{(2)}(X) = 0$ holds.

If M is compact and aspherical, then the Singer conjecture states that the L^2 cohomology should vanish except for the middle dimension, where in our case of four-manifolds, only the second L^2 cohomology is able to survive and $H^1_{(2)}(X)$ should vanish. This result has been verified for many classes of compact aspherical manifolds whose fundamental groups have a "hyperbolic" structure [18].

3. L^p analysis and estimates on Sobolev spaces

3.1. Sobolev spaces over covering spaces

Let E', E be vector bundles over a compact Riemannian manifold M, and let

$$l : C^\infty(M; E') \to C^\infty(M; E)$$

be a first-order elliptic differential operator.

Let us lift them over the universal covering space $X = \tilde{M}$ and introduce the lift of the L^2 inner product

$$\langle u, v\rangle = \int_X \big(u(x), v(x)\big) \,\mathrm{vol}$$

over X, which is Γ-invariant. Let l^* be the formal adjoint operator over X. We will use the Sobolev norms on sections of $E' \to X$ by

$$\begin{aligned}\langle u, v\rangle_{L^2_1} &= \langle u, v\rangle + \big\langle l(u), l(v)\big\rangle,\\ \langle u, v\rangle_{L^2_2} &= \langle u, v\rangle + \big\langle l(u), l(v)\big\rangle + \big\langle l^*l(u), l^*l(v)\big\rangle,\\ &\vdots\end{aligned}$$

whose spaces are given by taking the closure of $C_c^\infty(X; E')$. In other words, the inner products can be written as

$$\langle u, v\rangle_{L^2_k} = \sum_{j=0}^{k} \big\langle (l^*l)^j(u), v\big\rangle.$$

Similarly on $E \to X$, we equip with the Sobolev norms by use of

$$\begin{aligned}\langle w, x\rangle_{L^2_1} &= \langle w, x\rangle + \big\langle l^*(w), l^*(x)\big\rangle,\\ \langle w, x\rangle_{L^2_2} &= \langle w, x\rangle + \big\langle l^*(w), l^*(x)\big\rangle + \big\langle ll^*(w), ll^*(x)\big\rangle,\\ &\vdots\end{aligned}$$

Lemma 3.1. $\langle l(v), w\rangle_{L^2_k} = \langle v, l^*(w)\rangle_{L^2_k}$ *holds for all* $k \geq 0$.

Proof. It holds for $k = 0$. Suppose it holds up to $k - 1$. Since the equalities

$$\begin{aligned}\big\langle l(u), w\big\rangle_{L^2_k} &= \big\langle l(u), w\big\rangle + \big\langle l^*l(u), l^*(w)\big\rangle_{L^2_{k-1}}\\ &= \big\langle u, l^*(w)\big\rangle + \big\langle l(u), ll^*(w)\big\rangle_{L^2_{k-1}} = \big\langle u, l^*(w)\big\rangle_{L^2_k}\end{aligned}$$

hold by induction, the conclusion also holds for k. ∎

In the case when $l : L^2_k(X) \cong L^2_{k-1}(X)$ gives a linear isomorphism, we can replace the norms by

$$\langle u, v\rangle'_{L^2_k} = \big\langle (l^*l)^k u, v\big\rangle_{L^2}.$$

Then, $l : L^2_k(X) \cong L^2_{k-1}(X)$ is unitary with respect to this particular norm. These norms are equivalent:

$$C^{-1}\|\ \|_{L^2_k} \leq \|\ \|'_{L^2_k} \leq C\|\ \|_{L^2_k},$$

to the above Sobolev norms for some $C \geq 1$. This follows from the fact that there is a positive $\delta > 0$ with the bound

$$\delta\|u\|_{L^2} \leq \big\|l(u)\big\|_{L^2}, \quad \delta\|w\|_{L^2} \leq \big\|l^*(w)\big\|_{L^2},$$

where δ is independent of u and w.

For convenience, we recall the local Sobolev estimates over four-dimensional manifolds. By local compactness we mean that it is compact on $L^2_k(K)_0$ that is a restriction of the Sobolev spaces with support on K. Here K is a closure of an open and bounded subset in X. More precisely, $L^2_k(K)_0$ is a Sobolev closure of $C^\infty_c(\mathrm{int}\, K)$.

Hereinafter, we assume that a compact subset K is a compact smooth submanifold of codimension zero so it should have a smooth boundary in X if X is non-compact.

Lemma 3.2. (1) *The continuous embeddings $L^p_k \subset L^q_l$ hold locally if both $k \geq l$ and $k - 4/p \geq l - 4/q$ hold. They are also locally compact if the stronger inequalities $k > l$ and $k - 4/p > l - 4/q$ hold.*

(2) *The continuous embeddings $L^p_k \subset C^l$ hold locally if $k - 4/p > l$ holds.*

In particular, it is convenient for us to check the embeddings $L^2_k \subset L^4_{k-1}$.

Proof. We refer to [13, 14] for the proof. We also refer to [17] for a more detailed analysis of Sobolev spaces. ∎

Corollary 3.3. *The following local multiplications are continuous locally:*

(a) $L^2_k \times L^2_k \to L^2_k$ *for* $k \geq 3$ *and*

(b) $L^2_k \times L^2_k \to L^2_{k-1}$ *for* $k \geq 1$.

Proof. Let us take $u, v \in L^2_k$. For $k' \leq k$,

$$\nabla^{k'}(uv) = \sum_{a+b=k'} \nabla^a(u)\nabla^b(v)$$

holds. If $0 \leq k' < k$, then the estimates

$$\left\|\nabla^a(u)\nabla^b(v)\right\|_{L^2_{\mathrm{loc}}} \leq \left\|\nabla^a(u)\right\|_{L^4_{\mathrm{loc}}} \left\|\nabla^b(v)\right\|_{L^4_{\mathrm{loc}}} \leq C \left\|\nabla^a(u)\right\|_{(L^2_1)_{\mathrm{loc}}} \left\|\nabla^b(v)\right\|_{(L^2_1)_{\mathrm{loc}}}$$

hold by Lemma 3.2 (1). Thus, we obtain

$$\|uv\|_{(L^2_{k'})_{\mathrm{loc}}} \leq C\|u\|_{(L^2_k)_{\mathrm{loc}}} \|v\|_{(L^2_k)_{\mathrm{loc}}}.$$

This verifies (b).

Let us verify (a). Suppose $3 \leq k' = k$. Then, we obtain the estimate

$$\left\|\nabla^k(u)v\right\|_{(L^2)_{\mathrm{loc}}} \leq C\|v\|_{C^0}\|u\|_{(L^2_k)_{\mathrm{loc}}} \tag{3.1}$$

by Lemma 3.2 (1). Combining this result with (3.1), we have verified (a). ∎

3.2. L^p cohomology

Let (X, g) be a complete Riemannian manifold. For $p > 1$, let $L^p_k(X; \Lambda^m)$ be the Banach space of L^p_k differential m-forms on X, and let d be the exterior differential whose domain is $C^\infty_c(X; \Lambda^m)$.

Let us recall the following notions (see [19]).

(1) The (unreduced) L^p cohomology $H^{m,p}(X)_k$ is given by

$$\mathrm{Ker}\left\{d : L^p_k(X,\Lambda^m) \to L^p_{k-1}(X,\Lambda^{m+1})\right\}/ \\ \mathrm{im}\left\{d : L^p_{k+1}(X,\Lambda^{m-1}) \to L^p_k(X,\Lambda^m)\right\}.$$

(2) The reduced L^p cohomology $\bar{H}^{m,p}(X)_k$ is given by

$$\mathrm{Ker}\left\{d : L^p_k(X,\Lambda^m) \to L^p_{k-1}(X,\Lambda^{m+1})\right\}/ \\ \overline{\mathrm{im}\left\{d : L^p_{k+1}(X,\Lambda^{m-1}) \to L^p_k(X,\Lambda^m)\right\}},$$

where $\overline{\mathrm{im}}$ is the closure of the image.

There is a canonical surjection $H^{m,p}(X)_k \to \bar{H}^{m,p}(X)_k$, and its kernel

$$T^{m,p}_k := \mathrm{Ker}\left\{H^{m,p}(X)_k \to \bar{H}^{m,p}(X)_k\right\}$$

is called the torsion of L^p cohomology. The differential d has closed range if and only if the torsion $T^{m,p}_k = 0$ vanishes (see [25]).

Definition 3.1. The space $\mathfrak{H}^{m,p}(X)$ of L^p harmonic m-forms is given by

$$\mathrm{Ker}\left\{(d \oplus d^*) : L^p_k(X,\Lambda^m) \to L^p_{k-1}(X,\Lambda^{m+1} \oplus \Lambda^{m-1})\right\}.$$

Note that $\mathfrak{H}^{m,p}(X)$ is independent of the choice of k. It is well known that the space $\bar{H}^{m,2}(X)_k$ is isomorphic to L^2 harmonic m-forms which are independent of k.

For our case of the AHS complex, the second cohomology involves d^+ rather than d. Let $\dim X = 4$.

Lemma 3.4. *Suppose* $d : L^2_k(X;\Lambda^i) \to L^2_{k-1}(X;\Lambda^{i+1})$ *have closed range for any* $k \geq 1$ *and* $i = 0, 1$. *Then, the composition with the self-dual projection*

$$d^+ : L^2_k(X : \Lambda^1) \to L^2_{k-1}(X : \Lambda^2_+)$$

also has closed range for any $k \geq 1$.

Proof. The proof consists of three steps.

Step 1. Let H be a Hilbert space and $W \subset H$ a closed linear subspace. If a sequence $w_i \in W$ weakly converges to some $w \in H$, then $w \in W$. In fact, $\langle w, h\rangle = \lim_i \langle w_i, h\rangle = 0$ for any $h \in W^\perp$.

Let H_1 and H_2 be both Hilbert spaces and $W \subset H_1 \oplus H_2$ a closed linear subspace. Let us consider the projection $P : H_1 \oplus H_2 \to H_1$ and take a sequence $w_i = v^1_i + v^2_i \in W \subset H_1 \oplus H_2$. Suppose the sequence $P(w_i) = v^1_i \in H_1$ converges to some $v_1 \in H_1$ and the weak limit of w_i does not lie on W. Then, $\|v^2_i\| \to \infty$ must hold. In fact, if $\|v^2_i\|$ could be bounded, then v^2_i weakly converges to some $v_2 \in H_2$. In particular, w_i weakly converges to $v_1 + v_2$ which should lie in W as we have verified.

Step 2. Let us verify the conclusion for $k = 1$. It follows from Stokes' theorem that, for $\alpha \in L^2_1(X; \Lambda^1)$,

$$0 = \int_X d(\alpha) \wedge d(\alpha) = \|d^+(\alpha)\|^2_{L^2} - \|d^-(\alpha)\|^2_{L^2}.$$

Thus, we have the equality $\|d^+(\alpha)\|^2_{L^2} = \|d^-(\alpha)\|^2_{L^2}$.

Suppose a sequence $\alpha_i \in L^2_1(X; \Lambda^1)$ converges as $d^+(\alpha_i) \to a_+ \in L^2(X; \Lambda^2_+)$. Then, $\{d(\alpha_i)\}_i$ is a bounded sequence in L^2, since the equality $\|d(\alpha_i)\|^2_{L^2} = \|d^+(\alpha_i)\|^2_{L^2} + \|d^-(\alpha_i)\|^2_{L^2}$ holds. Hence, the bounded sequence has a weak limit $w\text{-}\lim_i d(\alpha_i) \to a \in L^2(X; \Lambda^2)$ such that the self-dual part of a coincides with a_+.

Let us apply Step 1 to $W := \operatorname{im} d(L^2_1(X; \Lambda^1))$, $v^1_i := d^+(\alpha_i) \in H_1 := L^2(X; \Lambda^+)$, and $v^2_i := d^-(\alpha_i) \in H_2 := L^2(X; \Lambda^-)$. Then, $a = d(\alpha) \in W$ for some $\alpha \in L^2_1(X; \Lambda^1)$; otherwise, $d^-(\alpha_i)$ should diverge in the L^2 norm. In particular, $a_+ = d^+(\alpha)$ holds and, hence, d^+ has closed range.

Step 3. Let us verify $k = 2$ case and assume that $\alpha_i \in L^2_2(X; \Lambda^1)$ satisfies the convergence $d^+(\alpha_i) \to a \in L^2_1(X; \Lambda^2_+)$. Then, there is some $\alpha \in L^2_1(X; \Lambda^1)$ with $d^+(\alpha) = a$ by Step 2.

We may assume $d^*(\alpha) = 0$ since $d : L^2_{k+1}(X) \to L^2_k(X; \Lambda^1)$ has closed range. Then, the elliptic estimate tells $\alpha \in L^2_2(X; \Lambda^1)$ and, hence, the $k = 2$ case follows.

We can proceed by induction such that the conclusion holds. ■

3.3. Examples of zero-torsion L^2 cohomology

There are several instances of zero-torsion L^p cohomology. See [31] for the $p \neq 2$ case.

Let us consider the case $p = 2$. Using the Hilbert space structure, there have been many examples with zero torsion discovered, some of which we present below.

3.3.1. Kähler hyperbolic manifolds. Let (M, ω) be a compact Kähler manifold and assume that the lift of the Kähler form $\tilde{\omega}$ over the universal covering space X represents zero in the second de Rham cohomology $H^2(X; \mathbb{R})$ such that it can be given as $\tilde{\omega} = d(\eta)$ for some $\eta \in C^\infty(X; \Lambda^1)$. Note that η cannot be $\Gamma = \pi_1(M)$ invariant, since then ω would be an exact form on M.

Lemma 3.5 ([18]). *Suppose $\|\eta\|_{L^\infty(X)} < \infty$ is finite. Then, the L^2 de Rham differentials have closed range.*

Moreover, (M, ω) satisfies the Singer conjecture.

See Section 1.3 and also Remark 2.11 on the Singer conjecture.

3.3.2. Zero torsion with positive scalar curvature. Let us present four-manifolds with positive scalar curvature whose universal covering spaces have zero torsion.

Lemma 3.6. *Let X and Y be complete Riemannian manifolds of dimension 2, where X is non-compact and Y is compact. Suppose the de Rham differentials have closed range on X. Then, the AHS complex over $X \times Y$ also has closed range.*

The following argument is quite straightforward and can be applied to more general cases.

Proof. The proof consists of three steps.

Step 1. It follows from Lemma 3.4 that d^+ also has closed range if d is the case on 1-forms. Thus, it is enough to verify closedness of d on both 0- and 1-forms.

Note that $C_c^\infty(X \times Y) \subset L^2(X \times Y)$ is dense and $L^2(X \times Y) = L^2(X) \otimes L^2(Y)$ holds, where the right-hand side is the Hilbert space tensor product.

Let $\{f_\lambda\}_\lambda$ and $\{u_\lambda\}_\lambda$ be the spectral decompositions of the Laplacian on the L^2 forms of degrees 0 and 1 over Y, respectively, where both f_λ and u_λ have the eigenvalues λ^2.

Step 2. Let $\Delta = d^* \circ d$ be the Laplacian acting on the space of L^2 functions on X. Then, by the open mapping theorem, its spectrum is contained in $[\varepsilon, \infty)$ for some positive $\varepsilon > 0$, because the de Rham differential has closed range over X, and X admits no non-zero L^2 harmonic function. Note that X is assumed to be non-compact. This implies that there is a positive constant $C > 0$ such that the uniform lower bound $\|dg\|_{L^2_j} \geq C\|g\|_{L^2_{j+1}}$ holds for any g. In fact, $\|dg\|^2_{L^2} = \langle \Delta g, g \rangle_{L^2} \geq \varepsilon \|g\|^2_{L^2}$ holds. Hence,

$$\|g\|^2_{L^2_1} = \|g\|^2_{L^2} + \|dg\|^2_{L^2} \leq (1 + \varepsilon^{-1})\|dg\|^2_{L^2}$$

holds. Then, we have

$$\|g\|^2_{L^2_{j+1}} = \|dg\|^2_{L^2_j} + \|g\|^2_{L^2} \leq \|dg\|^2_{L^2_j} + \varepsilon^{-1}\|dg\|^2_{L^2} = (1 + \varepsilon^{-1})\|dg\|^2_{L^2_j}.$$

Step 3. Let us consider the case of 0-forms. Suppose $\alpha = d(F) \in L^2_k(X \times Y; \Lambda^1)$ lies in the image

$$d\big(L^2_{k+1}(X \times Y)\big) \subset \big(d\big(L^2_{k+1}(X)\big) \otimes L^2_{k+1}(Y)\big) \oplus \big(L^2_{k+1}(X) \otimes d\big(L^2_{k+1}(Y)\big)\big),$$

where the right-hand side is the Hilbert space tensor product, which is defined as both $d(L^2_{k+1}(X))$ and $d(L^2_{k+1}(Y))$ are closed in L^2_k.

Decompose $F = \sum_\lambda g_\lambda \otimes f_\lambda$. Note that $\|df_\lambda\|^2_{L^2} = \lambda^2 \|f_\lambda\|^2_{L^2}$. Moreover, there is a positive constant $C > 0$ such that the uniform lower bound $\|dg_\lambda\|_{L^2_j} \geq C\|g_\lambda\|_{L^2_{j+1}}$ holds by Step 3.

Then, we have the equalities

$$\|\alpha\|^2_{L^2_k} = \Big\| d \sum_\lambda g_\lambda \otimes f_\lambda \Big\|^2_{L^2_k} = \sum_\lambda \sum_{j=0}^k \Big(\|dg_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2_{k-j}} + \|g_\lambda\|^2_{L^2_j} \|df_\lambda\|^2_{L^2_{k-j}} \Big).$$

By using the spectral weights for the Sobolev spaces, it is equal to

$$\sum_\lambda \sum_{j=0}^k \lambda^{2(k-j)} \big(\|dg_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2} + \|g_\lambda\|^2_{L^2_j} \|df_\lambda\|^2_{L^2} \big)$$

$$= \sum_\lambda \sum_{j=0}^k \lambda^{2(k-j)} \big(\|dg_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2} + \lambda^2 \|g_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2} \big)$$

$$= \sum_{\lambda} \sum_{j=0}^{k} \Big(\|dg_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2_{k-j}} + \|g_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2_{k-j+1}} \Big).$$

Then, it is bounded by the followings

$$C \sum_{\lambda} \sum_{j=0}^{k} \|g_\lambda\|^2_{L^2_{j+1}} \|f_\lambda\|^2_{L^2_{k-j}} + \sum_{\lambda \neq 0} \sum_{j=0}^{k} \|g_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2_{k-j+1}}$$
$$\geq C' \sum_{\lambda} \sum_{j=0}^{k+1} \|g_\lambda\|^2_{L^2_j} \|f_\lambda\|^2_{L^2_{k+1-j}} = C' \| \sum_{\lambda} g_\lambda \otimes f_\lambda \|^2_{L^2_{k+1}}$$

for another positive constant $C' > 0$. This verifies that d has closed range on 0-forms over $X \times Y$.

Step 4. Let us verify a general fact. Let $d : H \to W = W_1 \oplus W_2$ be a linear map between Hilbert spaces and suppose that both compositions with the projections $d_i : H \to W_i$ have closed range for $i = 1, 2$. Then, we claim that d itself has closed range.

To see this, we replace both W_1 and W_2 with the images $d_1(H)$ and $d_2(H)$, respectively, because the image of f is contained in $d_1(H) \oplus d_2(H)$. Hence, d_1 and d_2 can be assumed to be surjective.

Let $V := \ker d_1 \cap \ker d_2 \subset H$ be the intersection of their kernels. Then, we can restrict on the orthogonal complement $V^\perp \subset H$. Note that there is a positive constant $C > 0$ such that any element

$$u = u_1 + u_2 \in \big(V^\perp \cap \ker d_1\big) \oplus \big(V^\perp \cap \ker d_2\big)$$

admits a lower bound $\|d(u)\| \geq C\|u\|$ for some positive constant $C > 0$, since $d(u) = d(u_1) + d(u_2)$ with $\|d_1(u_2)\| \geq C\|u_2\|$ and $\|d_2(u_1)\| \geq C\|u_1\|$. Hence, we obtain the bound

$$\big\|d(u)\big\|^2 = \big\|d_1(u_2)\big\|^2 + \big\|d_2(u_1)\big\|^2 \geq C\big(\|u_2\|^2 + \|u_1\|^2\big) = C\|u\|^2.$$

This implies that any element $u \in V^\perp$ also admits a lower bound $\|d(u)\|^2 \geq C\|u\|^2$ for some positive constant $C > 0$. This verifies that d has closed range.

Step 5. Let us consider the case of 1-forms. To check closedness of the differential, we may assume that $u \in d(L^2_{k+1}(X \times Y))^\perp$ by using the inner product in Lemma 3.1 for $l = d \oplus (d^+)^*$.

Let us take an element $u \in L^2_k(X \times Y; \Lambda^0_X \oplus \Lambda^1_Y)$. We decompose

$$u = \sum_{\lambda} g_\lambda \otimes u_\lambda = \sum_{\lambda} g_\lambda \otimes (\alpha_\lambda + d^*\omega_\lambda),$$

where α_λ is a closed 1-form.

Sublemma 3.7. *α_λ is a harmonic form.*

Proof. In fact, the inner product

$$0 = \langle u, d(g \otimes \alpha)\rangle_{L^2_k} = \langle d^* u, g \otimes \alpha\rangle_{L^2_k}$$
$$= \sum_\lambda \langle g_\lambda \otimes d^*(u_\lambda), g \otimes \alpha\rangle_{L^2_k} = \sum_\lambda \langle g_\lambda, g\rangle_{L^2_k} \cdot \langle d^*(u_\lambda), \alpha\rangle_{L^2_k}$$
$$= \sum_\lambda \langle g_\lambda, g\rangle_{L^2_k} \cdot \langle d^*(\alpha_\lambda), \alpha\rangle_{L^2_k}$$

vanishes for any $g \otimes \alpha$. Hence, $d^*(\alpha_\lambda) = 0$ vanishes. ∎

Then,

$$du = \sum_\lambda dg_\lambda \otimes u_\lambda + \sum_\lambda g_\lambda \otimes du_\lambda \in L^2_{k-1}(X \times Y; \Lambda^1_X \otimes \Lambda^1_Y) \oplus L^2_{k-1}(X \times Y; \Lambda^0_X \otimes \Lambda^2_Y)$$
$$= \sum_\lambda dg_\lambda \otimes \alpha_\lambda + \sum_\lambda dg_\lambda \otimes d^*\omega_\lambda + \sum_\lambda g_\lambda \otimes dd^*\omega_\lambda.$$

It is sufficient to check closedness of the differential on each term above from Step 4. Let us consider the projection of the differential to the first term:

$$d^1 : u = \sum_\lambda g_\lambda \otimes \alpha_\lambda \in L^2_k(X \times Y; \Lambda^0_X \otimes \Lambda^1_Y)$$
$$\mapsto d^1 u := \sum_\lambda dg_\lambda \otimes \alpha_\lambda \in L^2_{k-1}(X \times Y; \Lambda^1_X \otimes \Lambda^1_Y).$$

Let $\mathcal{H}^1(Y)$ be the space of harmonic 1-forms on Y. Note that the restriction

$$d^1 : L^2_k(X) \otimes \mathcal{H}^1(Y) \to L^2_{k-1}(X; \Lambda^1_X) \otimes \mathcal{H}^1(Y)$$

has closed range, because $\mathcal{H}^1(Y)$ is finite-dimensional and the de Rham differential on X is assumed to have closed range.

Step 6. Let us verify closedness of the differential on the remaining case:

$$d^2 : u = \sum_\lambda g_\lambda \otimes d^*\omega_\lambda \in L^2_k(X \times Y; \Lambda^0_X \otimes \Lambda^1_Y)$$
$$\mapsto d^2 u := \sum_\lambda (dg_\lambda \otimes d^*\omega_\lambda + g_\lambda \otimes dd^*\omega_\lambda) \in L^2_{k-1}(X \times Y; \Lambda^1_X \otimes \Lambda^1_Y \oplus \Lambda^0_X \otimes \Lambda^2_Y).$$

Note the equalities

$$\|dd^*\omega_\lambda\|^2_{L^2} = \langle dd^*\omega_\lambda, dd^*\omega_\lambda\rangle = \langle d^*dd^*\omega_\lambda, d^*\omega_\lambda\rangle = \lambda^2 \|d^*\omega_\lambda\|^2_{L^2}.$$

Then, we have the estimates

$$\|du\|^2_{L^2_{k-1}}$$
$$= \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-1-j)} \|dg_\lambda\|^2_{L^2_j} \|d^*\omega_\lambda\|^2_{L^2} + \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-1-j)} \|g_\lambda\|^2_{L^2_j} \|dd^*\omega_\lambda\|^2_{L^2}$$

$$
\begin{aligned}
&= \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-1-j)} \|dg_\lambda\|^2_{L^2_j} \|d^*\omega_\lambda\|^2_{L^2} + \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-j)} \|g_\lambda\|^2_{L^2_j} \|d^*\omega_\lambda\|^2_{L^2} \\
&\geq C \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-1-j)} \|g_\lambda\|^2_{L^2_{j+1}} \|d^*\omega_\lambda\|^2_{L^2} + \sum_\lambda \sum_{j=0}^{k-1} \lambda^{2(k-j)} \sum_{j=0}^{k-1} \|g_\lambda\|^2_{L^2_j} \|d^*\omega_\lambda\|^2_{L^2} \\
&\geq C' \|u\|^2_{L^2_k}
\end{aligned}
$$

for some positive constants $C, C' > 0$.

Step 7. Let us consider the case

$$
u = \sum_\lambda v_\lambda \otimes f_\lambda \in L^2_k(X \times Y; \Lambda^1_X \otimes \Lambda^0_Y).
$$

Let us decompose $v_\lambda = \alpha_\lambda + d^*\omega_\lambda$ with $d\alpha_\lambda = 0$.

By a similar argument as Step 5, α_λ is a harmonic form. Moreover, we may assume $\alpha_0 = 0$, since $f_0 = 1$ is the constant function and, hence, $d(\alpha_0 \otimes f_0) = 0$ holds.

By the assumption, there is a positive constant $C > 0$ such that the estimate

$$
\|dd^*\omega\|_{L^2_j} \geq C \|d^*\omega\|_{L^2_{j+1}}
$$

holds. Then,

$$
\begin{aligned}
\|du\|^2_{L^2_{k-1}} &= \left\| \sum_\lambda dd^*\omega_\lambda \otimes f_\lambda + v_\lambda \otimes df_\lambda \right\|^2_{L^2_{k-1}} \\
&= \sum_\lambda \|dd^*\omega_\lambda \otimes f_\lambda\|^2_{L^2_{k-1}} + \|v_\lambda \otimes df_\lambda\|^2_{L^2_{k-1}} \\
&= \sum_\lambda \sum_{j=0}^{k-1} \|dd^*\omega_\lambda\|^2_{L^2_j} \cdot \|f_\lambda\|^2_{L^2_{k-1-j}} + \|v_\lambda\|^2_{L^2_j} \cdot \|df_\lambda\|^2_{L^2_{k-1-j}} \\
&\geq C \sum_\lambda \sum_{j=0}^{k-1} \|d^*\omega_\lambda\|^2_{L^2_{j+1}} \cdot \|f_\lambda\|^2_{L^2_{k-1-j}} + \sum_{\lambda \neq 0} \sum_{j=0}^{k-1} \lambda^{2(k-j)} \|v_\lambda\|^2_{L^2_j} \cdot \|f_\lambda\|^2_{L^2} \\
&\geq C \| \sum_\lambda v_\lambda \otimes f_\lambda \|^2_{L^2_k} = C \|u\|^2_{L^2_k}.
\end{aligned}
$$

Step 8. Let us consider the final case as follows, which is a linear combination of Steps 5, 6, and 7:

$$
u = \sum_\lambda v_\lambda \otimes f_\lambda + g_\lambda \otimes u_\lambda \in L^2_k(X \times Y; \Lambda^1_X \otimes \Lambda^0_Y \oplus \Lambda^0_X \otimes \Lambda^1_Y).
$$

Again, we obtain closedness of the differential by checking the property for each degree of the differential forms on $\Lambda^*(X) \otimes \Lambda^*(Y)$ by Step 4. The only remaining case to be checked is closedness of the image in $\Lambda^1(X) \otimes \Lambda^1(Y)$.

Let us consider the differential

$$\begin{aligned} d^1 : u = \sum_\lambda (\beta_\lambda + d^*\omega_\lambda) \otimes f_\lambda + g_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \\ \mapsto \sum_\lambda (\beta_\lambda + d^*\omega_\lambda) \otimes df_\lambda + dg_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda), \end{aligned}$$

where both α_λ and β_λ are harmonic 1-forms. Then, it follows from the equalities

$$\begin{aligned} \langle \beta_\lambda, d^*\omega_\lambda \rangle_{L^2_j} = \langle \beta_\lambda, dg_\lambda \rangle_{L^2_j} = \langle d^*\omega_\lambda, dg_\lambda \rangle_{L^2_j} = 0, \\ \langle \alpha_\lambda, d^*\mu_\lambda \rangle_{L^2_j} = \langle \alpha_\lambda, df_\lambda \rangle_{L^2_j} = \langle df_\lambda, d^*\mu_\lambda \rangle_{L^2_j} = 0 \end{aligned}$$

that

$$\|d^1 u\|^2_{L^2_{k-1}} = \sum_\lambda \left\| (\beta_\lambda + d^*\omega_\lambda) \otimes df_\lambda \right\|^2_{L^2_{k-1}} + \left\| dg_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \right\|^2_{L^2_{k-1}}$$

holds. The first term is bounded as

$$\sum_\lambda \left\| (\beta_\lambda + d^*\omega_\lambda) \otimes df_\lambda \right\|^2_{L^2_{k-1}} \geq C \sum_\lambda \left\| (\beta_\lambda + d^*\omega_\lambda) \otimes f_\lambda \right\|^2_{L^2_k}$$

for some positive constant $C > 0$ by Step 7. The second term is bounded as

$$\sum_\lambda \left\| dg_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \right\|^2_{L^2_{k-1}} \geq C \sum_\lambda \left\| g_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \right\|^2_{L^2_k}$$

for some positive constant $C > 0$ by Steps 5 and 6.

Since the equality

$$\begin{aligned} \sum_\lambda \left\| (\beta_\lambda + d^*\omega_\lambda) \otimes f_\lambda \right\|^2_{L^2_k} + \sum_\lambda \left\| g_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \right\|^2_{L^2_k} \\ = \sum_\lambda \left\| (\beta_\lambda + d^*\omega_\lambda) \otimes f_\lambda + g_\lambda \otimes (\alpha_\lambda + d^*\mu_\lambda) \right\|^2_{L^2_k} = \|u\|^2_{L^2_k} \end{aligned}$$

holds, this completes the proof for all cases. ■

The universal covering space of Σ_g for $g \geq 2$ is the upper half plane $\mathbf{H}^2$ equipped with the hyperbolic metric. It is well known that the differential d on $\mathbf{H}^2$ has closed range on any degree [12]. Then, by Lemma 3.6, $\Sigma_g \times S^2$ satisfies two conditions that the AHS complex over their universal covering space has closed range. Moreover, the Dirac operator is invertible, because S^2 admits a metric of positive scalar curvature.

Let us compute the Γ-index of the AHS complex over $\mathbf{H}^2 \times S^2$ with $\Gamma = \pi_1(\Sigma_g)$ actions. We denote by $H^*_\Gamma(\Sigma_g \times S^2)$ the L^2 cohomology group $H^*_{(2)}(\mathbf{H}^2 \times S^2)$ equipped with $\Gamma = \pi_1(\Sigma_g)$ action.

Lemma 3.8. *For any $g \geq 2$, the L^2 cohomology group $H^*_\Gamma(\Sigma_g \times S^2)$ is zero for $* = 0, 2$ and satisfies* $\dim_\Gamma H^1_\Gamma(\Sigma_g \times S^2) = 2g - 2$.

Proof. The proof consists of two steps.

Step 1. Any L^2 harmonic function over a complete non-compact manifold is zero and, hence, $H^0_\Gamma(\Sigma_g) = 0$. By the Hodge duality, any L^2 harmonic 2-forms on $\mathbf{H}^2$ are also zero. It follows from Atiyah's Γ-index theorem that

$$\dim_\Gamma H^1_\Gamma(\Sigma_g) = 2g - 2.$$

Since $H^1_\Gamma(\Sigma_g)$ is isomorphic to the space of L^2 harmonic 1-forms, we have the estimate

$$\dim_\Gamma H^1_\Gamma(\Sigma_g \times S^2) \geq 2g - 2.$$

Step 2. We claim that the above estimate is actually equal. Let $\alpha \in L^2(\mathbf{H}^2 \times S^2; \Lambda^1)$ be an L^2 harmonic 1-form, and decompose

$$\alpha = \alpha_1 + \alpha_2$$

with respect to $\Lambda^1_{\mathbf{H}^2 \times S^2} \cong \Lambda^1_{\mathbf{H}^2} \otimes \Lambda^0_{S^2} \oplus \Lambda^0_{\mathbf{H}^2} \otimes \Lambda^1_{S^2}$. Note that each component lies in $\alpha_i \in L^2_k(\mathbf{H}^2 \times S^2; \Lambda^1)$ for any $k \geq 0$. It follows from $d(\alpha) = d_1(\alpha) + d_2(\alpha) = 0$ that

$$0 = d_1(\alpha_1) \in L^2_k\big(\mathbf{H}^2 \times S^2; \Lambda^2_{\mathbf{H}^2} \otimes \Lambda^0\big) \quad \text{and} \quad 0 = d_2(\alpha_2) \in L^2_k\big(\mathbf{H}^2 \times S^2; \Lambda^0_{\mathbf{H}^2} \otimes \Lambda^2\big)$$

hold, where d_1 and d_2 are the differentials with respect to $\mathbf{H}^2$ and S^2-coordinates, respectively.

Note the isomorphism

$$L^2\big(\mathbf{H}^2 \times S^2; \Lambda^0_{\mathbf{H}^2} \otimes \Lambda^1_{S^2}\big) \cong L^2(\mathbf{H}^2) \otimes L^2(S^2; \Lambda^1).$$

Let $\{f_\lambda\}_\lambda$ be the spectral decomposition of $L^2(S^2)$, as in the proof of Lemma 3.6 with $Y = S^2$. We can decompose it as

$$\alpha_2 = \sum_\lambda k_\lambda \otimes d_2 f_\lambda$$

since $H^1(S^2) = 0$ holds.

Next, we decompose α_1 as $\alpha_1 = \sum_\lambda a_\lambda \otimes f_\lambda$, where each $a_\lambda \in L^2(\mathbf{H}^2; \Lambda^1)$. Since $d_1(\alpha_1) = 0$ vanishes, we also have $d_1 a_\lambda = 0$. Thus, we can write $a_\lambda = h_\lambda + d_1 g_\lambda$, where h_λ is an L^2 harmonic 1-form. Then, we have

$$\alpha_1 = \sum_\lambda (h_\lambda + d_1 g_\lambda) \otimes f_\lambda.$$

It follows from $d_2\alpha_1 + d_1\alpha_2 = 0$ that

$$\sum_\lambda \{-h_\lambda - d_1 g_\lambda + d_1 k_\lambda\} \otimes d_2 f_\lambda = 0. \tag{3.2}$$

We claim that the equality

$$\{-h_\lambda - d_1 g_\lambda + d_1 k_\lambda\} \otimes d_2 f_\lambda = 0 \tag{3.3}$$

holds. In fact, by applying d_2^* on both sides of (3.2), we obtain

$$0 = \sum_\lambda \{-h_\lambda - d_1 g_\lambda + d_1 k_\lambda\} \otimes d_2^* d_2 f_\lambda$$
$$= \sum_\lambda \{-h_\lambda - d_1 g_\lambda + d_1 k_\lambda\} \otimes \lambda \cdot f_\lambda.$$

Since $\{f_\lambda\}_\lambda$ consists of an orthonormal basis of $L^2(S^2)$, we obtain the equality (3.3).

This implies $h_\lambda = 0$ and $d_1(g_\lambda - k_\lambda) = 0$ for $\lambda \neq 0$. Hence, we can assume $g_\lambda = k_\lambda$ in the expression of α_1.

For the $\lambda = 0$ case, f_0 is clearly constant. Hence, α has the form

$$\alpha = (h + d_1 g) \otimes 1 + \sum_\lambda d(g_\lambda \otimes f_\lambda),$$

where h is an L^2 harmonic 1-form.

Since α is harmonic, both the equalities

$$d_1 g = 0 \quad \text{and} \quad \sum_\lambda d(g_\lambda \otimes f_\lambda) = d \sum_\lambda g_\lambda \otimes f_\lambda = 0$$

should hold. This implies that any L^2 harmonic 1-form on $\mathbf{H}^2 \times S^2$ can be given by tensoring an L^2 harmonic 1-form on $\mathbf{H}^2$ with a constant on S^2. ∎

3.4. Some estimates over non-compact four-manifolds

In order to apply L^p estimates over non-compact spaces, let us induce some basic inequalities. Let M be a compact four-manifold and let $X = \tilde{M}$ be the universal covering space with $\Gamma = \pi_1(M)$.

Lemma 3.9. *For $p \geq 2$, the global Sobolev embeddings hold:*

$$L^p_{i+1}(X) \subset L^{2p}_i(X).$$

Proof. Let $K \subset X$ be a fundamental domain. Then, the local Sobolev estimate gives the embedding $L^p_{i+1}(K) \subset L^{2p}_i(K)$ in Lemma 3.2 (1).

Now, we take $a \in L^p_{i+1}(X)$. Then, we have the estimate

$$\|a\|_{L^p_{i+1}(\gamma(K))} \geq c\|a\|_{L^{2p}_i(\gamma(K))},$$

where c is independent of $\gamma \in \Gamma$. Thus, we have the estimates

$$\|a\|^{2p}_{L^{2p}_i(X)} = \Sigma_{\gamma\in\Gamma}\|a\|^{2p}_{L^{2p}_i(\gamma(K))} \leq c\Sigma_{\gamma\in\Gamma}\|a\|^{2p}_{L^p_{i+1}(\gamma(K))}$$
$$\leq c\big(\Sigma_{\gamma\in\Gamma}\|a\|^p_{L^p_{i+1}(\gamma(K))}\big)^2 = c\|a\|^{2p}_{L^p_{i+1}(X)}.$$

See [14, Chapter 1, Theorem 3.4]. ∎

Corollary 3.10. *Let* $p = 2^l \geq 2$.

(1) *The embeddings hold:*

$$\mathfrak{H}^{m,p}(X) \supset \mathfrak{H}^{m,2}(X),$$

between the L^p *and* L^2 *harmonic* m*-forms.*

Proof. Let $p = 2^l$. It follows from Lemma 3.8 that the embeddings hold:

$$L^2_{i+l-1}(X) \subset L^{2^2}_{i+l-2}(X) \subset \cdots \subset L^p_i(X).$$

Then, the conclusion holds since L^p harmonic forms have finite Sobolev norms in all L^p_k. ∎

Remark 3.11. One may consider the converse embedding. So far, there has not been significant development of analysis, even though it is a quite basic subject.

3.5. L^p closedness

We assume that the de Rham differential has closed range on L^2 such that it admits the L^2 harmonic projection.

Let us take $a \in L^2_k(X; \Lambda^1)$ for some large $k \gg 1$. It follows from Lemma 3.9 that $a \in L^p_1(X; \Lambda^1) \cap L^{2p}_1(X; \Lambda^1)$ for $p = 2^l$ with $l \leq k - 1$. Suppose the conditions

(1) $\|a\|_{L^p_1(X)} \leq C$,

(2) $\|a\|_{L^p(K)} \geq \varepsilon_0$, and

(3) $d^*(a) = 0$,

hold for some constants C and $\varepsilon_0 > 0$, and a compact subset $K \subset X$. Let us denote the L^2 harmonic projection of a by $a_{\text{harm}} \in \mathfrak{H}^{1,2}$ and its L^2_1 norm by $\|a\|_{\text{harm}}$.

Lemma 3.12. (α) *Assume the above three conditions* (1), (2), *and* (3). *Then, at least one of the following criteria holds:*

- *the following estimates hold:* $\|a\|_{L^{2p}_1(X)} \leq c(\|d^+ a\|_{L^{2p}(X)} + \|a\|_{\text{harm}})$, *for some* $c > 0$ *independent of* a, *or*
- *there is a sequence* $\{a_i\}_i$ *as above that they weakly converge in* $L^p_1 \cap L^{2p}_1$ *to a non-zero element in* $\mathfrak{H}^{1,p} \cap \mathfrak{H}^{1,2p}$, *but not in* $\mathfrak{H}^{1,2}$.

(β) *Assume the above condition* (3). *Then, the estimate holds:*

$$\|a\|_{L^{2p}_1(X)} \leq c\big(\max\big\{\|d^+ a\|_{L^{2p}(X)}, \|a\|_{L^p_1(X)}\big\} + \|a\|_{\text{harm}}\big).$$

Proof. Let us verify (α). Assume that a family $\{a_i\}_i$ with the above conditions satisfies the property

$$\|a_i\|_{L^{2p}_1(X)} = \delta_i^{-1}\big(\|d^+(a_i)\|_{L^{2p}(X)} + \|a\|_{\text{harm}}\big)$$

for some $\delta_i \to 0$.

Let us divide this situation into two cases.

(a) Suppose $\|a_i\|_{L_1^{2p}(X)}$ are uniformly bounded. Then, $\{a_i\}_i$ weakly converges to some $a \in L_1^{2p}(X;\Lambda^1) \cap L_1^p(X;\Lambda^1)$ by condition (1) and the standard local elliptic estimate. Moreover, the equalities $d^+(a) = 0$, $a_{\text{harm}} = 0$, and $d^*(a) = 0$ hold by condition (3) since both the convergences $\|d^+(a_i)\|_{L^{2p}(X)}$ and $\|a_i\|_{\text{harm}} \to 0$ hold. Note that L^p spaces are reflective for $1 < p < \infty$.

The restriction $a_i|K$ strongly converges to $a|K$ in L^p. It follows from condition (2) above that a is non-zero and, hence, gives a non-trivial element in $\mathfrak{H}^{1,p} \cap \mathfrak{H}^{1,2p}$, but not in $\mathfrak{H}^{1,2}$ since the L^2 harmonic part of a is zero.

(b) Assume $\|a_i\|_{L_1^{2p}(X)} \to \infty$. Then, the following estimates hold:

$$\begin{aligned}\|a_i\|_{L_1^{2p}(X)} &\le c\big(\big\|d^+(a_i)\big\|_{L^{2p}(X)} + \|a_i\|_{L^{2p}(X)}\big)\\ &\le c\big\{\delta_i\|a_i\|_{L_1^{2p}(X)} + \|a_i\|_{L^{2p}(X)}\big\},\end{aligned}$$

where the first inequality comes from the elliptic estimate. In particular, the following inequality holds:

$$\|a_i\|_{L_1^{2p}(X)} \le c'\|a_i\|_{L^{2p}}.$$

It follows from Lemma 3.9 that the estimate

$$\|a_i\|_{L_1^{2p}(X)} \le c'\|a_i\|_{L^{2p}} \le c''\|a_i\|_{L_1^p}.$$

must hold. The left-hand side diverges while $\|a_i\|_{L_1^p}$ are uniformly bounded by condition (1). Therefore, case (b) does not happen.

Next, we consider (β). If the former conclusion of (α) holds, then we are done. Otherwise, we can take a decreasing sequence $\delta_i \to 0$ as in the above proof. Then, the same estimates as above give the inequality

$$\|a_i\|_{L_1^{2p}(X)} \le c\|a_i\|_{L_1^p(X)}.$$

The conclusion is just a combination of these cases. ■

Remark 3.13. For the purpose of our analysis of the covering-monopole map in Section 4, any $p > 2$ suffices, but the $p = 2$ case is not sufficient. Hereinafter, we will use β only.

3.6. Multiplication estimates

Let $X = \widetilde{M}$ be the universal covering space of a compact four-manifold M with $\pi_1(M) = \Gamma$, and let $K \subset X$ be a fundamental domain.

Lemma 3.14. *The multiplication*

$$L_k^2(X) \otimes L_k^2(X) \to L_m^2(X)$$

is bounded for $m \le k$, with $k \ge 3$, or $m < k$, with $k \ge 1$.

Proof. For the proof, we refer to [14, Chapter 1, Theorem 3.12]. We include the proof for convenience.

Let us take $a, b \in L^2_k(X)$ where $a = \sum_{\gamma \in \Gamma} a_\gamma$ with $a_\gamma \in L^2_k(\gamma(K))$. By Corollary 3.3, the local Sobolev multiplication gives the estimates

$$\|a_\gamma b_\gamma\|_{L^2_m} \leq C \|a_\gamma\|_{L^2_k} \|b_\gamma\|_{L^2_k},$$

where C is independent of $\gamma \in \pi_1(M)$. Therefore,

$$\begin{aligned}\|ab\|^2_{L^2_m(X)} &= \sum_\gamma \|a_\gamma b_\gamma\|^2_{L^2_m} \leq C \sum_\gamma \|a_\gamma\|^2_{L^2_k} \|b_\gamma\|^2_{L^2_k} \\ &\leq C \Big(\sum_\gamma \|a_\gamma\|_{L^2_k} \Big)^2 \Big(\sum_\gamma \|b_\gamma\|_{L^2_k} \Big)^2 \\ &= \|a\|^2_{L^2_k(X)} \|b\|^2_{L^2_k(X)}.\end{aligned}$$ ■

Lemma 3.15. *Let $m > 2$ and choose $0 < \varepsilon < 1$. Then, there is a constant $C = C_K$ independent of $\varepsilon > 0$ such that if two elements $a, b \in L^2_{2m}(X)$ with $\|a\|_{L^2_{2m}(X)} = \|b\|_{L^2_{2m}(X)} = 1$ satisfy uniform estimates*

$$\|a\|_{L^2_{2m}(\gamma K)}, \|b\|_{L^2_{2m}(\gamma K)} < \varepsilon$$

for all $\gamma \in \Gamma$, then the following estimate holds:

$$\|ab\|_{L^2_{2m}(X)} < C\varepsilon.$$

Proof. It follows from the local Sobolev embedding $L^2_{2m} \hookrightarrow C^m$ in Lemma 3.2 (2) that the estimates hold for all $\gamma \in \Gamma$:

$$\|a\|_{C^m(\gamma K)}, \|b\|_{C^m(\gamma K)} < C\varepsilon.$$

This implies the global estimates $\|a\|_{C^m(X)}, \|b\|_{C^m(X)} < C\varepsilon$.

Consider the absolute values of the derivatives:

$$\left|\nabla^l(ab)\right| \leq \Sigma^l_{s=0} \left|\nabla^s(a)\right| \left|\nabla^{l-s}(b)\right|$$

for $l \leq 2m$, where each component of the right-hand side satisfies the property that one of s or $l - s$ is less than or equal to m. Suppose $s \leq m$ holds. Then,

$$\left|\nabla^s(a)\right|^2 \left|\nabla^{l-s}(b)\right|^2 \leq C^2\varepsilon^2 \left|\nabla^{l-s}(b)\right|^2 \leq C^2\varepsilon^2 \big(\left|\nabla^s(a)\right|^2 + \left|\nabla^{l-s}(b)\right|^2 \big).$$

By the same argument, we can obtain the same estimate when $l - s \leq m$ holds. Therefore, in any case, the following estimate holds:

$$\left|\nabla^l(ab)\right|^2 \leq C'\varepsilon^2 \Sigma^l_{s=0} \big(\left|\nabla^s(a)\right|^2 + \left|\nabla^s(b)\right|^2 \big).$$

Now, we obtain the estimate by integration:

$$\|ab\|_{L^2_{2m}(X)} \leq C\varepsilon \big(\|a\|_{L^2_{2m}(X)} + \|b\|_{L^2_{2m}(X)} \big).$$ ■

Corollary 3.16. *There is a constant $C = C_K$ such that, for two elements $b \in L^2_m(X)$ and $a \in L^2_{2m}(X)$, if a satisfies the uniformly small estimate*

$$\|a\|_{L^2_{2m}(\gamma K)} < \varepsilon$$

for all $\gamma \in \Gamma$, then the following estimate holds:

$$\|ab\|_{L^2_m(X)} < C\varepsilon \|b\|_{L^2_m(X)}.$$

Proof. Consider the absolute values of the derivatives:

$$\left|\nabla^l(ab)\right| \leq \Sigma_{s=0}^l \left|\nabla^s(a)\right|\left|\nabla^{l-s}(b)\right|$$

for $l \leq m$. Then, the same argument as above gives the estimate

$$\left|\nabla^l(ab)\right|^2 \leq C'\varepsilon^2 \Sigma_{s=0}^l \left|\nabla^s(b)\right|^2.$$

Hence, we obtain

$$\|ab\|_{L^2_m(X)} \leq C\varepsilon \|b\|_{L^2_m(X)}.$$ ∎

Remark 3.17. Let $K \subset X$ be a fundamental domain and choose a finite set

$$\bar{\gamma} := \{\gamma_1, \ldots, \gamma_m\} \subset \pi_1(M) = \Gamma.$$

For a positive constant $\varepsilon > 0$, let us set

$$H'(\varepsilon, \bar{\gamma}) := \left\{w \in L^2_k(X; E) = H' : \|w\|_{L^2_k(\gamma K)} < \varepsilon, \ \gamma \notin \bar{\gamma}\right\}.$$

(1) For any r and the open ball $B_r \subset H'$ of radius r, there is some m such that the embedding $B_r \subset H'(\varepsilon, m) := \bigcup_{\bar{\gamma}\in\Gamma^m} H'(\varepsilon, \bar{\gamma})$ holds.

(2) By Lemma 3.15, there is a constant C such that the covering-SW map restricts:

$$F : H'(\varepsilon, \bar{\gamma}) \to H(C\varepsilon, \bar{\gamma}).$$

3.7. Locality of linear operators

Let $K \subset X$ be a compact subset. Recall the local Sobolev space $L^2_k(K)_0$ of Definition 1.1. Suppose $l : L^2_k(X) \to L^2_{k-1}(X)$ is a first-order differential operator and consider its restriction

$$l : L^2_k(K)_0 \to L^2_{k-1}(K)_0$$

between the Sobolev spaces on a compact subset K. Let us take an element

$$w \in l\big(L^2_k(X)\big) \cap L^2_{k-1}(K)_0$$

and ask when w lies in the image $l(L^2_k(K)_0)$. In general, this is not always the case. Later, when we consider properness of the covering-monopole map, we shall use projections to the Sobolev spaces on compact subsets. Here let us observe a general analytic property.

Let us introduce a K-*spill* $e(w) \in [0, \infty)$ by

$$e(w; K) = \inf_{v \in L^2_k(X)} \{\|v\|_{L^2_k(K^c)} : l(v) = w\}.$$

Let $K_0 \Subset K_1 \Subset K_2 \subset \cdots \subset X$ be exhaustion.

Lemma 3.18. *Suppose $l : L^2_k(X) \to L^2_{k-1}(X)$ is injective with closed range. Choose $w_i \in L^2_{k-1}(K_i)_0 \cap l(L^2_k(X))$, which converge to $w \in L^2_{k-1}(X)$.*

Then, the spills go to zero as

$$e(w_i; K_i) \to 0.$$

Proof. By the assumption, there are $v_i \in L^2_k(X)$ that converge to $v \in L^2_k(X)$ with $l(v_i) = w_i$ and $l(v) = w$. For small $\varepsilon > 0$, there is i_0 such that $\|w - w_i\|_{L^2_{k-1}(X)} < \varepsilon$ holds for $i \geq i_0$. Hence, the estimates $\|v - v_i\|_{L^2_k(X)} \leq C\varepsilon$ hold for a constant C.

Suppose there exists $\delta > 0$ with $e(w_i; K_i) \geq \delta$. Then, we have

$$\|v\|_{L^2_k(K_i^c)} \geq \|v_i\|_{L^2_k(K_i^c)} - \|v - v_i\|_{L^2_k(K_i^c)} \geq \delta - C\varepsilon > 0$$

for all sufficiently large i, which cannot happen since the L^2_k-norm of v is finite on X. Then, its L^2_k-norms on the complements of K_i should go to zero because $\{K_i\}_i$ exhausts X. ∎

3.8. More Sobolev estimates

Here, we verify the Sobolev estimate, which improves the original version to the most general way. Note that the estimate will not be used in later sections.

Lemma 3.19. *Suppose*

(1) $k - \frac{4}{p} \geq l - \frac{4}{q}$ *with* $k \geq l$, *and*

(2) $p \leq q$.

Then, the embeddings $L^p_k(X) \subset L^q_l(X)$ hold over the universal covering space $X = \tilde{M}$ of a compact four-manifold.

Proof. For the proof, we refer to [14, Chapter 1, Theorem 3.4]. We include the proof for convenience.

It follows from the assumption (1) that the local Sobolev embedding $(L^p_k)_{\mathrm{loc}} \subset (L^q_l)_{\mathrm{loc}}$ holds by Lemma 3.2 (1).

Take $a \in L^p_k(X)$. Then, we obtain

$$(\|a\|_{L^p_k(X)})^q = \left(\Sigma_{\gamma \in \Gamma} \|a\|^p_{L^p_k(\gamma(K))}\right)^{\frac{q}{p}} \geq c\left(\Sigma_{\gamma \in \Gamma} \|a\|^p_{L^q_l(\gamma(K))}\right)^{\frac{q}{p}}.$$

We want to verify the inequality

$$\left(\Sigma_{\gamma \in \Gamma} \|a\|^p_{L^q_l(\gamma(K))}\right)^{\frac{q}{p}} \geq \Sigma_{\gamma \in \Gamma} \|a\|^q_{L^q_l(\gamma(K))}.$$

The following sublemma completes the proof of the lemma.

Sublemma 3.20. *Let* $\{a_i\}_{i=0}^{\infty}$ *be a non-negative sequence. Then, the estimates*

$$\sum_i a_i \leq \Big(\sum_i a_i^{t^{-1}} \Big)^t$$

hold for $t \geq 1$.

This elementary fact follows from the sub-additivity of the function $x \mapsto x^{t^{-1}}$. ■

4. Properness of the monopole map

A metrically proper map between Hilbert spaces is defined by the property that the pre-image of a bounded set is also bounded. The method of constructing a finite-dimensional approximation requires, in addition, the property that the restriction on any bounded set is proper.

It is a characteristic of infinite dimensionality that there exists a metrically proper map which is not proper on each bounded set. For example, for an infinite-dimensional Hilbert space H, the distance function $d : H \to \mathbb{R}$ by $x \to \|x\|$ is metrically proper but is not proper on each bounded set, because the restriction $d : D \to [0, 1]$ on the unit disk $D \subset H$ is not proper.

A map is called *strongly proper* if it satisfies these two properties (see Definition 1.1). Both properties are satisfied for the monopole map over a compact four-manifold, because it is Fredholm.

In our case, the base space is non-compact, and we will verify the locally strong properness under the assumption of closedness of the AHS complex. This also works for the construction of a finite-dimensional approximation method.

In Section 4, we assume $k \geq 3$ whenever we write L^2_k. Moreover, we continue to assume that a compact subset $K \subset X$ is a compact smooth submanifold of codimension zero, possibly with smooth boundary.

Let M be a compact oriented smooth four-manifold and let $X = \tilde{M}$ be the universal covering space with $\pi_1(M) = \Gamma$.

Let us fix a spin^c structure on M and choose a solution (A_0, ψ_0) to the SW equations over M. Note that the pair (A_0, ψ_0) is smooth after gauge transform (see [30, Theorem 5.3.6]). Hereinafter, we always assume that the base solution (A_0, ψ_0) is smooth. We take their lift $(\tilde{A}_0, \tilde{\psi}_0)$ over X as a base point of the covering-monopole map.

In this section, we verify the following property.

Theorem 4.1. *Suppose the AHS complex has closed range over* X*. Then, the covering-monopole map is locally strongly proper in the sense that the map*

$$\begin{aligned} \tilde{\mu} : L^2_k(K; \tilde{S}^+)_0 \oplus L^2_k(X; \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^* \\ \to L^2_{k-1}(K; \tilde{S}^-)_0 \oplus L^2_{k-1}(X; \Lambda^2_+ \otimes i\mathbb{R}) \oplus H^1_{(2)}(X) \\ (\psi, a) \mapsto \big(D_{\tilde{A}_0 + a}\psi, d^+(a) - \sigma(\tilde{\psi}_0, \psi), [a] \big) \end{aligned}$$

is strongly proper for any compact subset $K \subset X$.

If the linearized operator of the covering-monopole map gives an isomorphism, then the AHS complex has closed range. Hence, in such a case, the conclusion holds. The stronger condition is required when the construction of Clifford C^*-algebra is involved.

Remark 4.2. The statement involves a mixture of spaces X and its subset K. This is because one can control the analytic behavior of differential forms by assuming closedness of the differentials, however there is no way to control it for the spinors. This is the reason why we have to be content with the restriction of the compactly supported spinors.

Proof of Theorem 4.1. The proof consists of three steps.

Step 1. Let us consider the properness on the restriction on bounded sets. The non-linear term is given by

$$c : (\psi, a) \mapsto \big(a\psi, \sigma(\psi)\big).$$

We claim that this is a compact mapping such that it maps a bounded set into a relatively compact subset.

Let us take another compact subset $K \Subset K'$ and let $\varphi : K' \to [0,1]$ be a smooth cut-off function with $\varphi|K \equiv 1$ that vanishes near the boundary of K'. Then, the multiplication by φ satisfies the inclusion

$$\varphi \cdot \operatorname{Ker} d^* \subset L^2_k(K'; \Lambda^1 \otimes i\mathbb{R})_0.$$

Then, c factors through the multiplication

$$(\mathrm{id}, \varphi) : L^2_k(K; \widetilde{S}^+)_0 \oplus \operatorname{Ker} d^* \to L^2_k(K; \widetilde{S}^+)_0 \oplus L^2_k(K'; \Lambda^1 \otimes i\mathbb{R}), \qquad (*)$$

$$c : L^2_k(K; \widetilde{S}^+)_0 \oplus L^2_k(K'; \Lambda^1 \otimes i\mathbb{R})_0 \to L^2_{k-1}(K; \widetilde{S}^-)_0 \oplus L^2_{k-1}(K'; \Lambda^2_+ \otimes i\mathbb{R})_0. \quad (**)$$

The former map (*) is linear and bounded. The second map (**) is compact as it factors through the inclusion $L^2_k(K')_0 \hookrightarrow L^2_{k-1}(K')_0$ by Lemma 3.14, and the last map is compact.

Step 2. To confirm properness of $\tilde{\mu}$ on the restriction on a bounded set, it is sufficient to confirm the metrical properness of its linear term, because the non-linear term is compact on the restriction of a bounded set by Step 1. The linear term is given by

$$(\psi, a) \mapsto \big(D_{\tilde{A}_0}\psi, d^+(a) - \sigma'(\tilde{\psi}_0, \psi), [a]\big),$$

where $\sigma'(\tilde{\psi}_0, \psi) = \sigma'(\tilde{\psi}_0, \psi) - \sigma(\psi)$.

It follows from the closedness of d^+ that $a \mapsto (d^+(a), [a])$ is injective and metrically proper (see the last paragraph of Step 2 in the proof of Lemma 4.8). Since the term $\sigma'(\tilde{\psi}_0, \psi)$ does not involve a, it is sufficient to confirm the properness of $D_{\tilde{A}_0}\psi$. Properness surely holds on any bounded sets over K as $D_{\tilde{A}_0}$ is an elliptic operator.

Step 3. Let us consider metrically properness. This follows from the combination of Lemma 4.8 and Proposition 4.10 with Lemma 4.11 which is a version of Lemma 4.5. These are all verified later in this section. ∎

We verify the globally strong properness for a particular class, which is stronger than locally strong properness.

Proposition 4.3. *Suppose the AHS complex has closed range over $X = \widetilde{M}$ whose second L^2 cohomology $H^+_{(2)}(X) = 0$ vanishes.*

If the metric on M has a positive scalar curvature, then the covering-monopole map is metrically proper and locally proper on each bounded set.

We present examples that satisfy the above conditions. Note that $S^2 \times \Sigma_g$ are the cases for all $g \geq 2$.

We have already seen the latter property above and, hence, need only to verify the metrically properness.

Our strategy is as follows. Assume that the AHS complex has closed range. Then, we verify the following:

(α) metrical properness for $k = 1$ under the assumption of existence of a metric of positive scalar curvature (Lemma 4.5),

(β) local metric properness for $k = 1$ (Lemma 4.11) under the assumption of (local) L^∞ bound,

(γ) (local) metric properness for $k \geq 1$ under the additional two assumptions of (local) L^∞ bound and (locally) metrical properness for $k = 1$ (Lemma 4.8), and

(δ) local L^∞ bound (Proposition 4.10).

Remark 4.4. Let $B \subset M$ be a small open ball. There exists a Riemannian metric g whose scalar curvature is positive except B [27]. One may assume that the lift $\widetilde{B} \subset X$ satisfies $\gamma(\widetilde{B}) \cap \widetilde{B} = \phi$ for all $\gamma \neq \mathrm{id} \in \Gamma := \pi_1(M)$. Let us set $\bar{B} := \bigcup_{\gamma \in \Gamma} \gamma(\widetilde{B})$. Assume that $\widetilde{\mu}$ could be metrically non-proper and choose $\widetilde{\mu}(x_i) = y_i$ such that $\|y_i\| \leq c < \infty$ while $\|x_i\| \to \infty$. Let φ be a cut-off function with $\varphi|\bar{B} \equiv 1$ and zero outside a small neighborhood of $\bar{B}$. Both two families $\widetilde{\mu}|\{(1-\varphi)x_i\}_i$ and $\widetilde{\mu}|\{\varphi x_i\}_i$ must be proper (see Lemma 4.5 below) and, hence, $\{x_i\}_i$ should be unbounded near the boundary of some $\gamma(\bar{B})$ and $\gamma \in \Gamma$.

4.1. Positive scalar curvature metric

Let us verify the metric properness of the covering-monopole map, in the case when the base manifold M admits a metric of positive scalar curvature and the AHS complex over $X = \widetilde{M}$ is closed.

Let us fix a reducible solution $(A_0, 0)$ to the SW equations over M and choose the pair as the base point of the covering-monopole map.

Lemma 4.5. *Suppose the AHS complex has closed range over $X = \widetilde{M}$. If M admits a Riemannian metric of positive scalar curvature, then the covering-monopole map*

$$\widetilde{\mu} : L^2_1\big((X, g); \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \cap \operatorname{Ker} d^* \to L^2\big((X, g); \widetilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X)$$
$$(\phi, a) \mapsto \big(F_{\widetilde{A}_0,0}(\phi, a), [a]\big)$$

is metrically proper, where $H^1_{(2)}(X)$ is the first L^2 cohomology group.

Proof. Let us set $\tilde{\mu}(\phi, a) = (\varphi, b, h)$ and denote $A = \tilde{A}_0 + a$.

Step 1. We have the pointwise equalities

$$(F_A\phi, \phi) = (F_A^+\phi, \phi) = \big((F_A^+ - \sigma(\phi))\phi + \sigma(\phi)\phi, \phi\big) = (b\phi, \phi) + \frac{|\phi|^4}{2}$$

since $F_A\phi = F_A^+\phi$ holds.

Suppose M admits a Riemannian metric of positive scalar curvature. Then, from the Weitzenböck formula

$$D_A^2(\phi) = \nabla_A^*\nabla_A(\phi) + \frac{\kappa}{4}\phi + \frac{F_A}{2}\phi,$$

it follows that the estimates

$$\big\| D_A(\phi)\big\|_{L^2}^2 + \|b\|_{L^2}\|\phi\|_{L^4}^2 \geq \delta\|\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^4}^4 \geq \frac{1}{4}\|\phi\|_{L^4}^4$$

hold for some positive $\delta > 0$. In particular, there is $c = c(\|\varphi\|_{L^2}, \|b\|_{L^2})$ such that the bound

$$\|\phi\|_{L^4} \leq c$$

holds. Using another estimate

$$\|\varphi\|_{L^2}^2 + \|b\|_{L^2}\|\phi\|_{L^4}^2 \geq \delta\|\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^4}^4 \geq \delta\|\phi\|_{L^2}^2,$$

we obtain the L^2 estimate

$$\|\phi\|_{L^2} \leq c'\big(\|\varphi\|_{L^2}, \|b\|_{L^2}, \delta\big).$$

Step 2. From the equality $d^+(a) = b + \sigma(\phi)$, it follows that

$$\|a\|_{L_1^2} \leq c\big(\|d^+(a)\|_{L^2} + \|a_{\text{harm}}\|_{L^2}\big) \leq c\big(\|b\|_{L^2} + \|\phi\|_{L^4}^2 + \|h\|_{L^2}\big).$$

Combining this with Step 1, we obtain

$$\|a\|_{L_1^2} \leq c\big(\|\varphi\|_{L^2}, \|b\|_{L^2}, \|h\|_{L^2}, \delta\big).$$

Step 3. It follows from the embedding $L_1^2 \subset L^4$ in Lemma 3.9 that

$$\|a\phi\|_{L^2} \leq \|a\|_{L^4}\|\phi\|_{L^4} \leq c\big(\|\varphi\|_{L^2}, \|b\|_{L^2}, \|h\|_{L^2}, \delta\big).$$

Then,

$$\big\| D_{\tilde{A}_0}(\phi)\big\|_{L^2} \leq \big\| D_A(\phi)\big\|_{L^2} + \|a\phi\|_{L^2} < \infty.$$

It follows from the elliptic estimate that in L_1^2, we have the bound

$$\|\phi\|_{L_1^2} \leq c\big(\|\varphi\|_{L^2}, \|b\|_{L^2}, \|h\|_{L^2}, \delta\big).$$

■

Remark 4.6. To induce higher Sobolev estimates, it is sufficient to obtain the estimate of $\|a\phi\|_{L^2_1}$. Certainly, we can obtain the estimate

$$\|a\phi\|_{L^2_k} \leq C_k \|a\|_{L^2_k} \|\phi\|_{L^2_k}$$

for $k \geq 3$, but it is not applied at $k = 1$. This forces us to use L^∞ estimates later, which leads to L^p analysis.

Example 4.7. Immediate examples of closed four-manifolds with positive scalar curvature metrics will be S^4 or $\Sigma_g \times S^2$ with their metrics $h + \varepsilon g$ with small $\varepsilon > 0$, where h and g are both the standard metrics. For the latter case, the AHS complex over their universal covering spaces has closed range by Lemma 3.6.

4.2. Regularity under L^∞ bounds

Let us take a solution (A_0, ψ_0) to the SW equations over M. One may assume that the solution is smooth. We consider the covering-monopole map with their lift $(\widetilde{A}_0, \widetilde{\psi}_0)$ as the base point.

Assume that the AHS complex has closed range and consider the monopole map

$$\widetilde{\mu} : L^2_k\big((X, g); \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}\big) \cap \operatorname{Ker} d^* \to L^2_{k-1}\big((X, g); \widetilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big) \oplus H^1(X)$$
$$(\phi, a) \mapsto \big(F_{\widetilde{A}_0, \widetilde{\psi}_0}(\phi, a), [a]\big).$$

It follows from Lemma 4.8, Proposition 4.10, and Lemma 4.11 below that $\widetilde{\mu}$ is metrically proper, under the conditions of $k \geq 3$ and closedness of the AHS complex.

Lemma 4.8. *Suppose that for any positive numbers $s, r > 0$, there is*

$$c = c(s, r, K, A_0, \psi_0) > 0$$

such that for any $k \geq 3$ and an element $(\phi, a) \in L^2_k((X, g); \widetilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \operatorname{Ker} d^*$ with $r := \|\widetilde{\mu}(a, \phi)\|_{L^2_{k-1}}$, the following conditions hold:

$$s := \big\|(\phi, a)\big\|_{L^\infty} < \infty, \quad \big\|(\phi, a)\big\|_{L^2_1} \leq c.$$

Then, there is a positive constant $c_k = c_k(s, r, K, A_0, \psi_0)$ such that the estimate $\|(\phi, a)\|_{L^2_k} \leq c_k$ holds.

Proof.

Step 1. Let us set $\widetilde{\mu}(\phi, a) = (\varphi, b, h)$. We check that the multiplication $a \cdot \phi$ is in $L^2_1(X)$ and bounded by $3sc$. First, $a \cdot \phi \in L^2(X)$ holds by the L^∞ bound

$$\|a \cdot \phi\|_{L^2} \leq \|a\|_{L^\infty} \|\phi\|_{L^2} \leq sc.$$

Note the equality $\nabla(a \cdot \phi) = \nabla(a)\phi + a\nabla(\phi)$. Then,

$$\big\|\nabla(a\phi)\big\|_{L^2} \leq \|\phi\|_{L^\infty} \big\|\nabla(a)\big\|_{L^2} + \|a\|_{L^\infty} \big\|\nabla(\phi)\big\|_{L^2} \leq 2sc.$$

Hence, we have $\|a \cdot \phi\|_{L^2_1} \leq 3sc$. Since ψ_0 is smooth over M which is compact, we have $\|a \cdot \tilde{\psi}_0\|_{L^2_1} \leq C(\psi_0)\|a\|_{L^2_1} \leq C(\psi_0)c$. Then, $D_{\tilde{A}_0}(\phi) \in L^2_1(X)$ holds, because the left-hand side of

$$\varphi = D_A(\phi) + a \cdot \tilde{\psi}_0 = D_{\tilde{A}_0}(\phi) + a \cdot \phi + a \cdot \tilde{\psi}_0$$

has an $L^2_{k-1}(X)$ norm less than r. Hence, the bound

$$\|\phi\|_{L^2_2} \leq C\big(r + C(\psi_0)c + 3sc + c\big) =: c'_2$$

holds by the elliptic estimate $\|\phi\|_{L^2_2} \leq C(\|D_{\tilde{A}_0}(\phi)\|_{L^2_1} + \|\phi\|_{L^2_1})$.

Step 2. Then, $\phi \in L^4_1(X)$ holds, because the embedding $L^2_2(X) \subset L^4_1(X)$ holds by Lemma 3.9. Let us denote $\sigma(\tilde{\psi}_0, \phi) =: \sigma(\phi) + l(\tilde{\psi}_0, \phi)$ (see Lemma 2.3). We obtain the estimates

$$\big\|\sigma(\phi)\big\|_{L^2} \leq \|\phi\|^2_{L^4} \leq \|\phi\|^2_{L^2_1} \leq c^2$$

and

$$\big\|l(\tilde{\psi}_0, \phi)\big\|_{L^2} \leq C(\psi_0)\|\phi\|_{L^2} \leq C(\psi_0)c.$$

For the derivatives, we have

$$\big\|\nabla\sigma(\phi)\big\|_{L^2} \leq \big\|\nabla(\phi)\big\|_{L^4}\|\phi\|_{L^4} \leq \|\phi\|_{L^4_1}\|\phi\|_{L^4} \leq C\|\phi\|_{L^2_2}\|\phi\|_{L^2_1} \leq Cc'_2c,$$
$$\big\|\nabla l(\tilde{\psi}_0, \phi)\big\|_{L^2} \leq C(\psi_0)\big(\|\phi\|_{L^2} + \|\nabla\phi\|_{L^2}\big) \leq C(\psi_0)\|\phi\|_{L^2_1} \leq C(\psi_0)c.$$

Hence, we have $\|\sigma(\tilde{\psi}_0, \phi)\|_{L^2_1} \leq C(\psi_0)cc'_2$. Then, the estimate

$$\|d^+(a)\|_{L^2_1(X)} \leq r + C(\psi_0)cc'_2$$

holds, because $b = d^+(a) - \sigma(\tilde{\psi}_0, \phi)$ has $L^2_{k-1}(X)$ norm that is less than r. Because the AHS complex has closed range, it follows from the open mapping theorem that there is a positive constant $C > 0$ such that the bound

$$\|a\|_{L^2_2} \leq C\big(\|d^+(a)\|_{L^2_1} + \|[a]\|_{L^2}\big)$$

holds for any $a \in L^2_2(X) \cap \operatorname{Ker} d^*$. Thus, the estimate holds:

$$\|a\|_{L^2_2} \leq C\big(r + C(\psi_0)cc'_2 + r\big) =: c_2.$$

Step 3. Let us verify $a \cdot \phi \in L^2_2(X)$. Note the equality

$$\nabla^2(a \cdot \phi) = \nabla^2(a) \cdot \phi + 2\nabla(a) \cdot \nabla(\phi) + a \cdot \nabla^2(\phi).$$

The L^2 norms of the first and last terms on the right-hand side are both bounded as $\|\nabla^2(a) \cdot \phi\|_{L^2}, \|a \cdot \nabla^2(\phi)\|_{L^2} \leq sc_2$. For the middle term, we have the estimates

$$\begin{aligned}\big\|\nabla(a) \cdot \nabla(\phi)\big\|_{L^2} &\leq \big\|\nabla(a)\big\|_{L^4}\big\|\nabla(\phi)\big\|_{L^4} \leq C\big\|\nabla(a)\big\|_{L^2_1}\big\|\nabla(\phi)\big\|_{L^2_1} \\ &\leq C\|a\|_{L^2_2}\|\phi\|_{L^2_2} \leq Cc_2^2,\end{aligned}$$

where we used Lemma 3.9. Hence, we have

$$\|a\cdot\phi\|_{L^2_2} \le \|a\cdot\phi\|_{L^2_1} + \left\|\nabla^2(a\cdot\phi)\right\|_{L^2} \le 3sc + 2sc_2 + Cc_2^2.$$

The remainder of the argument is parallel to Step 1. We have $\|D_{\widetilde{A}_0}(\phi)\|_{L^2_2} \le r + C_2(\psi_0)c_2 + 3sc + 2sc_2 + Cc_2^2$. Then, the estimate holds:

$$\begin{aligned}\|\phi\|_{L^2_3} &\le C_3\big(\|D_{\widetilde{A}_0}(\phi)\|_{L^2_2} + \|\phi\|_{L^2_2}\big)\\ &\le C_3\big(r + C_2(\psi_0)c_2 + 3sc + 2sc_2 + Cc_2^2 + c_2'\big) =: c_3'.\end{aligned}$$

By applying similar estimates in Step 2, we have $\sigma(\widetilde{\psi}_0,\phi) \in L^2_2(X)$ and, then, we obtain $d^+(a) \in L^2_2(X)$. Then, we have $\|a\|_{L^2_3(X)} \le c_3$.

Step 4. We have verified $a,\phi \in L^2_3(X)$. Now, we can use a simpler argument that the multiplication

$$L^2_3(X) \times L^2_3(X) \to L^2_3(X)$$

is continuous in Lemma 3.14 such that we can see the inclusion $a\cdot\phi \in L^2_3(X)$ immediately by the estimates

$$\|a\cdot\phi\|^2_{L^2_3(X)} \le C\|a\|^2_{L^2_3(X)}\|\phi\|^2_{L^2_3(X)} \le c_3^2.$$

Then, we repeat the latter part of Step 1 and Step 2. Then, we obtain the inclusions $\phi \in L^2_4(X)$ and $a \in L^2_4(X)$.

The remainder of the argument is parallel and we obtain the L^2_k bound of (a,ϕ) by a constant c_k. ∎

Remark 4.9. The above proof also verifies that one can restrict the functional spaces to $L^2_1(K;\widetilde{S}^+)_0 \oplus L^2_1(X;\Lambda^1\otimes i\mathbb{R})_0 \cap \operatorname{Ker} d^*$ and still obtain the same conclusion such that regularity on local metric properness holds.

4.3. L^∞ estimates

Let us take a solution (A_0,ψ_0) to the SW equations over M and consider the covering-monopole map with the base (A_0,ψ_0).

We take an element:

$$(\phi,a) \in L^2_k\big((K',g);\widetilde{S}^+\big) \oplus L^2_k\big((X,g);\Lambda^1\otimes i\mathbb{R}\big) \cap \operatorname{Ker} d^*,$$

and set $\widetilde{\mu}(\phi,a) = (\varphi,b,h)$ and $r = \|(\varphi,b,h)\|_{L^2_{k-1}}$, where $K' \subset X$ is a compact subset.

Proposition 4.10. *Suppose the AHS complex has closed range. Then, one has the L^∞ estimate $(\phi,a) \in L^\infty$ in terms of $r = \|(\varphi,b,h)\|_{L^2_{k-1}}$ and K' as*

$$\big\|(\phi,a)\big\|_{L^\infty(X)} \le C(r,K')$$

for some constant (r,K') that depends only on r and K'.

Proof. We verify the conclusion when the base solution is reducible $(A_0,0)$ at Step 2. The general case is verified at Step 3.

Step 1. We claim that the uniform estimate

$$\|a\|_{L^8_1} \le C\big(\|d^+(a)\|_{L^p(X)} + r\big)$$

holds for at least one of $p = 2, 4$ or 8.

It follows from Lemma 3.12 (β) that the inequality:

$$\|a\|_{L^8_1(X)} \le c\big(\max\{\|d^+a\|_{L^8(X)}, \|a\|_{L^4_1(X)}\} + \|a\|_{\text{harm}}\big).$$

By the same lemma again, we have another inequality:

$$\begin{aligned}\|a\|_{L^4_1(X)} &\le c\big(\max\{\|d^+a\|_{L^4(X)}, \|a\|_{L^2_1(X)}\} + \|a\|_{\text{harm}}\big)\\ &\le C\big(\max\{\|d^+a\|_{L^4(X)}, \|d^+(a)\|_{L^2}\} + \|a\|_{\text{harm}}\big),\end{aligned}$$

where we have used the assumption of closedness of the AHS complex for $k = 1$ at the second inequality. Thus, we verify the claim by combining these two estimates.

Step 2. It follows from Step 1 with the Sobolev estimate that the uniform estimate

$$\|a\|_{L^\infty} \le c\|a\|_{L^8_1} \le C\big(\|d^+(a)\|_{L^p(X)} + \|a_{\text{harm}}\| + r\big)$$

holds for at least one of $p = 2, 4$ or 8.

The Weitzenböck formula gives

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{1}{4}s + \frac{1}{2}F_A^+,$$

where s is the scalar curvature.

Now, suppose the base point is reducible $(A_0, 0)$ over M. Let Δ be the Laplacian on the functions. We have the pointwise estimate

$$\Delta|\phi|^2 \le \big\langle 2D_A^*\varphi - \frac{s}{2}\phi - \big(b + \sigma(\phi)\big)\phi, \phi\big\rangle$$

(see [4, p. 12]). Then,

$$\begin{aligned}\Delta|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 &= \big\langle 2D_{\tilde{A}_0}^*\varphi, \phi\big\rangle + \langle 2a\varphi, \phi\rangle - \langle b\phi, \phi\rangle\\ &\le 2\big(\|D_{\tilde{A}_0}^*\varphi\|_{L^\infty} + \|a\|_{L^\infty}\|\varphi\|_{L^\infty}\big)|\phi| + \|b\|_{L^\infty}|\phi|^2\end{aligned}$$

holds by use of the equality $\sigma(\phi)\phi = \frac{|\phi|^2}{2}\phi$.

By the assumption, there is a compact subset $K' \subset X$ such that ϕ has a compact support inside K'. Note that an a priori estimate:

$$\|\phi\|_{L^p(X)} = \|\phi\|_{L^p(K')} \le C\|\phi\|_{L^\infty(K')} = C\|\phi\|_{L^\infty(X)},$$

holds for some constant $C = C_{K'}$. Combining this estimate with the equality $d^+a = b + \sigma(\phi)$, we obtain

$$\begin{aligned}\|a\|_{L^\infty} &\le c\big(\|a_{\text{harm}}\| + \|b\|_{L^p} + \|\phi\|^2_{L^\infty} + r\big)\\ &\le c\big(\|a_{\text{harm}}\| + \|b\|_{L^2_2} + \|\phi\|^2_{L^\infty} + r\big)\\ &\le c\big(\|a_{\text{harm}}\| + \|b\|_{L^2_{k-1}} + \|\phi\|^2_{L^\infty} + r\big)\end{aligned}$$

by Lemma 3.9. For φ, we have the estimates

$$\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{L^8_1(X)} \leq C'\|\varphi\|_{L^2_3(X)} \leq C'\|\varphi\|_{L^2_{k-1}(X)}$$

again by Lemma 3.9. At the maximum of $|\phi|^2$, the value of $\Delta|\phi|^2$ is non-negative and, hence, we obtain

$$\|\phi\|^4_{L^\infty} \leq c\big(\|a_{\mathrm{harm}}\|, \|b\|_{L^2_{k-1}}, \|\varphi\|_{L^2_{k-1}}, r\big)\big(\|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty} + \|\phi\|^3_{L^\infty}\big).$$

Thus, we have L^∞ estimates of the pair (ϕ, a) by $(\|\varphi\|_{L^2_{k-1}}, \|b\|_{L^2_{k-1}}, \|h\|)$.

Step 3. Let us induce the L^∞ bound for the case of general base $[A_0, \psi_0]$. We will follow Steps 1 and 2.

It follows from Step 1 that the uniform estimates

$$\|a\|_{L^\infty} \leq c\|a\|_{L^8_1} \leq c\big(\|d^+a\|_{L^p} + \|a_{\mathrm{harm}}\| + r\big)$$

hold for at least one of $p = 2, 4$ or 8.

We set $\phi_0 := \widetilde{\psi}_0 + \phi$. We note that the bound

$$-c + \|\phi\|_{L^\infty} \leq \|\phi_0\|_{L^\infty} \leq c + \|\phi\|_{L^\infty}$$

holds since $\widetilde{\psi}_0$ is Γ-invariant. The Weitzenböck formula gives the pointwise estimate

$$\begin{aligned}\Delta|\phi_0|^2 &\leq \big\langle 2D^*_A D_A\phi_0 - \frac{s}{2}\phi_0 - \big(b + \sigma(\phi_0)\big)\phi_0, \phi_0\big\rangle \\ &= \big\langle 2D^*_A(\varphi) - \frac{s}{2}\phi_0 - \big(b + \sigma(\phi_0)\big)\phi_0, \phi_0\big\rangle.\end{aligned}$$

Now, we have the estimates

$$\begin{aligned}\Delta|\phi_0|^2 + \frac{s}{2}|\phi_0|^2 + \frac{|\phi_0|^4}{2} &\leq 2\big|D_A(\varphi)\big||\phi_0| + \big|\langle b\phi_0, \phi_0\rangle\big| \\ &\leq 2\big(\big\|D_{\widetilde{A}_0}(\varphi)\big\|_{L^\infty} + \|a\|_{L^\infty}\|\varphi\|_{L^\infty}\big)|\phi_0| + \|b\|_{L^\infty}|\phi_0|^2 \\ &\leq 2\big(\big\|D_{\widetilde{A}_0}(\varphi)\big\|_{L^2_3} + \|a\|_{L^\infty} r\big)|\phi_0| + r|\phi_0|^2 \\ &\leq 2r\big(c + \|a\|_{L^\infty}\big)|\phi_0| + r|\phi_0|^2,\end{aligned}$$

where we have used the estimates $\|b\|_{L^\infty} \leq c\|b\|_{L^8_1} \leq c\|b\|_{L^2_3} \leq r$ and the elliptic estimate.

By the assumption, there is a compact subset $K' \subset X$ and a constant $C = C_{K'}$ such that ϕ has a compact support inside K'. Note the estimates $\|\phi\|_{L^p(X)} \leq C\|\phi\|_{L^\infty(X)}$ and

$$\|\sigma(\widetilde{\psi}_0, \phi)\|_{L^p} \leq C\big(\|\phi\|_{L^p} + \|\phi\|^2_{L^{2p}}\big) \leq C\big(\|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty}\big).$$

Combining Step 1 with the equality $d^+a = b + \sigma(\widetilde{\psi}_0, \phi)$, we obtain

$$\begin{aligned}\|a\|_{L^\infty} &\leq c\big(\|b\|_{L^p} + \|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty} + r\big) \\ &\leq c\big(\|b\|_{L^2_2} + \|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty} + r\big)\end{aligned}$$

$$\leq c\big(\|b\|_{L^2_{k-1}} + \|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty} + r\big)$$
$$\leq c'\big(\|\phi\|_{L^\infty} + \|\phi\|^2_{L^\infty} + r\big)$$
$$\leq c'\big(\|\phi_0\|_{L^\infty} + \|\phi_0\|^2_{L^\infty} + r + 1\big).$$

We now combine the above estimates. At the maximum of $|\phi_0|^2$, the value of $\Delta|\phi_0|^2$ is non-negative, so we obtain

$$\begin{aligned}\|\phi_0\|^4_{L^\infty} &\leq 4r\big\{\big(c + \|a\|_{L^\infty}\big)\|\phi_0\|_{L^\infty} + \|\phi_0\|^2_{L^\infty}\big\}\\ &= 4r\big(c + \|\phi_0\|_{L^\infty}\big)\|\phi_0\|_{L^\infty} + \|a\|_{L^\infty}\|\phi_0\|_{L^\infty}\\ &\leq \big\{4r\big(c + \|\phi_0\|_{L^\infty}\big) + c'\big(\|\phi_0\|_{L^\infty} + \|\phi_0\|^2_{L^\infty} + r + 1\big)\big\}\|\phi_0\|_{L^\infty}\\ &\leq \big\{4r\|\phi_0\|_{L^\infty} + c'\big(\|\phi_0\|_{L^\infty} + \|\phi_0\|^2_{L^\infty} + r + 1\big)\big\}\|\phi_0\|_{L^\infty}.\end{aligned}$$

Therefore, we obtain the L^∞ estimate of the pair (ϕ, a) in terms of $(\|\varphi\|_{L^2_{k-1}}, \|b\|_{L^2_{k-1}}, \|h\|)$. ■

4.4. L^2_1 bounds

Let (A_0, ψ_0) be a smooth solution to the SW equations over M. Let $\tilde{\mu}$ be the covering-monopole map with the base (A_0, ψ_0). Recall the notation that $\tilde{A}_0$ and $\tilde{\psi}_0$ are both the lift on the universal covering space $X = \tilde{M}$.

Lemma 4.11. *Suppose the AHS complex has closed range over X and consider the restriction of the monopole map*

$$\tilde{\mu} : L^2_1(K; \tilde{S}^+)_0 \oplus L^2_1(X; \Lambda^1) \cap \operatorname{Ker} d^* \to L^2(K; \tilde{S}^-)_0 \oplus L^2(X; \Lambda^2_+; X) \oplus H^1(X)$$
$$(\phi, a) \mapsto \big(D_{\tilde{A}_0, \tilde{\psi}_0}(a, \phi), d^+(a) - \sigma(\tilde{\psi}_0, \phi), [a]\big)$$

for a compact subset $K \subset X$, where $H^1_{(2)}(X)$ is the first L^2 cohomology group.

For any positive numbers $s, r > 0$, there is a positive constant $C = C(r, s, K, A_0, \psi_0) > 0$ such that if an element (a, ϕ) in the domain satisfies the estimates

$$\tilde{\mu}(a, \phi)\|_{L^2} \leq r, \quad \|(a, \phi)\|_{L^\infty} \leq s,$$

then the estimate $\|(a, \phi)\|_{L^2_1} \leq C$ holds.

Proof. We follow the argument in Lemma 4.5. Let us denote $\tilde{\mu}(\phi, a) = (\varphi, b, h)$ by $\phi \in L^2_1(K)_0$.

Note the local Sobolev estimates

$$\|\phi\|_{L^2(K)_0} \leq C_K\|\phi\|_{L^3(K)_0} \leq C'_K\|\phi\|_{L^4(K)_0}.$$

Step 1. First, we suppose the base point is reducible $(A_0, 0)$. Following Step 1 in Lemma 4.5, we have the estimates

$$\begin{aligned}\big\|D_A(\phi)\big\|^2_{L^2(K)} + \|b\|_{L^2(X)}\|\phi\|^2_{L^4(K)} &\geq -\delta\|\phi\|^2_{L^2(K)} + \frac{1}{4}\|\phi\|^4_{L^4(K)}\\ &\geq -C_K^2\delta\|\phi\|^2_{L^4(K)} + \frac{1}{4}\|\phi\|^4_{L^4(K)}.\end{aligned}$$

In particular, there is $c = c(\|\varphi\|_{L^2}, \|b\|_{L^2}, \delta, K)$ such that the bound $\|\phi\|_{L^4(K)_0} \leq c$ holds and, hence, $\|\phi\|_{L^2(K)_0} \leq c'$.

The rest of the argument is the same as Steps 2 and 3 in Lemma 4.5 for this case.

Step 2. Let us consider the general case and choose a solution (A_0, ψ_0) to the SW equations over M. The L^∞ norm of the lift $\|\tilde{\psi}_0\|_{L^\infty} \leq c$ is finite, because ψ_0 is smooth and M is compact. Let us consider the equality $d^+(a) = b + \sigma(\tilde{\psi}_0, \phi)$. Recall that the support of ϕ is contained in K and, hence, the equality $d^+(a) = b$ holds on K^c.

Let us consider the equality $d^+(a) = b + \sigma(\tilde{\psi}_0, \phi)$. We have the estimates

$$\begin{aligned}
\|d^+(a)\|^2_{L^2(X)} &= \|d^+(a)\|^2_{L^2(K)} + \|d^+(a)\|^2_{L^2(K^c)} \\
&= \|b + \sigma(\tilde{\psi}_0, \phi)\|^2_{L^2(K)} + \|b\|^2_{L^2(K^c)} \\
&\leq \big(\|b\|_{L^2(K)} + \|\sigma(\tilde{\psi}_0, \phi)\|_{L^2(K)}\big)^2 + \|b\|^2_{L^2(K^c)} \\
&\leq 4\big(\|b\|^2_{L^2(K)} + \|\sigma(\tilde{\psi}_0, \phi)\|^2_{L^2(K)}\big) + \|b\|^2_{L^2(K^c)} \\
&\leq 4\|b\|^2_{L^2(X)} + C_K \|\sigma(\tilde{\psi}_0, \phi)\|^2_{L^\infty(K)} \\
&\leq 4\|b\|^2_{L^2(X)} + C'_K \big(\|\tilde{\psi}_0\|^2_{L^\infty} \|\phi\|^2_{L^\infty(K)} + \|\phi\|^4_{L^\infty(K)}\big) \\
&\leq C(r, s, K, \psi_0).
\end{aligned}$$

Hence, we obtain the L^2_1 bound

$$\|a\|_{L^2_1(X)} \leq c\big(\|d^+(a)\|_{L^2(X)} + \|a\|_{\text{harm}}\big) \leq C(r, s, K, \psi_0).$$

For ϕ, we have the estimates

$$\begin{aligned}
\|D_{\tilde{A}_0}(\phi)\|^2_{L^2} &\leq 4\|D_A(\phi)\|^2_{L^2} + 4\|a \cdot \phi\|^2_{L^2} = 4\|\varphi - a\tilde{\psi}_0\|^2_{L^2} + 4\|\phi\|^2_{L^\infty}\|a\|^2_{L^2} \\
&\leq 16\big(\|\varphi\|^2_{L^2} + \|\tilde{\psi}_0\|^2_{L^\infty}\|a\|^2_{L^2}\big) + 4\|\phi\|_{L^\infty}\|a\|^2_{L^2} \\
&\leq C(r, s, K, \psi_0)
\end{aligned}$$

(see the equalities above Lemma 2.3). By combining the elliptic estimate

$$\|\phi\|_{L^2_1} \leq C_{A_0,K}\big(\|\phi\|_{L^2} + \|D_{\tilde{A}_0}(\phi)\|_{L^2}\big)$$

with the bound $\|\phi\|_{L^2(K)} \leq C_K\|\phi\|_{L^\infty}$, we obtain the L^2_1 bound

$$\|\phi\|_{L^2_1} \leq C(r, s, K, \psi_0, A_0). \qquad \blacksquare$$

Remark 4.12. (1) The proof in Step 1 implies that the map $\tilde{\mu}$ above is metrically proper without the L^∞ condition if we use a reducible base solution.

(2) Note that we have restricted that spinors in the domain of the map are compactly supported on K. We compare this condition with Lemma 4.5, where we have not required such a condition, but, instead, we have assumed that the scalar curvature is positive.

4.5. Effect on smallness of local norms on 1-forms

We now consider the effect of local Sobolev norms effect on the global norm. It is a characteristic of non-compactness that a situation can happen where the local norm is small, but the total norm is quite large. Below, we induce a bound on the Sobolev norm under smallness of local norms on 1-forms.

Let $K \subset X$ be a fundamental domain. We take an element (ϕ, a) and set $\tilde{\mu}(\phi, a) = (\varphi, b, h)$. Recall that we have assumed $k \geq 3$ (see the third paragraph of Section 4).

Lemma 4.13. *Let $k \geq 4$. Let us choose a reducible base solution $(A_0, 0)$ over M. Suppose the AHS complex has closed range and the Dirac operator is invertible. Then, there is a small $\varepsilon_0 > 0$ and a positive constant $C > 0$ such that if the local bound $\|a\|_{L^2_{k-1}(\gamma(K))} < \varepsilon_0$ holds for any $\gamma \in \Gamma$, then the pair (ϕ, a) satisfies the bound*

$$\big\|(\phi, a)\big\|_{L^2_k(X)} \leq C(r + r^2),$$

where $r := \|(\varphi, b, h)\|_{L^2_{k-1}(X)}$.

Proof. Let us denote $A = \tilde{A}_0 + a$. Let us check that the estimate

$$\|a \cdot \phi\|_{L^2_{k-1}(X)} \leq C\varepsilon_0 \|\phi\|_{L^2_{k-1}(X)}$$

holds. It follows from Lemma 3.14 that the estimate $\|a \cdot \phi\|_{L^2_{k-1}(\gamma(K))} \leq C\|a\|_{L^2_{k-1}(\gamma(K))} \cdot \|\phi\|_{L^2_{k-1}(\gamma(K))}$ holds for some positive constant $C > 0$. Hence, the following estimate holds:

$$\begin{aligned}
\|a \cdot \phi\|^2_{L^2_{k-1}(X)} &= \sum_{\gamma \in \Gamma} \|a \cdot \phi\|^2_{L^2_{k-1}(\gamma(K))} \\
&\leq C^2 \sum_{\gamma \in \Gamma} \|a\|^2_{L^2_{k-1}(\gamma(K))} \cdot \|\phi\|^2_{L^2_{k-1}(\gamma(K))} \\
&\leq C^2 \varepsilon_0^2 \sum_{\gamma \in \Gamma} \|\phi\|^2_{L^2_{k-1}(\gamma(K))} = C^2 \varepsilon_0^2 \|\phi\|^2_{L^2_{k-1}(X)}.
\end{aligned}$$

Then, we obtain the estimates

$$\begin{aligned}
\|\phi\|_{L^2_k(X)} &\leq C' \big\|D_{\tilde{A}_0}(\phi)\big\|_{L^2_{k-1}(X)} \leq C'\big(\big\|D_A(\phi)\big\|_{L^2_{k-1}(X)} + \|a \cdot \phi\|_{L^2_{k-1}(X)}\big) \\
&\leq C'\big(r + C\varepsilon_0 \|\phi\|_{L^2_{k-1}(X)}\big).
\end{aligned}$$

In particular, if $\varepsilon_0 > 0$ is sufficiently small, then we have

$$\|\phi\|_{L^2_k(X)} \leq C'' r.$$

Next, it follows from Lemma 3.9 that

$$\big\|\sigma(\phi)\big\|_{L^2_{k-1}(X)} \leq C' \|\phi\|^2_{L^4_{k-1}(X)} \leq C\|\phi\|^2_{L^2_k(X)}.$$

Hence, we obtain

$$\|d^+(a)\|_{L^2_{k-1}(X)} \le r + \|\sigma(\phi)\|_{L^2_{k-1}(X)} \le r + C\|\phi\|^2_{L^2_k(X)} \le r + Cr^2$$

as $d^+(a) + \sigma(\phi)$ has the L^2_{k-1} norm lower than r. It follows from the assumption that

$$\|a\|_{L^2_k(X)} \le C\big(\|d^+(a)\|_{L^2_{k-1}(X)} + \|a_{\mathrm{harm}}\|\big) \le Cr(1+r). \qquad \blacksquare$$

5. Approximation by finite-dimensional spaces

5.1. Fredholm map

Let H' and H be two Hilbert spaces and consider a Fredholm map:

$$F = l + c : H' \to H,$$

whose linear part l is Fredholm and where c is compact on each bounded set. For our purpose later, we restrict the domain by the Sobolev space. A method of finite-dimensional approximation has been developed for a metrically proper and Fredholm map [4, 32]. It is applied to the monopole map when the underlying space X is a compact four-manifold.

Below, we introduce an equivariant version of this type of approximation on a non-linear map over the covering space $X = \tilde{M}$ of a compact four-manifold with the action of the fundamental group $\Gamma = \pi_1(M)$.

Let $E \to X$ be a vector bundle. Let us say that a smooth map $c : H' = L^2_k(X; E) \to H$ is *locally compact* on each bounded set if its restriction $c|L^2_k(K, E)_0 \cap D$ is compact, where $K \subset X$ is a compact subset and $D \subset H'$ is a bounded set.

5.2. Technical estimates

We apply the results in this subsection to construct a finite-dimensional approximation method in the next subsection. In particular, Lemma 5.1 below is applied to the Dirac operator in the case of the covering-monopole map.

Let

$$D : L^2_k(X; E') \to L^2_{k-1}(X; E)$$

be a first-order elliptic differential operator that is Γ-invariant. Let $K \subset X$ be a compact subset and consider the restriction

$$D : L^2_k(K; E)_0 \to L^2_{k-1}(K; F)_0.$$

Lemma 5.1. *Suppose $D : L^2_k(X; E') \to L^2_{k-1}(X; E)$ has closed range. Then, D is surjective if and only if $D^* : L^2_k(X : E) \to L^2_{k-1}(X : E')$ is injective with closed range.*

Proof. The proof consists of two steps.

Step 1. Let us check that the formal adjoint

$$D^* : L^2_k(X; E) \to L^2_{k-1}(X; E')$$

is injective with closed range.

If $w \in L^2_k(X; E)$ with $D^*(w) = 0$ holds, then

$$\langle D^*(w), u\rangle_{L^2_{k-1}} = \langle w, D(u)\rangle_{L^2_{k-1}} = 0$$

follows for all $u \in L^2_k(X : E')$. This implies $w = 0$ by the surjectivity of D.

For any $w \in L^2_k$, there is $u' \in L^2_{k+1}$ such that $w = D(u')$ holds. Then

$$\begin{aligned} \| D^*(w)\|^2_{L^2_{k-1}} &= \langle D^*(w), D^*(w)\rangle_{L^2_{k-1}} \\ &= \langle D^* D(u'), D^* D(u')\rangle_{L^2_{k-1}} \geq C\|u'\|^2_{L^2_{k+1}} \geq C\| D(u')\|^2_{L^2_k} \\ &= C\|w\|^2_{L^2_k}. \end{aligned}$$

Hence, D^* also has closed range.

Step 2. Conversely suppose $D : L^2_k(X; E') \to L^2_{k-1}(X; E)$ has closed range but is not surjective. Then, there is $0 \neq u \in L^2_{k-1}(X; E)$ with

$$\langle D(w), u\rangle_{L^2_{k-1}} = \langle w, D^*(u)\rangle_{L^2_{k-1}} = 0$$

for any $w \in L^2_k(X; E')$. This implies $u \in L^2_k(X; E)$ with $D^*(u) = 0$ as D^* is elliptic.

Combining this with Step 1, it follows under the closedness of D that D is surjective if and only if D^* is injective. ■

Remark 5.2. In general, $D : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$ is not necessarily surjective, even if $D : L^2_k(X; E') \to L^2_{k-1}(X; E)$ is surjective. Later, we will verify that it is asymptotically surjective in some sense.

Lemma 5.3 below tells us that, under some condition, the vectors w_i distribute in some high spectral region in the co-kernel of D.

Lemma 5.3. *Suppose $D : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$ has closed range. Moreover, assume that a sequence $w_i \in L^2_k(X; E)$ with $\|w_i\|_{L^2_k} = 1$ satisfies the condition*

$$\lim_{i\to\infty} \sup_{v\in B\cap \operatorname{im} D} \left|\langle w_i, v\rangle_{L^2_{k-1}}\right| = 0,$$

where $B \subset L^2_{k-1}(X; E)$ is the unit ball. Then, $w_i - P_0(w_i)$ converges to 0 in L^2_{k-1}, where P_0 is the spectral projection to the harmonic space of DD^*.

In particular, $w_i \to 0$ holds in L^2_{k-1} if D is surjective with closed range.

Proof. The proof consists of three steps.

Step 1. Assume that there is a sequence $w_i \in L^2_k(X; E)$ with $\|w_i\|_{L^2_k} = 1$ and

$$\sup_{v\in B\cap \operatorname{im} D} \left|\langle w_i, v\rangle_{L^2_{k-1}}\right| < \varepsilon_i \to 0.$$

Then, any $f \in L^2_k(X;E)$ with $\|D(f)\|_{L^2_{k-1}} = 1$ satisfies the bounds

$$\langle D^*(w_i), f\rangle_{L^2_{k-1}} = \langle w_i, D(f)\rangle_{L^2_{k-1}} < \varepsilon_i$$

by Lemma 3.1.

Step 2. Let P be the spectral projection of DD^* on $L^2(X;E)$. We claim that it is sufficient to confirm the convergence

$$\lim_{i\to\infty} \|P_{[\lambda^2,\mu^2]}(w_i)\|_{L^2_{k-1}} < \varepsilon_i \to 0 \tag{5.1}$$

for any $0 < \lambda < \mu$. Then, it follows from the equality $\|D^*w\|^2_{L^2_{k-1}} = \langle w, DD^*w\rangle_{L^2_{k-1}}$ that the estimate

$$\|w\|^2_{L^2_k} = \|D^*w\|^2_{L^2_{k-1}} + \|w\|^2_{L^2} \geq \mu^2\|w\|^2_{L^2_{k-1}} + \|w\|^2_{L^2} \tag{5.2}$$

holds for any element $w \in \operatorname{im} P_{[\mu^2,\infty)}$. This implies that the L^2_{k-1} norm of the projection to high spectra of w_i must be small if $\|w_i\|_{L^2_k} = 1$ holds. Then, combining this with the two properties (5.1) and (5.2), we obtain the convergence

$$\lim_{i\to\infty} \|P_{[\lambda^2,\infty)}(w_i)\|_{L^2_{k-1}} < \varepsilon_i \to 0 \tag{5.3}$$

for any $\lambda > 0$. By the diagonal method, there is a decreasing sequence $0 < \lambda_i \to 0$ such that convergence holds:

$$\lim_{i\to\infty} \|P_{[\lambda_i^2,\infty)}(w_i)\|_{L^2_{k-1}} \to 0. \tag{5.4}$$

Noting that DD^* has a gap in its spectrum around zero, we choose $\lambda^2 > 0$ in this gap. Then, $P_{[\lambda_i^2,\infty)}(w_i) = w_i - P_0(w_i)$ and, hence, its L^2_{k-1}-norm goes to zero by (5.3). This implies the conclusion.

Step 3. Let us verify the claim in Step 2. We suppose the contrary and assume that there is a constant with the uniform bound

$$\lim_{i\to\infty} \|P_{[\lambda^2,\mu^2]}(w_i)\|_{L^2_{k-1}} \geq \varepsilon_0 > 0.$$

We set $f = D^*P_{[\lambda^2,\mu^2]}(w_i)$. Then, we have the bound

$$\|Df\|_{L^2_{k-1}} \leq \mu^2\|w_i\|_{L^2_{k-1}} \leq \mu^2\|w_i\|_{L^2_k} \leq \mu^2.$$

Then, we have the estimates

$$\begin{aligned}\langle w_i, Df\rangle_{L^2_{k-1}} &= \langle w_i, DD^*P_{[\lambda^2,\mu^2]}(w_i)\rangle_{L^2_{k-1}} = \langle P_{[\lambda^2,\mu^2]}(w_i), DD^*P_{[\lambda^2,\mu^2]}(w_i)\rangle_{L^2_{k-1}}\\ &\geq \lambda^2\|P_{[\lambda^2,\mu^2]}(w_i)\|^2_{L^2_{k-1}} \geq \lambda^2\varepsilon_0.\end{aligned}$$

This contradicts to the assumption. Thus, combined with Step 2, we obtain the first statement.

The last statement follows by Lemma 5.1. ■

Let us consider the restriction

$$D : L^2_k(K;E')_0 \to L^2_{k-1}(K;E)_0$$

and set

$$\partial_K := D\big(L^2_k(K;E')_0\big)^{\perp} \cap L^2_{k-1}(K;E)_0.$$

Lemma 5.4. *Suppose $D : L^2_k(X;E') \to L^2_{k-1}(X;E)$ is surjective. Assume that a sequence $u_i \in L^2_k(K;E')_0$ with $\|u_i\|_{L^2_k} = 1$ satisfies*

$$\big\| P_K(u_i) - u_i \big\|_{L^2_{k-1}} \to 0,$$

where $P_K : L^2_{k-1}(K;E)_0 \to \partial_K$ is the orthogonal projection. Then, $\|u_i\|_{L^2_{k-1}} \to 0$ holds. Moreover,

$$\| P_{[0,\lambda]} u_i \|_{L^2_k} \to 0$$

holds for any $\lambda > 0$, where $P_{[0,\lambda]}$ is in Lemma 5.3.

Proof. Take any $v \in L^2_k(K;E)_0$. Then, we have the estimate

$$\begin{aligned} \big|\langle v, D^* u_i\rangle_{L^2_{k-1}}\big| &= \big|\langle Dv, u_i\rangle_{L^2_{k-1}}\big| = \big|\big\langle Dv, (1-P_K)u_i\big\rangle_{L^2_{k-1}}\big| \\ &\le \big\|(1-P_K)u_i\big\|_{L^2_{k-1}} \|v\|_{L^2_k} \to 0. \end{aligned}$$

Hence, $D^* u_i$ weakly converges to zero.

Assume that a subsequence of $\{\|u_i\|_{L^2_{k-1}}\}$ is uniformly bounded from below. For simplicity of notation, we assume $\|u_i\|_{L^2_{k-1}} \ge \varepsilon > 0$. Then, from Rellich's lemma a subsequence of $\{u_i\}_i$ converges to a non-zero element $u \in L^2_{k-1}(K;E)_0$ with $D^* u = 0$. This cannot happen by Lemma 5.1 since D is assumed to be surjective. Hence, $\|u_i\|_{L^2_{k-1}} \to 0$ holds.

Then, one must see the property $\|P_{[0,\lambda]}u_i\|_{L^2_k} \to 0$ for any $\lambda > 0$, since we have the estimates $\|P_{[0,\lambda]}u_i\|_{L^2_k} \le \lambda \|P_{[0,\lambda]}u_i\|_{L^2_{k-1}} \to 0$. ∎

5.3. Finite-dimensional approximations

To apply a method of finite-dimensional approximation, we need to induce a kind of properness on the image of the projection.

Let $F = l + c : H' \to H$ be a metrically proper map between Hilbert spaces. Then, there is a proper and increasing function $g : [0,\infty) \to [0,\infty)$ such that the following lower bound holds:

$$g\big(\|F(m)\|\big) \ge \|m\|. \tag{5.5}$$

Later, we analyze a family of maps of the form $F_i : W_i' \to W_i$, where W_i' and W_i are both finite-dimensional linear spaces. We also say that the family of maps is *metrically proper*, in the sense that there are positive numbers $r_i, s_i \to \infty$ such that the inclusion

$$F_i^{-1}(B_{s_i} \cap W_i) \subset B_{r_i} \cap W_i'$$

holds, where B_{s_i}, B_{r_i} are the open balls with radii s_i and r_i, respectively.

Lemma 5.5. *Let $F = l + c : H' \to H$ be a metrically proper map. Suppose l is surjective and c is compact on each bounded set. Then, for any $r > 0$ and $\delta_0 > 0$, there is a finite-dimensional linear subspace $W'_0 \subset H'$ such that for any linear subspace $W'_0 \subset W' \subset H'$,*

$$\mathrm{pr} \circ F : B_r \cap W' \to W$$

also satisfies the bound

$$f\big(\|\mathrm{pr} \circ F(m)\|\big) \geq \|m\|$$

for any $m \in B_r \cap W'$, where $W = l(W')$, pr *is the orthogonal projection to W, $f(x) := g(x + \delta_0)$, and g is in (5.5).*

Moreover, the following estimate holds:

$$\sup_{m \in B_r \cap W'} \big\|F(m) - \mathrm{pr} \circ F(m)\big\| \leq \delta_0.$$

Proof. Take any positive constant $\delta_0 > 0$. Let $C \subset H$ be the closure of the image $c(B_r)$, which is compact. Then, there is a finite number of points $w_1, \ldots, w_m \in c(B_r)$ such that their δ_0 neighborhoods cover C. Choose $w'_i \in H'$ such that $l(w'_i) = w_i$ hold for $1 \leq i \leq m$, and let W'_0 be the linear span of these w'_i.

The restriction $\mathrm{pr} \circ F : B_r \cap W'_0 \to W_0$ satisfies the equality

$$\mathrm{pr} \circ F = l + \mathrm{pr} \circ c,$$

where $W_0 = l(W'_0)$. Notice the equalities $\mathrm{pr} \circ F(w'_i) = F(w'_i)$ for $1 \leq i \leq m$. Then, for any $m \in B_r \cap W'_0$, there is some w'_i with $\|c(m) - c(w'_i)\| \leq \delta_0$, and the estimate $\|F(m) - \mathrm{pr} \circ F(m)\| \leq \delta_0$ holds.

Since g is increasing, we obtain the estimates

$$g\big(\|\mathrm{pr} \circ F(m)\| + \delta_0\big) \geq g\big(\|F(m)\|\big) \geq \|m\|.$$

The function $f(x) = g(x + \delta_0)$ satisfies the desired property.

For any other linear subspace $W'_0 \subset W' \subset H'$, the same property holds for $\mathrm{pr} \circ F : B_r \cap W' \to W$ with $W = l(W')$. ∎

Remark 5.6. Note that if l is not injective, then $l^{-1}(W')$ is already infinite-dimensional, in the case of infinite covering-monopole map, because the kernel is infinite-dimensional.

Remark 5.7. In the case of the covering-monopole map we analyzed, the domain is not the full Sobolev space, but its closed linear subspace $L^2_k(K; \tilde{S}^+)_0 \oplus L^2_k(X; \Lambda^1) \cap \mathrm{Ker}\, d^*$. Moreover, the target space is the sum of the Sobolev space with the first L^2 cohomology group. Nevertheless, the content in Section 5 works for this case, as the linearized map splits into the sum of the Dirac operator with d^+ and the harmonic projection.

5.3.1. Compactly supported Sobolev space. Let

$$F = l + c : L^2_k(X; E') \to L^2_{k-1}(X; E)$$

be a smooth map between Sobolev spaces, where l is a first-order differential operator and c is pointwise and locally compact on each bounded set.

Local compactness on each bounded set means that the restriction on $L^2_k(K; E')_0 \cap B$ is compact, where $B \subset L^2_k(X; E')$ is a bounded set and $K \subset X$ is a compact subset.

In Section 5.3.1, we assume that l has closed range and that the restriction

$$F : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$$

is metrically proper.

Consider the splitting

$$L^2_{k-1}(K; E)_0 = l\big(L^2_k(K; E')_0\big) \oplus \partial_K,$$

where ∂_K is the orthogonal complement of $l(L^2_k(K; E')_0)$. We denote by

$$\text{pr}_K : L^2_{k-1}(K; E)_0 \to l\big(L^2_k(K; E)_0\big)$$

the orthogonal projection. For any closed linear subspace $W \subset L^2_{k-1}(K; E)_0$, we also denote by

$$\text{pr}_W : L^2_{k-1}(K; E)_0 \to W$$

the orthogonal projection.

Let $S \subset L^2_k(K; E')_0$ be the unit sphere and consider the closure of the image as

$$\overline{c(S)} \subset l\big(L^2_k(K; E')_0\big) \oplus \partial_K = L^2_{k-1}(K; E)_0.$$

We say that c is quadratic if $c(av) = a^2 c(v)$ holds for any $a \in \mathbb{R}$ and $v \in L^2_k(K; E')_0$.

Lemma 5.8. *Assume, moreover, that c is quadratic. If the $w_1 \neq 0$ component in $l(L^2_k(K; E')_0)$ is non-zero for any element $w = (w_1, w_2) \in \overline{c(S)}$, then the composition*

$$\text{pr}_K \circ F : L^2_k(K; E')_0 \to l\big(L^2_k(K; E')_0\big) \subset L^2_{k-1}(K; E)_0$$

is metrically proper.

Proof. Since $\overline{c(S)}$ is compact,

$$r = \inf_{w \in \overline{c(S)}} \|w_1\| > 0$$

is positive. Then, for any $u \in L^2_k(K; E')_0$, the estimate $\|\text{pr}_K(c(u))\| \geq r\|u\|^2_{L^2_k}$ holds because c is quadratic. On the other hand, $\|l(u)\|_{L^2_{k-1}} \leq C\|u\|_{L^2_k}$ holds for some C. Hence, we obtain the lower bound

$$\|\text{pr}_K \circ F(u)\|_{L^2_{k-1}} \geq r\|u\|^2_{L^2_k} - C\|u\|_{L^2_k}.$$

The conclusion follows. ∎

Of course, it is unrealistic to expect that such assumption could happen. Therefore, we state a modified version.

Let $B_r \subset L^2_k(K;E')_0$ be the open ball with radius r. Take a finite-dimensional linear subspace $U_r \subset \partial_K$ and denote the orthogonal projection by

$$P_r : L^2_{k-1}(K;E)_0 \to l\big(L^2_k(K;E')_0\big) \oplus U_r \subset L^2_{k-1}(K;E)_0.$$

Lemma 5.9. *Suppose F is locally metrically proper. Moreover, suppose that l has closed range and c is locally compact on each bounded set. Then, for each $r > 0$, there is a finite-dimensional linear subspace $U_r \subset \partial_K$ such that the composition*

$$P_r \circ F : B_r \to l\big(L^2_k(K;E')_0\big) \oplus U_r$$

is still metrically proper.

Proof. The proof is very much in the same spirit as Lemma 5.5. We fix $\delta > 0$, which is independent of $r > 0$. Then, for this $\delta > 0$ and $r > 0$, we take a sufficiently many but finite set of points $\{p_1, \dots, p_m\} \subset B_r$ and denote the finite-dimensional linear subspace spanned by $c(p_i)$ as $\widetilde{W}_r$. Then, we can assume that $\widetilde{W}_r \subset L^2_{k-1}(K;E)_0$ satisfies the bound $d(\widetilde{W}_r, c(B_r)) < \delta$ as c is locally compact on each bounded set. Hence, any element $w \in \mathrm{pr}_{\partial_K}(\overline{c(B_r)})$ is at most δ away from $U_r := \mathrm{pr}_{\partial_K}(\widetilde{W}_r)$, where pr_{∂_K} is the orthogonal projection to ∂_K.

Let us consider the linear plane

$$L_r := l\big(L^2_k(K;E')_0\big) + \widetilde{W}_r = l\big(L^2_k(K;E')_0\big) \oplus U_r.$$

Then, we obtain the bound

$$\delta \geq \sup_{m \in B_r} \big\| F(m) - \mathrm{pr}_{L_r} \circ F(m) \big\|, \tag{5.6}$$

where the right-hand side depends on $r > 0$, but δ is independent of it. Hence, the conclusion follows from the estimate (5.6) with metric properness of F. ∎

Corollary 5.10. *Suppose F, l, and c satisfy the conditions in Lemma 5.9. There are finite-dimensional linear subspaces $W'_r \subset L^2_k(K;E)_0$ and $U_r \subset \partial_K$ with a linear map*

$$l' : L^2_k(K;E')_0 \to l\big(L^2_k(K;E')_0\big) \oplus U_r$$

such that the following hold.

(a) *The composition of l' with the projection pr_K to the first component coincides with l as*

$$l = \mathrm{pr}_K \circ l' : W'_r \to l(W'_r).$$

(b) *Let $\mathrm{pr}_{W_r} : L^2_{k-1}(K,F)_0 \to W_r := l'(W_r)$ be the orthogonal projection. Then,*

$$\mathrm{pr}_{W_r} \circ F : W'_r \cap B_r \to W_r$$

is proper (see Lemma 5.5).

(c) *If* $l : L^2_k(X;E) \hookrightarrow L^2_{k-1}(X;F)$ *is injective, then the estimates*

$$\|l(v)\| \le \|l'(v)\| \le 3\|l(v)\|$$

hold for any $v \in L^2_k(K;E)_0$.

The same properties hold if one takes a larger but still finite-dimensional linear subspace $\tilde{W}'_r$ *with* $W'_r \subset \tilde{W}'_r \subset L^2_k(K;E')_0$.

Proof. Let $\{p_1,\dots,p_m\} \subset B_r$ and U_r be as in the proof of Lemma 5.9 such that $F(p_i) \in l(L^2_k(K;E')_0) \oplus U_r$ holds. Let $W'_r \subset L^2_k(K;E')_0$ be the finite-dimensional linear subspace spanned by $\{p_1,\dots,p_m\}$. One may assume the estimate

$$d\big(\mathrm{pr}_K\big(c(B_r)\big), l(W'_r)\big) < \delta \tag{5.7}$$

by adding extra points, if necessarily.

Let us introduce a linear map as follows. Let $f : [0,\infty)^2 \to [0,\infty)$ be a smooth map with

$$f(r,s) = \begin{cases} s & r \ge s, \\ 2r & 2r \le s. \end{cases}$$

Then, we define $l' : W'_r \to l(L^2_k(K;E')_0) \oplus U_r$ by the linear extension of the map

$$l'(p_i) = l(p_i) + f\big(\|l(p_i)\|, \|\mathrm{pr}_{U_r} \circ c(p_i)\|\big)\frac{\mathrm{pr}_{U_r} \circ c(p_i)}{1 + \|\mathrm{pr}_{U_r} \circ c(p_i)\|}.$$

We require a slightly complicated formula for the second term because the norm $\|\mathrm{pr}_K \circ c(p_i)\|$ can grow more than linearly. Clearly, both (a) and (c) are satisfied.

There is a proper increasing map $g : [0,\infty) \to [0,\infty)$ independent of r such that

$$\max\big(\|\mathrm{pr}_K \circ F(v)\|_{L^2_{k-1}}, \|\mathrm{pr}_{U_r} \circ c(v)\|_{L^2_{k-1}}\big) \ge g\big(\|v\|_{L^2_k}\big)$$

holds for $v \in D_r \cap W'_r$, since F is metrically proper. Hence, (b) follows by combination with (5.7). ∎

5.3.2. Asymptotic surjection. In this section, we assume the restriction $F|L^2_k(K;E)_0$ is metrically proper on any compact subset, $l : L^2_k(X;E) \to L^2_{k-1}(X;F)$ is surjective, and c is locally compact on each bounded set.

One can obtain a finite-dimensional approximation of F as Corollary 5.10, but the linear map l' may be quite different from l. In this subsection, we will use a larger compact subset $K \subset L$ in the target space such that l' surely approximates l.

We want to use $P_r \circ F : W'_r \cap B_r \to l(L^2_k(K;E')_0) \oplus U_r$ in Lemma 5.9 as an asymptotic approximation of F instead of using F itself. As above, we have to use a pair of compact subsets to approximate its linearized operator. Note that the choice of these linear subspaces W'_r or U_r heavily depends on the compact subset $K \subset X$. Thus, we denote P^K_r instead of P_r above.

Let us consider two compact subsets $K \subset L \subset X$ and let $\partial_L \subset L^2_{k-1}(L;E)_0$ be the orthogonal complement of $l(L^2_k(L,E')_0)$.

Lemma 5.11. *We fix K. Then, for any $\varepsilon > 0$, there is another compact subset $L \supset K$ such that the orthogonal projection*

$$P : L^2_{k-1}(K; E)_0 \to \partial_L$$

satisfies the estimate

$$\|P\| \leq \varepsilon.$$

Proof. Let us choose a compact subset $L \subset X$ such that it admits a smooth cut-off function $\varphi : L \to [0, 1]$ with the following properties:

(1) $\varphi | K \equiv 1$,

(2) $\varphi | L^c \equiv 0$, and

(3) $\|\nabla(\varphi)\|_{L^2_{k-1}} < \delta$, where $\delta > 0$ is sufficiently small.

There is a constant C such that for any $u \in L^2_{k-1}(K; E)_0$, there is $v \in l^{-1}(u) \subset L^2_k(X; E')$ with $\|v\|_{L^2_k} \leq C\|u\|_{L^2_{k-1}}$.

It follows from the equality

$$u = \varphi l(v) = l(\varphi v) - [l, \varphi]v$$

that the estimates

$$\big\|P(u)\big\|_{L^2_{k-1}} \leq \big\|[l, \varphi]v\big\|_{L^2_{k-1}} \leq \delta\|v\|_{L^2_k} \leq C\delta\|u\|_{L^2_{k-1}}$$

hold. ∎

Let L be a compact subset and apply Corollary 5.10 to L as

$$l' : L^2_k(L; E')_0 \to l\big(L^2_k(L; E')_0\big) \oplus U^L_r$$

with $W'_r \subset L^2_k(L; E)_0$ and a proper map

$$\mathrm{pr}_{W_r} \circ F : W'_r \cap B_r \to W_r \subset L^2_{k-1}(L, E)_0.$$

Corollary 5.12. *We fix K. Then, for any $\varepsilon > 0$, there is another compact subset $L \supset K$ such that the operator norm of the restriction satisfies the estimate*

$$\big\|(l - l')|L^2_k(K, E')_0\big\| < \varepsilon. \tag{5.8}$$

Moreover, the restriction

$$\mathrm{pr}_{W_r} \circ F : W'_r \cap B_r \cap L^2_k(K; E')_0 \to W_r \subset l\big(L^2_k(L; E')_0\big) \oplus U^L_r$$

satisfies the estimate

$$\sup_{m \in W'_r \cap B_r \cap L^2_k(K; E')_0} \big\|\mathrm{pr}_{W_r} \circ F(m) - F(m)\big\|_{L^2_{k-1}} < \varepsilon. \tag{5.9}$$

The same properties hold if one takes a larger but still finite-dimensional linear subspace $L^2_k(L, E')_0 \supset \widetilde{W}'_r \supset W'_r$.

Proof. The first estimate (5.8) follows from Corollary 5.10 with Lemma 5.11. Moreover, through combination with Lemma 5.5, we can obtain the second estimate (5.9). ∎

5.4. Finitely approximable maps

So far, we have fixed a compact subset $K \subset X$. Our final aim is to approximate a non-linear map between Sobolev spaces over X by a family of maps between finite-dimensional linear subspaces that are included in exhausting compactly supported functional spaces.

Consider a map $F = l + c : L^2(X; E') \to L^2_{k-1}(X; E)$. Throughout 5.4, we assume that $l : L^2_k(X, E') \cong L^2_{k-1}(X, E)$ is a linear isomorphism and c is pointwise and locally compact on each bounded set. We also assume that F is locally metrically proper. Note that these conditions satisfy the properties we have assumed in Subsections 5.3.1 and 5.3.2.

Recall the locally strong properness of a map in Definition 1.1. Note that the above properties are satisfied if F is locally strongly proper, l is isomorphic, and c is pointwise and locally compact on each bounded set.

In particular, it follows from Theorem 4.1 that the covering-monopole map satisfies the above properties if the AHS complex has closed range and l is isomorphic. This is equivalent to the two properties that, if the AHS complex has closed range and the Dirac operator is invertible, then a reducible solution exists and the fundamental group of X is infinite.

Let us consider a family of maps:

$$F_i : W_i' \to W_i,$$

where $W_i' \subset H'$ and $W_i \subset H$ are both finite-dimensional linear subspaces whose respective unions

$$\bigcup_{i=0}^{\infty} W_i' \subset H', \quad \bigcup_{i=0}^{\infty} W_i \subset H$$

are dense. We denote by $B'_{r_i} \subset W_i'$ and $B_{s_i} \subset W_i$ the open balls with radii r_i and s_i, respectively.

Let us say that the family $\{F_i\}_i$ is *asymptotically proper* on H' if there are two sequences $s_0 < s_1 < \cdots \to \infty$ and $r_0 < r_1 < \cdots \to \infty$ such that the following embeddings hold:

$$F_i^{-1}(B_{s_i}) \subset B'_{r_i}.$$

To obtain a better approximation as in Corollary 5.12, we can take a larger compact subset.

Let us apply this notion to $H' = L^2_k(X; E')$ and $H = L^2_{k-1}(X; E)$. Let

$$F = l + c : L^2_k(X; E') \to L^2_{k-1}(X; E)$$

be a locally strongly proper map, where l is a first-order elliptic differential operator and c is pointwise and locally compact on each bounded set. Suppose l is an isomorphism. Let us restate Corollary 5.12 in terms of a family of maps.

Corollary 5.13. *There exists an exhaustion $\bigcup_i K_i = X$ by compact subsets and families of finite-dimensional linear subspaces:*

$$W_i' \subset L^2_k(K_i; E')_0 \quad \textit{and} \quad W_i \subset L^2_{k-1}(K_i; E')_0$$

such that the following holds. For any compact subset $K \subset X$, the limit of the operator norms of the restriction

$$\lim_{i \to \infty} \left\| (l - l_i) | W_i' \cap L_k^2(K, E')_0 \right\| = 0$$

holds. Moreover, the restriction approximates F as

$$\lim_{i \to \infty} \sup_{m \in W_i' \cap B_{r_i} \cap L_k^2(K, E')_0} \left\| F_i(m) - F(m) \right\|_{L^2_{k-1}} = 0,$$

where $F_i := \mathrm{pr}_{W_i} \circ F : W_i' \to W_i$ is an asymptotically proper family with linear isomorphisms $l_i : W_i' \cong W_i$ and $\mathrm{pr}_{W_i} : H \to W_i$ is the orthogonal projection.

Let $K_1 \Subset \cdots \Subset K_i \Subset K_{i+1} \subset X$ be an exhaustion of X by compact subsets and $E', E \to X$ be vector bundles over X. Then, we have an increasing family of Sobolev spaces:

$$L_k^2(K_i; E')_0 \subset L_k^2(K_{i+1}; E')_0 \subset \cdots \subset L_k^2(X; E').$$

Let $F = l + c : L^2(X; E') \to L^2_{k-1}(X; E)$ be a smooth map, where l is a first-order differential operator and c is the non-linear term which is a pointwise operator.

Let F_i and $\mathrm{pr}_{W_i} : L^2_{k-1}(X; E) \to W_i$ be in Corollary 5.13, which considers a situation in which a compact subset K is fixed. Now, we use whole families of such approximations over K_i and select well-approximated maps as below.

First, let us state a weaker version of the approximation.

Lemma 5.14. *Let W_i' be as in Corollary 5.13. For any $v' \in L_k^2(X; E')$, there is an approximation $v_i' \in W_i'$ with $v_i' \to v'$ in $L_k^2(X; E')$ such that the convergence*

$$\lim_i \left\| F_i(v_i') - F(v') \right\|_{L^2_{k-1}} = 0$$

holds, where $F_i = \mathrm{pr}_{W_i} \circ F : W_i' \to W_i$.

Proof. Let $v_i' \in L_k^2(K_i; E')_0$ be any approximation with $v_i' \to v'$. By Corollary 5.13, $\lim_i \| F_i(v_{i_0}') - F(v_{i_0}') \|_{L^2_{k-1}} = 0$ holds for each i_0. Since F is continuous, for any $\varepsilon > 0$, there is i_0 such that the estimate $\| F(v_{i_0}') - F(v') \| < \varepsilon$ holds.

Note that we can regard $v_{i_0}' \in W_i'$ for $i \geq i_0$. Then, we replace the approximation of v' such that we obtain the desired estimate by applying the triangle inequality. ∎

Let $B_{r_i}' \subset W_i'$ and $B_{s_i} \subset W_i$ be the open balls with radii r_i and s_i, respectively.

Definition 5.1 ([26]). Let $F = l + c : H' \to H$ be a smooth map, where l is the linear part and c is its non-linear term.

We say that F is weakly finitely approximable, if there is an increasing family of finite-dimensional linear subspaces $W_0' \subset W_1' \subset \cdots \subset W_i' \subset \cdots \subset H'$, an asymptotically proper family $\{F_i\}_i$, and linear isomorphisms $l_i : W_i' \cong W_i$ such that

(1) their union $\bigcup_{i \geq 0} W_i' \subset H'$ is dense;

(2) the inclusion $F_i^{-1}(B_{s_i}) \subset B'_{r_i}$ holds, where

$$F_i := \mathrm{pr}_i \circ F : W_i' \to W_i = l_i(W_i)$$

and $\mathrm{pr}_i : H \to W_i$ is the orthogonal projection;

(3) for each i_0, the norm converges:

$$\lim_{i \to \infty} \sup_{m \in B'_{r_{i_0}}} \left\| F(m) - F_i(m) \right\|_{L^2_{k-1}} = 0;$$

(4) the operator norm of the restriction converges:

$$\lim_{i \to \infty} \left\| (l - l_i)|W'_{i_0} \right\| = 0;$$

(5) the uniform bounds $C^{-1}\|l\| \leq \|l_i\| \leq C\|l\|$ hold on their norms, where C is independent of i.

Later, we will introduce a finite approximability, below Definition 6.1.

Let us introduce two variations:

(A) Suppose both H' and H admit linear isometric actions by a group Γ and assume that F is Γ-equivariant. Then, we say that F is *weakly finitely* Γ*-approximable* if, moreover, for the above family $\{W_i'\}_i$, the union

$$\bigcup_i \{\gamma(W_i') \cap W_i'\} \subset H'$$

is dense for any $\gamma \in \Gamma$.

Note that the above family $\{F_i\}_i$ satisfies convergence for any $\gamma \in \Gamma$:

$$\lim_{i \to \infty} \sup_{m \in B'_{r_i} \cap \gamma^{-1}(B'_{r_i})} \left\| \gamma F_i(m) - F_i(\gamma m) \right\| = 0$$

because the following estimate holds:

$$\begin{aligned} \left\| \gamma F_i(m) - F_i(\gamma m) \right\| &\leq \left\| \gamma F(m) - \gamma F_i(m) \right\| + \left\| \gamma F(m) - F_i(\gamma m) \right\| \\ &= \left\| F(m) - F_i(m) \right\| + \left\| F(\gamma m) - F_i(\gamma m) \right\|. \end{aligned}$$

Let us take $\gamma \in \Gamma$ and consider the γ shift of the weakly finite approximation data

$$\gamma(W_i'), \quad \gamma^*(F_i), \quad \gamma^*(l_i).$$

This shift gives another weakly finite approximation of F.

(B) Suppose $F = l + c : L^2_k(X; E') \to L^2_{k-1}(X; E)$ consists of a first-order differential operator and c is the non-linear term by some pointwise operation.

Let us say that a weakly finite approximation of F is *adapted* if there is an exhaustion $K_1 \subset \cdots \subset K_i \subset K_{i+1} \subset \cdots \subset X$ by compact subsets such that the following inclusions both hold:

$$W_i' \subset L^2_k(K_i; E')_0, \quad W_i \subset L^2_{k-1}(K_i; E)_0.$$

Hereinafter, we always assume that any weakly finite approximation of F is adapted whenever F is a map between Sobolev spaces. Note that if a group Γ acts on X, then the Γ orbit $\Gamma(K_i) = X$ coincides with X for all sufficiently large i. Hereinafter, we assume this property for any i.

Proposition 5.15. *Let*

$$F = l + c : H' = L^2_k(X; E') \to H = L^2_{k-1}(X; E)$$

be a Γ*-equivariant locally strongly proper map, where* l *is a first-order elliptic differential operator and* c *is pointwise and locally compact on each bounded set. Suppose* l *is isomorphic.*

Then, there is an adapted family of finite-dimensional linear subspaces $\{W_i'\}_i$ *that weakly finitely* Γ*-approximates* F.

Proof. Take an exhaustion of X by compact subsets $K_1 \subset \cdots \subset K_i \subset K_{i+1} \subset X$. It follows from Corollary 5.13 that there are finite-dimensional linear subspaces $W_i' \subset L^2_k(K_i; E')_0$ and $W_i \subset L^2_{k-1}(K_i; E)_0$ with positive numbers $s_i, r_i > 0$ such that the family of maps:

$$F_i := \mathrm{pr}_{W_i} \circ F : B'_{r_i} \to W_i,$$

is asymptotically proper and satisfies the inclusion $F_i^{-1}(B_{s_i}) \subset B'_{r_i}$. Moreover, the restrictions satisfy the convergences

$$\lim_{i\to\infty} \|F_i - F\|_{B'_{r_{i_0}}} = 0 \quad \text{and} \quad \lim_{i\to\infty} \left\|(l - l_i)|W'_{i_0}\right\| = 0.$$

These properties also hold if W_i' is replaced by any other finite-dimensional linear subspace $\widetilde{W}_i'$ that contains W_i'. Thus, we assume that the union

$$\bigcup_{i\geq 1} W_i' \subset L^2_k(X; E')$$

is dense and, hence, $\bigcup_i W_i \subset L^2_{k-1}(X; E)$ is also dense because l is assumed to be isomorphic.

Let us consider the Γ action. Let us replace W_i' by the span of $\Gamma(W_i') \cap L^2_k(K_i; E')_0$ and denote it by W_i''. Note that W_i'' contains W_i' and is also finite-dimensional. Moreover, the corresponding F_i and l_i still give the weakly finite approximation data.

Take any γ and i. Then, there is some $j \geq i$ such that $\gamma^{-1}(W_i') \subset L^2_k(K_j; E')_0$. So $\gamma^{-1}(W_i') \subset W_j''$. Hence, the inclusion

$$\gamma(W_j'') \cap W_j'' \supset W_i'$$

holds. This implies that the union of the left-hand side with respect to j is dense in $L^2_k(X; E')$. This gives a weakly Γ-finite approximation of F. ∎

Remark 5.16. The above argument implies that by adding more linear planes if necessarily, one may assume Γ-invariance of the union

$$\gamma\Big(\bigcup_i W_i\Big) = \bigcup_i W_i$$

for any $\gamma \in \Gamma$.

In other words, for any γ and i, there is some $i' \geq i$ such that the inclusion $\gamma(W_i) \subset W_{i'}$ holds.

5.5. Sliding end phenomena

In general, $F_i | B_{r_i} \cap W_i'$ may not converge to F in operator topology if there is a difference between the images of $\mathrm{pr} \circ c(L^2_k(K; E')_0) \subset L^2_{k-1}(K; E)_0$ and $l(L^2_k(K; E')_0)$.

Example 5.17. Let us give a simple example. Let $H' = H = l^2(\mathbb{Z})$ and consider $F = l + c : H' \to H$, where $l(\{a_i\}_i) = \{a_{i-1}\}_i$ is the shift and $c(\{a_i\}) = \{b_i\}_i$ with $b_i = f(a_i)$.

We set $V_{m,n} = \{\{a_i\}_i : a_i = 0 \text{ for } i \leq m \text{ or } n \leq i\}$. Then, $l : V_{m,n} \cong V_{m+1,n+1}$ and the restriction $\mathrm{pr} \circ F - F : V_{m,n} \to V_{m,n+1}$ satisfy $(\mathrm{pr} \circ F - F)(\{a_i\}_i) = -f(a_m)$. Therefore, $\mathrm{pr} \circ F$ pushes bubbling $f(a_m)$ off as $m \to -\infty$.

Let us introduce a sliding end quantity. Let $K_1 \Subset \cdots \Subset K_i \Subset K_{i+1} \Subset X$ be an exhaustion of X by compact subsets and let

$$\mathrm{pr}_i : L^2_{k-1}(K_i; E)_0 \to l\big(L^2_k(K_i; E')_0\big)$$

be the orthogonal projections.

We introduce a *sliding end quantity* $b(F) \in [0, C_0]$ given by

$$b(F) := \inf_{\{K_i\}_i} \lim_{i \to \infty} b(F)_i,$$

where

$$b(F)_i = \sup_{v \in L^2_k(K_i)_0} \big\{ \big\| (1 - \mathrm{pr}_i)\big(c(v)\big) \big\|_{L^2_{k-1}} : \|v\|_{L^2_k(K_i)_0} \leq 1 \big\}.$$

6. Infinite-dimensional Bott periodicity

Let H be a Hilbert space. Higson, Kasparov, and Trout constructed the Clifford C^*-algebra $S\mathfrak{C}(H)$ of H and verified Bott periodicity $\beta : K(C_0(\mathbb{R})) \cong K(S\mathfrak{C}(H))$ by use of approximations by finite-dimensional linear subspaces. If a discrete group Γ acts on H linearly and isometrically, then its construction induces the equivariant Bott periodicity:

$$\beta : K\big(C_0(\mathbb{R} \rtimes \Gamma)\big) \cong K\big(S\mathfrak{C}(H) \rtimes \Gamma\big),$$

where the crossed product is full.

Remark 6.1. Even though the C^*-algebra in the right-hand side is generally quite "huge", its K-theory has the same size as $K_*(C^*\Gamma)$. In non-commutative geometry, it is conjectured (the Baum–Connes conjecture) and actually verified for many classes of groups that the K-theory of the reduced group C^*-algebra $C_r^*\Gamma$ is isomorphic to the K-homology of the classifying space $B\Gamma$ if Γ is torsion free. In particular, $K_*(C^*\Gamma)$ is isomorphic to $K_*(B\Gamma)$ if Γ is a torsion-free amenable group [21]. Notice that $K_*(X)$ is rationally isomorphic to $H_*(X;\mathbb{Q})$ for a CW complex X.

6.1. Asymptotic unitary maps

Let $l : H' \cong H$ be a linear isomorphism between Hilbert spaces.

Definition 6.1 ([26]). Let H' and H be Hilbert spaces and let $l : H' \cong H$ be a linear isomorphism. Then, l is asymptotically unitary if, for any $\varepsilon > 0$, there is a finite-dimensional linear subspace $V \subset H'$ such that the restriction

$$l : V^{\perp} \cong l(V^{\perp})$$

satisfies the estimate

$$\|(l - \bar{l})|V^{\perp}\| < \varepsilon$$

on its operator norm, where $\bar{l}$ is the unitary of the polar decomposition of $l : H' \cong H$.

Let $F = l + c : H' \to H$ be weakly finitely approximable with $F_i : W_i' \to W_i$ (see Definition 5.1). In [26], we have introduced *finite approximability* on F.

Definition 6.2. (1) F is finitely approximable if, moreover, l is asymptotically unitary.

(2) F is strongly finitely approximable if it is finitely approximable with the same $l_i = l$ and $c_i = \mathrm{pr}_i \circ c$ such that

$$\lim_{i\to\infty} \sup_{m\in B'_{r_i}} \left\|(1 - \mathrm{pr}_i) \circ c(m)\right\| = 0$$

holds, where $\mathrm{pr}_i : H \to W_i$ is the orthogonal projection.

(3) F is asymptotically finitely approximable if there is a stratification by infinite-dimensional Hilbert subspaces:

$$H_1' \subset H_2' \subset \cdots \subset H'$$

with $W_i' \subset H_i'$ such that the restriction of the linear part $l|H_i'$ on H_i' is asymptotically unitary for each i.

Remark 6.2. In (3) above, when we consider the Γ action, we do not generally require Γ-invariance on each H_i'. Note that, by definition, the union $\bigcup_i H_i' \subset H'$ is dense.

Suppose H' and H are the Sobolev spaces such that $l : L_k^2(X;E') \cong L_{k-1}^2(X;E)$ is an elliptic operator that gives an isomorphism. Recall the Sobolev norm introduced in Section 3. We denote by P the spectral projection of $l^* \circ l$, where l is regarded as an unbounded operator on $L^2(X; E')$ and l^* is the formal adjoint operator. We mostly regard l as a bounded operator between Sobolev spaces and, hence, l^* is the adjoint operator between them, unless otherwise stated.

The following lemma 6.3 is a key to inducing asymptotic unitarity for an elliptic operator.

Lemma 6.3. *Let l be as above. Then, the operator $l : H' \cong H$ satisfies the property that, for any $\varepsilon > 0$, there is $\lambda_0 \gg 1$ such that the operator norm of the restriction of $l^* \circ l$ on $P[\lambda_0, \infty) \subset H'$ satisfies the estimate*

$$\big\| (l - \bar{l}) | P[\lambda_0, \infty) \big\| < \varepsilon, \tag{6.1}$$

where $^-$ is the unitary of the polar decomposition.

In particular, the self-adjoint operator

$$U := \bar{l}^* \circ l = \sqrt{l^* l} : H' \cong H'$$

satisfies the estimate

$$\big\| (U - \mathrm{id}) \big| P[\lambda_0, \infty) \big\| < \varepsilon. \tag{6.2}$$

Proof. The latter statement (6.2) follows from the former (6.1).

Let us verify the former property (6.1). We set

$$P_N(c) = \frac{c^N - 1}{c - 1}$$

for $c > 1$. Notice the equalities $cP_{N-1}(c) + 1 = P_N(c)$.

If u is an eigenvector vector with $l^* l(u) = \lambda^2 u$, then the formulas

$$\|u\|^2_{L^2_k} = P_k(\lambda^2) \|u\|^2_{L^2}$$

hold for all $k \geq 0$. We can check this by induction as

$$\begin{aligned} \langle u, u \rangle_{L^2_k} &= \big\langle l(u), l(u) \big\rangle_{L^2_{k-1}} + \langle u, u \rangle_{L^2} \\ &= \big\langle l^* l(u), u \big\rangle_{L^2_{k-1}} + \langle u, u \rangle_{L^2} \\ &= \lambda^2 \langle u, u \rangle_{L^2_{k-1}} + \|u\|^2_{L^2}. \end{aligned}$$

In particular, if $u \in L^2_k(X; E')$ with $\|u\|_{L^2_k} = 1$ lies in the image of the spectral projection to $[\lambda_0^2, \infty)$ on $l^* \circ l$, then $\|u\|_{L^2}$ is sufficiently small for large $\lambda_0 \gg 1$. Then, from

$$\langle u, u \rangle_{L^2_k} = \big\langle l(u), l(u) \big\rangle_{L^2_{k-1}} + \langle u, u \rangle_{L^2},$$

it follows that l is close to preserve the norms. ∎

Let $l : L^2_k(X; E') \cong L^2_{k-1}(X; E)$ be as above and let $K \subset X$ be a compact subset. Consider the restriction

$$l : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0.$$

Proposition 6.4. *$l : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$ is asymptotically unitary. In particular, $U := \bar{l}^{-1} \circ l$ is an asymptotic identity.*

Proof. The proof consists of three steps.

Step 1. We restate that, for any $\varepsilon > 0$, there is a finite-dimensional linear subspace $V \subset L^2_k(K; E')_0$ such that the restriction

$$l : V^\perp \cap L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$$

satisfies

$$\|(l - \bar{l})|V^\perp \cap L^2_k(K; E')_0\| < \varepsilon.$$

In particular, we obtain the estimate

$$\|(U - \mathrm{id})|V^\perp \cap L^2_k(K; E')_0\| < \varepsilon.$$

We will verify this in Steps 2 and 3.

Note that an eigenvalue can have infinite multiplicity on $L^2(X)$, when X is non-compact. For such cases, the above estimate does not hold.

Step 2. Let $P[0, \lambda] : L^2(X; E') \to L^2(X; E')$ be the spectral projection of $l^* \circ l$ and let $B_K \subset L^2_k(K; E')_0$ be the unit ball.

We claim that $P[0, \lambda](B_K) \subset L^2_k(X; E')$ is relatively compact for every $\lambda > 0$.

In fact, the inclusion $L^2_{k+1}(X; E') \to L^2_k(X; E')$ is compact by Rellich's lemma.

Since the bounded map $P[0, \lambda] : L^2_k(X; E') \to L^2_k(X; E')$ extends to a bounded map $P[0, \lambda] : L^2_k(X; E') \to L^2_{k+1}(X; E')$, the former map factors through the last one. Then, the composition is relatively compact. This verifies the claim.

Step 3. Let us take an orthonormal basis $\{u_i\}_i \subset L^2_k(K; E')_0$ and set $u^1_i = P[0, \lambda](u_i)$ with $u^2_i = u_i - u^1_i \in P(\lambda, \infty)(B_K)$. It follows from Step 2 that a subsequence of $\{u^1_i\}_i$ converges in L^2_k. In particular, for any $\varepsilon > 0$, there is a finite-dimensional vector space V' spanned by $\{u^1_{i_1}, \dots, u^1_{i_m}\}$ for some $\{i_1, \dots, i_m\}$ such that

$$\|(1 - \mathrm{pr}_{V'})u^1_i\|_{L^2_k} < \varepsilon$$

holds for all i.

Let $V \subset L^2_k(K; E')_0$ be a finite-dimensional vector space spanned by $\{u_{i_1}, \dots, u_{i_m}\}$. Then, the inclusion

$$V \subset V' \oplus P[\lambda, \infty)$$

holds. Moreover, for any i, there is $u'_i \in L^2_k(X; E)$ with $\|u^1_i - (u'_i)^1\|_{L^2_k} < \varepsilon$ such that

$$u'_i \in V' \oplus P[\lambda, \infty)$$

holds. Then, the conclusion follows by Lemma 6.3. ∎

In particular, if we apply Proposition 6.4 to the monopole map over a compact four-manifold, we obtain the following.

Corollary 6.5. *Let $F = l + c : H' \to H$ be the monopole map over a compact oriented four-manifold M with $b^1(M) = 0$, such that the Fredholm index of l is zero. Then, F is strongly finitely approximable.*

Proof. This follows from [4] with Proposition 6.4. ∎

Remark 6.6. (1) In the case of the covering-monopole map:

$$\begin{aligned}\tilde{\mu} : L^2_k\big((X,g);\tilde{S}^+\big) \oplus L^2_k\big((X,g);\Lambda^1 \otimes i\mathbb{R}\big) \cap \operatorname{Ker} d^* \\ \to L^2_{k-1}\big((X,g);\tilde{S}^- \oplus \Lambda^2_+ \otimes i\mathbb{R}\big) \oplus H^1_{(2)}(X) \\ (\phi, a) \mapsto \big(F_{\tilde{A}_0,\tilde{\psi}_0}(\psi, a), [a]\big),\end{aligned}$$

the target space is the sum of the Sobolev space with $H^1_{(2)}$, where the latter space is infinite-dimensional if not zero.

By Hodge theory, $\operatorname{Ker} d^*$ decomposes as $d^*(L^2_{k+1}(X;\Lambda^2_+)) \oplus H^1_{(2)}(X)$ and, hence,

$$\bigcup_i d^*\big(L^2_{k+1}(K_i;\Lambda^2_+)_0\big) \oplus H^1_{(2)}(X)$$

is dense in $\operatorname{Ker} d^*$.

The restriction of the linearized map on the harmonic part is, in fact, isometry. Hence, the covering-monopole map is also asymptotically unitary over compactly supported Sobolev spaces.

(2) In the case when $l : L^2_k(X) \cong L^2_{k-1}(X)$ gives a linear isomorphism, we can use the Sobolev norms by

$$\langle u, v\rangle_{L^2_k} = \big\langle (l^* l)^k u, v\big\rangle_{L^2}.$$

Then, $l : L^2_k(X) \cong L^2_{k-1}(X)$ is unitary with respect to this particular norm. (See the paragraph below the proof of Lemma 3.1.)

Now consider the case of a covering-monopole map.

Corollary 6.7. *Assume the conditions in Proposition 5.15. Then, F can admit an asymptotic Γ-approximation.*

Proof. Let us take an exhaustion by compact subsets $\bigcup_i K_i = X$. It follows from Proposition 5.15 that there is a family of finite-dimensional linear subspaces $W'_i \subset L^2(K_i, E')$ that satisfies the conditions in Definition 5.1. We may assume that it is adapted such that $W'_i \subset L^2_k(K_i, E')_0$.

To obtain an asymptotically Γ-finite approximation, we set $H'_i := L^2_k(K_i; E')_0$ with $H' := L^2_k(X; E')$ in Definition 6.2. Then, from Proposition 6.4, the restriction

$$l : L^2_k(K; E')_0 \to L^2_{k-1}(K; E)_0$$

is asymptotically unitary on each compact subset $K \subset X$. ∎

Remark 6.8. Let us describe some functional analytic aspect of a differential operator acting on Sobolev spaces. Let $\tilde{l}_i : L^2_k(K_i; E')_0 \to L^2_{k-1}(K_i; E)_0$ be the restriction of $l : L^2_k(X; E') \to L^2_{k-1}(X; E)$ such that the equality $\tilde{l}_i^* \tilde{l}_i = \mathrm{pr}_{L^2_k(K_i)_0} \circ \tilde{l}_{i+1}^* \tilde{l}_{i+1}$ holds on $L^2_k(K_i; E')_0$, where $\tilde{l}_i^* : L^2_{k-1}(K_i; E')_0 \to L^2_k(K_i; E)_0$ is the adjoint operator between these Hilbert spaces.

We claim that the restrictions of the self-adjoint operators below satisfy the equality

$$\tilde{l}_i^* \tilde{l}_i | W'_{i_0} = \tilde{l}_{i+1}^* \tilde{l}_{i+1} | W'_{i_0} \quad \text{for } i_0 < i.$$

Note that l is assumed to be a first-order differential operator. Let $\varphi_i \in C_c^\infty(K_{i+1})$ be a cut-off function with $\varphi_i | K_{i_0} \equiv 1$ and $\varphi_i | K_i^c \equiv 0$. Consider the equalities among the inner product values as

$$\begin{aligned} \langle \tilde{l}_{i+1}^* \tilde{l}_{i+1}(v), v' \rangle &= \langle \tilde{l}_{i+1}(v), \tilde{l}_{i+1}(v') \rangle = \langle \tilde{l}_i(v), \tilde{l}_i(\varphi_i v') \rangle \\ &= \langle \tilde{l}_i^* \tilde{l}_i(v), \varphi_i v' \rangle = \langle \varphi_i \tilde{l}_i^* \tilde{l}_i(v), v' \rangle \end{aligned}$$

for any unit vectors $v \in W'_{i_0} \subset L^2_k(K_{i_0}; E')_0$ and $v' \in L^2_k(K_{i+1}; E')_0$. Hence, the equality $\tilde{l}_{i+1}^* \tilde{l}_{i+1} = \varphi_i \tilde{l}_i^* \tilde{l}_i$ holds on W'_{i_0}. In particular,

$$\langle \tilde{l}_{i+1}^* \tilde{l}_{i+1}(v), v'' \rangle = 0$$

vanishes for any $v'' \in L^2_k(K_i; E')_0^\perp \cap L^2_k(K_{i+1}; E')_0$. Thus, if we decompose $v' = u + v'' \in L^2_k(K_{i+1}; E')_0$ with $u \in L^2_k(K_i; E')_0$, then the equality

$$\langle \tilde{l}_{i+1}^* \tilde{l}_{i+1}(v), v' \rangle = \langle \tilde{l}_{i+1}^* \tilde{l}_{i+1}(v), u \rangle = \langle \tilde{l}_i^* \tilde{l}_i(v), u \rangle$$

holds, which verifies the claim.

The above argument implies the inclusion

$$\tilde{l}_j^* \tilde{l}_j \big(L^2_k(K_{i-1}; E')_0 \big) \subset L^2_k(K_i; E')_0$$

for any $j \geq i$.

6.2. Induced Clifford C^*-algebras

We recall the construction of the induced Clifford C^*-algebras in [26]. Assume that $F = l + c : H' \to H$ is finitely approximable as in Definition 6.2 with respect to the data $W'_0 \subset \cdots \subset W'_i \subset \cdots \subset H'$ with open disks $B'_{r_i} \subset W'_i$ and $B_{s_i} \subset W_i$, and $F_i = l_i + c_i : W'_i \to W_i$.

Let $S_r := C_0(-r, r) \subset S$ be the set of continuous functions on $(-r, r)$ vanishing at infinity and consider the following C^*-subalgebras:

$$S_{r_i} \mathfrak{C}(B'_{r_i}) := S_{r_i} \hat{\otimes} C_0\big(B'_{r_i}, \mathrm{Cl}(W'_i)\big).$$

Since the inclusion $F_i^{-1}(B_{s_i}) \subset B'_{r_i}$ holds, it induces a $*$-homomorphism:

$$F_i^* : S_{s_i} \mathfrak{C}(B_{s_i}) \to S_{r_i} \mathfrak{C}(B'_{r_i})$$

given by $F_i^*(h)(v') := \bar{l}_i^{-1}(h(F_i(v')))$. Denote its image by

$$S_{r_i}\mathfrak{C}_{F_i}(B'_{r_i}) := F_i^*\big(S_{s_i}\mathfrak{C}(B_{s_i})\big),$$

which is a C^*-subalgebra with the norm $\| \ \|_{S_{r_i}\mathfrak{C}_{F_i}}$.

Let us say that a family of elements $\alpha_i \in S_{r_i}\mathfrak{C}_{F_i}(B'_{r_i})$ is *F-compatible* if there is an element $u_{i_0} \in S_{s_{i_0}}\mathfrak{C}(B_{s_{i_0}})$ such that

$$\alpha_i = F_i^*(u_i) \in S_{r_i}\mathfrak{C}_{F_i}(B'_{r_i})$$

holds for any $i \geq i_0$, where $u_i = \beta(u_{i_0}) \in S_{s_i}\mathfrak{C}(B_{s_i})$ with the standard Bott map β introduced in [22]. For an F-compatible sequence $\alpha = \{\alpha_i\}_{i \geq i_0}$, the limit

$$\big\|\{\alpha_i\}_i\big\| := \lim_{j\to\infty}\lim_{i\to\infty} \|\alpha_i | B'_{r_j}\|$$

exists because both F_i and l_i converge weakly. Moreover, both F_i^* and β are $*$-homomorphisms between C^*-algebras and, hence, both are norm-decreasing.

Definition 6.3. Let F be finitely approximable. The induced Clifford C^*-algebra is given by

$$S\mathfrak{C}_F(H') = \overline{\big\{\{\alpha_i\}_i ; F\text{-compatible sequences}\big\}},$$

which is obtained by the norm closure of all F-compatible sequences, where the norm is the one above.

6.3. Induced maps on Clifford C^* algebras

When $H = E$ is finite-dimensional, $S\mathfrak{C}(E)$ is given by

$$C_0(\mathbb{R})\hat{\otimes}C_0\big(E, \mathrm{Cl}(E)\big),$$

where $\mathrm{Cl}(E)$ is the complex Clifford algebra of E. Let E' and E be two finite-dimensional Euclidean spaces and let

$$F = l + c : E' \to E$$

be a proper map, where l is its linear part. Assume that $l : E' \cong E$ is an isomorphism and let $\bar{l} := l\sqrt{l^*l}^{-1} : E' \cong E$ be the unitary corresponding to l. There is a natural pull-back $F^* : S\mathfrak{C}(E) \to S\mathfrak{C}(E')$ which is induced from

$$F^* : C_0\big(E, \mathrm{Cl}(E)\big) \to C_0\big(E', \mathrm{Cl}(E')\big)$$

by $u \to v' \to \bar{l}^*(u(F(v')))$, where $\bar{l} : \mathrm{Cl}(E') \cong \mathrm{Cl}(E)$ is the canonical extension of $\bar{l}$ between the Clifford algebras. When $F = l + c : H' \to H$ is a map between infinite-dimensional Hilbert spaces, we typically cannot obtain such a general induced map as F^* between $S\mathfrak{C}(H)$, in general.

Lemma 6.9 ([26]). *Let $F = l + c : H' \to H$ be a strongly finitely approximable map. Then, it induces a $*$-homomorphism:*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H').$$

Let us apply K-theory. The above F^* induces a homomorphism:

$$F^* : \mathbb{Z} \cong K\big(S\mathfrak{C}(H)\big) \to K\big(S\mathfrak{C}(H')\big) \cong \mathbb{Z}.$$

It is given by the multiplication of an integer that we call the *degree* of F as follows:

$$F^* = \deg F \times : \mathbb{Z} \to (\deg F)\mathbb{Z}.$$

Remark 6.10. We can replace the condition of linear isomorphism of l with a zero Fredholm index [26, Remark 5.4].

For finitely approximable F, we constructed a variant $S\mathfrak{C}_F(H)$ of $S\mathfrak{C}(H)$ in [26]. In fact, its construction can be straightforwardly generalized to apply and obtain the C^*-algebra.

Lemma 6.11. *Let F be asymptotically finitely approximable as in Definition 6.2. Then, there is a C^*-algebra $S\mathfrak{C}_F(H')$.*

If $H' = E'$ and $H = E$ are both finite-dimensional, then an asymptotically finitely approximable map is finitely approximable, and $S\mathfrak{C}_F(E')$ is given by the image of the induced map

$$F^* : S\mathfrak{C}(E) \cong S\mathfrak{C}_F(E') = F^*\big(S\mathfrak{C}(E)\big) \subset S\mathfrak{C}(E')$$

in the standard sense in basic algebraic topology.

The following property has been verified for the class of finitely approximable maps in [26]. However, the proof is parallel to the case for a broader class of asymptotically finitely approximable maps.

Proposition 6.12. *Let $F = l + c : H' \to H$ be a finitely approximable map. Then, F induces a $*$-homomorphism:*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H').$$

If a discrete group Γ acts on both H' and H linearly and isometrically, then F^* is Γ-equivariant.

In particular, F induces a homomorphism

$$F^* : K_*\big(S\mathfrak{C}(H) \rtimes \Gamma\big) \cong K_{*+1}\big(C^*(\Gamma)\big) \to K\big(S\mathfrak{C}_F(H') \rtimes \Gamma\big),$$

where the isomorphism above is given by [22].

For convenience, we now quickly describe how to construct the induced $*$-homomorphism. Take an element $v \in S\mathfrak{C}(H)$ and its approximation $v = \lim_{i\to\infty} v_i$ with $v_i \in S_{s_i}\mathfrak{C}(B_{s_i}) = C_0(-s_i, s_i)\hat{\otimes} C_0(B_{s_i}, \mathrm{Cl}(W_i))$.

Consider the induced $*$-homomorphism

$$F_i^* : S_{s_i}\mathfrak{C}(B_{s_i}) \to S_{r_i}\mathfrak{C}(B'_{r_i})$$

and denote its image by $S_{r_i}\mathfrak{C}_{F_i}(B'_{r_i}) := F_i^*(S_{s_i}\mathfrak{C}(B_{s_i}))$.

Let $u_i = \beta(v_{i_0}) \in S_{s_i}\mathfrak{C}(B_{s_i})$ be the image of the standard Bott map for some i_0. Then, the family $\{F_i^*(u_i)\}_{i \geq i_0}$ determines an element in $S\mathfrak{C}_F(H')$, which gives a $*$-homomorphism

$$F^* : S_{s_{i_0}}\mathfrak{C}(B_{s_{i_0}}) \to S\mathfrak{C}_F(H')$$

since both F_i^* and β are $*$-homomorphisms. Note that the composition of the two $*$-homomorphisms,

$$S_{s_{i_0}}\mathfrak{C}(B_{s_{i_0}}) \xrightarrow{\beta} S_{s_{i'_0}}\mathfrak{C}(B_{s_{i'_0}}) \xrightarrow{F^*} S\mathfrak{C}_F(H'),$$

coincides with $F^* : S_{s_{i_0}}\mathfrak{C}(B_{s_{i_0}}) \to S\mathfrak{C}_F(H')$.

Take two sufficiently large $i'_0 \geq i_0 \gg 1$ such that the estimate $\|\beta(v_{i_0}) - v'_{i_0}\| < \varepsilon$ holds for a small $\varepsilon > 0$. Set $u'_i = \beta(v_{i'_0}) \in S_{s_i}\mathfrak{C}(B_{s_i})$ for $i \geq i'_0$. Since F^* is norm-decreasing, the estimate $\|F_i^*(u_i) - F_i^*(u'_i)\| < \varepsilon$ holds for any $i \geq i'_0$. Hence, the estimate $\|F^*(v_{i_0}) - F^*(v'_{i_0})\| < \varepsilon$ holds.

Thus, we obtain the assignment $v \to \lim_{i_0 \to \infty} F^*(v_{i_0})$, which gives a Γ-equivariant $*$-homomorphism:

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H'),$$

where $\{v_i\}_i$ is any approximation of v.

7. *K*-theoretic higher degree

Let H' and H be two Hilbert spaces on which Γ acts linearly and isometrically, and let $F = l + c : H' \to H$ be a Γ equivariant smooth map such that $l : H' \cong H$ gives a linear isomorphism.

Assume that F is asymptotically Γ-finitely approximable such that there is a family of finite-dimensional linear subspaces $W'_0 \subset W'_1 \subset \cdots \subset W'_i \subset \cdots \subset H'$ with dense union, $F_i : W'_i \to W_i = l_i(W'_i)$ with the inclusions $F_i^{-1}(B_{s_i}) \subset B'_{r_i}$, and two convergences to both F and l (see Definition 6.2 and Definition 5.1).

Our basic idea is to pull back functions on $W_i = l_i(W'_i)$ by F_i and combine them. Consider the induced $*$-homomorphism $F_i^* : S\mathfrak{C}(W_i) \to S\mathfrak{C}(W'_i)$ by

$$F_i^*(f \hat{\otimes} h)(v) = f \hat{\otimes} \bar{l}_i^{-1}\big(h\big(F_i(v)\big)\big),$$

where $S\mathfrak{C}(W_i) = C_0(\mathbb{R}) \hat{\otimes} C_0(W_i, \mathrm{Cl}(W_i))$.

We shall give the equivariant degree of the covering-monopole map as a homomorphism between K-groups of the C^*-algebras.

Theorem 7.1. *Let $F = l + c : H' \to H$ be the covering-monopole map. Assume that the linearized operator is an isomorphism. Furthermore, assume that the AHS complex has closed range over the universal covering space. Then, F induces the equivariant $*$-homomorphism*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H').$$

In particular, it induces a map on K-theory as

$$F^* : K_*\big(C^*(\Gamma)\big) \to K_*\big(S\mathfrak{C}_F(H') \rtimes \Gamma\big).$$

We call this as the *higher degree of the covering-monopole map*.

Proof. It follows from Theorem 4.1 that the covering-monopole map is locally strongly proper.

Then, by Proposition 5.15, it is weakly Γ-finitely approximable. By Corollary 6.7, F is actually asymptotically Γ-finitely approximable.

Then, the conclusion follows from Proposition 6.12. ∎

Finally, we describe the case of the monopole map over a compact manifold.

Proposition 7.2. *Let $F : H' \to H$ be the monopole map over a compact oriented four-manifold M with $b^1(M) = 0$ and $b^+(M) \geq 1$. Then, F induces a $*$-homomorphism*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H').$$

Moreover, the induced map

$$F^* : K\big(S\mathfrak{C}(H)\big) \cong \mathbb{Z} \to K\big(S\mathfrak{C}(H')\big) \cong \mathbb{Z}$$

is given by multiplication by the degree 0 SW invariant.

Proof. Suppose the index is non-zero. Then, we simply put the map as zero.

Suppose the index is equal to zero. If l is linearly isomorphic, then the conclusion follows from Corollary 6.5 with Lemma 6.9. If the Fredholm index of l is zero, then the same conclusion follows from Remark 6.10. The K-theoretic degree of F coincides with the degree 0 SW invariant by a theorem in [4]. ∎

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References

[1] M. F. Atiyah, *K-Theory. Lecture Notes by D. W. Anderson*. W. A. Benjamin, New York, 1967 Zbl 0159.53302 MR 0224083

[2] M. F. Atiyah, The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pp. 73–84, Univ. Tokyo Press, Tokyo, 1969 Zbl 0193.52302 MR 0254864

[3] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*. Ergeb. Math. Grenzgeb. (3) 4, Springer, Berlin, 1984 Zbl 0718.14023 MR 749574

[4] S. Bauer and M. Furuta, A stable cohomotopy refinement of Seiberg-Witten invariants. I. *Invent. Math.* **155** (2004), no. 1, 1–19 Zbl 1050.57024 MR 2025298

[5] C. Bohr, On the signatures of even 4-manifolds. *Math. Proc. Cambridge Philos. Soc.* **132** (2002), no. 3, 453–469 Zbl 1007.57017 MR 1891683

[6] S. Chern, On curvature and characteristic classes of a Riemann manifold. *Abh. Math. Sem. Univ. Hamburg* **20** (1955), 117–126 Zbl 0066.17003 MR 75647

[7] A. Connes, *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[8] A. Connes and N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), no. 2, 101–106 Zbl 0717.46062 MR 1065438

[9] S. K. Donaldson, An application of gauge theory to four-dimensional topology. *J. Differential Geom.* **18** (1983), no. 2, 279–315 Zbl 0507.57010 MR 710056

[10] S. K. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.* **26** (1987), no. 3, 397–428 Zbl 0683.57005 MR 910015

[11] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*. Oxford Math. Monogr., Oxford University Press, New York, 1990 Zbl 0820.57002 MR 1079726

[12] H. Donnelly, The differential form spectrum of hyperbolic space. *Manuscripta Math.* **33** (1981), no. 3-4, 365–385 Zbl 0464.58020 MR 612619

[13] D. G. Ebin, *Espace des métriques riemanniennes et mouvement des fluides via les variétés d'applications*. Centre de Mathématiques de l'École Polytechnique et Université Paris VII, 1972 Zbl 0242.58002 MR 0464298

[14] J. Eichhorn, *Global Analysis on Open Manifolds*. Nova Science Publishers, New York, 2007 Zbl 1188.58001 MR 2343536

[15] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*. 2nd edn., Math. Sci. Res. Inst. Publ. 1, Springer, New York, 1991 Zbl 0559.57001 MR 1081321

[16] M. Furuta, Monopole equation and the $\frac{11}{8}$-conjecture. *Math. Res. Lett.* **8** (2001), no. 3, 279–291 Zbl 0984.57011 MR 1839478

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Classics Math., Springer, Berlin, 2001 Zbl 1042.35002 MR 1814364

[18] M. Gromov, Kähler hyperbolicity and L_2-Hodge theory. *J. Differential Geom.* **33** (1991), no. 1, 263–292 Zbl 0719.53042 MR 1085144

[19] M. Gromov, Asymptotic invariants of infinite groups. In *Geometric Group Theory, Vol. 2 (Sussex, 1991)*, pp. 1–295, London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press, Cambridge, 1993 Zbl 0841.20039 MR 1253544

[20] M. Gromov and H. B. Lawson Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Publ. Math. Inst. Hautes Études Sci.* (1983), no. 58, 83–196 Zbl 0538.53047 MR 720933

[21] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.* **144** (2001), no. 1, 23–74 Zbl 0988.19003 MR 1821144

[22] N. Higson, G. Kasparov, and J. Trout, A Bott periodicity theorem for infinite-dimensional Euclidean space. *Adv. Math.* **135** (1998), no. 1, 1–40 Zbl 0911.46040 MR 1617411

[23] T. Kato, ASD moduli spaces over four-manifolds with tree-like ends. *Geom. Topol.* **8** (2004), 779–830 Zbl 1064.57022 MR 2087070

[24] T. Kato, Spectral analysis on tree like spaces from gauge theoretic view points. In *Discrete Geometric Analysis*, pp. 113–129, Contemp. Math. 347, Amer. Math. Soc., Providence, RI, 2004 Zbl 1072.14052 MR 2077033

[25] T. Kato, Asymptotically quasiconformal four manifolds. *J. Math. Soc. Japan* **64** (2012), no. 2, 423–487 Zbl 1250.57043 MR 2916075

[26] T. Kato, Induced map on K theory for certain Γ-equivariant maps between Hilbert spaces. *J. Noncommut. Geom.*, to appear. DOI 10.4171/JNCG/442

[27] J. L. Kazdan and F. W. Warner, Scalar curvature and conformal deformation of Riemannian structure. *J. Differential Geometry* **10** (1975), 113–134 Zbl 0296.53037 MR 365409

[28] D. Kotschick, Signatures, monopoles and mapping class groups. *Math. Res. Lett.* **5** (1998), no. 1-2, 227–234 Zbl 0905.57010 MR 1617905

[29] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994), no. 4, 455–481 Zbl 0853.57021 MR 1280122

[30] J. W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*. Math. Notes 44, Princeton University Press, Princeton, NJ, 1996 Zbl 0846.57001 MR 1367507

[31] P. Pansu, L^p-cohomology of symmetric spaces. In *Geometry, Analysis and Topology of Discrete Groups*, pp. 305–326, Adv. Lect. Math. (ALM) 6, Int. Press, Somerville, MA, 2008 Zbl 1159.43003 MR 2464400

[32] A. Schwarz, On the homotopic topology of Banach spaces. *Dokl. Akad. Nauk SSSR* **154** (1964), 61–63; translation in *Soviet Math. Dokl.* **5** (1964), 57–59 Zbl 0181.39904 MR 0160095

[33] C. H. Taubes, Gauge theory on asymptotically periodic 4-manifolds. *J. Differential Geom.* **25** (1987), no. 3, 363–430 Zbl 0615.57009 MR 882829

[34] J. A. Wolf, On the classification of hermitian symmetric spaces. *J. Math. Mech.* **13** (1964), 489–495 Zbl 0245.32011 MR 0160850

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