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An analog of the Krein–Milman theorem for certain non-compact convex sets

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Abstract. We make a contribution towards extending the remarkable Krein–Milman analog result of K. Thomsen and L. Li, in which a certain non-compact convex set is shown to be generated by its extreme points.

1. Introduction

The classical Krein–Milman theorem states that any convex and compact subset of a locally convex topological space is the closure of the convex hull of its extreme points. The Krein–Milman theorem has many applications in different areas of mathematics, e.g., dynamical systems, operator algebras, etc. A strikingly new result resembling the Krein–Milman theorem, but without the compactness, was obtained by K. Thomsen in [3] and (in a stronger form) by L. Li in [1], for the convex set of unital positive linear maps on $C[0,1]$. These maps are also called Markov operators on $C[0,1]$.

Although this set is closed in the topology of pointwise convergence (the strong operator topology), it is not compact. In spite of this, Thomsen and Li succeeded in showing that this closed convex set is the closed convex hull of its extreme points. These are of course the unital algebra homomorphisms.

In this paper, we study the possibility of extending this result. To be precise, we investigate the approximation problem for a Markov operator on $C[0,1]$ leaving a certain subspace invariant, which corresponds to the space of continuous affine functions on the cone of traces on a certain subhomogeneous C^*-algebra. We want to use an average of homomorphisms on $C[0,1]$ to do an approximation, additionally requiring that the average also leaves the subspace invariant.

In one case, we have succeeded in making these homomorphisms themselves leave the subspace invariant—they are exactly extreme points. We present such an approximation for the subspace of $C[0,1]$ arising from a Razak C^*-algebra, namely, the following C^*-algebra (see [2]):

$$R(a,k)=\left\{f\in C[0,1]\otimes \mathrm{M}_m \mid f(0)=\begin{pmatrix} W\otimes \mathrm{id}_a & \\ & 0\otimes \mathrm{id}_k\end{pmatrix},\ f(1)=W\otimes \mathrm{id}_{a+k},\ W\in \mathrm{M}_n\right\},$$

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where a, k, m, and n are non-zero natural numbers. Such a C^*-algebra is non-unital and stably projectionless. The space of continuous affine functions on the tracial cone of $R(a,k)$ is isomorphic to the subspace $C[0,1]_{(a,k)}$ of $C[0,1]$ given by

$$C[0,1]_{(a,k)} = \left\{ f \in C[0,1] \mid f(0) = \frac{a}{a+k} f(1) \right\}$$

(see [2, Proposition 2.1]). Note that, for ease of notation, we consider complex-valued functions.

The first main result of this paper is the following theorem (of Li type).

Theorem 1.1. *Given any finite subset $F \subset C[0,1]_{(a,k)}$ and $\varepsilon > 0$, there is an integer $N > 0$ with the following property: for any unital positive linear map ϕ on $C[0,1]$ which preserves $C[0,1]_{(a,k)}$, there are N unital homomorphisms $\phi_1, \phi_2, \dots, \phi_N$ from $C[0,1]$ to $C[0,1]$ such that each ϕ_i leaves $C[0,1]_{(a,k)}$ invariant (and hence $\frac{1}{N}\sum_{i=1}^N \phi_i(f)$ will also do the same job) and*

$$\left\| \phi(f) - \frac{1}{N}\sum_{i=1}^{N} \phi_i(f) \right\| < \varepsilon$$

for all $f \in F$.

For the case of different subspaces $C[0,1]_{(a,k)}$ and $C[0,1]_{(b,k)}$, there is no unital positive linear map on $C[0,1]$ which sends $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$ if $b < a$; see Remark 3.8. So, we only need to deal with the case $b > a$, and a theorem of Thomsen type is obtained.

Theorem 1.2. *Given any finite subset $F \subset C[0,1]_{(a,k)}$, $\varepsilon > 0$, and any unital positive linear map ϕ on $C[0,1]$ which sends $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$ with $b > a$, there are N unital homomorphisms $\phi_1, \phi_2, \dots, \phi_N : C[0,1] \to C[0,1]$ such that $\frac{1}{N}\sum_{i=1}^N \phi_i(f)$ belongs to $C[0,1]_{(b,k)}$ for all $f \in C[0,1]_{(a,k)}$ and*

$$\left\| \phi(f) - \frac{1}{N}\sum_{i=1}^{N} \phi_i(f) \right\| < \varepsilon$$

for all $f \in F$.

To achieve these results, we keep tracking the approximation process of Li's theorem in [1] and Thomsen's theorem in [3], and the crucial point is that we must argue if we are able to choose proper eigenvalue maps to define homomorphisms such that their average preserves the subspace. The existence of such a choice relies on an analysis of the measures induced by the point evaluations of a given Markov operator at 0 and 1.

The paper is organized as follows. Section 2 contains some preliminaries on Markov operators and basic properties of the subspace $C[0,1]_{(a,k)}$ of $C[0,1]$. In Section 3, concrete analyses of the measures induced by evaluations of a given Markov operator at 0 and 1 are given, and based on this the proofs of Theorems 1.1 and 1.2 are presented.

2. Preliminaries

Definition 2.1. A Markov operator T from $C(X)$ to $C(Y)$, where X and Y are compact Hausdorff spaces, is a unital positive linear map.

In S. Razak's paper [2], he considered certain stably projectionless building blocks—necessarily non-unital. The space of continuous affine functions on this building block's tracial cone is a non-unital subspace of $C[0,1]$; see [2, Proposition 2.1]. Therefore, we consider Markov operators on $C[0,1]$ which preserve this subspace. Fix a positive integer a and a positive integer k, and denote by $C[0,1]_{(a,k)}$ the subspace of $C[0,1]$:

$$C[0,1]_{(a,k)} = \left\{ f \in C[0,1] \mid f(0) = \frac{a}{a+k} f(1) \right\}.$$

Next, we shall see some examples of Markov operators on $C[0,1]$ which preserve this subspace.

- Example 1: let $\lambda : [0,1] \to [0.1]$ be a continuous function with $\lambda(0) = 0$ and $\lambda(1) = 1$, and define a Markov operator T from $C[0,1]$ to $C[0,1]$ by $T(f) = f \circ \lambda$. Then, T sends functions in $C[0,1]_{(a,k)}$ to $C[0,1]_{(a,k)}$.
- Example 2: let $\lambda_1, \lambda_2 : [0,1] \to [0,1]$ be two continuous functions such that $\lambda_1(0) = 0$, $\lambda_1(1) = 1$, and $\lambda_2(0) = 1$, $\lambda_2(1) = 0$, and let k_1, k_2 be natural numbers with $k_1 > k_2$. Define a Markov operator T on $C[0,1]$ as follows:

$$T(f) = \frac{k_1 f \circ \lambda_1 + k_2 f \circ \lambda_2}{k_1 + k_2}.$$

  Then, T sends functions in $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$, where $b = \frac{k_1 a + k_2 a + k_2 k}{k_1 - k_2} \geq 0$ (one can choose suitable k_1, k_2 to guarantee b being an integer).
- Example 3: in general, one has similar examples involving more points. With λ_1, λ_2 as above, as well as choosing $\lambda_3 : [0,1] \to [0,1]$ by $\lambda_3(t) = 1/2$, then, for any natural numbers k_1, k_2, one can define a Markov operator T as follows:

$$T(f)(t) = \frac{k_1 f \circ \lambda_1 + k_2 f \circ \lambda_2 + s(t) f \circ \lambda_3}{k_1 + k_2 + s(t)},$$

  where $s(t) = (\frac{k_1 a}{k+a} + k_2)(1-t) + (\frac{k_2 a}{k+a} + k_1)t$. One can verify that T sends functions in $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$ for some $b > 0$. This process can continue to involve more points in $[0,1]$.

The following direct sum decomposition holds.

Lemma 2.2. *$C[0,1] = C[0,1]_{(a,k)} \oplus \mathbb{C}$ as vector space direct sums, where $\mathbb{C}$ denotes the multiple of the constant function* 1.

Proof. Suppose $f \in C[0,1]$ and let $\lambda = ((a+k)f(0) - af(1))/k$, $g(x) = f(x) - \lambda$. Then, $f(x) = \lambda + g(x)$ and $g(x) \in C[0,1]_{(a,k)}$. Next, we show the decomposition is unique. Assume that $f(x) = g_1(x) + \lambda_1 = g_2(x) + \lambda_2$. Then, $\lambda_1 - \lambda_2 = g_1(x) - g_2(x)$ for any $x \in [0, 1]$. It follows that $\lambda_1 - \lambda_2 = g_1(0) - g_2(0) = g_1(1) - g_2(1)$ for any $x \in [0, 1]$. But $g_1(0) - g_2(0) = \frac{a}{a+k}(g_1(1) - g_2(1))$. Since $a \neq 0$, we have $\lambda_1 = \lambda_2$ and $g_1(x) = g_2(x)$. ∎

Looking at things from the opposite point of view, one might consider positive linear maps on these subspaces which can be extended to Markov operators on $C[0, 1]$. Based on the direct sum decomposition, any positive linear map ϕ from $C[0, 1]_{(a,k)}$ to $C[0, 1]$ can be extended naturally to a unital linear map $\widetilde{\phi}$ from $C[0, 1]$ to $C[0, 1]$, given by $\widetilde{\phi}(f) = \lambda + \phi(g)$. Moreover, this algebraic extension needs to be positive.

Definition 2.3. A positive linear map ϕ from $C[0, 1]_{(a,k)}$ to $C[0, 1]$ is called positively extendible if $\widetilde{\phi}$ is still positive.

It is not hard to see that if ϕ is positively extendible, then it must be a contraction. However, the converse is not true, even in the case that ϕ is of norm one.

Remark 2.4. Fix an $x_0 \in (0, 1)$ and define a map ϕ from $C[0, 1]_{(a,k)}$ to $C[0, 1]$ as $\phi(g)(x) = g(x_0)\frac{a+kx}{a+k}$ for any $g \in C[0, 1]_{(a,k)}$. It is obvious that ϕ is positive and linear. Moreover, ϕ has norm one. First, for any $g \in C[0, 1]_{(a,k)}$, $\|\phi(g)\| = \sup_{x\in[0,1]} |\phi(g)(x)| = \sup_{x\in[0,1]} |g(x_0)\frac{a+kx}{a+k}| \leq \|g\|$. Next, we construct a function $g_1(x)$ by

$$g_1(x) = \begin{cases} \frac{a}{a+k} + \frac{(1-\frac{a}{a+k})x}{x_0}, & x \in [0, x_0], \\ 1, & x \in (x_0, 1]; \end{cases}$$

then $\|\phi(g_1)\| = \sup_{x\in[0,1]} |\phi(g_1)(x)| = \sup_{x\in[0,1]} |g_1(x_0)\frac{a+kx}{a+k}| = \|g_1\| = 1$. However, the natural extension $\widetilde{\phi}$ is not positive. Take

$$f(x) = \begin{cases} 0, & x \in [0, x_0], \\ \frac{k}{1-x_0}(x - 1) + k, & x \in (x_0, 1]; \end{cases}$$

then $f(x) \geq 0$. Consider its direct sum decomposition $f = \lambda + g$, where

$$\lambda = \frac{(a + k)f(0) - af(1)}{k} = -a, \quad g(x) = \begin{cases} a, & x \in [0, x_0], \\ \frac{k}{1-x_0}(x - 1) + a + k, & x \in (x_0, 1]. \end{cases}$$

Then, $\widetilde{\phi}(f)(x) = \lambda + \phi(g) = -a + a\frac{a+kx}{a+k} = \frac{ak(x-1)}{a+k} \leq 0$.

Definition 2.5. For each $1 \geq \delta > 0$, the lower test function $e_\delta \in C[0, 1]_{(a,k)}$ is the continuous function which has value 1 on $[\delta, 1]$ and value $a/(k + a)$ at 0, and is linear on the interval $[0, \delta]$. Denote by L the set of all such lower test functions. For each $1 > \sigma \geq 0$, the upper test function $\gamma_\sigma \in C[0, 1]_{(a,k)}$ is the continuous function which has value 1 on $[0, 1 - \sigma]$ and value $(a + k)/a$ at 1, and is linear on the interval $[1 - \sigma, 1]$. Denote by S the set of all such upper test functions.

Proposition 2.6. *Let there be given a positive linear map ϕ from $C[0,1]_{(a,k)}$ to $C[0,1]$. Then, ϕ is positively extendible if and only if the following inequalities hold:*

$$\inf_{\gamma\in S}\{\phi(\gamma)\} \geq 1 \geq \sup_{e\in L}\{\phi(e)\}.$$

Proof. First, suppose ϕ is positively extendible; then $\widetilde{\phi}$ is positive. For any $\gamma \in S$, $e \in L$, $\gamma > 1 > e$, $\widetilde{\phi}(\gamma) \geq 1 \geq \widetilde{\phi}(e)$, then $\phi(\gamma) \geq 1 \geq \phi(e)$, and $\inf_{\gamma\in S}\{\phi(\gamma)\} \geq 1 \geq \sup_{e\in L}\{\phi(e)\}$.

Conversely, suppose $\inf_{\gamma\in S}\{\phi(\gamma)\} \geq 1 \geq \sup_{e\in L}\{\phi(e)\}$, and let us show that $\widetilde{\phi}(f) = \lambda + \phi(g)$ is positive. For any positive $f \in C[0,1]$ with the decomposition $f = \lambda + g$, we need to show $\widetilde{\phi}(f) = \lambda + \phi(g) \geq 0$. Thus, we need to prove $\phi(g) \geq -\lambda$ if $g \geq -\lambda$.

Case I. If $\lambda = 0$, then $f \in C[0,1]_{(a,k)}$. We have $\widetilde{\phi}(f) = \phi(f) \geq 0$ since ϕ is positive.

Case II. If $\lambda < 0$, then $-\lambda > 0$. Then, we can find a $\gamma_\sigma \in S$ such that $(-\lambda)\gamma_\sigma \leq g$. Since ϕ is positive, one has $\phi(g) \geq \phi(-\lambda\gamma_\sigma) = -\lambda\phi(\gamma_\sigma) \geq -\lambda$.

Case III. If $\lambda > 0$, then $-\lambda < 0$. Then, we can find a $e_\delta \in L$ such that $g \geq -\lambda e_\delta$. Since $\sup_{0<\delta\leq 1}\phi(e_\delta) \leq 1$, one has $-\lambda\phi(e_\delta) \geq -\lambda$. Therefore, $\phi(g) \geq -\lambda\phi(e_\delta) \geq -\lambda$.

Hence, ϕ is positively extendible. ∎

3. Approximation results on [0, 1]

Given a Markov operator between $C[0,1]$ leaving the subspace $C[0,1]_{(a,k)}$ invariant, we want to approximate it by an average of homomorphisms on $C[0,1]$ and additionally require that the average also leaves the subspace invariant. Since the sub-homogeneity in consideration arises at 0 and 1, we need to investigate the measures induced by the point evaluations of a Markov operator at the endpoints 0 and 1.

Lemma 3.1. *Let there be given a unital positive linear map ϕ on $C[0,1]$ which preserves $C[0,1]_{(a,k)}$. Then, the measures induced by the evaluations of ϕ at 0 and 1 actually concentrate on 0 and 1, respectively. In other words, $\phi(f)(0) = f(0)$ and $\phi(f)(1) = f(1)$ for all $f \in C[0,1]$.*

Proof. For any fixed $y \in [0,1]$, $f \mapsto \phi(f)(y)$ gives a positive Borel probability measure on $[0,1]$, say μ_y. Thus, $\phi(f)(y) = \int_0^1 f\,d\mu_y$.

Then, for all $g \in C[0,1]_{(a,k)}$, $\phi(g)(0) = \int_0^1 g\,d\mu_0$, and $\phi(g)(1) = \int_0^1 g\,d\mu_1$, and since $\phi(g)(0) = \frac{a}{a+k}\phi(g)(1)$, one has

$$\int_0^1 g\,d\mu_0 = \frac{a}{a+k}\int_0^1 g\,d\mu_1.$$

For any $\delta > 0$, choose a finite δ-dense subset $\{x_1, x_2, \ldots, x_n\} \subset [0,1]$ with $x_1 = 0$, $x_n = 1$. Then, for every $x \in [0,1]$, there is an x_i in the finite subset above such that $\operatorname{dist}(x, x_i) < \delta$. Then, there exists a partition of $[0,1]$, denoted by $\{X_1, X_2, \ldots, X_n\}$, with each X_i being a connected Borel set, satisfying the following conditions:

(1) $x_i \in X_i$, $i = 1, 2, \ldots, n$;

(2) $[0,1] = \bigcup_{i=1}^{n} X_i$, $X_i \cap X_j = \varnothing$ if $i \neq j$;

(3) $\operatorname{dist}(x, x_i) < \delta$ if $x \in X_i$.

For the above fixed partition $\{X_i\}_{i=1}^{n}$, there exists a $\delta_0 > 0$ such that $[0, \delta_0] \subseteq X_1$. Choose a function $e_\delta \in C[0,1]_{(a,k)}$ as follows:

$$e_{\delta_0} = \begin{cases} \frac{a}{a+k} + \frac{1-\frac{a}{a+k}}{\delta_0} x, & x \in [0, \delta_0], \\ 1, & x \in (\delta_0, 1]. \end{cases}$$

Then,

$$\begin{aligned} \phi(e_{\delta_0})(0) = \int_0^1 e_{\delta_0}\, d\mu_0 &= \int_{X_1} e_{\delta_0}\, d\mu_0 + \mu_0(X_2) + \mu_0(X_3) + \cdots + \mu_0(X_n) \\ &= \int_{X_1} e_{\delta_0}\, d\mu_0 + 1 - \mu_0(X_1) = 1 + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0, \\ \phi(e_{\delta_0})(1) = \int_0^1 e_{\delta_0}\, d\mu_1 &= \int_{X_1} e_{\delta_0}\, d\mu_1 + \mu_1(X_2) + \mu_1(X_3) + \cdots + \mu_1(X_n) \\ &= \int_{X_1} e_{\delta_0}\, d\mu_1 + 1 - \mu_1(X_1) = 1 + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_1. \end{aligned}$$

Since

$$\phi(e_{\delta_0})(0) = \frac{a}{a+k} \phi(e_{\delta_0})(1),$$

one has

$$1 + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0 = \frac{a}{a+k} \Big(1 + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_1 \Big).$$

Then,

$$\frac{k}{a+k} + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0 = \frac{a}{a+k} \int_{X_1} (e_{\delta_0} - 1)\, d\mu_1. \tag{3.1}$$

Since $a/(a+k) \leq e_{\delta_0} \leq 1$ and μ_0, μ_1 are positive measures, one has that $-k/(a+k) \leq \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0 \leq 0$ and $-k/(a+k) \leq \int_{X_1} (e_{\delta_0} - 1)\, d\mu_1 \leq 0$. Therefore,

$$0 \leq \frac{k}{a+k} + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0 \leq \frac{k}{a+k}.$$

Then, the left-hand side of equation (3.1) is ≥ 0 and the right-hand side is ≤ 0. Hence,

$$\frac{k}{a+k} + \int_{X_1} (e_{\delta_0} - 1)\, d\mu_0 = \frac{a}{a+k} \int_{X_1} (e_{\delta_0} - 1)\, d\mu_1 = 0.$$

Then, we have $\int_{X_1} (1 - e_{\delta_0})\, d\mu_0 = k/(a+k)$ and $0 \leq \int_{X_1} (1 - e_{\delta_0})\, d\mu_0 \leq k/(a+k)\mu_0(X_1)$. Hence, $\mu_0(X_1) \geq 1$, and so $\mu_0(X_1) = 1$. Since δ is arbitrary, this shows that $\mu_0(\{0\}) = 1$.

In a similar way, for the partition above, there exists a δ_1 such that $[\delta_1, 1] \subseteq X_n$. One can choose a function $\gamma_{\delta_1} \in C[0,1]_{(a,k)}$ as follows:

$$\gamma_{\delta_1} = \begin{cases} 1, & x \in [0, \delta_1), \\ 1 + \frac{\frac{a+k}{a} - 1}{1-\delta_1}(x - \delta_1), & x \in (\delta_1, 1]. \end{cases}$$

Then,

$$\begin{aligned} \phi(\gamma_{\delta_1})(0) &= \int_0^1 \gamma_{\delta_1}\, d\mu_0 = \int_{X_n} \gamma_{\delta_1}\, d\mu_0 + \mu_0(X_1) + \mu_0(X_2) + \cdots + \mu_0(X_{n-1}) \\ &= 1 - \mu_0(X_n) + \int_{X_n} \gamma_{\delta_1}\, d\mu_0 = 1 + \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_0, \\ \phi(\gamma_{\delta_1})(1) &= \int_0^1 \gamma_{\delta_1}\, d\mu_1 = \int_{X_n} \gamma_{\delta_1}\, d\mu_1 + \mu_1(X_1) + \mu_1(X_2) + \cdots + \mu_1(X_{n-1}) \\ &= 1 - \mu_1(X_n) + \int_{X_n} \gamma_{\delta_1}\, d\mu_1 = 1 + \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1. \end{aligned}$$

Since

$$\frac{a+k}{a}\phi(\gamma_{\delta_1})(0) = \phi(\gamma_{\delta_1})(1),$$

one has

$$\begin{aligned} \frac{a+k}{a}\Big(1 + \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_0\Big) &= 1 + \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1, \\ \frac{k}{a} + \frac{a+k}{a}\int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_0 &= \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1. \end{aligned} \tag{3.2}$$

While $0 \leq \gamma_{\delta_1} - 1 \leq k/a$ and $0 \leq \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_0 \leq k/a$, $0 \leq \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1 \leq k/a$, the left-hand side of equation (3.2) is $\geq k/a$ and the right-hand side is $\leq k/a$. Hence,

$$\frac{k}{a} + \frac{a+k}{a}\int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_0 = \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1 = \frac{k}{a}.$$

Then, we have

$$\frac{k}{a} \leq \int_{X_n} (\gamma_{\delta_1} - 1)\, d\mu_1 \leq \frac{k}{a}\mu_1(X_n);$$

then $\mu_1(X_n) \geq 1$, so $\mu_1(X_n) = 1$. Since δ is arbitrary, $\mu_1(\{1\}) = 1$.

Hence, $\phi(f)(0) = f(0)$ and $\phi(f)(1) = f(1)$ for all $f \in C[0,1]$. ∎

Corollary 3.2. *Let $\phi : C[0,1] \to C[0,1]$ be a Markov operator which preserves the subspace $C[0,1]_{(a,k)}$, defined by $\phi(f) = f \circ \lambda$ for some continuous $\lambda : [0,1] \to [0,1]$. Then, $\lambda(0) = 0$ and $\lambda(1) = 1$.*

Proof. Choose some injective function f and apply the lemma above. ∎

Next, we proceed to prove Theorem 1.1 (which is of Li type).

Proof of Theorem 1.1. The proof is inspired by Li's proof in [1]; the thing is we need more accurate analysis for the endpoints. We spell it out in full detail for the convenience of the readers.

Step I. For all $f \in F$ and $y \in [0,1]$, we approximate $\phi(f)(y)$ by a finite sum of point evaluations of f with continuous coefficients.

For any $\varepsilon > 0$, there is a δ_0 such that for any $x_1, x_2 \in [0,1]$, if $\mathrm{dist}(x_1, x_2) < \delta_0$, then

$$\left|f(x_1) - f(x_2)\right| < \frac{\varepsilon}{4}$$

for all $f \in F$. Choose a finite subset $\{x_1, x_2, \dots, x_n\} \subset [0,1]$ which is δ_0-dense in $[0,1]$ with $x_1 = 0, x_n = 1$. Then, for every $x \in [0,1]$, there is an x_i in the finite subset such that $\mathrm{dist}(x, x_i) < \delta_0$. Choose a partition of $[0,1]$, denoted by $\{X_1, X_2, \dots, X_n\}$, with each X_i being a connected Borel set, satisfying the following conditions:

(1) $x_i \in X_i, i = 1, 2, \dots, n$;

(2) $[0,1] = \bigcup_{i=1}^n X_i$, $X_i \cap X_j = \varnothing$ for $i \neq j$;

(3) $\mathrm{dist}(x, x_i) < \delta_0$ if $x \in X_i$.

Then, for any probability measure μ on $[0,1]$, there are non-negative numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\left|\mu(f) - \sum_{i=1}^n \lambda_i f(x_i)\right| < \frac{\varepsilon}{4} \quad \text{for all } f \in F.$$

Actually, we have

$$\left|\mu(f) - \sum_{i=1}^n \mu(X_i) f(x_i)\right| < \left|\sum_{i=1}^n \int_{X_i} \big(f(x) - f(x_i)\big)\, d\mu\right| \le \sum_{i=1}^n \frac{\varepsilon}{4}\mu(X_i) = \frac{\varepsilon}{4}.$$

So one may choose $\lambda_i = \mu(X_i)$.

For any fixed $y \in [0,1]$, $f \mapsto \phi(f)(y)$ is a probability measure on [0,1], and thus from above, there are non-negative numbers $\lambda_{1y}, \lambda_{2y}, \dots, \lambda_{ny}$ with $\sum_{i=1}^n \lambda_{iy} = 1$ such that

$$\left|\phi(f)(y) - \sum_{i=1}^n \lambda_{iy} f(x_i)\right| < \frac{\varepsilon}{4} \quad \text{for all } f \in F.$$

By continuity of $\phi(f)$, this estimation holds in a neighborhood of y. Since $[0,1]$ is compact, we can find a finite open cover $\{V_j : j = 1, 2, \dots, R\}$ of $[0,1]$, such that

(1) $0 \in V_1, 0 \notin \bigcup_{j=2}^R V_j$, $1 \in V_R$, $1 \notin \bigcup_{j=1}^{R-1} V_j$,

(2) $y_j \in V_j$, $y_1 = 0$, $y_R = 1$, $j = 2, \dots, R-1$.

Then, one has

$$\left|\phi(f)(y) - \sum_{i=1}^n \lambda_{iy_j} f(x_i)\right| < \frac{\varepsilon}{4} \quad \text{for all } y \in V_j \text{ and } f \in F.$$

Let $\{h_j\}_{j=1}^R$ be a partition of unity subordinate to $\{V_j\}_{j=1}^R$. Define $\lambda_i(y) = \sum_{j=1}^R \lambda_{iy_j} h_j(y)$. Then, $\lambda_i \in C[0,1], \lambda_i(0) = \lambda_{i0} = \mu_0(X_i), \lambda_i(1) = \lambda_{i1} = \mu_1(X_i)$, and

$$\sum_{i=1}^n \lambda_i(y) = \sum_{i=1}^n \left(\sum_{j=1}^R \lambda_{iy_j} h_j(y) \right) = \sum_{j=1}^R h_j(y) = 1.$$

Hence,

$$\left| \phi(f)(y) - \sum_{i=1}^n \lambda_i(y) f(x_i) \right| < \frac{\varepsilon}{4}$$

for all $y \in [0,1]$ and $f \in F$.

Step II. We approximate the finite sum of point evaluations above by a linear map w on $C[0,1]$ defined as an integral of the composition with some continuous function $h(y,t)$ from $[0,1] \times [0,1]$ to $[0,1]$.

Let there be given a $\delta > 0$ to be used later with $5n\delta \sup_{f \in F} \|f\| < \varepsilon/4$.

First, we define continuous maps $G_0, G_1, \ldots, G_n : [0,1] \to [0,1]$ by

$$G_0(y) = 0, \quad G_j(y) = \sum_{i=1}^j \lambda_i(y), \quad j = 1, 2, \ldots, n.$$

For each $y \in [0,1]$, these points $\{G_i(y)\}_{i=0}^n$ give rise to a partition of $[0,1]$. Moreover, for each j, we define

$$f_j(y) = \min \left\{ G_{j-1}(y) + \delta; \frac{G_{j-1}(y) + G_j(y)}{2} \right\}$$

and

$$g_j(y) = \max \left\{ G_j(y) - \delta; \frac{G_{j-1}(y) + G_j(y)}{2} \right\}.$$

To define $h(y,t)$, we only need to define $h(y,t)$ on each $[0,1] \times [G_{j-1}(y), G_j(y)]$; let us denote by $h_j(y,t)$ this restriction. For our purpose, we choose the following $h_j(y,t)$:

$$h_j(y,t) = \begin{cases} \frac{x_j(t - G_{j-1}(y))}{\delta}, & t \in [G_{j-1}(y), f_j(y)], \\ \min\left(x_j, \frac{x_j(G_j(y) - G_{j-1}(y))}{2\delta}\right), & t \in \left[f_j(y), g_j(y)\right], \\ \frac{x_j(G_j(y) - t)}{\delta}, & t \in \left[g_j(y), G_j(y)\right]. \end{cases}$$

Then, $h_j(y,t)$ satisfies that, for any $y \in [0,1]$,

$$\left| h_j(y,t_1) - h_j(y,t_2) \right| \le \frac{x_j |t_1 - t_2|}{\delta},$$

and $h_j(y,t) : [0,1] \times [G_{j-1}(y), G_j(y)] \to [0,1]$ is continuous. Then, $h(y,t) : [0,1] \times [0,1] \to [0,1]$ is continuous and

$$\left| h(y,t_1) - h(y,t_2) \right| \le \frac{|t_1 - t_2|}{\delta}.$$

Define $w : C[0,1] \to C[0,1]$ by $w(f)(y) = \int_0^1 f(h(y,t))dt$ for $f \in C[0,1]$, $y \in [0,1]$. Then, for all $f \in F$, one has

$$\begin{aligned}
&\left|\sum_{i=1}^{n} \lambda_i(y) f(x_i) - w(f)(y)\right| \\
&\quad = \left|\sum_{i=1}^{n} \int_{G_{i-1}(y)}^{G_i(y)} f(x_i)dt - \int_0^1 f\big(h(y,t)\big)dt\right| \\
&\quad \le \sum_{i=1}^{n} \left|\int_{G_{i-1}(y)}^{G_i(y)} f(x_i) - f\big(h(y,t)\big)dt\right| \\
&\quad = \sum_{i=1}^{n} \left|\left(\int_{G_{i-1}(y)}^{f_i(y)} + \int_{f_i(y)}^{g_i(y)} + \int_{g_i(y)}^{G_i(y)}\right)\big(f(x_i) - f\big(h(y,t)\big)\big)dt\right| \\
&\quad \le \sum_{i=1}^{n} \Big(2\delta \sup_{f\in F} \|f\| + 0 + 2\delta \sup_{f\in F} \|f\|\Big) \\
&\quad = 4n\delta \sup_{f\in F} \|f\| < \frac{\varepsilon}{4}.
\end{aligned}$$

Step III. Finally, we shall choose N continuous maps on [0, 1] to define the homomorphisms. Such maps come from $h(y,t)$ by specifying N values of t. First, we shall choose these maps such that their average approximates the map w above.

Choose an integer $N_1 > 0$ with $1/N_1 < \delta\delta_0$, and choose specified values of t as $t_j = j/N_1 \in [0, 1]$, $j = 1, 2, \ldots, N_1$. Then, the linear map w can be approximated by the average of the homomorphisms induced by $h(y, t_j)$, $j = 1, \ldots, N_1$.

This is shown as follows: set

$$w(f)(y) = \sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} f\big(h(y,t)\big)dt;$$

then

$$\begin{aligned}
\left|w(f)(y) - \frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big)\right| &= \left|\sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} f\big(h(y,t)\big) - f\big(h(y,t_j)\big)dt\right| \\
&< \sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} \frac{\varepsilon}{4} = \frac{\varepsilon}{4}.
\end{aligned}$$

for all $f \in F$, where $|f(h(y,t)) - f(h(y,t_j))| < \varepsilon/4$, since $|h(y,t) - h(y,t_j)| < \delta_0$ (note that $|t - t_j| \le 1/N_1 < \delta\delta_0$).

Next, we make more delicate choices of maps to get new homomorphisms such that each of them leaves the subspace invariant, and hence the average will also do the same job.

By Lemma 3.1, we know that, for $\phi(f)(0)$, the coefficients are

$$\lambda_1(0) = 1, \quad \lambda_2(0) = \cdots = \lambda_n(0) = 0.$$

Similarly, for $\phi(f)(1)$, one has

$$\lambda_1(1) = \lambda_2(1) = \cdots = \lambda_{n-1}(1) = 0, \quad \lambda_n(1) = 1.$$

Therefore, $h(0,t) = 0$ and

$$h(1,t) = \begin{cases} \frac{t}{\delta}, & t \in [0,\delta], \\ 1, & t \in [\delta, 1-\delta], \\ \frac{1-t}{\delta}, & t \in [1-\delta, 1]. \end{cases}$$

If we can choose new t_j such that $h(0,t_j) = 0$ and $h(1,t_j) = 1$, then the corresponding homomorphisms will fit our purpose. Choose those j such that $\delta \leq j/N_1 \leq 1-\delta$; i.e., $\delta N_1 \leq j \leq (1-\delta)N_1$. Denote by N the number of such j; then $N = \lfloor (1-\delta)N_1 \rfloor - \lceil \delta N_1 \rceil + 1$.

We are going to show that the average of these N homomorphisms can approximate the average of the original N_1 homomorphisms:

$$\begin{aligned} &\left| \frac{1}{N_1} \sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{1}{N} \sum_{j=\lceil \delta N_1 \rceil}^{\lfloor (1-\delta)N_1 \rfloor} f\big(h(y,t_j)\big) \right| \\ &= \frac{1}{N_1} \left| \sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{N_1}{N} \sum_{j=\lceil \delta N_1 \rceil}^{\lfloor (1-\delta)N_1 \rfloor} f\big(h(y,t_j)\big) \right| \\ &= \frac{1}{N_1} \left| \sum_{j=1}^{\lceil N_1\delta \rceil - 1} f\big(h(y,t_j)\big) + \sum_{\lfloor N_1(1-\delta) \rfloor + 1}^{N_1} f\big(h(y,t_j)\big) \right. \\ &\qquad \left. + \left(1 - \frac{N_1}{N}\right) \sum_{j=\lceil N_1\delta \rceil}^{\lfloor N_1(1-\delta) \rfloor} f\big(h(y,t_j)\big) \right| \\ &\leq \frac{1}{N_1} \Big(N_1\delta \sup_{f\in F} \|f\| + (N_1\delta + 1) \sup_{f\in F} \|f\| + (2N_1\delta + 1) \sup_{f\in F} \|f\| \Big) \\ &\leq 5\delta \sup_{f\in F} \|f\| \leq \frac{\varepsilon}{4}. \end{aligned}$$

Note that the above estimation holds since

$$\begin{aligned} N_1 - N &= N_1 - \big(\lfloor (1-\delta)N_1 \rfloor - \lceil \delta N_1 \rceil + 1 \big) = N_1 - 1 - \big(\big\lfloor (1-\delta)N_1 \big\rfloor - \lceil \delta N_1 \rceil \big) \\ &\leq N_1 - 1 - \big[(1-\delta)N_1 - 1 - (\delta N_1 + 1) \big] = 2\delta N_1 + 1. \end{aligned}$$

For those new j, let us define $\phi_j : C[0,1] \to C[0,1]$ by $\phi_j(f(y)) = f(h(y,t_j))$. Then,

$$\begin{aligned}
&\left|\phi(f)(y) - \frac{1}{N}\sum_{j=\lceil\delta N_1\rceil}^{\lfloor(1-\delta)N_1\rfloor}\phi_j(f)(y)\right| \\
&= \left|\phi(f)(y) - \frac{1}{N}\sum_{j=\lceil\delta N_1\rceil}^{\lfloor(1-\delta)N_1\rfloor} f\big(h(y,t_j)\big)\right| \\
&\leq \left|\phi(f)(y) - \sum_{i=1}^{n}\lambda_i(y)f(x_i)\right| + \left|\sum_{i=1}^{n}\lambda_i(y)f(x_i) - w(f)(y)\right| \\
&\quad + \left|w(f)(y) - \frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big)\right| + \left|\frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{1}{N}\sum_{j=\lceil\delta N_1\rceil}^{\lfloor(1-\delta)N_1\rfloor} f\big(h(y,t_j)\big)\right| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon.
\end{aligned}$$

∎

Remark 3.3. From the proof above, one can see that, for any integer $M_1 \geq N_1$, there are a corresponding integer M and M homomorphisms such that the average of these M morphisms also meets the requirements.

Hence, under the assumption of Theorem 1.1, there is a sequence of positive integers $\{L_j\}_{j=1}^{\infty}$ with a large enough lower bound, and there are L_j corresponding morphisms for each j, such that the average of these L_j morphisms meets the requirements.

Consider the C^*-algebras

$$\begin{aligned}
A &= \big\{f \in C\big([0,1], M_n\big) \mid f(0) = \mathrm{diag}(d \otimes \mathrm{id}_a, 0 \otimes \mathrm{id}_k),\ f(1) = d \otimes \mathrm{id}_{a+k}\big\}, \\
B &= \big\{f \in C\big([0,1], M_m\big) \mid f(0) = \mathrm{diag}(e \otimes \mathrm{id}_a, 0 \otimes \mathrm{id}_k),\ f(1) = e \otimes \mathrm{id}_{a+k}\big\},
\end{aligned}$$

where d and e are matrices of the appropriate sizes. Theorem 1.1 can be used to build up $*$-homomorphisms between certain C^*-algebras.

Corollary 3.4. *With A a C^*-algebra as above, for any $\varepsilon > 0$, and any finite subset $F \subseteq \mathrm{AffT}\,A = C[0,1]_{(a,k)}$, there is an integer $N > 0$ such that, for any C^*-algebra B of the form above, with generic fiber size N times the generic fiber size of A, and any unital positive linear map ξ on $C[0,1]$ which preserves $C[0,1]_{(a,k)}$, there is a $*$-homomorphism ϕ from A to B such that*

$$\big\|\xi(f) - \mathrm{AffT}\,\phi(f)\big\| < \varepsilon$$

for all $f \in F$.

Proof. We take N as in Theorem 1.1 and the corresponding N continuous maps $h(y,t_1)$, $\ldots, h(y,t_N)$ on $[0,1]$. By the constructions of the function $h(y,t_j)$, there are unitary matrices U_0 and U_1 such that, for each $g \in A$,

$$U_0\,\mathrm{diag}\big(g\big(h(0,t_1)\big), \ldots, g\big(h(0,t_N)\big)\big)U_0^* = \mathrm{diag}(e \otimes \mathrm{id}_a, 0 \otimes \mathrm{id}_k)$$

and

$$U_1 \operatorname{diag}\big(g\big(h(1,t_1)\big),\dots,g\big(h(1,t_N)\big)\big)U_1^* = e \otimes \mathrm{id}_{a+k},$$

for some matrix e. By choosing a continuous path of unitaries $U(t)$ connecting U_0 and U_1, one can define $\phi : A \to B$ as

$$\phi(g)(y) = U(y) \operatorname{diag}\big(g \circ h(y,t_1),\dots,g \circ h(y,t_N)\big)U^*(y).$$

Keep in mind the correspondence $f = (tr \otimes \delta_t)(g)$ (see [2, Proposition 2.1]), and, applying Theorem 1.1, one has that

$$\big\|\xi(f) - \mathrm{AffT}\,\phi(f)\big\| < \varepsilon$$

for all $f \in F$. ∎

Now, we consider the case involving different subspaces.

Lemma 3.5. *Let μ be a Borel probability measure on $[0,1]$. Then, for any $x \in [0,1]$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $\mu(B^0(x,\delta)) < \varepsilon$ (where $B^0(x,\delta)$ denotes the open ball centered at x with radius δ, but excluding the center x).*

Proof. Let $D_k = (B(x,1/k) \setminus B(x,1/(1+k))) \cap [0,1]$, where $B(x,1/k)$ refers to the open ball centered at x with radius $1/k$. Set $S_n = \sum_{k=1}^{n} \mu(D_k)$, then S_n is increasing and bounded above, so $\{S_n\}$ converges. Then, for any $\varepsilon > 0$, there exists $N > 0$ such that $\sum_{k=N}^{\infty} \mu(D_k) < \varepsilon$. Hence, there exists $\delta = 1/N$ such that $\mu(B^0(x,\delta)) < \varepsilon$, since $B^0(x,\delta) = \bigcup_{k=N}^{\infty} D_k$. ∎

Examples in Section 2 show that the measures induced by evaluations of a Markov operator at 0 and 1 actually could involve as many points as you want, so we investigate the behavior of induced measures with respect to a given partition of $[0,1]$ coming from an approximation.

Lemma 3.6. *Given a unital positive linear map ϕ from $C[0,1]$ to $C[0,1]$ which sends $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$, denote by μ_0 and μ_1 the measures induced by evaluations of ϕ at 0 and 1. Let there be given a partition $\{X_1, X_2, \dots, X_n\}$ of $[0,1]$, where X_i is a connected Borel set (i.e., an interval) and $0 \in X_1$, $1 \in X_n$.*

Then, one has the following distribution of μ_0 and μ_1 with respect to the partition:

$$\mu_0(X_i) = \frac{b}{b+k}\mu_1(X_i) \quad (i = 2,\dots,n-1), \tag{3.3}$$

$$\frac{a}{a+k}\mu_0(X_1) + \mu_0(X_n) = \frac{b}{b+k}\left(\frac{a}{a+k}\mu_1(X_1) + \mu_1(X_n)\right). \tag{3.4}$$

Proof. The first relation is shown as follows. Choose a continuous function which is almost supported on X_i $(i = 2,\dots,n-1)$, and then apply ϕ. Comparing the evaluations at 0 and 1, one can get the relation.

For fixed $i = 2, \dots, n-1$, set $a_i = \sup\{x : x \in X_i\}$, $b_i = \inf\{x : x \in X_i\}$. We only spell it out in one case; a similar proof works for the other cases.

Let us focus on the case $X_i = [b_i, a_i]$. For all $\varepsilon > 0$, by Lemma 3.5, there exists a $\delta > 0$ such that

$$\mu_0\big((b_i - \delta, b_i)\big) < \varepsilon, \mu_1\big((b_i - \delta, b_i)\big) < \varepsilon$$

and

$$\mu_0\big((a_i, a_i + \delta)\big) < \varepsilon, \mu_1\big((a_i, a_i + \delta)\big) < \varepsilon.$$

We choose a function $g_i(x)$ as follows:

$$g_i(x) = \begin{cases} 0, & x \in (b_i - \delta, a_i + \delta)^c, \\ \frac{x - b_i}{\delta} + 1, & x \in (b_i - \delta, b_i), \\ -\frac{x - a_i}{\delta} + 1, & x \in (a_i, a_i + \delta), \\ 1, & x \in [b_i, a_i]; \end{cases}$$

then

$$\phi(g_i)(0) = \int_{[0,1]} g_i \, d\mu_0 = \mu_0(X_i) + \int_{(b_i - \delta, b_i)} g_i \, d\mu_0 + \int_{(a_i, a_i + \delta)} g_i \, d\mu_0$$

and

$$\phi(g_i)(1) = \int_{[0,1]} g_i \, d\mu_1 = \mu_1(X_i) + \int_{(b_i - \delta, b_i)} g_i \, d\mu_1 + \int_{(a_i, a_i + \delta)} g_i \, d\mu_1.$$

Let us take

$$\varepsilon_{i0} = \int_{(b_i - \delta, b_i)} g_i \, d\mu_0 + \int_{(a_i, a_i + \delta)} g_i \, d\mu_0$$

and

$$\varepsilon_{i1} = \int_{(b_i - \delta, b_i)} g_i \, d\mu_1 + \int_{(a_i, a_i + \delta)} g_i \, d\mu_1;$$

then $\varepsilon_{i0} < 2\varepsilon, \varepsilon_{i1} < 2\varepsilon$ and

$$\frac{\mu_0(X_i) + \varepsilon_{i0}}{\mu_1(X_i) + \varepsilon_{i1}} = \frac{b}{b+k}$$

(since $\phi(g_i) \in C[0,1]_{(b,k)}$).

Hence,

$$\left| \mu_0(X_i) - \frac{b}{b+k} \mu_1(X_i) \right| < 4\varepsilon,$$

and since ε is arbitrary, one concludes that

$$\mu_0(X_i) = \frac{b}{b+k} \mu_1(X_i), \quad i = 2, \dots, n-1.$$

Next, we prove the second relation. We use similar ideas; i.e., we choose a function which is almost supported on X_1 and X_n, apply ϕ, and then compare the evaluations.

For $i = 1$ and $i = n$, we know that $0 = \inf\{x : x \in X_1\}$, $1 = \sup\{x : x \in X_n\}$, and suppose that $a_1 = \sup\{x : x \in X_1\}$, $b_n = \inf\{x : x \in X_n\}$. Consider the case $X_1 = [0, a_1]$, $X_n = [b_n, 1]$. For all $\varepsilon > 0$, by Lemma 3.5, there exists a $\delta > 0$ such that

$$\mu_0\big((b_n - \delta, b_n)\big) < \varepsilon, \quad \mu_1\big((b_n - \delta, b_n)\big) < \varepsilon$$

and

$$\mu_0\big((a_1, a_1 + \delta)\big) < \varepsilon, \quad \mu_1\big((a_1, a_1 + \delta)\big) < \varepsilon.$$

We choose a function $g(x)$ as follows:

$$g(x) = \begin{cases} \frac{a}{k+a}, & x \in [0, a_1], \\ -\frac{a(x-a_1)}{\delta(a+k)} + \frac{a}{a+k}, & x \in (a_1, a_1 + \delta), \\ 0, & x \in [a_1 + \delta, b_n - \delta], \\ \frac{x-b_n}{\delta} + 1, & x \in (b_n - \delta, b_n), \\ 1, & x \in [b_n, 1]. \end{cases}$$

Then,

$$\phi(g)(0) = \int_{[0,1]} g\, d\mu_0 = \mu_0(X_n) + \frac{a}{a+k}\mu_0(X_1) + \int_{(a_1,a_1+\delta)} g\, d\mu_0 + \int_{(b_n-\delta,b_n)} g\, d\mu_0$$

and

$$\phi(g)(1) = \int_{[0,1]} g\, d\mu_1 = \mu_1(X_n) + \frac{a}{a+k}\mu_1(X_1) + \int_{(a_1,a_1+\delta)} g\, d\mu_1 + \int_{(b_n-\delta,b_n)} g\, d\mu_1.$$

Let us take

$$\varepsilon_0 = \int_{(a_1,a_1+\delta)} g\, d\mu_0 + \int_{(b_n-\delta,b_n)} g\, d\mu_0$$

and

$$\varepsilon_1 = \int_{(a_1,a_1+\delta)} g\, d\mu_1 + \int_{(b_n-\delta,b_n)} g\, d\mu_1.$$

Then,

$$\varepsilon_0 < \frac{2a+k}{a+k}\varepsilon, \quad \varepsilon_1 < \frac{2a+k}{a+k}\varepsilon.$$

Moreover, one has

$$\frac{\mu_0(X_n) + \frac{a}{a+k}\mu_0(X_1) + \varepsilon_0}{\mu_1(X_n) + \frac{a}{a+k}\mu_1(X_1) + \varepsilon_1} = \frac{b}{b+k}$$

(since $\phi(g) \in C[0,1]_{(b,k)}$).

Then,

$$\left|\frac{a}{a+k}\mu_0(X_1)+\mu_0(X_n)-\frac{b}{b+k}\left(\frac{a}{a+k}\mu_1(X_1)+\mu_1(X_n)\right)\right|<r\varepsilon$$

for some r, since ε is arbitrary, one has

$$\frac{a}{a+k}\mu_0(X_1)+\mu_0(X_n)=\frac{b}{b+k}\left(\frac{a}{a+k}\mu_1(X_1)+\mu_1(X_n)\right).$$

Similar proofs go through in the other cases. ∎

Corollary 3.7. *With the same assumption as above, one has*

$$\mu_0(X_1)=\frac{b}{b+k}\mu_1(X_1)+\frac{a+k}{b+k}, \tag{3.5}$$

$$\mu_0(X_n)=\frac{b}{b+k}\mu_1(X_n)-\frac{a}{b+k}. \tag{3.6}$$

Proof. By adding (3.3) over $i=2,\dots,n-1$ and (3.4), we have

$$\begin{aligned}\sum_{i=2}^{n-1}\mu_0(X_i)+\frac{a}{a+k}\mu_0(X_1)+\mu_0(X_n)\\ =\sum_{i=2}^{n-1}\frac{b}{b+k}\mu_1(X_i)+\frac{b}{b+k}\left(\frac{a}{a+k}\mu_1(X_1)+\mu_1(X_n)\right).\end{aligned}$$

Then, we add $\mu_0(X_1)+\frac{b}{b+k}\mu_1(X_1)$ to both sides of the equation above, and since $\mu_0([0,1])=1$, $\mu_1([0,1])=1$, we get

$$1+\frac{a}{a+k}\mu_0(X_1)+\frac{b}{b+k}\mu_1(X_1)=\frac{b}{b+k}+\mu_0(X_1)+\frac{b}{b+k}\frac{a}{a+k}\mu_1(X_1).$$

Then, one can solve $\mu_0(X_1)$ to get (3.5). Equation (3.6) follows from (3.4) and (3.5). ∎

Remark 3.8. By (3.5), we have

$$\mu_0(X_1)=\frac{b}{b+k}\mu_1(X_1)+\frac{a+k}{b+k}\geq\frac{a+k}{b+k}$$

and $\mu_0(X_1)\leq 1$; thus $b\geq a$. In other words, if $b<a$, there is no unital positive linear map on $C[0,1]$ which sends $C[0,1]_{(a,k)}$ to $C[0,1]_{(b,k)}$.

Lemma 3.9. *Given $\mu_0(X_i)$ and $\mu_1(X_i)$ $(i=1,\dots,n)$ as above in Lemma 3.6, for any $\eta>0$, there exist rational numbers $0\leq r_1,\dots,r_n\leq 1$ and $0\leq s_1,\dots,s_n\leq 1$ which add up to 1, respectively, such that*

$$0\leq|r_i-\mu_0(X_i)|\leq\eta,\quad 0\leq|s_i-\mu_1(X_i)|\leq\eta,\quad i=1,2,\dots,n.$$

Moreover, the relations among $\mu_0(X_i)$ and $\mu_1(X_i)$ hold for these r_i and s_i; i.e.,

$$r_i = \frac{b}{b+k} s_i \quad (i = 2, \dots, n-1), \tag{3.7}$$

$$\frac{a}{a+k} r_1 + r_n = \frac{b}{b+k}\left(\frac{a}{a+k} s_1 + s_n\right). \tag{3.8}$$

Proof. For all $\eta > 0$, take a rational approximation s_n for $\mu_1(X_n)$ with $0 \le s_n - \mu_1(X_n) \le \eta$. Then, take rational approximations s_i for $\mu_1(X_i)$ with $|\mu_1(X_i) - s_i| \le \eta$ for $1 \le i \le n-1$, such that $s_1 + \cdots + s_{n-1} = 1 - s_n$. Then, take corresponding $r_i = \frac{b}{b+k} s_i$ for $2 \le i \le n-1$; thus r_i $(2 \le i \le n-1)$ approximates $\mu_0(X_i)$ based on the relation between $\mu_0(X_i)$ and $\mu_1(X_i)$ $(2 \le i \le n-1)$. Set

$$r_n = \frac{b}{b+k} s_n - \frac{a}{b+k},$$

which is non-negative since $s_n \ge \mu_1(X_n) \ge a/b$, and then r_n approximates $\mu_0(X_n)$ based on (3.6). Set

$$r_1 = 1 - (r_2 + \cdots + r_{n-1}) - r_n;$$

then based on (3.5) r_1 approximates $\mu_0(X_1)$. Moreover, one can verify all of the data fit in the requirement (3.8). ∎

Corollary 3.10. *For any positive integer N, any collection of N points $\{x_i \in (0,1) \mid 1 \le i \le N\}$, and any integers k_1, k_n, m_1, m_n satisfying the relation (3.8), one has*

$$k_1 f(0) + \sum_{i=1}^{N} b l_i f(x_i) + k_n f(1) = \frac{b}{b+k}\left(m_1 f(0) + \sum_{i=1}^{N} (b+k) l_i f(x_i) + m_n f(1)\right)$$

for all $f \in C[0,1]_{(a,k)}$, where $l_1, \dots, l_N$ are arbitrarily chosen positive integers.

Proof. Since $f \in C[0,1]_{(a,k)}$, one has $f(0) = \frac{a}{a+k} f(1)$. Then, the left-hand side of the above equals $(\frac{a}{a+k} k_1 + k_n) f(1) + \sum_{i=1}^{N} b l_i f(x_i)$. Hence, it coincides with the right-hand side by the relation (3.8). ∎

The rest of the paper will be devoted to the proof of Theorem 1.2; before we start, let us make some explanations and comments.

Remark 3.11. (1) The basic strategy is the same as the proof of Theorem 1.1, which consists of two essential issues. One is we define properly a continuous function $h(y,t) : [0,1] \times [0,1] \to [0,1]$ which provides necessary eigenvalue maps later by specifying some values of the second parameter t at $y = 0$ and $y = 1$. The other one is we must specify certain values of t (as many as possible) to meet two requirements; namely, $h(0, t_j)$ and $h(1, t_j)$ together must guarantee that the average of corresponding homomorphisms fits the compatibility of subspaces, as well as the purpose of approximation.

(2) However, due to different sub-homogeneity, evaluations of a Markov operator at endpoints lead to a more refined measure distribution, which causes some technical complexity when we try to realize the two issues above. To be precise, when we define $h(y,t)$ on $[0,1]\times[0,1]$, we take a partition of second interval $[0,1]$ which comes from the natural measure representation, but, in this case, at endpoints 0 and 1, the non-degenerate $\mu_0(X_i)$ and $\mu_1(X_i)$ are not equalized, which is very inconvenient for later analysis. We first make some adjustments to equalize them somehow. The other trouble is to make right choices of eigenvalue maps. We need to involve not only 0 and 1, but also the points x_i from certain approximation net. On one hand, to guarantee the compatibility of subspaces, we need to choose those t_j such that all chosen $h(0,t_j)$ and $h(1,t_j)$ satisfy the corresponding relation coming from the measure distribution. To achieve this, we have to drop some $h(0,t)$ and $h(1,t)$ in corresponding proportion such that the remaining ones satisfy the required relation. But we cannot drop too much; otherwise it will violate the approximation purpose, which is controlled by taking some small enough parameter δ.

(3) Those technical arrangements we did are not complicated; the thing is to spell out full detail costs expressions, which might cover the idea. So we put some figures during the proof to demonstrate the idea and convince people.

Now we proceed to prove Theorem 1.2 (which is of Thomsen type).

Proof of Theorem 1.2. We have divided the proof into four steps.

Step I. The first step is exactly the same as the first step of the proof of Theorem 1.1. To avoid redundancy, we skip this but still use the same notation there. In particular, take those points $x_i\in[0,1]$ and functions λ_i on $[0,1]$ for all $i=1,\dots,n$. Recall that we already have

$$\sum_{i=1}^{n}\lambda_i(y)=1 \quad \text{and} \quad \left|\phi(f)(y)-\sum_{i=1}^{n}\lambda_i(y)f(x_i)\right|<\varepsilon/4, \quad \forall y\in[0,1],\ \forall f\in F.$$

Step II. In a similar way, we approximate the finite sum of point evaluations by a linear map w on $C[0,1]$, defined as the integral of the composition with some continuous function $h(y,t)$ from $[0,1]\times[0,1]$ to $[0,1]$, and $h(y,t)$ is formulated based on a partition of the second $[0,1]$.

Recall in *Step I* that $\lambda_i(0)=\mu_0(X_i)$ and $\lambda_i(1)=\mu_1(X_i)$, $i=1,\dots,n$. By Lemma 3.6 and Corollary 3.7, we know

$$\begin{aligned}
\lambda_1(0)&=\frac{b}{b+k}\lambda_1(1)+\frac{a+k}{b+k},\\
\lambda_n(0)&=\frac{b}{b+k}\lambda_n(1)-\frac{a}{b+k},\\
\lambda_i(0)&=\frac{b}{b+k}\lambda_i(1), \quad i=2,\dots,n-1,
\end{aligned}$$

which imply that

$$\lambda_1(0)>\lambda_1(1), \quad \lambda_i(0)\le\lambda_i(1), \quad i=2,\dots,n.$$

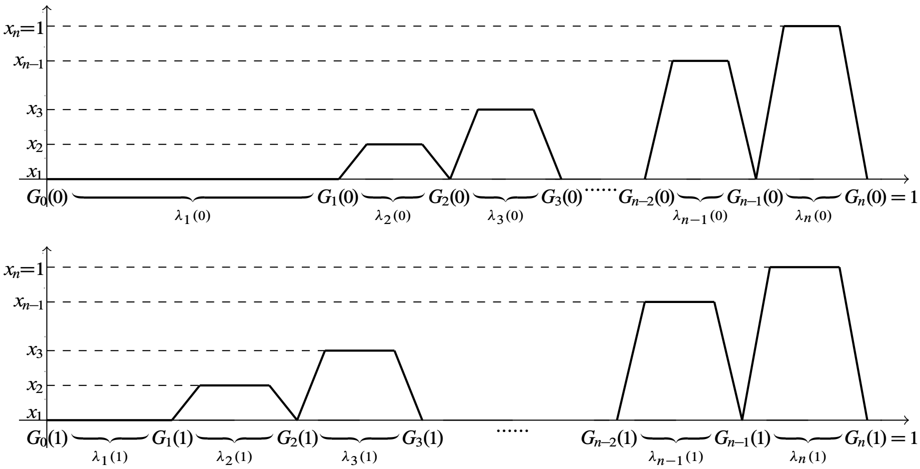

Figure 1. Naive choice of $h(0,t)$ and $h(1,t)$.

Later on, $h(0,t)$ and $h(1,t)$ are built up as some piecewise linear functions. All pieces of domain form a partition of the second [0, 1]. If we still use the naive partition of the second [0, 1] as before, namely, the successive intervals of length $\lambda_i(0)$ and $\lambda_i(1)$, $i = 1, \ldots, n$, then we have the naive choice of $h(0,t)$ and $h(1,t)$ as in Figure 1 above. Such a choice is not good for later analysis, instead we would better have somehow equalized pieces for the domain of $h(0,t)$ and $h(1,t)$. To achieve this, we make some technical adjustments such that the real choice of $h(0,t)$ and $h(1,t)$ looks like in Figure 2. Namely, in Figure 1, we move the first interval of length $\lambda_1(1)$ to be the last one and equalize the other $\lambda_i(0)$ and $\lambda_i(1)$ by borrowing an additional piece from $\lambda_1(0)$ to $\lambda_1(1)$. So we take the corresponding partition of [0, 1] after the equalization; see Figure 2.

For $y \in (0,1)$, we try to do similar things for the domain of $h(y,t)$ and $h(1,t)$, but it may happen that $\lambda_1(y) \le \lambda_1(1)$ or $\lambda_i(1) \le \lambda_i(y)$ for some i. Then, we do equalizations of piecewise domains whenever it is possible and necessary and do nothing otherwise.

So, we define

$$
\begin{aligned}
l_1(y) &= \lambda_2(y),\\
l_2(y) &= \max\left\{0, \min\left\{\lambda_1(y) - \lambda_1(1), \lambda_2(1) - \lambda_2(y)\right\}\right\},\\
&\vdots\\
l_{2i-3}(y) &= \lambda_i(y), \quad i = 2, \ldots, n,\\
l_{2i-2}(y) &= \max\left\{0, \min\left\{\lambda_1(y) - \lambda_1(1) - l_2(y) - \cdots - l_{2i-4}(y), \lambda_i(1) - \lambda_i(y)\right\}\right\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 2, \ldots, n,\\
&\vdots\\
l_{2n-1}(y) &= 1 - \sum_{j=1}^{2n-2} l_j(y),\\
l_{2n}(y) &= 0.
\end{aligned}
$$

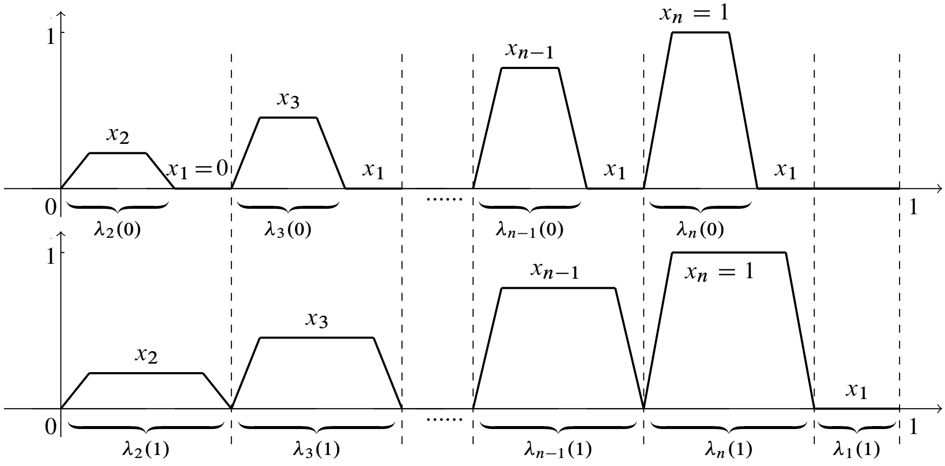

Figure 2. Real choice of $h(0, t)$ and $h(1, t)$.

Note that we have

$$\sum_{i=1}^{n} l_{2i}(y) + l_{2n-1}(y) = 1 - \sum_{j=2}^{n} l_{2j-3}(y) = 1 - \sum_{j=2}^{n} \lambda_j(y) = \lambda_1(y), \tag{3.9}$$

and we get what we expect for the piecewise domains of $h(0, t)$ and $h(1, t)$:

$$\begin{aligned}
l_1(0) &= \mu_0(X_2), & l_1(1) &= \mu_1(X_2),\\
l_2(0) &= \mu_1(X_2) - \mu_0(X_2), & l_2(1) &= 0,\\
&\vdots & &\\
l_{2n-3}(0) &= \mu_0(X_n), & l_{2n-3}(1) &= \mu_1(X_n),\\
l_{2n-2}(0) &= \mu_1(X_n) - \mu_0(X_n), & l_{2n-2}(1) &= 0,\\
l_{2n-1}(0) &= \mu_1(X_1), & l_{2n-1}(1) &= \mu_1(X_1),\\
l_{2n}(0) &= 0. & l_{2n}(1) &= 0.
\end{aligned}$$

Next define $G_1, \ldots, G_{2n} : [0, 1] \to [0, 1]$ by

$$G_j(y) = \sum_{i=1}^{j} l_i(y), \quad j = 1, 2, \ldots, 2n.$$

Then, for $j = 2k$, we have the consistency that

$$G_j(0) = G_j(1) = \sum_{i=2}^{k+1} \mu_1(X_i).$$

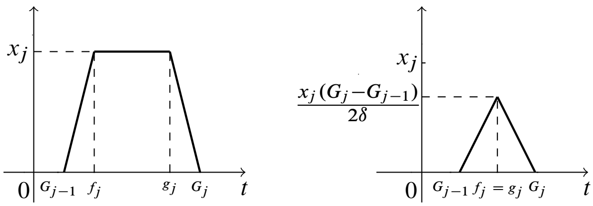

Figure 3. Graph of $h_j(y,t)$.

Let $\delta > 0$ be a rational number. For each $y \in [0,1]$, the points $\{G_i(y)\}_i$, $i = 1, \dots, 2n$, give rise to a partition of [0, 1]. Moreover, for each j, we define

$$f_j(y) = \min\left\{G_{j-1}(y) + \delta, \frac{G_{j-1}(y) + G_j(y)}{2}\right\}$$

and

$$g_j(y) = \max\left\{G_j(y) - \delta, \frac{G_{j-1}(y) + G_j(y)}{2}\right\}.$$

To define $h(y,t)$, we only need to define $h(y,t)$ on each $[0,1] \times [G_{j-1}(y), G_j(y)]$. Let us denote by $h_j(y,t)$ this restriction. For our purpose, we choose the following $h_j(y,t)$:

$$h_j(y,t) = \begin{cases} \frac{z_j(t-G_{j-1}(y))}{\delta}, & t \in \big[G_{j-1}(y), f_j(y)\big], \\ \min\Big(z_j, \frac{z_j(G_j(y)-G_{j-1}(y))}{2\delta}\Big), & t \in \big[f_j(y), g_j(y)\big], \\ \frac{z_j(G_j(y)-t)}{\delta}, & t \in \big[g_j(y), G_j(y)\big], \end{cases}$$

where $z_j = x_i$ if $j = 2i-3$ $(i = 2, \dots, n)$, $z_j = x_1 = 0$ if $j = 2i$ $(i = 1, \dots, n-1)$, and $z_{2n-1} = x_1 = 0$. The graph of $h_j(y,t)$ is shown in Figure 3 (depending on δ). Moreover, we can choose δ being further small enough later such that the tent case does not appear for $h(0,t)$ and $h(1,t)$; i.e., they really enjoy the shape shown in Figure 2.

Then, $h_j(y,t)$ satisfies that, for any $y \in [0,1]$,

$$\big|h_j(y,t_1) - h_j(y,t_2)\big| \le \frac{|t_1 - t_2|}{\delta},$$

and $h_j(y,t) : [0,1] \times [G_{j-1}(y), G_j(y)] \to [0,1]$ is continuous. Then, $h(y,t) : [0,1] \times [0,1] \to [0,1]$ is continuous and

$$\big|h(y,t_1) - h(y,t_2)\big| \le \frac{|t_1 - t_2|}{\delta}.$$

Define $w : C[0,1] \to C[0,1]$ by

$$w(f)(y) = \int_0^1 f\big(h(y,t)\big)dt,$$

where $f \in C[0,1]$, $y \in [0,1]$.

Then, for all $f \in F$, due to calculations before (especially (3.9)), we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} \lambda_i(y) f(x_i) - w(f)(y)\right| &= \left|\sum_{j=1}^{2n} l_j(y) f(z_j) - w(f)(y)\right| \\
&= \left|\sum_{j=1}^{2n} \int_{G_{j-1}(y)}^{G_j(y)} f(z_j)dt - \int_0^1 f\big(h(y,t)\big)dt\right| \\
&\leq \sum_{j=1}^{2n} \left|\int_{G_{j-1}(y)}^{G_j(y)} f(z_i) - f\big(h(y,t)\big)dt\right| \\
&= \sum_{j=1}^{2n} \left|\int_{G_{j-1}(y)}^{f_j(y)} + \int_{f_j(y)}^{g_j(y)} + \int_{g_j(y)}^{G_j(y)} f(z_j) - f\big(h(y,t)\big)dt\right| \\
&\leq \sum_{j=1}^{2n} \Big(2\delta \sup_{f\in F} \|f\| + 0 + 2\delta \sup_{f \in F} \|f\|\Big) \\
&= 8n\delta \sup_{f\in F} \|f\| < \frac{\varepsilon}{4}.
\end{aligned}$$

The last inequality holds because of the choice of δ to be made later.

Step III. We shall choose N_1 continuous maps $h(y,t_j)$ on $[0,1]$ by specifying N_1 values of t so that the corresponding average of homomorphisms approximates the map w above on F.

Among all X_i, $i = 2, \dots, n-1$ (keeping in mind that they are intervals), there might be some ones degenerating, i.e., with μ_0-measure zero. Let τ be the number of all non-degenerating ones. Denote them by $X_{i_1}, \dots, X_{i_\tau}$. Then, these ones also enjoy $\mu_1(X_i) \neq 0$ by (3.3).

For some counting convenience later, we approximate these $\mu_0(X_i)$ and $\mu_1(X_i)$ by rational numbers which still keep the relations among $\mu_0(X_i)$ and $\mu_1(X_i)$. By Lemma 3.9, for the tolerance δ/n, there exist rational numbers $0 \leq r_1, r_{i_1}, \dots, r_{i_\tau}, r_n \leq 1$, and $0 \leq s_1, s_{i_1}, \dots, s_{i_\tau}, s_n \leq 1$ such that

$$r_1 + \sum_{i=1}^{\tau} r_{i_\iota} + r_n = 1; \quad 0 \leq |r_i - \mu_0(X_i)| \leq \frac{\delta}{n} \ (i = 1, i_1, \dots, i_\tau, n),$$

$$s_1 + \sum_{i=1}^{\tau} s_{i_\iota} + s_n = 1; \quad 0 \leq |s_i - \mu_1(X_i)| \leq \frac{\delta}{n} \ (i = 1, i_1, \dots, i_\tau, n),$$

$$\frac{r_{i_\iota}}{s_{i_\iota}} = \frac{b}{b+k}, \quad \iota = 1, \dots, \tau,$$

and

$$\frac{a}{a+k} r_1 + r_n = \frac{b}{b+k}\left(\frac{a}{a+k} s_1 + s_n\right).$$

Choose an integer $N_1 > 0$ such that $1/N_1 < \delta\delta_0$ and $N_1\delta$, $N_1 s_i$, $N_1 r_i$ $(i = 1, i_1, \ldots, i_\tau, n)$ are integers. Let $t_j = j/N_1 \in [0, 1]$, $j = 1, 2, \ldots, N_1$.

To save notation, rewrite $N_1 s_i$, $N_1 r_i$ still as s_i, r_i $(i = 1, i_1, \ldots, i_\tau, n)$. Then,

$$\sum_{\iota=1}^{\tau} s_{i_\iota} + s_n + s_1 = \sum_{\iota=1}^{\tau} r_{i_\iota} + r_n + r_1 = N_1$$

and

$$0 \le \left| r_i - \mu_0(X_i) N_1 \right| \le N_1 \frac{\delta}{n} \quad (i = 1, i_1, \ldots, i_\tau, n), \tag{3.10}$$

$$0 \le \left| s_i - \mu_1(X_i) N_1 \right| \le N_1 \frac{\delta}{n} \quad (i = 1, i_1, \ldots, i_\tau, n), \tag{3.11}$$

$$\frac{r_{i_\iota}}{s_{i_\iota}} = \frac{b}{b+k}, \quad \iota = 1, \ldots, \tau, \tag{3.12}$$

and

$$\frac{a}{a+k} r_1 + r_n = \frac{b}{b+k}\left(\frac{a}{a+k} s_1 + s_n\right). \tag{3.13}$$

Define $\phi_j : C[0,1] \to C[0,1]$ by $\phi_j(f)(y) = f(h(y, t_j))$. Then,

$$w(f)(y) = \int_0^1 f\big(h(y,t)\big) dt = \sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} f\big(h(y,t)\big) dt$$

and

$$\begin{aligned} \left| w(f)(y) - \frac{1}{N_1} \sum_{j=1}^{N_1} \phi_j(f)(y) \right| &= \left| \sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} f\big(h(y,t)\big) - f\big(h(y,t_j)\big) dt \right| \\ &< \sum_{j=1}^{N_1} \int_{t_{j-1}}^{t_j} \frac{\varepsilon}{4} = \frac{\varepsilon}{4} \end{aligned}$$

for all $f \in F$, where $|f(h(y,t)) - f(h(y,t_j))| < \varepsilon/4$, because $|h(y,t) - h(y,t_j)| < \delta_0$ (note that $|t - t_j| \le 1/N_1 < \delta\delta_0$).

Step IV. We shall choose N new maps (as many as possible) from the N_1 maps above to guarantee that the corresponding average of homomorphisms fits the compatibility of subspaces, and also the new average of these N guys approximates the average of the original N_1 homomorphisms. We have to involve points other than 0 and 1. By Corollary 3.10, to fit compatibility of subspaces, we must choose such points in proportion as well as keep the proportion between the numbers of 0 and 1 chosen for values of $h(0,t)$ and $h(1,t)$. Recall the graph of $h(0,t)$ and $h(1,t)$ shown in Figure 2; roughly speaking, we will not choose those j/N_1 whose function value lies in the slant part of the graph, instead we choose as many as possible those j/N_1 whose function value lies in the horizontal part.

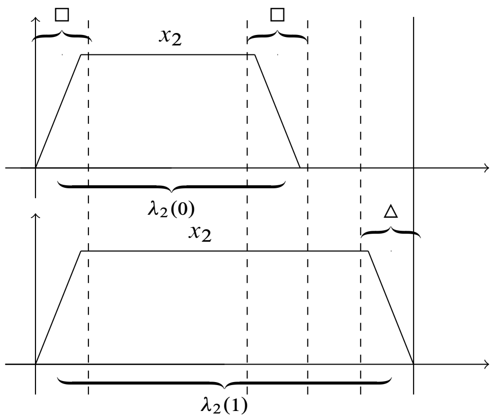

Figure 4. A simple explanation of choosing/dropping $h(0, t_j)$ and $h(1, t_j)$ on one piece of domain.

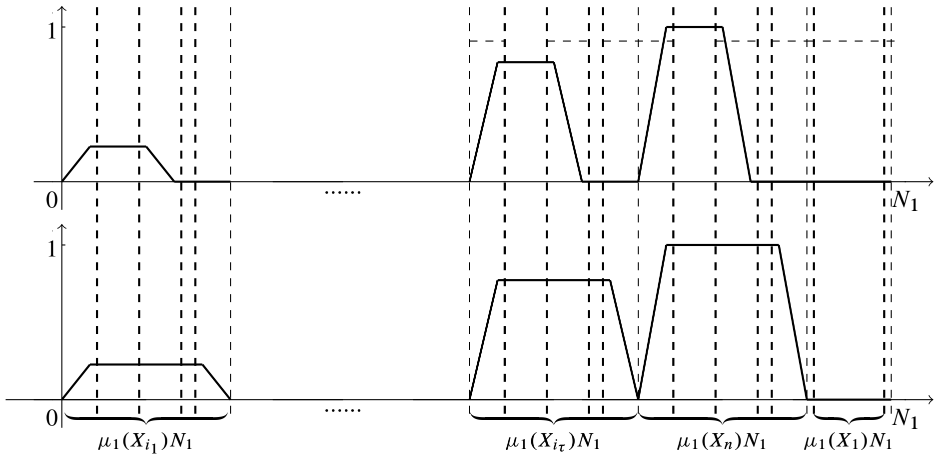

Figure 5. Process of choosing/dropping $h(0, t_j)$ and $h(1, t_j)$.

The crucial point is that we must keep corresponding relations for our choices. This can be done if we drop a little bit more than the slant part, which is explained for one piece of domain of $h(0, t)$ and $h(1, t)$ as in Figure 4. Namely, we drop those j/N_1 lying in the two □ parts and the △ part simultaneously for $h(0, t)$ and $h(1, t)$ such that the remaining j/N_1 satisfy the required proportion $b/(b + k)$, which by a simple calculation amount to a certain proportion of □ and △. Similarly corresponding operations can be done for all pieces of domains of $h(0, t)$ and $h(1, t)$. Finally, the slant parts are controlled by the parameter δ; i.e., if δ is small enough, then the slant parts will be small, and so are □ and △. Hence, we can also meet the purpose of approximation. The global process of choosing/dropping is shown in Figure 5. The idea is somehow straightforward, but the full detail in the following might be tedious.

Based on *Step III*, define new integers

$$S_{i_0} = 0, \quad S_{i_\gamma} = \sum_{\iota=1}^{\gamma} s_{i_\iota} \ (\gamma = 1, \ldots, \tau), \quad S_n = S_{i_\tau} + s_n$$

and

$$R_{i_1} = r_{i_1}, \quad R_{i_\gamma} = S_{i_{\gamma-1}} + r_{i_\gamma} \ (\gamma = 1, \ldots, \tau), \quad R_n = S_{i_\tau} + r_n.$$

Then,

$$0 \le \left| S_{i_\gamma} - \left[\sum_{\iota=1}^{\gamma} \mu_1(X_{i_\iota}) \right] N_1 \right| \le N_1 \delta \quad (\gamma = 1, \ldots, \tau),$$

$$0 \le \left| S_n - \left[\sum_{\iota=1}^{\tau} \mu_1(X_{i_\iota}) + \mu_1(X_n) \right] N_1 \right| \le N_1 \delta,$$

$$0 \le \left| R_{i_\gamma} - \left[\sum_{\iota=1}^{\gamma-1} \mu_1(X_{i_\iota}) + \mu_0(X_{i_\gamma}) \right] N_1 \right| \le N_1 \delta \quad (\gamma = 1, \ldots, \tau),$$

and

$$0 \le \left| R_n - \left[\sum_{\iota=1}^{\tau} \mu_1(X_{i_\iota}) + \mu_0(X_n) \right] N_1 \right| \le N_1 \delta.$$

To ensure the compatibility of subspaces, we drop some functions $h(y, t_j)$ for which $h(0, t_j) \neq x_{i_\iota}$ when $j \in (S_{i_{\iota-1}}, R_{i_\iota}]$, $h(1, t_j) \neq x_{i_\iota}$ when $j \in (S_{i_{\iota-1}}, S_{i_\iota}]$ $(\iota = 1, \ldots, \tau)$, $h(0, t_j) \neq 1$ when $j \in (S_{i_\tau}, R_n]$, and $h(1, t_j) \neq 1$ when $j \in (S_{i_\tau}, S_n]$.

Assume that we throw out m_{i_ι} functions $h(0, t_j)$ for $j \in (S_{i_{\iota-1}}, R_{i_\iota}]$ $(\iota = 1, \ldots, \tau)$, m_n functions $h(0, t_j)$ for $j \in (S_{i_\tau}, R_n]$, and m_1 functions $h(0, t_j)$ for the remaining j; z_{i_ι} functions $h(1, t_j)$ for $j \in (S_{i_{\iota-1}}, S_{i_\iota}]$ $(\iota = 1, \ldots, \tau)$, z_n functions $h(1, t_j)$ for $j \in (S_{i_\tau}, S_n]$, and z_1 functions $h(1, t_j)$ for $j \in (S_n, N_1]$.

By Corollary 3.10, to achieve our goal, we need to throw out functions in proportion so that the remaining ones could satisfy the relations (3.12) and (3.13). So we need to require

$$\frac{m_{i_\iota}}{z_{i_\iota}} = \frac{b}{b+k} \quad (\iota = 1, \ldots, \tau), \tag{3.14}$$

$$\frac{a}{a+k} m_1 + m_n = \frac{b}{b+k} \left(\frac{a}{a+k} z_1 + z_n \right). \tag{3.15}$$

It might happen that $\mu_0(X_n) = 0$ or $\mu_0(X_n) \neq 0$; we exhibit our concrete choices in both cases. We always choose the rational approximation $s_n \ge \mu_1(X_n)$ and $s_1 \le \mu_1(X_1)$, which can be done by the proof of Lemma 3.9.

Case I: $\mu_0(X_n) = 0$. Of course, we take $r_n = 0$ and $s_n = \mu_1(X_n) = a/b$. Then, take $m_n = 0$. Let us assume

$$m_{i_\iota} = 4\delta N_1 b, \qquad z_{i_\iota} = 4\delta N_1 (b+k) \quad (\iota = 1, \ldots, \tau),$$

$$m_n = 0, \qquad z_n = t,$$
$$m_1 = 4\delta N_1 k\tau + t, \quad z_1 = 0.$$

By (3.15), we know that

$$\frac{a}{a+k}(4\delta N_1 k\tau + t) = \frac{b}{b+k}t,$$

so $t = \frac{4\delta N_1 a\tau(b+k)}{b-a}$. Then, we take

$$m_{i_\iota} = 4\delta N_1(b-a)b, \quad z_{i_\iota} = 4\delta N_1(b-a)(b+k) \quad (\iota = 1, \dots, \tau),$$
$$m_n = 0, \quad z_n = 4\delta N_1 a\tau(b+k),$$
$$m_1 = 4\delta N_1(a+k)\tau b, \quad z_1 = 0.$$

To ensure $r_{i_\iota} > m_{i_\iota}$, $s_{i_\iota} > z_{i_\iota}$, $\iota = 1, \dots, \tau$, and $r_1 > m_1$, $s_n > z_n$, as well as the purpose of approximation, we require that δ satisfies

$$8n\delta(b+k)b \sup_{f\in F} \|f\| \le \frac{\varepsilon}{4},$$
$$5\delta b(b-a) \le \mu_0(X_{i_\iota}) \quad (\iota = 1, 2, \dots, \tau),$$
$$5\delta(a+k)b\tau \le \mu_0(X_1),$$

and

$$5\delta(b+k)a\tau \le \mu_1(X_n).$$

By (3.10) and (3.11), we have

$$r_{i_\iota} > 4\delta N_1(b-a)b \quad (\iota = 1, 2, \dots, \tau),$$
$$r_1 > 4\delta N_1(a+k)b\tau,$$
$$s_{i_\iota} > 4\delta N_1(b-a)(b+k) \quad (\iota = 1, 2, \dots, \tau),$$

and

$$s_n > 4\delta N_1(b+k)a\tau.$$

Consider the set Λ_1 of integers j which belong to one of the following intervals:

$$S_{i_{\gamma-1}} + 2\delta N_1(b-a)b + 1 \le j \le R_{i_\gamma} - 2\delta N_1(b-a)b \quad (\gamma = 1, \dots, \tau),$$
$$R_{i_\gamma} + 2\delta N_1(b-a)k + 1 \le j \le S_{i_\gamma} - 2\delta N_1(b-a)k \quad (\gamma = 1, \dots, \tau),$$
$$S_{i_\tau} + 2\delta N_1(b+k)a\tau + 1 \le j \le S_n - 2\delta N_1(b+k)a\tau,$$

and

$$S_n + 1 \le j \le N_1.$$

Set $D_1 = \{\frac{j}{N_1} \mid j \in \Lambda_1\}$ and $N = |D_1|$. Then, by the construction of $h(y,t)$, point evaluations of h at the points in D_1 have the following distributions:

$$h(0,t_j) = h(1,t_j) = x_{i_\gamma},$$
$$\text{if } S_{i_{\gamma-1}} + 2\delta N_1(b-a)b + 1 \le j \le R_{i_\gamma} - 2\delta N_1(b-a)b \quad (\gamma = 1, \dots, \tau),$$

$$
\begin{aligned}
& h(0,t_j)=0, \quad h(1,t_j)=x_{i_\gamma}, \\
& \qquad \text{if } R_{i_\gamma}+2\delta N_1(b-a)k+1 \le j \le S_{i_\gamma}-2\delta N_1(b-a)k \quad (\gamma=1,\ldots,\tau), \\
& h(0,t_j)=0, \quad h(1,t_j)=1, \\
& \qquad \text{if } S_{i_\tau}+2\delta N_1(b+k)a\tau+1 \le j \le S_n-2\delta N_1(b+k)a\tau, \\
& h(0,t_j)=0=h(1,t_j), \\
& \qquad \text{if } S_n+1 \le j \le N_1.
\end{aligned}
$$

Note that $N = N_1(1-4\delta\tau(b+k)b)$ and that one has

$$
\begin{aligned}
\frac{1}{N}\sum_{d=1}^{N}\phi_d(f)(0) &= \frac{1}{N}\sum_{d=1}^{N} f\big(h(0,t_d)\big) \\
&= \frac{1}{N}\left(\sum_{\iota=1}^{\tau}(r_{i_\iota}-m_{i_\iota})f(x_{i_\iota})+(r_1-m_1)f(0)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{N}\sum_{d=1}^{N}\phi_d(f)(1) &= \frac{1}{N}\sum_{d=1}^{N} f\big(h(1,t_d)\big) \\
&= \frac{1}{N}\left(\sum_{\iota=1}^{\tau}(s_{i_\iota}-z_{i_\iota})f(x_{i_\iota})+(s_n-z_n)f(1)+s_1f(0)\right).
\end{aligned}
$$

It follows from Corollary 3.10 that

$$
\frac{1}{N}\sum_{d=1}^{N}\phi_d(f)(0) = \frac{b}{b+k}\frac{1}{N}\sum_{d=1}^{N}\phi_d(f)(1).
$$

Next, we show that the average of these N homomorphisms approximates the average of the original N_1 homomorphisms:

$$
\begin{aligned}
&\left|\frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big)-\frac{1}{N}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right| \\
&\quad = \frac{1}{N_1}\left|\sum_{j=1}^{N_1} f\big(h(y,t_j)\big)-\frac{1}{(1-4\delta\tau(b+k)b)}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right| \\
&\quad = \frac{1}{N_1}\left|\sum_{d\notin D_1} f\big(h(y,t_d)\big)+\left(1-\frac{1}{1-4\delta\tau(b+k)b}\right)\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right| \\
&\quad \le \frac{1}{N_1}\Big(N_1 4\delta\tau(b+k)b\sup_{f\in F}\|f\| + N_1 4\delta\tau(b+k)b\sup_{f\in F}\|f\|\Big) \\
&\quad = 8\delta\tau(b+k)b\sup_{f\in F}\|f\| \le \frac{\varepsilon}{4} \quad (\text{since } \tau \le n-2).
\end{aligned}
$$

Case II: $\mu_0(X_n) \neq 0$. In this case, we throw out $m_n = 4\delta N_1(b-a)b$ functions $h(0, t_j)$, and let us assume

$$\begin{aligned} m_{i_\iota} &= 4\delta N_1(b-a)b, & z_{i_\iota} &= 4\delta N_1(b-a)(b+k) \quad (\iota = 1, \dots, \tau), \\ m_n &= 4\delta N_1(b-a)b, & z_n &= 4\delta N_1(b-a)b + t, \\ m_1 &= 4\delta N_1 k\tau(b-a) + t, & z_1 &= 0. \end{aligned}$$

By (3.11), we know that

$$\frac{a}{a+k}\big[4\delta N_1 k\tau(b-a) + t\big] + 4\delta N_1(b-a)b = \frac{b}{b+k}\big[4\delta N_1(b-a)b + t\big],$$

so $t = 4\delta N_1(a\tau(b+k) + b(a+k))$. Then, we take

$$\begin{aligned} m_{i_\iota} &= 4\delta N_1(b-a)b, & z_{i_\iota} &= 4\delta N_1(b-a)(b+k) \quad (\iota = 1, \dots, \tau), \\ m_n &= 4\delta N_1(b-a)b, & z_n &= 4\delta N_1(a\tau + b)(b+k), \\ m_1 &= 4\delta N_1(a+k)(a+\tau)b, & z_1 &= 0. \end{aligned}$$

Similarly, we require that δ satisfies

$$\begin{aligned} 8\delta n(k+b)b \sup_{f\in F} \|f\| &\le \frac{\varepsilon}{4}, \\ 5\delta(b-a)b &< \mu_0(X_{i_\iota}) \quad (\iota = 1, 2, \dots, \tau), \\ 5\delta(b-a)b &< \mu_0(X_n), \\ 5\delta(a+k)(b+b\tau) &< \mu_0(X_1), \end{aligned}$$

and

$$5\delta(b+k)(a\tau + b) < \mu_1(X_n).$$

By (3.10) and (3.11), we have

$$\begin{aligned} r_{i_\iota} &> 4\delta N_1(b-a)b \quad (\iota = 1, 2, \dots, \tau), \\ r_n &> 4\delta N_1(b-a)b, \\ r_1 &> 4\delta N_1(a+k)(1+\tau)b, \\ s_{i_\iota} &> 4\delta N_1(b-a)(b+k) \quad (\iota = 1, 2, \dots, \tau), \end{aligned}$$

and

$$s_n > 4\delta N_1(b+k)(a\tau + b).$$

Consider the set Λ_2 of integers j which belong to one of the following intervals:

$$\begin{aligned} S_{i_{\gamma-1}} + 2\delta N_1(b-a)b + 1 \le j &\le R_{i_\gamma} - 2\delta N_1(b-a)b \quad (\gamma = 1, \dots, \tau), \\ R_{i_\gamma} + 2\delta N_1(b-a)k + 1 \le j &\le S_{i_\gamma} - 2\delta N_1(b-a)k \quad (\gamma = 1, \dots, \tau), \\ S_{i_\tau} + 2\delta N_1(b-a)b + 1 \le j &\le R_n - 2\delta N_1(b-a)b, \\ R_n + 2\delta N_1(ab\tau + ak\tau + bk + ba) + 1 \le j &\le S_n - 2\delta N_1(ab\tau + ak\tau + bk + ba), \end{aligned}$$

and

$$S_n + 1 \le j \le N_1.$$

Set $D_2 = \{\frac{j}{N_1} \mid j \in \Lambda_2\}$ and $N = |D_2|$. Then, by the construction of $h(y,t)$, point evaluations of h at the points in D_2 have the following distributions:

$$\begin{aligned}
&h(0,t_j) = h(1,t_j) = x_{i_\gamma},\\
&\quad \text{if } S_{i_{\gamma-1}} + 2\delta N_1(b-a)b + 1 \le j \le R_{i_\gamma} - 2\delta N_1(b-a)b \quad (\gamma = 1,\dots,\tau),\\
&h(0,t_j) = 0, \quad h(1,t_j) = x_{i_\gamma},\\
&\quad \text{if } R_{i_\gamma} + 2\delta N_1(b-a)k + 1 \le j \le S_{i_\gamma} - 2\delta N_1(b-a)k \quad (\gamma = 1,\dots,\tau),\\
&h(0,t_j) = h(1,t_j) = 1,\\
&\quad \text{if } S_{i_\tau} + 2\delta N_1(b-a)b + 1 \le j \le R_n - 2\delta N_1(b-a)b,\\
&h(0,t_j) = 0, \quad h(1,t_j) = 1,\\
&\quad \text{if } R_n + 2\delta N_1(ab\tau + ak\tau + bk + ba) + 1 \le j \le S_n - 2\delta N_1(ab\tau + ak\tau + bk + ba),\\
&h(0,t_j) = 0 = h(1,t_j),\\
&\quad \text{if } S_n + 1 \le j \le N_1.
\end{aligned}$$

Note that $N = N_1(1 - 4\delta(1+\tau)(k+b)b)$ and that one has

$$\frac{1}{N}\sum_{d=1}^{N} \phi_d(f)(0) = \frac{1}{N}\left(\sum_{\iota=1}^{\tau}(r_{i_\iota} - m_{i_\iota}) f(x_{i_\iota}) + (r_n - m_n) f(1) + (r_1 - m_1) f(0)\right)$$

and

$$\frac{1}{N}\sum_{d=1}^{N} \phi_d(f)(1) = \frac{1}{N}\left(\sum_{\iota=1}^{\tau}(s_{i_\iota} - z_{i_\iota}) f(x_{i_\iota}) + (s_n - z_n) f(1) + s_1 f(0)\right).$$

It follows from Corollary 3.10 that

$$\frac{1}{N}\sum_{d=1}^{N} \phi_d(f)(0) = \frac{b}{b+k}\,\frac{1}{N}\sum_{d=1}^{N} \phi_d(f)(1).$$

Next, we show that the average of these N homomorphisms approximates the average of the original N_1 homomorphisms:

$$\begin{aligned}
&\left|\frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{1}{N}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right|\\
&\quad = \frac{1}{N_1}\left|\sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{1}{1 - 4\delta(1+\tau)(k+b)b}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right|\\
&\quad = \frac{1}{N_1}\left|\sum_{d \notin D_2} f\big(h(y,t_d)\big) + \left(1 - \frac{1}{1 - 4\delta(1+\tau)(k+b)b}\right)\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right|
\end{aligned}$$

$$\leq \frac{1}{N_1}\Big(N_1 4\delta(1+\tau)(k+b)b \sup_{f\in F}\|f\| + N_1 4\delta(1+\tau)(k+b)b \sup_{f\in F}\|f\|\Big)$$

$$= 8\delta(1+\tau)(k+b)b \sup_{f\in F}\|f\| \leq \frac{\varepsilon}{4} \quad (\text{since } \tau \leq n-2).$$

In other words, no matter in which case, we can always find N functions $h(y, t_d)$, $d = 1, \ldots, N$, as required.

Finally, let us define $\phi_d : C[0,1] \to C[0,1]$ by $\phi_d(f(y)) = f(h(y,t_d))$ $(d = 1, \ldots, N)$. Then,

$$\begin{aligned}
&\left|\phi(f)(y) - \frac{1}{N}\sum_{d=1}^{N}\phi_j(f)(y)\right| \\
&= \left|\phi(f)(y) - \frac{1}{N}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right| \\
&\leq \left|\phi(f)(y) - \sum_{i=1}^{n}\lambda_i(y)f(x_i)\right| + \left|\sum_{i=1}^{n}\lambda_i(y)f(x_i) - w(f)(y)\right| \\
&\quad + \left|w(f)(y) - \frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big)\right| + \left|\frac{1}{N_1}\sum_{j=1}^{N_1} f\big(h(y,t_j)\big) - \frac{1}{N}\sum_{d=1}^{N} f\big(h(y,t_d)\big)\right| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon. \qquad \blacksquare
\end{aligned}$$

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