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Rapid decay and polynomial growth for bicrossed products

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Abstract. We study the rapid decay property and polynomial growth for duals of bicrossed products coming from a matched pair of a discrete group and a compact group.

1. Introduction

In the breakthrough paper [4], Haagerup showed that the norm of the reduced C^*-algebra $C_r^*(\mathbb{F}_N)$ of the free group on N-generators $\mathbb{F}_N$ can be controlled by the Sobolev l^2-norms associated to the word length function on $\mathbb{F}_N$. This is a striking phenomenon which actually occurs in many more cases. Jolissaint recognized this phenomenon, called Rapid Decay (or property (RD)), and studied it in a systematic way in [6]. Property (RD) has now many applications. Let us mention the remarkable one concerning K-theory. Property (RD) allowed Jolissaint [5] to show that the K-theory and $C_r^*(\Gamma)$ equal the K-theory of subalgebras of rapidly decreasing functions on Γ (Jolissaint did attribute this result to Connes). This result was then used by V. Lafforgue in his approach to the Baum–Connes conjecture via Banach KK-theory [8, 9].

In this paper, we view discrete quantum groups as duals of compact quantum groups. The theory of compact quantum groups has been developed by Woronowicz [12–14]. Property (RD) for discrete quantum groups has been introduced and studied by Vergnioux [11]. Later, in [2], it has been refined in order to fit in the context of non-unimodular discrete quantum groups.

In this paper, we study the permanence of property (RD) under the bicrossed product construction. This construction was initiated by Kac [7] in the context of finite quantum groups and was extensively studied later by many authors in different settings. The general construction, for locally compact quantum groups, was developed by Vaes–Vainerman [10]. The case of bicrossed products coming from a matched pair of classical locally compact groups is discussed by Baaj–Skandalis–Vaes [1] in which they provide interesting concrete examples of bicrossed products. In the context of compact quantum groups given by matched pairs of classical groups, an easier approach, that we will follow, was given by Fima–Mukherjee–Patri [3].

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Following [3], the bicrossed product construction associates to a compact quantum group $\mathbb{G}$, called the bicrossed product, to a matched pair (Γ, G) of a discrete group Γ and a compact group G (see Section 2.2). Given a length function l on the set of equivalence classes $\text{Irr}(\mathbb{G})$ of irreducible unitary representations of $\mathbb{G}$, one can associate in a canonical way, as explained in Proposition 4.2, a pair of length functions (l_Γ, l_G) on Γ and $\text{Irr}(G)$, respectively. Such a pair satisfies some compatibility relations and every pair of length functions (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ satisfying those compatibility relations will be called matched (see Definition 4.1). Any matched pair (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ allows one to reconstruct a canonical length function on $\text{Irr}(\mathbb{G})$. The main result of the present paper is the following.

Theorem 1. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G. Denote by $\mathbb{G}$ the bicrossed product. The following are equivalent.*

(1) *$\widehat{\mathbb{G}}$ has property (RD).*

(2) *There exists a matched pair of length functions (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ such that both (Γ, l_Γ) and $(\widehat{G}, l_G)$ have (RD).*

For amenable discrete groups, property (RD) is equivalent to polynomial growth [6] and the same occurs for discrete quantum groups [11]. Hence, for the compact classical group G, one has that $(\widehat{G}, l_G)$ has (RD) if and only if it has polynomial growth. Note that a bicrossed product of a matched pair (Γ, G) is co-amenable if and only if Γ is amenable [3]. The following theorem shows the permanence of polynomial growth under the bicrossed product construction.

Theorem 2. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G. Denote by $\mathbb{G}$ the bicrossed product. The following are equivalent.*

(1) *$\widehat{\mathbb{G}}$ has polynomial growth.*

(2) *There exists a matched pair of length functions (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ such that both (Γ, l_Γ) and $(\widehat{G}, l_G)$ have polynomial growth.*

The main ingredient to prove Theorems 1 and 2 is the classification of the irreducible unitary representation of a bicrossed product and the fusion rules.

The paper is organized as follows. Section 2 is a preliminary section in which we introduce our notations. In Section 3, we classify the irreducible unitary representations of a bicrossed product and describe their fusion rules. Finally, in Section 5, we prove Theorems 1 and 2.

2. Preliminaries

2.1. Notations

For a Hilbert space H, we denote by $\mathcal{U}(H)$ its unitary group and by $\mathcal{B}(H)$ the C^*-algebra of bounded linear operators on H. When H is finite dimensional, we denote by Tr the unique trace on $\mathcal{B}(H)$ such that $\text{Tr}(1) = \dim(H)$. We use the same symbol $\otimes$

for the tensor product of Hilbert spaces, the unitary representations of compact quantum groups, and the minimal tensor product of C^*-algebras. For a compact quantum group G, we denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible unitary representations and by $\mathrm{Rep}(G)$ the class of finite dimensional unitary representations. We will often denote by $[u]$ the equivalence class of an irreducible unitary representation u. For $u \in \mathrm{Rep}(G)$, we denote by $\chi(u)$ its character; i.e., viewing $u \in \mathcal{B}(H) \otimes C(G)$ for some finite dimensional Hilbert space H, one has $\chi(u) := (\mathrm{Tr} \otimes \mathrm{id})(u) \in C(G)$. We denote by $\mathrm{Pol}(G)$ the unital C^*-algebra obtained by taking the span of the coefficients of irreducible unitary representation, by $C_m(G)$ the enveloping C^*-algebra of $\mathrm{Pol}(G)$, and by $C(G)$ the C^*-algebra generated by the GNS construction of the Haar state on $C_m(G)$. We also denote by $\varepsilon : C_m(G) \to \mathbb{C}$ the counit and we use the same symbol $\varepsilon \in \mathrm{Irr}(G)$ to denote the trivial representation and its class in $\mathrm{Irr}(G)$. In the entire paper, the word representation means a unitary and finite dimensional representation.

2.2. Compact bicrossed products

In this section, we follow the approach and the notations of [3].

Let (Γ, G) be a pair of a countable discrete group Γ and a second countable compact group G with a left action $\alpha : \Gamma \to \mathrm{Homeo}(G)$ of Γ on the compact space G by homeomorphisms and a right action $\beta : G \to S(\Gamma)$ of G on the discrete space Γ, where $S(\Gamma)$ is the Polish group of bijections of Γ, with the topology being the one of pointwise convergence, i.e., the smallest one for which the evaluation maps $S(\Gamma) \to \Gamma$, $\sigma \mapsto \sigma(\gamma)$ are continuous, for all $\gamma \in \Gamma$, where Γ has the discrete topology. Here, α is a group homomorphism and β is an antihomomorphism. The pair (Γ, G) is called a matched pair if $\Gamma \cap G = \{e\}$, with e being the common unit for both G and Γ, and if the actions α and β satisfy the following matched pair relations:

$$\begin{cases} \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_g(\gamma)}(h), \\ \beta_g(\gamma\mu) = \beta_{\alpha_\mu(g)}(\gamma)\beta_g(\mu), \\ \alpha_\gamma(e) = \beta_g(e) = e, \end{cases} \quad \forall g, h \in G,\ \gamma, \mu \in \Gamma. \tag{2.1}$$

We also write $\gamma \cdot g := \beta_g(\gamma)$. From now on, we assume that (Γ, G) is matched. It is shown in [3, Proposition 3.2] that β is automatically continuous. By continuity of β and compactness of G, every β orbit is finite. Moreover, the sets $G_{r,s} := \{g \in G : r \cdot g = s\}$ are clopen (see [3, Section 2.1]). Let $v_{rs} = 1_{G_{r,s}} \in C(G)$ be the characteristic function of $G_{r,s}$. It is shown in [3, Section 2.1] that, for all β-orbits $\gamma \cdot G \in \Gamma/G$, the unitary $v_{\gamma \cdot G} := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes v_{rs} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes C(G)$ is a unitary representation of G as well as a magic unitary, where $e_{rs} \in \mathcal{B}(l^2(\gamma \cdot G))$ are the canonical matrix units and the Haar probability measure ν on G is α-invariant.

It is shown in [3, Theorem 3.4] that there exists a unique compact quantum group $\mathbb{G}$, called the bicrossed product of the matched pair (Γ, G), such that $C(\mathbb{G}) = \Gamma_\alpha \ltimes C(G)$ is the reduced C^*-algebraic crossed product, generated by a copy of $C(G)$ and the unitaries u_γ, $\gamma \in \Gamma$, and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is the unique unital $*$-homomorphism satisfying $\Delta|_{C(G)} = \Delta_G$ (the comultiplication on $C(G)$) and $\Delta(u_\gamma) = \sum_{r \in \gamma \cdot G} u_\gamma v_{\gamma r} \otimes u_r$, for all $\gamma \in \Gamma$. It is also shown that the Haar state on $\mathbb{G}$ is a trace and is given by the formula $h(u_\gamma F) = \delta_{\gamma,1} \int_G F\, d\nu$, for all $\gamma \in \Gamma$ and $F \in C(G)$.

3. Representation theory of bicrossed products

3.1. Classification of irreducible representations

In this section, we classify the irreducible representations of a bicrossed product. Let (Γ, G) be a matched pair of a discrete countable group Γ and a second countable compact group G with actions α, β.

For $\gamma \in \Gamma$, we denote by $G_\gamma := G_{\gamma,\gamma}$ the stabilizer of γ for the action $\beta : \Gamma \curvearrowleft G$. Note that G_γ is an open (hence closed) subgroup of G, and hence of finite index: its index is $|\gamma \cdot G|$. We view $C(G_\gamma) = v_{\gamma\gamma} C(G) \subset C(G)$ as a non-unital C^*-subalgebra. Let us denote by ν the Haar probability measure on G and note that $\nu(G_\gamma) = \frac{1}{|\gamma \cdot G|}$ so that the Haar probability measure ν_γ on G_γ is given by $\nu_\gamma(A) = |\gamma \cdot G| \nu(A)$ for all Borel subset A of G_γ.

For $\gamma \in \Gamma$, we fix a section, still denoted by γ, $\gamma : \gamma \cdot G \to G$ of the canonical surjection $G \to \gamma \cdot G : g \mapsto \gamma \cdot g$. This means that $\gamma : \gamma \cdot G \to G$ is an injective map such that $\gamma \cdot \gamma(r) = r$, for all $r \in \gamma \cdot G$. We choose the section γ such that $\gamma(\gamma) = 1$, for all $\gamma \in \Gamma$. For $r, s \in \gamma \cdot G$, we denote by $\psi^\gamma_{r,s}$ the ν-preserving homeomorphism of G defined by $\psi^\gamma_{r,s}(g) = \gamma(r) g \gamma(s)^{-1}$. It follows from our choices that $\psi^\gamma_{\gamma,\gamma} = \mathrm{id}$, for all $\gamma \in \Gamma$. Moreover, for all $g \in G$, one has $\psi^\gamma_{r,s}(g) \in G_\gamma$ if and only if $g \in G_{r,s}$. It follows that $\psi^\gamma_{r,r}$ is an isomorphism and a homeomorphism from G_r to G_γ intertwining the Haar probability measures.

Let $u : G_\gamma \to \mathcal{U}(H)$ be a unitary representation of G_γ and view u as a continuous function $G \to \mathcal{B}(H)$ which is zero outside G_γ, i.e., a partial isometry in $\mathcal{B}(H) \otimes C(G)$ such that $uu^* = u^*u = \mathrm{id}_H \otimes v_{\gamma\gamma}$. Define, for $r, s \in \gamma \cdot G$, the partial isometry $u_{r,s} := u \circ \psi^\gamma_{r,s} := (g \mapsto u(\psi^\gamma_{r,s}(g))) \in \mathcal{B}(H) \otimes C(G)$ and note that $u^*_{r,s} u_{r,s} = u_{r,s} u^*_{r,s} = \mathrm{id}_H \otimes 1_{G_{r,s}}$. In the sequel, we view $u_{r,s} \in \mathcal{B}(H) \otimes C(G) \subset \mathcal{B}(H) \otimes C(\mathbb{G})$ and we define

$$\gamma(u) := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u_{r,s} \in \mathcal{B}\big(l^2(\gamma \cdot G)\big) \otimes \mathcal{B}(H) \otimes C(\mathbb{G}),$$

where we recall that e_{rs}, for $r, s \in \gamma \cdot G$, are the matrix units associated to the canonical orthonormal basis of $l^2(\gamma \cdot G)$.

The irreducible unitary representations of $\mathbb{G}$ are described as follows.

Theorem 3.1. *The following hold.*

1. *For all* $\gamma \in \Gamma$ *and* $u \in \mathrm{Rep}(G_\gamma)$, *one has* $\gamma(u) \in \mathrm{Rep}(\mathbb{G})$.
2. *The character of* $\gamma(u)$ *is* $\chi(\gamma(u)) = \sum_{r \in \gamma \cdot G} u_r v_{rr} \chi(u) \circ \psi^\gamma_{r,r}$.
3. *For all* $\gamma, \mu \in \Gamma$, $u \in \mathrm{Rep}(G_\gamma)$, *and* $w \in \mathrm{Rep}(G_\mu)$, *one has*

$$\dim\big(\mathrm{Mor}_{\mathbb{G}}\left(\gamma(u), \mu(w)\right)\big) = \delta_{\gamma \cdot G, \mu \cdot G} \dim\big(\mathrm{Mor}_{G_\gamma}(u, w \circ \psi^\mu_{\gamma,\gamma})\big).$$

(4) *For all* $\gamma \in \Gamma$ *and* $u \in \mathrm{Rep}(G_\gamma)$, *one has* $\overline{\gamma(u)} \simeq \gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})$ *(which makes sense since* $\alpha_{\gamma^{-1}} : G_{\gamma^{-1}} \to G_\gamma$ *is a group isomorphism and a homeomorphism).*

(5) $\gamma(u)$ *is irreducible if and only if* u *is irreducible. Moreover, for any irreducible unitary representation* u *of* $\mathbb{G}$, *there exists* $\gamma \in \Gamma$ *and an irreducible representation* v *of* G_γ *such that* $u \simeq \gamma(v)$.

Proof. (1) Writing $\gamma(u) = \sum_{r,s} e_{r,s} \otimes V_{r,s}$, where $V_{r,s} := (1 \otimes u_r v_{rs}) u_{r,s} \in \mathcal{B}(H) \otimes C(\mathbb{G})$, it suffices to check that, for all $r, s \in \gamma \cdot G$, one has $(\mathrm{id} \otimes \Delta)(V_{r,s}) = \sum_{t \in \gamma \cdot G} (V_{r,t})_{12} (V_{t,s})_{13}$. We first claim that, for all $r, s \in \gamma \cdot G$, $(\mathrm{id} \otimes \Delta)(u_{r,s}) = \sum_{t \in \gamma \cdot G} (u_{r,t})_{12} (u_{t,s})_{13}$. To check our claim, first recall that, for all $r, s \in \gamma \cdot G$, one has $\psi^\gamma_{r,s}(g) \in G_\gamma$ if and only if $r \cdot g = s$. Let $r, s \in \gamma \cdot G$ and $g, h \in G$. For $t = r \cdot g \in \gamma \cdot G$, one has

$$\begin{aligned} u_{r,s}(gh) &= u\big(\gamma(r) g \gamma(t)^{-1} \gamma(t) h \gamma(s)^{-1}\big) \\ &= u\big(\psi^\gamma_{r,t}(g) \psi^\gamma_{t,s}(h)\big) = \begin{cases} u_{r,t}(g) u_{t,s}(h) & \text{if } r \cdot gh = s, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since we also have $u_{t,s}(h) = 0$ whenever $r \cdot gh \neq s$, we find, in both cases, that $u_{r,s}(gh) = u_{r,t}(g) u_{t,s}(h)$. Now, for $t \neq r \cdot g$ we have $u_{r,t}(g) = 0$, so the following formula holds for any $r, s \in \gamma \cdot G$ and any $g, h \in G$:

$$v_{r,t}(g) u_{r,s}(gh) = u_{r,t}(g) u_{t,s}(h).$$

Hence, for all $r, s, t \in \gamma \cdot G$, $(1 \otimes v_{r,t} \otimes 1)(\mathrm{id} \otimes \Delta)(u_{r,s}) = (u_{r,t})_{12} (u_{t,s})_{13}$. Using this, we find

$$\begin{aligned} \sum_{t \in \gamma \cdot G} (V_{r,t})_{12} (V_{t,s})_{13} &= \sum_t (1 \otimes u_r v_{rt} \otimes 1)(u_{r,t})_{12} (1 \otimes 1 \otimes u_t v_{ts})(u_{t,s})_{13} \\ &= \sum_t (1 \otimes u_r v_{rt} \otimes u_t v_{ts})(u_{r,t})_{12} (u_{t,s})_{13} \\ &= \left(1 \otimes \left(\sum_t u_r v_{rt} \otimes u_t v_{ts}\right)\right)(\mathrm{id} \otimes \Delta)(u_{r,s}). \end{aligned}$$

Since v_γ is a unitary representation of G and a magic unitary, we also have

$$\Delta(u_r v_{rs}) = \sum_{t,t'} (u_r v_{rt} \otimes u_t)(v_{rt'} \otimes v_{t's}) = \sum_t u_r v_{rt} \otimes u_t v_{ts}.$$

This shows that $\gamma(u)$ is a representation of $\mathbb{G}$. We now check that $\gamma(u)$ is unitary. As before, since, for all $r, s \in \gamma \cdot G$, one has $\psi^\gamma_{r,s}(g) \in G_\gamma$ if and only if $r \cdot g = s$ and because u is a unitary representation of G_γ, we have, for all $r, t \in \gamma \cdot G$, $(1 \otimes v_{rt}) u_{r,t} u^*_{r,t} = 1 \otimes v_{rt}$. Hence,

$$\begin{aligned} \sum_{t \in \gamma \cdot G} V_{r,t} V^*_{s,t} &= \sum_t (1 \otimes u_r)(1 \otimes v_{rt}) u_{r,t} u^*_{s,t} (1 \otimes v_{st})(1 \otimes u^*_s) \\ &= \delta_{r,s} (1 \otimes u_r) \left(\sum_t (1 \otimes v_{rt}) u_{r,t} u^*_{r,t} \right) (1 \otimes u^*_r) \end{aligned}$$

$$= \delta_{r,s}(1 \otimes u_r)\Big(\sum_t (1 \otimes v_{rt})\Big)(1 \otimes u_r^*)$$
$$= \delta_{r,s}.$$

A similar computation shows that $\sum_{t \in \gamma \cdot G} V_{t,r}^* V_{t,s} = \delta_{r,s}$.

(2) The character of $\gamma(u)$ is given by

$$\begin{aligned}\chi\big(\gamma(u)\big) &= \sum_{r \in \gamma \cdot G} (\mathrm{Tr} \otimes \mathrm{id})(V_{r,r}) \\ &= \sum_r u_r v_{rr} (\mathrm{Tr} \otimes \mathrm{id})(u_{r,r}) \\ &= \sum_r u_r v_{rr} \chi(u) \circ \psi_{r,r}^{\gamma}.\end{aligned}$$

(3) Let $\gamma, \mu \in \Gamma$ and u, w be representations of G_γ and G_μ, respectively. Since the Haar measure on G is invariant under the action α and the homeomorphisms $\psi_{r,r}^{\gamma}$ and $\psi_{r,r}^{\mu}$, we find, by the character formula in (2) and the crossed-product relations,

$$\begin{aligned}&\dim\big(\mathrm{Mor}\big(\gamma(u), \mu(w)\big)\big) \\ &\quad = h\big(\chi\big(\gamma(u)\big)\chi\big(\mu(w)\big)^*\big) \\ &\quad = \sum_{r \in \gamma \cdot G, s \in \mu \cdot G} h\big(u_{rs^{-1}} \alpha_s\big(v_{rr} v_{ss} \chi(u) \circ \psi_{r,r}^{\gamma}\big(\chi(w) \circ \psi_{s,s}^{\mu}\big)^*\big)\big) \\ &\quad = \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_G \alpha_r\big(v_{rr}\big(\chi(u) \circ \psi_{r,r}^{\gamma}\big)\big(\overline{\chi(w)} \circ \psi_{r,r}^{\mu}\big)\big)\, d\nu \\ &\quad = \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_r} \big(\chi(u) \circ \psi_{r,r}^{\gamma}\big)\big(\chi(\overline{w}) \circ \psi_{r,r}^{\mu}\big)\, d\nu \\ &\quad = \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{\gamma,\gamma}^{\mu})^{-1}\big(\chi(\overline{w}) \circ \psi_{r,r}^{\mu} \circ (\psi_{r,r}^{\gamma})^{-1} \circ (\psi_{\gamma,\gamma}^{\mu})^{-1}\big)\, d\nu.\end{aligned}$$

Now, note that $\psi_{r,r}^{\mu} \circ (\psi_{r,r}^{\gamma})^{-1} \circ (\psi_{\gamma,\gamma}^{\mu})^{-1} = \mathrm{Ad}(h)$, where $h = \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1}$. Moreover, $\mu \cdot h = \mu$ since

$$\mu \cdot \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1} = r \cdot \gamma(r)^{-1}\mu(\gamma)^{-1} = \gamma \cdot \mu(\gamma)^{-1} = \mu.$$

Hence, $h \in G_\mu$. Since the characters of finite dimensional unitary representations of a group Λ are central functions, i.e., invariant under $\mathrm{Ad}(\lambda)$ for all $\lambda \in \Lambda$, we have $\chi(\overline{w}) \circ \psi_{r,r}^{\mu} \circ (\psi_{r,r}^{\gamma})^{-1} \circ (\psi_{\gamma,\gamma}^{\mu})^{-1} = \chi(\overline{w}) \circ \mathrm{Ad}(h) = \chi(\overline{w})$. Hence,

$$\begin{aligned}\dim\big(\mathrm{Mor}\big(\gamma(u), \mu(w)\big)\big) &= \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{\gamma,\gamma}^{\mu})^{-1} \chi(\overline{w})\, d\nu \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{\gamma,\gamma}^{\mu})^{-1} \chi(\overline{w})\, d\nu_\mu\end{aligned}$$

$$
\begin{aligned}
&= \delta_{\gamma\cdot G,\mu\cdot G} \dim\big(\mathrm{Mor}_{G_\mu}\big(u \circ (\psi^{\mu}_{\gamma,\gamma})^{-1}, w\big)\big) \\
&= \delta_{\gamma\cdot G,\mu\cdot G} \int_{G_\gamma} \chi(u)\chi(\overline{w} \circ \psi^{\mu}_{\gamma,\gamma})\, d\nu_\mu \\
&= \delta_{\gamma\cdot G,\mu\cdot G} \dim\big(\mathrm{Mor}_{G_\gamma}(u, w \circ \psi^{\mu}_{\gamma,\gamma})\big).
\end{aligned}
$$

(4) Note that, by the bicrossed product relations, we have, for all $\gamma \in \Gamma$ and $g \in G$, $(\gamma \cdot g)^{-1} = \gamma^{-1} \cdot \alpha_\gamma(g)$. Hence, $v_{\gamma^{-1}\gamma^{-1}} \circ \alpha_\gamma = v_{\gamma\gamma}$ and $(\gamma \cdot G)^{-1} = \gamma^{-1} \cdot G$. In particular, $\alpha_\gamma : G_\gamma \to G_{\gamma^{-1}}$ is a homeomorphism and, by the bicrossed product relations, one has, for all $g \in G_\gamma$ and $h \in G$, $\alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\gamma\cdot g}(h) = \alpha_\gamma(g)\alpha_\gamma(h)$ so that $\alpha_\gamma : G_\gamma \to G_{\gamma^{-1}}$ is also a group homomorphism.

For $r \in \gamma \cdot G$, one has $\gamma^{-1} \cdot \alpha_\gamma(\gamma(r)) = (\gamma \cdot \gamma(r))^{-1} = r^{-1} = \gamma^{-1} \cdot \gamma^{-1}(r^{-1})$. This implies that, for all $\gamma \in \Gamma$, there exists a map $\eta_\gamma : \gamma \cdot G \to G_{\gamma^{-1}}$ such that, for all $r \in \gamma \cdot G$, one has $\alpha_\gamma(\gamma(r)) = \eta_\gamma(r)\gamma^{-1}(r^{-1})$.

Let now $r \in \gamma \cdot G$ and $g \in G_r$. One has, using the bicrossed product relations, that $e = \alpha_r(\gamma(r)\gamma(r)^{-1}) = \alpha_\gamma(\gamma(r))\alpha_r(\gamma(r)^{-1})$, and hence

$$
\begin{aligned}
(\alpha_\gamma \circ \psi^{\gamma}_{r,r})(g) &= \alpha_\gamma\big(\gamma(r)\big)\alpha_r(g)\alpha_r\big(\gamma(r)^{-1}\big) \\
&= \alpha_\gamma\big(\gamma(r)\big)\alpha_r(g)\big(\alpha_\gamma\big(\gamma(r)\big)\big)^{-1} \\
&= \eta_\gamma(r)\big(\psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \circ \alpha_r\big)(g)\big(\eta_\gamma(r)\big)^{-1}.
\end{aligned}
$$

Hence, for all $\gamma \in \Gamma$, if $w \in \mathrm{Rep}(G_{\gamma^{-1}})$, since $\chi(w) \in C(G_{\gamma^{-1}})$ is central we have

$$
\chi(w) \circ \alpha_\gamma \circ \psi^{\gamma}_{r,r}(g) = \chi(w) \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \circ \alpha_r(g) \quad \text{for all } r \in \gamma \cdot G,\ g \in G_r.
$$

Since, as we have seen above, $\gamma^{-1} \cdot G = (\gamma \cdot G)^{-1}$ and because $\chi(\overline{u} \circ \alpha_{\gamma^{-1}}) = \chi(\overline{u}) \circ \alpha_{\gamma^{-1}}$, we find, by the character formula in (2),

$$
\chi\big(\gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})\big) = \sum_{r\in\gamma\cdot G} u_{r^{-1}} v_{r^{-1}r^{-1}} \chi(\overline{u}) \circ \alpha_{\gamma^{-1}} \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}}.
$$

It then follows from the crossed-product relations and the discussion above that

$$
\begin{aligned}
\chi\big(\gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})\big) &= \sum_{r\in\gamma\cdot G} u_{r^{-1}} v_{r^{-1}r^{-1}} \chi(\overline{u}) \circ \alpha_{\gamma^{-1}} \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \\
&= \sum_{r\in\gamma\cdot G} \big(\chi(\overline{u}) \circ \alpha_{\gamma^{-1}} \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \circ \alpha_r\big)(v_{r^{-1}r^{-1}} \circ \alpha_r) u_{r^{-1}} \\
&= \sum_{r\in\gamma\cdot G} \chi(\overline{u}) \circ \psi^{\gamma}_{r,r} v_{rr} u_r^* = \sum_{r\in\gamma\cdot G} \big(\chi(u) \circ \psi^{\gamma}_{r,r} v_{rr}\big)^* u_r^* \\
&= \chi\big(\gamma(u)\big)^*.
\end{aligned}
$$

(5) From the statement on irreducibility following from (3), it suffices, by the general theory, to show that the linear span X of coefficients of representations of the form $\gamma(u)$, $\gamma \in \Gamma$ and u being an irreducible unitary representation of G_γ, is a dense subset of $C(\mathbb{G})$.

Note that, for all $\gamma \in \Gamma$, the relation $1 = \sum_{r \in \gamma \cdot G} v_{\gamma r}$ implies that any function in $C(G)$ is a sum of continuous functions with support in $G_{\gamma,r} := \{g \in G : \gamma \cdot g = r\}$, for $r \in \gamma \cdot G$. Moreover, since $G_{\gamma,r} = (\psi^{\gamma}_{\gamma,r})^{-1}(G_\gamma)$, any continuous function on G with support in $G_{\gamma,r}$ is of the form $F \circ \psi^{\gamma}_{\gamma,r}$, where $F \in C(G_\gamma)$. Since the linear span of coefficients of irreducible unitary representation of G_γ is dense in $C(G_\gamma)$, it suffices to show that, for any $\gamma \in \Gamma$, for any irreducible unitary representation of G_γ, $u : G_\gamma \to \mathcal{U}(H)$, any coefficient $u_{ij} \in C(G_\gamma) = v_{\gamma\gamma}C(G) \subset C(G)$ satisfies $u_\gamma u_{ij} \in X$. But this is obvious since one has

$$u_\gamma u_{ij} = u_\gamma v_{\gamma\gamma} u_{i,j} = u_\gamma v_{\gamma\gamma} u_{i,j} \circ \psi^{\gamma}_{\gamma,\gamma} = \gamma(u)_{\gamma,\gamma,i,j} \in X. \qquad \blacksquare$$

Finally, the fusion rules are described as follows.

Let $\gamma, \mu \in \Gamma$ and let $u : G_\gamma \to \mathcal{U}(H_u)$, $v : G_\mu \to \mathcal{U}(H_v)$ be unitary representations of G_γ and G_μ, respectively. For any $r \in (\gamma \cdot G)(\mu \cdot G)$, we define the r-twisted tensor product of u and v: $u \underset{r}{\otimes} v$, as a unitary representation of G_r on $K_r \otimes H_u \otimes H_v$, where

$$K_r := \operatorname{Span}\big(\{e_s \otimes e_t : s \in \gamma \cdot G \text{ and } t \in \mu \cdot G \text{ such that } st = r\}\big) \subset l^2(\gamma \cdot G) \otimes l^2(\mu \cdot G).$$

For $g \in G$, we define

$$\begin{aligned}(u \underset{r}{\otimes} v)(g) = \sum_{\substack{s,s' \in \gamma \cdot G \\ t,t' \in \mu \cdot G \\ st = r = s't'}} & e_{ss'} \otimes e_{tt'} \otimes v_{ss'}\big(\alpha_t(g)\big) v_{tt'}(g) u\big(\psi^{\gamma}_{s,s'}\big(\alpha_t(g)\big)\big) \otimes v\big(\psi^{\mu}_{t,t'}(g)\big) \\ & \in \mathcal{U}(K_r \otimes H_u \otimes H_v).\end{aligned}$$

Theorem 3.2. *The following hold.*

(1) *For all $\gamma, \mu \in \Gamma$, all $r \in (\gamma \cdot G)(\mu \cdot G)$, and all u, v finite dimensional unitary representations of G_γ, G_μ, respectively, the element $u \underset{r}{\otimes} v$ is a unitary representation of G_r.*

(2) *The character of $u \underset{r}{\otimes} w$ is*

$$\chi(u \underset{r}{\otimes} v) = \sum_{s \in \gamma \cdot G,\, t \in \mu \cdot G,\, st = r} (v_{ss} \circ \alpha_t) v_{tt} \big(\chi(u) \circ \psi^{\gamma}_{s,s} \circ \alpha_t\big)\big(\chi(v) \circ \psi^{\mu}_{t,t}\big).$$

(3) *For all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and all u, v, w unitary representations of G_{γ_1}, G_{γ_2}, and G_{γ_3}, respectively, the number* $\dim(\mathrm{Mor}_{\mathbb{G}}(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w)))$ *is equal to*

$$\begin{cases} \dfrac{1}{|\gamma_1 \cdot G|} \displaystyle\sum_{r \in \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G)} \dim\big(\mathrm{Mor}_{G_r}(u \circ \psi^{\gamma_1}_{r,r}, v \underset{r}{\otimes} w)\big) & \text{if } \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let us observe that, by the bicrossed product relations, we have, for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$,

$$\gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset \Leftrightarrow \gamma_1 \cdot G \subset (\gamma_2 \cdot G)(\gamma_3 \cdot G).$$

Proof. (1) Put $w = u \underset{r}{\otimes} v$ and let $g, h \in G_r$. Then, $w(gh)$ is equal to

$$\sum_{s,s'\in\gamma\cdot G,\ t,t'\in\mu\cdot G,\ st=s't'=r} e_{ss'}\otimes e_{tt'}\otimes v_{ss'}\big(\alpha_t(gh)\big)v_{tt'}(gh)u\big(\psi^{\gamma}_{s,s'}\big(\alpha_t(gh)\big)\big)\otimes v\big(\psi^{\mu}_{t,t'}(gh)\big).$$

Since $v_{ty}(g) \neq 0$ precisely when $t \cdot g = y$, the factor

$$v_{ss'}\big(\alpha_t(gh)\big)v_{tt'}(gh)u\big(\psi^{\gamma}_{s,s'}\big(\alpha_t(gh)\big)\big) \otimes v\big(\psi^{\mu}_{t,t'}(gh)\big)$$

is equal to

$$\begin{aligned}&\sum_{x\in\gamma\cdot G, y\in\mu\cdot G} v_{sx}\big(\alpha_t(g)\big)v_{xs'}\big(\alpha_{t\cdot g}(h)\big)v_{ty}(g)v_{yt'}(h)u\big(\psi^{\gamma}_{s,x}\big(\alpha_t(g)\big)\big)u\big(\psi^{\gamma}_{x,s'}\big(\alpha_{t\cdot g}(h)\big)\big)\\ &\qquad\otimes v\big(\psi^{\mu}_{t,y}(g)\big)v\big(\psi^{\mu}_{y,t'}(h)\big)\\ &= \sum_{x\in\gamma\cdot G, y\in\mu\cdot G} v_{sx}\big(\alpha_t(g)\big)v_{xs'}\big(\alpha_y(h)\big)v_{ty}(g)v_{yt'}(h)u\big(\psi^{\gamma}_{s,x}\big(\alpha_t(g)\big)\big)u\big(\psi^{\gamma}_{x,s'}\big(\alpha_y(h)\big)\big)\\ &\qquad\otimes v\big(\psi^{\mu}_{t,y}(g)\big)v\big(\psi^{\mu}_{y,t'}(h)\big).\end{aligned}$$

Moreover, since, for all $g \in G_r$ and all s, t such that $st = r$, one has, whenever $t \cdot g = y$ and $s \cdot \alpha_t(g) = x$, that $xy = (s \cdot \alpha_t(g))(t \cdot g) = (st) \cdot g = r \cdot g = r$. It follows that the only non-zero terms in the last sum are for $x \in \gamma \cdot G$ and $y \in \mu \cdot G$ such that $xy = r$. By the properties of the matrix units, we see immediately that $w(gh) = w(g)w(h)$. To end the proof of (1), it suffices to check that $w(1) = 1$, which is clear, and that $w(g)^* = w(g^{-1})$ for all $g \in G_r$. So let $g \in G_r$. One has

$$\begin{aligned}w(g)^* = \sum_{s,s'\in\gamma\cdot G,\ t,t'\in\mu\cdot G,\ st=r=s't'} & e_{ss'} \otimes e_{tt'} \otimes v_{s's}\big(\alpha_{t'}(g)\big)v_{t't}(g)u\big(\big(\psi^{\gamma}_{s',s}\big(\alpha_{t'}(g)\big)\big)^{-1}\big)\\ &\otimes v\big(\big(\psi^{\mu}_{t',t}(g)\big)^{-1}\big).\end{aligned}$$

Note that, for all $t, t' \in \Gamma$ and all $g \in G$, one has $v_{s's}(g) = v_{ss'}(g^{-1})$. Also, using the bicrossed product relations, one finds that $\alpha_r(g)^{-1} = \alpha_{r\cdot g}(g^{-1})$ for all $r \in \Gamma$ and $g \in G$. In particular, $v_{s's}(\alpha_{t'}(g))v_{t't}(g) = v_{ss'}(\alpha_t(g^{-1}))v_{tt'}(g^{-1})$ and, when $t' \cdot g = t$, one has $\psi^{\gamma}_{s',s}(\alpha_{t'}(g))^{-1} = \psi^{\gamma}_{s,s'}(\alpha_t(g^{-1}))$. It follows immediately that $w(g)^* = w(g^{-1})$.

(2) Is a direct computation.

(3) One has $\dim(\mathrm{Mor}_{\mathbb{G}}(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w))) = h(\chi(\gamma_1(u))^*\chi(\gamma_2(v))\chi(\gamma_3(w)))$ which is equal to

$$\begin{aligned}&\sum_{r\in\gamma_1\cdot G,\ s\in\gamma_2\cdot G,\ t\in\gamma_3\cdot G} h\big(\chi(\overline{u}) \circ \psi^{\gamma_1}_{r,r} v_{rr} u_r^* u_s v_{ss}\chi(v) \circ \psi^{\gamma_2}_{s,s} u_t v_{tt}\chi(w) \circ \psi^{\gamma_3}_{t,t}\big)\\ &= \sum_{r,s,t} h\big(u_{r^{-1}st}\alpha_{t^{-1}s^{-1}r}\big(\chi(\overline{u}) \circ \psi^{\gamma_1}_{r,r} v_{rr}\big)\alpha_{t^{-1}}\big(v_{ss}\chi(v) \circ \psi^{\gamma_2}_{s,s}\big)v_{tt}\chi(w) \circ \psi^{\gamma_3}_{t,t}\big)\\ &= \sum_{r\in\gamma_1\cdot G}\ \sum_{s\in\gamma_2\cdot G,\ t\in\gamma_3\cdot G,\ st=r} \int_G \chi(\overline{u}) \circ \psi^{\gamma_1}_{r,r} v_{rr}\alpha_{t^{-1}}\big(v_{ss}\chi(v) \circ \psi^{\gamma_2}_{s,s}\big)v_{tt}\chi(w) \circ \psi^{\gamma_3}_{t,t}\, d\nu\end{aligned}$$

$$= \sum_{r\in\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)} \frac{1}{|r\cdot G|}\int_{G_r} \chi(\overline{u})\circ\psi^{\gamma_1}_{r,r}\chi(v \underset{r}{\otimes} w)\, d\nu_r$$
$$= \frac{1}{|\gamma_1\cdot G|}\sum_{r\in\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)} \dim\left(\mathrm{Mor}_{G_r}(u\circ\psi^{\gamma_1}_{r,r}, v\underset{r}{\otimes} w)\right).$$

Note that, whenever $\gamma_1\cdot G\cap((\gamma_2\cdot G)(\gamma_3\cdot G))=\emptyset$, there are no non-zero terms in the sum above. ■

3.2. The induced representation

In this section, we explain how the induced representation may be viewed as a particular twisted tensor product.

For $\gamma\in\Gamma$ and $u: G_\gamma\to\mathcal{U}(H)$ is a unitary representation of G_γ, we define the induced representation

$$\mathrm{Ind}_\gamma^G(u) := \varepsilon_{G_{\gamma^{-1}}} \underset{1}{\otimes} u : G\to\mathcal{U}\big(l^2(\gamma\cdot G)\otimes H\big);\ g\mapsto \sum_{r,s\in\gamma\cdot G} e_{rs}\otimes v_{rs}(g)u\big(\psi^\gamma_{rs}(g)\big).$$

It follows from Theorem 3.2 that $\mathrm{Ind}_\gamma^G(u)$ is indeed a unitary representation of G. We collect some elementary and well-known facts about this representation in the following proposition. Note that, in property (3), we use the symbol $\mathrm{Res}^G_{G_\gamma}(u)$ for $u\in\mathrm{Rep}(G)$ to denote the restriction of u to a representation of G_γ. Hence, property (3) motivates the name induced representation for the representation $\mathrm{Ind}_\gamma^G(u)$.

Proposition 3.3. *The following hold.*

(1) *For all $\gamma\in\Gamma$ and all $u\in\mathrm{Rep}(G_\gamma)$, one has*

$$\chi\big(\mathrm{Ind}_\gamma^G(u)\big)(g) = \sum_{r\in\gamma\cdot G} v_{rr}(g)\chi(u)\big(\psi^\gamma_{rr}(g)\big) \quad \text{for all } g\in G.$$

(2) *For all $\gamma\in\Gamma$ and all $u,v\in\mathrm{Rep}(G_\gamma)$, one has $u\simeq v \implies \mathrm{Ind}_\gamma^G(u)\simeq\mathrm{Ind}_\gamma^G(v)$.*

(3) *For all $\gamma\in\Gamma$, $u\in\mathrm{Rep}(G)$, and $v\in\mathrm{Rep}(G_\gamma)$, one has* $\dim(\mathrm{Mor}_G(u,\mathrm{Ind}_\gamma^G(v))) = \dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(u),v))$.

Proof. (1) It is obvious, by definition of $\mathrm{Ind}_\gamma^G(u)$.

(2) If $u\simeq v$, then $\chi(u)=\chi(v)$. Hence, $\chi(\mathrm{Ind}_\gamma^G(u))=\chi(\mathrm{Ind}_\gamma^G(v))$ by (1). It follows that $\mathrm{Ind}_\gamma^G(u)\simeq\mathrm{Ind}_\gamma^G(v)$.

(3) Let $\gamma\in\Gamma$, $u\in\mathrm{Rep}(G)$, and $v\in\mathrm{Rep}(G_\gamma)$. One has

$$\dim\left(\mathrm{Mor}_G\left(u,\mathrm{Ind}_\gamma^G(v)\right)\right) = \sum_{r\in\gamma\cdot G}\int_G \chi(\overline{u})v_{rr}\chi(v)\circ\psi^\gamma_{rr}\,d\nu$$
$$= \frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma\cdot G}\int_{G_r}\chi(\overline{u})\chi(v)\circ\psi^\gamma_{rr}\,d\nu_\gamma.$$

Since $\psi^{\gamma}_{rr} : G_r \to G_\gamma$ is a Haar probability preserving a homeomorphism, we obtain

$$\dim\big(\mathrm{Mor}_G\big(u, \mathrm{Ind}_\gamma^G(v)\big)\big) = \frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma\cdot G}\int_{G_\gamma}\chi(\overline{u})\circ(\psi^{\gamma}_{rr})^{-1}\chi(v)\,d\nu_\gamma.$$

Finally, since, for all $g\in G$, $\chi(\overline{u})\circ(\psi^{\gamma}_{rr})^{-1}(g) = \chi(\overline{u})(g)$ (because $\chi(\overline{u})$ is a central function on G), it follows that

$$\begin{aligned}\dim\big(\mathrm{Mor}_G\big(u, \mathrm{Ind}_\gamma^G(v)\big)\big) &= \frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma\cdot G}\int_{G_\gamma}\chi(\overline{u})\chi(v)\,d\nu_\gamma\\ &= \dim\big(\mathrm{Mor}_{G_\gamma}\big(\mathrm{Res}^G_{G_\gamma}(u), v\big)\big).\end{aligned}$$ ∎

4. Length functions

Recall that, given a compact quantum group $\mathbb{H}$, a function $l : \mathrm{Irr}(\mathbb{H}) \to [0,\infty)$ is called a *length function on* $\mathrm{Irr}(\mathbb{H})$ if $l([\epsilon]) = 0$, $l(\overline{x}) = l(x)$, that $l(x) \le l(y) + l(z)$ whenever $x \subset y\otimes z$. A length function on a discrete group Λ is a function $l : \Lambda \to [0,\infty)$ such that $l(1) = 0$, $l(r) = l(r^{-1})$, and $l(rs) \le l(r) + l(s)$ for all $r, s\in\Lambda$.

Let (Γ, G) be a matched pair with a bicrossed product $\mathbb{G}$. In view of the description of the irreducible representations of $\mathbb{G}$, the fusion rules, and the contragredient representation, it is clear that, to get a length function on $\mathrm{Irr}(\mathbb{G})$, we need a family of maps $l_\gamma : \mathrm{Irr}(G_\gamma) \to [0,+\infty[$, for $\gamma\in\Gamma$, satisfying the hypothesis of the following definition.

Definition 4.1. Let (Γ, G) be a matched pair and let $l : \mathrm{Irr}(G) \to [0,+\infty[$ and $l_\Gamma : \Gamma \to [0,+\infty[$ be length functions. The pair (l, l_Γ) is *matched* if, for all $\gamma\in\Gamma$, there exists a function $l_\gamma : \mathrm{Irr}(G_\gamma) \to [0,+\infty[$ such that

(i) $l_1 = l$ and $l_\gamma(\varepsilon_{G_\gamma}) = l_\Gamma(\gamma)$;

(ii) for any $\gamma\in\Gamma$, $r\in\gamma\cdot G$, and $x\in\mathrm{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_r([u^x\circ\psi^{\gamma}_{r,r}])$;

(iii) for any $\gamma\in\Gamma$ and $x\in\mathrm{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_{\gamma^{-1}}([\overline{u^x}\circ\alpha_{\gamma^{-1}}])$;

(iv) for any $\gamma_1,\gamma_2,\gamma_3\in\Gamma$, $x\in\mathrm{Irr}(G_{\gamma_1})$, $y\in\mathrm{Irr}(G_{\gamma_2})$, and $z\in\mathrm{Irr}(G_{\gamma_3})$, if $\gamma_3\in(\gamma_1\cdot G)(\gamma_2\cdot G)$ and

$$\dim\mathrm{Mor}_{G_r}(u^z\circ\psi^{\gamma_3}_{r,r}, u^x\otimes_r u^y) \ne 0 \tag{4.1}$$

for some $r\in\gamma_3\cdot G$, then

$$l_{\gamma_3}(z) \le l_{\gamma_1}(x) + l_{\gamma_2}(y). \tag{4.2}$$

The next proposition shows that our notion of a matched pair for length functions is the good one, as expected.

Proposition 4.2. *Let* (Γ, G) *be a matched pair with a bicrossed product* $\mathbb{G}$.

(1) *If l is a length function on* $\mathrm{Irr}(\mathbb{G})$*, then the maps* $l_G : \mathrm{Irr}(G) = \mathrm{Irr}(G_1) \to [0, +\infty[$, $x \mapsto l([1(x)])$*, and* $l_\Gamma : \Gamma \to [0, +\infty[$, $\gamma \mapsto l([\gamma(\varepsilon_{G_\gamma})])$ *are length functions and the pair* (l_Γ, l_G) *is matched.*

(2) *If* l_Γ *is any* β*-invariant length function on* Γ*, then the map* $l' : \mathrm{Irr}(\mathbb{G}) \mapsto [0, +\infty[$, $[\gamma(u^x)] \mapsto l_\Gamma(\gamma)$ *is a well-defined length function on* $\mathrm{Irr}(\mathbb{G})$.

(3) *If* (l_Γ, l_G) *is a matched pair of length functions on* $(\Gamma, \mathrm{Irr}(G))$*, then* l_Γ *is* β*-invariant and the maps* $l, \widetilde{l} : \mathrm{Irr}(\mathbb{G}) \to [0, +\infty[$, $l([\gamma(u^x)]) := l_\gamma(x)$*, and* $\widetilde{l}([\gamma(u^x)]) := l_\gamma(x) + l_\Gamma(\gamma)$ *are well-defined length functions.*

Proof. (1) Since $1(\varepsilon_G)$ is the trivial representation of $\mathbb{G}$, one has $l_\Gamma(1) = 0$. Let $\gamma, \mu \in \Gamma$ and note that $\gamma\mu \in (\gamma \cdot G)(\mu \cdot G)$. Moreover,

$$\begin{aligned}\dim\big(\mathrm{Mor}(\varepsilon_{G_{\gamma\mu}}, \varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu})\big) &= \int_{G_{\gamma\mu}} \chi(\varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu})\, d\nu_{G_{\gamma\mu}} \\ &= |\gamma\mu \cdot G| \sum_{s \in \gamma\cdot G,\, t \in \mu\cdot G, st=\gamma\mu} \int_{G_{\gamma\mu}} (v_{ss} \circ \alpha_t) v_{tt}\, d\nu \\ &= |\gamma\mu \cdot G| \sum_{s \in \gamma\cdot G,\, t \in \mu\cdot G, st=\gamma\mu} \nu\big(\alpha_{t^{-1}}(G_s) \cap G_t \cap G_{\gamma\mu}\big) \\ &\geq \nu\big(\alpha_{\mu^{-1}}(G_\gamma) \cap G_\mu \cap G_{\gamma\mu}\big).\end{aligned}$$

Hence, since $\alpha_{\mu^{-1}}(G_\gamma) \cap G_\mu \cap G_{\gamma\mu}$ is open and non-empty (it contains 1), we deduce that

$$\dim\big(\mathrm{Mor}(\varepsilon_{G_{\gamma\mu}}, \varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu})\big) > 0.$$

So, $\varepsilon_{G_{\gamma\mu}} \subset \varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu}$ and, by the fusion rules of $\mathbb{G}$ in Theorem 3.2,

$$(\gamma\mu)(\varepsilon_{G_{\gamma\mu}}) \subset \gamma(\varepsilon_{G_\gamma}) \otimes \mu(\varepsilon_{G_\mu}).$$

Hence, since l is a length function,

$$l_\Gamma(\gamma\mu) = l\big(\big[\gamma\mu(\varepsilon_{G_{\gamma\mu}})\big]\big) \leq l\big(\big[\gamma(\varepsilon_{G_\gamma})\big]\big) + l\big(\big[\mu(\varepsilon_{G_\mu})\big]\big) = l_\Gamma(\gamma) + l_\Gamma(\mu).$$

Finally, note that, by point (4) of Theorem 3.1, for all $\gamma \in \Gamma$, one has $\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}}) \simeq \overline{\gamma(\varepsilon_G)}$. Hence,

$$l_\Gamma(\gamma^{-1}) = l\big(\big[\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}})\big]\big) = l\big(\big[\overline{\gamma(\varepsilon_G)}\big]\big) = l\big(\big[\gamma(\varepsilon_G)\big]\big) = l_\Gamma(\gamma).$$

So, l_Γ is a length function on Γ. It is obvious that l_G is a length function on $\mathrm{Irr}(G)$. Let us prove that the pair (l_Γ, l_G) is matched. Indeed, defining $l_\gamma : \mathrm{Irr}(G_\gamma) \to [0, +\infty[$ by $l_\gamma(x) = l([\gamma(u^x)])$, point (i) of Definition 4.1 is clear while point (ii) follows from point (3) of Theorem 3.1, since it implies $[\gamma(u^x)] = [r(u^x \circ \psi^r_{\gamma,\gamma})]$, and thus

$$l_\gamma(x) = l\big(\big[\gamma(u^x)\big]\big) = l\big(\big[r(u^x \circ \psi^r_{\gamma,\gamma})\big]\big) = l_r\big(\big[u^x \circ \psi^r_{\gamma,\gamma}\big]\big).$$

Next, by point (4) of Theorem 3.1, we have $\overline{[\gamma(u^x)]} = [\gamma^{-1}(\overline{u^x}) \circ \alpha_{\gamma^{-1}}]$. Thus,

$$l_\gamma(x) = l\big(\overline{[\gamma(u^x)]}\big) = l\big([\gamma^{-1}(\overline{u^x}) \circ \alpha^{-1}]\big) = l_{\gamma^{-1}}\big([\overline{u^x} \circ \alpha^{-1}]\big),$$

which proves point (ii) of Definition 4.1. Finally, for point (iv), the fusion rules in Theorem 3.2 imply

$$\begin{aligned} &\dim \mathrm{Mor}\big(\gamma_3(u^z), \gamma_1(u^x) \otimes \gamma_2(u^y)\big) \\ &\quad = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma_3 \cdot G} \dim \mathrm{Mor}_{G_r}(u^z \circ \psi^{\gamma_3}_{r,r}, u^x \otimes_r u^y). \end{aligned} \tag{4.3}$$

If $\dim \mathrm{Mor}_{G_r}(u^z \circ \psi^{\gamma_3}_{r,r}, u^x \otimes_r u^y) \neq 0$ for some $r \in \gamma_3 \cdot G$, then (4.3) is also non-zero, which means by irreducibility of $\gamma_3(u^z)$ that $[\gamma_3(u^z)] \subseteq [\gamma_1(u^x)] \otimes [\gamma_2(u^y)]$. Hence, since l is a length function on $\mathrm{Irr}(\mathbb{G})$,

$$l_{\gamma_3}(z) = l\big([\gamma_3(u^z)]\big) \le l\big([\gamma_1(u^x)]\big) + l\big([\gamma_2(u^y)]\big) = l_{\gamma_1}(x) + l_{\gamma_2}(y).$$

(2) Since l_Γ is β-invariant, the map l' is well defined by Theorem 3.1. It is clear that $l'(\varepsilon_{\mathbb{G}}) = 0$ and, by point (4) (and (5)) of Theorem 3.1 and since l' is a length function, we also have that $l'(z) = l'(\overline{z})$ for all $z \in \mathrm{Irr}(\mathbb{G})$. Let now $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, $x \in \mathrm{Irr}(G_{\gamma_1})$, $y \in \mathrm{Irr}(G_{\gamma_2})$, and $z \in \mathrm{Irr}(G_{\gamma_3})$ be such that $\gamma_1(u^x) \subset \gamma_2(u^y) \otimes \gamma_3(u^z)$. Then, by point (3) in Theorem 3.2, there exist $r \in \gamma_1 \cdot G$, $s \in \gamma_2 \cdot G$, and $t \in \gamma_3 \cdot G$ such that $r = st$ (and $u^x \circ \psi^{\gamma_1}_{r,r} \subset u^y \underset{r}{\otimes} u^z$). Then,

$$\begin{aligned} l'\big([\gamma_1(u^x)]\big) &= l_\Gamma(\gamma_1) = l_\Gamma(r) \le l_\Gamma(s) + l_\Gamma(t) \\ &= l_\Gamma(\gamma_2) + l_\Gamma(\gamma_3) = l'\big([\gamma_2(u^y)]\big) + l'\big([\gamma_3(u^z)]\big). \end{aligned}$$

(3) Let (l_Γ, l_G) be a matched pair of length functions. By points (i) and (ii) of Definition 4.1, we have, for all $\gamma \in \Gamma$ and all $r \in \gamma \cdot G$, $l_\Gamma(\gamma) = l_\gamma(\varepsilon_{G_\gamma}) = l_r([\varepsilon_{G_\gamma} \circ \psi^\gamma_{r,r}]) = l_r(\varepsilon_{G_r}) = l_\Gamma(r)$. Hence, l_Γ is β-invariant. By assertion (2) we just proved above, we get a length function l' on $\mathrm{Irr}(\mathbb{G})$. Now, it is clear from Definition 4.1, the fusion rules, and the adjoint representation of a bicrossed product (point (3) of Theorem 3.2 and point (4) of Theorem 3.1) that $l : [\gamma(u^x)] \mapsto l_\gamma(x)$ is a length function on $\mathrm{Irr}(\mathbb{G})$. Since $\widetilde{l} = l + l'$, $\widetilde{l}$ is also a length function on $\mathrm{Irr}(\mathbb{G})$. ∎

5. Rapid decay and polynomial growth

In this section, we study property (RD) and polynomial growth for (the dual of) bicrossed products.

5.1. Generalities

We use the notion of property (RD) developed by Vergnioux in [11] (see also [2]) and recall the definition below. Since we are only dealing with Kac algebras, we recall the definition of the Fourier transform and rapid decay only for Kac algebras.

Let $\mathbb{H}$ be a compact quantum group. We use the notation $l^\infty(\widehat{\mathbb{H}}) := \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})} \mathcal{B}(H_x)$ to denote the l^∞ direct sum. The c_0 direct sum is denoted by $c_0(\widehat{\mathbb{H}}) \subset l^\infty(\widehat{\mathbb{H}})$ and the algebraic direct sum is denoted by $c_c(\widehat{\mathbb{H}}) \subset c_0(\widehat{\mathbb{H}})$. An element $a \in c_c(\widehat{\mathbb{H}})$ is said to have finite support and its finite support is denoted by $\mathrm{Supp}(a) := \{x \in \mathrm{Irr}(\mathbb{H}) : ap_x \neq 0\}$, where p_x, for $x \in \mathrm{Irr}(\mathbb{H})$, denotes the central minimal projection of $l^\infty(\widehat{\mathbb{H}})$ corresponding to the block $\mathcal{B}(H_x)$.

For a compact quantum group $\mathbb{H}$ which is always supposed to be of Kac type and $a \in C_c(\widehat{\mathbb{H}})$, we define its Fourier transform as

$$\mathcal{F}_{\mathbb{H}}(a) = \sum_{x\in\mathrm{Irr}(\mathbb{H})} \dim(x)(\mathrm{Tr}_x \otimes \mathrm{id})\big(u^x(ap_x \otimes 1)\big) \in \mathrm{Pol}(\mathbb{H})$$

and its "Sobolev 0-norm" by $\|a\|^2_{\mathbb{H},0} = \sum_{x\in\mathrm{Irr}(\mathbb{H})} \dim(x)\,\mathrm{Tr}_x((a^*a)p_x)$.

Given a length function $l : \mathrm{Irr}(\mathbb{H}) \to [0,\infty)$, consider the element $L = \sum_{x\in\mathrm{Irr}(\mathbb{H})} l(x)p_x$ which is affiliated to $c_0(\widehat{\mathbb{H}})$. Let q_n denote the spectral projections of L associated to the interval $[n, n+1)$.

The pair $(\widehat{\mathbb{H}}, l)$ is said to have

- *polynomial growth* if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ one has
$$\sum_{x\in\mathrm{Irr}(\mathbb{H}),\, k\le l(x)<k+1} \dim(x)^2 \le P(k);$$
- *property* (RD) if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k c_c(\widehat{\mathbb{H}})$ we have $\|\mathcal{F}(a)\|_{C(\mathbb{H})} \le P(k)\|a\|_{\mathbb{H},0}$.

Finally, $\widehat{\mathbb{H}}$ is said to have *polynomial growth* (resp. *property* (RD)) if there exists a length function l on $\mathrm{Irr}(\mathbb{H})$ such that $(\widehat{\mathbb{H}}, l)$ has polynomial growth (resp. property (RD)).

It is known from [11] that if $(\widehat{\mathbb{H}}, l)$ has polynomial growth, then $(\widehat{\mathbb{H}}, l)$ has a rapid decay and the converse also holds when we assume $\mathbb{H}$ to be co-amenable. Moreover, it is also shown in [11] that duals of compact connected real Lie groups have polynomial growth. The fact that polynomial growth implies (RD) can easily be deduced from the following lemma.

Lemma 5.1. *Let $\mathbb{H}$ be a CQG, $F \subset \mathrm{Irr}(\mathbb{H})$ a finite subset, and $a \in l^\infty(\widehat{\mathbb{H}})$ with $ap_x = 0$, for all $x \notin F$. Then,*

$$\big\|\mathcal{F}_{\mathbb{H}}(a)\big\| \le 2\sqrt{\sum_{x\in F}\dim(x)^2}\,\|a\|_{\mathbb{H},0}.$$

Proof. One can copy the proof of Proposition 4.2, assertion (a), in [2] or the proof of Proposition 4.4, assertion (ii), in [11]. ■

5.2. Technicalities

Let (Γ, G) be a matched pair with actions (α, β) and denote by $\mathbb{G}$ the bicrossed product.

Recall that $\mathrm{Irr}(\mathbb{G}) = \sqcup_{\gamma \in I} \mathrm{Irr}(G_\gamma)$, where $I \subset \Gamma$ is a complete set of representatives for Γ/G. For $\gamma \in I$ and $x \in \mathrm{Irr}(G_\gamma)$, we denote by $\gamma(x)$ the corresponding element in $\mathrm{Irr}(\mathbb{G})$. If a complete set of representatives of $\mathrm{Irr}(G_\gamma)$, $x \in \mathrm{Irr}(G_\gamma)$ is given by $u^x \in \mathcal{B}(H_x) \otimes C(G_\gamma)$, then a representative for $\gamma(x)$ is given by

$$u^{\gamma(x)} := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u^x \circ \psi_{r,s} \in \mathcal{B}\big(l^2(\gamma \cdot G)\big) \otimes C(\mathbb{G}).$$

The following lemma gives a way of obtaining an element $\widetilde{a} \in c_c(\widehat{G})$ from an $a \in c_c(\widehat{G_\gamma})$ in a suitable way so that they are compatible with the Fourier transforms.

Lemma 5.2. *Let $\gamma \in \Gamma$ and $a \in c_c(\widehat{G}_\gamma)$. Define $\widetilde{a} \in c_c(\widehat{G})$ by*

$$\widetilde{a} p_y = \sum_{x \in \mathrm{supp}(a),\ y \subset \mathrm{Ind}_\gamma^G(x)} \frac{\dim(x)}{\dim(y)} \sum_{i=1}^{\dim(\mathrm{Mor}_G(y, \mathrm{Ind}_\gamma^G(x)))} (S_i^y)^* (e_{\gamma\gamma} \otimes a p_x) S_i^y,$$

where $S_i^y \in \mathrm{Mor}(y, \mathrm{Ind}_\gamma^G(x))$ is a basis of isometries with pairwise orthogonal images. The following hold.

(1) *If (l_Γ, l) is a matched pair of length functions on $(\Gamma, \mathrm{Irr}(G))$, then, for all $y \in \mathrm{supp}(\widetilde{a})$, one has*

$$l(y) \leq \max\big(\{l_\gamma(x) : x \in \mathrm{supp}(a)\}\big) + l_\Gamma(\gamma),$$

where $(l_\gamma)_{\gamma \in \Gamma}$ is any family of maps realizing the compatibility relations of Definition 4.1.

(2) $\mathcal{F}_{G_\gamma}(a) = v_{\gamma\gamma} \mathcal{F}_G(\widetilde{a})$.

(3) $\|\widetilde{a}\|_{G,0} \leq \|a\|_{G_\gamma,0}$.

Proof. (1) Since any $y \in \mathrm{supp}(\widetilde{a})$ is such that $y \subset \mathrm{Ind}_\gamma^G(x) = \varepsilon_{G_{\gamma^{-1}}} \underset{1}{\otimes} x$ for some $x \in \mathrm{supp}(a)$, it follows that any $y \in \mathrm{supp}(\widetilde{a})$ satisfies

$$l(y) = l_1(y) \leq l_{\gamma^{-1}}(\varepsilon_{G_{\gamma^{-1}}}) + l_\gamma(x) = l_\Gamma(\gamma^{-1}) + l_\gamma(x) = l_\Gamma(\gamma) + l_\gamma(x)$$

for some $x \in \mathrm{supp}(a)$.

(2) One has

$$v_{\gamma\gamma} \mathcal{F}_G(\widetilde{a}) = v_{\gamma\gamma} \sum_y \dim(y)(\mathrm{Tr}_y \otimes \mathrm{id})(u^y \widetilde{a} p_y \otimes 1)$$

$$= v_{\gamma\gamma} \sum_{x \in \mathrm{supp}(a),\ y \subset \mathrm{Ind}_\gamma^G(x)} \sum_{i=1}^{\dim(\mathrm{Mor}(y, \mathrm{Ind}_\gamma^G(x)))} \dim(x)(\mathrm{Tr}_y \otimes \mathrm{id})\big(u^y \big((S_i^y)^* (e_{\gamma\gamma} \otimes a p_x) S_i^y\big) \otimes 1\big)$$

$$= v_{\gamma\gamma} \sum_{x,y,i} \dim(x)(\mathrm{Tr}_y \otimes \mathrm{id})\big(\big((S_i^y)^* \otimes 1\big) \mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes a p_x \otimes 1)(S_i^y \otimes 1)\big)$$

$$= v_{\gamma\gamma} \sum_{x,y,i} \dim(x)(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x} \otimes \mathrm{id})\big(\mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes ap_x \otimes 1)\big(S_i^y(S_i^y)^* \otimes 1\big)\big)$$
$$= v_{\gamma\gamma} \sum_{x\in\mathrm{supp}(a)} \dim(x)(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x} \otimes \mathrm{id})\big(\mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes ap_x \otimes 1)\big)$$
$$= v_{\gamma\gamma} \sum_{x\in\mathrm{supp}(a)} \dim(x)(\mathrm{Tr}_x \otimes \mathrm{id})(u^x ap_x \otimes 1) = \mathcal{F}_{G_\gamma}(a),$$

where, in the 3rd equation, we use the fact that $(S_i^y)^* \in \mathrm{Mor}(\mathrm{Ind}_\gamma^G(x), y)$ and, in the last equation, we use the definition of the representation $\mathrm{Ind}_\gamma^G(u^x)$.

(3) One has

$$\|\tilde{a}\|_{G,0}^2 = \sum_y \dim(y)\,\mathrm{Tr}_y(\tilde{a}^*\tilde{a}p_y)$$
$$= \sum_{x\in\mathrm{supp}(a),\ y\subset\mathrm{Ind}_\gamma^G(x)} \sum_{i,j=1}^{\dim(\mathrm{Mor}(y,\mathrm{Ind}_\gamma^G(x)))} \dim(y)\frac{\dim(x)^2}{\dim(y)^2}$$
$$\times \mathrm{Tr}_y\left((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_j^y)^*(e_{\gamma\gamma}\otimes ap_x)S_j^y\right)$$
$$= \sum_{x,y,i} \dim(x)\left(\frac{\dim(x)}{\dim(y)}\right)\mathrm{Tr}_y\left((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y\right).$$

Since, for all y, i, $S_i^y(S_i^y)^*$ is a projection, one has $S_i^y(S_i^y)^* \leq 1$. Hence,

$$\mathrm{Tr}_y\left((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y\right) \leq \mathrm{Tr}_y\left((S_i^y)^*(e_{\gamma\gamma}\otimes a^*ap_x)S_i^y\right).$$

Moreover, by Proposition 3.3, one has $y \subset \mathrm{Ind}_\gamma^G(x)$ if and only if

$$\dim\left(\mathrm{Mor}_{G_\gamma}\left(\mathrm{Res}_{G_\gamma}^G(y), x\right)\right) = \dim\left(\mathrm{Mor}_G\left(y, \mathrm{Ind}_\gamma^G(x)\right)\right) \neq 0.$$

Since x is irreducible, we find that $y \subset \mathrm{Ind}_\gamma^G(x) \Leftrightarrow x \subset \mathrm{Res}_{G_\gamma}^G(y)$. In particular, for any $y \subset \mathrm{Ind}_\gamma^G(x)$, one has $\dim(x) \leq \dim(y)$. Hence,

$$\|\tilde{a}\|_{G,0}^2 \leq \sum_{x,y,i} \dim(x)\,\mathrm{Tr}_y\left((S_i^y)^*(e_{\gamma\gamma}\otimes a^*ap_x)S_i^y\right)$$
$$= \sum_{x,y,i} \dim(x)\,\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}\left(e_{\gamma\gamma}\otimes a^*ap_x(S_i^y)^*S_i^y\right)$$
$$= \sum_{x\in\mathrm{supp}(a)} \dim(x)\,\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}(e_{\gamma\gamma}\otimes a^*ap_x)$$
$$= \sum_{x\in\mathrm{supp}(a)} \dim(x)\,\mathrm{Tr}_x(a^*ap_x) = \|a\|_{G_\gamma,0}^2. \qquad \blacksquare$$

Lemma 5.3. *Let* (l_Γ, l) *be a matched pair of length functions on* $(\Gamma, \mathrm{Irr}(G))$. *If* $(\widehat{G}, l)$ *has polynomial growth, then there exist* $C > 0$ *and* $N \in \mathbb{N}$ *such that*

- $\|\mathcal{F}_G(a)\| \leq C(k+1)^N \|a\|_{G,0}$ *for all* $a \in c_c(\widehat{G})$ *with* $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : l(x) < k+1\}$*;*
- $|\gamma \cdot G| \dim(x) \leq C(l_\Gamma(\gamma) + l_\gamma(x) + 1)^N$ *for all* $\gamma \in \Gamma$*,* $x \in \mathrm{Irr}(G_\gamma)$*;*
- *for all* $\gamma \in \Gamma$*,* $\sum_{x \in \mathrm{Irr}(G_\gamma), l_\gamma(x) < k+1} \dim(x)^2 \leq C^2(k + l_\Gamma(\gamma) + 1)^{2N}$.

Proof. Let $P \in \mathbb{R}[X]$ be such that $\sum_{x \in \mathrm{Irr}(G),\, k \leq l(x) < k+1} \dim(x)^2 \leq P(k)$ for all $k \in \mathbb{N}$ and let $C_1 > 0$ and $N_1 \in \mathbb{N}$ be such that $P(k) \leq C_1(k+1)^{N_1}$ for all $k \in \mathbb{N}$. By Lemma 5.1, one has, for all $a \in c_c(\widehat{G})$, with $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : k \leq l(x) < k+1\}$,

$$\left\|\mathcal{F}_G(a)\right\| \leq 2\sqrt{P(k)}\|a\|_{G,0} \leq \sqrt{C_1}(k+1)^{\frac{N_1}{2}}\|a\|_{G,0}.$$

Now, suppose that $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : l(x) < k+1\}$ so that $a \in q_k c_c(\widehat{G})$, where $q_k = \sum_{j=0}^k p_j$ and $p_j = \sum_{x \in \mathrm{Irr}(G),\, k \leq l(x) < k+1}$. One has

$$\begin{aligned}\left\|\mathcal{F}_G(a)\right\| = \sum_{j=0}^{k} \left\|\mathcal{F}_G(ap_j)\right\| &\leq \sum_{j=0}^{k} \sqrt{C_1}(j+1)^{\frac{N_1}{2}}\|a\|_{G,0} \\ &\leq \sqrt{C_1}(k+1)^{\frac{N_1}{2}+1}\|a\|_{G,0}. \end{aligned} \tag{5.1}$$

Now, let $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G_\gamma)$. By Proposition 3.3, one has

$$\begin{aligned}|\gamma \cdot G| \dim(x) = \dim\left(\mathrm{Ind}_\gamma^G(x)\right) &= \sum_{y \in \mathrm{Irr}(G)} \dim\left(\mathrm{Mor}_G\left(y, \mathrm{Ind}_\gamma^G(x)\right)\right)\dim(y) \\ &= \sum_{y \in \mathrm{Irr}(G),\, y \subset \mathrm{Ind}_\gamma^G(x)} \dim\left(\mathrm{Mor}_{G_\gamma}\left(\mathrm{Res}_{G_\gamma}^G(y), x\right)\right)\dim(y).\end{aligned}$$

Note that $\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}_{G_\gamma}^G(y), x)) \leq \dim(y)$ for all x, y. Moreover, since $\mathrm{Ind}_\gamma^G(x) \simeq \varepsilon_{G_{\gamma^{-1}}} \underset{1}{\otimes} x$ and the pair (l_Γ, l) is matched, one has

$$\left\{y \in \mathrm{Irr}(G), y \subset \mathrm{Ind}_\gamma^G(x)\right\} \subset \left\{y \in \mathrm{Irr}(G) : l(y) \leq l_\Gamma(\gamma) + l_\gamma(x)\right\}.$$

Hence,

$$\begin{aligned}|\gamma \cdot G| \dim(x) &\leq \sum_{y \in \mathrm{Irr}(G), l(y) < l_\Gamma(\gamma) + l_\gamma(x) + 1} \dim(y)^2 \\ &= \sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)} \sum_{y \in \mathrm{Irr}(G),\, j \leq l(y) < j+1} \dim(y)^2 \\ &\leq \sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)} P(j) \leq C_1 \sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)} (j+1)^{N_1} \\ &\leq C_1\left(l_\Gamma(\gamma) + l_\gamma(x) + 1\right)^{N_1+1}. \end{aligned} \tag{5.2}$$

It follows from (5.1) and (5.2) that $C := \mathrm{Max}(C_1, \sqrt{C_1})$ and $N := N_1 + 1$ do the job.

Let us show the last point. Fix $\gamma \in \Gamma$ and let $F \subset \mathrm{Irr}(G_\gamma)$ be a finite subset. Define $p_F \in c_c(\widehat{G}_\gamma)$ by $p_F = \sum_{x\in F} p_x$ and note that $\mathcal{F}_{G_\gamma}(p_F) = \sum_{x\in F} \dim(x)\chi(x)$ and $\|a\|^2_{G_\gamma,0} = \sum_{x\in F} \dim(x)^2$. Suppose that $F \subset \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$. Using Lemma 5.2 and the first part of the proof, we find, since $\widetilde{p_F} \in c_c(\widehat{G})$ with $\mathrm{supp}(\widetilde{p_F}) \subset \{x \in \mathrm{Irr}(G) : l(x) < l_\Gamma(\gamma) + k + 1\}$,

$$\begin{aligned}\Big\|\sum_{x\in F}\dim(x)\chi(x)\Big\|^2 = \big\|\mathcal{F}_{G_\gamma}(p_F)\big\|^2 = \big\|v_{\gamma\gamma}\mathcal{F}_G(\widetilde{p_F})\big\|^2 \le \big\|\mathcal{F}_G(\widetilde{p_F})\big\|^2 \\ \le C^2\big(k + l_\Gamma(\gamma) + 1\big)^{2N}\|\widetilde{p_F}\|^2_{G,0} \\ \le C^2\big(k + l_\Gamma(\gamma) + 1\big)^{2N}\|p_F\|^2_{G_\gamma,0} \\ = C^2\big(k + l_\Gamma(\gamma) + 1\big)^{2N}\sum_{x\in F}\dim(x)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\Big(\sum_{x\in F}\dim(x)^2\Big)^2 = \Big(\sum_{x\in F}\dim(x)\chi(x)(1)\Big)^2 \le \Big\|\sum_{x\in F}\dim(x)\chi(x)\Big\|^2_{C(G)} \\ \le C^2\big(k + l_\Gamma(\gamma) + 1\big)^{2N}\sum_{x\in F}\dim(x)^2.\end{aligned}$$

Hence, for all non-empty finite subsets $F \subset \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$, one has $\sum_{x\in F}\dim(x)^2 \le C^2(k + l_\Gamma(\gamma) + 1)^{2N}$. The last assertion follows. ∎

5.3. Polynomial growth for bicrossed product

We start with the following result.

Theorem 5.4. *Suppose that (l_G, l_Γ) is a matched pair of length functions on (Γ, G). If both (Γ, l_Γ) and $(\widehat{G}, l_G)$ have polynomial growth, then $(\mathbb{G}, \widetilde{l})$ has polynomial growth.*

Proof. Let $I \subset \Gamma$ be a complete set of representatives for Γ/G so that $\mathrm{Irr}(\mathbb{G}) = \sqcup_{\gamma\in I}\mathrm{Irr}(G_\gamma)$. Let $k \ge 1$ and define

$$F_k := \big\{z \in \mathrm{Irr}(\mathbb{G}) : \widetilde{l}(z) < k\big\} \subset \sqcup_{\gamma\in I_k} T_{\gamma,k},$$

where $I_k := \{\gamma \in \Gamma : l_\Gamma(\gamma) < k+1\} \cap I$ and $T_{\gamma,k} := \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$. Since (Γ, l_Γ) has polynomial growth, there exists a polynomial P_1 such that, for all $k \in \mathbb{N}$, $|I_k| \le P_1(k)$. Moreover, since $(\widehat{G}, l_G)$ has polynomial growth, we can apply Lemma 5.3 to get $C > 0$ and $N \in N$ such that, for all $k \in \mathbb{N}$ and all $\gamma \in I_k$, one has $\sum_{x\in T_{\gamma,k}}\dim(x)^2 \le C^2(2k+2)^{2N}$ and $|\gamma\cdot G| = |\gamma\cdot G|\dim(\varepsilon_G) \le C(2k+3)^N$. Hence, for all $k \ge 1$,

$$\begin{aligned}\sum_{z\in F_k}\dim(z)^2 = \sum_{\gamma\in I_k}|\gamma\cdot G|^2\sum_{x\in T_{\gamma,k}}\dim(x)^2 \le C^2(2k+2)^{2N}\sum_{\gamma\in I_k}|\gamma\cdot G|^2 \\ \le C^4(2k+2)^{2N}(2k+3)^{2N}|I_k| \\ \le C^4(2k+2)^{2N}(2k+3)^{2N}P_1(k).\end{aligned}$$

∎

To complete the proof of Theorem 2, we need the following proposition.

Proposition 5.5. *Assume that there exists a length function l on* $\mathrm{Irr}(\mathbb{G})$ *such that* $(\widehat{\mathbb{G}}, l)$ *has polynomial growth and consider the matched pair of length functions* (l_Γ, l_G) *associated to l given in Proposition 4.2. Then,* (Γ, l_Γ) *and* $(\widehat{G}, l_G)$ *both have polynomial growth.*

Proof. Assume that $(\widehat{\mathbb{G}}, l)$ has polynomial growth. Since the map $\mathrm{Irr}(G) \to \mathrm{Irr}(\mathbb{G})$, $x \mapsto 1(x)$ is injective, dimension preserving, and length preserving (by definition of l_G), it is clear that $(\widehat{G}, l_G)$ has polynomial growth. Let us show that (Γ, l_Γ) also has polynomial growth. Let P be a polynomial witnessing (RD) for $(\widehat{\mathbb{G}}, l)$ and $k \in \mathbb{N}$. Define $F_k := \{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$. One has, for all $k \in \mathbb{N}$,

$$\begin{aligned}|F_k| &= \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} 1 \leq \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} |\gamma \cdot G|^2 \\ &= \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} \dim\left([\gamma(\varepsilon_G)]\right)^2 \leq \sum_{z \in \mathrm{Irr}(\mathbb{G}),\, k \leq l(z) < k+1} \dim(z)^2 \leq P(k).\end{aligned}$$ ∎

5.4. Rapid decay for bicrossed product

Recall that $l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\gamma \cdot G \in \Gamma/G} \bigoplus_{x \in \mathrm{Irr}(G_\gamma)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$. Let us denote by $p_{\gamma(x)}$ the central projection of $l^\infty(\widehat{\mathbb{G}})$ corresponding to the block $\mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$ and define, for $\gamma \cdot G \in \Gamma/G$, the central projection

$$p_\gamma := \sum_{x \in \mathrm{Irr}(G_\gamma)} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}}).$$

Note that $p_\gamma l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{x \in \mathrm{Irr}(G_\gamma)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x) \simeq \mathcal{B}(l^2(\gamma \cdot G)) \otimes L(G_\gamma)$, where $L(G_\gamma) = \bigoplus_{x \in \mathrm{Irr}(G_\gamma)} \mathcal{B}(H_x)$ is the group von-Neumann algebra of G_γ (which is also the multiplier C^*-algebra of $C_r^*(G_\gamma) = \bigoplus^{c_0}_{x \in \mathrm{Irr}(G_\gamma)} \mathcal{B}(H_x)$). Using this identification, we define $\pi_\gamma : c_0(\widehat{\mathbb{G}}) \to \mathcal{B}(l^2(\gamma \cdot G)) \otimes C_r^*(G_\gamma) \subset c_0(\widehat{\mathbb{G}})$ to be the $*$-homomorphism given by $\pi_\gamma(a) = a p_\gamma$, for all $a \in c_0(\widehat{\mathbb{G}})$. We also write, for $a \in c_0(\widehat{\mathbb{G}})$, $\pi_\gamma(a) = \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes \pi^\gamma_{r,s}(a)$, where we recall that (e_{rs}) are the matrix units associated to the canonical orthonormal basis $(e_r)_{r \in \gamma \cdot G}$ of $l^2(\gamma \cdot G)$ and $\pi^\gamma_{r,s} : c_0(\widehat{\mathbb{G}}) \to C_r^*(G_\gamma)$ is the completely bounded map defined by $\pi^\gamma_{r,s} := (\omega_{e_s,e_r} \otimes \mathrm{id}) \circ \pi_\gamma$ and $\omega_{e_s,e_r} \in \mathcal{B}(l^2(\gamma \cdot G))$, $\omega_{e_s,e_r}(T) = \langle T e_s, e_r \rangle$.

We start with the following result.

Theorem 5.6. *Let* (l_Γ, l_G) *be a matched pair of length functions on* $(\Gamma, \mathrm{Irr}(G))$*. Suppose that* $(\widehat{G}, l_G)$ *has polynomial growth and* (Γ, l_Γ) *has (RD). Then,* $(\widehat{\mathbb{G}}, \widetilde{l})$ *has (RD).*

Proof. Let $a \in c_c(\widehat{\mathbb{G}})$ and write $a = \sum_{\gamma \in S} \sum_{x \in T_\gamma} a p_{\gamma(x)}$, where $S \subset I$ and $T_\gamma \subset \mathrm{Irr}(G_\gamma)$ are finite subsets.

Claim. *The following hold.*

(1) $\mathcal{F}_{\mathbb{G}}(a) = \sum_{\gamma \in S} |\gamma \cdot G| (\sum_{r,s \in \gamma \cdot G} u_r v_{rs} \mathcal{F}_{G_\gamma}(\pi^\gamma_{s,r}(a)) \circ \psi^\gamma_{r,s})$.

(2) $\|a\|_{\mathbb{G},0}^2 = \sum_{\gamma\in S} |\gamma\cdot G|(\sum_{r,s\in\gamma\cdot G} \|\pi_{r,s}^{\gamma}(a)\|_{G_\gamma,0}^2)$.

Proof of the claim. (1) A direct computation gives

$$\begin{aligned}\mathcal{F}_{\mathbb{G}}(a) &= \sum_{\gamma\in S,\, x\in T_\gamma} |\gamma\cdot G|\dim(x)\big(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}\otimes \mathrm{id}\big)\big(\gamma(u^x)ap_x\otimes 1\big)\\ &= \sum_{\gamma\in S,\, x\in T_\gamma} |\gamma\cdot G|\dim(x)\sum_{r,s\in\gamma\cdot G} u_r v_{rs}(\mathrm{Tr}_x\otimes\mathrm{id})\big(u^x\circ\psi_{r,s}^{\gamma}\pi_{s,r}^{\gamma}(a)p_x\otimes 1\big)\\ &= \sum_{\gamma\in S}|\gamma\cdot G|\sum_{r,s\in\gamma\cdot G} u_r v_{rs}\mathcal{F}_{G_\gamma}\big(\pi_{s,r}^{\gamma}(a)\big)\circ\psi_{r,s}^{\gamma}.\end{aligned}$$

(2) Since π_γ is a $*$-homomorphism, we have $\pi_{r,s}^{\gamma}(a^*a) = \sum_{t\in\gamma\cdot G}\pi_{t,r}^{\gamma}(a)^*\pi_{t,s}^{\gamma}(a)$. Hence,

$$\begin{aligned}\|a\|_{\mathbb{G},0}^2 &= \sum_{\gamma\in S,\, x\in T_\gamma}|\gamma\cdot G|\dim(x)\sum_{r,s\in\gamma\cdot G}(\mathrm{Tr}_x\otimes\mathrm{id})\big(\pi_{s,r}^{\gamma}(a)^*\pi_{r,s}^{\gamma}(a)\big)\\ &= \sum_{\gamma\in S}|\gamma\cdot G|\sum_{r,s\in\gamma\cdot G}\big\|\pi_{r,s}^{\gamma}(a)\big\|_{G_\gamma,0}^2. \qquad\blacksquare\end{aligned}$$

Let us now prove the theorem. Let $b = \sum_{\gamma\in S'}\sum_{t,t'\in\gamma\cdot G} u_t v_{tt'}F_\gamma\circ\psi_{t,t'}^{\gamma}\in C(\mathbb{G})$, where $F_\gamma\in C(G_\gamma)$ and $S'\subset I$ is a finite subset. For all $r\in\Gamma$, we denote by γ_r the unique element in I such that $\gamma_r\cdot G = r\cdot G$. We may re-order the sums and write

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|\Bigg(\sum_{s\in r\cdot G} u_r v_{rs}\mathcal{F}_{G_{\gamma_r}}\big(\pi_{s,r}^{\gamma_r}(a)\big)\circ\psi_{r,s}^{\gamma_r}\Bigg)$$

and

$$b = \sum_{t\in\Gamma} u_t 1_{S'\cdot G}(t)\Bigg(\sum_{t'\in t\cdot G} v_{tt'}F_{\gamma_t}\circ\psi_{t,t'}^{\gamma_t}\Bigg).$$

Also, $\|a\|_{\mathbb{G},0}^2 = \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|(\sum_{s\in r\cdot G}\|\pi_{r,s}^{\gamma_r}(a)\|_{G_{\gamma_r},0}^2)$. Then, $\|\mathcal{F}_{\mathbb{G}}(a)b\|_{2,h_{\mathbb{G}}}^2$ is equal to

$$\begin{aligned}&\Bigg\|\sum_{r,t\in\Gamma} u_{rt}1_{S\cdot G}(r)1_{S'\cdot G}(t)\\ &\qquad\times|r\cdot G|\Bigg(\sum_{s\in r\cdot G,t'\in t\cdot G} v_{rs}\circ\alpha_t\mathcal{F}_{G_{\gamma_r}}\big(\pi_{s,r}^{\gamma_r}(a)\big)\circ\psi_{r,s}^{\gamma_r}\circ\alpha_t v_{tt'}F_{\gamma_t}\circ\psi_{t,t'}^{\gamma_t}\Bigg)\Bigg\|_{2,h_{\mathbb{G}}}^2\\ &= \sum_{x\in\Gamma}\Bigg\|\sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r)1_{S'\cdot G}(t)\\ &\qquad\times|r\cdot G|\Bigg(\sum_{s\in r\cdot G,t'\in t\cdot G} v_{rs}\circ\alpha_t\mathcal{F}_{G_{\gamma_r}}\big(\pi_{s,r}^{\gamma_r}(a)\big)\circ\psi_{r,s}^{\gamma_r}\circ\alpha_t v_{tt'}F_{\gamma_t}\circ\psi_{t,t'}^{\gamma_t}\Bigg)\Bigg\|_2^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in\Gamma} \Bigg\| \sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) \\
&\quad \times |r\cdot G| \Bigg(\sum_{s\in r\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t\Bigg)\Bigg(\sum_{t'\in t\cdot G} v_{tt'}F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'}\Bigg)\Bigg\|_2^2 \\
&\le \sum_{x}\Bigg(\sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) \\
&\quad \times |r\cdot G| \Bigg\| \sum_{s\in r\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t\Bigg\|_\infty \Bigg\|\sum_{t'\in t\cdot G} v_{tt'}F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'}\Bigg\|_2\Bigg)^2 \\
&= \sum_{x}\Bigg(\sum_{\substack{r,t\in\Gamma\\ rt=x}} \Bigg(1_{S\cdot G}(r) \\
&\quad \times |r\cdot G| \Bigg\| \sum_{s\in r\cdot G} v_{rs} \mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\Bigg\|_\infty\Bigg)\Bigg(1_{S'\cdot G}(t)\Bigg\|\sum_{t'\in t\cdot G} v_{tt'}F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'}\Bigg\|_2\Bigg)\Bigg)^2 \\
&= \|\psi * \phi\|^2_{l^2(\Gamma)},
\end{aligned}
$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote, respectively, the L^2-norm and the supremum norm on $C(G)$, and $\psi,\phi:\Gamma\to\mathbb{R}_+$ are finitely supported functions defined by

$$
\begin{aligned}
\psi(r) &:= 1_{S\cdot G}(r)|r\cdot G|\Bigg\|\sum_{s\in r\cdot G} v_{rs}\mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\Bigg\|_\infty, \\
\phi(t) &:= 1_{S'\cdot G}(t)\Bigg\|\sum_{t'\in t\cdot G} v_{tt'}F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'}\Bigg\|_2.
\end{aligned}
$$

Note that $\|\phi\|^2_{l^2(\Gamma)} = \|b\|^2_{2,h_{\mathbb{G}}}$. Moreover, one has, since $\psi^{\gamma}_{r,s}: G_{r,s}\to G_\gamma$ is a homeomorphism,

$$
\begin{aligned}
\|\psi\|^2_{l^2(\Gamma)} &= \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^2\Bigg\|\sum_{s\in r\cdot G} v_{rs}\mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\Bigg\|^2_\infty \\
&\le \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G}\big\|v_{rs}\mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\circ\psi^{\gamma_r}_{r,s}\big\|^2_\infty \\
&= \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G}\big\|\mathcal{F}_{G_{\gamma_r}}\big(\pi^{\gamma_r}_{s,r}(a)\big)\big\|^2_{C(G_{\gamma_r})}.
\end{aligned}
$$

For $k\in\mathbb{N}$, let

$$
\begin{aligned}
p_k &= \sum_{\gamma\in I,\, x\in\mathrm{Irr}(G_\gamma):\, k\le l(\gamma(x))<k+1} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}}), \\
p_k^{G_\gamma} &= \sum_{x\in\mathrm{Irr}(G_\gamma):\, k\le l_{G_\gamma}(x)<k+1} p_x \in l^\infty(\widehat{G}_\gamma)
\end{aligned}
$$

and suppose from now on that $a \in p_k c_c(\widehat{\mathbb{G}})$. Hence, we must have $S \subset \{\gamma \in \Gamma : l_\Gamma(\gamma) < k+1\}$ and, for all $\gamma \in S$, $T_\gamma \subset \{x \in \mathrm{Irr}(G_\gamma) : l_{G_\gamma}(x) < k+1\}$. Hence, for all $\gamma \in S$ and all $r, s \in \gamma \cdot G$, one has $\pi^\gamma_{r,s}(a) \in q^\gamma_k c_c(\widehat{G}_\gamma)$, where $q^\gamma_k = \sum_{j=0}^k p_j^{G_\gamma}$.

Since $(\widehat{G}, l_G)$ has polynomial growth, there exist $C > 0$ and $N \in \mathbb{N}$ satisfying the properties of Lemma 5.3. In particular, one has, for all $\gamma \in \Gamma$, $|\gamma \cdot G| \leq C(2l_\Gamma(\gamma)+1)^N$. Moreover, since $S \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$ and l_Γ is β-invariant, it follows that $S \cdot G \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$. By Lemma 5.2 (and Lemma 5.3), we deduce that

$$\begin{aligned}
\|\psi\|^2_{l^2(\Gamma)} &\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\| v_{\gamma_r\gamma_r}\mathcal{F}_G\left(\widetilde{\pi^{\gamma_r}_{s,r}(a)}\right)\right\|^2 \\
&\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\|\mathcal{F}_G\left(\widetilde{\pi^{\gamma_r}_{s,r}(a)}\right)\right\|^2 \\
&\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} C^2\left(k + l_\Gamma(\gamma_r)+1\right)^{2N} \left\|\widetilde{\pi^{\gamma_r}_{s,r}(a)}\right\|^2_{G,0} \\
&\leq C^2(2k+2)^{2N} \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\|\pi^{\gamma_r}_{s,r}(a)\right\|^2_{G_{\gamma_r},0} \\
&\leq C^4(2k+3)^{4N} \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G| \sum_{s\in r\cdot G} \left\|\pi^{\gamma_r}_{s,r}(a)\right\|^2_{G_{\gamma_r},0} \\
&= C^4(2k+3)^{4N}\|a\|^2_{\mathbb{G},0}.
\end{aligned}$$

Since (Γ, l_Γ) has (RD), let $C_2 > 0$ and $N_2 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, for all function ξ on Γ supported on $\{g \in \Gamma : l_\Gamma(g) < k+1\}$, we have

$$\|\xi * \eta\|_{l^2(\Gamma)} \leq C_2(k+1)^{N_2}\|\xi\|_{l^2(\Gamma)}\|\eta\|_{l^2(\Gamma)}.$$

Note that ψ is supported on $S \cdot G$ and $S \cdot G \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$. Hence, it follows from the preceding computations that

$$\begin{aligned}
\left\|\mathcal{F}_{\mathbb{G}}(a)b\right\|^2_{2,h_{\mathbb{G}}} &\leq \|\psi * \phi\|^2_{l^2(\Gamma)} \leq C_2^2(k+1)^{2N_2}\|\psi\|_{l^2(\Gamma)}\|\phi\|_{l^2(\Gamma)} \\
&\leq C^4(2k+3)^{4N}C_2^2(k+1)^{2N_2}\|a\|^2_{\mathbb{G},0}\|b\|^2_{2,h_{\mathbb{G}}} \\
&= \left(P(k)\|a\|^2_{\mathbb{G},0}\|b\|_{2,h_{\mathbb{G}}}\right)^2,
\end{aligned}$$

where $P(X) = C^2C_2^2(2X+3)^{2N}(X+1)^{N_2}$. This concludes the proof of Theorem 5.6. ∎

To complete the proof of Theorem 1, we need the following proposition.

Proposition 5.7. *Assume that there exists a length function l on* $\mathrm{Irr}(\mathbb{G})$ *such that $(\widehat{\mathbb{G}}, l)$ has (RD) and consider the matched pair of length functions (l_Γ, l_G) associated to l given in Proposition 4.2. Then, (Γ, l_Γ) has (RD) and $(\widehat{G}, l_G)$ has polynomial growth.*

Proof. Suppose that $(\widehat{\mathbb{G}}, l)$ has (RD). The fact that $(\widehat{G}, l_G)$ has (RD) follows from the general theory (since $C(G) \subset C(\mathbb{G})$ intertwines the comultiplication and the associated injection $\mathrm{Irr}(G) \to \mathrm{Irr}(\mathbb{G})$, actually given by $(x \mapsto 1(x))$, preserves the length functions). Let us show that (Γ, l_Γ) has (RD). Let $k \in \mathbb{N}$ and $\xi : \Gamma \to \mathbb{C}$ be a finitely supported function with support in $\{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$. Define $\tilde{\xi} \in c_c(\widehat{\mathbb{G}})$ by

$\widetilde{\xi} = \sum_{\gamma \in I} \frac{1}{|\gamma \cdot G|} (\sum_{r \in \gamma \cdot G} \xi(r) e_{rr}) p_{\gamma(1)}$, where we recall $e_{rs} \in \mathcal{B}(l^2(\gamma \cdot G))$ for $r, s \in \gamma \cdot G$ are the matrix units associated to the canonical orthonormal basis. Then,

$$\mathcal{F}_{\mathbb{G}}(\widetilde{\xi}) = \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r)(\mathrm{Tr}_{l^2(\gamma \cdot G)} \otimes \mathrm{id})\big(u^{\gamma(1)}(e_{rr} \otimes 1)\big) = \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr}$$

and also

$$\begin{aligned} \|\widetilde{\xi}\|^2_{\mathbb{G},0} &= \sum_{\gamma \in I} |\gamma \cdot G| \, \mathrm{Tr}_{l^2(\gamma \cdot G)} \Bigg(\sum_{r \in \gamma \cdot G} \frac{|\xi(r)|^2}{|\gamma \cdot G|^2} e_{rr} \Bigg) \\ &= \sum_{\gamma \in I} \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 \leq \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 = \|\xi\|_2^2. \end{aligned}$$

Since ξ is supported in $\{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$ and l_Γ is β-invariant, it follows that $\mathrm{supp}(\widetilde{\xi}) \subset \{z \in \mathrm{Irr}(\mathbb{G}) : k \leq l(z) < k+1\}$. Hence, denoting by P a polynomial witnessing (RD) for $(\widehat{\mathbb{G}}, l)$, we have

$$\Big\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \Big\| \leq P(k) \|\xi\|_2.$$

Denote by Ψ the unital $*$-homomorphism $\Psi : C(\mathbb{G}) = \Gamma \ltimes C(G) \to C_r^*(\Gamma)$ such that $\Psi(u_\gamma F) = \lambda_\gamma F(1)$ for all $\gamma \in \Gamma$ and $F \in C(G)$. Since Ψ has norm one, denoting by $\lambda(\xi) \in C_r^*(\Gamma)$ the convolution operator by ξ, we have

$$\begin{aligned} \|\lambda(\xi)\| &= \Big\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) \lambda_r \Big\| = \Big\| \Psi\Big(\sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \Big) \Big\| \\ &\leq \Big\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \Big\| \leq P(k) \|\xi\|_2. \end{aligned}$$

This concludes the proof. ∎

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