Strict quantization of coadjoint orbits

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Abstract. For every semisimple coadjoint orbit $\hat{\mathcal{O}}$ of a complex connected semisimple Lie group $\hat{G}$, we obtain a family of $\hat{G}$-invariant products $\hat{*}_\hbar$ on the space of holomorphic functions on $\hat{\mathcal{O}}$. For every semisimple coadjoint orbit $\mathcal{O}$ of a real connected semisimple Lie group G, we obtain a family of G-invariant products $*_\hbar$ on a space $\mathcal{A}(\mathcal{O})$ of certain analytic functions on $\mathcal{O}$ by restriction. $\mathcal{A}(\mathcal{O})$, endowed with one of the products $*_\hbar$, is a G-Fréchet algebra, and the formal expansion of the products around $\hbar = 0$ determines a formal deformation quantization of $\mathcal{O}$, which is of Wick type if G is compact. Our construction relies on an explicit computation of the canonical element of the Shapovalov pairing between generalized Verma modules and complex analytic results on the extension of holomorphic functions.

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1. Introduction

The quantization problem in physics asks how to associate a quantum system to a classical mechanical system such that the classical system can be recovered from the quantum system in a classical limit. Since both systems can be studied by their observable algebras, a first step is to quantize the classical observable algebra. This algebra is usually the Poisson algebra $\mathscr{C}^\infty(M)$ of smooth functions on a Poisson manifold M. The observable algebra of a quantum mechanical system is some non-commutative $*$-algebra $\mathcal{A}$, which in many cases is obtained from a C^*-algebra. In a second step, the states of the quantum mechanical system can be obtained as normalized positive linear functionals on $\mathcal{A}$. To define their superposition, one has to represent $\mathcal{A}$ on a (pre-)Hilbert space so that the superposition of

two vector states can be defined as the vector state corresponding to the sum of the two vectors.

Formal deformation quantization, as introduced in [2], has proven to be a fruitful theory for studying some aspects of the quantization problem. One views Planck's constant $\hbar$ as a formal parameter $\hbar$ and tries to find the so-called *formal star products* $\star$ on $\mathcal{A} = \mathscr{C}^\infty(M)[[\hbar]]$, which may be thought of as the infinite jet of a full solution to the quantization problem at $\hbar = 0$. These star products are associative $\mathbb{C}[[\hbar]]$-bilinear products for which $1 \in \mathscr{C}^\infty(M)$ is a unit and which satisfy the correct classical limit. To be more precise, if $f, g \in \mathscr{C}^\infty(M)$ and $f \star g = \sum_{r=0}^\infty \hbar^r C_r(f, g)$ with operators $C_r : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, then one requires C_0 to be the pointwise multiplication, $C_0(f, g) = fg$, and the quantization to be in the direction of the Poisson bracket, $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$. Usually one also requires the C_r to be bidifferential operators so that $\star$ is local and can be restricted to open subsets of M. Using formal power series means on the one hand that we cannot substitute $\hbar$ with the real value of Planck's constant as required for direct physical applications, but on the other hand that we can transfer the quantization problem to algebra by neglecting analytic aspects, such as convergence of the power series. Consequently, many powerful tools become available for its study, and existence and classification results were obtained in [5, 13, 17, 33] for symplectic manifolds, whereas in the more general case of Poisson manifolds they follow from Kontsevich's formality theorem [27]. One can also study formal star products that are equivariant with respect to the action of a Lie group, where the classification follows for example from [14].

A complete solution of the quantization problem consists of a Hilbert space H together with a quantization map that associates a quantum observable, usually a self-adjoint operator on H, to any classical observable. This motivates the definition of a *strict quantization* [29, 31, 32, 37], which is some field of "nice" *-algebras $\mathcal{A}_\hbar$ (over $\mathbb{C}$) depending "nicely" on a parameter $\hbar$ ranging over some subset of $\mathbb{C}$, with $\mathcal{A}_0$ being a completion of the classical observable algebra and the deformation being in the direction of the Poisson bracket. However, strict quantizations are much harder to understand than formal deformation quantizations. There are many examples of strict quantizations in different contexts, and therefore there are several ways to formalize the above definition, i.e., specifying the parameter set and what "nice" actually means. No general existence results are known, and a classification seems completely hopeless due to the increased complexity.

There are two prominent constructions of strict quantizations. The first is due to Rieffel [37] who, using oscillatory integrals, deforms the product on a Fréchet algebra endowed with an isometric action of $\mathbb{R}^d$. If the original algebra is a C^*-algebra, then Rieffel constructs a C^*-algebraic quantization. A generalization to negatively curved Kählerian Lie groups can be found in [6]. The second construction, due to Natsume, Nest, and Peter [32], essentially glues convergent versions of the Weyl product on charts to obtain a C^*-algebraic quantization. However, both methods work only for some symplectic manifolds and fail for example for the 2-sphere with its SO(3)-invariant symplectic structure [38].

They also make a crucial use of the finite dimensionality of the classical mechanical system, so it remains unclear how to apply them to quantum field theories, despite such field theories fitting into the framework of formal deformation quantization.

Another approach to strict quantization was proposed by Beiser and Waldmann in [3, 4, 40]. They start with formal deformation quantizations, which are well understood, and try to find subalgebras on which the formal power series converge. Such subalgebras are usually defined using additional geometric structures and can be completed with respect to a topology in which the product is continuous. This approach was carried out explicitly for star products of exponential type on possibly infinite-dimensional vector spaces [39], for the linear Poisson structure on the dual of a Lie algebra [16], and for the hyperbolic disc $\mathbb{D}^n$ using an invariant star product obtained via phase space reduction [28]. See also [41] for a survey. In this paper, we extend this approach to semisimple coadjoint orbits of connected semisimple Lie groups, which gives a much larger class of geometrically interesting examples.

Coadjoint orbits play an important role in different areas of mathematics. In the representation theory of unitary Lie groups they appear, e.g., in the Kirillov orbit method [26], while in symplectic geometry they are related to momentum maps. Basic examples of coadjoint orbits are hyperbolic discs and complex projective spaces, including the 2-sphere. Any coadjoint orbit $\mathcal{O}$ of a Lie group G has a canonical G-invariant symplectic form, and if $\mathcal{O}$ is semisimple and G is compact, connected, and semisimple, then there is a unique compatible G-invariant complex structure that makes $\mathcal{O}$ a Kähler manifold.

Constructions of star products on coadjoint orbits are due to many authors [1, 8–10, 18, 24, 25, 36]. In this article, we focus on semisimple coadjoint orbits of connected semisimple Lie groups and the algebraic construction of Alekseev–Lachowska [1]. The canonical element F_λ of the Shapovalov pairing between certain generalized Verma modules satisfies an associativity equation generalizing that of a Drinfel'd twist. This twist induces a formal product for holomorphic functions on a complex orbit and a formal star product for smooth functions on a real orbit, and those products are compatible by restriction. It is very convenient that we can pass from one setting to the other: we will mainly work in the complex setting, which is more convenient for obtaining continuity estimates, and restrict to the real setting only in the very end.

Our first result uses methods developed by Ostapenko [35] to obtain an explicit formula for the canonical element of the Shapovalov pairing for a semisimple Lie algebra $\mathfrak{g}$.

Main Theorem I. *The Shapovalov pairing* $\langle \, \cdot \, , \, \cdot \, \rangle_\lambda^{\sim} \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathbb{C}$ *is nondegenerate if* $\lambda \in \widetilde{\Lambda}$*, and in this case its canonical element* $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \, \widehat{\otimes} \, \mathscr{U}(\tilde{\mathfrak{n}}^-)$ *is given by*

$$F_\lambda = \sum_{w \in \widetilde{W}} p_\lambda^w(\alpha_w)^{-1} \tilde{\pi}_\lambda^+(X_w) \otimes \tilde{\pi}_\lambda^-(Y_w). \tag{1.1}$$

The notation is explained in detail in Section 3. For now, it suffices to mention that the Shapovalov pairing is a pairing between the universal enveloping algebras of two nilpotent Lie subalgebras $\tilde{\mathfrak{n}}^\pm$ of $\mathfrak{g}$, depending on a parameter $\lambda \in \mathfrak{g}^*$. The sum is over a set of words

$\widetilde{W}$ related to the root system of $\mathfrak{g}$, the $p_\lambda^w(\alpha_w)$ are non-zero coefficients which are defined by an explicit formula, X_w and Y_w are elements of $\mathscr{U}\mathfrak{g}$, and $\widetilde{\pi}_\lambda^\pm$ maps these elements to $\mathscr{U}(\widetilde{\mathfrak{n}}^\pm)$. The element $F_\hbar$, which induces the star product, is obtained by rescaling λ, and doing so the coefficients $p_{i\lambda/\hbar}^w(\alpha_w)^{-1}$ will depend rationally on $\hbar$, with a countable set of poles P that accumulate only at 0. It seems as if explicit formulas for deformation quantizations received special attention by various authors, and (1.1) provides such a formula that works in great generality.

As mentioned above, the formal expansion of $F_\hbar$ induces formal products in complex and real settings. Furthermore, we also obtain a family of actual (non-formal) products for holomorphic polynomial functions in the complex setting and for polynomial functions in the real setting, parametrized by $\mathbb{C} \setminus P$, since only finitely many elements of the infinite sum defining $F_\hbar$ are non-zero on polynomials. All these products are G-invariant, and under some conditions on the Cartan subalgebra used in the construction they are also Hermitian, meaning that $\overline{f *_\hbar g} = \overline{g} *_{\overline{\hbar}} \overline{f}$. In the real setting and for a compact semisimple connected Lie group G, the formal star product is of Wick type [23] with respect to the Kähler complex structure on the coadjoint orbit, meaning that it derives the first argument only in holomorphic directions and the second argument only in antiholomorphic directions.

The next major step after constructing the star product is to use the explicit formulas to prove its continuity in the complex setting with respect to the topology of locally uniform convergence. This topology is locally convex and we can extend the product to a continuous product on the completion of the holomorphic polynomials. Using methods from analytic geometry, we identify this completion with the space of holomorphic functions.

Main Theorem II. *For any semisimple coadjoint orbit $\widehat{\mathcal{O}}$ of a connected semisimple complex Lie group G, there is a family of products $\hat{*}_\hbar \colon \mathrm{Hol}(\widehat{\mathcal{O}}) \times \mathrm{Hol}(\widehat{\mathcal{O}}) \to \mathrm{Hol}(\widehat{\mathcal{O}})$ for $\hbar \in \mathbb{C} \setminus P$, where every product $\hat{*}_\hbar$ is G-invariant and continuous with respect to the topology of locally uniform convergence. The dependence of $\hat{*}_\hbar$ on $\hbar$ is holomorphic.*

This result is certainly interesting in its own right. However, as mentioned above, we can also restrict it to real coadjoint orbits $\mathcal{O} \subseteq \widehat{\mathcal{O}}$. Denote by $\mathcal{A}(\mathcal{O})$ the class of functions on $\mathcal{O}$ that extend to holomorphic functions on $\widehat{\mathcal{O}}$ (if a function extends, its extension is unique), which contains the polynomials. We define the topology of extended locally uniform convergence on $\mathcal{A}(\mathcal{O})$ by saying that a sequence of functions in $\mathcal{A}(\mathcal{O})$ converges if the corresponding sequence of extensions converges locally uniformly so that $\mathcal{A}(\mathcal{O})$ is homeomorphic to $\mathrm{Hol}(\widehat{\mathcal{O}})$.

Main Theorem III. *For any semisimple coadjoint orbit $\mathcal{O}$ of a connected semisimple real Lie group G, there is a family of products $*_\hbar \colon \mathcal{A}(\mathcal{O}) \times \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\mathcal{O})$ for $\hbar \in \mathbb{C} \setminus P$, where every product $*_\hbar$ is G-invariant and continuous with respect to the topology of extended locally uniform convergence. The dependence of $*_\hbar$ on $\hbar$ is holomorphic. The formal expansion of $*_\hbar$ around 0 is a formal star product deforming the G-invariant symplectic form of $\mathcal{O}$.*

For the hyperbolic disc, the quantum algebra $(\mathcal{A}(\mathbb{D}^n), *_\hbar)$ agrees with the algebra obtained in [28] while, for the 2-sphere, $(\mathcal{A}(\mathbb{S}^2), *_\hbar)$ is the algebra considered in [15].

Since we constructed a quantization of the holomorphic functions on a complex coadjoint orbit and the restriction $\mathrm{Hol}(\hat{\mathcal{O}}) \to \mathcal{A}(\mathcal{O})$ is an isomorphism, the quantizations of different real orbits with the same complexification are related.

Main Theorem IV. *If $\mathcal{O}$ and $\mathcal{O}'$ are coadjoint orbits of real semisimple connected Lie groups with the same complexification and through one common semisimple element, then the algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ and $(\mathcal{A}(\mathcal{O}'), *'_\hbar)$ are isomorphic.*

This isomorphism generalizes the classical Wick rotation, which can be interpreted as an isomorphism between the polynomial algebras $\mathrm{Pol}(\mathbb{CP}^n)$ and $\mathrm{Pol}(\mathbb{D}^n)$. However, this isomorphism does not necessarily respect the star involutions with which the algebras $\mathcal{A}(\mathcal{O})$ are equipped. In other words, the algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}')$ are isomorphic as algebras, but not necessarily as $*$-algebras.

In order to apply our quantization to physics, we should represent the Fréchet algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ on a Hilbert space. Given a positive linear functional, we can use the GNS representation to do so. For a formal star product of Wick type all point evaluation functionals are formally positive. However, formal positivity means only that the first non-vanishing order is positive and therefore, as in this case, might not survive the passage to strict products (where the contribution of higher orders can dominate the contribution of the first order). For certain coadjoint orbits we will prove that point evaluations stay positive.

One aspect that we do not discuss in this work is the relation to geometric or Berezin–Toeplitz quantization [8–10, 36]. These theories construct a quantization by studying holomorphic sections of a quantizing line bundle over the manifold M. This line bundle needs to satisfy some integrality condition, which for compact M means that only countably many values of $\hbar$, accumulating at 0, are allowed. The algebra $\mathscr{C}^\infty(M)$ is, in the limit $\hbar \to 0$, approximated by finite dimensional matrix algebras. The construction of Alekseev–Lachowska coincides with another more geometric construction of star products on semisimple coadjoint orbits by Karabegov [15, 25] if $\hbar$ is not a pole. However, Karabegov's construction still makes sense at the poles, where it coincides with (a variant of) the Berezin–Toeplitz quantization [25]. In this sense, our infinite dimensional Fréchet algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ interpolate between the finite dimensional Berezin–Toeplitz algebras. It could be very interesting to study this in greater detail.

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In Section 2, we recall some well-known facts about coadjoint orbits. This includes the realizability of coadjoint orbits as orbits of matrix Lie groups and a characterization of invariant multidifferential operators on homogeneous spaces. In Section 3, we introduce the Shapovalov pairing of (generalized) Verma modules and derive an explicit formula for its canonical element. From this, we obtain a product for holomorphic polynomials on complex coadjoint orbits. In Section 4, we show that this product is continuous with respect to the topology of locally uniform convergence so that we can extend it to the com-

pletion, which consists of all holomorphic functions on the orbit. Finally, we restrict our results to real coadjoint orbits in Section 5. We will determine additional properties of the star products obtained in this way (e.g., being of Wick type or of standard ordered type), study positive linear functionals, and investigate isomorphisms of the algebras obtained for different real forms of the same complex coadjoint orbit. In Appendix A, we give some remaining proofs and more details on complex structures.

Notation

In the whole paper, G is either a real or complex Lie group, $\mathfrak{g}$ denotes the Lie algebra of G, and $\mathscr{U}\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$. In Sections 3 and 4, G is always complex. In Section 5, G refers to a real Lie group and $\hat{G}$ refers to a complexification of G. K denotes a compact real Lie group. Coadjoint orbits through $\lambda \in \mathfrak{g}^*$ are denoted by $\mathcal{O}_\lambda$.

We write $\mathscr{C}^\infty(M)$ for the smooth complex-valued functions on a manifold M. If M is a real manifold, $\mathrm{T}M$ denotes its (real) tangent bundle (so sections of $\mathrm{T}M$ are derivations of the algebra of real-valued smooth functions on M). The complexification of $\mathrm{T}M$ is denoted by $\mathrm{T}^\mathbb{C}M$ (so sections of $\mathrm{T}^\mathbb{C}M$ are derivations of $\mathscr{C}^\infty(M)$). If M is a complex manifold, then the holomorphic tangent bundle is denoted by $\mathrm{T}^{(1,0)}M$.

2. Preliminaries

In this section, we summarize some results that are needed in the rest of this article: we review the definition of coadjoint orbits and their realizability as orbits of matrix Lie groups in Section 2.1. In Section 2.2, we introduce invariant multidifferential operators on homogeneous spaces.

2.1. Coadjoint orbits

Let G be a real or complex Lie group with Lie algebra $\mathfrak{g}$. We denote the *adjoint action* of G on $\mathfrak{g}$ by $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$. For any $g \in G$, $\mathrm{Ad}_g := \mathrm{Ad}(g)$ is the tangent map of the conjugation $G \ni x \mapsto gxg^{-1} \in G$ by g. Its differential $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{end}(\mathfrak{g})$ is given by the Lie bracket, $\mathrm{ad}_X(Y) = [X, Y]$. The *coadjoint action* $\mathrm{Ad}^*\colon G \to \mathrm{End}(\mathfrak{g}^*)$ of G on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is defined by $\mathrm{Ad}_g^* \xi = \xi \circ \mathrm{Ad}_{g^{-1}}$ for $\xi \in \mathfrak{g}^*$.

The *coadjoint orbit* $\mathcal{O}_\lambda$ of G through an element $\lambda \in \mathfrak{g}^*$ is defined as

$$\mathcal{O}_\lambda = \{\xi \in \mathfrak{g}^* \mid \xi = \mathrm{Ad}_g^* \lambda \text{ for some } g \in G\}. \tag{2.1}$$

It is well known that $\mathcal{O}_\lambda \cong G/G_\xi$, where $\xi \in \mathcal{O}_\lambda$ is any point on the coadjoint orbit and $G_\xi = \{g \in G \mid \mathrm{Ad}_g^* \xi = \xi\}$ is the *stabilizer subgroup* of ξ. If G is a real (complex) Lie group, there is a unique smooth (complex) manifold structure on G/G_ξ that makes the projection $\pi\colon G \to G/G_\xi$ a smooth (holomorphic) submersion, and we use it to define the structure of a smooth (complex) manifold on $\mathcal{O}_\lambda$. It does not depend on the choice of $\xi \in \mathcal{O}_\lambda$.

Fix a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and let C_{ij}^k be the structure constants with respect to this basis, i.e., $[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k$. In this case, $\{f, g\}(\xi) = \sum_{i,j,k=1}^n C_{ij}^k \xi(e_k) \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j}$ defines a linear Poisson structure on $\mathfrak{g}^*$, where $f, g \in \mathscr{C}^\infty(\mathfrak{g}^*)$ and the e_i are viewed as global linear coordinates on $\mathfrak{g}^*$. The following proposition is well known; see, e.g., [11, Example 1.1.3].

Proposition 2.1. *If the Lie group G is connected, then the coadjoint orbits of G are precisely the symplectic leaves of this linear Poisson structure. In particular, all connected Lie groups with the same Lie algebra have the same coadjoint orbits.*

Corollary 2.2. *If the Lie group G is semisimple and connected, then G and its image under* $\mathrm{Ad} \colon G \to \mathrm{End}(\mathfrak{g})$ *have the same coadjoint orbits.*

Proof. Since $\mathfrak{g}$ is semisimple, it has trivial center and therefore $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{end}(\mathfrak{g})$ is injective. Consequently, G and its image in $\mathrm{End}(\mathfrak{g})$ have the same Lie algebra. Since both are connected, the result follows by applying the previous proposition. ∎

It is easy to show that not only G and its image under Ad have the same coadjoint orbits, but also $\mathrm{Ad} \colon G \to \mathrm{End}(\mathfrak{g})$ intertwines the actions of G and its image on the coadjoint orbits. Since the image of G under Ad is a matrix Lie group, we can therefore, when studying coadjoint orbits of connected semisimple Lie groups, assume without loss of generality that such a Lie group is a matrix Lie group. Using the argument provided in [19, Theorem 9], we can even assume that G is a closed matrix Lie group.

For $X \in \mathfrak{g}$, denote the *fundamental vector field* of X for the coadjoint action by $X_{\mathcal{O}_\lambda}|_\xi := \frac{d}{dt}|_{t=0} \mathrm{Ad}^*_{\exp(-tX)} \xi$, where $\xi \in \mathcal{O}_\lambda$. Note that the map $\mathfrak{g}/\mathfrak{g}_\xi \to T_\xi \mathcal{O}_\lambda$, $X \mapsto X_{\mathcal{O}_\lambda}|_\xi$ is an isomorphism, where $\mathfrak{g}_\xi$ denotes the Lie algebra of G_ξ. Consequently,

$$\omega_{\mathrm{KKS}}(X_{\mathcal{O}_\lambda}, Y_{\mathcal{O}_\lambda})|_\xi = \xi([X, Y]) \tag{2.2}$$

determines a well-defined 2-form on $\mathcal{O}_\lambda$, which is called the Kirillov–Kostant–Souriau form. One can show that ω_{KKS} is symplectic and G-invariant. By symplectic we mean that ω_{KKS} is closed and that $\omega_{\mathrm{KKS}}|_\xi \colon T_\xi \mathcal{O}_\lambda \times T_\xi \mathcal{O}_\lambda \to \Bbbk$ is $\Bbbk$-bilinear, antisymmetric, and non-degenerate for all $\xi \in \mathcal{O}_\lambda$, where $\Bbbk$ is either $\mathbb{R}$ or $\mathbb{C}$, depending on whether G is real or complex.

For a semisimple Lie algebra $\mathfrak{g}$, the Killing form $B \colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ is non-degenerate, giving an isomorphism $\flat \colon \mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto X^\flat := B(X, \cdot)$. We denote its inverse by $\sharp \colon \mathfrak{g}^* \to \mathfrak{g}$. In the complex case, we say that $\lambda \in \mathfrak{g}^*$ is semisimple if $\mathrm{ad}_{\lambda^\sharp} \in \mathfrak{end}(\mathfrak{g})$ is diagonalisable and in the real case $\lambda \in \mathfrak{g}^*$ is semisimple if the complex linear extension of λ to the complexification of $\mathfrak{g}$ is semisimple. A coadjoint orbit $\mathcal{O}_\lambda$ is semisimple if λ is semisimple.

Proposition 2.3. *Let G be a complex connected semisimple Lie group and let $\lambda \in \mathfrak{g}^*$ be semisimple. Then G_λ is connected.*

Proof. The Lie algebra spanned by $\lambda^\sharp$ integrates to a connected commutative Lie subgroup T' of G, and since $\lambda^\sharp$ is semisimple, all elements of T' are diagonalisable in the

adjoint representation. There is a smallest closed complex Lie group T containing T', that can be obtained as follows: take the closure of T' (which is a real Lie group), take the Lie algebra of this closure (which is a real Lie subalgebra of $\mathfrak{g}$), take the complex Lie algebra spanned by it, integrate this Lie algebra to a connected Lie subgroup of G, and possibly repeat these steps. T is still connected and commutative, and all its elements are diagonalisable in the adjoint representation, so T is a complex torus in G. Its centralizer is exactly G_λ, and centralizers of tori are connected. ∎

Note that the statement is also true for a real compact connected semisimple Lie group K, but might fail if the compactness assumption is dropped.

We denote the smooth functions on G that are invariant under the action of G_λ from the right by $\mathscr{C}^\infty(G)^{G_\lambda}$. That is, $f \in \mathscr{C}^\infty(G)^{G_\lambda}$ if and only if $f \in \mathscr{C}^\infty(G)$ and $f(gg') = f(g)$ for all $g \in G$ and $g' \in G_\lambda$. There is an algebra isomorphism

$$\pi^* \colon \mathscr{C}^\infty(G/G_\lambda) \to \mathscr{C}^\infty(G)^{G_\lambda}, \quad f \mapsto \pi^* f := f \circ \pi \tag{2.3}$$

and, for a complex Lie group, this isomorphism restricts to an isomorphism on holomorphic functions. We denote the inverse by $\pi_* \colon \mathscr{C}^\infty(G)^{G_\lambda} \to \mathscr{C}^\infty(G/G_\lambda)$.

Remark 2.4. This article is written mainly from a differential geometric perspective. Note, however, that any complex connected semisimple Lie group G has a unique structure of an algebraic group; see [34, Theorem 6.3 and the preceding corollary in Chapter 1]. Any holomorphic representation of G is polynomial. Consequently, if G is realized as a subgroup of $\mathrm{GL}_N(\mathbb{C})$, it is automatically closed. The coadjoint action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a morphism of algebraic varieties, and coadjoint orbits of G are smooth subvarieties of $\mathfrak{g}^*$. A coadjoint orbit of G is closed in the Zariski topology if and only if it is semisimple; see [12, Theorem 5.4]. In particular, semisimple coadjoint orbits of complex connected semisimple Lie groups are affine algebraic varieties.

Note that this is not necessarily true for real connected semisimple Lie groups (not even if they are linear). It is still true that real connected semisimple linear Lie groups and their coadjoint orbits are connected components (with respect to the usual topology) of affine algebraic varieties.

2.2. Invariant holomorphic k-differential operators

In the whole subsection, G is a complex Lie group, H is a closed complex Lie subgroup of G, and $k \geq 1$ is an integer. We present some results on holomorphic G-invariant k-differential operators on the homogeneous space G/H; in particular, we construct a bijection between the set $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\otimes k})^H$ and the set of such operators. The results seem to be well known, but proofs are hard to find in the literature.

A k-differential operator D (see Appendix A.1 for a short review of the definition) on a manifold M endowed with an action of a Lie group G is said to be invariant under G if $\phi_g^*(D\vec{f}) = D((\phi_g^*)^{\times k}\vec{f})$ for all $\vec{f} \in \mathscr{C}^\infty(M)^k$ and all $g \in G$. Here $\phi_g \colon M \to M$ is the diffeomorphism of M given by the action of a fixed element $g \in G$, and the upper

star denotes the pullback. We write $k\text{-DiffOp}^G_{\mathcal{H}}(M)$ for the space of holomorphic G-invariant k-differential operators on a complex manifold M. A k-differential operator on G is said to be left-invariant if it is invariant with respect to the left action $\mathrm{L}: G \times G \to G$, $(g, g') \mapsto gg' =: \mathrm{L}_g(g')$.

Let M be a complex manifold with complex structure $I: \mathrm{T}M \to \mathrm{T}M$. For a vector field $V \in \Gamma^{\infty}(\mathrm{T}M)$ its holomorphic part is $V^{(1,0)} = \frac{1}{2}(V - \mathrm{i}\,IV) \in \Gamma^{\infty}(\mathrm{T}^{(1,0)}M)$. Let $\mathfrak{g}$ be the Lie algebra of G. For any $X \in \mathfrak{g}$ define the *left-invariant vector field*

$$X^{\mathrm{left}}\big|_g := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g \exp(tX) \in \Gamma^{\infty}(\mathrm{T}G). \tag{2.4}$$

Its holomorphic part $X^{\mathrm{left},(1,0)} = \frac{1}{2}(X^{\mathrm{left}} - \mathrm{i}(\mathrm{i}X)^{\mathrm{left}}) \in \Gamma^{\infty}(\mathrm{T}^{(1,0)}G)$ induces a holomorphic left-invariant 1-differential operator $f \mapsto X^{\mathrm{left},(1,0)} f$ on G. Since $(\,\cdot\,)^{\mathrm{left},(1,0)}: \mathfrak{g} \to \Gamma^{\infty}(\mathrm{T}^{(1,0)}G)$ is a Lie algebra homomorphism, it induces an algebra homomorphism $(\,\cdot\,)^{\mathrm{left},(1,0)}: \mathscr{U}\mathfrak{g} \to \mathrm{DiffOp}^G_{\mathcal{H}}(G)$.

In the following, we extend various maps to k-fold products and still denote them by the same symbol,

$$\mathrm{Ad}_g: (\mathscr{U}\mathfrak{g})^{\otimes k} \to (\mathscr{U}\mathfrak{g})^{\otimes k},$$
$$u_1 \otimes \cdots \otimes u_k \mapsto \mathrm{Ad}_g u_1 \otimes \cdots \otimes \mathrm{Ad}_g u_k, \tag{2.5a}$$
$$\pi^*: \mathscr{C}^{\infty}(G/H)^k \to \left(\mathscr{C}^{\infty}(G)^H\right)^k,$$
$$(f_1, \ldots, f_k) \mapsto (\pi^* f_1, \ldots, \pi^* f_k), \tag{2.5b}$$
$$(\,\cdot\,)^{\mathrm{left},(1,0)}: (\mathscr{U}\mathfrak{g})^{\otimes k} \to k\text{-DiffOp}^G_{\mathcal{H}}(G),$$
$$u_1 \otimes \cdots \otimes u_k \mapsto \left((f_1, \ldots, f_k) \mapsto u_1^{\mathrm{left},(1,0)} f_1 \cdots u_k^{\mathrm{left},(1,0)} f_k\right). \tag{2.5c}$$

Proposition 2.5. *The map* $(\,\cdot\,)^{\mathrm{left},(1,0)}: (\mathscr{U}\mathfrak{g})^{\otimes k} \to k\text{-DiffOp}^G_{\mathcal{H}}(G)$ *is an isomorphism.*

Proof. See Appendix A.1. ∎

Next, we want to describe holomorphic G-invariant k-differential operators on the homogeneous space G/H. Let H be a closed Lie subgroup of G with Lie algebra $\mathfrak{h}$, and let $\mathscr{U}\mathfrak{g} \cdot \mathfrak{h} \subseteq \mathscr{U}\mathfrak{g}$ be the left ideal generated by $\mathfrak{h}$. Note that $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\otimes k}$ is isomorphic to $(\mathscr{U}\mathfrak{g})^{\otimes k}/I$, where $I = I_1 + \cdots + I_k$ and $I_i = (\mathscr{U}\mathfrak{g})^{\otimes(i-1)} \otimes \mathscr{U}\mathfrak{g} \cdot \mathfrak{h} \otimes (\mathscr{U}\mathfrak{g})^{\otimes(k-i)}$ is a left ideal in $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Introduce the set

$$U_{\mathrm{inv}} = \left\{\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k} \mid [\vec{u}] \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\otimes k} \text{ is } H\text{-invariant}\right\}$$
$$= \left\{\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k} \mid \mathrm{Ad}_h \vec{u} - \vec{u} \in I \text{ for all } h \in H\right\}. \tag{2.6}$$

Here the action of H on $(\mathscr{U}\mathfrak{g})^{\otimes k}$ is the diagonal action defined in (2.5a).

Lemma 2.6. *Let* $\vec{u} \in U_{\mathrm{inv}}$, $\vec{v} \in I$, *and* $\vec{f} \in (\mathscr{C}^{\infty}(G)^H)^k$. *Then*

$$\vec{v}^{\,\mathrm{left},(1,0)} \vec{f} = 0 \quad \text{and} \quad \vec{u}^{\,\mathrm{left},(1,0)} \vec{f} \in \mathscr{C}^{\infty}(G)^H. \tag{2.7}$$

Proof. Let $Y \in \mathfrak{h}$ and $f \in \mathscr{C}^\infty(G)^H$. Then we compute

$$(Y^{\text{left}} f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(g \exp(tY)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g) = 0.$$

Using that $Y^{\text{left},(1,0)} = \frac{1}{2}(Y^{\text{left}} - \mathrm{i}(\mathrm{i}Y)^{\text{left}})$, this implies that $Y^{\text{left},(1,0)} f = 0$, and therefore also $\vec{v}^{\text{left},(1,0)} \vec{f} = 0$ for all $\vec{v} \in I$ and $\vec{f} \in (\mathscr{C}^\infty(G)^H)^k$. If $X \in \mathfrak{g}$, then

$$(X^{\text{left}} f)(gh) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(gh \exp(tX)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(g \exp(t \, \mathrm{Ad}_h X)\big)$$
$$= \big((\mathrm{Ad}_h X)^{\text{left}} f\big)(g)$$

for all $f \in \mathscr{C}^\infty(G)^H$, $g \in G$, and $h \in H$. Consequently, we obtain $(X^{\text{left},(1,0)} f)(gh) = ((\mathrm{Ad}_h X)^{\text{left},(1,0)} f)(g)$, and extending to the universal enveloping algebra and to tensor products yields $(\vec{u}^{\text{left},(1,0)} \vec{f})(gh) = ((\mathrm{Ad}_h \vec{u})^{\text{left},(1,0)} \vec{f})(g)$ for all $\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ and $\vec{f} \in (\mathscr{C}^\infty(G)^H)^k$. If $\vec{u} \in U_{\text{inv}}$, then together with the first part we obtain

$$\big(\vec{u}^{\text{left},(1,0)} \vec{f}\big)(gh) = \big((\mathrm{Ad}_h \vec{u})^{\text{left},(1,0)} \vec{f}\big)(g)$$
$$= \big(\vec{u}^{\text{left},(1,0)} \vec{f}\big)(g) + \big((\mathrm{Ad}_h \vec{u} - \vec{u})^{\text{left},(1,0)} \vec{f}\big)(g)$$
$$= \big(\vec{u}^{\text{left},(1,0)} \vec{f}\big)(g). \qquad \blacksquare$$

Because of this lemma we can define

$$\widetilde{\Psi} : U_{\text{inv}} \to \mathrm{Map}\big(\mathscr{C}^\infty(G/H)^k, \mathscr{C}^\infty(G/H)\big), \quad \widetilde{\Psi}(\vec{u})\vec{f} = \pi_*\big(\vec{u}^{\text{left},(1,0)}(\pi^* \vec{f})\big).$$

Since π^* and π_* are algebra homomorphisms, it follows that $\widetilde{\Psi}(\vec{u})$ and $\vec{u}^{\text{left},(1,0)}$ satisfy essentially the same commutation relations with the operator that multiplies a component by a smooth function. Consequently $\widetilde{\Psi}(\vec{u})$ is k-differential and of the same order as $\vec{u}^{\text{left},(1,0)}$ (see the definition of k-differential operators given in Definition A.1). Moreover, $\widetilde{\Psi}(\vec{u})$ is G-invariant, because π^* and π_* are G-equivariant and $\vec{u}^{\text{left},(1,0)}$ is G-invariant. Since $\pi : G \to G/H$ is a holomorphic map, it follows that $\widetilde{\Psi}(\vec{u})$ is holomorphic, and $\widetilde{\Psi}$ really maps into $k\text{-DiffOp}_{\mathscr{H}}^G(G/H)$. The map $\widetilde{\Psi}$ descends to a map

$$\Psi : \big((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\otimes k}\big)^H \to k\text{-DiffOp}_{\mathscr{H}}^G(G/H) \tag{2.8}$$

because $\widetilde{\Psi}(I) = 0$ according to the previous lemma.

Proposition 2.7. *The map Ψ defined in* (2.8) *is an isomorphism.*

Proof. The proof is given in Appendix A.1. $\qquad \blacksquare$

The last result of this subsection gives a description of the k-differential operator $\Psi([\vec{u}])$ on the coadjoint orbit without using extensions to G. Let S be the antipode of $\mathscr{U}\mathfrak{g}$ and extend the Lie algebra homomorphism $\mathfrak{g} \ni X \mapsto X_{\mathcal{O}_\lambda} \in \Gamma^\infty(\mathrm{T}\mathcal{O}_\lambda)$ defined just before (2.2) to an algebra homomorphism $\mathscr{U}\mathfrak{g} \to \mathrm{DiffOp}(\mathcal{O}_\lambda)$.

Proposition 2.8. *Let $\mathcal{O}_\lambda \cong G/G_\lambda$ be a coadjoint orbit. If $\vec{u} = u_1 \otimes \cdots \otimes u_k \in U_{\mathrm{inv}}$ and $\vec{f} = (f_1, \ldots, f_k) \in \mathscr{C}^\infty(\mathcal{O}_\lambda)^k$, then*

$$\Psi\big([\vec{u}]\big)\, \vec{f}(\mathrm{Ad}_g^* \lambda)$$
$$= \big(S(\mathrm{Ad}_g\, u_1)\big)_{\mathcal{O}_\lambda}^{(1,0)} f_1(\mathrm{Ad}_g^* \lambda) \cdot \ldots \cdot \big(S(\mathrm{Ad}_g\, u_k)\big)_{\mathcal{O}_\lambda}^{(1,0)} f_k(\mathrm{Ad}_g^* \lambda). \qquad (2.9)$$

Proof. Defining the Lie algebra homomorphism $(\cdot)^{\mathrm{right}} \colon \mathfrak{g} \to \Gamma^\infty(\mathrm{T}G)$, $X \mapsto X^{\mathrm{right}}$ with $X^{\mathrm{right}}|_g := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(-tX)g$ and extending to $\mathscr{U}\mathfrak{g}$ as before, one checks that

$$u^{\mathrm{left}} f(g) = X_1^{\mathrm{left}} \cdots X_j^{\mathrm{left}} f(g)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t_1}\Big|_{t_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}t_j}\Big|_{t_j=0} f\big(g \exp(t_1 X_1) \cdots \exp(t_j X_j)\big)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t_1}\Big|_{t_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}t_j}\Big|_{t_j=0} f\big(\exp(t_1 \,\mathrm{Ad}_g\, X_1) \cdots \exp(t_j \,\mathrm{Ad}_g\, X_j)g\big)$$
$$= (-\mathrm{Ad}_g\, X_j)^{\mathrm{right}} \cdots (-\mathrm{Ad}_g\, X_1)^{\mathrm{right}} f(g) = \big(S(\mathrm{Ad}_g\, u)\big)^{\mathrm{right}} f(g)$$

for $u = X_1 \cdots X_j \in \mathscr{U}\mathfrak{g}$. Similarly, $u^{\mathrm{left},(1,0)} f(g) = (S(\mathrm{Ad}_g\, u))^{\mathrm{right},(1,0)} f(g)$. Furthermore, we have

$$X^{\mathrm{right}}(\pi^* f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^* f\big(\exp(-tX)g\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(\mathrm{Ad}_{\exp(-tX)}^* \mathrm{Ad}_g^* \lambda\big)$$
$$= X_{\mathcal{O}_\lambda} f(\mathrm{Ad}_g^* \lambda) = \pi^*(X_{\mathcal{O}_\lambda} f)(g)$$

for all $X \in \mathfrak{g}$, implying that $X^{\mathrm{right},(1,0)} \circ \pi^* = \pi^* \circ X_{\mathcal{O}_\lambda}^{(1,0)}$ and therefore that $u^{\mathrm{right},(1,0)} \circ \pi^* = \pi^* \circ u_{\mathcal{O}_\lambda}^{(1,0)}$ for all $u \in \mathscr{U}\mathfrak{g}$. Finally,

$$\Psi\big([\vec{u}]\big)\, \vec{f}(\mathrm{Ad}_g^* \lambda)$$
$$= \big(\vec{u}^{\mathrm{left},(1,0)} \pi^* \vec{f}\big)(g)$$
$$= u_1^{\mathrm{left},(1,0)}(\pi^* f_1)(g) \cdot \ldots \cdot u_k^{\mathrm{left},(1,0)}(\pi^* f_k)(g)$$
$$= \big(S(\mathrm{Ad}_g\, u_1)\big)^{\mathrm{right},(1,0)}(\pi^* f_1)(g) \cdot \ldots \cdot \big(S(\mathrm{Ad}_g\, u_k)\big)^{\mathrm{right},(1,0)}(\pi^* f_k)(g)$$
$$= \big(S(\mathrm{Ad}_g\, u_1)\big)_{\mathcal{O}_\lambda}^{(1,0)} f_1(\mathrm{Ad}_g^* \lambda) \cdot \ldots \cdot \big(S(\mathrm{Ad}_g\, u_k)\big)_{\mathcal{O}_\lambda}^{(1,0)} f_k(\mathrm{Ad}_g^* \lambda). \qquad \blacksquare$$

3. Quantizing complex coadjoint orbits

In this section, we construct a formal associative product for holomorphic functions on a semisimple coadjoint orbit of a complex connected semisimple Lie group and a strict associative product for polynomials. These products are induced by a twist, which is constructed using the Shapovalov pairing between generalized Verma modules. For the convenience of the reader, we first consider the special case of regular semisimple orbits in Section 3.1, where we introduce the Shapovalov pairing between Verma modules and

compute its canonical element. In Section 3.2, we generalize these results to non-regular semisimple orbits. In Section 3.3, we describe the induced formal and strict products in detail. We consider an example in Section 3.4.

Later, in Section 5, we will use the results of this section to obtain star products on semisimple coadjoint orbits of real connected semisimple Lie groups. From the example considered in this section, we will then obtain strict quantizations of the hyperbolic disc and the complex projective space.

3.1. Verma modules and the Shapovalov pairing

In this subsection, we introduce the Shapovalov pairing between Verma modules. In case this pairing is non-degenerate, we derive an explicit formula for its canonical element, following [35]. A similar formula in the more general setting of quantum groups was obtained recently in [30]. The results allow us to quantize regular orbits.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Recall that a root is a non-zero element $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}^\alpha := \{X \in \mathfrak{g} \mid \mathrm{ad}_H X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ contains a non-zero element. Denote the set of roots by Δ and choose an ordering (i.e., a subset Δ^+ of positive roots such that, setting $\Delta^- := -\Delta^+$, we have $\Delta^+ \cup \Delta^- = \Delta$, $\Delta^+ \cap \Delta^- = \emptyset$, and such that if the sum of positive roots is a root, then it is positive). Denote the simple roots (i.e., elements of Δ^+ that cannot be written as a sum of two elements of Δ^+) by Σ. Let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ be the nilpotent Lie subalgebras of $\mathfrak{g}$ spanned by the positive and negative root spaces, respectively, and define $\mathfrak{b}^+ := \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{n}^-$ (the direct sum is as vector spaces, the Lie algebra structure on $\mathfrak{b}^\pm \subseteq \mathfrak{g}$ is obtained by restriction from $\mathfrak{g}$).

Note that 0 is not a root. However, it is convenient to introduce the notation $\mathfrak{g}^0 := \mathfrak{h}$. Then $\mathfrak{g}$ is $(\Delta \cup \{0\})$-graded, in the sense that $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}^\alpha$ and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ for any $\alpha, \beta \in \Delta \cup \{0\}$. Consequently, the tensor algebra $\mathrm{T}\mathfrak{g}$ is $\mathbb{Z}\Delta$-graded, where the so-called root lattice $\mathbb{Z}\Delta$ is the set of linear combinations of roots. The two-sided ideal generated by elements of the form $X \otimes Y - Y \otimes X - [X, Y]$ with $X, Y \in \mathfrak{g}$ is homogeneous and therefore the universal enveloping algebra $\mathscr{U}\mathfrak{g} = \mathrm{T}\mathfrak{g}/\langle X \otimes Y - Y \otimes X - [X, Y]\rangle$ is also $\mathbb{Z}\Delta$-graded. Denote the degree of a homogeneous element $w \in \mathscr{U}\mathfrak{g}$ by $d(w) \in \mathbb{Z}\Delta$.

Given a linear functional $\lambda \in \mathfrak{h}^*$, the formula $H \rhd z = \lambda(H)z$ makes $\mathbb{C}$ a left $\mathfrak{h}$-module and, since $\mathfrak{h}$ is commutative, also a right $\mathfrak{h}$-module. We can extend this to a left or right $\mathfrak{b}^\pm$-module by noting that $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$ and letting $\mathfrak{n}^\pm$ act trivially. Denote the corresponding left $\mathscr{U}(\mathfrak{b}^\pm)$-module by $\mathbb{C}_\lambda^\pm$ and the right $\mathscr{U}(\mathfrak{b}^-)$-module by $\mathbb{C}_\lambda^*$. Define the *Verma modules*

$$M_\lambda := \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda^+, \quad M_\lambda^- := \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\mathfrak{b}^-)} \mathbb{C}_{-\lambda}^-, \quad M_\lambda^* := \mathbb{C}_\lambda^* \otimes_{\mathscr{U}(\mathfrak{b}^-)} \mathscr{U}\mathfrak{g}. \quad (3.1)$$

Note that M_λ and M_λ^- are left $\mathscr{U}\mathfrak{g}$-modules, whereas M_λ^* is a right $\mathscr{U}\mathfrak{g}$-module. M_λ is the most general left $\mathscr{U}\mathfrak{g}$-module of highest weight λ, meaning that any other left $\mathscr{U}\mathfrak{g}$-module of highest weight λ can be obtained as a quotient of M_λ. M_λ^- is the most general left $\mathscr{U}\mathfrak{g}$-module of lowest weight $-\lambda$.

There are canonical isomorphisms $M_\lambda^* \otimes_{\mathscr{U} \mathfrak{g}} M_\lambda \cong \mathbb{C}_\lambda^* \otimes_{\mathscr{U}(\mathfrak{b}^-)} \mathscr{U} \mathfrak{g} \otimes_{\mathscr{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda \cong \mathbb{C}_\lambda^* \otimes_{\mathscr{U}\mathfrak{h}} \mathbb{C}_\lambda \cong \mathbb{C}$ since the left and right $\mathfrak{h}$-module structures on $\mathbb{C}$ coincide.

Definition 3.1. The pairing $\langle \, \cdot \, , \, \cdot \, \rangle'_\lambda \colon M_\lambda^* \times M_\lambda \to \mathbb{C}$ defined by $(x, y) \mapsto x \otimes_{\mathscr{U}\mathfrak{g}} y$ is called the *Shapovalov pairing* between M_λ^* and M_λ.

In the following, it will be convenient to have alternative descriptions of M_λ, M_λ^-, and M_λ^*. Let $\{X_1, \ldots, X_k\}$ be a basis of $\mathfrak{n}^+$, $\{Y_1, \ldots, Y_k\}$ a basis of $\mathfrak{n}^-$, and $\{H_1, \ldots, H_r\}$ a basis of $\mathfrak{h}$. Since $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ (as vector spaces), the Poincaré–Birkhoff–Witt theorem implies that

$$\{Y^I H^J X^K \mid I, K \in \mathbb{N}_0^k, \ J \in \mathbb{N}_0^r\} \quad \text{and} \quad \{X^K H^J Y^I \mid I, K \in \mathbb{N}_0^k, \ J \in \mathbb{N}_0^r\}$$

are bases for $\mathscr{U} \mathfrak{g}$. Here we use the multi-index notation $Y^I := Y_1^{I_1} \cdots Y_k^{I_k}$ (and similarly for H and X). Define maps

$$\begin{aligned} &\pi_\lambda^- \colon \mathscr{U} \mathfrak{g} \to \mathscr{U}(\mathfrak{n}^-), \\ &\qquad \pi_\lambda^-(Y^I H^J X^K) := \lambda(H_1)^{J_1} \cdots \lambda(H_r)^{J_r} Y^I \delta_{K,0}, \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &\pi_\lambda^+ \colon \mathscr{U} \mathfrak{g} \to \mathscr{U}(\mathfrak{n}^+), \\ &\qquad \pi_\lambda^+(X^K H^J Y^I) := \bigl(-\lambda(H_1)\bigr)^{J_1} \cdots \bigl(-\lambda(H_r)\bigr)^{J_r} X^K \delta_{I,0}, \end{aligned} \tag{3.2b}$$

$$\begin{aligned} &\pi_\lambda^* \colon \mathscr{U} \mathfrak{g} \to \mathscr{U}(\mathfrak{n}^+), \\ &\qquad \pi_\lambda^*(Y^I H^J X^K) := \lambda(H_1)^{J_1} \cdots \lambda(H_r)^{J_r} X^K \delta_{I,0}, \end{aligned} \tag{3.2c}$$

where $\delta_{K,0}$ is 1 if $K = (0, \ldots, 0)$ and is 0 otherwise. Note that $\pi_\lambda^\pm$ and π_λ^* are independent of the choice of bases. Fix non-zero vectors $1 \in \mathbb{C}_\lambda^\pm$ and $1 \in \mathbb{C}_\lambda^*$ (thinking of $\mathbb{C}$ as a vector space, this choice is not canonical).

Lemma 3.2. *The maps $\cdot \otimes 1 \colon \mathscr{U}(\mathfrak{n}^-) \to M_\lambda$, $v \mapsto v \otimes 1$ and $\cdot \otimes 1 \colon \mathscr{U}(\mathfrak{n}^+) \to M_\lambda^-$, $u \mapsto u \otimes 1$ define isomorphisms of left $\mathscr{U}(\mathfrak{n}^-)$-modules and $\mathscr{U}(\mathfrak{n}^+)$-modules, respectively. The map $1 \otimes \cdot \colon \mathscr{U}(\mathfrak{n}^+) \to M_\lambda^*$, $u \mapsto 1 \otimes u$ defines an isomorphism of right $\mathscr{U}(\mathfrak{n}^+)$-modules. The $\mathscr{U} \mathfrak{g}$-module structures on $\mathscr{U}(\mathfrak{n}^\pm)$ obtained by transferring the module structures on the Verma modules with these isomorphisms are given explicitly by*

$$\rhd_\lambda^- \colon \mathscr{U} \mathfrak{g} \times \mathscr{U}(\mathfrak{n}^-) \to \mathscr{U}(\mathfrak{n}^-), \quad (w, v) \mapsto w \rhd_\lambda^- v := \pi_\lambda^-(wv), \tag{3.3a}$$

$$\rhd_\lambda^+ \colon \mathscr{U} \mathfrak{g} \times \mathscr{U}(\mathfrak{n}^+) \to \mathscr{U}(\mathfrak{n}^+), \quad (w, u) \mapsto w \rhd_\lambda^+ u := \pi_\lambda^+(wu), \tag{3.3b}$$

$$\lhd_\lambda^* \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U} \mathfrak{g} \to \mathscr{U}(\mathfrak{n}^+), \quad (u, w) \mapsto u \lhd_\lambda^* w := \pi_\lambda^*(uw). \tag{3.3c}$$

Furthermore, $S(w \rhd_\lambda^+ u) = S(u) \lhd_\lambda^* S(w)$, where S denotes the antipode of $\mathscr{U} \mathfrak{g}$. Or, in other words, $S \colon \mathscr{U}(\mathfrak{n}^+) \to \mathscr{U}(\mathfrak{n}^+)$ is an isomorphism from the left $\mathscr{U} \mathfrak{g}$-module $(\mathscr{U}(\mathfrak{n}^+), \rhd_\lambda^+)$ to the right $\mathscr{U} \mathfrak{g}$-module $(\mathscr{U}(\mathfrak{n}^+), \lhd_\lambda^*)$ over the map $S \colon \mathscr{U} \mathfrak{g} \to \mathscr{U} \mathfrak{g}$.

Proof. One checks easily that the maps $M_\lambda \to \mathscr{U}(\mathfrak{n}^-)$, $w \otimes z1 \mapsto z \cdot \pi_\lambda^-(w)$ and $M_\lambda^- \to \mathscr{U}(\mathfrak{n}^+)$, $w \otimes z1 \mapsto z \cdot \pi_\lambda^+(w)$ as well as $M_\lambda^* \to \mathscr{U}(\mathfrak{n}^+)$, $z1 \otimes w \mapsto z \cdot \pi_\lambda^*(w)$ are all well defined and inverses of the maps in the statement of the lemma. Consequently, we have $w \rhd_\lambda^- v = (\cdot \otimes 1)^{-1}(wv \otimes 1) = \pi_\lambda^-(wv)$, and (3.3b) and (3.3c) follow similarly. Finally,

$\pi_\lambda^* \circ S = S \circ \pi_\lambda^+$, so $S(w \rhd_\lambda^+ u) = S \circ \pi_\lambda^+(wu) = \pi_\lambda^* \circ S(wu) = \pi_\lambda^*(S(u)S(w)) = S(u) \lhd_\lambda^* S(w)$. ∎

The pairing of the left $\mathscr{U}\mathfrak{g}$-modules $(\mathscr{U}(\mathfrak{n}^\pm), \rhd_\lambda^\pm)$ obtained from the Shapovalov pairing by composing with the isomorphisms $(\mathscr{U}(\mathfrak{n}^-), \rhd_\lambda^-) \xrightarrow{\cdot \otimes 1} M_\lambda$ and $(\mathscr{U}(\mathfrak{n}^+), \rhd_\lambda^+) \xrightarrow{S} (\mathscr{U}(\mathfrak{n}^+), \lhd_\lambda^*) \xrightarrow{1 \otimes \cdot} M_\lambda^*$ of the previous lemma is

$$\langle \cdot, \cdot \rangle_\lambda \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}(\mathfrak{n}^-) \to \mathbb{C}, \quad (u, v) \mapsto \langle u, v \rangle_\lambda := \langle 1 \otimes S(u), v \otimes 1 \rangle_\lambda'. \quad (3.4)$$

In order to compute $\langle u, v \rangle_\lambda$ for $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$, one needs to write $S(u)v \in \mathscr{U}\mathfrak{g}$ in the form $\sum_i v_i' h_i' u_i'$ with $u_i' \in \mathscr{U}(\mathfrak{n}^+)$, $v_i' \in \mathscr{U}(\mathfrak{n}^-)$, and $h_i' \in \mathscr{U}\mathfrak{h}$. The pairing is then given by summing $\lambda(h_i')$ for those summands that have $v_i' = u_i' = 1$. This is made more precise in the next lemma. Define $\pi_\lambda := \pi_\lambda^- \circ \pi_\lambda^* = \pi_\lambda^* \circ \pi_\lambda^- \colon \mathscr{U}\mathfrak{g} \to \mathbb{C}$, where $\mathbb{C}$ is identified with $\mathbb{C}1 \subseteq \mathscr{U}(\mathfrak{n}^\pm)$ and we have implicitly used the inclusion $\mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}\mathfrak{g}$ when composing the maps.

Lemma 3.3. *For $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$, the pairing $\langle \cdot, \cdot \rangle_\lambda$ defined in (3.4) can be computed as*

$$\langle u, v \rangle_\lambda = \pi_\lambda(S(u)v). \quad (3.5)$$

It is $\mathscr{U}\mathfrak{g}$-invariant, in the sense that $\langle w \rhd_\lambda^+ u, v \rangle_\lambda = \langle u, S(w) \rhd_\lambda^- v \rangle_\lambda$ for $u \in \mathscr{U}(\mathfrak{n}^+)$, $v \in \mathscr{U}(\mathfrak{n}^-)$, and $w \in \mathscr{U}\mathfrak{g}$. The pairing respects the degree d defined in the beginning of this section, meaning that $\langle u, v \rangle_\lambda = 0$ for homogeneous elements $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ with $d(u) \neq -d(v)$. Furthermore, if $d(u) = -d(v)$, then

$$\langle u, v \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)} = S(u) \rhd_\lambda^- v \quad \text{and} \quad \langle u, v \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)} = S(v) \rhd_\lambda^+ u. \quad (3.6)$$

Proof. By definition $\langle u, v \rangle_\lambda = 1 \otimes_{\mathscr{U}(\mathfrak{b}^-)} S(u)v \otimes_{\mathscr{U}(\mathfrak{b}^+)} 1$. So to prove (3.5) it suffices to check that $1 \otimes_{\mathscr{U}(\mathfrak{b}^-)} w \otimes_{\mathscr{U}(\mathfrak{b}^+)} 1 = \pi_\lambda(w)$ for all $w \in \mathscr{U}\mathfrak{g}$, which one can easily verify on the basis $\{Y^I H^J X^K \mid I, K \in \mathbb{N}_0^k, J \in \mathbb{N}_0^r\}$. The $\mathscr{U}\mathfrak{g}$-invariance follows by noting that $\langle \cdot, \cdot \rangle_\lambda'$ is $\mathscr{U}\mathfrak{g}$-invariant, meaning $\langle xw, y \rangle_\lambda' = \langle x, wy \rangle_\lambda'$ for $x \in M_\lambda^*$ and $y \in M_\lambda$, and using the isomorphisms of the previous lemma. For homogeneous $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ with $d(u) \neq -d(v)$, it follows that $S(u)v$ is also homogeneous of degree $d(u) + d(v) \neq 0$ and therefore $\pi_\lambda(S(u)v) = 0$. Finally, if $d(u) = -d(v)$, then $d(S(u)v) = 0$ and $\langle u, v \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)} = \pi_\lambda(S(u)v)1_{\mathscr{U}(\mathfrak{n}^-)} = \pi_\lambda^-(S(u)v) = S(u) \rhd_\lambda^- v$, implying the first equality of (3.6). The second one follows from applying S on both sides of $\langle u, v \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)} = \pi_\lambda(S(u)v)1_{\mathscr{U}(\mathfrak{n}^+)} = \pi_\lambda^*(S(u)v) = S(\pi_\lambda^+(S(v)u)) = S(S(v) \rhd_\lambda^+ u)$. ∎

If the pairing $\langle \cdot, \cdot \rangle_\lambda$ is non-degenerate, we can pick bases $\{u_i\}_{i \in \mathbb{N}}$ of $\mathscr{U}(\mathfrak{n}^+)$ and $\{v_j\}_{j \in \mathbb{N}}$ of $\mathscr{U}(\mathfrak{n}^-)$ consisting of homogeneous elements with respect to d and satisfying $\langle u_i, v_j \rangle_\lambda = \delta_{ij}$. Then the element $F_\lambda := \sum_{i=1}^\infty u_i \otimes v_i \in \mathscr{U}(\mathfrak{n}^+) \hat{\otimes} \mathscr{U}(\mathfrak{n}^-)$ is called the *canonical element* of the pairing. It is independent of the choice of bases. By $\mathscr{U}(\mathfrak{n}^+) \hat{\otimes} \mathscr{U}(\mathfrak{n}^-)$ we mean the completion of the tensor product with respect to the $\mathbb{Z}\Delta$-grading d defined in the beginning of this subsection, which is needed to make sense of the infinite sum. The following lemma is a standard statement when working with canonical elements.

Lemma 3.4. *Assume that $\langle \, \cdot \, , \, \cdot \, \rangle_\lambda$ is non-degenerate, and let $F_\lambda = \sum_{i=1}^{\infty} u_i \otimes v_i \in \mathscr{U}(\mathfrak{n}^+) \widehat{\otimes} \mathscr{U}(\mathfrak{n}^-)$ be its canonical element. Then*

$$\sum_{i=1}^{\infty} u_i \langle u, v_i \rangle_\lambda = u \quad \text{and} \quad \sum_{i=1}^{\infty} v_i \langle u_i, v \rangle_\lambda = v \tag{3.7}$$

hold for all $u \in \mathscr{U}(\mathfrak{n}^+)$ and all $v \in \mathscr{U}(\mathfrak{n}^-)$, and F_λ is uniquely determined by this property.

Note that $\langle u, v_i \rangle$ and $\langle u_i, v \rangle$ are non-zero for only finitely many indices i so that the sums in (3.7) are both finite. The pairing $\langle \, \cdot \, , \, \cdot \, \rangle_\lambda$ is non-degenerate precisely when the Verma modules are irreducible, but we will not need this below. In order to determine F_λ explicitly, we need to introduce some more notation.

Denote the Killing form of $\mathfrak{g}$ by B. Since $\mathfrak{g}$ is semisimple, B is non-degenerate on $\mathfrak{g}$. Extending linear functionals on $\mathfrak{h}$ by 0 on the root spaces $\mathfrak{g}^\alpha$, we may view $\mathfrak{h}^*$ as a subspace of $\mathfrak{g}^*$. Since B restricts to zero on $\mathfrak{h} \times \mathfrak{g}^\alpha$ for any $\alpha \in \Delta$, it follows that B is non-degenerate on $\mathfrak{h}$ and that the maps $^\flat \colon \mathfrak{g} \to \mathfrak{g}^*$ and $^\sharp \colon \mathfrak{g}^* \to \mathfrak{g}$ defined in Section 2.1 restrict to mutually inverse isomorphisms $^\flat \colon \mathfrak{h} \to \mathfrak{h}^*$ and $^\sharp \colon \mathfrak{h}^* \to \mathfrak{h}$. For $\alpha, \beta \in \mathfrak{h}^*$, let $(\alpha, \beta) := B(\alpha^\sharp, \beta^\sharp)$.

Denote the positive roots by $\alpha_1, \ldots, \alpha_k$. For every positive root $\alpha_i \in \Delta^+$ choose elements $X_i := X_{\alpha_i} \in \mathfrak{g}^{\alpha_i}$ and $Y_i := Y_{\alpha_i} = X_{-\alpha_i} \in \mathfrak{g}^{-\alpha_i}$ such that $B(X_i, Y_i) = 1$. Then we have $[X_i, Y_i] = \alpha_i^\sharp$ since, for all $H \in \mathfrak{h}$,

$$B([X_i, Y_i], H) = B(X_i, [Y_i, H]) = \alpha_i(H)B(X_i, Y_i) = \alpha_i(H) = B(\alpha_i^\sharp, H)$$

and B is non-degenerate on $\mathfrak{h}$. Note that $[\alpha_i^\sharp, X_i] = \alpha_i(\alpha_i^\sharp)X_i = (\alpha_i, \alpha_i)X_i$ and similarly $[\alpha_i^\sharp, Y_i] = -(\alpha_i, \alpha_i)Y_i$, so $X_i' = 2(\alpha_i, \alpha_i)^{-1}X_i$, $Y_i' = Y_i$, and $H_i' = 2(\alpha_i, \alpha_i)^{-1}\alpha_i^\sharp$ satisfy the commutation relations $[X_i', Y_i'] = H_i'$, $[H_i', X_i'] = 2X_i'$, and $[H_i', Y_i'] = -2Y_i'$ of the usual generators of $\mathfrak{sl}_2(\mathbb{C})$, the special linear Lie algebra in 2 dimensions.

Let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ be the *half-sum* of all positive roots. Denote non-negative integral linear combinations of positive roots by $\mathbb{N}_0 \Delta^+$. For $\lambda \in \mathfrak{h}^*$ fixed and $\mu \in \mathfrak{h}^*$ define the number

$$p_\lambda(\mu) := \frac{1}{2}(\mu, \mu) - (\rho, \mu) - (\lambda, \mu). \tag{3.8}$$

Recall that for a representation $\varrho \colon \mathfrak{g} \to V$ and $\mu \in \mathfrak{h}^*$ we define $V^\mu := \{v \in V \mid \varrho(H)v = \mu(H)v \text{ for all } H \in \mathfrak{h}\}$. If $V^\mu \neq \{0\}$, then we call μ a *weight* and any $v \in V^\mu$ is called a *weight vector* of weight μ. V is called a weight module if $V = \bigoplus_{\mu \in \mathfrak{h}^*} V^\mu$. A highest weight module is a weight module generated by a vector $v \in V$ satisfying $X_\alpha v = 0$ for all $\alpha \in \Delta^+$. It is said to be of highest weight μ if $v \in V^\mu$.

Lemma 3.5 (Ostapenko [35]). *Let V be a highest weight module of highest weight λ, assume that $\mu \in \mathbb{N}_0 \Delta^+$, and let $v \in V^{\lambda - \mu}$. Then*

$$- p_\lambda(\mu)v = \sum_{\alpha \in \Delta^+} Y_\alpha X_\alpha v. \tag{3.9}$$

Proof. Choose an orthonormal basis $\{H_1, \ldots, H_r\}$ of $\mathfrak{h}$ with respect to the Killing form. The Casimir element

$$c = \sum_{\alpha \in \Delta^+} (X_\alpha Y_\alpha + Y_\alpha X_\alpha) + \sum_{i=1}^{r} H_i H_i = \sum_{\alpha \in \Delta^+} (2 Y_\alpha X_\alpha + \alpha^\sharp) + \sum_{i=1}^{r} H_i H_i$$

acts as a scalar on V because V is generated by a highest weight vector and c is central in $\mathscr{U}\mathfrak{g}$. Evaluating it on a highest weight vector, the $Y_\alpha X_\alpha$-part vanishes and we obtain that c acts as multiplication by $\sum_{\alpha \in \Delta^+} (\alpha, \lambda) + \sum_{i=1}^{r} \lambda(H_i)\lambda(H_i) = (2\rho, \lambda) + (\lambda, \lambda)$. Therefore

$$(2\rho, \lambda)v + (\lambda, \lambda)v = 2 \sum_{\alpha \in \Delta^+} Y_\alpha X_\alpha v + (2\rho, \lambda - \mu)v + (\lambda - \mu, \lambda - \mu)v$$

holds for any $v \in V^{\lambda - \mu}$, and rearranging this equation proves the lemma. ∎

Let W be the set of words with letters from $\{1, \ldots, k\}$. For any $w = (w_1, \ldots, w_{|w|}) \in W$, we define $w^{\mathrm{opp}} := (w_{|w|}, \ldots, w_1)$, $w_{i\cdots j} := (w_i, \ldots, w_j)$, $X_w := X_{w_1} \cdots X_{w_{|w|}} \in \mathscr{U}(\mathfrak{n}^+)$, $Y_w := Y_{w_1} \cdots Y_{w_{|w|}} \in \mathscr{U}(\mathfrak{n}^-)$, and $\alpha_w := \alpha_{w_1} + \cdots + \alpha_{w_{|w|}}$. We use $w_{i \cdots j} := \emptyset$ if $j < i$, $X_\emptyset := 1$, $Y_\emptyset := 1$, and $\alpha_\emptyset := 0$. Furthermore, let

$$p_\lambda^w(\mu) := \prod_{i=0}^{|w|-1} p_\lambda(\mu - \alpha_{w_1 \cdots i}). \tag{3.10}$$

We call a set T of words a *tree* if $w = (w_1, \ldots, w_{|w|}) \in T$ implies that $w_{1 \cdots i} \in T$ for all $i = 0, \ldots, |w| - 1$ and $(w_1, w_2, \ldots, w_{|w|-1}, x) \in T$ for all $x \in \{1, \ldots, k\}$. See Figure 1 for a visualization of a tree. For a tree T we denote by $\max T$ the set of elements $w \in T$ such that $w \neq w'_{1 \cdots i}$ for any $w' \in T$ and any $i \in \{0, \ldots, |w'| - 1\}$. Finally, a tree is said to be *μ-admissible* if $p_\lambda(\mu - \alpha_w) \neq 0$ for all $w \in T \setminus \max T$, or equivalently if $p_\lambda^w(\mu) \neq 0$ for all $w \in T$.

Lemma 3.6 (Ostapenko [35]). *Let V be a highest weight module of highest weight λ, assume that $\mu \in \mathbb{N}_0 \Delta^+$, and let $v \in V^{\lambda - \mu}$. Then*

$$v = \sum_{w \in \max T} (-1)^{|w|} p_\lambda^w(\mu)^{-1} Y_w X_{w^{\mathrm{opp}}} v \tag{3.11}$$

holds for every μ-admissible tree T.

Proof. Apply the previous lemma repeatedly. ∎

Lemma 3.7. *Let V be a lowest weight module of lowest weight $-\lambda$, assume that $\mu \in \mathbb{N}_0 \Delta^+$, and let $v \in V^{-\lambda + \mu}$. Then $\sum_{\alpha \in \Delta^+} X_\alpha Y_\alpha v = -p_\lambda(\mu)v$, and*

$$v = \sum_{w \in \max T} (-1)^{|w|} p_\lambda^w(\mu)^{-1} X_w Y_{w^{\mathrm{opp}}} v \tag{3.12}$$

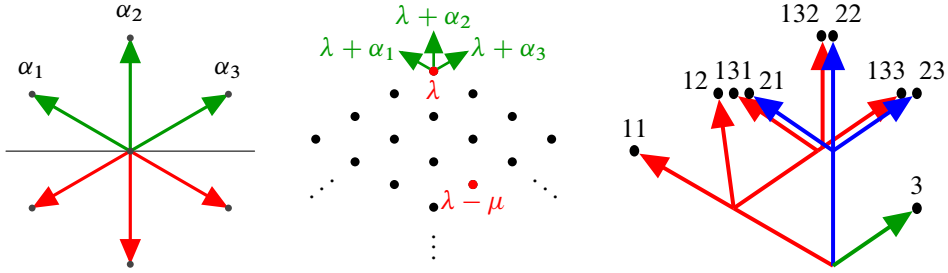*holds for every μ-admissible tree T.*

Figure 1. Left: The roots of $\mathfrak{sl}_3(\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_3(\mathbb{C})$ is 2-dimensional and there are six 1-dimensional root spaces. The picture shows the real subspace of $\mathfrak{h}^*$ spanned by the roots. The positive roots are denoted by α_1, α_2, and α_3 and drawn in green; negative roots are drawn in red. Middle: The weights in a highest weight module of highest weight λ. The picture shows again the real subspace of $\mathfrak{h}^*$ spanned by the roots. Weights are indicated by black dots, and $\mu = 3\alpha_1 + 2\alpha_3$. Note that since λ is a highest weight, the spaces $V^{\lambda+\alpha_1}$, $V^{\lambda+\alpha_2}$, and $V^{\lambda+\alpha_3}$ must all be trivial. Right: Visualization of the tree $T = \{\emptyset, 1, 2, 3, 11, 12, 13, 21, 22, 23, 131, 132, 133\}$. The elements of $\max T = \{3, 11, 12, 21, 22, 23, 131, 132, 133\}$ are indicated by black dots. Words starting with a 1 are colored red, words starting with a 2 blue, and words starting with a 3 green.

Proof. Similar to the proof of Lemmas 3.5 and 3.6. ∎

Define the set

$$\Lambda := \{\lambda \in \mathfrak{h}^* \mid p_\lambda(\mu) \neq 0 \; \forall \mu \in \mathbb{N}_0 \Delta^+ \setminus \{0\}\}. \tag{3.13}$$

Proposition 3.8. *The Shapovalov pairing* $\langle \cdot, \cdot \rangle_\lambda \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}(\mathfrak{n}^-) \to \mathbb{C}$ *is non-degenerate for* $\lambda \in \Lambda$*, and in this case its canonical element* $F_\lambda \in \mathscr{U}(\mathfrak{n}^+) \widehat{\otimes} \mathscr{U}(\mathfrak{n}^-)$ *is given by*

$$F_\lambda = \sum_{w \in W} p_\lambda^w (\alpha_w)^{-1} X_w \otimes Y_w = \sum_{w \in W} \prod_{i=1}^{|w|} p_\lambda \big(\alpha_{w_{i \cdots |w|}}\big)^{-1} X_w \otimes Y_w. \tag{3.14}$$

Proof. We check that F_λ satisfies the property given in Lemma 3.4. We decompose $v \in \mathscr{U}(\mathfrak{n}^-)$ as $v = \sum_{\mu \in \mathbb{N}_0 \Delta^+} v_{-\mu}$, where $v_{-\mu}$ is homogeneous of degree $-\mu$ with respect to the $\mathbb{Z}\Delta$-grading. For $\mu \in \mathbb{N}_0 \Delta^+$ let W_μ be the set of words $w \in W$ satisfying $\alpha_w = \mu$. Then

$$\sum_{w \in W} p_\lambda^w (\alpha_w)^{-1} Y_w \langle X_w, v \rangle_\lambda = \sum_{w \in W} p_\lambda^w (\alpha_w)^{-1} Y_w \vartriangleright_\lambda^- S(X_w) \vartriangleright_\lambda^- v_{-\alpha_w}$$

$$= \sum_{\mu \in \mathbb{N}_0 \Delta^+} \sum_{w \in W_\mu} (-1)^{|w|} p_\lambda^w (\alpha_w)^{-1} Y_w \vartriangleright_\lambda^- X_{w^{\mathrm{opp}}} \vartriangleright_\lambda^- v_{-\mu}$$

$$= \sum_{\mu \in \mathbb{N}_0 \Delta^+} v_{-\mu} = v.$$

The first equality holds because $Y_w \langle X_w, v \rangle_\lambda = Y_w \vartriangleright_\lambda^- (\langle X_w, v_{-\alpha_w} \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)}) = Y_w \vartriangleright_\lambda^- S(X_w) \vartriangleright_\lambda^- v_{-\alpha_w}$ by Lemma 3.3. The second equality is true by basic manipula-
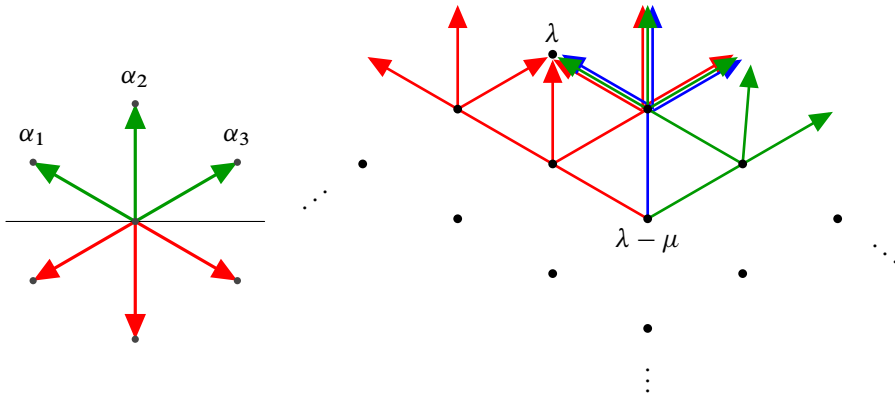
Figure 2. The tree T used in the proof of Proposition 3.8 for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mu = 2\alpha_1 + \alpha_3$. Elements of the tree starting with 1, 2 and 3 are colored red, blue, and green, respectively. Note that all weight spaces of maximal elements of this tree are trivial, except for V^λ. All non-maximal weight spaces are non-trivial.

tions. The third equality follows from Lemma 3.6 because we can rewrite the sum over all $w \in W_\mu$ as a sum over $\max T$ for a μ-admissible tree T (see Figure 2) as follows: define

$$T := \{\emptyset\} \cup \{w \in W \mid \exists w' \in W_\mu \text{ and } 0 \leq i \leq |w'| - 1 \text{ s.t. } w_{1 \cdots |w|-1} = w'_{1 \cdots i}\},$$

which is the smallest tree containing W_μ. Since $\lambda \in \Lambda$, this tree is μ-admissible, and clearly $W_\mu \subseteq \max T$. Let $w \in \max T$. Then either $\alpha_w = \mu$, so that $w \in W_\mu$, or there does not exist $w' \in W_\mu$ and $i \in \{0, \ldots, |w'|\}$ with $w = w'_{1 \cdots i}$, so that $\mu - \alpha_w \notin \mathbb{N}_0 \Delta^+$, and therefore $X_{w^{\mathrm{opp}}} \triangleright_\lambda^- v_{-\mu} = 0$.

Similarly, for $u = \sum_{\mu \in \mathbb{N}_0 \Delta^+} u_\mu \in \mathscr{U}(\mathfrak{n}^+)$ with $d(u_\mu) = \mu$ we compute that

$$\sum_{w \in W} p_\lambda^w(\mu)^{-1} X_w \langle u, Y_w \rangle_\lambda = \sum_{w \in W} p_\lambda^w(\mu)^{-1} X_w \triangleright_\lambda^+ S(Y_w) \triangleright_\lambda^+ u_{\alpha_w}$$

$$= \sum_{\mu \in \mathbb{N}_0 \Delta^+} \sum_{w \in W_\mu} (-1)^{|w|} p_\lambda^w(\mu)^{-1} X_w \triangleright_\lambda^+ Y_{w^{\mathrm{opp}}} \triangleright_\lambda^+ u_\mu$$

$$= \sum_{\mu \in \mathbb{N}_0 \Delta^+} u_\mu = u,$$

using $X_w \langle u, Y_w \rangle_\lambda = X_w \triangleright_\lambda^+ (\langle u_{\alpha_w}, Y_w \rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)}) = X_w \triangleright_\lambda^+ S(Y_w) \triangleright_\lambda^+ u_{\alpha_w}$ and that the sum over $w \in W_\mu$ can be rewritten as a sum over maximal elements of a tree T in a similar way as before. ∎

Using the inclusion $\mathscr{U}(\mathfrak{n}^+) \widehat{\otimes} \mathscr{U}(\mathfrak{n}^-) \to (\mathscr{U}\mathfrak{g})^{\widehat{\otimes} 2}$ and passing to the quotient, we can map the element F_λ from (3.14) to $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\widehat{\otimes} 2}$. Note that $\mathscr{U}\mathfrak{g} \cdot \mathfrak{h}$ is a homogeneous ideal in $\mathscr{U}\mathfrak{g}$ with respect to the degree d, so the quotient $\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h}$ is still graded.

The completed tensor product is defined with respect to this grading. The action of $\mathfrak{h}$ on $(\mathscr{U}\mathfrak{g})^{\otimes 2}$ given by $H \triangleright (w \otimes w') = \mathrm{ad}_H w \otimes w' + w \otimes \mathrm{ad}_H w'$ with $H \in \mathfrak{h}$ and $w, w' \in \mathscr{U}\mathfrak{g}$ stays well defined on the quotient and preserves the degree, so it extends uniquely to a continuous action on the completed tensor product. Denote the coproduct of the Hopf algebra $\mathscr{U}\mathfrak{g}$ by Δ. It is defined by extending the assignment $\mathfrak{g} \ni X \mapsto X \otimes 1 + 1 \otimes X \in \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$ to an algebra homomorphism $\Delta \colon \mathscr{U}\mathfrak{g} \to \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$.

Proposition 3.9 (Alekseev–Lachowska [1]). *Let $\lambda \in \Lambda$. Then $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\widehat{\otimes} 2}$ is $\mathfrak{h}$-invariant and satisfies*

$$\big((\mathrm{id} \otimes \Delta)F_\lambda\big)1 \otimes F_\lambda = \big((\Delta \otimes \mathrm{id})F_\lambda\big)F_\lambda \otimes 1 \tag{3.15}$$

in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\widehat{\otimes} 3}$.

Proof. See the proof of Theorem 3.23. ∎

Using the results of Section 2.2, elements of $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{h})^{\otimes 2})^H$ determine bidifferential operators on a complex coadjoint orbit for which $\mathfrak{g}_\lambda = \mathfrak{h}$. Such orbits are of maximal dimension among all coadjoint orbits and called *regular*. Note that H is automatically connected by Proposition 2.3, so $\mathfrak{h}$-invariance of F_λ implies H-invariance, but F_λ is only an element of the completed tensor product. So applying the construction from Section 2.2 naively gives a sum of bidifferential operators of increasing orders. To make sense of this sum, we can either introduce a formal parameter $\hbar$ in the construction in such a way that we obtain a formal power series of bidifferential operators, or we can restrict ourselves to applying these operators to some class of polynomials, for which only finitely many of the bidifferential operators appearing in the sum give a non-zero contribution.

We will now proceed as follows: in Section 3.2, we generalize the construction of F_λ to work for arbitrary stabilizers $\mathfrak{g}_\lambda$ (and not just $\mathfrak{h}$). In Section 3.3, we will give details on how to construct bidifferential operators out of F_λ, both in the formal and polynomial settings mentioned above.

3.2. Generalization to non-regular orbits

The aim of this subsection is to generalize the results of the last subsection to non-regular semisimple coadjoint orbits. To achieve this, we need to replace $\mathfrak{h}$ by a possibly larger stabilizer $\mathfrak{g}_\lambda$ and define a generalization of the Shapovalov pairing. When this pairing is non-degenerate, we derive an explicit formula for its canonical element, which satisfies (3.15).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra acting under the coadjoint action, i.e., the action dual to the adjoint action, on its dual $\mathfrak{g}^*$. We assume that $\lambda \in \mathfrak{g}^*$ is semisimple (as defined in Section 2.1) with stabilizer $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \mathrm{ad}_X^* \lambda = 0\}$. Fix a Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$ (which is possible since λ is semisimple) and denote the corresponding root system by Δ. Since any $H \in \mathfrak{h}$ commutes with $\lambda^\sharp$, it follows that $\mathrm{ad}_H^* \lambda = \lambda([-H, \cdot]) = -B(\lambda^\sharp, [H, \cdot]) = -B([\lambda^\sharp, H], \cdot) = 0$, so $\mathfrak{h} \subseteq \mathfrak{g}_\lambda$. We let

$$\Delta' := \big\{\alpha \in \Delta \mid (\alpha, \lambda) = 0\big\} \quad \text{and} \quad \widehat{\Delta} := \big\{\alpha \in \Delta \mid (\alpha, \lambda) \neq 0\big\} = \Delta \setminus \Delta'.$$
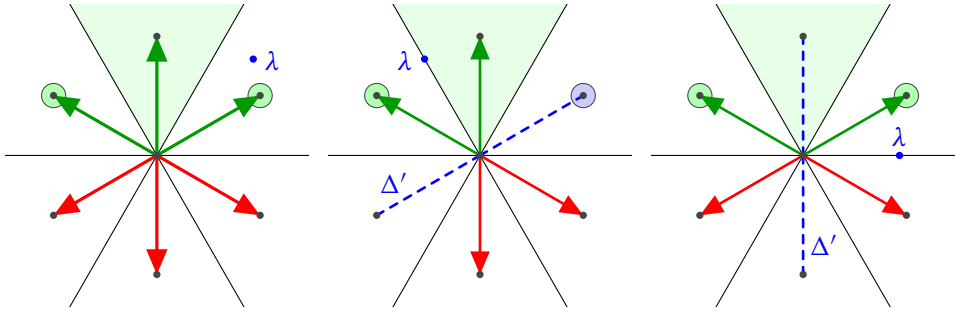
Figure 3. Invariant and non-invariant orderings. As in the left picture of Figure 1, the roots of $\mathfrak{sl}_3(\mathbb{C})$ are shown. Simple roots are encircled. Roots in Δ' are drawn with blue dashed lines. Roots in $\widehat{\Delta}$ are drawn in green if they are positive, and in red if they are negative. The fundamental Weyl chamber has a light green background. A regular orbit of $SL_3(\mathbb{C})$ is shown on the left; the other two pictures are of non-regular orbits. In the right picture, the ordering on Δ is not invariant, since adding the negative root in Δ' (the lower blue dashed line) to one of the positive roots (a green arrow) gives a negative root (a red arrow). The ordering in the middle picture is invariant and standard, the ordering in the left picture is invariant, but not standard. It would be standard if λ was in the fundamental Weyl chamber.

One checks easily that $\mathfrak{g}_\lambda = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}^\alpha$. Given an ordering on Δ with $\Delta^\pm$ being the set of positive resp. negative roots, define $\widehat{\Delta}^\pm = \Delta^\pm \cap \widehat{\Delta}$ and $(\Delta')^\pm = \Delta^\pm \cap \Delta'$. Furthermore, let $\tilde{\mathfrak{n}}^\pm := \bigoplus_{\alpha \in \widehat{\Delta}^\pm} \mathfrak{g}^\alpha$ and $\tilde{\mathfrak{b}}^\pm := \mathfrak{g}_\lambda \oplus \tilde{\mathfrak{n}}^\pm$.

Definition 3.10. An ordering of Δ is called *invariant* if, for any $\alpha \in \widehat{\Delta}^+$ and $\beta \in \Delta'$ such that $\alpha + \beta$ is again a root, this root $\alpha + \beta$ is in $\widehat{\Delta}^+$.

Note that since the sum of two roots in Δ' is again in Δ' (if it is a root), it is automatic that $\alpha + \beta \in \widehat{\Delta}$. The important part of the previous definition is that $\alpha + \beta$ should again be positive. See Figure 3 for an example of invariant and non-invariant orderings.

Lemma 3.11. *An ordering of Δ is invariant if and only if $\alpha + \beta \in \widehat{\Delta}^+$ holds for any $\alpha, \beta \in \Delta^+$ with $\alpha + \beta \in \Delta$.*

In the condition of the lemma, it is automatic that $\alpha + \beta$ is positive and the important part is that it lies in $\widehat{\Delta}$.

Proof. Assume the condition of the lemma is false, i.e., $\alpha, \beta \in \widehat{\Delta}^+$ and $\alpha + \beta \in \Delta \setminus \widehat{\Delta}^+$. Since $\alpha + \beta$ is positive, we must then have $\alpha + \beta \in \Delta'$. Consequently, $\alpha + (-(\alpha + \beta)) = -\beta \notin \widehat{\Delta}^+$, so the ordering is not invariant.

Conversely, if the ordering is not invariant, then we can find $\alpha \in \widehat{\Delta}^+$ and $\beta \in \Delta'$ such that $\alpha + \beta \in \Delta \setminus \widehat{\Delta}^+$. Then we must have $\alpha + \beta \in \widehat{\Delta}^-$ and therefore $\alpha + (-(\alpha + \beta)) = -\beta \notin \widehat{\Delta}^+$, so the condition of the lemma is not fulfilled. ∎

Intuitively the invariance of an ordering means that roots in Δ' are close to being simple or more precisely that they are linear combinations of simple roots in Δ'. Indeed,

if $\alpha \in (\Delta')^+$, then α is a non-negative linear combination of simple roots. By the lemma at least one of those simple roots, say σ, must be in Δ', so $\alpha = \sigma$ or $\alpha - \sigma \in (\Delta')^+$ and we can apply induction.

Corollary 3.12. *If the ordering of Δ is invariant, then $\tilde{\mathfrak{n}}^\pm$ and $\tilde{\mathfrak{b}}^\pm$ are both Lie subalgebras of $\mathfrak{g}$. Moreover, $[\mathfrak{g}_\lambda, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$ and $[\mathfrak{g}_\lambda, \tilde{\mathfrak{b}}^\pm] \subseteq \tilde{\mathfrak{b}}^\pm$.*

Proof. The condition in the previous lemma says precisely that $[\tilde{\mathfrak{n}}^\pm, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$, i.e., that $\tilde{\mathfrak{n}}^\pm$ is a Lie subalgebra of $\mathfrak{g}$. The defining property of an invariant ordering means that $[\mathfrak{g}_\lambda, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$. The statements for $\tilde{\mathfrak{b}}^\pm$ are then clear. ∎

Definition 3.13. We say an ordering is *standard* if there is a set $S \subseteq \mathbb{C} \setminus \{0\}$, closed under addition and satisfying $S \cap (-S) = \emptyset$, $S \cup (-S) = \mathbb{C} \setminus \{0\}$ such that $\alpha \in \hat{\Delta}$ is positive if and only if $(\alpha, \lambda) \in S$.

Standard invariant orderings exist always since we can construct them as follows. First, take any ordering on the set Δ' (meaning a subset $(\Delta')^+$ such that if the sum of two elements of $(\Delta')^+$ is in Δ', then it is in $(\Delta')^+$ and such that for $(\Delta')^- := -(\Delta')^+$ we have $(\Delta')^+ \cup (\Delta')^- = \Delta'$ and $(\Delta')^+ \cap (\Delta')^- = \emptyset$). Then choose a set S that is closed under addition and satisfies $S \cap (-S) = \emptyset$ and $S \cup (-S) = \mathbb{C} \setminus \{0\}$, e.g., $S = \{z \in \mathbb{C} \setminus \{0\} \mid \mathrm{Re}(z) > 0 \text{ or } z \in i\mathbb{R}^+\}$. Let $\alpha \in \Delta$ be positive if $\alpha \in (\Delta')^+$ or $(\alpha, \lambda) \in S$.

For real coadjoint orbits standard invariant orderings are the ones which induce star products of pseudo Wick type (under some further assumptions, see Proposition 5.21) and therefore the orderings we are mainly interested in. However, the construction below works also for other (possibly non-standard) invariant orderings.

Before generalizing the results of the last subsection, we would like to mention the following technical lemma for later use.

Lemma 3.14. *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\lambda \in \mathfrak{g}^*$ be semisimple, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$. Assume that we have chosen an invariant ordering defining sets Δ^+, $\hat{\Delta}$, and Δ' as above. Then there is a constant $M \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the sum of m positive roots in $\hat{\Delta}^+$ and at least Mm positive roots in $(\Delta')^+$ is not in $\mathbb{N}_0 \hat{\Delta}^+$.*

Proof. Label the simple roots by $\sigma_1, \ldots, \sigma_r$ such that the first r' simple roots $\sigma_1, \ldots, \sigma_{r'}$ are in Δ' and the remaining simple roots are in $\hat{\Delta}$. Label all roots in $\hat{\Delta}^+$ by $\alpha_1, \ldots, \alpha_{\tilde{k}}$. Then there are unique non-negative integers $c_j^i \in \mathbb{N}_0$ such that $\alpha_j = \sum_{i=1}^r c_j^i \sigma_i$. Set $M' = \max_{j \in \{1, \ldots, \tilde{k}\}} \sum_{i=1}^{r'} c_j^i$, $M'' = \max_{j \in \{1, \ldots, \tilde{k}\}} \sum_{i=r'+1}^r c_j^i$, and $M = M'M'' + 1$.

Since $\alpha_j \in \hat{\Delta}^+$, we have $\sum_{i=r'+1}^r c_j^i \geq 1$ for any $j \in \{1, \ldots, \tilde{k}\}$, and $\sum_{i=1}^{r'} c_j^i \leq M' \leq M' \sum_{i=r'+1}^r c_j^i$. Note that any element $\beta \in \mathbb{N}_0 \hat{\Delta}^+$ can be written uniquely as $\beta = \sum_{i=1}^r \beta^i \sigma_i$ with $\beta^i \in \mathbb{N}_0$, and the coefficients satisfy the same inequality

$$\sum_{i=1}^{r'} \beta^i \leq M' \sum_{i=r'+1}^r \beta^i.$$

Recall that any root in $(\Delta')^+$ is a linear combination of simple roots in $(\Delta')^+$. So if $\sum_{i=1}^r d^i \sigma_i \in (\Delta')^+$, then $d^i = 0$ for all $i = r' + 1, \ldots, r$. Therefore, if γ is the sum of m roots from $\hat{\Delta}^+$ and at least Mm roots from $(\Delta')^+$, and $\gamma = \sum_{i=1}^r \gamma^i \sigma_i$, then $M' \sum_{i=r'+1}^r \gamma^i \le M'M''m < Mm \le \sum_{i=1}^{r'} \gamma^i$, so γ cannot be in $\mathbb{N}_0 \hat{\Delta}^+$. ∎

For a regular coadjoint orbit we have $\Delta' = \emptyset$. Consequently, $\hat{\Delta} = \Delta$, $\mathfrak{g}_\lambda = \mathfrak{h}$, $\tilde{\mathfrak{n}}^+ = \mathfrak{n}^+$, and $\tilde{\mathfrak{n}}^- = \mathfrak{n}^-$. In this case, every ordering is invariant, and the generalized Shapovalov pairing, that we will introduce now, coincides with the Shapovalov pairing introduced in the last subsection. Since $\mathfrak{g}_\lambda = \mathfrak{h}$ when $\Delta' = \emptyset$, we usually denote an element of $\mathfrak{g}_\lambda$ by H.

Let $\lambda \in \mathfrak{g}_\lambda^*$ be the restriction of $\lambda \in \mathfrak{g}^*$ to $\mathfrak{g}_\lambda$. Then $\lambda([H', H]) = \mathrm{ad}_H^* \lambda(H') = 0$ for all $H, H' \in \mathfrak{g}_\lambda$, so $H \triangleright z = \lambda(H)z$ makes $\mathbb{C}$ a left or right $\mathfrak{g}_\lambda$-module. Extending trivially along $\tilde{\mathfrak{n}}^\pm$ gives a left or right $\tilde{\mathfrak{b}}^\pm$-module, and we denote the corresponding left $\mathscr{U}(\tilde{\mathfrak{b}}^\pm)$-module by $\tilde{\mathbb{C}}_\lambda^\pm$ and the right $\mathscr{U}(\tilde{\mathfrak{b}}^-)$-module by $\tilde{\mathbb{C}}_\lambda^*$. Define the *generalized Verma modules*

$$\tilde{M}_\lambda = \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^+)} \tilde{\mathbb{C}}_\lambda^+,$$
$$\tilde{M}_\lambda^- = \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^-)} \tilde{\mathbb{C}}_{-\lambda}^*,$$
$$\tilde{M}_\lambda^* = \tilde{\mathbb{C}}_\lambda^* \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^-)} \mathscr{U}\mathfrak{g},$$
(3.16)

where $\tilde{M}_\lambda$ and $\tilde{M}_\lambda^-$ are left $\mathscr{U}\mathfrak{g}$-modules and $\tilde{M}_\lambda^*$ is a right $\mathscr{U}\mathfrak{g}$-module. Most of the results of the previous subsection have obvious analogues in this setting.

Let $\{X_1, \ldots, X_{\tilde{k}}\}$ be a basis of $\tilde{\mathfrak{n}}^+$, $\{Y_1, \ldots, Y_{\tilde{k}}\}$ a basis of $\tilde{\mathfrak{n}}^-$, and $\{H_1, \ldots, H_{\tilde{r}}\}$ a basis of $\mathfrak{g}_\lambda$. Since $\mathfrak{g} = \tilde{\mathfrak{n}}^+ \oplus \mathfrak{g}_\lambda \oplus \tilde{\mathfrak{n}}^-$, the Poincaré–Birkhoff–Witt theorem implies that

$$\{Y^I H^J X^K \mid I, K \in \mathbb{N}_0^{\tilde{k}},\ J \in \mathbb{N}_0^{\tilde{r}}\} \quad \text{and} \quad \{X^K H^J Y^I \mid I, K \in \mathbb{N}_0^{\tilde{k}},\ J \in \mathbb{N}_0^{\tilde{r}}\}$$

are bases for $\mathscr{U}\mathfrak{g}$. Define maps

$$\tilde{\pi}_\lambda^- : \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^-),$$
$$\tilde{\pi}_\lambda^-(Y^I H^J X^K) := \lambda(H_1)^{J_1} \cdots \lambda(H_{\tilde{r}})^{J_{\tilde{r}}} Y^I \delta_{K,0},$$
(3.17a)

$$\tilde{\pi}_\lambda^+ : \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^+),$$
$$\tilde{\pi}_\lambda^+(X^K H^J Y^I) := \big(-\lambda(H_1)\big)^{J_1} \cdots \big(-\lambda(H_{\tilde{r}})\big)^{J_{\tilde{r}}} X^K \delta_{I,0},$$
(3.17b)

$$\tilde{\pi}_\lambda^* : \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^+),$$
$$\tilde{\pi}_\lambda^*(Y^I H^J X^K) := \lambda(H_1)^{J_1} \cdots \lambda(H_{\tilde{r}})^{J_{\tilde{r}}} X^K \delta_{I,0}.$$
(3.17c)

Note that they are compatible with the maps π_λ^-, π_λ^+, and π_λ^* in the sense that $\tilde{\pi}_\lambda^- \circ \pi_\lambda^- = \tilde{\pi}_\lambda^-$, $\tilde{\pi}_\lambda^+ \circ \pi_\lambda^+ = \tilde{\pi}_\lambda^+$, and $\tilde{\pi}_\lambda^* \circ \pi_\lambda^* = \tilde{\pi}_\lambda^*$. On the left-hand sides, we are implicitly using the inclusion $\mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}\mathfrak{g}$. Note that this inclusion is not a $\mathscr{U}\mathfrak{g}$-module map.

Lemma 3.15. *The maps $\cdot \otimes 1 : \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \tilde{M}_\lambda$, $v \mapsto v \otimes 1$ and $\cdot \otimes 1 : \mathscr{U}(\tilde{\mathfrak{n}}^+) \to \tilde{M}_\lambda^-$, $u \mapsto u \otimes 1$ define isomorphisms of left $\mathscr{U}(\tilde{\mathfrak{n}}^-)$-modules and $\mathscr{U}(\tilde{\mathfrak{n}}^+)$-modules, respectively. The map $1 \otimes \cdot : \mathscr{U}(\tilde{\mathfrak{n}}^+) \to M_\lambda^*$, $u \mapsto 1 \otimes u$ is an isomorphism of right $\mathscr{U}(\tilde{\mathfrak{n}}^+)$-modules. The $\mathscr{U}\mathfrak{g}$-module structures on $\mathscr{U}(\tilde{\mathfrak{n}}^\pm)$ obtained by transferring the module*

structures on the generalized Verma modules with these isomorphisms are given explicitly by

$$\widetilde{\rhd}_\lambda^- : \mathcal{U}\mathfrak{g} \times \mathcal{U}(\tilde{\mathfrak{n}}^-) \to \mathcal{U}(\tilde{\mathfrak{n}}^-), \quad (w, v) \mapsto w \,\widetilde{\rhd}_\lambda^-\, v := \tilde{\pi}_\lambda^-(wv), \tag{3.18a}$$

$$\widetilde{\rhd}_\lambda^+ : \mathcal{U}\mathfrak{g} \times \mathcal{U}(\tilde{\mathfrak{n}}^+) \to \mathcal{U}(\tilde{\mathfrak{n}}^+), \quad (w, u) \mapsto w \,\widetilde{\rhd}_\lambda^+\, u := \tilde{\pi}_\lambda^+(wu), \tag{3.18b}$$

$$\widetilde{\lhd}_\lambda^* : \mathcal{U}(\tilde{\mathfrak{n}}^+) \times \mathcal{U}\mathfrak{g} \to \mathcal{U}(\tilde{\mathfrak{n}}^+), \quad (u, w) \mapsto u \,\widetilde{\lhd}_\lambda^*\, w := \tilde{\pi}_\lambda^*(uw). \tag{3.18c}$$

Furthermore, $S(w \,\widetilde{\rhd}_\lambda^+\, u) = S(u) \,\widetilde{\lhd}_\lambda^*\, S(w)$, *where S denotes the antipode of $\mathcal{U}\mathfrak{g}$.*

Proof. Similar to the proof of Lemma 3.2. ∎

Note that since $\mathcal{U}(\tilde{\mathfrak{n}}^\pm)$ is a $\mathcal{U}\mathfrak{g}$-module, we must have

$$\tilde{\pi}_\lambda^\pm\big(w\tilde{\pi}_\lambda^\pm(w')\big) = w \,\widetilde{\rhd}_\lambda^\pm\, (w' \,\widetilde{\rhd}_\lambda^\pm\, 1) = (ww') \,\widetilde{\rhd}_\lambda^\pm\, 1 = \tilde{\pi}_\lambda^\pm(ww') \tag{3.19}$$

and

$$\tilde{\pi}_\lambda^*\big(\tilde{\pi}_\lambda^*(w)w'\big) = \tilde{\pi}_\lambda^*(ww') \tag{3.20}$$

for all $w, w' \in \mathcal{U}\mathfrak{g}$. This implies that the map $\tilde{\pi}_\lambda^\pm|_{\mathcal{U}(\mathfrak{n}^\pm)} : \mathcal{U}(\mathfrak{n}^\pm) \to \mathcal{U}(\tilde{\mathfrak{n}}^\pm)$ is a $\mathcal{U}\mathfrak{g}$-module homomorphism (with respect to the module structures given by $\rhd_\lambda^\pm$ and $\widetilde{\rhd}_\lambda^\pm$). Indeed, for the plus case we have

$$\tilde{\pi}_\lambda^+(w \rhd_\lambda^+ u) = \tilde{\pi}_\lambda^+ \pi_\lambda^+(wu) = \tilde{\pi}_\lambda^+(wu) = \tilde{\pi}_\lambda^+(w\tilde{\pi}_\lambda^+ u) = w \,\widetilde{\rhd}_\lambda^+\, \tilde{\pi}_\lambda^+ u$$

for all $w \in \mathcal{U}\mathfrak{g}$ and $u \in \mathcal{U}(\mathfrak{n}^+)$ and the minus case is similar. Define $\mathfrak{g}_\lambda^\pm := \bigoplus_{\alpha \in (\Delta')^\pm} \mathfrak{g}^\alpha = \mathfrak{g}_\lambda \cap \mathfrak{n}^\pm$. Note that $\mathcal{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^\pm = \{w \rhd_\lambda^\pm X \mid w \in \mathcal{U}\mathfrak{g}, X \in \mathfrak{g}_\lambda^\pm\}$ is a $\mathcal{U}\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{n}^\pm)$. Since $\tilde{\pi}_\lambda^\pm$ is a map of $\mathcal{U}\mathfrak{g}$-modules and vanishes on $\mathfrak{g}_\lambda^\pm$, $\mathcal{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^\pm$ is in its kernel.

Lemma 3.16. *The induced maps $\tilde{\pi}_\lambda^\pm : \mathcal{U}(\mathfrak{n}^\pm)/\mathcal{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^\pm \to \mathcal{U}(\tilde{\mathfrak{n}}^\pm)$ are isomorphisms of $\mathcal{U}\mathfrak{g}$-modules.*

Proof. It is easy to check that the quotient map induced by the inclusion $\mathcal{U}(\tilde{\mathfrak{n}}^\pm) \to \mathcal{U}(\mathfrak{n}^\pm)$ defines an inverse. ∎

As before, there are isomorphisms $\tilde{M}_\lambda^* \otimes_{\mathcal{U}\mathfrak{g}} \tilde{M}_\lambda \cong \widetilde{\mathbb{C}}_\lambda^- \otimes_{\mathcal{U}(\tilde{\mathfrak{b}}^-)} \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\tilde{\mathfrak{b}}^+)} \widetilde{\mathbb{C}}_\lambda \cong \widetilde{\mathbb{C}}_\lambda^- \otimes_{\mathcal{U}(\mathfrak{g}_\lambda)} \widetilde{\mathbb{C}}_\lambda \cong \mathbb{C}$, which we use to define the Shapovalov pairings $\langle \cdot, \cdot \rangle_\lambda^{\sim'} : \tilde{M}_\lambda^* \times \tilde{M}_\lambda \to \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle_\lambda^{\sim'} := x \otimes y$ and

$$\langle \cdot, \cdot \rangle_\lambda^\sim : \mathcal{U}(\tilde{\mathfrak{n}}^+) \times \mathcal{U}(\tilde{\mathfrak{n}}^-) \to \mathbb{C},$$
$$\langle u, v \rangle_\lambda^\sim = \big\langle 1 \otimes S(u), v \otimes 1 \big\rangle_\lambda^{\sim'} = 1 \otimes S(u)v \otimes 1. \tag{3.21}$$

In the same way as in Lemma 3.3, one proves that this pairing can be computed by

$$\langle u, v \rangle_\lambda^\sim = \pi_\lambda\big(S(u)v\big). \tag{3.22}$$

Note that $\tilde{\pi}_\lambda^- \circ \tilde{\pi}_\lambda^* = \tilde{\pi}_\lambda^* \circ \tilde{\pi}_\lambda^- = \pi_\lambda^* \circ \pi_\lambda^- = \pi_\lambda$, so there is no need to introduce a $\tilde{\pi}_\lambda$.

Lemma 3.17. *Let $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$. Then $\langle \tilde{\pi}^+_\lambda u, \tilde{\pi}_\lambda v \rangle^\sim_\lambda = \langle u, v \rangle_\lambda$. In particular, $\langle \,\cdot\,, \cdot\, \rangle_\lambda|_{\mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}\mathfrak{g} \cdot \mathfrak{g}^-_\lambda} = \langle \,\cdot\,, \cdot\, \rangle_\lambda|_{\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}^+_\lambda \times \mathscr{U}(\mathfrak{n}^-)} = 0$.*

Proof. Using (3.19) twice, we compute

$$
\begin{aligned}
\langle \tilde{\pi}^+_\lambda u, \tilde{\pi}^-_\lambda v \rangle^\sim_\lambda &= \pi_\lambda \big(S(\tilde{\pi}^+_\lambda u) \tilde{\pi}^-_\lambda v \big) = \tilde{\pi}^*_\lambda \circ \tilde{\pi}^-_\lambda \big(\tilde{\pi}^*_\lambda (Su) \tilde{\pi}^-_\lambda v \big) \\
&= \tilde{\pi}^*_\lambda \circ \tilde{\pi}^-_\lambda \big(\tilde{\pi}^*_\lambda (Su) v \big) = \tilde{\pi}^-_\lambda \circ \tilde{\pi}^*_\lambda \big(\tilde{\pi}^*_\lambda (Su) v \big) \\
&= \tilde{\pi}^-_\lambda \circ \tilde{\pi}^*_\lambda \big(S(u) v \big) = \pi_\lambda \big(S(u) v \big) = \langle u, v \rangle_\lambda. \qquad \blacksquare
\end{aligned}
$$

Define the set

$$
\tilde{\Lambda} = \big\{ \lambda \in \mathfrak{h}^* \mid p_\lambda(\mu) \neq 0 \ \forall \mu \in \mathbb{N}_0 \hat{\Delta}^+ \setminus \{0\} \big\}. \tag{3.23}
$$

Furthermore, let $\tilde{W}$ be the set of words $w \in W$ such that $\alpha_{w_i \cdots |w|} \in \mathbb{N}_0 \hat{\Delta}^+$ for all $i = 1, \dots, |w|$. Since $\tilde{\pi}^+_\lambda(X_w) = 0$ and $\tilde{\pi}^-_\lambda(Y_w) = 0$ for $w \in W \setminus \tilde{W}$, the following theorem is not surprising.

Theorem 3.18. *Let $\lambda \in \tilde{\Lambda}$. Then the Shapovalov pairing $\langle \,\cdot\,, \cdot\, \rangle^\sim_\lambda \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathbb{C}$ is non-degenerate. Its canonical element $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \hat{\otimes} \mathscr{U}(\tilde{\mathfrak{n}}^-)$ is given by*

$$
\begin{aligned}
F_\lambda &= \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w) \\
&= \sum_{w \in \tilde{W}} \prod_{i=1}^{|w|} p_\lambda(\alpha_{w_i \cdots |w|})^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w). \tag{3.24}
\end{aligned}
$$

Proof. It suffices to prove that $\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle \tilde{\pi}^+_\lambda(X_w), \tilde{v} \rangle^\sim_\lambda = \tilde{v}$ for all $\tilde{v} \in \mathscr{U}(\tilde{\mathfrak{n}}^-)$ and that $\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \langle \tilde{u}, \tilde{\pi}^-_\lambda(Y_w) \rangle^\sim_\lambda = \tilde{u}$ for all $\tilde{u} \in \mathscr{U}(\tilde{\mathfrak{n}}^+)$ by using an analogue of Lemma 3.4. Let $v \in \mathscr{U}(\mathfrak{n}^-)$ be the image of $\tilde{v}$ under the inclusion $\mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathscr{U}(\mathfrak{n}^-)$ so that $\tilde{\pi}^-_\lambda(v) = \tilde{v}$. Assume that $v = \sum_{\mu \in \mathbb{N}_0 \hat{\Delta}^+} v_{-\mu}$ is the weight decomposition of v. Then

$$
\begin{aligned}
&\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle \tilde{\pi}^+_\lambda(X_w), \tilde{v} \rangle^\sim_\lambda \\
&\quad = \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle X_w, v \rangle_\lambda \\
&\quad = \tilde{\pi}^-_\lambda \Big(\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} Y_w \langle X_w, v_{-\alpha_w} \rangle_\lambda \Big) \\
&\quad = \tilde{\pi}^-_\Lambda \Big(\sum_{\mu \in \mathbb{N}_0 \hat{\Delta}^+} \sum_{w \in \tilde{W}_\mu} (-1)^{|w|} p^w_\lambda(\alpha_w)^{-1} Y_w \vartriangleright^-_\lambda X_{w^{\mathrm{opp}}} \vartriangleright^-_\lambda v_{-\mu} \Big),
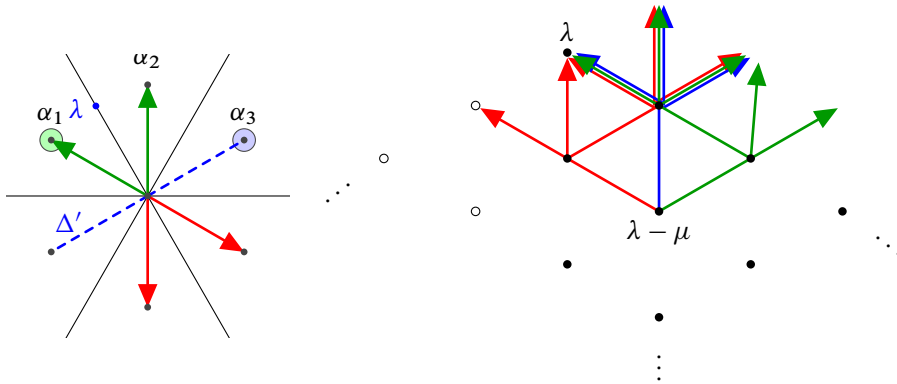\end{aligned}
$$

Figure 4. The tree T used in the proof of Theorem 3.18 for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mu = 2\alpha_1 + \alpha_3$. Compare this with Figure 2. Elements of the tree starting with 1, 2, and 3 are colored red, blue, and green, respectively. Only the weight spaces marked with filled dots are non-trivial (but might have a different dimension than in the case where $\Delta' = \emptyset$), and all weight spaces marked with circles only contain 0. In particular, the weight spaces at maximal elements of the tree are trivial, except for V^λ. All non-maximal weight spaces are non-trivial.

where $\widetilde{W}_\mu = \{w \in \widetilde{W} \mid \alpha_w = \mu\}$. We claim that there is an admissible tree T and $v' \in \mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^-$ such that

$$\sum_{w \in \widetilde{W}_\mu} (-1)^{|w|} p_\lambda^w (\alpha_w)^{-1} Y_w \triangleright_\lambda^- X_{w^{\mathrm{opp}}} \triangleright_\lambda^- v_{-\mu}$$

$$= v' + \sum_{w \in \max T} (-1)^{|w|} p_\lambda^w (\alpha_w)^{-1} Y_w \triangleright_\lambda^- X_{w^{\mathrm{opp}}} \triangleright_\lambda^- v_{-\mu},$$

which would finish the proof by using Lemma 3.6. Indeed, let

$$T = \{\emptyset\} \cup \{w \in W \mid \exists w' \in \widetilde{W}_\mu \text{ and } 0 \le i \le |w'| - 1 \text{ s.t. } w_{1 \cdots |w| - 1} = w'_{1 \cdots i}\}$$

be the smallest tree containing $\widetilde{W}_\mu$ (see Figure 4). Since $\lambda \in \widetilde{\Lambda}$, this tree is admissible. Furthermore, $\widetilde{W}_\mu \subseteq \max T$ and any element $w \in \max T$ satisfies exactly one of the following two conditions. Either $\alpha_w = \mu$ so that $w \in \widetilde{W}_\mu$ appears in the sum on the left-hand side of the above equation or $\mu - \alpha_w \notin \mathbb{N}_0 \widehat{\Delta}^+$ so that $X_{w^{\mathrm{opp}}} v_{-\mu}$ would have to be of weight $\alpha_w - \mu \notin -\mathbb{N}_0 \widehat{\Delta}^+$ and does therefore either vanish or lie in $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^-$. The statement for $\tilde{u}$ is proven similarly. ∎

Using the inclusions $\mathscr{U}(\tilde{\mathfrak{n}}^\pm) \to \mathscr{U}\mathfrak{g}$ and the projection $\mathscr{U}\mathfrak{g} \to \mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$, we map F_λ to $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}2}$. Note that, as before, $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$ is a homogeneous ideal in $\mathscr{U}\mathfrak{g}$, so the grading of $\mathscr{U}\mathfrak{g}$ stays well defined on the quotient. The action of $\mathfrak{g}_\lambda$ on $(\mathscr{U}\mathfrak{g})^{\otimes 2}$ also passes to the quotient and extends to a continuous action on the completed tensor product.

Theorem 3.19 (Alekseev–Lachowska [1]). *Let $\lambda \in \widetilde{\Lambda}$. Then $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}2}$ is $\mathfrak{g}_\lambda$-invariant and satisfies*

$$\big((\mathrm{id} \otimes \Delta)F_\lambda\big)1 \otimes F_\lambda = \big((\Delta \otimes \mathrm{id})F_\lambda\big)F_\lambda \otimes 1 \tag{3.25}$$

in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}3}$.

Proof. The $\mathfrak{g}$-invariance of the Shapovalov pairing (proven similarly as in Lemma 3.3) implies that $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \widehat{\otimes} \mathscr{U}(\tilde{\mathfrak{n}}^-)$ is also $\mathfrak{g}$-invariant. Then $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}2}$ is $\mathfrak{g}_\lambda$-invariant since the map $\mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}2}$ is $\mathfrak{g}_\lambda$-equivariant. Equation (3.25) is proven in [1, Section 4]. ∎

It will be convenient in the following to write F_λ as a sum of elements that are all invariant under $\mathfrak{g}_\lambda$.

Lemma 3.20. *Let $\lambda \in \widetilde{\Lambda}$. Then there is a partition of $\widetilde{W}$ into finite subsets $\widetilde{W}_\ell$, $\ell \in \mathbb{N}_0$ such that*

$$F_{\lambda,\ell} := \sum_{w \in \widetilde{W}_\ell} p_\lambda^w(\alpha_w)^{-1}\tilde{\pi}_\lambda^+(X_w) \otimes \tilde{\pi}_\lambda^-(Y_w) \tag{3.26}$$

is $\mathfrak{g}_\lambda$-invariant.

Proof. It will be convenient to introduce a different grading d' on $\mathfrak{g}$, for which $\mathfrak{g}_\lambda$ is of degree 0. To this end, let $\mathfrak{h}$ and the root spaces of simple roots in Δ' be of degree 0, and let the root spaces of simple roots in $\widehat{\Delta}$ be of degree 1. Since any root is a unique linear combination of simple roots, this assignment extends to a grading on $\mathfrak{g}$. More explicitly, if $\sigma_1, \ldots, \sigma_r \in \Delta$ are the simple roots, with $\sigma_1, \ldots, \sigma_{r'} \in \Delta'$, then the root space of a root $\alpha = \sum_{i=1}^r c^i \sigma_i$ is of degree $d'(\alpha) = \sum_{i=r'+1}^r c^i$. Since $\mathfrak{g}_\lambda$ is spanned by $\mathfrak{h}$ and the root spaces of roots in Δ', and since the invariance of the ordering implies that any root in Δ' is a linear combination of simple roots in Δ', it follows that every element of $\mathfrak{g}_\lambda$ is homogeneous of degree 0. This grading is coarser than the grading given by d, in the sense that the graded components with respect to the new grading d' are direct sums of the graded components with respect to d. The restrictions of the maps $\tilde{\pi}_\lambda^\pm$ to $\mathscr{U}(\mathfrak{n}^\pm)$ are homogeneous of degree 0 with respect to (the restriction of) the $\mathbb{Z}$-grading on $\mathscr{U}\mathfrak{g}$ induced by d'.

For $w \in W$ set $d'(w) := d'(\alpha_{w_1}) + \cdots + d'(\alpha_{w_{|w|}})$, and define $\widetilde{W}_\ell := \{w \in \widetilde{W} \mid d'(w) = \ell\}$. It follows from Lemma 3.14 that $\widetilde{W}_\ell$ is finite for every ℓ. The elements $F_{\lambda,\ell}$ defined from $\widetilde{W}_\ell$ as in (3.26) have a nice description in terms of the grading d'. Since all graded components of $\tilde{\mathfrak{n}}^+$ resp. $\tilde{\mathfrak{n}}^-$ are of degree ≥ 1 resp. ≤ -1, d' induces a grading of $\mathscr{U}(\tilde{\mathfrak{n}}^+) \otimes \mathscr{U}(\tilde{\mathfrak{n}}^-)$ by $\mathbb{N}_0 \times (-\mathbb{N}_0)$. Using the homogeneity of $\tilde{\pi}_\lambda^\pm$, it follows directly from the definition of $\widetilde{W}_\ell$ that $F_{\lambda,\ell}$ is precisely the component of F_λ of degree $(\ell, -\ell)$ with respect to this grading. Since $\mathfrak{g}_\lambda$ is of degree 0, the action of $\mathfrak{g}_\lambda$ on $\mathscr{U}(\tilde{\mathfrak{n}}^+) \otimes \mathscr{U}(\tilde{\mathfrak{n}}^-)$ preserves the graded components, and the $\mathfrak{g}_\lambda$-invariance of F_λ implies that all the graded components $F_{\lambda,\ell}$ must also be $\mathfrak{g}_\lambda$-invariant. ∎

3.3. The induced formal and strict products

In this subsection, we construct associative products from the element F_λ obtained at the end of the last subsection. We will rescale λ in order to introduce a parameter playing the role of Planck's constant in the construction. Then we would like to use the results of Section 2.2 to obtain bidifferential operators from (the rescaled) F_λ. However, since F_λ is only in the completed tensor product, applying these results naively would give a sum of bidifferential operators of increasing orders and we have to deal with its convergence. There are essentially two solutions to this problem: firstly, we can take a formal expansion in the parameter $\hbar$, which will give us a well-defined power series of bidifferential operators of increasing order. Secondly, we can restrict ourselves to applying these operators only to some polynomial functions, for which only finitely many terms of the infinite sum give a non-zero contribution. We discuss both approaches in detail, starting with the formal one.

Let us first introduce the rescaling. Define the set

$$P_\lambda = \{0\} \cup \{\hbar \in \mathbb{C} \setminus \{0\} \mid \mathrm{i}\lambda/\hbar \notin \widetilde{\Lambda}\}, \tag{3.27}$$

and for $\hbar \in \mathbb{C} \setminus P_\lambda$ set $F_\hbar := F_{\mathrm{i}\lambda/\hbar}$ and $F_{\hbar,\ell} := F_{\mathrm{i}\lambda/\hbar,\ell}$, where $F_{\mathrm{i}\lambda/\hbar}$ was computed in Theorem 3.19 and $F_{\mathrm{i}\lambda/\hbar,\ell}$ was defined in Lemma 3.20. Note that $\mathfrak{g}_{\mathrm{i}\lambda/\hbar} = \mathfrak{g}_\lambda$, so $F_\hbar \in ((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\widehat{\otimes}2})^{\mathfrak{g}_\lambda}$ holds for all $\hbar \in \mathbb{C} \setminus P_\lambda$. Furthermore, the projections $\widetilde{\pi}^\pm_{\mathrm{i}\lambda/\hbar}|_{\mathscr{U}(\mathfrak{n}^\pm)}\colon \mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}(\widetilde{\mathfrak{n}}^\pm)$ are independent of $\hbar$, which one can easily see from their definition in (3.17).

Proposition 3.21. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$. Fix an invariant ordering on Δ, and assume that $(\lambda, \mu) \neq 0$ for all $\mu \in \mathbb{N}_0 \widehat{\Delta}^+$ satisfying $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$. Then the set P_λ is countable and accumulates only at zero.*

Proof. From the definition of P_λ we obtain

$$P_\lambda = \{0\} \cup \{\hbar \in \mathbb{C} \setminus \{0\} \mid p_{\mathrm{i}\lambda/\hbar}(\mu) = 0 \text{ for some } \mu \in \mathbb{N}_0\widehat{\Delta}^+ \setminus \{0\}\}.$$

Under our assumptions, the function $\hbar \mapsto p_{\mathrm{i}\lambda/\hbar}(\mu) = \frac{1}{2}(\mu, \mu) - (\rho, \mu) - \frac{\mathrm{i}}{\hbar}(\lambda, \mu)$ has the only root $\mathrm{i}(\lambda, \mu)/(\frac{1}{2}(\mu, \mu) - (\rho, \mu))$ if $\frac{1}{2}(\mu, \mu) - (\rho, \mu) \neq 0$ and no root otherwise. Therefore P_λ is countable since $\mathbb{N}_0\widehat{\Delta}^+ \setminus \{0\}$ is countable. Furthermore, P_λ accumulates only at zero since

$$\left| \frac{\mathrm{i}(\lambda, \mu)}{\frac{1}{2}(\mu, \mu) - (\rho, \mu)} \right| \leq \frac{\|\lambda\|\|\mu\|}{\frac{1}{2}\|\mu\|^2 - \|\mu\|\|\rho\|} = \frac{\|\lambda\|}{\frac{1}{2}\|\mu\| - \|\rho\|}$$

if $\|\mu\| > 2\|\rho\|$. Note that there are only finitely many elements $\mu \in \mathbb{N}_0\widehat{\Delta}^+$ with $\|\mu\| \leq 2\|\rho\|$. ∎

Remark 3.22. If the ordering in the previous proposition is standard, then any element $\mu \in \mathbb{N}_0\widehat{\Delta}^+$ automatically satisfies $(\lambda, \mu) \neq 0$: for all $\alpha \in \widehat{\Delta}^+$ we have $(\lambda, \alpha) \in S$ and

since S is closed under addition this implies $(\lambda, \mu) \in S$ for all $\mu \in \mathbb{N}_0 \widehat{\Delta}^+$. Note that $0 \notin S$, so in particular $(\lambda, \mu) \neq 0$.

Note also that $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$ implies $\|\mu\| \leq 2\|\rho\|$, so there can only be finitely many elements $\mu \in \mathbb{N}_0 \Delta$ satisfying $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$. Among those are all simple roots and the element 2ρ. However, simple roots which are in $\mathbb{N}_0 \widehat{\Delta}$ are by definition not orthogonal to λ. An example of an element that is not a simple root and not 2ρ in the case of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ with root system as in Figure 1 is $\mu = \alpha_1 + \alpha_2$.

We say that $F_\hbar$ depends rationally on $\hbar$ if all the $F_{\hbar,\ell}$ depend rationally on $\hbar$. This makes sense since $F_{\hbar,\ell}$ takes values in a finite dimensional subspace of $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}$ that is independent of $\hbar$.

Theorem 3.23 (Alekseev–Lachowska [1]). *Let $\lambda \in \mathfrak{h}^*$ and assume that P_λ is countable. Then $F_\hbar$ depends rationally on $\hbar$, with no pole at zero. In particular, the Taylor series expansion of $F_\hbar$ around 0 makes sense, and it gives an element $F \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}[\![\hbar]\!]$, where the tensor product is the usual (not completed) tensor product. Furthermore, F satisfies (3.25) in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 3}[\![\hbar]\!]$ and is $\mathfrak{g}_\lambda$-invariant.*

Proof. As mentioned before, $\mathfrak{g}_{i\lambda/\hbar}$ and $\widetilde{\pi}^{\pm}_{i\lambda/\hbar}|_{\mathscr{U}(\mathfrak{n}^\pm)} : \mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}(\widetilde{\mathfrak{n}}^\pm)$ are independent of $\hbar$, so only the coefficients $p^w_{i\lambda/\hbar}(\alpha_w)^{-1}$ in the formula for $F_{i\lambda/\hbar}$ obtained in Theorem 3.18 depend on $\hbar$. Since they are products of elements of the form

$$p_{i\lambda/\hbar}(\mu)^{-1} = \left(\frac{1}{2}(\mu, \mu) - (\rho, \mu) - \left(\frac{i\lambda}{\hbar}, \mu \right) \right)^{-1} = \frac{\hbar}{\left(\frac{1}{2}(\mu, \mu) - (\rho, \mu) \right)\hbar - (i\lambda, \mu)}$$

with $\mu \in \mathbb{N}_0 \widehat{\Delta}^+ \setminus \{0\}$, their dependence on $\hbar$ is rational without a pole at zero. (Observe that $\frac{1}{2}(\mu, \mu) - (\rho, \mu)$ and $(i\lambda, \mu)$ cannot vanish simultaneously since P_λ is assumed to be countable.) Consequently, we may take the Taylor expansion of $F_{i\lambda/\hbar}$ around $\hbar = 0$. To see that this yields an element in the usual tensor product, note that the formal expansion of $p_{i\lambda/\hbar}(\mu)^{-1}$ is a multiple of $\hbar$ unless $(\lambda, \mu) = 0$. Now

$$p^w_{i\lambda/\hbar}(\alpha_w)^{-1} = \prod_{i=1}^{|w|} p_{i\lambda/\hbar}(\alpha_{w_{i\cdots|w|}})^{-1},$$

and if the formal expansions of both $p_{i\lambda/\hbar}(\alpha_{w_{i\cdots|w|}})^{-1}$ and $p_{i\lambda/\hbar}(\alpha_{w_{i+1\cdots|w|}})^{-1}$ are not multiples of $\hbar$, then $(\lambda, \alpha_{w_i}) = 0$, i.e., $\alpha_{w_i} \in \Delta'$. However, Lemma 3.14 ensures that this cannot happen too often: if M is the constant obtained in that lemma, then at least $\lceil |w|/(M+1) \rceil$ many elements in the formal expansion of $p^w_{i\lambda/\hbar}(\alpha_w)^{-1}$ are multiples of $\hbar$, so this expansion is of order at least $\hbar^{\lceil |w|/(M+1) \rceil}$. Consequently, only finitely many words contribute to a given order in $\hbar$ so that we do not need to complete the tensor product. Since every $F_\hbar$ satisfies (3.25) and is $\mathfrak{g}_\lambda$-invariant, this is also true for the formal expansion F. ∎

Let us now apply this theorem to quantize complex coadjoint orbits. Let G be a complex connected semisimple Lie group with coadjoint orbit $\mathcal{O}_\lambda$ through a semisimple element $\lambda \in \mathfrak{g}^*$. Pick a Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$. Choose an invariant ordering for which P_λ is countable (e.g., a standard invariant ordering).

By Proposition 2.3 we know that G_λ is connected. Therefore the $\mathfrak{g}_\lambda$-invariance of the elements F and $F_\hbar$ constructed previously implies their G_λ-invariance. Consequently, we can apply the results of Section 2.2 in order to obtain holomorphic G-invariant bidifferential operators on $\mathcal{O}_\lambda \cong G/G_\lambda$. Define the formal product

$$\star : \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!] \times \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!] \to \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!], \quad (f, g) \mapsto f \star g := \Psi(F)(f, g), \tag{3.28}$$

and note that this product is well defined since the previous theorem asserts that $F \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}[\![\hbar]\!]$.

Proposition 3.24. *The product $\star$ is associative and restricts to a product*

$$\star : \mathrm{Hol}(\mathcal{O}_\lambda)[\![\hbar]\!] \times \mathrm{Hol}(\mathcal{O}_\lambda)[\![\hbar]\!] \to \mathrm{Hol}(\mathcal{O}_\lambda)[\![\hbar]\!] \tag{3.29}$$

on power series of holomorphic functions. Moreover, $\star$ is G-invariant, in the sense that $(g \rhd f_1) \star (g \rhd f_2) = g \rhd (f_1 \star f_2)$ holds for all $g \in G$ and $f_1, f_2 \in \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!]$.

Proof. It is a standard argument that the twist condition (3.25) translates into associativity of the induced product. That $\star$ restricts to power series of holomorphic functions and is G-invariant is immediate since the image of Ψ consists of holomorphic G-invariant bidifferential operators. $\blacksquare$

In order to define strict star products from $F_\hbar$ directly, i.e., without taking a formal power series expansion, we need to ensure that $\Psi(F_\hbar)$ is well defined. To do that we introduce polynomials on the coadjoint orbit. It will turn out that only finitely many elements of the infinite sum defining $F_\hbar$ contribute non-trivially when $\Psi(F_\hbar)$ is applied to polynomials.

Recall from Section 2.1 that we may assume without loss of generality that G is a closed complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$. We fix a way to realize G as such a matrix Lie group once and for all. In particular, the Lie algebra $\mathfrak{g}$ of G is realized as a complex Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$.

Definition 3.25 (Polynomials on $\mathcal{O}_\lambda$). Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a complex coadjoint orbit. Then

$$\mathrm{Pol}(\mathcal{O}_\lambda)$$
$$= \{p : \mathcal{O}_\lambda \to \mathbb{C} \mid p = P|_{\mathcal{O}_\lambda} \text{ for some holomorphic polynomial } P \text{ on } \mathfrak{g}^*\} \tag{3.30}$$

is called the algebra of polynomials on $\mathcal{O}_\lambda$.

Recall that the symmetric algebra $\mathrm{S}\mathfrak{g}$ of $\mathfrak{g}$ is isomorphic (as an algebra) to the algebra $\mathrm{Pol}(\mathfrak{g}^*)$ of polynomials on $\mathfrak{g}^*$. The isomorphism sends an element $X_1 \vee \cdots \vee X_j \in \mathrm{S}^j \mathfrak{g}$ to $\xi \mapsto \xi(X_1) \cdots \xi(X_j)$.

Definition 3.26 (Polynomials on G). For a complex linear Lie group G, the algebra of polynomials $\mathrm{Pol}(G)$ is the unital complex subalgebra of $\mathscr{C}^\infty(G)$ generated by the functions $P_{ij} : G \to \mathbb{C}, g \mapsto g_{ij}$.

Polynomials on a complex Lie group G are holomorphic. In the case of semisimple connected Lie groups, both the Lie group itself and the coadjoint orbit are affine algebraic varieties (see Remark 2.4) and our definition of polynomials coincides with the definition of regular functions on algebraic varieties. If G is connected and semisimple, then the definition of polynomials on G is independent of the way in which G is realized as a linear group, which can be proven as outlined in Appendix A.2.

Proposition 3.27. *Assume that the complex linear Lie group G is semisimple and connected. Then $\pi^*\colon \mathrm{Hol}(\mathcal{O}_\lambda) \cong \mathrm{Hol}(G/G_\lambda) \to \mathrm{Hol}(G)^{G_\lambda}$ restricts to an isomorphism*

$$\pi^*\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}.$$

Proof. Since the Lie algebra $\mathfrak{g}$ is semisimple, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e., every element of $\mathfrak{g}$ can be written as a sum of commutators. Consequently, the trace of any element of $\mathfrak{g}$ is zero. Therefore any element in a sufficiently small neighborhood of the identity of G must have determinant 1, and consequently G is a Lie subgroup of $\mathrm{SL}_N(\mathbb{C})$.

Let $E_{ij} \in \mathfrak{gl}_N(\mathbb{C})$ be the matrix that is 1 at position (i, j) and 0 otherwise. Extend λ to a linear functional $\tilde{\lambda} \in \mathfrak{gl}_N(\mathbb{C})^*$. For an element $X \in \mathfrak{g} = \mathrm{S}^1\mathfrak{g}$, which we identify with a polynomial on $\mathfrak{g}^*$, we compute

$$\pi^*\big(X\big|_{\mathcal{O}_\lambda}\big)(g) = X\big|_{\mathcal{O}_\lambda}(\pi(g)) = X\big|_{\mathcal{O}_\lambda}\big(\mathrm{Ad}_g^* \lambda\big) = X\big|_{\mathcal{O}_\lambda}\big(\lambda(g^{-1} \cdot g)\big)$$
$$= \lambda\big(g^{-1} X g\big) = \sum_{i,j} \tilde{\lambda}\big((g^{-1} X g)_{ij} E_{ij}\big) = \sum_{i,j,k,\ell} (g^{-1})_{ik} g_{\ell j} X_{k\ell} \tilde{\lambda}(E_{ij}).$$

Since $\det g = 1$, we can write $(g^{-1})_{ik}$ as a polynomial in the entries of g so that $\pi^*(X|_{\mathcal{O}_\lambda})$ itself is a polynomial in the entries of g. Since $\mathrm{Pol}(\mathcal{O}_\lambda)$ is generated by $X|_{\mathcal{O}_\lambda}$ and π^* is an algebra homomorphism, it follows that $\pi^* p \in \mathrm{Pol}(G)$ for any $p \in \mathrm{Pol}(\mathcal{O}_\lambda)$. Injectivity of π^* is immediate. Surjectivity is harder to prove. One can either use methods from algebraic geometry (making use of Remark 2.4; see for example [22, Chapter 12]) or work in a more differential geometric setting using G-finite functions as outlined in Appendix A.2. ∎

Recall the degree d' introduced in the proof of Lemma 3.20.

Lemma 3.28. *For any polynomial $p \in \mathrm{Pol}(\mathrm{GL}_N(\mathbb{C}))$, there is a constant $N_p \in \mathbb{N}$ such that $u^{\mathrm{left},(1,0)} p = v^{\mathrm{left},(1,0)} p = 0$ holds for any $u \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \subseteq \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ of degree d' greater than N_p and any $v \in \mathscr{U}(\tilde{\mathfrak{n}}^-) \subseteq \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ of degree d' smaller than $-N_p$.*

Proof. Using the Leibniz rule, we may assume that $p = P_{k\ell}$ in the notation of Definition 3.26. Let $E_{ij} \in \mathfrak{gl}_N(\mathbb{C})$ be the matrix that is 1 at position (i, j) and 0 otherwise. It is easy to check that $E_{ij}^{\mathrm{left}} P_{k\ell} = \delta_{j\ell} P_{ki}$ and therefore

$$X^{\mathrm{left}} P_{k\ell} = \bigg(\sum_{i,j} X_{ij} E_{ij}\bigg)^{\mathrm{left}} P_{k\ell} = \sum_i X_{i\ell} P_{ki} \quad \text{for all } X \in \mathfrak{gl}_N(\mathbb{C}).$$

Since $P_{k\ell}$ is holomorphic, this implies that also $X^{\text{left},(1,0)} P_{k\ell} = X^{\text{left}} P_{k\ell} = \sum_i X_{i\ell} P_{ki}$. Consequently, if $u = u_1 \cdots u_M \in \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ with $u_1, \ldots, u_M \in \mathfrak{gl}_N(\mathbb{C})$, then

$$
\begin{aligned}
u^{\text{left},(1,0)} P_{k\ell} &= \sum_{i_M} (u_1 \cdots u_{M-1})^{\text{left},(1,0)} (u_M)_{i_M \ell} P_{k i_M} \\
&= \sum_{i_{M-1}, i_M} (u_1 \cdots u_{M-2})^{\text{left},(1,0)} (u_{M-1})_{i_{M-1} i_M} (u_M)_{i_M \ell} P_{k i_{M-1}} = \cdots \\
&= \sum_{i_1, \ldots, i_M} (u_1)_{i_1 i_2} \cdots (u_{M-1})_{i_{M-1} i_M} (u_M)_{i_M \ell} P_{k i_1} = \sum_i (u_1 \cdots u_M)_{i\ell} P_{ki}.
\end{aligned}
$$

Since ad_X is nilpotent for any $X \in \tilde{\mathfrak{n}}^+$, it follows that $0 = (\operatorname{ad} X)_s = \operatorname{ad}(X_s)$ for $X \in \tilde{\mathfrak{n}}^+$, where the index s stands for the semisimple part of the Jordan decomposition. Since $\mathfrak{g}$ is semisimple, this implies $X_s = 0$, so every $X \in \tilde{\mathfrak{n}}^+$ is realized by a nilpotent matrix. It follows from Engel's theorem that any matrix Lie algebra consisting of nilpotent matrices is nilpotent as an algebra, so there exists a constant $M \in \mathbb{N}$ such that products of M or more elements of $\tilde{\mathfrak{n}}^+$ vanish. Therefore, if u is a product of at least M elements of $\tilde{\mathfrak{n}}^+$, the above calculation shows that $u^{\text{left}} P_{k\ell} = 0$. If M' is an upper bound for the degree d' of elements of $\tilde{\mathfrak{n}}^+$, then we can set $N_{P_{k\ell}} := MM'$. It is easy to check that this constant also works for $\tilde{\mathfrak{n}}^-$. ∎

Corollary 3.29. *For all $p, q \in \operatorname{Pol}(\mathcal{O}_\lambda)$ and all $\hbar \in \mathbb{C} \setminus P_\lambda$, the sum $\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q)$ is finite, and $\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) \in \operatorname{Pol}(\mathcal{O}_\lambda)$.*

Proof. Proposition 3.27 implies that $\pi^* p$ and $\pi^* q$ are polynomials. By Lemma 3.20 the components $F_{\hbar,\ell}$ are of degree $(\ell, -\ell)$, and Lemma 3.28 implies that only finitely many summands of $\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\text{left},(1,0)}(\pi^* p, \pi^* q)$ are non-zero. Its proof shows that

$$
\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\text{left},(1,0)}(\pi^* p, \pi^* q)
$$

is again a polynomial. The components $F_{\hbar,\ell}$ are $\mathfrak{g}_\lambda$-invariant and therefore, since G_λ is connected by Proposition 2.3, also G_λ-invariant. Applying Lemma 2.6, we obtain that $\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\text{left},(1,0)}(\pi^* p, \pi^* q)$ is G_λ-invariant. Then Proposition 3.27 yields that

$$
\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) = \sum_{\ell=0}^\infty \pi_* \left(F_{\hbar,\ell}^{\text{left},(1,0)}(\pi^* p, \pi^* q) \right)
$$

is a polynomial. ∎

Corollary 3.30. *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a complex connected semisimple Lie group G with Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$ and that one has chosen an invariant ordering. Then for any $\hbar \in \mathbb{C} \setminus P_\lambda$,*

$$
*_\hbar \colon \operatorname{Pol}(\mathcal{O}_\lambda) \times \operatorname{Pol}(\mathcal{O}_\lambda) \to \operatorname{Pol}(\mathcal{O}_\lambda), \quad (p,q) \mapsto p *_\hbar q := \sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) \quad (3.31)
$$

defines an associative and G-invariant product (where G-invariant means that $(g \triangleright p) *_\hbar$ $(g \triangleright q) = g \triangleright (p *_\hbar q)$ holds for any $g \in G$ and $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$). For $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, $p *_\hbar q$ depends rationally on $\hbar$, and the formal expansion of $*_\hbar$ around $\hbar = 0$ coincides with the formal product $\star$.

Proof. As in the formal case, it is a standard argument to show that (3.25) implies the associativity of $*_\hbar$. Since the codomain of Ψ consists of G-invariant bidifferential operators, it is clear that $*_\hbar$ is G-invariant. Since the dependence of $F_\hbar$ on $\hbar$ is rational without pole at 0, it follows that $*_\hbar$ also depends rationally on $\hbar$ without pole at 0, and since $\star$ was constructed from the formal expansion of $F_\hbar$, it coincides with the formal expansion of $*_\hbar$. ∎

Remark 3.31. When considering $\Psi(F_{\hbar,\ell})$, we may leave out the projections $\widetilde{\pi}_\lambda^\pm$ in the formula for $F_{\hbar,\ell}$ from Lemma 3.20 to obtain the same result. Indeed, by Lemma 3.16 the difference of $F_{\hbar,\ell}$ and

$$F'_{\hbar,\ell} := \sum_{w \in \widetilde{W}_\ell} p_{\mathrm{i}\lambda/\hbar}^w (\alpha_w)^{-1} X_w \otimes Y_w \in \mathscr{U}(\mathfrak{n}^+) \otimes \mathscr{U}(\mathfrak{n}^-) \tag{3.32}$$

is an element in the ideal $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda \otimes \mathscr{U}\mathfrak{g} + \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$ and therefore contained in the kernel of Ψ by Lemma 2.6.

Recall that we obtained a condition for P_λ being countable in Proposition 3.21 and that this condition is satisfied in particular when the ordering is standard; see Remark 3.22.

Proposition 3.32. *Assume that P_λ is countable. Then the first order commutator of $\star$ coincides with the Poisson bracket induced by the KKS form ω_{KKS} defined in (2.2).*

Proof. Note that the formal expansion of

$$p_{\mathrm{i}\lambda/\hbar}(\mu)^{-1} = \left(\frac{1}{2}(\mu, \mu) - (\rho, \mu) - \frac{\mathrm{i}}{\hbar}(\lambda, \mu) \right)^{-1}$$

$$= \mathrm{i}\hbar \left(\frac{\mathrm{i}\hbar}{2}(\mu, \mu) - \mathrm{i}\hbar(\rho, \mu) + (\lambda, \mu) \right)^{-1}$$

is of order $\hbar$ if $(\lambda, \mu) \neq 0$. It follows from Theorem 3.18 that the element F is the formal expansion of

$$\sum_{\substack{w \in \widetilde{W} \\ |w| \leq 1}} p_{\mathrm{i}\lambda/\hbar}^w (\alpha_w)^{-1} \widetilde{\pi}_\lambda^+(X_w) \otimes \widetilde{\pi}_\lambda^-(Y_w) + \sum_{\substack{w \in \widetilde{W} \\ |w| \geq 2}} p_{\mathrm{i}\lambda/\hbar}^w (\alpha_w)^{-1} \widetilde{\pi}_\lambda^+(X_w) \otimes \widetilde{\pi}_\lambda^-(Y_w).$$

Using that the words $w \in \widetilde{W}$ with $|w| \leq 1$ are precisely the empty word and the one-letter words (ℓ) with $\alpha_\ell \in \widehat{\Delta}^+$, i.e., $(\lambda, \alpha_\ell) \neq 0$, it follows that the first sum expands to $1 + \mathrm{i}\hbar \sum_{\alpha \in \widehat{\Delta}_+} (\lambda, \alpha)^{-1} X_\alpha \otimes Y_\alpha + \mathcal{O}(\hbar^2)$. Let us argue why the formal expansion of the second sum is of order $\hbar^2$. By definition $p_{\mathrm{i}\lambda/\hbar}^w (\alpha_w)^{-1} = \prod_{i=1}^{|w|} p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_i \cdots |w|})^{-1}$. Since,

by definition of $\widetilde{W}$, we have $\alpha_{w_{|w|}} \in \widehat{\Delta}^+$, it is clear that the formal expansions of all summands with $(\lambda, \alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}) \neq 0$ are of order $\hbar^2$ (because both $p_{i\lambda/\hbar}(\alpha_{(w_{|w|-1}, w_{|w|})})^{-1}$ and $p_{i\lambda/\hbar}(\alpha_{w_{|w|}})^{-1}$ are of order $\hbar$). So assume that $(\lambda, \alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}) = 0$, in which case $\alpha_{w_{|w|-1}} \in \widehat{\Delta}^+$ and, by invariance of the ordering, $\alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}$ is not a root. Therefore $X_{w_{|w|-1}} X_{w_{|w|}} = X_{w_{|w|}} X_{w_{|w|-1}}$, and if $w' = (w_1, \ldots, w_{|w|-2}, w_{|w|}, w_{|w|-1})$ is the word obtained form w by switching the last two letters, then $X_w = X_{w'}$. Similarly, $Y_w = Y_{w'}$. Furthermore, by definition of α_w, we have $\alpha_{w_{i\cdots|w|}} = \alpha_{w'_{i\cdots|w'|}}$ for all $i < |w|$ and

$$p^w_{i\lambda/\hbar}(\alpha_w)^{-1} + p^{w'}_{i\lambda/\hbar}(\alpha_{w'})^{-1}$$
$$= \left(p_{i\lambda/\hbar}(\alpha_{w_{|w|}})^{-1} + p_{i\lambda/\hbar}(\alpha_{w_{|w|-1}})^{-1}\right) \prod_{i=1}^{|w|-1} p_{i\lambda/\hbar}(\alpha_{w_{i\cdots|w|}})^{-1}.$$

But under our assumptions $(\alpha_{w_{|w|}}, \lambda)^{-1} + (\alpha_{w_{|w|-1}}, \lambda)^{-1} = 0$, and therefore the formal expansion of $p_{i\lambda/\hbar}(\alpha_{w_{|w|}})^{-1} + p_{i\lambda/\hbar}(\alpha_{w_{|w|-1}})^{-1}$ is $i\hbar(\alpha_{w_{|w|}}, \lambda)^{-1} + i\hbar(\alpha_{w_{|w|-1}}, \lambda)^{-1} + \mathcal{O}(\hbar^2) = \mathcal{O}(\hbar^2)$. Consequently, the summands which could potentially be of order $\hbar$ in the sum over $w \in \widetilde{W}$ with $|w| \geq 2$ cancel out, and this sum is therefore of order $\hbar^2$ as claimed.

To conclude the proof, note that antisymmetrizing the first order gives indeed

$$F^{\mathrm{antisym}}_{(1)} = i \sum_{\alpha \in \widehat{\Delta}^+} \lambda(\alpha^\sharp)^{-1} (X_\alpha \otimes Y_\alpha - Y_\alpha \otimes X_\alpha)$$
$$= i \sum_{\alpha \in \widehat{\Delta}} \lambda([X_\alpha, Y_\alpha])^{-1} X_\alpha \otimes Y_\alpha = i\pi_{\mathrm{KKS}},$$

where π_{KKS} denotes the Poisson tensor associated to the KKS symplectic form. ∎

We conclude this subsection by saying a bit more about the directions in which $\star$ and $*_\hbar$ differentiate.

Lemma 3.33. *For any $\xi = \mathrm{Ad}^*_g \lambda \in \mathcal{O}_\lambda$, the subspaces*

$$L_{+,\xi} = \mathrm{span}\left\{(\mathrm{Ad}_g X_\alpha)_{\mathcal{O}_\lambda}|_\xi, \ \alpha \in \widehat{\Delta}^+\right\} \subseteq T_\xi \mathcal{O}_\lambda, \tag{3.33a}$$

$$L_{-,\xi} = \mathrm{span}\left\{(\mathrm{Ad}_g X_\alpha)_{\mathcal{O}_\lambda}|_\xi, \ \alpha \in \widehat{\Delta}^-\right\} \subseteq T_\xi \mathcal{O}_\lambda \tag{3.33b}$$

are independent of the choice of $g \in G$.

Proof. Any two choices $g, g' \in G$ differ by an element of G_λ; that is, $g' = gx$ with $x \in G_\lambda$. So it suffices to prove that $\mathrm{span}\{\mathrm{Ad}_x X_\alpha, \ \alpha \in \widehat{\Delta}^\pm\} = \mathrm{span}\{X_\alpha, \ \alpha \in \widehat{\Delta}^\pm\}$. This follows from the invariance of the ordering and the connectedness of G_λ. ∎

Therefore the distributions L_+ and L_- in $T\mathcal{O}_\lambda$ spanned by $L_{+,\xi}$ and $L_{-,\xi}$, respectively, are well defined.

Corollary 3.34. *The star product $*_\hbar$ derives the first argument only in the directions of $L^{(1,0)}_+$ and the second argument only in the directions of $L^{(1,0)}_-$.*

Proof. This follows from the explicit formula for $F_\hbar$ obtained in Theorem 3.18, from Remark 3.31 and Proposition 2.8. ∎

3.4. Examples

In this subsection, we derive formulas for $F_\hbar$ in the case $G = \mathrm{SL}_{1+n}(\mathbb{C})$ for the largest non-trivial stabilizer G_λ. When restricting to real coadjoint orbits in Section 5.4, this example allows us to obtain quantizations of complex projective spaces and hyperbolic discs.

Example 3.35 ($\mathrm{SL}_{1+n}(\mathbb{C})$). Let $G = \mathrm{SL}_{1+n}(\mathbb{C})$ be the Lie group of matrices with determinant 1. Its Lie algebra $\mathfrak{g} = \mathfrak{sl}_{1+n}(\mathbb{C})$ consists of matrices with trace 0. Number the rows and columns of a matrix $X \in \mathfrak{g}$ by $0, \ldots, n$. Let $\lambda \colon \mathfrak{g} \to \mathbb{C}$, $X \mapsto -irX_{0,0}$, where $r \in \mathbb{C}$. Using that the Killing form B satisfies $B(X, Y) = 2(n+1)\operatorname{tr}(XY)$, where tr is the usual (not normalized) matrix trace, it follows that $\lambda^\sharp$ is a multiple of the diagonal matrix $\operatorname{diag}(n, -1, \ldots, -1)$, and therefore

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{sl}_{1+n}(\mathbb{C}) \mid X_{0,i} = X_{i,0} = 0 \text{ for } 1 \leq i \leq n\}, \tag{3.34a}$$

$$G_\lambda = \{g \in \mathrm{SL}_{1+n}(\mathbb{C}) \mid g_{0,i} = g_{i,0} = 0 \text{ for } 1 \leq i \leq n\}. \tag{3.34b}$$

We choose the Cartan subalgebra $\mathfrak{h}$ consisting of the diagonal matrices in $\mathfrak{g}$. The roots are then given by $\alpha_{i,j} = L_i - L_j$ for $0 \leq i, j \leq n$ with $i \neq j$, where $L_i \in \mathfrak{h}^*$, $L_i(X) = X_{i,i}$. If we let the roots $\alpha_{i,j}$ with $i < j$ be positive, then the simple roots are $\alpha_{0,1}, \alpha_{1,2}, \ldots, \alpha_{n-1,n}$. As before, denote the matrix with entry 1 at position (i, j) by $E_{i,j}$, and define $X_{i,j} := E_{i,j} \in \mathfrak{g}^{\alpha_{i,j}}$ and $Y_{i,j} := E_{j,i} \in \mathfrak{g}^{\alpha_{j,i}} = \mathfrak{g}^{-\alpha_{i,j}}$. Note that

$$B(X_{i,j}, Y_{i,j}) = 2(n+1)\operatorname{tr}(X_{i,j} Y_{i,j}) = 2(n+1),$$

so we use a normalization different from that in Section 3.1.

If $n = 1$, it is easy to simplify the formula for $F_\hbar$ obtained in Theorem 3.18: there is only one positive root $\alpha = \alpha_{0,1}$, and there is a unique word w_ℓ of a given length $\ell \in \mathbb{N}_0$. Note that $\lambda = -ir\alpha/2$ and $\rho = \alpha/2$, so $p_{i\lambda/\hbar}(m\alpha) = \frac{1}{2}m^2(\alpha, \alpha) - \frac{1}{2}m(\alpha, \alpha) - \frac{1}{2\hbar}mr(\alpha, \alpha) = \frac{1}{4}m(m - 1 - \frac{r}{\hbar})$. Therefore

$$p_{i\lambda/\hbar}^{w_\ell}(\alpha_{w_\ell}) = \prod_{m=1}^{\ell} \frac{4}{m(m - 1 - \frac{r}{\hbar})} = \frac{(-4)^\ell}{\ell! \frac{r}{\hbar}(\frac{r}{\hbar} - 1) \cdots (\frac{r}{\hbar} - (\ell - 1))}.$$

We set $X := X_{0,1}$ and $Y := Y_{0,1}$. Since $B(X, Y) = 4$, we have to plug the normalized elements $X/2$ and $Y/2$ into (3.24) and obtain

$$F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{(-1)^\ell}{\ell! \frac{r}{\hbar}(\frac{r}{\hbar} - 1) \cdots (\frac{r}{\hbar} - (\ell - 1))} X^\ell \otimes Y^\ell. \tag{3.35}$$

This result was already obtained in [1, Example 4.16]. For arbitrary n, we compute the canonical element of the Shapovalov pairing directly, instead of simplifying (3.24).

Proposition 3.36. *For $G = \mathrm{SL}_{1+n}(\mathbb{C})$, the same λ, and the same ordering as above, one has*

$$F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{(-1)^\ell}{\ell! \frac{r}{\hbar} \left(\frac{r}{\hbar} - 1 \right) \cdots \left(\frac{r}{\hbar} - (\ell - 1) \right)} (X_{0,1} \otimes Y_{0,1} + \cdots + X_{0,n} \otimes Y_{0,n})^\ell. \quad (3.36)$$

Proof. The Lie algebras $\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{n}}^-$ are commutative Lie algebras spanned by $X_{0,1}, \ldots,$ $X_{0,n}$ and $Y_{0,1}, \ldots, Y_{0,n}$, respectively, so $\{X^I := X_{0,1}^{I_1} \cdots X_{0,n}^{I_n} \mid I \in \mathbb{N}_0^n\}$ and $\{Y^J := Y_{0,1}^{J_1} \cdots Y_{0,n}^{J_n} \mid J \in \mathbb{N}_0^n\}$ are bases of $\mathscr{U}(\tilde{\mathfrak{n}}^+)$ and $\mathscr{U}(\tilde{\mathfrak{n}}^-)$. The Lie algebra $\mathfrak{n}^+$ is spanned by $X_{i,j}$ with $i < j$ and we can view X^I also as an element of $\mathscr{U}(\mathfrak{n}^+)$. Then $\tilde{\pi}_\lambda^+(X^I) = X^I$ and similarly $\tilde{\pi}_\lambda^-(Y^J) = Y^J$. Consequently, $\langle X^I, Y^J \rangle_{\mathrm{i}\lambda/\hbar}^{\sim} = \langle X^I, Y^J \rangle_{\mathrm{i}\lambda/\hbar}$. For degree reasons, the bases $\{X^I\}$ and $\{Y^J\}$ are orthogonal, meaning that $\langle X^I, Y^J \rangle_{\mathrm{i}\lambda/\hbar} = 0$ for $I \neq J$. Indeed, X^I and Y^J are homogeneous with respect to the degree d defined in the beginning of Section 3.1, $d(X^I) = I_1 d(X_{0,1}) + \cdots + I_n d(X_{0,n}) = I_1 \alpha_{0,1} + \cdots + I_n \alpha_{0,n}$, and $d(Y^J) = -(J_1 \alpha_{0,1} + \cdots + J_n \alpha_{0,n})$. Since the $\alpha_{0,i}$ are linearly independent, Lemma 3.3 implies the claimed orthogonality.

Therefore it suffices to determine the normalization $\langle X^I, Y^I \rangle_{\mathrm{i}\lambda/\hbar}$. Define $H_i := [X_{0,i}, Y_{0,i}] = E_{0,0} - E_{i,i}$. Given a multi-index $I \in \mathbb{N}_0^n$, we can form a sequence that starts with I_1 many 1's, then has I_2 many 2's, $\ldots$, then I_n many n's. Denote the kth element of this sequence by $I_{\{k\}}$. Introduce the projection $(\,\cdot\,)_0$ to $\mathscr{U}\mathfrak{h}$ in the decomposition $\mathscr{U}\mathfrak{g} = \mathscr{U}\mathfrak{h} \oplus (\mathfrak{n}^- \cdot \mathscr{U}\mathfrak{g} + \mathscr{U}\mathfrak{g} \cdot \mathfrak{n}^+)$ so that $\pi_\lambda(u) = \lambda((u)_0)$. Then we claim that

$$(X^I Y^I)_0 = I! \prod_{\ell=0}^{|I|-1} \left(H_{I_{\{\ell\}}} - \ell \right). \quad (3.37)$$

To see that this formula implies the proposition, note that

$$\langle X^I, Y^I \rangle_{\mathrm{i}\lambda/\hbar} = \pi_{\mathrm{i}\lambda/\hbar} \left(S(X^I) Y^I \right) = (-1)^{|I|} \left(\frac{\mathrm{i}}{\hbar} \lambda \right) \left((X^I Y^I)_0 \right)$$

and that $\frac{\mathrm{i}}{\hbar} \lambda(H_i) = \frac{r}{\hbar}$ for all $i = 1, \ldots, n$. So

$$F_\hbar = \sum_{I \in \mathbb{N}_0^n} \frac{1}{\langle X^I, Y^I \rangle_{\mathrm{i}\lambda/\hbar}} X^I \otimes Y^I$$

$$= \sum_{I \in \mathbb{N}_0^n} \frac{(-1)^{|I|}}{I! \frac{r}{\hbar} \left(\frac{r}{\hbar} - 1 \right) \cdots \left(\frac{r}{\hbar} - (|I| - 1) \right)} X^I \otimes Y^I$$

and an application of the multinomial theorem gives (3.36).

It remains to prove (3.37). For $n = 1$ this is the statement of [15, Lemma 5.2]. Note that this also means that

$$Z := X_{0,n}^{I_n} Y_{0,n}^{I_n} - I_n! H_n (H_n - 1) \cdots (H_n - I_n + 1) \in \mathscr{U}\left(\mathrm{span}\{X_{0,n}, Y_{0,n}, H_n\} \right)$$

satisfies $(Z)_0 = 0$. We proceed by induction and assume that (3.37) holds for $n-1$. Writing $I_- = (I_1, \ldots, I_{n-1}, 0)$ and noting that $[H_n, X_{0,i}] = X_{0,i}$ for $1 \le i \le n-1$, we compute

$$
\begin{aligned}
(X^I Y^I)_0 &= \big(X^{I_-} X_{0,n}^{I_n} Y_{0,n}^{I_n} Y^{I_-}\big)_0 \\
&= \big(X^{I_-} \big(I_n! H_n (H_n - 1) \cdots (H_n - I_n + 1) + Z\big) Y^{I_-}\big)_0 \\
&= I_n! \big((H_n - |I_-|)\big(H_n - |I_-| - 1\big) \cdots \big(H_n - |I_-| - I_n + 1\big) X^{I_-} Y^{I_-}\big)_0 \\
&\quad + (X^{I_-} Z Y^{I_-})_0 \\
&= I_n! \big(H_n - |I_-|\big)\big(H_n - |I_-| - 1\big) \cdots \big(H_n - |I_-| - I_n + 1\big)(X^{I_-} Y^{I_-})_0 \\
&\quad + (X^{I_-} Z Y^{I_-})_0.
\end{aligned}
$$

Since $(Z)_0 = 0$ and $d(Z) = d(X_{0,n}^{I_n} Y_{0,n}^{I_n} - I_n! H_n (H_n - 1) \cdots (H_n - I_n + 1)) = 0$, we can write $Z = Y_{0,n} Z' X_{0,n}$ for some $Z' \in \mathcal{U}(\mathrm{span}\{X_{0,n}, Y_{0,n}, H_n\})$. Since $Y_{0,n} \in \mathfrak{g}^{\alpha_{n,0}}$, any commutator of $Y_{0,n}$ with elements of $\mathfrak{g}^{\alpha_{0,1}}, \ldots, \mathfrak{g}^{\alpha_{0,n-1}}$ has degree d equal to $L_n - \sum_{i=0}^{n-1} c_i L_i$ for some $c_i \in \mathbb{Z}$, so it must either be 0 or in a negative root space. Therefore $(X^{I_-} Z Y^{I_-})_0 = 0$, and the claim follows by applying the induction hypothesis to the first summand in the equation above. ∎

Corollary 3.37. *Let $G = \mathrm{SL}_{1+n}(\mathbb{C})$ and let λ be as above, but choose the opposite ordering, for which $\alpha_{i,j}$ with $i > j$ is positive. Then*

$$
F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{1}{\ell! \frac{r}{\hbar}\big(\frac{r}{\hbar} + 1\big) \cdots \big(\frac{r}{\hbar} + (\ell - 1)\big)} (Y_{0,1} \otimes X_{0,1} + \cdots + Y_{0,n} \otimes X_{0,n})^\ell. \quad (3.38)
$$

Proof. The only change in the computation above is that the roles of $X_{0,i}$ and $Y_{0,i}$ are swapped. Now $[Y_{0,i}, X_{0,i}] = E_{i,i} - E_{0,0}$, so $\frac{i}{\hbar}\lambda([Y_{0,i}, X_{0,i}]) = -\frac{r}{\hbar}$, which means that r changes sign. ∎

4. Continuity

In this section, we extend the product $*_\hbar \colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ obtained in Corollary 3.30 to a product $*_\hbar \colon \mathrm{Hol}(\mathcal{O}_\lambda) \times \mathrm{Hol}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\mathcal{O}_\lambda)$ on all holomorphic functions on the coadjoint orbit, that is continuous with respect to the topology of locally uniform convergence. More precisely, we prove the following theorem.

Theorem 4.1. *Let $\mathcal{O}_\lambda$ be a complex semisimple coadjoint orbit of a complex semisimple connected Lie group G. Then for any $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*_\hbar$ on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the topology of locally uniform convergence and extends to a continuous and G-invariant product $*_\hbar \colon \mathrm{Hol}(\mathcal{O}_\lambda) \times \mathrm{Hol}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\mathcal{O}_\lambda)$ on the space of all holomorphic functions on $\mathcal{O}_\lambda$.*

The proof of this theorem proceeds as follows: in Section 4.1, we prove the continuity of $*_\hbar$ with respect to a topology that we call the reduction topology and in Section 4.3 we

prove that the reduction topology coincides with the topology of locally uniform convergence. Consequently, $*_\hbar$ extends to the completion of the space of polynomials on $\mathcal{O}_\lambda$. Using the results of Section 4.2, we prove in Section 4.3 that this completion is the space $\mathrm{Hol}(\mathcal{O}_\lambda)$ of all holomorphic functions on $\mathcal{O}_\lambda$.

In the whole section, we assume that the complex connected semisimple Lie group G is concretely realized as a complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$ for some $N \in \mathbb{N}$, as explained in Section 2.1. In particular, since G is semisimple, it is a closed submanifold of $\mathbb{C}^{N \times N}$.

4.1. Continuity in the reduction topology

In this subsection, we prove the continuity of the star product $*_\hbar$ with respect to a topology that we call the reduction topology, defined below. Recall that a sequence of functions $f_i : X \to \mathbb{C}$ on a topological space X is said to be locally uniformly convergent if for every $x \in X$ there is a neighborhood $U \subseteq X$ such that f_i converges uniformly to f on U, i.e., $\lim_{i \to \infty} \sup_{y \in U} |f_i(y) - f(y)| = 0$. In this work, X will always be a manifold. Then the topology of locally uniform convergence coincides with the topology of compact convergence (for every compact subset $K \subseteq X$, f_i converges uniformly on K), and is therefore a locally convex topology, defined by the seminorms $\|f\|_K := \sup_K |f|$.

Denote the ideal of polynomials in $\mathrm{Pol}(\mathbb{C}^{N \times N})$ whose restriction to G vanishes by $\mathcal{I}(G)$.

Definition 4.2 (Reduction topology). The topology $\mathcal{T}_{\mathrm{lc}}$ of locally uniform convergence on the space $\mathrm{Pol}(\mathbb{C}^{N \times N})$ of polynomials on $\mathbb{C}^{N \times N}$ induces a quotient topology on the space $\mathrm{Pol}(G) \cong \mathrm{Pol}(\mathbb{C}^{N \times N})/\mathcal{I}(G)$ of polynomials on G, and we call the subspace topology on the space $\mathrm{Pol}(\mathcal{O}_\lambda) \cong \mathrm{Pol}(G)^{G_\lambda}$ of polynomials on the coadjoint orbit $\mathcal{O}_\lambda$ the reduction topology.

In Section 4.3, we will prove that the reduction topology coincides with the topology of locally uniform convergence on $\mathcal{O}_\lambda$.

This topology is convenient for obtaining continuity estimates for $*_\hbar$, since we gave a description of $\Psi(F_\hbar)$ via bidifferential operators on G in Section 2.2. Since we assume that the Lie group G is concretely realized as a complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$, its Lie algebra $\mathfrak{g}$ is realized as a Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$. Considering the element $F'_{\hbar,\ell}$ defined in (3.32) as an element of $\mathscr{U}(\mathfrak{gl}_N(\mathbb{C})) \otimes \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$, we let

$$*'_\hbar : \mathrm{Pol}(\mathbb{C}^{N \times N}) \times \mathrm{Pol}(\mathbb{C}^{N \times N}) \to \mathrm{Pol}(\mathbb{C}^{N \times N}),$$

$$(p, q) \mapsto p *'_\hbar q := \sum_{\ell=0}^{\infty} (F'_{\hbar,\ell})^{\mathrm{left},(1,0)}(p, q), \qquad (4.1)$$

which is well defined because Lemma 3.28 implies that the sum over ℓ is finite and that $(F'_{\hbar,\ell})^{\mathrm{left},(1,0)}(p, q)$ is again a polynomial. Note that $*'_\hbar$ is (in general) not associative since $\sum_{\ell=0}^{\infty} F'_{\hbar,\ell}$ satisfies (3.25) only after passing to the quotient. However, since $F'_{\hbar,\ell}$ lies in the subspace $\mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$, it induces a product on $\mathrm{Pol}(G) \cong \mathrm{Pol}(\mathbb{C}^{N \times N})/\mathcal{I}(G)$.

As in Remark 3.31, it follows that the restriction of this product to $\mathrm{Pol}(G)^{G_\lambda} \cong \mathrm{Pol}(\mathcal{O}_\lambda)$ coincides with $*_\hbar$.

Theorem 4.3. *For $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*'_\hbar$ on $\mathrm{Pol}(\mathbb{C}^{N \times N})$ is continuous with respect to the topology of locally uniform convergence $\mathcal{T}_{\mathrm{lc}}$.*

Before proving this theorem in the rest of this section, we would like to note the following consequence, which motivates the definition of the reduction topology given above.

Corollary 4.4. *For $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*_\hbar$ on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the reduction topology.*

Proof. This follows immediately from the previous theorem and the construction of the reduction topology. ∎

Remark 4.5. It is interesting to point out that the proof of Theorem 4.3 will not use anything about the actual Lie algebra structure but semisimplicity and the form of the element $F_\hbar$. In fact, we only need that the coefficients of $F_\hbar$ behave like $p_\lambda^w(\alpha_w) \approx |w|^2$ for large $|w|$. The rest of the proof consists in counting terms and checking that there are not too many.

The strategy to prove Theorem 4.3 is as follows. We first introduce a different locally convex topology that is better suited for obtaining continuity estimates. Then we prove that this topology is equivalent to the topology of locally uniform convergence and we prove the continuity of $*'_\hbar$ with respect to this topology.

Set $m = N^2$. Let $B = \{b_1, \ldots, b_m\}$ be the standard basis of $\mathbb{C}^m$ and denote the dual basis of $(\mathbb{C}^m)^*$ by $B^* = \{b_1^*, \ldots, b_m^*\}$. Elements of $\mathrm{Pol}(\mathbb{C}^m) \cong \mathrm{S}((\mathbb{C}^m)^*)$ (where S denotes the symmetric tensor algebra) can be written uniquely in the form $\sum_{I \in \mathbb{N}_0^m} a_I b_I^*$. Here $I \in \mathbb{N}_0^m$ is a multi-index, $b_I^* = (b_1^*)^{\vee I_1} \vee \cdots \vee (b_m^*)^{\vee I_m}$, and only finitely many of the coefficients $a_I \in \mathbb{C}$ are non-zero. For any $R \in \mathbb{R}^+$ define a norm

$$\left\|\!\!\left\| \sum_{I \in \mathbb{N}_0^m} a_I b_I^* \right\|\!\!\right\|_R := \sum_{I \in \mathbb{N}_0^m} |a_I| R^{|I|}. \tag{4.2}$$

Note that these norms coincide with the T_0-norms with respect to the basis B^*, studied for example in [40]. We denote the locally convex topology given by endowing $\mathrm{Pol}(\mathbb{C}^m) \cong \mathrm{S}((\mathbb{C}^m)^*)$ with the seminorms $\|\!|\cdot|\!\|_R$ by $\mathcal{T}_{\|\cdot\|}$. This topology can equivalently be defined by the countable set of norms $\|\!|\cdot|\!\|_R$ with $R \in \mathbb{N}$.

Note that $\|\!|\cdot|\!\|_R$ is submultiplicative with respect to the classical product:

$$\left\|\!\!\left\| \left(\sum_{I \in \mathbb{N}_0^m} a_I b_I^* \right) \vee \left(\sum_{J \in \mathbb{N}_0^m} a'_J b_J^* \right) \right\|\!\!\right\|_R = \left\|\!\!\left\| \sum_{I,J \in \mathbb{N}_0^m} a_I a'_J b_I^* \vee b_J^* \right\|\!\!\right\|_R$$

$$\leq \sum_{I,J \in \mathbb{N}_0^m} |a_I| |a'_J| R^{|I|+|J|}$$

$$= \left(\sum_{I \in \mathbb{N}_0^m} |a_I| R^{|I|} \right) \left(\sum_{J \in \mathbb{N}_0^m} |a'_J| R^{|J|} \right)$$

$$= \left\| \left\| \sum_{I \in \mathbb{N}_0^m} a_I b_I^* \right\| \right\|_R \left\| \left\| \sum_{J \in \mathbb{N}_0^m} a'_J b_J^* \right\| \right\|_R .$$

Proposition 4.6. *The topologies $\mathcal{T}_{\|\cdot\|}$ and $\mathcal{T}_{lc}$ coincide.*

Proof. Assume that $p = \sum_{I \in \mathbb{N}_0^m} a_I b_I^* \in \mathrm{Pol}(\mathbb{C}^m)$ is a polynomial. Given a compact subset $K \subseteq \mathbb{C}^m$, choose $R \in \mathbb{R}$ such that $|z| \leq R$ holds for all $z \in K$. Then on the one hand we have

$$\|p\|_K = \max_{z \in K} |p(z)| \leq \sum_{I \in \mathbb{N}_0^m} |a_I| R^{|I|} = \|p\|_R.$$

On the other hand, if $D_R = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_i| \leq R \text{ for } 1 \leq i \leq m\} \subseteq \mathbb{C}^m$ denotes a closed polydisc of radius R, then Cauchy's integral formula yields

$$|a_I| = \frac{1}{I!} |\partial_I p(0)| = \frac{1}{(2\pi)^m} \left| \int_{|z_i|=R} \frac{p(z)}{z^{I+(1,\ldots,1)}} \, dz^I \right|$$

$$\leq \max_{z \in D_R} |p(z)| \frac{R^m}{R^{|I+(1,\ldots,1)|}} = \frac{1}{R^{|I|}} \max_{z \in D_R} |p(z)|.$$

Applying this estimate for a polydisc of radius $2mR$ yields

$$\|p\|_R = \sum_{I \in \mathbb{N}_0^m} |a_I| R^{|I|} \leq \sum_{I \in \mathbb{N}_0^m} \frac{1}{(2mR)^{|I|}} R^{|I|} \max_{z \in D_{2mR}} |p(z)|$$

$$\leq \max_{z \in D_{2mR}} |p(z)| \sum_{I \in \mathbb{N}_0^m} (2m)^{-|I|} \leq 2 \max_{z \in D_{2mR}} |p(z)| = 2\|p\|_{D_{2mR}}.$$

Consequently, we can estimate any norm of $\mathcal{T}_{\|\cdot\|}$ by a seminorm of $\mathcal{T}_{lc}$ and vice versa, so the topologies $\mathcal{T}_{\|\cdot\|}$ and $\mathcal{T}_{lc}$ coincide. ∎

Because of the previous proposition, we can and will work with the norms $\|\cdot\|_R$ instead of the seminorms $\|\cdot\|_K$ in the following. To obtain continuity estimates, we need to estimate the coefficients $p_\lambda(\mu)$ defined in (3.8).

Lemma 4.7 (Estimates for p_λ). *For any fixed compact set $K \subseteq \mathfrak{h}^*$ there are constants $C > 0$ and M such that $p_\lambda(\alpha_w)$ defined in (3.8) satisfies*

$$|p_\lambda(\alpha_w)| \geq C|w|^2 \tag{4.3}$$

for all words $w \in W$ of length $|w| \geq M$ and all $\lambda \in K$.

Proof. Assume that the positive roots $\alpha_1, \ldots, \alpha_k \in \Delta^+$ are ordered in such a way that $\alpha_1, \ldots, \alpha_r$ are the simple roots. Write $\alpha_w = \sum_{i=1}^r c_{w,i} \alpha_i$ as a linear combination of simple roots, where $c_{w,i} \in \mathbb{N}_0$ satisfy $|w| \leq \sum_{i=1}^r c_{w,i} \leq c|w|$ with c depending only on

the root system. Since $(\rho, \alpha_i) > 0$ for all $1 \leq i \leq r$ we can choose $c_\rho, C_\rho \in \mathbb{R}^+$ such that $c_\rho \leq (\rho, \alpha_i) \leq C_\rho$ holds for all $1 \leq i \leq r$. Similarly, there is $C' \in \mathbb{R}^+$ with $|(\lambda, \alpha_i)| \leq C'$ for all $\lambda \in K$ and $1 \leq i \leq r$. Then

$$(\alpha_w, \alpha_w) \geq \frac{1}{(\rho, \rho)} (\alpha_w, \rho)^2 = \frac{1}{(\rho, \rho)} \left(\sum_{i=1}^r (c_{w,i} \alpha_i, \rho) \right)^2$$

$$\geq \frac{c_\rho^2}{(\rho, \rho)} \left(\sum_{i=1}^r c_{w,i} \right)^2 \geq \frac{c_\rho^2}{(\rho, \rho)} |w|^2$$

and for all $\lambda \in K$ we obtain

$$\left| (\rho + \lambda, \alpha_w) \right| \leq \sum_{i=1}^r c_{w,i} \left(|(\rho, \alpha_i)| + |(\lambda, \alpha_i)| \right)$$

$$\leq (C_\rho + C') \sum_{i=1}^r c_{w,i} \leq c(C_\rho + C')|w|.$$

Setting $C := \frac{1}{4(\rho,\rho)} c_\rho^2$, $C_1 := c(C_\rho + C')$, and $M := \frac{C_1}{C}$, and assuming $|w| \geq M$, we obtain

$$\left| p_\lambda(\alpha_w) \right| \geq \frac{1}{2} (\alpha_w, \alpha_w) - \left| (\rho + \lambda, \alpha_w) \right|$$

$$\geq 2C|w|^2 - C_1|w| \geq 2C|w|^2 - C|w|^2 = C|w|^2. \qquad \blacksquare$$

Corollary 4.8 (Estimates for p_λ^w). *Fix $\lambda \in \mathfrak{h}^*$. For any compact set $K \subseteq \mathbb{C} \setminus P_\lambda$ there is a constant $C_p > 0$ such that $p_{i\lambda/\hbar}^w(\alpha_w)$ defined in (3.8) satisfies*

$$|p_{i\lambda/\hbar}^w(\alpha_w)^{-1}| \leq \frac{C_p^{|w|}}{(|w|!)^2} \tag{4.4}$$

for all words $w \in \widetilde{W}$ and all $\hbar \in K$.

Proof. Note that $K' = \{ i\lambda/\hbar \mid \hbar \in K \}$ is a compact subset of $\widetilde{\Lambda}$. Let M and C be the constants obtained by applying the previous lemma to K', so $|p_{\lambda'}(\alpha_w)| \geq C|w|^2$ for all $w \in W$ with $|w| \geq M$ and all $\lambda' \in K'$. Since $i\lambda/\hbar \in \widetilde{\Lambda}$, we have $\min_{w \in \widetilde{W}, |w| < M} |p_{i\lambda/\hbar}(\alpha_w)| > 0$ for all $\hbar \in K$. Since this quantity depends continuously on $\hbar$, the minimum for $\hbar \in K$ exists and must also be positive. Hence we may decrease the constant C such that $|p_{i\lambda/\hbar}(\alpha_w)| \geq C|w|^2$ also holds for the finitely many words $w \in \widetilde{W}$ with $|w| < M$. Consequently, $|p_{i\lambda/\hbar}(\alpha_w)| \geq C|w|^2$ holds for all words $w \in \widetilde{W}$. Setting $C_p := 1/C$, the corollary follows by rearranging. $\blacksquare$

We have now collected all the results needed to prove Theorem 4.3.

Proof of Theorem 4.3. First, we note that it suffices to prove the existence of a constant M such that for any multi-indices $I, J \in \mathbb{N}_0^m$ we have $\left\| \left\| b_I^* *_\hbar' b_J^* \right\| \right\|_R \leq (RM)^{|I|+|J|}$.

Indeed, this statement implies the continuity of $*_\hbar'$ since for $p = \sum_{I \in \mathbb{N}_0^m} p_I b_I^*$ and $q = \sum_{I \in \mathbb{N}_0^m} q_I b_I^*$ we estimate

$$
\begin{aligned}
\||p *_\hbar' q\||_R &= \left\|\left| \sum_{I \in \mathbb{N}_0^m} p_I b_I^* *_\hbar' \sum_{J \in \mathbb{N}_0^m} q_J b_J^* \right|\right\|_R \\
&\leq \sum_{I \in \mathbb{N}_0^m} \sum_{J \in \mathbb{N}_0^m} |p_I||q_J| \||b_I^* *_\hbar' b_J^*\||_R \\
&\leq \sum_{I \in \mathbb{N}_0^m} \sum_{J \in \mathbb{N}_0^m} |p_I||q_J| (RM)^{|I|+|J|} \\
&= \left\|\left| \sum_{I \in \mathbb{N}_0^m} p_I b_I^* \right|\right\|_{RM} \left\|\left| \sum_{J \in \mathbb{N}_0^m} q_J b_J^* \right|\right\|_{RM} \\
&= \||p\||_{RM} \||q\||_{RM}.
\end{aligned}
$$

Using the notation $I_{\{j\}}$ introduced in the proof of Proposition 3.36 we estimate

$$
\begin{aligned}
\||b_I^* *_\hbar' b_J^*\||_R &= \left\|\left| \sum_{\ell=0}^\infty (F_{\hbar,\ell}')^{\mathrm{left},(1,0)}(b_I^*, b_J^*) \right|\right\|_R \\
&\leq \left\|\left| \sum_{w \in \widetilde{W}} p_{i\lambda/\hbar}^w (\alpha_w)^{-1} (X_w \otimes Y_w)^{\mathrm{left},(1,0)}(b_I^*, b_J^*) \right|\right\|_R \\
&\overset{(1)}{\leq} \sum_{w \in \widetilde{W}} \left| p_{i\lambda/\hbar}^w (\alpha_w)^{-1} \right| \\
&\quad \cdot \sum_{w_{(1)},\ldots,w_{(|I|)}} \sum_{w_{(1)}',\ldots,w_{(|J|)}'} \left\|\left| X_{w_{(1)}}^{\mathrm{left},(1,0)} b_{I_{\{1\}}}^* \right|\right\|_R \cdots \left\|\left| X_{w_{(|I|)}}^{\mathrm{left},(1,0)} b_{I_{\{|I|\}}}^* \right|\right\|_R \\
&\quad \cdot \left\|\left| Y_{w_{(1)}'}^{\mathrm{left},(1,0)} b_{J_{\{1\}}}^* \right|\right\|_R \cdots \left\|\left| Y_{w_{(|J|)}'}^{\mathrm{left},(1,0)} b_{J_{\{|J|\}}}^* \right|\right\|_R \\
&\overset{(2)}{\leq} \sum_{w \in \widetilde{W}} \frac{C_p^{|w|}}{(|w|!)^2} |I|^{|w|} |J|^{|w|} C^{2|w|} R^{|I|+|J|} \\
&\overset{(3)}{\leq} R^{|I|+|J|} \sum_{\ell=0}^\infty (k C_p C^2)^\ell \frac{|I|^\ell |J|^\ell}{(\ell!)^2} \\
&\overset{(4)}{\leq} R^{|I|+|J|} \sum_{\ell=0}^\infty \frac{(k^{1/2} C_p^{1/2} C |I|)^\ell}{\ell!} \sum_{\ell'=0}^\infty \frac{(k^{1/2} C_p^{1/2} C |J|)^{\ell'}}{\ell'!} \\
&\leq R^{|I|+|J|} e^{k^{1/2} C_p^{1/2} C |I|} e^{k^{1/2} C_p^{1/2} C |J|} \\
&= \left(R e^{k^{1/2} C_p^{1/2} C} \right)^{|I|+|J|}.
\end{aligned}
$$

The sum $\sum_{w_{(1)},\dots,w_{(|I|)}}$ introduced in (1) is over all partitions of $w \in \widetilde{W}$ into words $w_{(1)}$, $\dots, w_{(|I|)}$. To be more precise, consider a partition $P_1, \dots, P_{|I|}$ of $\{1, \dots, |w|\}$ into $|I|$ many subsets. To $P_i = \{p_{i,1}, \dots, p_{i,j_i}\}$ with $p_{i,1} < \cdots < p_{i,j_i}$, we associate the word $w_{(i)} = w_{p_{i,1}} w_{p_{i,2}} \cdots w_{p_{i,j_i}}$. Then we sum over all partitions. The other sum is defined similarly. We also used submultiplicativity of $\||\cdot\||_R$ in this step. To justify (2), we note that, for any $Z \in \mathfrak{gl}_N(\mathbb{C})$, $Z^{\text{left},(1,0)} b_i^*$ is of degree 1 so that $X_{w_{(\ell)}}^{\text{left},(1,0)} b_{I_{\{\ell\}}}^*$ is of degree 1. Defining $C := \max_{i \in \{1,\dots,m\}, \alpha \in \Delta} \||X_\alpha^{\text{left},(1,0)} b_i^*\||_1$, we obtain

$$\left\|\left\| X_{w_{(\ell)}}^{\text{left},(1,0)} b_{I_{\{\ell\}}}^* \right\|\right\|_R \leq C^{|w_{(\ell)}|} R.$$

The sum over $w_{(1)}, \dots, w_{(|I|)}$ has $|I|^{|w|}$ many terms, since for each letter of w we can choose in which of the $|I|$ many sets we want to have it. The same holds true for the other sum. In (3) we used that there are at most $k^{|w|}$ many words of a given length $|w|$ in $\widetilde{W}$ and (4) holds, because we just added some positive extra terms. ∎

Remark 4.9. For a fixed compact set $K \subseteq \mathbb{C} \setminus P_\lambda$ the proof above shows that there is a constant $M \in \mathbb{R}^+$ such that for any $\hbar \in K$ we have

$$\||p *_\hbar' q\||_R \leq \||p\||_{RM} \||q\||_{RM} \tag{4.5}$$

since Corollary 4.8 gives uniform estimates for all $\hbar \in K$.

4.2. Stein manifolds and extension of holomorphic functions

In this subsection, we discuss extension properties of holomorphic functions on closed complex submanifolds of Stein manifolds or, more generally, on analytic subsets of Stein manifolds. We will use the results in the next subsection to identify the reduction topology with the topology of locally uniform convergence and to determine the completion of the space of polynomials with respect to this topology.

Since analytic subsets in a Stein manifold are a very natural setting to prove the extendability results, we formulate them in this generality (even though we only need the case of closed submanifolds most of the time). The content of this subsection has been known for long and can be found, e.g., in the textbook [21].

Recall that, for a complex manifold M, we denote the vector space of holomorphic functions on M by $\text{Hol}(M)$.

Definition 4.10 (Holomorphic convex hull). For a compact subset K of a complex manifold M we define its *holomorphic convex hull* to be the set

$$\widehat{K}_M = \Big\{ z \in M \mid |f(z)| \leq \sup_K |f| \text{ for all } f \in \text{Hol}(M) \Big\}. \tag{4.6}$$

Definition 4.11 (Stein manifold). A complex manifold M of dimension n is said to be *Stein* if

(i)   for any compact subset $K \subseteq M$ its holomorphic convex hull $\widehat{K}_M$ is compact,

(ii)   for every $z \in M$ there are functions $f_1, \ldots, f_n \in \mathrm{Hol}(M)$ that form a coordinate system around z.

Stein manifolds should be thought of as domains of holomorphicity for holomorphic functions of several complex variables. Clearly, $\mathbb{C}^n$ is Stein.

Definition 4.12. A subset $V \subseteq M$ of a complex manifold is called *analytic*, if for every point $z \in M$ there is a neighborhood $U \subseteq M$ of z such that there is a family of holomorphic functions $f_j \in \mathrm{Hol}(U)$, indexed by j in some index set J, such that

$$V \cap U = \{z \in U \mid f_j(z) = 0 \text{ for all } j \in J\}. \tag{4.7}$$

Example 4.13. Any closed complex submanifold M of $\mathbb{C}^n$ is an analytic subset. Indeed, around any $z \in M$ we can find a submanifold chart, that is a neighborhood U and coordinates $z = (z_1, \ldots, z_n)$ such that $M \cap U$ is given by the vanishing of the first $n - \dim M$ coordinates. Therefore we can take $f_j = z_j$ for $j = 1, \ldots, n - \dim M$ in Definition 4.12. Around any $z \notin M$, there is a neighborhood U such that $U \cap M = \emptyset$ and we may pick $f_1 = 1$ in Definition 4.12.

Definition 4.14. A function $f \colon V \to \mathbb{C}$ on an analytic subset $V \subseteq M$ of a complex manifold is called holomorphic, if for every point $z \in V$ there is a neighborhood $U \subseteq M$ of z and a holomorphic function $g \in \mathrm{Hol}(U)$ such that $g|_{U \cap V} = f|_{U \cap V}$.

Example 4.15. If V is a closed complex submanifold of $\mathbb{C}^n$ as in Example 4.13, then this definition of a holomorphic function coincides with the usual definition. Indeed, in any submanifold chart (U, z) as in Example 4.13, a holomorphic function on $U \cap V$ can be extended constantly along the first $n - \dim M$ variables to a holomorphic function on U. The reverse implication is clear.

Proposition 4.16. *Let V be an analytic subset of a Stein manifold M. Then $\mathrm{Hol}(V)$ endowed with the topology of locally uniform convergence is a Fréchet space.*

Proof. It follows from the definition of analytic subsets that V is closed. Therefore the restriction of any compact exhaustion of M to V gives a compact exhaustion K_i of V. The seminorms $\|f\|_{K_i} = \sup_{K_i} |f|$ define a countable system of seminorms inducing the topology of locally uniform convergence. The completeness of $\mathrm{Hol}(V)$ with respect to this topology is a non-trivial result and proved in [21, Theorem 7.4.9]. ∎

The crucial property of an analytic subset V of a Stein manifold is the following extendability property for any holomorphic function on V.

Theorem 4.17 (Extendability of holomorphic functions). *Let V be an analytic subset of a Stein manifold M. Any holomorphic function $f \in \mathrm{Hol}(V)$ can be extended to a holomorphic function $f \in \mathrm{Hol}(M)$. In other words, the restriction map $\mathrm{Hol}(M) \to \mathrm{Hol}(V)$ is surjective.*

Proof. See [21, Theorem 7.4.8]. ∎

For an analytic subset V of a complex manifold M we denote the subspace of $\mathrm{Hol}(M)$ consisting of functions that vanish on V by $\mathcal{I}(V)$. Note that the restriction map $\mathrm{Hol}(M) \to \mathrm{Hol}(V)$ descends to a map on the quotient, $r\colon \mathrm{Hol}(M)/\mathcal{I}(V) \to \mathrm{Hol}(V)$. This map is clearly injective by definition of $\mathcal{I}(V)$, and if M is Stein it is surjective by the previous theorem.

Corollary 4.18. *Assume that M is Stein and that $V \subseteq M$ is an analytic subset. If $\mathrm{Hol}(M)/\mathcal{I}(V)$ is endowed with the quotient topology of the topology of locally uniform convergence and $\mathrm{Hol}(V)$ is endowed with the topology of locally uniform convergence, then the map $r\colon \mathrm{Hol}(M)/\mathcal{I}(V) \to \mathrm{Hol}(V)$ is a homeomorphism.*

Proof. We know that r is bijective, so it only remains to prove the continuity of r and r^{-1}. Both $\mathrm{Hol}(M)$ and $\mathrm{Hol}(V)$ are Fréchet spaces (for $\mathrm{Hol}(M)$ this is well known, for $\mathrm{Hol}(V)$ it is the statement of Proposition 4.16). Since $\mathcal{I}(V)$ is closed, $\mathrm{Hol}(M)/\mathcal{I}(V)$ is also a Fréchet space. Clearly, the locally uniform convergence of a sequence $f_i \in \mathrm{Hol}(M)$ implies the locally uniform convergence of the sequence of restrictions $f_i|_V \in \mathrm{Hol}(V)$, so the map r is continuous. The statement then follows from the open mapping theorem for Fréchet spaces. ∎

4.3. Characterizing the reduction topology

In this subsection, we show that the reduction topology on $\mathcal{O}_\lambda$ as defined in Section 4.1 is the topology of locally uniform convergence and that the completion of the space of polynomials $\mathrm{Pol}(\mathcal{O}_\lambda)$ on $\mathcal{O}_\lambda$ with respect to this topology is exactly the space of holomorphic functions $\mathrm{Hol}(\mathcal{O}_\lambda)$ on $\mathcal{O}_\lambda$.

Proposition 4.19. *The reduction topology $\mathcal{T}_{\mathrm{red}}$ on $\mathcal{O}_\lambda$ coincides with the topology of locally uniform convergence.*

Proof. By the assumption at the beginning of this section (see also Section 2.1), G is a closed complex submanifold of $\mathbb{C}^{N \times N}$, hence an analytic subset by Example 4.13. Applying Corollary 4.18 yields that the quotient topology on $\mathrm{Hol}(G)$ induced by the topology of locally uniform convergence on $\mathbb{C}^{N \times N}$ is precisely the topology of locally uniform convergence on G.

By Definition 4.2 the reduction topology is the restriction of this topology to the subspace of right G_λ-invariant holomorphic functions. Using that this subspace is closed, and that a sequence $f_i \in \mathrm{Hol}(\mathcal{O}_\lambda)$ converges locally uniformly if and only if the sequence $\pi^*(f_i) \in \mathrm{Hol}(G)^{G_\lambda}$ converges locally uniformly, one can easily check that the reduction topology coincides with the topology of locally uniform convergence on $\mathrm{Hol}(\mathcal{O}_\lambda)$. ∎

Finally, we would like to determine the completion $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda)$ of $\mathrm{Pol}(\mathcal{O}_\lambda)$ with respect to the topology of locally uniform convergence.

Proposition 4.20. *We have $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda) = \mathrm{Hol}(\mathcal{O}_\lambda)$.*

Proof. It is clear that $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda) \subseteq \mathrm{Hol}(\mathcal{O}_\lambda)$, since $\mathrm{Pol}(\mathcal{O}_\lambda) \subseteq \mathrm{Hol}(\mathcal{O}_\lambda)$ and the limit of a locally uniformly convergent sequence of holomorphic functions is again holomorphic.

The other inclusion is easy to see if one uses that semisimple coadjoint orbits are affine algebraic varieties; see Remark 2.4. In particular, they are analytic subsets of the Stein manifold $\mathfrak{g}^*$ and therefore we can use Theorem 4.17 to extend any $f \in \mathrm{Hol}(\mathcal{O}_\lambda)$ to a holomorphic function $\tilde{f} \in \mathrm{Hol}(\mathfrak{g}^*)$, which can be approximated by polynomials. Restricting these approximating polynomials to $\mathcal{O}_\lambda$ gives a sequence of polynomials in $\mathrm{Pol}(\mathcal{O}_\lambda)$ converging locally uniformly to f.

Alternatively, we know that G is a closed submanifold of the Stein manifold $\mathbb{C}^{N \times N}$, so the same argument yields that any $f \in \mathrm{Hol}(G)$ can be approximated by some $p_n \in \mathrm{Pol}(G)$. Assume that $f \in \mathrm{Hol}(G)^{G_\lambda}$. Let K_λ be a maximal compact subgroup of G_λ. Averaging p_n over K_λ gives a sequence $p'_n \in \mathrm{Pol}(G)^{K_\lambda}$ that converges locally uniformly to f. Now p'_n is even G_λ-invariant since the action of G is holomorphic, so $\pi_* p'_n \in \mathrm{Pol}(\mathcal{O}_\lambda)$ converges to $\pi_* f \in \mathrm{Hol}(\mathcal{O}_\lambda)$. ∎

We are now able to prove the main theorem stated in the introduction to this section.

Proof of Theorem 4.1. From Section 4.1, we know that the product $*_\hbar$ is continuous with respect to the reduction topology. We showed in Proposition 4.19 that the reduction topology coincides with the topology of locally uniform convergence on $\mathcal{O}_\lambda$. The previous proposition shows that the completion of $\mathrm{Pol}(\mathcal{O}_\lambda)$ in this topology is $\mathrm{Hol}(\mathcal{O}_\lambda)$. Finally, G-invariance of the product on the completion is clear since the action of G on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the topology of locally uniform convergence. ∎

We close this section by the following proposition, which asserts that the dependence of $*_\hbar$ on $\hbar$ is holomorphic.

Proposition 4.21 (Holomorphic dependence on $\hbar$). *For two fixed holomorphic functions $p, q \in \mathrm{Hol}(\mathcal{O}_\lambda)$ and $x \in \mathcal{O}_\lambda$ the map $\mathbb{C} \setminus P_\lambda \to \mathbb{C}, \hbar \mapsto p *_\hbar q(x)$ is holomorphic.*

Proof. By construction of $*_\hbar$ in Section 3, the map $\mathbb{C} \setminus P_\lambda \to \mathbb{C}, \hbar \mapsto p' *_\hbar q'(x)$ is rational for $p', q' \in \mathrm{Pol}(\mathcal{O}_\lambda)$. Assume that p_n, q_n are sequences of polynomials on $\mathcal{O}_\lambda$ such that $p_n \to p$ and $q_n \to q$ locally uniformly. Since the estimates of Section 4.1 are locally uniform in $\hbar$ (see Remark 4.9), it follows that $p_n *_\hbar q_n \to p *_\hbar q$ locally uniformly in $\hbar$. But clearly the evaluation at x is continuous so that $\hbar \mapsto p *_\hbar q(x)$ is a locally uniform limit of rational functions and therefore holomorphic. ∎

5. Quantizing real coadjoint orbits

We have seen in the previous sections how to construct (formal and strict) quantizations of complex coadjoint orbits. In this section, we will use these results to obtain (formal and strict) quantizations of real coadjoint orbits.

In Sections 5.1 and 5.2, we collect some preliminary results on the complexification of a real coadjoint orbit $\mathcal{O}_\lambda$ and a real Lie group G. We define a certain class of analytic functions that we denote by $\mathcal{A}(\mathcal{O}_\lambda)$ and $\mathcal{A}(G)$. In Section 5.3, we construct a quantization of real orbits by restricting the quantization of a complexification. We discuss the examples of complex projective spaces and hyperbolic discs in Section 5.4. Finally, we show

that point evaluation functionals are positive for certain coadjoint orbits in Section 5.5 and compare the quantum algebras obtained for coadjoint orbits of real Lie groups with the same complexification in Section 5.6. Most results in the later subsections follow almost directly from the results in the complex case.

From now on, all complex Lie groups and Lie algebras will be denoted with a hat and letters without decoration will be used to denote real objects. We will also use hats for maps between complex objects, e.g., we rename the map defined in (2.8) to $\widehat{\Psi}$.

5.1. Complexification

In this subsection, we define the complexification of a real coadjoint orbit $\mathcal{O}_\lambda$ and a real Lie group G and show how they are related.

For a real Lie algebra $\mathfrak{g}$, denote the space of real-valued real-linear functionals on $\mathfrak{g}$ by $\mathfrak{g}^*$. As before, $\hat{\mathfrak{g}}^*$ denotes the space of complex-valued complex-linear functionals on a complex Lie algebra $\hat{\mathfrak{g}}$. In the following, we will always assume that $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of $\mathfrak{g}$. In this case, any element of $\mathfrak{g}^*$ has a unique extension to an element of $\hat{\mathfrak{g}}^*$. We will perform this extension implicitly whenever necessary, without mentioning it. For example, in the following proposition, the coadjoint orbit $\widehat{\mathcal{O}}_\lambda$ is really the coadjoint orbit through the extension of $\lambda \in \mathfrak{g}^*$ to an element of $\hat{\mathfrak{g}}^*$.

Proposition 5.1. *Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a coadjoint orbit of a real connected Lie group, and assume that $\hat{\mathfrak{g}}$ is the complexification of $\mathfrak{g}$. Then $\mathcal{O}_\lambda$ is a submanifold of a unique complex coadjoint orbit $\widehat{\mathcal{O}}_\lambda \subseteq \hat{\mathfrak{g}}^*$ of a complex connected Lie group with Lie algebra $\hat{\mathfrak{g}}$. The tangent space $T_\xi \widehat{\mathcal{O}}_\lambda$ of this orbit $\widehat{\mathcal{O}}_\lambda$ is the complexification of $T_\xi \mathcal{O}_\lambda$ for every $\xi \in \mathfrak{g}^*$.*

Proof. By Proposition 2.1 the coadjoint orbit $\mathcal{O}_\lambda$ is the symplectic leaf through λ of the linear Poisson structure on $\mathfrak{g}^*$ defined just before Proposition 2.1. Similarly, the coadjoint orbits in $\hat{\mathfrak{g}}^*$ are symplectic leaves of the linear Poisson structure on $\hat{\mathfrak{g}}^*$, and the symplectic leaf containing $\lambda \in \hat{\mathfrak{g}}^*$ contains the whole orbit $\mathcal{O}_\lambda$. This proves the existence and uniqueness of $\widehat{\mathcal{O}}_\lambda$.

As in Section 2.1, we can identify $T_\xi \mathcal{O}_\lambda$ with $\mathfrak{g}/\mathfrak{g}_\xi$ (as real vector spaces) and $T_\xi \widehat{\mathcal{O}}_\lambda$ with $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\xi$ (as complex vector spaces) for all $\xi \in \mathcal{O}_\lambda$. Therefore $T_\xi \widehat{\mathcal{O}}_\lambda$ is indeed the complexification of $T_\xi \mathcal{O}_\lambda$. ∎

We refer to the complex coadjoint orbit $\widehat{\mathcal{O}}_\lambda$ of the previous proposition as the *complexification* of $\mathcal{O}_\lambda$. We will show how to realize it explicitly as the coadjoint orbit of some Lie group $\widehat{G}$.

Definition 5.2. Let G be a real Lie group. A *complexification* of G is a complex connected Lie group $\widehat{G}$ together with an embedding $\iota: G \to \widehat{G}$ such that the corresponding Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$ and such that the map $T_e \iota: \mathfrak{g} \to \hat{\mathfrak{g}}$ corresponds to the injection $X \mapsto X \otimes 1$ under this isomorphism.

Note that a complexification according to this definition may fail to exist or may not be unique, if it exists. See the paragraph after Proposition 5.8 for an example of a Lie group with non-unique complexification. For a connected semisimple Lie group G a com-

plexification exists if and only if the group can be realized as a linear group: existence for linear Lie groups is shown below, and the reverse implication follows since semisimplicity of G implies semisimplicity of the complexification and complex connected semisimple Lie groups are always matrix Lie groups; see Remark 2.4. There is a different notion of a universal complexification that does always exist, but that does not enjoy the property that $\hat{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathbb{C}$. We will not use universal complexifications in this paper.

Proposition 5.3. *If G is a real connected closed linear Lie group, then it admits a complexification $\hat{G}$.*

Proof. We may assume that both G and its Lie algebra $\mathfrak{g}$ are realized by real matrices. Then the complexification $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is a Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$. We can use the exponential map to construct an immersed complex Lie subgroup $\hat{G}$ of $\mathrm{GL}_N(\mathbb{C})$ containing G as a subgroup and having $\hat{\mathfrak{g}}$ as Lie algebra; see, e.g., [20, Chapter 5.9]. Since G is a closed subgroup of $\mathrm{GL}_N(\mathbb{C})$, it is also a closed subgroup of $\hat{G}$. ∎

Note that we did not claim that $\hat{G}$ is a closed subgroup of $\mathrm{GL}_N(\mathbb{C})$. For semisimple Lie groups this follows automatically from Remark 2.4.

Lemma 5.4. *Let G be a real connected Lie group with complexification $\hat{G}$ and let $\mathcal{O}_\lambda$ be a coadjoint orbit of G with complexification $\hat{\mathcal{O}}_\lambda$. Then $\hat{\mathcal{O}}_\lambda$ is a coadjoint orbit of $\hat{G}$ and the embedding $\iota: G \to \hat{G}$ descends to an embedding $\mathcal{O}_\lambda \cong G/G_\lambda \to \hat{G}/\hat{G}_\lambda \cong \hat{\mathcal{O}}_\lambda$.*

Proof. Since $\hat{G}$ is connected and has the Lie algebra $\hat{\mathfrak{g}}$, it follows from Proposition 2.1 that its coadjoint orbit through λ is $\hat{\mathcal{O}}_\lambda$. We identify G with a subgroup of $\hat{G}$. Since the coadjoint action of $\hat{G}$ on $\hat{\mathfrak{g}}$ is holomorphic, $\hat{G}_\lambda$ is a complexification of $G_\lambda = \hat{G}_\lambda \cap G$. So the map ι descends to a map $\mathcal{O}_\lambda \cong G/G_\lambda \to \hat{G}/\hat{G}_\lambda \cong \hat{\mathcal{O}}_\lambda$ that is still injective. To see that it is an embedding, note that the actions of G_λ and $\hat{G}_\lambda$ on $\hat{G}$ are proper and free, so $\hat{G}$ is a principal G_λ resp. $\hat{G}_\lambda$ bundle over $\hat{G}/G_\lambda$ resp. $\hat{G}/\hat{G}_\lambda$. This implies first that $G/G_\lambda \to \hat{G}/G_\lambda$ is still an embedding and then that $G/G_\lambda \to \hat{G}/\hat{G}_\lambda$ also is. ∎

5.2. Polynomials and analytic functions

In this subsection, we introduce polynomials $\mathrm{Pol}(\mathcal{O}_\lambda)$ and a certain class of analytic functions $\mathcal{A}(\mathcal{O}_\lambda)$ on a real coadjoint orbit $\mathcal{O}_\lambda$. $\mathcal{A}(\mathcal{O}_\lambda)$ consists of restrictions of holomorphic functions on the complexification. In analogy to the complex case, $\mathcal{A}(\mathcal{O}_\lambda)$ is the completion of $\mathrm{Pol}(\mathcal{O}_\lambda)$ with respect to some locally convex topology.

All our polynomials are complex-valued. So for a real finite dimensional vector space V we define $\mathrm{Pol}(V)$ to be the unital complex subalgebra of $\mathscr{C}^\infty(V)$ generated by the linear maps. (Remember that $\mathscr{C}^\infty(V)$ consists of smooth functions $V \to \mathbb{C}$.) So $\mathrm{Pol}(V) \cong \mathrm{S}(V_\mathbb{C}^*)$, where $V_\mathbb{C}^*$ is the complexification of $V^* = \{\phi : V \to \mathbb{R}, \phi \text{ linear}\}$.

Definition 5.5 (Polynomials). Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a real connected Lie group G with Lie algebra $\mathfrak{g}$. Then

$$\mathrm{Pol}(\mathcal{O}_\lambda) = \left\{ p : \mathcal{O}_\lambda \to \mathbb{C} \mid p = P\big|_{\mathcal{O}_\lambda} \text{ for some polynomial } P \text{ on } \mathfrak{g}^* \right\} \tag{5.1}$$

is called the algebra of polynomials on $\mathcal{O}_\lambda$.

Note that polynomials on a complex orbit $\widehat{\mathcal{O}}_\lambda$ were assumed to be holomorphic and do therefore not coincide with polynomials on the underlying real orbit. We will always use holomorphic polynomials on complexifications, so this will hopefully not cause any confusion.

Denote the ideal of polynomials on $\mathfrak{g}^*$ resp. $\hat{\mathfrak{g}}^*$ vanishing on $\mathcal{O}_\lambda$ resp. $\widehat{\mathcal{O}}_\lambda$ by $\mathcal{I}(\mathcal{O}_\lambda)$ resp. $\mathcal{I}(\widehat{\mathcal{O}}_\lambda)$. It is clear that the maps $\mathrm{Pol}(\mathfrak{g}^*)/\mathcal{I}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ and $\mathrm{Pol}(\hat{\mathfrak{g}}^*)/\mathcal{I}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda)$ are isomorphisms. We would now like to relate polynomials on $\mathcal{O}_\lambda$ and $\widehat{\mathcal{O}}_\lambda$.

Proposition 5.6. *Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a real coadjoint orbit with complexification $\widehat{\mathcal{O}}_\lambda \subseteq \hat{\mathfrak{g}}^*$. Then the restriction map $(\,\cdot\,)|_{\mathcal{O}_\lambda} \colon \mathscr{C}^\infty(\widehat{\mathcal{O}}_\lambda) \to \mathscr{C}^\infty(\mathcal{O}_\lambda)$ restricts to an isomorphism $(\,\cdot\,)|_{\mathcal{O}_\lambda} \colon \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$.*

Proof. Since restriction to V is a bijection between complex linear maps $V \otimes \mathbb{C} \to \mathbb{C}$ and real linear maps $V \to \mathbb{C}$ for any finite dimensional real vector space V, it follows that the restriction map $\mathrm{Pol}(\hat{\mathfrak{g}}^*) \to \mathrm{Pol}(\mathfrak{g}^*)$ is an isomorphism. If we can prove that the restriction map $\mathcal{I}(\widehat{\mathcal{O}}_\lambda) \to \mathcal{I}(\mathcal{O}_\lambda)$ is also an isomorphism, then we are done since $\mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \cong \mathrm{Pol}(\hat{\mathfrak{g}}^*)/\mathcal{I}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\mathfrak{g}^*)/\mathcal{I}(\mathcal{O}_\lambda) \cong \mathrm{Pol}(\mathcal{O}_\lambda)$ would be an isomorphism.

Since any map vanishing on $\widehat{\mathcal{O}}_\lambda$ vanishes in particular on $\mathcal{O}_\lambda \subseteq \widehat{\mathcal{O}}_\lambda$, the restriction map $\mathcal{I}(\widehat{\mathcal{O}}_\lambda) \to \mathcal{I}(\mathcal{O}_\lambda)$ is well defined and it is injective since it is the restriction of the injective map $\mathrm{Pol}(\hat{\mathfrak{g}}^*) \to \mathrm{Pol}(\mathfrak{g}^*)$. So we only need to prove surjectivity, meaning that if a polynomial p on $\mathfrak{g}^*$ vanishes on $\mathcal{O}_\lambda$, then its unique extension to a polynomial $\hat{p}$ on $\hat{\mathfrak{g}}^*$ vanishes on $\widehat{\mathcal{O}}_\lambda$. Since $\widehat{\mathcal{O}}_\lambda$ is a complex submanifold of $\hat{\mathfrak{g}}^*$, the restriction of $\hat{p}$ to $\widehat{\mathcal{O}}_\lambda$ is holomorphic. As such, it is determined by its derivatives (of all orders) at λ. It is even determined by its derivatives in the direction of $T_\lambda \mathcal{O}_\lambda$ since $T_\lambda \widehat{\mathcal{O}}_\lambda$ is the complexification of $T_\lambda \mathcal{O}_\lambda$. But all these derivatives vanish since the restriction of $\hat{p}$ to $\mathcal{O}_\lambda$ vanishes. ∎

Definition 5.7. Let G be a linear real Lie group. Its algebra of polynomials $\mathrm{Pol}(G)$ is the unital complex subalgebra of $\mathscr{C}^\infty(G)$ generated by the functions $P_{ij} \colon G \to \mathbb{C}, g \mapsto g_{ij}$.

In contrast to the complex case, the algebra of polynomials $\mathrm{Pol}(G)$ may depend on the way in which G is realized as a linear group, even in the semisimple case. We will give an instructive example after stating the following proposition, which can be proven in much the same way as Proposition 5.6.

Proposition 5.8. *Let $G \subseteq \mathrm{GL}_N(\mathbb{R})$ be a linear connected Lie group with complexification $\widehat{G} \subseteq \mathrm{GL}_N(\mathbb{C})$. Then the restriction map $(\,\cdot\,)|_G \colon \mathscr{C}^\infty(\widehat{G}) \to \mathscr{C}^\infty(G)$ restricts to an isomorphism $(\,\cdot\,)|_G \colon \mathrm{Pol}(\widehat{G}) \to \mathrm{Pol}(G)$.*

The reason why the algebra of polynomials $\mathrm{Pol}(G)$ may depend on the linear structure of G is essentially that G may not have a unique complexification. Consider the linear semisimple Lie group $\mathrm{SL}_3(\mathbb{R}) \subseteq \mathrm{GL}_3(\mathbb{R})$, which has $\mathrm{SL}_3(\mathbb{C})$ as a complexification. The images of $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$ under Ad are again semisimple Lie groups. Furthermore, $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R})) \cong \mathrm{SL}_3(\mathbb{R})$ since $\mathrm{SL}(3, \mathbb{R})$ has trivial center, and $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{C})) \cong \mathrm{SL}_3(\mathbb{C})/\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ is a complexification of $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R}))$. From the previous proposition we obtain $\mathrm{Pol}(\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R}))) \cong \mathrm{Pol}(\mathrm{SL}_3(\mathbb{C})/\{1, e^{2\pi i/3}, e^{4\pi i/3}\}) \to \mathrm{Pol}(\mathrm{SL}_3(\mathbb{C})) \cong$

$\mathrm{Pol}(\mathrm{SL}_3(\mathbb{R}))$, where the map in the middle is not surjective, since there are polynomials on $\mathrm{SL}_3(\mathbb{C})$ that are not constant on $\{1, \mathrm{e}^{2\pi \mathrm{i}/3}, \mathrm{e}^{4\pi \mathrm{i}/3}\}$.

We denote the inverses of the isomorphisms in Propositions 5.6 and 5.8 by

$$\hat{\cdot}: \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \quad \text{and} \quad \hat{\cdot}: \mathrm{Pol}(G) \to \mathrm{Pol}(\widehat{G}). \tag{5.2}$$

Lemma 5.9. *Let G be a real connected linear Lie group with complexification $\widehat{G}$, and let $\lambda \in \mathfrak{g}^*$ be such that $\widehat{G}_\lambda$ is connected. If $f \in \mathrm{Pol}(\widehat{G})$ satisfies $f|_G \in \mathrm{Pol}(G)^{G_\lambda}$, then $f \in \mathrm{Pol}(\widehat{G})^{\widehat{G}_\lambda}$.*

Proof. Let f be as in the statement of the lemma. Since $f|_G = (g \triangleright f)|_G$ holds for all $g \in G_\lambda$, it follows from the injectivity of $(\cdot)|_G$ that $f = g \triangleright f$, so $f \in \mathrm{Pol}(\widehat{G})^{G_\lambda}$. Therefore f is in particular invariant under $\mathfrak{g}_\lambda$ and thus also under $\hat{\mathfrak{g}}_\lambda$ since the action is holomorphic. Since $\widehat{G}_\lambda$ is connected, we obtain that f is $\widehat{G}_\lambda$-invariant. ∎

Corollary 5.10. *Let G be a real connected semisimple linear Lie group with complexification $\widehat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. In this case the restriction map*

$$(\cdot)|_G: \mathrm{Pol}(\widehat{G})^{\widehat{G}_\lambda} \to \mathrm{Pol}(G)^{G_\lambda}$$

is an isomorphism.

Proof. G_λ is connected by Proposition 2.3, so this is an immediate consequence of Proposition 5.8 and Lemma 5.9. ∎

Corollary 5.11. *Let G be a real connected semisimple linear Lie group with complexification $\widehat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. Then the map $\pi^*: \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}$ is an isomorphism.*

Proof. From Propositions 5.6, 3.27, and Corollary 5.10 it follows that the composition

$$\mathrm{Pol}(\mathcal{O}_\lambda) \xrightarrow{\hat{\cdot}} \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \xrightarrow{\widehat{\pi}^*} \mathrm{Pol}(\widehat{G})^{\widehat{G}_\lambda} \xrightarrow{(\cdot)|_G} \mathrm{Pol}(G)^{G_\lambda}$$

is an isomorphism, and this composition is π^*. ∎

Corollary 5.12. *Let G be a real connected semisimple linear Lie group with complexification $\widehat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. Then the following diagram commutes and all arrows are isomorphisms:*

$$\begin{array}{ccc} \mathrm{Pol}(\widehat{G})^{\widehat{G}_\lambda} & \underset{\widehat{\pi}^*}{\overset{\widehat{\pi}_*}{\rightleftarrows}} & \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \\ \hat{\cdot} \Big\updownarrow (\cdot)|_G & & \hat{\cdot} \Big\updownarrow (\cdot)|_{\mathcal{O}_\lambda} \\ \mathrm{Pol}(G)^{G_\lambda} & \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} & \mathrm{Pol}(\mathcal{O}_\lambda) \end{array} \qquad . \tag{5.3}$$

Next, we want to introduce a class of analytic functions, that becomes the closure of the polynomials with respect to a certain locally convex topology. To this end, assume that

$\mathcal{O}_\lambda$ is a coadjoint orbit with complexification $\widehat{\mathcal{O}}_\lambda$ and that G is a real connected Lie group with complexification $\widehat{G}$. Then define

$$\mathcal{A}(\mathcal{O}_\lambda) = \mathrm{im}\left((\,\cdot\,)\big|_{\mathcal{O}_\lambda} : \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \to \mathscr{C}^\infty(\mathcal{O}_\lambda)\right) \tag{5.4}$$

and

$$\mathcal{A}(G) = \mathrm{im}\left((\,\cdot\,)\big|_G : \mathrm{Hol}(\widehat{G}) \to \mathscr{C}^\infty(G)\right). \tag{5.5}$$

Note that an element $f \in \mathcal{A}(\mathcal{O}_\lambda)$ determines a unique element $\hat{f} \in \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$: existence follows by definition of $\mathcal{A}(\mathcal{O}_\lambda)$ and $\hat{f}$ is determined by all its derivatives at λ. Since the complexification of $\mathrm{T}_\lambda \mathcal{O}_\lambda$ is just $\mathrm{T}_\lambda \widehat{\mathcal{O}}_\lambda$ (see Lemma 5.4), it suffices to take derivatives in the direction of $\mathrm{T}_\lambda \mathcal{O}_\lambda$. But these derivatives are determined by f. A similar reasoning holds for G and $\widehat{G}$. We obtain a commuting square that is similar to the square for polynomials obtained in Corollary 5.12.

Proposition 5.13. *The following diagram is commutative and all arrows are isomorphisms:*

$$
\begin{array}{ccc}
\mathrm{Hol}(\widehat{G})^{\widehat{G}_\lambda} & \xleftarrow[\widehat{\pi}^*]{\widehat{\pi}_*} & \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \\
{\scriptstyle\wedge}\big\uparrow\big\downarrow{\scriptstyle(\cdot)|_G} & & {\scriptstyle\wedge}\big\uparrow\big\downarrow{\scriptstyle(\cdot)|_{\mathcal{O}_\lambda}} \\
\mathcal{A}(G)^{G_\lambda} & \xleftarrow[\pi^*]{\pi_*} & \mathcal{A}(\mathcal{O}_\lambda)
\end{array}
\tag{5.6}
$$

Proof. We know from Section 2.1 that $\widehat{\pi}^* : \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Hol}(\widehat{G})^{\widehat{G}_\lambda}$ is an isomorphism. In the previous paragraph, we explained that $\hat{\ } : \mathcal{A}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$ and $\hat{\ } : \mathcal{A}(G) \to \mathrm{Hol}(\widehat{G})$ are isomorphisms and as in Lemma 5.9 it follows that the same is true for $\hat{\ } : \mathcal{A}(G)^{G_\lambda} \to \mathrm{Hol}(\widehat{G})^{\widehat{G}_\lambda}$. Composing these isomorphisms, we obtain that $\pi^* : \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(G)^{G_\lambda}$ is an isomorphism. ∎

Since $\mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \subseteq \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$, it follows that $\mathrm{Pol}(\mathcal{O}_\lambda) \subseteq \mathcal{A}(\mathcal{O}_\lambda)$. We can define a topology $\mathcal{T}_{\widehat{\mathrm{lu}}}$ of *extended locally uniform convergence* on $\mathcal{A}(\mathcal{O}_\lambda)$ as follows: a sequence $f_n \in \mathcal{A}(\mathcal{O}_\lambda)$ converges to some $f \in \mathcal{A}(\mathcal{O}_\lambda)$ if and only if the sequence $\hat{f}_n \in \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$ converges locally uniformly to $\hat{f} \in \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$. Clearly, the maps $\hat{\ } : \mathcal{A}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$ and $(\,\cdot\,)|_{\mathcal{O}_\lambda} : \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$ are both homeomorphisms. From Proposition 4.20 it follows that the closure of $\mathrm{Pol}(\mathcal{O}_\lambda)$ with respect to the topology of extended locally uniform convergence is $\mathcal{A}(\mathcal{O}_\lambda)$.

5.3. Formal and strict star products on real coadjoint orbits

In a sense all constructions in Sections 2, 3, and 4 are compatible with the restriction to real forms. In this subsection, we want to make this statement precise. In particular, we will show that we can restrict formal and strict products from a complexification $\widehat{\mathcal{O}}_\lambda$ of a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a real connected semisimple Lie group G to formal and strict star products on $\mathcal{O}_\lambda$. These star products can—as before—be computed by applying fundamental vector fields or by passing to the Lie group by using the maps π^* and π_*.

We will determine when the star products on $\mathcal{O}_\lambda$ are of (pseudo) Wick type or of standard ordered type.

Proposition 5.14. *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group G. By Lemma 5.4 it has a complexification $\widehat{\mathcal{O}}_\lambda$, and there are strict products $\hat{*}_\hbar\colon \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \times \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda)$ with extensions $\hat{*}_\hbar\colon \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \times \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Hol}(\widehat{\mathcal{O}}_\lambda)$ constructed in Corollary 3.30 and Theorem 4.1, where $\hbar \in \mathbb{C} \setminus P_\lambda$. These products restrict to G-invariant strict products*

$$*_\hbar\colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda) \quad and \quad *_\hbar\colon \mathcal{A}(\mathcal{O}_\lambda) \times \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda) \quad (5.7)$$

for all $\hbar \in \mathbb{C} \setminus P_\lambda$. For fixed $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, the dependence of $p *_\hbar q$ on $\hbar$ is rational with no pole at zero, and, for fixed $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$ and $x \in \mathcal{O}_\lambda$, the dependence of $f *_\hbar g(x)$ on $\hbar$ is holomorphic. Both products are continuous with respect to the topology of extended locally uniform convergence defined at the end of Section 5.2.

Proof. Since the restriction maps $\mathrm{Pol}(\widehat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ and $\mathrm{Hol}(\widehat{\mathcal{O}}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$ are both homeomorphisms (with respect to the topology of locally uniform convergence on the domains and the topology of extended locally uniform convergence on the codomains), the statement follows trivially from the corresponding statements for $\hat{*}_\hbar$, obtained in Corollary 3.30, Theorem 4.1, and Proposition 4.21. ∎

We would like to compute these star products without passing to the complexification. The construction of bidifferential operators from Section 2.2 works completely similarly in the real setting. Recall that our differential operators act on complex-valued functions, and therefore any complex vector field $\Gamma^\infty(\mathrm{T}^{\mathbb{C}} M)$ defines a first-order differential operator on M.

Proposition 5.15. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$, and let $\hat{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. The map*

$$(\,\cdot\,)^{\mathrm{left}}\colon (\mathcal{U}\hat{\mathfrak{g}})^{\otimes k} \to k\text{-}\mathrm{DiffOp}^G(G)$$

obtained by extending $\hat{\mathfrak{g}} \ni X \mapsto X^{\mathrm{left}} \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}} G)$ to an algebra homomorphism $\mathcal{U}\hat{\mathfrak{g}} \to \mathrm{DiffOp}^G(G)$ and further to tensor products as in (2.5c) is an isomorphism. If H is a closed Lie subgroup of G, then the map

$$\Psi\colon \big((\mathcal{U}\hat{\mathfrak{g}}/\mathcal{U}\hat{\mathfrak{g}} \cdot \hat{\mathfrak{h}})^{\otimes k}\big)^H \to k\text{-}\mathrm{DiffOp}^G(G/H), \quad \Psi\big([\vec{u}]\big)(\vec{f}) = \pi_*\big(\vec{u}^{\mathrm{left}}(\pi^* \vec{f})\big) \quad (5.8)$$

is also an isomorphism.

Proof. With the obvious modifications, the proofs of Propositions 2.5 and 2.7 given in Appendix A.1 apply also to the real situation. ∎

To be consistent with the notation of this chapter, we denote the map defined in (2.8) by $\widehat{\Psi}$.

Lemma 5.16. *Let G be a real Lie group with closed subgroup H and assume that the complex Lie group $\widehat{G}$ is a complexification of G and contains a complex closed subgroup $\widehat{H}$ that is a complexification of H. The maps $(\,\cdot\,)^{\mathrm{left}}$ and Ψ are compatible with the maps $(\,\cdot\,)^{\mathrm{left},(1,0)}$ and $\widehat{\Psi}$ in the sense that the diagrams*

$$
\begin{array}{ccc}
\mathrm{Hol}(\widehat{U})^k & \xrightarrow{(\cdot)|_U^{\times k}} & \mathscr{C}^\infty(U)^k \\
{\scriptstyle \vec{u}^{\mathrm{left},(1,0)}}\downarrow & & \downarrow{\scriptstyle \vec{u}^{\mathrm{left}}} \\
\mathrm{Hol}(\widehat{U}) & \xrightarrow[(\cdot)|_U]{} & \mathscr{C}^\infty(U)
\end{array}
\quad and \quad
\begin{array}{ccc}
\mathrm{Hol}(\widehat{V})^k & \xrightarrow{(\cdot)|_V^{\times k}} & \mathscr{C}^\infty(V)^k \\
{\scriptstyle \widehat{\Psi}([\vec{v}])}\downarrow & & \downarrow{\scriptstyle \Psi([\vec{v}])} \\
\mathrm{Hol}(\widehat{V}) & \xrightarrow[(\cdot)|_V]{} & \mathscr{C}^\infty(V)
\end{array}
\tag{5.9}
$$

commute for all open subsets $\widehat{U} \subseteq \widehat{G}$ and $\widehat{V} \subseteq \widehat{G}/\widehat{H}$, with $U := \widehat{U} \cap G$ and $V := \widehat{V} \cap G/H$, and all elements $\vec{u} \in (\mathscr{U}\widehat{\mathfrak{g}})^{\otimes k}$ and $\vec{v} \in ((\mathscr{U}\widehat{\mathfrak{g}}/\mathscr{U}\widehat{\mathfrak{g}} \cdot \widehat{\mathfrak{h}})^{\otimes k})^{\widehat{H}}$.

Proof. The commutativity of the second diagram follows easily from commutativity of the first, since the restrictions are compatible with π^* and π_*. To prove commutativity of the first diagram, assume that $k = 1$ and $\vec{u} = X \in \widehat{\mathfrak{g}} \subseteq \mathscr{U}\widehat{\mathfrak{g}}$. The tangent map of a holomorphic function commutes with the multiplication by i. We compute

$$
\begin{aligned}
X^{\mathrm{left},(1,0)} f(g) &= \frac{1}{2}\big(\mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(X) - \mathrm{i}\mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(\mathrm{i}X) \big) \\
&= \mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(X) = X^{\mathrm{left}} f\big|_U(g)
\end{aligned}
$$

for $f \in \mathrm{Hol}(\widehat{U})$ and $g \in U$. The general case follows from this computation by the way in which $(\,\cdot\,)^{\mathrm{left},(1,0)}$ and $(\,\cdot\,)^{\mathrm{left}}$ are extended to $(\mathscr{U}\widehat{\mathfrak{g}})^{\otimes k}$. ∎

Corollary 5.17. *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group G. For $\hbar \in \mathbb{C} \setminus P_\lambda$ and $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, the product $*_\hbar \colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ defined in Proposition 5.14 can be computed by*

$$
p *_\hbar q = \sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q). \tag{5.10}
$$

Proof. The previous lemma implies

$$
p *_\hbar q = (\hat{p} \,\hat{*}_\hbar\, \hat{q})|_{\mathcal{O}_\lambda} = \sum_{\ell=0}^\infty \widehat{\Psi}(F_{\hbar,\ell})(\hat{p}, \hat{q})|_{\mathcal{O}_\lambda} = \sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q).
$$

Note that the sum over ℓ is finite by Corollary 3.29. ∎

Theorem 5.18. *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group G. The product $\star \colon \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!] \times \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!] \to \mathscr{C}^\infty(\mathcal{O}_\lambda)[\![\hbar]\!]$ defined by $f \star g = \Psi(F)(f,g)$ where F was obtained in Theorem 3.23 is a G-invariant formal star product. In particular, it is associative and deforms the KKS symplectic form on $\mathcal{O}_\lambda$. Furthermore, $p \star q$ coincides with the formal power series expansion of $p *_\hbar q$ around $\hbar = 0$ for $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, and $f \star g = \hat{f} \,\hat{\star}\, \hat{g}|_{\mathcal{O}_\lambda}$ for $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$.*

Proof. It is immediate from the definition of F and Ψ that every order of $\star$ is given by a G-invariant bidifferential operator. Since F is the formal power series expansion of $F_\hbar$ around $\hbar = 0$ and $p *_\hbar q$ is rational with no pole at 0 for $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, it follows that $p \star q$ coincides with the formal power series expansion of $p *_\hbar q$. The compatibility with $\hat{\star}$ is immediate from Lemma 5.16. Since bidifferential operators are uniquely determined by their behavior on $\mathrm{Pol}(\mathcal{O}_\lambda) \subseteq \mathcal{A}(\mathcal{O}_\lambda)$, the compatibility with $\hat{\star}$ implies that $\star$ is associative and, using Proposition 3.32, that it deforms the KKS symplectic form. ∎

Recall that we proved in Corollary 3.34 that the product $\hat{*}_\hbar$ separates variables with respect to the distributions L_+ and L_-, which we call $\hat{L}_+$ and $\hat{L}_-$ in this section. In the real case, those distributions may have further properties. They can be real or the holomorphic and antiholomorphic tangent spaces with respect to a complex structure. Before giving further details, let us make the following definitions.

Definition 5.19 (Standard ordered type). A star product $*_\hbar$ on a symplectic manifold M is said to be of *standard ordered type* if there are two Lagrangian distributions $L_1, L_2 \subseteq \mathrm{T}M$ spanning the real tangent bundle $\mathrm{T}M$ of M such that the first argument of the star product is derived only in directions of L_1 and the second argument only in directions of L_2.

Definition 5.20 ((Pseudo) Wick type). A star product $*_\hbar$ on a complex manifold M that is also symplectic is said to be of *pseudo Wick type* if the first argument is derived only in holomorphic directions and the second argument only in antiholomorphic directions. A star product of pseudo Wick type on a Kähler manifold is said to be of *Wick type*.

For formal star products of Wick type and with respect to the usual $*$-involution given by complex conjugation, point evaluations are positive linear functionals, which is not necessarily the case for formal star products of pseudo Wick type. Note that the situation is more complicated for strict star products, as we shall see in Section 5.5.

Let us briefly recall some results on the existence of invariant complex structures on coadjoint orbits. See Appendix A.3 for more details. Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a real connected semisimple Lie group G with Lie algebra $\mathfrak{g}$, and assume that G_λ is compact. Choose a real Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$. Since $\mathfrak{h} \subseteq \mathfrak{g}_\lambda$, it follows that $\mathfrak{h}$ is compact (meaning that it integrates to a subgroup of G with compact closure). Then there are G-invariant complex structures on $\mathcal{O}_\lambda$, and these structures are in bijection to invariant orderings of $\hat{\Delta}$ (we say an ordering on $\hat{\Delta}$ is invariant if it is the restriction of an invariant ordering of Δ as defined in Definition 3.10) as follows. Recall that $\mathrm{T}_\lambda^{\mathbb{C}} \mathcal{O}_\lambda \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda \cong \bigoplus_{\alpha \in \hat{\Delta}} \mathfrak{g}^\alpha$. So given an invariant ordering we can define a map $I_\lambda : \mathrm{T}_\lambda^{\mathbb{C}} \mathcal{O}_\lambda \to \mathrm{T}_\lambda^{\mathbb{C}} \mathcal{O}_\lambda$ by letting $I_\lambda X_\alpha = \mathrm{i} X_\alpha$ if $\alpha \in \hat{\Delta}^+$, and $I_\lambda X_\alpha = -\mathrm{i} X_\alpha$ if $\alpha \in \hat{\Delta}^-$. The map I_λ extends G-invariantly to an endomorphism I of the complexified tangent bundle $\mathrm{T}^{\mathbb{C}} \mathcal{O}_\lambda$ and restricts to an endomorphism of the real tangent bundle $\mathrm{T}\mathcal{O}_\lambda$; thus it defines a complex structure.

If G is compact, there is a unique ordering that makes $\mathcal{O}_\lambda$ with the complex structure I and the KKS symplectic form ω_{KKS} a Kähler manifold. This ordering is characterized by $\alpha \in \hat{\Delta}$ being positive iff $(\alpha, \mathrm{i}\lambda) > 0$. In particular, it is standard. See Appendix A.3 for more details.

Proposition 5.21. *For a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a real connected semisimple linear Lie group G, the product $*_\hbar$ obtained in Proposition 5.14*

(i)   *has poles $P_\lambda \subseteq \mathbb{R}$ if $\mathfrak{h}$ is compact,*

(ii)   *is of pseudo Wick type if G_λ is compact and the same ordering is used in the construction of the star product and the definition of the complex structure,*

(iii) *is of standard ordered type with poles $P_\lambda \subseteq i\mathbb{R}$ if $i\mathfrak{h} \subseteq \hat{\mathfrak{g}}$ is compact.*

In particular, if G is compact and, in the construction of $*_\hbar$, one chooses the ordering that makes $\mathcal{O}_\lambda$ with the induced complex structure I a Kähler manifold, then $*_\hbar$ is of Wick type.

Proof. Roots take purely imaginary values on a compact Lie subalgebra of $\mathfrak{h}$. Since $\lambda \in \mathfrak{g}^*$ is by definition real on $\mathfrak{h} \subseteq \hat{\mathfrak{h}}$, it follows that $(\lambda, \mu) \in i\mathbb{R}$ if $\mathfrak{h}$ is compact and $(\lambda, \mu) \in \mathbb{R}$ if $i\mathfrak{h}$ is compact. Since $\frac{1}{2}(\mu, \mu) - (\rho, \mu) \in \mathbb{R}$, this implies that the roots (with respect to $\hbar$) of $p_{i\lambda/\hbar}(\mu) = \frac{1}{2}(\mu, \mu) - (\rho, \mu) - \frac{i}{\hbar}(\lambda, \mu)$ are real if $\mathfrak{h}$ is compact and purely imaginary if $i\mathfrak{h}$ is compact.

Recall the definition of the distributions L_+ and L_-, which we denote by $\hat{L}_+$ and $\hat{L}_-$ in this section, made just after Lemma 3.33. Restricting them to $\mathcal{O}_\lambda \subseteq \hat{\mathcal{O}}_\lambda$ gives two distributions $L_+, L_- \subseteq \mathrm{T}^\mathbb{C}\mathcal{O}_\lambda$ of the complexified tangent bundle. An analogue of Proposition 2.8 in the real case and the explicit formula for $F_\hbar$ from Theorem 3.18 together with Remark 3.31 show that $\star$ derives the first argument only in directions of L_+ and the second argument only in directions of L_-.

Assume that $\mathfrak{g}_\lambda$ is compact. The holomorphic tangent space $\mathrm{T}_\lambda^{(1,0)}\mathcal{O}_\lambda$ is, under the isomorphism $\mathrm{T}_\lambda^\mathbb{C}\mathcal{O}_\lambda \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda$, spanned by $X_\alpha - iI_\lambda X_\alpha$ for $\alpha \in \hat{\Delta}$. If I_λ is defined using the ordering chosen in the construction of $*_\hbar$ as described above, then $X_\alpha - iI_\lambda X_\alpha = X_\alpha - i \cdot iX_\alpha = 2X_\alpha$ if $\alpha \in \hat{\Delta}^+$, and $X_\alpha - iI_\lambda X_\alpha = X_\alpha - i \cdot (-i)X_\alpha = 0$ if $\alpha \in \hat{\Delta}^-$, so $\mathrm{T}_\lambda^{(1,0)}\mathcal{O}_\lambda = \mathrm{span}\{(X_\alpha)_{\mathcal{O}_\lambda}|_\lambda, \alpha \in \hat{\Delta}^+\}$. This coincides exactly with $L_+|_\lambda$, and by G-invariance it follows that L_+ coincides with $\mathrm{T}^{(1,0)}\mathcal{O}_\lambda$. Similarly, L_- coincides with $\mathrm{T}^{(0,1)}\mathcal{O}_\lambda$. Therefore $\star$ is of pseudo Wick type.

If $i\mathfrak{h}$ is compact, then every ad_H for $H \in \mathfrak{h}$ is self-adjoint. Since they are all commuting we can find simultaneous eigenvectors in $\mathfrak{g}$ of all ad_H (without complexifying $\mathfrak{g}$). But then we can pick X_α and Y_α to lie in $\mathfrak{g}$ so that $L_1 = L_+ \cap \mathfrak{g}$ and $L_2 = L_- \cap \mathfrak{g}$ are Lagrangian distributions satisfying Definition 5.19.   ∎

Remark 5.22. Assume that $\mathfrak{g}_\lambda$ is compact as in part (ii) of the previous proposition. If one uses different invariant orderings in the construction of the star product and in the definition of a complex structure, then the distributions L_+ and L_- may both contain holomorphic and antiholomorphic directions. Since we are mainly interested in star products of (pseudo) Wick type (these are the ones for which we would hope to find positive linear functionals on the star product algebra; see Section 5.5), we will usually assume that the two orderings agree.

5.4. Examples: complex projective spaces and hyperbolic discs

Recall that we have computed the canonical element of the Shapovalov pairing for $SL_{1+n}(\mathbb{C})$ and a certain choice of λ in Section 3.4. Let us now specialize this result to the real forms $SU(1 + n)$ and $SU(1, n)$.

Example 5.23 ($\mathbb{CP}^n$). The coadjoint orbit of $SU(1 + n)$ through the point $\lambda \colon \mathfrak{su}_{1+n} \to \mathbb{R}$, $X \mapsto -irX_{0,0}$ with $r \in \mathbb{R}^+$ is the complex projective space $\mathbb{CP}^n$. $SL_{1+n}(\mathbb{C})$ is a complexification of $SU(1 + n)$. Using the notation $\hat{\mathfrak{h}}$ for the Cartan subalgebra of $\mathfrak{sl}_{1+n}(\mathbb{C})$ introduced in Section 3.4, we obtain a compact Cartan subalgebra $\mathfrak{h} := \mathfrak{su}_{1+n} \cap \hat{\mathfrak{h}}$ of $\mathfrak{su}_{1+n}$. Proposition A.10 tells us that the Kähler complex structure is defined by the ordering of $\hat{\Delta}$ for which $\alpha \in \hat{\Delta}^+$ iff $(i\lambda, \alpha) > 0$. This ordering is the restriction of the ordering on Δ for which all $\alpha_{i,j}$ with $i < j$ are positive. Therefore the element $F_\hbar$ from Proposition 3.36 induces a Wick type star product on $\mathbb{CP}^n$. This product has poles at $\{\frac{1}{n}r \mid n \in \mathbb{N}\}$.

Example 5.24 ($\mathbb{D}^n$). Denote the complex hyperbolic disc in n dimensions by $\mathbb{D}^n$. Recall that $SU(1, n)$ denotes the group of isometries of the indefinite scalar product $g(v, w) = -v_0 w_0 + \sum_{i=1}^{n} v_i w_i$ on $\mathbb{R}^{1+n}$. The coadjoint orbit of $SU(1, n)$ through $\lambda \colon \mathfrak{su}_{1,n} \to \mathbb{R}$, $X \mapsto -irX_{0,0}$ with $r \in \mathbb{R}^+$ is the hyperbolic disc $\mathbb{D}^n$. $SL_{1+n}(\mathbb{C})$ is a complexification of $SU(1, n)$. Again, $\mathfrak{h} := \mathfrak{su}_{1,n} \cap \hat{\mathfrak{h}}$ defines a compact Cartan subalgebra of $\mathfrak{su}_{1,n}$. Now all roots are non-compact so that according to Corollary A.11 the Kähler complex structure is defined by the ordering on $\hat{\Delta}$ for which $\alpha \in \hat{\Delta}^+$ iff $(i\lambda, \alpha) < 0$. This ordering is the restriction of the ordering on Δ for which all $\alpha_{i,j}$ with $i > j$ are positive. Therefore the element $F_\hbar$ from Corollary 3.37 induces a Wick type star product on $\mathbb{D}^n$. This product has poles at $\{-\frac{1}{n}r \mid n \in \mathbb{N}\}$.

Remark 5.25. A star product of Wick type on the hyperbolic disc was also studied in [28], where it was obtained from a star product of Wick type on $\mathbb{C}^{1+n}$ using phase space reduction. This product coincides with the star product obtained in Example 5.24. To see this, one checks that monomials of degree 1 generate the star product algebra so that it suffices to compare the two formulas for a degree 1 monomial and an arbitrary monomial. But for a degree 1 monomial only very few summands are non-zero in both constructions and one can explicitly check that the expressions agree.

5.5. Positive linear functionals

In this subsection, we prove that for certain coadjoint orbits and certain values of $\hbar$ the point evaluation functionals of the star product algebras constructed in Section 5.3 are positive. In order to have a meaningful notion of positivity we need a star involution on $(\mathcal{A}(\mathcal{O}_\lambda), *_\hbar)$. Of course, this star involution should be the restriction of the complex conjugation of $\mathscr{C}^\infty(\mathcal{O}_\lambda)$, but we need to prove that this restriction is well defined.

Assume that $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of a Lie algebra $\mathfrak{g}$. The complex conjugation $\overline{\cdot} \colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $X \otimes z \mapsto X \otimes \overline{z}$ is an antilinear involution on $\hat{\mathfrak{g}}$. Then $\overline{\cdot} \colon \hat{\mathfrak{g}}^* \to \hat{\mathfrak{g}}^*$, $\phi \mapsto \overline{\phi} := \overline{\cdot} \circ \phi \circ \overline{\cdot}$ defines an antilinear involution on $\hat{\mathfrak{g}}^*$. Note that on the right-hand

side, we first apply the involution of $\hat{\mathfrak{g}}$, then ϕ, and then the complex conjugation of $\mathbb{C}$. Therefore the right-hand side defines a complex linear functional $\overline{\phi} \in \hat{\mathfrak{g}}^*$. The map $\phi \mapsto \overline{\phi}$ is antilinear.

Lemma 5.26. *Let $G \subseteq \mathrm{GL}_N(\mathbb{R})$ be a real linear Lie group with complexification $\hat{G} \subseteq \mathrm{GL}_N(\mathbb{C})$, assume that $\lambda \in \mathfrak{g}^*$, and let $\hat{\mathcal{O}}_\lambda$ be the coadjoint orbit of $\hat{G}$ through λ. Then the map $\overline{\cdot}: \hat{\mathfrak{g}}^* \to \hat{\mathfrak{g}}^*$ restricts to an antilinear involution $\overline{\cdot}: \hat{\mathcal{O}}_\lambda \to \hat{\mathcal{O}}_\lambda$.*

Proof. Note that since $\lambda \in \mathfrak{g}^*$ we have $\overline{\lambda} = \lambda$. Therefore we compute

$$\overline{\mathrm{Ad}_g^* \lambda} = \overline{\lambda \circ \mathrm{Ad}_{g^{-1}}} = \overline{\lambda} \circ \overline{\cdot} \circ \mathrm{Ad}_{g^{-1}} \circ \overline{\cdot} = \lambda \circ \mathrm{Ad}_{\overline{g}^{-1}} = \mathrm{Ad}_{\overline{g}}^* \lambda.$$

Here $\overline{g}$ denotes the entrywise complex conjugate of $g \in \hat{G}$. Since the exponential map $\hat{\mathfrak{g}} \to \hat{G}$ commutes with the complex conjugation, it follows that $\hat{G}$ is closed under entrywise complex conjugation, and therefore $\overline{g} \in \hat{G}$ and $\mathrm{Ad}_{\overline{g}}^* \lambda \in \hat{\mathcal{O}}_\lambda$. This proves that $\overline{\cdot}$ restricts to $\hat{\mathcal{O}}_\lambda$, and the restriction is clearly still an antilinear involution. ∎

Note that $\mathrm{T}_\xi \overline{\cdot} \circ I_\xi = (I_{\overline{\xi}})^{-1} \circ \mathrm{T}_\xi \overline{\cdot}$ holds for $\xi \in \hat{\mathfrak{g}}^*$, where $\mathrm{T}_\xi \overline{\cdot}: \mathrm{T}_\xi \hat{\mathfrak{g}}^* \to \mathrm{T}_{\overline{\xi}} \hat{\mathfrak{g}}^*$ is the tangent map to the complex conjugation of $\hat{\mathfrak{g}}^*$ and $I_\xi: \mathrm{T}_\xi \hat{\mathfrak{g}}^* \to \mathrm{T}_\xi \hat{\mathfrak{g}}^*$ is the complex structure at ξ. Since the complex structure I and the complex conjugation $\overline{\cdot}$ of $\mathcal{O}_\lambda$ are both obtained by restriction from $\hat{\mathfrak{g}}^*$, they satisfy the same relation.

For any $f \in \mathrm{Hol}(\hat{\mathcal{O}}_\lambda)$ consider the function $f^* := \overline{\cdot} \circ f \circ \overline{\cdot}$, where the left $\overline{\cdot}$ is the complex conjugation of $\mathbb{C}$ and the right $\overline{\cdot}$ is the antilinear involution obtained in the previous lemma. Denote the complex structure of $\mathbb{C}$ by J, and identify the tangent space of $\mathbb{C}$ with $\mathbb{C}$. Then

$$\mathrm{T}_\xi f^* \circ I_\xi = \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} \circ I_\xi = \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ I_{\overline{\xi}}^{-1} \circ \mathrm{T}_\xi \overline{\cdot}$$
$$= \overline{\cdot} \circ J^{-1} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} = J \circ \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} = J \circ \mathrm{T}_\xi f^*$$

shows that f^* is holomorphic. Since $\overline{\cdot}$ restricts to the identity on $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$, it follows that $f^*|_{\mathcal{O}_\lambda} = \overline{f|_{\mathcal{O}_\lambda}}$. Consequently, the restriction of $^*: \mathrm{Hol}(\hat{\mathcal{O}}_\lambda) \to \mathrm{Hol}(\hat{\mathcal{O}}_\lambda)$ to $\mathcal{A}(\mathcal{O}_\lambda)$ is just the complex conjugation $\overline{\cdot}: \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$. In other words, the complex conjugation is well defined on $\mathcal{A}(\mathcal{O}_\lambda)$.

Proposition 5.27. *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a connected semisimple real Lie group G. Assume that the Cartan subalgebra $\mathfrak{h}$ used in the construction of a star product $*_\hbar$ is compact. Then $\overline{f *_\hbar g} = \overline{g} *_{\overline{\hbar}} \overline{f}$ holds for all $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$.*

Proof. As in the proof of Proposition 5.21 one argues that since $\mathfrak{h}$ is compact the coefficients $p_{i\lambda}^w(\alpha_w)$ are real and more generally $\overline{p_{i\lambda/\hbar}^w(\alpha_w)} = p_{i\lambda/\overline{\hbar}}^w(\alpha_w)$. From (A.3) we obtain that $\overline{X_\alpha \otimes Y_\alpha} = Y_\alpha \otimes X_\alpha = \tau(X_\alpha \otimes Y_\alpha)$ for both a compact and a non-compact root $\alpha \in \hat{\Delta}^+$, and the same formula holds when α is replaced by a word $w \in \widetilde{W}$. Here $\overline{\cdot}$ is the complex conjugation of $\hat{\mathfrak{g}}$ with respect to $\mathfrak{g}$, extended to $(\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2}$, and $\tau: (\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2} \to (\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2}$ is the flip of the two tensor factors. Note that τ stays well defined on $(\mathscr{U}\hat{\mathfrak{g}}/\mathscr{U}\hat{\mathfrak{g}} \cdot \hat{\mathfrak{g}}_\lambda)^{\otimes 2}$, and

therefore the formula for $F_\hbar$ obtained in Theorem 3.18, Remark 3.31, and the computations above imply $\overline{F_{\hbar,\ell}} = \tau(F_{\bar\hbar,\ell})$. Consequently,

$$\overline{f *_\hbar g} = \sum_{\ell=0}^{\infty} \overline{\Psi(F_{\hbar,\ell})(f,g)} = \sum_{\ell=0}^{\infty} \Psi(\overline{F_{\hbar,\ell}})(\bar f, \bar g)$$
$$= \sum_{\ell=0}^{\infty} \Psi(\tau(F_{\bar\hbar,\ell}))(\bar f, \bar g) = \sum_{\ell=0}^{\infty} \Psi(F_{\bar\hbar,\ell})(\bar g, \bar f) = \bar g *_{\bar\hbar} \bar f$$

holds for all $f, g \in \mathrm{Pol}(\mathcal{O}_\lambda)$ and extends to $\mathcal{A}(\mathcal{O}_\lambda)$ by continuity. ∎

A linear functional ϕ on a *-algebra $\mathscr{A}$ is said to be positive if $\phi(a^*a) \geq 0$ for all $a \in \mathscr{A}$. In the following, we formulate our results for the star algebra $\mathscr{A}_\hbar := (\mathcal{A}(\mathcal{O}_\lambda), *_\hbar, \overline{\;\cdot\;})$ but would like to point out that they also hold for $(\mathrm{Pol}(\mathcal{O}_\lambda), *_\hbar, \overline{\;\cdot\;})$.

Theorem 5.28. *Assume that $\mathcal{O}_\lambda$ is a semisimple coadjoint orbit of a real connected semisimple Lie group G. Assume further that $\mathfrak{h}$ is a compact Cartan subalgebra and that all roots (with respect to the complexification $\hat{\mathfrak{h}}$ of $\mathfrak{h}$) in $\hat\Delta$ are non-compact. Let $*_\hbar$ be the star product constructed with respect to the ordering for which $\alpha \in \hat\Delta$ is positive if and only if $(\alpha, i\lambda) < 0$. Then there is a constant $M > 0$ such that for all $\xi \in \mathcal{O}_\lambda$ and $\hbar \in (0, M) \setminus P_\lambda$ the point evaluation at ξ is a positive linear functional $\mathrm{ev}_\xi \colon \mathscr{A}_\hbar \to \mathbb{C}$.*

Proof. Since $(\alpha, i\lambda) < 0$ for all $\alpha \in \hat\Delta^+$, it follows that $-i(\lambda, \mu) > 0$ holds for all $\mu \in \mathbb{N}_0 \hat\Delta^+ \setminus \{0\}$. There are only finitely many $\mu \in \mathbb{N}_0 \hat\Delta^+$ with $(\rho, \mu) - \frac{1}{2}(\mu, \mu) > 0$; thus we can choose $M > 0$ such that $-\frac{i}{\hbar}(\lambda, \mu) > (\rho, \mu) - \frac{1}{2}(\mu, \mu)$ holds for all $\mu \in \mathbb{N}_0 \hat\Delta^+ \setminus \{0\}$ and $\hbar \in (0, M) \setminus P_\lambda$. But this says precisely that $p_{i\lambda/\hbar}(\mu) > 0$, and therefore $p_{i\lambda/\hbar}^w(\alpha_w) > 0$ for all $w \in \tilde W$. For a non-compact root we have $\overline{X_\alpha} = Y_\alpha$ according to (A.3b). Consequently, if $g \in G$ is such that $\xi = \mathrm{Ad}_g^*(\lambda)$, then

$$\mathrm{ev}_\xi(f *_\hbar \bar f) = \sum_{\ell=0}^{\infty} \Psi\left(\sum_{w \in \tilde W_\ell} p_{i\lambda/\hbar}^w(\alpha_w)^{-1} \tilde\pi^+(X_w) \otimes \tilde\pi^-(Y_w) \right)(f, \bar f)(\xi)$$
$$= \sum_{\ell=0}^{\infty} \sum_{w \in \tilde W_\ell} p_{i\lambda/\hbar}^w(\alpha_w)^{-1} X_w^{\mathrm{left}}(\pi^* f)(g) \cdot Y_w^{\mathrm{left}}(\pi^* \bar f)(g)$$
$$= \sum_{\ell=0}^{\infty} \sum_{w \in \tilde W_\ell} p_{i\lambda/\hbar}^w(\alpha_w)^{-1} X_w^{\mathrm{left}}(\pi^* f)(g) \cdot \overline{X_w^{\mathrm{left}}(\pi^* f)(g)}$$
$$\geq 0$$

holds for all $f \in \mathcal{A}(\mathcal{O}_\lambda)$. ∎

Example 5.29 ($\mathbb{D}^n$). It is straightforward to check that the choices made to quantize the hyperbolic disc in Example 5.24 are such that $\mathfrak{h}$ is compact such that every root in $\hat\Delta$ is

non-compact, and such that $\alpha \in \widehat{\Delta}$ is positive iff $(\alpha, i\lambda) < 0$. Therefore the previous theorem implies the existence of a constant $M > 0$ such that all point evaluation functionals are positive if $\hbar \in (0, M)$.

We can prove a stronger result by using the formula for $F_\hbar$ derived in Corollary 3.37. If $\hbar \in (0, \infty)$, then all the coefficients appearing in this formula are positive, and so point evaluations are positive for all $\hbar \in (0, \infty)$.

Note that a similar proof does not work for $\mathbb{CP}^n$ since some of the coefficients in (3.36) are negative. Indeed, one can use the appearing negative coefficients to show that no point evaluation functional is positive on $\mathbb{CP}^n$ for $\hbar \in (0, \infty) \setminus P_\lambda$.

5.6. A generalized Wick rotation

In this subsection, we want to state an immediate corollary of the construction in the previous sections. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two real semisimple Lie algebras with the same complexification $\hat{\mathfrak{g}}$. Assume that $\lambda \in \mathfrak{g}_1^* \cap \mathfrak{g}_2^*$, where we view $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ as subspaces of $\hat{\mathfrak{g}}^*$. Denote the coadjoint orbits in $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ through λ by $\mathcal{O}_\lambda^1$ and $\mathcal{O}_\lambda^2$, respectively. There is an isomorphism $\mathrm{Pol}(\mathcal{O}_\lambda^1) \to \mathrm{Pol}(\mathcal{O}_\lambda^2)$ given by composing the map $\mathrm{Pol}(\mathcal{O}_\lambda^1) \ni p \mapsto \hat{p} \in \mathrm{Pol}(\widehat{\mathcal{O}}_\lambda)$ with the restriction to $\mathcal{O}_\lambda^2$. Here $\widehat{\mathcal{O}}_\lambda$ is the complex extension of $\mathcal{O}_\lambda$. It turns out that this isomorphism is still an isomorphism of both the uncompleted and completed quantum algebras.

Theorem 5.30. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two real semisimple Lie algebras with a common complexification $\hat{\mathfrak{g}}$ and assume that $\lambda \in \mathfrak{g}_1^* \cap \mathfrak{g}_2^*$ is semisimple. Then the algebras $(\mathrm{Pol}(\mathcal{O}_\lambda^1), *_\hbar^1)$ and $(\mathrm{Pol}(\mathcal{O}_\lambda^2), *_\hbar^2)$, and also the algebras $(\mathcal{A}(\mathcal{O}_\lambda^1), *_\hbar^1)$ and $(\mathcal{A}(\mathcal{O}_\lambda^2), *_\hbar^2)$, constructed with respect to the same Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_1 \cap \mathfrak{g}_2$ and the same ordering, are isomorphic.*

Proof. Both algebras are isomorphic to $(\mathrm{Pol}(\widehat{\mathcal{O}}_\lambda), \hat{*}_\hbar)$ or $(\mathrm{Hol}(\widehat{\mathcal{O}}_\lambda), \hat{*}_\hbar)$. ∎

Example 5.31 ($\mathbb{CP}^n$ and $\mathbb{D}^n$). We know from Examples 5.23 and 5.24 that $\mathbb{CP}^n$ and $\mathbb{D}^n$ are coadjoint orbits of the Lie groups $\mathrm{SU}(1 + n)$ and $\mathrm{SU}(1, n)$ through the same element, and that $\mathrm{SL}_{1+n}(\mathbb{C})$ is a common complexification. So the previous proposition implies that the star product algebras on $\mathbb{CP}^n$ and $\mathbb{D}^n$ are isomorphic if we choose the same ordering in the construction of the star products.

The ordering that induces a Kähler complex structure on $\mathbb{CP}^n$ induces the complex structure on $\mathbb{D}^n$ that is the opposite of the Kähler complex structure. Therefore the associated star product on $\mathbb{D}^n$ is of pseudo Wick type with respect to this opposite complex structure and, therefore, of anti-Wick type for the Kähler complex structure. (A star product is of anti-Wick type if the first argument is derived in antiholomorphic directions and the second argument is derived in holomorphic ones.) Consequently, the algebra $\mathcal{A}(\mathbb{CP}^n)$ with the Wick type star product is isomorphic to the algebra $\mathcal{A}(\mathbb{D}^n)$ with the anti-Wick type star product is product. Similarly, the algebra $\mathcal{A}(\mathbb{CP}^n)$ with the anti-Wick type star product is isomorphic to the algebra $\mathcal{A}(\mathbb{D}^n)$ with the Wick type star product.

One can also construct an isomorphism between the Wick type star product for $\hbar$ and the anti-Wick type star product for $-\hbar$, both on the hyperbolic disc and the complex projective space. Composing with these isomorphisms shows that the Wick type star product for $\hbar$ on $\mathbb{CP}^n$ is isomorphic to the Wick type star product for $-\hbar$ on $\mathbb{D}^n$.

Note that Theorem 5.30 only gives an algebra homomorphism between $\mathrm{Pol}(\mathcal{O}_\lambda^1)$ and $\mathrm{Pol}(\mathcal{O}_\lambda^2)$, or between $\mathcal{A}(\mathcal{O}_\lambda^1)$ and $\mathcal{A}(\mathcal{O}_\lambda^2)$. If we view these algebras as *-algebras with the star involution considered in Section 5.5, then they are in general not *-isomorphic! One can see this for example by proving that the point evaluation functionals on $\mathbb{CP}^n$ are not positive for $\hbar \in (0, \infty) \setminus P_\lambda$.

A. Proofs, G-finite functions, complex structures

In Appendix A.1, we prove Propositions 2.5 and 2.7. In Appendix A.2, we prove Proposition 3.27 using the concept of G-finite functions. Finally, we recall some facts about complex structures on coadjoint orbits in Appendix A.3.

A.1. Proofs of Propositions 2.5 and 2.7

Let M be a manifold. For $f \in \mathscr{C}^\infty(M)$ we define $M_f : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, $f' \mapsto f f'$, and $M_f^i = \mathrm{id}^{\times(i-1)} \times M_f \times \mathrm{id}^{\times(k-i)} : \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)^k$.

Definition A.1. Let M be a manifold. For a multi-index $K \in \mathbb{Z}^k$ with $K = (K_1, \ldots, K_k)$ we define $k\text{-}\mathrm{DiffOp}_K(M) = \{0\}$ if some $K_i < 0$ and otherwise we define inductively

$$k\text{-}\mathrm{DiffOp}_K(M) = \big\{ D : \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M) \mid M_f \circ D - D \circ M_f^i \in k\text{-}\mathrm{DiffOp}_{K-E_i}(M)$$
$$\text{for all } f \in \mathscr{C}^\infty(M) \text{ and } 1 \leq i \leq k \big\}. \tag{A.1}$$

Here $(K - E_i)_j = K_j - \delta_{ij}$, where δ_{ij} is 1 if $i = j$ and 0 otherwise. Elements of $k\text{-}\mathrm{DiffOp}_K(M)$ are called k-*differential operators of degree* K. A map $D : \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)$ is said to be a k-*differential operator* if it is a k-differential operator of some degree K. The space of k-differential operators is denoted by $k\text{-}\mathrm{DiffOp}(M)$.

It follows that a k-differential operator is local in every argument so that it can be restricted to any open subset. In a chart $U \subseteq M$ with local coordinates $(x^1, \ldots, x^n)$, a k-differential operator D of degree K can be written as

$$D(f_1, \ldots, f_k) = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} c_{I_1, \ldots, I_k} \partial_x^{I_1} f_1 \cdot \ldots \cdot \partial_x^{I_k} f_k, \tag{A.2}$$

where $c_{I_1, \ldots, I_k} \in \mathscr{C}^\infty(M)$ and $c_{I_1, \ldots, I_k} = 0$ if $|I_i| > K_i$ for some $1 \leq i \leq k$. For a multi-index $J \in \mathbb{N}_0^n$ we used $\partial_x^J := \partial_{x^1}^{J_1} \cdots \partial_{x^n}^{J_n}$ and $\partial_{x^i} := \frac{\partial}{\partial x^i}$. Conversely, an operator

$$D : \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)$$

that has this form in any chart is k-differential of order K. A k-differential operator D on a complex manifold M is holomorphic if, in local holomorphic coordinates $(z^1, \ldots, z^n)$, we have

$$D(f_1, \ldots, f_k) = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} c_{I_1, \ldots, I_k} \partial_z^{I_1} f_1 \cdot \cdots \cdot \partial_z^{I_k} f_k$$

with all $c_{I_1, \ldots, I_k}$ being holomorphic. Here $\partial_z^J = \partial_{z^1}^{J_1} \cdots \partial_{z^n}^{J_n}$ and $\partial_{z^i} = \frac{\partial}{\partial z^i}$. Equivalently, D is holomorphic if D maps $\mathrm{Hol}(U)^k$ into $\mathrm{Hol}(U)$ and $D|_U \circ M_f^i - M_f \circ D|_U = 0$ for all open subsets $U \subseteq M$ and all antiholomorphic functions f on U. We write $k\text{-DiffOp}_{\mathcal{H}}(M)$ for the space of holomorphic k-differential operators.

We say a k-differential operator is of order $K \in \mathbb{Z}^k$ at a point $p \in M$ if, when written in a local chart U around p as in (A.2), we have $c_{I_1, \ldots, I_k}(p) = 0$ whenever $|I_j| > K_j$ for some $1 \le j \le k$.

If $I_1, \ldots, I_k, J, K \in \mathbb{N}_0^n$ are all multi-indices, we write $J \le K$ if $J_i \le K_i$ for all $1 \le i \le n$. If $X_1, \ldots, X_n \in \mathfrak{g}$, then we use X^J as a shorthand for $X_1^{J_1} \cdots X_n^{J_n} \in \mathcal{U}\mathfrak{g}$ and $X^{I_1 \otimes \cdots \otimes I_k}$ as a shorthand for $X^{I_1} \otimes \cdots \otimes X^{I_k} \in (\mathcal{U}\mathfrak{g})^{\otimes k}$.

Proof of Proposition 2.5. Choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. It follows from the Poincaré–Birkhoff–Witt theorem that $\{X^{I_1 \otimes \cdots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n\}$ is a basis of $(\mathcal{U}\mathfrak{g})^{\otimes k}$. Moreover, $\{X_1^{\mathrm{left},(1,0)}|_e, \ldots, X_n^{\mathrm{left},(1,0)}|_e\}$ is a basis of the tangent space $\mathrm{T}_e^{(1,0)} G$ and we can choose a complex chart U around e with local coordinates $(z^1, \ldots, z^n)$ such that $\partial_{z^i}|_e = X_i^{\mathrm{left},(1,0)}|_e$.

Assume that $\vec{u} = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} c_{I_1, \ldots, I_k} X^{I_1 \otimes \cdots \otimes I_k} \ne 0$ with only finitely many $c_{I_1, \ldots, I_k} \ne 0$. Choose $I_1, \ldots, I_k$ in such a way that $c_{I_1, \ldots, I_k} \ne 0$ and $c_{J_1, \ldots, J_k} = 0$ whenever $I_i \le J_i$ and $(I_1, \ldots, I_k) \ne (J_1, \ldots, J_k)$. For $\vec{f} = (z^{I_1}, \ldots, z^{I_k}) \in \mathscr{C}^\infty(U)^{\times k}$ we compute $\vec{u}^{\mathrm{left},(1,0)} \vec{f}(e) = I_1! \cdots I_k! c_{I_1, \ldots, I_k} \ne 0$. So $\vec{u}^{\mathrm{left},(1,0)} \ne 0$ and $(\,\cdot\,)^{\mathrm{left},(1,0)}$ is injective.

Note that $(X^{I_1})^{\mathrm{left},(1,0)} f_1 \cdot \cdots \cdot (X^{I_k})^{\mathrm{left},(1,0)} f_k = \partial_z^{I_1} f_1 \cdot \cdots \cdot \partial_z^{I_k} f_k + D'(f_1, \ldots, f_k)$, where D' is a holomorphic k-differential operator whose order at e is strictly smaller than $(|I_1|, \ldots, |I_k|)$. For any holomorphic k-differential operator D we can therefore, by induction, find coefficients $c_{I_1, \ldots, I_k} \in \mathbb{C}$, only finitely many of which are non-zero such that

$$D(f_1, \ldots, f_k)(e) = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} c_{I_1, \ldots, I_k} (X^{I_1})^{\mathrm{left},(1,0)} f_1(e) \cdot \cdots \cdot (X^{I_k})^{\mathrm{left},(1,0)} f_k(e)$$

holds for all $f_1, \ldots, f_k \in \mathscr{C}^\infty(G)$. In other words, D and the differential operator $\sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} (c_{I_1, \ldots, I_k} X^{I_1 \otimes \cdots \otimes I_k})^{\mathrm{left},(1,0)}$ agree at e. So if D is also left-invariant, then these operators agree everywhere on G, proving surjectivity. ∎

The proof of Proposition 2.7 is similar. We need the following lemma to simplify the local calculations.

Lemma A.2. *Let G be a complex Lie group with Lie algebra $\mathfrak{g}$, and assume that H is a closed complex Lie subgroup of G with Lie algebra $\mathfrak{h}$. Given a basis $B = \{X_1, \ldots, X_n\}$*

of $\mathfrak{g}$ such that $B' = \{X_{n-r+1}, \ldots, X_n\}$ is a basis of $\mathfrak{h}$ one can choose a neighborhood U of e in G and complex coordinates $z = (z^1, \ldots, z^n)$ on U such that

(i)   for any $g \in U$ its fiber $gH \cap U$ is given locally as $(\{z(g)\} + \{0\} \times \mathbb{C}^r) \cap z(U)$,

(ii) the left-invariant holomorphic vector fields agree with coordinate vector fields at $e \in G$; that is $X_i^{\mathrm{left},(1,0)}|_e = \partial_{z^i}|_e$.

Proof. It is well known that $\pi \colon G \to G/H$ is a principal bundle. Therefore we can choose a local trivialization $\chi \colon \pi^{-1}(V) \to V \times H$ on a small neighborhood V of eH in G/H. Choosing coordinates on V (after possibly shrinking V first) and on a neighborhood W of the identity in H, we obtain coordinates z' on $U := \chi^{-1}(V \times W) \subseteq G$ satisfying property (i). Since all $X_i^{\mathrm{left},(1,0)}$ are linearly independent, we can write $X_i^{\mathrm{left},(1,0)}|_e = A_{ij} \partial_{(z')^j}|_e$ for some invertible matrix A and since $X_i^{\mathrm{left},(1,0)}$ is tangential to $H \subseteq G$ for $i > n - r$, it follows that $A_{ij} = 0$ for $i > n - r$, $j \leq n - r$. Then the coordinates $z := (A^{-1})^{\mathrm{T}} z'$ satisfy both properties of the lemma. ∎

Let $\pi \colon G \to G/H$. Given coordinates as in the previous lemma, we may identify $\pi(U)$ locally with $\{(z^1(g), \ldots, z^{n-r}(g), 0, \ldots, 0) \mid g \in U\}$. Then $(z^1, \ldots, z^{n-r})$ descend to coordinates on $\pi(U)$ and π is, with respect to these coordinates, given by the projection to the first $n - r$ coordinates.

Lemma A.3. *The map Ψ from Proposition 2.7 is injective.*

Proof. Let $r = \dim \mathfrak{h}$ and $n = \dim \mathfrak{g} \geq r$. We can choose a basis $B = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $B' = \{X_{n-r+1}, \ldots, X_n\}$ is a basis of $\mathfrak{h}$. Recall from the proof of Proposition 2.5 that $\{X^{I_1 \otimes \cdots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n\}$ is a basis of $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Furthermore,

$$\left\{ X^{I_1 \otimes \cdots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n, (I_i)_j > 0 \text{ for some } 1 \leq i \leq k \text{ and some } j > n - r \right\}$$

is a basis of the ideal I defined just before Lemma 2.6 and

$$\left\{ X^{I_1 \otimes \cdots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n, (I_i)_j = 0 \text{ for all } 1 \leq i \leq k, j > n - r \right\}$$
$$= \left\{ X^{I_1 \otimes \cdots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^{n-r} \right\}$$

is a basis of a complement C of I in $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Injectivity of Ψ means that 0 is the only element of C on which Ψ vanishes.

So to prove that Ψ is injective, it suffices to find, for any non-zero

$$\vec{u} = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k} X^{I_1 \otimes \cdots \otimes I_k} \in C,$$

some open subset $U \subseteq G/H$ and some k-tuple of functions $\vec{f} \in \mathscr{C}^\infty(U)^k$ such that $\Psi([\vec{u}])(\vec{f}) \neq 0$. Fix $\vec{u} \in C \setminus \{0\}$ and assume that $I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}$ are chosen such that $c_{I_1, \ldots, I_k} \neq 0$ and such that for any multi-indices $J_1, \ldots, J_k \in \mathbb{N}_0^{n-r}$ satisfying $I_i \leq J_i$ and $(I_1, \ldots, I_k) \neq (J_1, \ldots, J_k)$ we have $c_{J_1, \ldots, J_k} = 0$. Choose coordinates $z = (z^1, \ldots, z^n)$ around e on G as in the previous lemma, and note that, as described just after this lemma,

$(z^1, \ldots, z^{n-r})$ descend to coordinates $(y^1, \ldots, y^{n-r})$ on G/H. Set $\vec{f} = (y^{I_1}, \ldots, y^{I_k})$ so that $\pi^* \vec{f} = (z^{I_1}, \ldots, z^{I_k})$. This implies that

$$\Psi([\vec{u}])(\vec{f})(eH) = \vec{u}^{\text{left},(1,0)}(\pi^* \vec{f})(e) = I_1! \cdots I_k! c_{I_1, \ldots, I_k} \neq 0. \qquad \blacksquare$$

Lemma A.4. *The map Ψ from Proposition* 2.7 *is surjective.*

Proof. We claim that for any holomorphic k-differential operator D on G/H we can find $\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ such that

$$\vec{u}^{\text{left},(1,0)}(\pi^* \vec{f})(e) = \pi^*(D\vec{f})(e)$$

holds for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$. We prove this claim by induction on the order $K \in \mathbb{Z}^k$ of D at eH. If $K_i < 0$ for some $1 \leq i \leq k$, then $D = 0$ and we can use $\vec{u} = 0$. For the induction step, assume that the claim is already proven for every holomorphic k-differential operator of order strictly smaller than K at eH. Choose coordinates $z = (z^1, \ldots, z^n)$ around e on G as in Lemma A.2 and denote the coordinates on G/H induced by $(z^1, \ldots, z^{n-r})$ by $y := (y^1, \ldots, y^{n-r})$. Locally, we can write

$$D(f_1, \ldots, f_k) = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k} \cdot \partial_y^{I_1} f_1 \cdot \cdots \cdot \partial_y^{I_k} f_k$$

with $c_{I_1, \ldots, I_k} \in \mathscr{C}^\infty(G/H)$ satisfying $c_{I_1, \ldots, I_k}(eH) = 0$ whenever $|I_i| > K_i$ for some $1 \leq i \leq k$. Define a holomorphic k-differential operator D_G on G by

$$D_G(f_1', \ldots, f_k') = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} (c_{I_1, \ldots, I_k} \circ \pi) \cdot \partial_z^{I_1} f_1' \cdot \cdots \cdot \partial_z^{I_k} f_k',$$

so that $D_G(\pi^* \vec{f})(e) = \pi^*(D\vec{f})(e)$. Set

$$\vec{u}_1 := \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k}\big(\pi(e)\big) X^{I_1} \otimes \cdots \otimes X^{I_k} \in (\mathscr{U}\mathfrak{g})^{\otimes k}.$$

Note that $D_G' := D_G - \vec{u}_1^{\text{left},(1,0)}$ has a strictly smaller order than D_G at e since $X_i^{\text{left},(1,0)}|_e = \partial_{z^i}|_e$. There are functions $c_{I_1, \ldots, I_k}' \in \mathscr{C}^\infty(G)$ such that we can express D_G' in local coordinates as

$$D_G'(f_1', \ldots, f_k') = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^n} c_{I_1, \ldots, I_k}' \cdot \partial_z^{I_1} f_1' \cdot \cdots \cdot \partial_z^{I_k} f_k'.$$

We obtain a k-differential operator D' on G/H of strictly smaller order than D at eH by letting

$$D'(f_1, \ldots, f_k) = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k}'(\cdot, 0) \partial_y^{I_1} f_1 \cdot \cdots \cdot \partial_y^{I_k} f_k.$$

It fulfils $D_G'(\pi^* \vec{f})(e) = \pi^*(D' \vec{f})(e)$. Using the induction hypothesis, we find $\vec{u}' \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ such that $\vec{u}'^{\text{left},(1,0)}(\pi^* \vec{f})(e) = \pi^*(D' \vec{f})(e)$. Now

$$(\vec{u}_1 + \vec{u}')^{\text{left},(1,0)}(\pi^* \vec{f})(e) = (D_G - D_G')(\pi^* \vec{f})(e) + \pi^*(D' \vec{f})(e)$$

$$= \pi^*(D\,\vec{f})(e) - \pi^*(D'\,\vec{f})(e) + \pi^*(D'\,\vec{f})(e)$$
$$= \pi^*(D\,\vec{f})(e),$$

proving the claim.

Assume that D is in addition left-invariant. Writing $L_g : G/H \to G/H$ also for the action of $g \in G$ on G/H, we compute

$$\vec{u}^{\text{left},(1,0)}(\pi^*\vec{f})(g) = L_g^*\vec{u}^{\text{left},(1,0)}(\pi^*\vec{f})(e) = \vec{u}^{\text{left},(1,0)}(L_g^{\times k})^*(\pi^*\vec{f})(e)$$
$$= \vec{u}^{\text{left},(1,0)}\pi^*((L_g^{\times k})^*\vec{f})(e) = \pi^*(D(L_g^{\times k})^*\vec{f})(e)$$
$$= \pi^*(L_g^* D\,\vec{f})(e) = L_g^*\pi^*(D\,\vec{f})(e) = \pi^*(D\,\vec{f})(g).$$

Thus $\vec{u}^{\text{left},(1,0)}(\pi^*\vec{f}) = \pi^*(D\,\vec{f})$ holds for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$. Finally, we need to show that $\vec{u}$ has the correct invariance properties under the adjoint action of H. Define $R_g : G \to G$, $R_{g'}(g) := gg'$. Since $R_h^*\pi^*(D\,\vec{f}) = \pi^*(D\,\vec{f})$ for all $h \in H$, we obtain $R_h^*\vec{u}^{\text{left},(1,0)}\pi^*\vec{f} = \vec{u}^{\text{left},(1,0)}\pi^*\vec{f}$ and therefore

$$(\text{Ad}_h\,\vec{u})^{\text{left},(1,0)}(\pi^*\vec{f})(g) = (\vec{u}^{\text{left},(1,0)}\pi^*\vec{f})(gh)$$
$$= R_h^*\vec{u}^{\text{left},(1,0)}\pi^*\vec{f}(g) = \vec{u}^{\text{left},(1,0)}\pi^*\vec{f}(g)$$

for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$ and all $g \in G$, where the first equality follows as in the proof of Lemma 2.6. This means that $(\text{Ad}_h\,\vec{u} - \vec{u})^{\text{left},(1,0)}(\pi^*\vec{f}) = 0$ for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$, and therefore the proof of injectivity implies $\text{Ad}_h\,\vec{u} - \vec{u} \in I$, or in other words $\vec{u} \in U_{\text{inv}}$.   ∎

A.2. G-finite functions

In this subsection, we introduce G-finite functions on a Lie group G and use them to prove Proposition 3.27. The definition of G-finite functions uses only abstract properties of the Lie group G and is therefore independent of whether G is explicitly realized by matrices or not. For complex semisimple connected Lie groups a function is G-finite if and only if it is a polynomial, and therefore G-finite functions give a characterization of polynomials that is independent of the representation.

Definition A.5 (G-finite functions). Let M be a manifold with an action of a Lie group G. Then $f \in \mathscr{C}^\infty(M)$ is said to be G-finite if the vector space $\text{span}\{g \triangleright f \mid g \in G\}$ is finite dimensional. We denote the space of G-finite functions on M by $\text{Fin}^G(M)$ or just by $\text{Fin}(M)$ if G is clear from the context.

Here $g \triangleright f$ denotes the smooth function on M defined by $(g \triangleright f)(m) = f(g^{-1} \triangleright m)$. Below, we use this definition only for $M = G$ and the action L or for $M = \widehat{\mathcal{O}}_\lambda$ and the coadjoint action, and will therefore not mention these actions explicitly.

Lemma A.6. *Let G be a real or complex matrix Lie group and let $\mathcal{O}_\lambda$ be a coadjoint orbit of G. Then polynomials on G are G-finite, and polynomials on $\mathcal{O}_\lambda$ are also G-finite.*

Proof. Let $P_{ij}\colon G \to \mathbb{C}$, $X \mapsto X_{ij}$, and call such polynomials elementary in this proof. We compute $(g \triangleright P_{ij})(h) = P_{ij}(g^{-1}h) = \sum_k (g^{-1})_{ik} h_{kj} = \sum_k (g^{-1})_{ik} P_{kj}(h)$ for $g \in G$, so $g \triangleright P_{ij}$ is a linear combination of some elementary polynomials. If $p = P_{i_1 j_1} \cdots P_{i_n j_n} \in$ Pol(G) is a product of n elementary polynomials, then $g \triangleright p$ is in the linear span of products of n many elementary polynomials, which is a finite dimensional space. The statement for arbitrary polynomials follows by taking linear combinations.

The action of G on Pol$(\mathcal{O}_\lambda)$ is obtained by restricting the adjoint action of G on $S\mathfrak{g} \cong$ Pol$(\mathfrak{g}^*)$. The adjoint action preserves the degree of a symmetric tensor, so span$\{\mathrm{Ad}_g X \mid g \in G\}$ is finite dimensional for any $X \in S\mathfrak{g}$, and therefore span$\{g \triangleright p \mid g \in G\}$ is finite dimensional for any $p \in \mathrm{Pol}(\mathcal{O}_\lambda)$. ∎

Proposition A.7. *Let G be a complex semisimple connected Lie group with coadjoint orbit $\mathcal{O}_\lambda$. Then G-finite holomorphic functions on $\mathcal{O}_\lambda$ are polynomials.*

Proof. Hol$(\mathcal{O}_\lambda)$ is isomorphic to Hol$(G)^{G_\lambda}$ as a G-module. The restriction to a maximal compact Lie subgroup $K \subseteq G$ is an injective K-module homomorphism to $\mathrm{L}^2(K)$, the square-integrable functions on K with respect to the left-invariant Haar measure so that we may view Hol$(\mathcal{O}_\lambda)$ as a K-submodule of $\mathrm{L}^2(K)$. In particular, it is completely reducible as a K-module and therefore also as a G-module. Each irreducible module of highest weight ν appears only finitely many times in $\mathrm{L}^2(K)$ and thus also in Hol$(\mathcal{O}_\lambda)$.

The scalar product of $\mathrm{L}^2(K)$ is K-invariant and therefore any irreducible modules of different highest weights are orthogonal. Restricting the scalar product to Hol$(\mathcal{O}_\lambda)$ gives that Hol$(\mathcal{O}_\lambda)^\nu$ is orthogonal to Hol$(\mathcal{O}_\lambda)^{\nu'}$ if $\nu \neq \nu'$.

Assume that $f \in \mathrm{Fin}(\mathcal{O}_\lambda)$ is holomorphic and not in Pol$(\mathcal{O}_\lambda)$. We can without loss of generality assume that $f \in \mathrm{Fin}(\mathcal{O}_\lambda)^\nu$ for some weight ν. (Indeed, we can write $f = \sum_\mu f^\mu$ with $f^\mu \in \mathrm{Fin}(\mathcal{O}_\lambda)^\mu$ and only finitely many f^μ are non-zero because f is G-finite. One of these f^μ is not in Pol$(\mathcal{O}_\lambda)$.) We can choose f orthogonal to Pol$(\mathcal{O}_\lambda)^\nu$ (which is finite dimensional) and therefore orthogonal to Pol$(\mathcal{O}_\lambda)$. However, this space is dense in Hol$(\mathcal{O}_\lambda)$ because polynomials on K are dense in $\mathrm{L}^2(K)$. So $f = 0$, a contradiction. ∎

Corollary A.8. *Let G be a complex semisimple connected Lie group. Then the pullback $\pi^*\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}$ is an isomorphism.*

Proof. We have seen in the proof of Proposition 3.27 that π^* is well defined and injective, so it only remains to show that π^* is surjective. Any element $f \in \mathrm{Pol}(G)^{G_\lambda}$ is G-finite by Lemma A.6. Then its image under the G-equivariant isomorphism $\pi_*\colon \mathrm{Hol}(G)^{G_\lambda} \to$ Hol$(\mathcal{O}_\lambda)$ is also G-finite because finite dimensionality of span$\{g \triangleright f \mid g \in G\}$ implies finite dimensionality of span$\{g \triangleright \pi_* f \mid g \in G\} = \mathrm{span}\{\pi_*(g \triangleright f) \mid g \in G\}$. The previous proposition implies that the G-finite element $\pi_* f \in \mathrm{Pol}(\mathcal{O}_\lambda)$ is a polynomial. It is mapped to f by π^*. ∎

With similar methods as in this subsection one can prove that G-finite functions on a complex semisimple connected Lie group G coincide with polynomials on G. Since the

definition of G-finite functions does not depend on a representation of G as a linear group, it follows that our definition of polynomials in Definition 3.26 is indeed independent of the representation. The same result is true for a compact semisimple connected Lie group K.

A.3. Complex structures on real coadjoint orbits

We have seen in Section 2.1 that a coadjoint orbit of a real Lie group G always admits a G-invariant symplectic structure; in particular its dimension is even. In this subsection, we will see that a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a connected semisimple real Lie group G admits a G-invariant complex structure if G_λ is compact and that the set of such complex structures is in bijection to invariant orderings. If G is compact, then there is a unique G-invariant complex structure that makes $\mathcal{O}_\lambda$ a Kähler manifold. If G is not compact, then $\mathcal{O}_\lambda$ might or might not admit a Kähler structure. All results of this subsection are classical and well known; see for example [7] for a summary.

Let G be a real connected semisimple Lie group. Assume that $\lambda \in \mathfrak{g}^*$ is semisimple and that G_λ is compact. Then any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ containing $\lambda^\sharp$ is contained in $\mathfrak{g}_\lambda$ and therefore compact. As usual, we denote the complexification of $\mathfrak{g}$ by $\hat{\mathfrak{g}}$ and let $\overline{\cdot}$ be the complex conjugation of $\hat{\mathfrak{g}}$ with respect to $\mathfrak{g}$.

Recall that a root $\alpha \in \mathfrak{h}^*$ is called *compact* if the Killing form B is negative definite on $\mathfrak{g} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$, and *non-compact* if it is positive definite. (The root spaces $\mathfrak{g}^\alpha$ are subspaces of the complexification $\hat{\mathfrak{g}}$ of $\mathfrak{g}$.) We can always choose $X_\alpha \in \mathfrak{g}^\alpha$ such that $B(X_\alpha, X_{-\alpha}) = 1$ and if $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, then $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ (see [7, Section 3]). In this case,

$$-X_{-\alpha} = \overline{X_\alpha} \quad \text{and} \quad \mathrm{i}(X_\alpha + X_{-\alpha}), X_\alpha - X_{-\alpha} \in \mathfrak{g} \quad \text{if } \alpha \text{ is compact,} \tag{A.3a}$$

$$X_{-\alpha} = \overline{X_\alpha} \quad \text{and} \quad \mathrm{i}(X_\alpha - X_{-\alpha}), X_\alpha + X_{-\alpha} \in \mathfrak{g} \quad \text{if } \alpha \text{ is non-compact.} \tag{A.3b}$$

Recall that $\widehat{\Delta}$ is the set of roots that are not orthogonal to λ.

Theorem A.9. *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a real connected semisimple Lie group G. Assume that G_λ is compact, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$. Then G-invariant complex structures on $\mathcal{O}_\lambda$ are in bijection with invariant orderings of $\widehat{\Delta}$ (i.e., choices of positive roots $\widehat{\Delta}^+$ that arise as $\widehat{\Delta}^+ = \widehat{\Delta} \cap \Delta^+$ from an invariant ordering of Δ as defined in Definition 3.10).*

Sketch. Introduce $\mathfrak{m} = \bigoplus_{\alpha \in \widehat{\Delta}} \mathfrak{g}^\alpha \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda$. Since taking fundamental vector fields (see Section 2.1) gives an isomorphism $\mathfrak{g}/\mathfrak{g}_\lambda \to T_\lambda \mathcal{O}_\lambda$, $\mathfrak{m}$ is isomorphic to the complexified tangent space $T_\lambda^{\mathbb{C}} \mathcal{O}_\lambda$ and $\mathfrak{g} \cap \mathfrak{m}$ is isomorphic to $T_\lambda \mathcal{O}_\lambda$.

Given an invariant ordering of $\widehat{\Delta}$ (see Definition 3.10), define $I : \mathfrak{m} \to \mathfrak{m}$ by extending $X_\alpha \mapsto \mathrm{i} X_\alpha$ if $\alpha \in \widehat{\Delta}^+$ and $X_\alpha \mapsto -\mathrm{i} X_\alpha$ if $\alpha \in \widehat{\Delta}^-$ linearly. Clearly, $I^2 = -\mathrm{id}$. For both a compact and a non-compact root α, I restricts to an endomorphism of $\mathfrak{g} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$, from which it follows that I restricts to a map $\mathfrak{g} \cap \mathfrak{m} \to \mathfrak{g} \cap \mathfrak{m}$, squaring to $-\mathrm{id}$. To prove that it extends to a G-invariant almost complex structure on $\mathcal{O}_\lambda$, it suffices to prove that I is G_λ-invariant. By applying the analogue of Proposition 2.3 for compact connected

semisimple Lie groups to a maximally compact subgroup of G containing G_λ, it follows that G_λ is connected, and it suffices to prove that I is $\mathfrak{g}_\lambda$-invariant, in the sense that $I([A, B]) = [A, I(B)]$ holds for all $A \in \mathfrak{g}_\lambda$ and $B \in \mathfrak{m}$. This identity holds for $A \in \mathfrak{h}$ since I preserves the root spaces. So we only need to check it for $A = X_\alpha$ and $B = X_\beta$ with $\alpha \in \Delta'$ and $\beta \in \widehat{\Delta}$, which is equivalent to the invariance of the ordering. Finally, one uses that $\alpha + \beta$ is positive if $\alpha, \beta \in \widehat{\Delta}$ are positive to compute that the Nijenhuis torsion of I vanishes, so I is indeed a complex structure.

Vice versa, a G-invariant complex structure I on $\mathcal{O}_\lambda$ determines a $\mathfrak{g}_\lambda$-invariant map $I \colon \mathfrak{m} \to \mathfrak{m}$ with $I^2 = -\mathrm{id}$ by restricting to the tangent space at λ and complexifying. In particular, I is $\mathfrak{h}$-invariant and, therefore, preserves the root spaces, so $X_\alpha \mapsto \mathrm{i}c_\alpha X_\alpha$ with $c_\alpha = \pm 1$. Since I preserves the real tangent space, we must have $c_\alpha = -c_{-\alpha}$. The Nijenhuis torsion of the complex structure vanishes, which implies that $\widehat{\Delta}^+ = \{\alpha \in \widehat{\Delta} \mid c_\alpha = 1\}$ defines an ordering. Finally, invariance under the whole Lie algebra $\mathfrak{g}_\lambda$ gives that this ordering is invariant. ∎

Proposition A.10. *Any coadjoint orbit $\mathcal{O}_\lambda$ of a compact connected semisimple Lie group K has a unique K-invariant complex structure I that makes $(\mathcal{O}_\lambda, I, \omega_{\mathrm{KKS}})$ a Kähler manifold, and this complex structure corresponds to an ordering for which $\alpha \in \widehat{\Delta}$ is positive if and only if $(\alpha, \mathrm{i}\lambda) > 0$.*

Note that α attains purely imaginary values on $\mathfrak{k}$, whereas λ attains real values. Therefore $(\alpha, \mathrm{i}\lambda) \in \mathbb{R}$. The ordering for which $\alpha \in \widehat{\Delta}$ is positive if $(\alpha, \mathrm{i}\lambda) > 0$ is standard (see Section 3.2).

Proof. Since K is compact, it follows that any root is compact. Given a K-invariant complex structure I, we associate the (not necessarily positive definite) metric $g(v, w) = \omega_{\mathrm{KKS}}(v, Iw)$ and $\mathcal{O}_\lambda$ is a Kähler manifold if g is positive definite. Since I and ω_{KKS} are K-invariant, so is g and we may check positive definiteness on $T_\lambda \mathcal{O}_\lambda$. Identifying $T_\lambda^{\mathbb{C}} \mathcal{O}_\lambda$ with $\mathfrak{m}$ as in the proof of the previous proposition and extending g complex linearly, we compute that $g(X_\alpha, X_\beta) = \omega_{\mathrm{KKS}}(X_\alpha, I X_\beta) = c_\beta \lambda([X_\alpha, X_\beta])$ for all $\alpha, \beta \in \widehat{\Delta}$. This expression is non-zero only if $\alpha = -\beta$, and in this case $g(X_\alpha, X_{-\alpha}) = -\mathrm{i}c_\alpha \lambda(\alpha^\sharp) = -\mathrm{i}c_\alpha \cdot (\alpha, \lambda)$. Then

$$g\big(\mathrm{i}(X_\alpha + X_{-\alpha}), \mathrm{i}(X_\alpha + X_{-\alpha})\big) = 2\mathrm{i}c_\alpha \cdot (\alpha, \lambda)$$

and

$$g(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = 2\mathrm{i}c_\alpha \cdot (\alpha, \lambda).$$

So g is positive definite if and only if $c_\alpha = 1$ for all $\alpha \in \widehat{\Delta}$ with $(\alpha, \mathrm{i}\lambda) > 0$. ∎

Note that the situation is more complicated if G is non-compact, but G_λ is compact, since we may then have both compact and non-compact roots. The condition for g being positive definite then becomes $c_\alpha = 1$ if either α is a compact root and $(\alpha, \mathrm{i}\lambda) > 0$ or if α is a non-compact root and $(\alpha, \mathrm{i}\lambda) < 0$. If these conditions define an invariant ordering, then $\mathcal{O}_\lambda$ has a G-invariant Kähler structure (which is automatically unique). One can give

more explicit criteria for when the conditions above define an invariant ordering (see [7]) but we only need the following easy case.

Corollary A.11. *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a connected semisimple Lie group G. Assume that G_λ is compact and that $\mathfrak{h}$ is a Cartan subalgebra containing $\lambda^\sharp$. If all roots in $\widehat{\Delta}$ are non-compact, then $(\mathcal{O}_\lambda, I, \omega_{\mathrm{KKS}})$ is a Kähler manifold, where I is the complex structure corresponding to the ordering for which $\alpha \in \widehat{\Delta}$ is positive if and only if $(\alpha, i\lambda) < 0$.*

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