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L^p coarse Baum–Connes conjecture and K-theory for L^p Roe algebras

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Abstract. In this paper, we verify the L^p coarse Baum–Connes conjecture for spaces with finite asymptotic dimension for $p \in [1, \infty)$. We also show that the K-theory of L^p Roe algebras is independent of $p \in (1, \infty)$ for spaces with finite asymptotic dimension.

1. Introduction

An elliptic differential operator on a closed manifold is Fredholm. The celebrated Atiyah–Singer index theorem computes the Fredholm index [1, 2]. In the recent 40 years, the Atiyah–Singer index theorem has been vastly generalized to the higher index theory [42, 49]. There are two most important cases. For a manifold carrying a proper cocompact group action, the Baum–Connes assembly map defines a higher index in the K-theory of the group C^*-algebra [22, 27]. For an open manifold without group actions, the coarse Baum–Connes assembly map defines a higher index in the K-theory of the Roe algebra of the manifold [35].

The Baum–Connes conjecture [3] and the coarse Baum–Connes conjecture [19, 44] give algorithms to compute the higher indices using K-homology. The K-homology is local and much more computable. In recent years, the L^p version of the Baum–Connes and coarse Baum–Connes conjectures is studied. The motivation for using Banach algebras is that they are more flexible than C^*-algebras. The traditional C^*-algebraic method [22] is very difficult for dealing with groups with property (T) (these groups admit no proper isometric actions on Hilbert spaces). Actually a lot of interesting groups, e.g., hyperbolic group, may have property (T). Lafforgue invented the Banach KK-theory and verified the Baum–Connes conjecture for a large class of groups with property (T) [24]. In [48], Guoliang Yu proved that hyperbolic groups always admit proper isometric actions on ℓ^p spaces. In [23], Kasparov and Yu proved that the L^p Baum–Connes conjecture is true for groups with a proper isometric action on ℓ^p space.

In [26], Benben Liao and Guoliang Yu proved that the K-theory of L^p group algebras is independent of p for a large class of groups, e.g., hyperbolic groups. Their proof relies on Lafforgue's results on the Baum–Connes conjecture [24] and L^p property (RD) for the group.

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Yeong–Chyuan Chung developed a quantitative K-theory for Banach algebras [6] and applied this theory to compute K-theory of L^p crossed products [7]. Chung showed that the L^p Baum–Connes conjecture for G with coefficient in $C(X)$ is true if the dynamical system $G \curvearrowright X$ has finite dynamical complexity, a property introduced by Guentner–Willett–Yu [17] and obtained a partial answer about p-independence of L^p crossed products.

Motivated by Liao-Yu and Chung's result, we ask the following question: Is the K-theory of L^p Roe algebras $B^p(X)$ independent of p? The main theorem of the paper provides a positive answer to this question for the spaces with finite asymptotic dimension.

Theorem 1.1 (see Theorem 5.22). *Let X be a proper metric space. If X has finite asymptotic dimension, then $K_*(B^p(X))$ does not depend on p for $p \in (1, \infty)$.*

The proof of the theorem relies on the L^p coarse Baum–Connes conjecture. The key ingredient is the Mayer–Vietoris argument. A coarse geometric Mayer–Vietoris sequence in K-theory was formulated by Higson–Roe–Yu [20]. In [46], Guoliang Yu invented the quantitative K-theory and a quantitative Mayer–Vietoris sequence, and he verified the coarse Baum–Connes conjecture for spaces with finite asymptotic dimension. The quantitative K-theory is a refined version of the classical operator K-theory. It encodes more geometric information, and it is a powerful tool to compute the K-theory of Roe algebras or other C^*-algebras coming from geometry. The quantitative K-theory has been generalized to general geometric C^*-algebras by Oyono–Oyono and Yu [29–31], to Banach algebras by Yeong–Chyuan Chung [6], and to groupoids by Clement Dell'Aiera [11]. It has many important applications in dynamical systems [7,17] and coarse geometry [8,25]. In this paper, by a similar argument of quantitative K-theory for L^p algebras, we prove the following result.

Theorem 1.2 (see Theorem 4.6). *For any $p \in [1, \infty)$, the L^p coarse Baum–Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

The result is very similar to Chung's result on the Baum–Connes conjecture with a coefficient for dynamical systems with finite dynamical complexity [7]. His result is for dynamical systems or transformation groupoids, while our result is for coarse geometry or coarse groupoids.

We want to emphasize that the results in this paper do not need the condition of bounded geometry. For the similar result for spaces with bounded geometry, we could generalize the result to spaces with finite decomposition complexity, introduced by Erik Guentner, Romain Tessera, and Guoliang Yu [15, 16]. Our method also works for uniform L^p Roe algebras. We will study the results in a separate paper.

The paper is organized in the following order. In Section 2, we recall the concept of L^p Roe algebras, L^p localization algebras, and L^p coarse Baum–Connes conjecture. In Section 3, we study the quantitative K-theory for L^p algebras. In Section 4, we prove that the L^p Baum–Connes conjecture is true for spaces with finite asymptotic dimension for $p \in [1, \infty)$. In Section 5, we prove that the K-theory of L^p Roe algebras is independent of $p \in (0, 1)$ for spaces with finite asymptotic dimensions. In the end, we raise some open problems for future study.

2. L^p coarse Baum–Connes conjecture

Let X be a proper metric space, $p \in [1, \infty)$. Recall that a metric space is called proper if every closed ball is compact. The proper metric space is a separable space the since compact metric space is separable.

Definition 2.1. An L^p-X-module is an L^p space $E_X^p = \ell^p(Z_X) \otimes \ell^p = \ell^p(Z_X, \ell^p)$ equipped with a natural pointwise multiplication action of $C_0(X)$ by restricting to Z_X, where Z_X is a countable dense subset in X, $\ell^p = \ell^p(\mathbb{N})$, and $C_0(X)$ is the algebra of all complex-valued continuous functions on X which vanish at infinity.

We notice that this action can be extended naturally to the algebra of all bounded Borel functions on X.

2.1. L^p Roe algebra

Definition 2.2. Let E_X^p be an L^p-X-module, E_Y^p an L^p-Y-module, and $T : E_X^p \to E_Y^p$ a bounded linear operator. The *support* of T, denoted by $\mathrm{supp}(T)$, consists of all points $(x, y) \in X \times Y$ such that $\chi_V T \chi_U \neq 0$ for all open neighborhoods U of x and V of y, where χ_U and χ_V are the characteristic functions of U and V, respectively.

Please note that the support defined in [42] is the inverse of ours.

We give some properties of the support; the proof can be obtained in much the same way as in [42, Chapter 4].

Remark 2.3. Let E_X^p be an L^p-X-module, E_Y^p an L^p-Y-module, and E_Z^p an L^p-Z-module. Let $R, S : E_X^p \to E_Y^p$ and $T : E_Y^p \to E_Z^p$ be bounded linear operators. Then

(1) $\mathrm{supp}(R + S) \subseteq \mathrm{supp}(R) \cup \mathrm{supp}(S)$;

(2) $\mathrm{supp}(TS) \subseteq \mathrm{cl}(\mathrm{supp}(S) \circ \mathrm{supp}(T)) = \mathrm{cl}(\{(x, z) \in X \times Z : \exists\, y \in Y \text{ s.t. } (x, y) \in \mathrm{supp}(S), (y, z) \in \mathrm{supp}(T)\})$, where "cl" means closure;

(3) if the coordinate projections $\pi_Y : \mathrm{supp}(T) \to Y$ and $\pi_Z : \mathrm{supp}(T) \to Z$ are proper maps or coordinate projections $\pi_X : \mathrm{supp}(S) \to X$ and $\pi_Y : \mathrm{supp}(S) \to Y$ are proper maps, then $\mathrm{supp}(TS) \subseteq \mathrm{supp}(S) \circ \mathrm{supp}(T)$;

(4) let $F = \mathrm{supp}(S)$; then for any compact subset K of X, respectively Y, we have $S\chi_K = \chi_{K \circ F} S \chi_K$, $\chi_K S = \chi_K S \chi_{F \circ K}$, where $K \circ F := \{y \in Y : \text{there is } x \in K \text{ such that } (x, y) \in F\}$, $F \circ K := \{x \in X : \text{there is } y \in K \text{ such that } (x, y) \in F\}$.

Definition 2.4. Let E_X^p be an L^p-X-module and T a bounded linear operator acting on E_X^p.

(1) The *propagation* of T is defined to be $\mathrm{prop}(T) = \sup\{d(x, y) : (x, y) \in \mathrm{supp}(T)\}$;

(2) T is said to be *locally compact* if $\chi_K T$ and $T \chi_K$ are compact operators for any compact subset K of X.

By Remark 2.3, we have the following properties of propagation.

Remark 2.5. Let E_X^p be an L^p-X-module and let $T, S : E_X^p \to E_X^p$ be bounded linear operators. Then

(1) $\mathrm{prop}(T + S) \le \max\{\mathrm{prop}(T), \mathrm{prop}(S)\}$;

(2) $\mathrm{prop}(TS) \le \mathrm{prop}(T) + \mathrm{prop}(S)$.

Definition 2.6. Let E_X^p be an L^p-X-module. The L^p Roe algebra of E_X^p, denoted by $B^p(E_X^p)$, is defined to be the norm closure of the algebra of all locally compact operators acting on E_X^p with finite propagations.

A Borel map f from a proper metric space X to another proper metric space Y is called *coarse* if (1) f is proper, i.e., the inverse image of any bounded set is bounded; (2) for every $R > 0$, there exists $R' > 0$ such that $d(f(x), f(y)) \le R'$ for all $x, y \in X$ satisfying $d(x, y) \le R$.

Lemma 2.7. *Let f be a continuous coarse map, E_X^p an L^p-X-module, and E_Y^p an L^p-Y-module. Then for any $\varepsilon > 0$, there exist an isometric operator $V_f : E_X^p \to E_Y^p$ and a contractive operator $V_f^+ : E_Y^p \to E_X^p$ with $V_f^+ V_f = I$ such that*

$$\mathrm{supp}(V_f) \subseteq \big\{(x, y) \in X \times Y : d\big(f(x), y\big) \le \varepsilon\big\},$$
$$\mathrm{supp}(V_f^+) \subseteq \big\{(y, x) \in Y \times X : d\big(f(x), y\big) \le \varepsilon\big\}.$$

Proof. Let Z_X, Z_Y be the dense subsets of X and Y for defining E_X^p and E_Y^p, respectively, as in Definition 2.1.

There exists a Borel cover $\{Y_i\}_i$ of Y such that

(1) $Y_i \cap Y_j = \emptyset$ if $i \ne j$;

(2) $\mathrm{diameter}(Y_i) \le \varepsilon$ for all i;

(3) each Y_i has nonempty interior.

Condition (3) implies that $Y_i \cap Z_Y$ is a countable set. Thus if $f^{-1}(Y_i) \cap Z_X \ne \emptyset$, then there exist an isometric operator $V_i : \ell^p(f^{-1}(Y_i) \cap Z_X) \otimes \ell^p \to \ell^p(Y_i \cap Z_Y) \otimes \ell^p$ and a contractive operator $V_i^+ : \ell^p(Y_i \cap Z_Y) \otimes \ell^p \to \ell^p(f^{-1}(Y_i) \cap Z_X) \otimes \ell^p$ such that $V_i^+ V_i = \chi_{f^{-1}(Y_i) \cap Z_X} \otimes I$. If $f^{-1}(Y_i) \cap Z_X = \emptyset$, then let $V_i = V_i^+ = 0$. Define

$$V_f = \bigoplus_i V_i : \bigoplus_i \ell^p\big(f^{-1}(Y_i) \cap Z_X\big) \otimes \ell^p \to \bigoplus_i \ell^p(Y_i \cap Z_Y) \otimes \ell^p,$$
$$V_f^+ = \bigoplus_i V_i^+ : \bigoplus_i \ell^p(Y_i \cap Z_Y) \otimes \ell^p \to \bigoplus_i \ell^p\big(f^{-1}(Y_i) \cap Z_X\big) \otimes \ell^p.$$

Then V_f is an isometric operator, V_f^+ is a contractive operator, and $V_f^+ V_f = I$. Condition (2) together with the construction of V_f and V_f^+ implies that

$$\mathrm{supp}(V_f) \subseteq \big\{(x, y) \in X \times Y : d\big(f(x), y\big) \le \varepsilon\big\},$$
$$\mathrm{supp}(V_f^+) \subseteq \big\{(y, x) \in Y \times X : d\big(f(x), y\big) \le \varepsilon\big\}.$$

■

Lemma 2.8. *Let f, E_X^p, and E_Y^p be as in Lemma 2.7. Then the pair (V_f, V_f^+) gives rise to a homomorphism $\mathrm{ad}((V_f, V_f^+)) : B^p(E_X^p) \to B^p(E_Y^p)$ defined by*

$$\mathrm{ad}\big((V_f, V_f^+)\big)(T) = V_f T V_f^+$$

for all $T \in B^p(E_X^p)$.

Moreover, the map $\mathrm{ad}((V_f, V_f^+))_*$ induced by $\mathrm{ad}((V_f, V_f^+))$ on K-theory depends only on f and not on the choice of the pair (V_f, V_f^+).

Proof. Obviously, $\mathrm{ad}(V_f, V_f^+)$ is a contractive homomorphism; thus we just need to show that if T has finite propagation and is locally compact, then $\mathrm{ad}((V_f, V_f^+))(T)$ has these properties too.

Assume first that T has finite propagation. Let ε be as in Lemma 2.7; then $d(f(x), y) \le \varepsilon$ and $d(f(x'), y') \le \varepsilon$ for any $(x, y) \in \mathrm{supp}(V_f)$ and $(y', x') \in \mathrm{supp}(V_f^+)$. Let $(y_1, y_2) \in \mathrm{supp}(V_f T V_f^+)$. By Remark 2.3 part (3), we have that

$$\mathrm{supp}(V_f T V_f^+) \subseteq \mathrm{supp}(V_f) \circ \mathrm{supp}(T) \circ \mathrm{supp}(V_f^+).$$

Hence there exist $x_1, x_2 \in X$ such that $(x_1, y_1) \in \mathrm{supp}(V_f)$, $(x_1, x_2) \in \mathrm{supp}(T)$, and $(y_2, x_2) \in \mathrm{supp}(V_f^+)$; then

$$d(y_1, y_2) \le d\big(y_1, f(x_1)\big) + d\big(f(x_1), f(x_2)\big) + d\big(f(x_2), y_2\big) \le 2\varepsilon + d\big(f(x_1), f(x_2)\big).$$

Since f is coarse and T has finite propagation, we have that $d(y_1, y_2)$ is smaller than some constant for all $(y_1, y_2) \in \mathrm{supp}(V_f T V_f^+)$. This completes the proof of finite propagation.

Now assume that T is locally compact. Let K be a compact subset of Y and let $F = \mathrm{supp}(V_f)$. By Remark 2.3 (4), we have that

$$\chi_K V_f T V_f^+ = \chi_K V_f \chi_{F \circ K} T V_f^+.$$

Since f is a proper map and X is a proper space, we know that $F \circ K$ is a compact subset in X. Then $\chi_{F\circ K} T$ is a compact operator, and hence $\chi_K V_f \chi_{F\circ K} T V_f^+$ is a compact operator. The case of $V_f T V_f^+ \chi_K$ is similar. Thus $\mathrm{ad}((V_f, V_f^+))(T)$ is locally compact.

Let (V_1, V_1^+) and (V_2, V_2^+) be two pairs of operators satisfying the conditions of Lemma 2.7; then we just need to prove

$$\mathrm{ad}\big((V_1, V_1^+)\big)_* = \mathrm{ad}\big((V_2, V_2^+)\big)_* : K_*\big(B^p(E_X^p)\big) \to K_*\big(B^p(E_Y^p)\big).$$

Let

$$U = \begin{pmatrix} I - V_1 V_1^+ & V_1 V_2^+ \\ V_2 V_1^+ & I - V_2 V_2^+ \end{pmatrix};$$

then $U^2 = I$ and

$$\begin{pmatrix} \mathrm{ad}\big((V_1, V_1^+)\big)(T) & 0 \\ 0 & 0 \end{pmatrix} = U \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{ad}\big((V_2, V_2^+)\big)(T) \end{pmatrix} U.$$

Thus $\mathrm{ad}((V_1, V_1^+))_* = \mathrm{ad}((V_2, V_2^+))_*$. ∎

Corollary 2.9. *For different L^p-X-modules E_X^p and $E_X'^p$, $B^p(E_X^p)$ is non-canonically isomorphic to $B^p(E_X'^p)$, and $K_*(B^p(E_X^p))$ is canonically isomorphic to $K_*(B^p(E_X'^p))$.*

For convenience, we replace $B^p(E_X^p)$ by $B^p(X)$ representing the L^p Roe algebra of X.

2.2. L^p localization algebra and L^p K-homology

Definition 2.10. Let X be a proper metric space. The L^p *localization algebra* of X, denoted by $B_L^p(X)$, is defined to be the norm closure of the algebra of all bounded and uniformly norm-continuous functions f from $[0,\infty)$ to $B^p(X)$ such that

$$\operatorname{prop}\big(f(t)\big) \text{ is uniformly bounded and } \operatorname{prop}\big(f(t)\big) \to 0 \quad \text{as } t \to \infty.$$

The *propagation* of f is defined to be $\max\{\operatorname{prop}(f(t)) : t \in [0,\infty)\}$.

Let f be a uniformly continuous coarse map from a proper metric space X to another proper metric space Y. Let $\{\varepsilon_k\}_k$ be a sequence of positive numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. By Lemma 2.7, for each ε_k, there exist an isometric operator V_k from an L^p-X-module E_X^p to an L^p-Y-module E_Y^p and a contractive operator V_k^+ from an L^p-Y-module E_Y^p to an L^p-X-module E_X^p such that $V_k^+ V_k = I$ and

$$\operatorname{supp}(V_k) \subseteq \big\{(x,y) \in X \times Y : d\big(f(x),y\big) \le \varepsilon_k\big\},$$
$$\operatorname{supp}(V_k^+) \subseteq \big\{(y,x) \in Y \times X : d\big(f(x),y\big) \le \varepsilon_k\big\}.$$

For $t \in [0,\infty)$, define

$$V_f(t) = R(t-k)(V_k \oplus V_{k+1})R^*(t-k),$$
$$V_f^+(t) = R(t-k)(V_k^+ \oplus V_{k+1}^+)R^*(t-k)$$

for all $k \le t \le k+1$, where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

$V_f(t)$ is an operator from $E_X^p \oplus E_X^p$ to $E_Y^p \oplus E_Y^p$, and $V_f^+(t)$ is an operator from $E_Y^p \oplus E_Y^p$ to $E_X^p \oplus E_X^p$ such that $\|V_f(t)\| \le 4$, $\|V_f^+(t)\| \le 4$, and $V_f^+(t)V_f(t) = I$ for all $t \in [0,\infty)$.

Lemma 2.11. *Let f and $\{\varepsilon_k\}_k$ be as above; then the pair $(V_f(t), V_f^+(t))$ induces a homomorphism $\operatorname{Ad}((V_f, V_f^+))$ from $B_L^p(X)$ to $B_L^p(Y) \otimes M_2(\mathbb{C})$ defined by*

$$\operatorname{Ad}\big((V_f, V_f^+)\big)(u)(t) = V_f(t)\big(u(t) \oplus 0\big)V_f^+(t)$$

for any $u \in B_L^p(X)$ and $t \in [0,\infty)$ such that

$$\operatorname{prop}\big(\operatorname{Ad}\big((V_f, V_f^+)\big)(u)(t)\big) \le \sup_{(x,y)\in \operatorname{supp}(u(t))} d\big(f(x), f(y)\big) + 2\varepsilon_k + 2\varepsilon_{k+1}$$

for all $t \in [k, k+1]$.

Moreover, the induced map $\operatorname{Ad}((V_f, V_f^+))_*$ on K-theory depends only on f and not on the choice of the pairs $\{(V_k, V_k^+)\}$ in the construction of $V_f(t)$ and $V_f^+(t)$.

Proof. For any $u \in B_L^p(X)$, $\mathrm{Ad}((V_f, V_f^+))(u)$ is bounded and uniformly norm-continuous in t although V_f and V_f^+ are not norm-continuous. By the same ways as the proof of Lemma 2.8, we can obtain that $\mathrm{Ad}((V_f, V_f^+))(u)(t)$ is locally compact when $u(t)$ is locally compact for each t and $\mathrm{Ad}((V_f, V_f^+))_*$ does not depend on the choice of the pair (V_f, V_f^+).

Thus we just need to consider $\mathrm{prop}(\mathrm{Ad}((V_f, V_f^+))(u)(t))$ for which $\mathrm{prop}(u(t))$ is uniformly finite and $\mathrm{prop}(u(t)) \to 0$ as $t \to \infty$. By Lemma 2.3 (4), we know that

$$\mathrm{prop}\left(V_k u(t) V_k^+\right) \le \sup\left\{d\left(f(x), f(y)\right) : (x, y) \in \mathrm{supp}\left(u(t)\right)\right\} + 2\varepsilon_k,$$
$$\mathrm{prop}\left(V_k u(t) V_{k+1}\right) \le \sup\left\{d\left(f(x), f(y)\right) : (x, y) \in \mathrm{supp}\left(u(t)\right)\right\} + \varepsilon_k + \varepsilon_{k+1}.$$

Thus by Remark 2.5, we have

$$\mathrm{prop}\left(\mathrm{Ad}\left((V_f, V_f^+)\right)(u)(t)\right) \le \sup\left\{d\left(f(x), f(y)\right) : (x, y) \in \mathrm{supp}\left(u(t)\right)\right\} + 2\varepsilon_k + 2\varepsilon_{k+1}.$$

Therefore, $\mathrm{prop}(\mathrm{Ad}((V_f, V_f^+))(u)(t))$ is uniformly finite since f is a coarse map, and $\mathrm{prop}(\mathrm{Ad}((V_f, V_f^+))(u)(t)) \to 0$ as $t \to \infty$ since f is a uniformly continuous map and $\varepsilon_k \to 0$. ∎

Definition 2.12. The ith L^p *K-homology* of X is defined to be $K_i(B_L^p(X))$.

2.3. Obstruction group

Let X be a proper metric space. Now consider the evaluation-at-zero homomorphism

$$e_0 : B_L^p(X) \to B^p(X),$$

which induces a homomorphism on K-theory:

$$e_0 : K_*\left(B_L^p(X)\right) \to K_*\left(B^p(X)\right).$$

Let C be a locally finite and uniformly bounded cover for X. The *nerve space* N_C associated to C is defined to be the simplicial complex whose set of vertices equals C and where a finite subset $\{U_0, \dots, U_n\} \subseteq C$ spans an n-simplex in N_C if and only if $\bigcap_{i=0}^n U_i \neq \emptyset$. Endow N_C with the ℓ^1*-metric*, i.e., the path metric whose restriction to each simplex $\{U_0, \dots, U_n\}$ is given by

$$d\left(\sum_{i=0}^n t_i U_i, \sum_{i=0}^n s_i U_i\right) = \sum_{i=0}^n |t_i - s_i|.$$

The distance of two points which are in different connected components is defined to be ∞ by convention.

A sequence of locally finite and uniformly bounded covers $\{C_k\}_{k=0}^\infty$ of metric space X is called an *anti-Čech system* of X [36] if there exists a sequence of positive numbers $R_k \to \infty$ such that, for each k,

(1) every set $U \in C_k$ has diameter less than or equal to R_k;
(2) any set of diameter R_k in X is contained in some member of C_{k+1}.
An anti-Čech system always exists [36].

By the property of the anti-Čech system, for every pair $k_2 > k_1$, there exists a simplicial map $i_{k_1k_2}$ from $N_{C_{k_1}}$ to $N_{C_{k_2}}$ such that $i_{k_1k_2}$ maps a simplex $\{U_0, \ldots, U_n\}$ in $N_{C_{k_1}}$ to a simplex $\{U'_0, \ldots, U'_n\}$ in $N_{C_{k_2}}$ satisfying $U_i \subseteq U'_i$ for all $0 \leq i \leq n$. Thus $i_{k_1k_2}$ gives rise to the following inductive systems of groups:

$$\mathrm{ad}\big((V_{i_{k_1k_2}}, V^+_{i_{k_1k_2}})\big)_* : K_*\big(B^p(N_{C_{k_1}})\big) \to K_*\big(B^p(N_{C_{k_2}})\big),$$
$$\mathrm{Ad}\big((V_{i_{k_1k_2}}, V^+_{i_{k_1k_2}})\big)_* : K_*\big(B^p_L(N_{C_{k_1}})\big) \to K_*\big(B^p_L(N_{C_{k_2}})\big).$$

The following conjecture is called the L^p coarse Baum–Connes conjecture.

Conjecture 2.13. *Let X be a proper metric space and $\{C_k\}_{k=0}^{\infty}$ an anti-Čech system of X; then the evaluation-at-zero homomorphism*

$$e_0 : \lim_{k\to\infty} K_*\big(B^p_L(N_{C_k})\big) \to \lim_{k\to\infty} K_*\big(B^p(N_{C_k})\big) \cong K_*\big(B^p(X)\big)$$

is an isomorphism.

For each $p \in [1, \infty)$, the group $\lim_{k\to\infty} K_*(B^p_L(N_{C_k}))$ is the L^p *coarse K-homology of X* (refer to [19, Definition 2.1]). Moreover, it is not difficult to see that the L^p coarse Baum–Connes conjecture for X does not depend on the choice of the anti-Čech system.

Let $B^p_{L,0}(X) = \{f \in B^p_L(X) : f(0) = 0\}$. There exists an exact sequence:

$$0 \to B^p_{L,0}(X) \to B^p_L(X) \to B^p(X) \to 0.$$

Thus we have the following reduction.

Lemma 2.14. *Let X be a proper metric space and $\{C_k\}_{k=0}^{\infty}$ an anti-Čech system of X; then the L^p coarse Baum–Connes conjecture is true if and only if*

$$\lim_{k\to\infty} K_*\big(B^p_{L,0}(N_{C_k})\big) = 0.$$

For obvious reason $\lim_{k\to\infty} K_*(B^p_{L,0}(N_{C_k}))$ is called the obstruction group to the L^p coarse Baum–Connes conjecture.

3. Controlled obstructions: $\mathrm{QP}_{\delta,N,r,k}(X)$, $\mathrm{QU}_{\delta,N,r,k}(X)$

The controlled obstructions QP and QU for the coarse Baum–Connes conjecture were introduced by Guoliang Yu [46]. In this section, we will introduce and study the L^p version of QP and QU, which can be considered as a controlled version of $K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$ and $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$. We will follow the notation in [46]. One may refer to [6, 29] for more detail about the controlled K-theory for C^*-algebras and L^p-algebras.

3.1. Fundamental concept and property

Definition 3.1 ([6]). Let A be a unital Banach algebra. For $0 < \delta < 1/100$, $N \geq 1$, we define the following: (1) an element e in A is called (δ, N)-*idempotent* if $\|e^2 - e\| < \delta$ and $\max\{\|e\|, \|I - e\|\} \leq N$; (2) an element u in A is called (δ, N)-*invertible* if $\|u\| \leq N$, and there exists $v \in A$ with $\|v\| \leq N$ such that $\max\{\|uv - I\|, \|vu - I\|\} < \delta$, where I is the unit of A. Such v is called a (δ, N)-inverse of u.

Let X be a proper metric space and let $B^p_{L,0}(X)^+$ be the Banach algebra obtained from $B^p_{L,0}(X)$ by adjoining an identity I.

Definition 3.2. Let $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, k and n be nonnegative integers. Define $\mathrm{QP}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous functions f from $[0,1]^k$ to $B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that

(1) $f(t)$ is an (δ, N)-idempotent and $\mathrm{prop}(f(t)) \leq r$ for all $t \in [0,1]^k$;

(2) $\|f(t) - e_m\| < \delta$ for all $t \in \mathrm{bd}([0,1]^k)$, the boundary of $[0,1]^k$ in $\mathbb{R}^k$, where $e_m = I \oplus \cdots \oplus I \oplus 0 \oplus \cdots \oplus 0$ with m identities;

(3) $\pi(f(t)) = e_m$, where π is the canonical homomorphism from $B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C})$.

Definition 3.3. Let $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, and let $\mathrm{QP}_{\delta,N,r,k}(X)$ be defined as the direct limit of $\mathrm{QP}_{\delta,N,r,k}(B^P_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ under the embedding: $p \to p \oplus 0$.

Definition 3.4. Let $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, k and n be nonnegative integers. Define $\mathrm{QU}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous functions u from $[0,1]^k$ to $B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that there exists a continuous function $v : [0,1]^k \to B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ satisfying that

(1) $u(t)$ is a (δ, N)-invertible with a (δ, N)-inverse $v(t)$ such that $\max\{\mathrm{prop}(u(t)), \mathrm{prop}(v(t))\} \leq r$ for all $t \in [0,1]^k$;

(2) $\|u(t) - I\| < \delta$ and $\|v(t) - I\| < \delta$ for all $t \in \mathrm{bd}([0,1]^k)$;

(3) $\pi(u(t)) = \pi(v(t)) = I$, where the map π is the canonical homomorphism from $B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C})$.

Such v is called a (δ, N, r)-*inverse* of u.

Definition 3.5. Let $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, and let $\mathrm{QU}_{\delta,N,r,k}(X)$ be defined as the direct limit of $\mathrm{QU}_{\delta,N,r,k}(B^P_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ under the embedding: $u \to u \oplus I$.

Definition 3.6. Let $e_1, e_2 \in \mathrm{QP}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$. We say e_1 is (δ, N, r)-*equivalent* to e_2 if there exists a continuous homotopy $a(t')$ in $\mathrm{QP}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ for $t' \in [0,1]$, such that $a(0) = e_1$ and $a(1) = e_2$. Such homotopy is called a (δ, N, r)-*homotopy*.

Notice that (1) any $e \in \mathrm{QP}_{\delta,N,r,k}(X)$ is (δ', N', r)-equivalent to some f for which $f(t) = \pi(f)$ for all $t \in \mathrm{bd}([0,1]^k)$; (2) if e_1 is (δ, N, r)-equivalent to e_2 and $e_1(t) = \pi(e_1)$,

$e_2(t) = \pi(e_2)$ for all $t \in \mathrm{bd}([0,1]^k)$, then there exists a homotopy $a(t')$ in $\mathrm{QP}_{\delta'',N'',r,k}(X)$ such that $a(0) = e_1$, $a(1) = e_2$, and $a(t')(t) = \pi(a(t'))$ for all $t \in \mathrm{bd}([0,1]^k)$, where δ', δ'' depend only on δ, N; N', N'' depend only on N.

Definition 3.7. Let u_1, u_2 be two elements in $\mathrm{QU}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$. We say that u_1 is (δ, N, r)-equivalent to u_2 if there exists a continuous homotopy $w(t')$ in $\mathrm{QU}_{\delta,N,r,k}(B^p_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ for $t' \in [0,1]$ such that $w(0) = u_1$ and $w(1) = u_2$. This equivalence induces an equivalent relation in $\mathrm{QU}_{\delta,N,r,k}(X)$.

The following lemma tells us that $\mathrm{QP}_{\delta,N,r,k}(X)$ can be considered as a controlled version of $K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$.

Lemma 3.8. *Let* $0 < \delta < 1/100$ *and let* χ *be a function such that* $\chi(x) = 1$ *for* $\mathrm{Re}(x) > 1/2$ *and* $\chi(x) = 0$ *for* $\mathrm{Re}(x) < 1/2$. *Then*

(1) *for any* $e \in \mathrm{QP}_{\delta,N,r,k}(X)$, $\chi(e)$ *is an idempotent and defines an element* $[\chi(e)] \in K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$*;*

(2) *for any two elements* $e_1, e_2 \in \mathrm{QP}_{\delta,N,r,k}(X)$ *satisfying that* e_1 *is* (δ, N, r)*-equivalent to* e_2, $[\chi(e_1)] = [\chi(e_2)]$ *in* $K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$*;*

(3) *for any* $0 < \delta < 1/100$, *every element in* $K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$ *can be represented as* $[\chi(e_1)] - [\chi(e_2)]$, *where* $e_1, e_2 \in \mathrm{QP}_{\delta,N,r,k}(X)$ *for some* $N \geq 1$ *and* $r > 0$.

Proof. (1) and (2) are straightforward by holomorphic functional calculus and the definition of (δ, N, r)-equivalence. To prove (3), for any $[p] - [q] \in K_0(B^p_{L,0}(X) \otimes C_0((0,1)^k))$, where $p, q \in (B^p_{L,0} \otimes C_0((0,1)^k))^+ \otimes M_n(\mathbb{C})$, let $N = \|p\| + \|1-p\| + 1$. By approximation argument, there exists $r > 0$ and $e_1 \in (B^p_{L,0} \otimes C((0,1)^k))^+ \otimes M_n(\mathbb{C})$ such that $\mathrm{prop}(e_1) < r$ and $\|e_1 - p\| < \frac{\delta}{4N}$. Thus we have $e_1 \in \mathrm{QP}_{\delta,N,r,k}(X)$. Now we just need to prove $[\chi(e_1)] = [p]$. Let $e(t') = t'e_1 + (1-t')p$ for $t' \in [0,1]$. We have $\|e^2(t') - e(t')\| < \delta$. Thus $\chi(e(t'))$ is a continuous homotopy of projections between $\chi(e(0)) = p$ and $\chi(e(1)) = \chi(e_1)$. ■

The following lemma tells us that $\mathrm{QU}_{\delta,N,r,k}(X)$ can be considered as a controlled version of $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$.

Lemma 3.9. *Let* $0 < \delta < 1/100$. *Then*

(1) *for any* $u \in \mathrm{QU}_{\delta,N,r,k}(X)$, u *is an invertible element and defines an element* $[u]$ *in* $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$*;*

(2) *if* u_1 *is* (δ, N, r)*-equivalent to* u_2 *in* $\mathrm{QU}_{\delta,N,r,k}(X)$, *then one has* $[u_1] = [u_2]$ *in* $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$*;*

(3) *for any* $0 < \delta < 1/100$, *every element in* $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$ *can be represented as* $[u]$, *where* $u \in \mathrm{QU}_{\delta,N,r,k}(X)$ *for some* $N \geq 1$ *and* $r > 0$.

Proof. (1) is true since the set of invertible elements in Banach algebra is open.

(2) is true by the definition of (δ, N, r)-equivalence. To prove (3), we assume that $[u'] \in K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$. Let $N = \|u'\| + \|u'^{-1}\| + 1$; then there exists $r > 0$ and $u, v \in B^p_{L,0}(X) \otimes C_0((0,1)^k) \otimes M_n(\mathbb{C})$ such that $\|u - u'\| < \frac{\delta}{2N}$, $\|v - u'_{-1}\| < \frac{\delta}{2N}$ and $\mathrm{prop}(u) < r$, $\mathrm{prop}(v) < r$. We have $\|u\| \le N$, $\|v\| \le N$ and $\|uv - I\| < \delta$, $\|vu - I\| < \delta$. Hence $u \in \mathrm{QU}_{\delta,N,r,k}(X)$. Let $w(t) = tu + (1-t)u'$ for $t \in [0,1]$. We have $\|w(t)u'^{-1} - I\| < \delta < 1/100$. Thus $w(t)u'^{-1}$ is an invertible element, so is $w(t)$. Therefore $[u] = [u']$ in $K_1(B^p_{L,0}(X) \otimes C_0((0,1)^k))$. ∎

Lemma 3.10 ([6, Lemma 2.29]). *If e is (δ, N, r)-equivalent to f by a homotopy $e_{t'}$ ($t' \in [0,1]$) in $\mathrm{QP}_{\delta,N,r,k}(X)$, then there exists $\alpha_N > 0, m \in \mathbb{N}$ such that $e \oplus I_m \oplus 0_m$ is $(2\delta, 3N, r)$-equivalent to $f \oplus I_m \oplus 0_m$ by an α_N-Lipschitz homotopy, where α_N depends only on N and not on e, f, δ, r; and m depends only on $\delta, N, e_{t'}$.*

Proof. There exists a partition $0 = t'_0 < t'_1 < \cdots < t'_m = 1$ such that

$$\|e_{t'_i} - e_{t'_{i-1}}\| < \inf_{t' \in [0,1]} \frac{\delta - \|e^2_{t'} - e_{t'}\|}{2N + 1}.$$

For each t', we have a Lipschitz $(\delta, 3N, r)$-homotopy between $I \oplus 0$ and $e_{t'} \oplus (1 - e_{t'})$ given by combining the linear homotopy connecting $I \oplus 0$ to $(e_{t'} - e^2_{t'}) \oplus 0$ and the homotopy

$$(e_{t'} \oplus 0) + R^*(s)\big((1 - e_{t'}) \oplus 0\big)R(s),$$

where $R(s) = \begin{pmatrix} \cos(\pi s/2) & \sin(\pi s/2) \\ -\sin(\pi s/2) & \cos(\pi s/2) \end{pmatrix}$. Obviously, the linear homotopy between $e_{t'_{i-1}}$ and $e_{t'_i}$ is Lipschitz for all i. Then

$$\begin{pmatrix} e_{t'_0} & & \\ & I_m & \\ & & 0_m \end{pmatrix} \simeq \begin{pmatrix} e_{t'_0} & & & & & \\ & I & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix} \simeq \begin{pmatrix} e_{t'_0} & & & & & \\ & I - e_{t'_1} & & & & \\ & & e_{t'_1} & & & \\ & & & \ddots & & \\ & & & & I - e_{t'_m} & \\ & & & & & e_{t'_m} \end{pmatrix}$$

$$\simeq \begin{pmatrix} e_{t'_0} & & & & & \\ & I - e_{t'_0} & & & & \\ & & e_{t'_1} & & & \\ & & & \ddots & & \\ & & & & I - e_{t'_{m-1}} & \\ & & & & & e_{t'_m} \end{pmatrix} \simeq \begin{pmatrix} I & & & & & \\ & 0 & & & & \\ & & I & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & e_{t'_m} \end{pmatrix}$$

$$\simeq \begin{pmatrix} e_{t'_m} & & \\ & I_m & \\ & & 0_m \end{pmatrix},$$

where $\simeq$ represents $(2\delta, 3N, r)$-equivalence by Lipschitz homotopy. ∎

We remark that we have a result for QU similar to the above lemma, i.e., homotopy implies Lipschitz homotopy.

The following lemma tells us that homotopy equivalence of two quasi-invertible elements implies homotopy equivalence of their quasi-inverses.

Lemma 3.11. *Let u_1, u_2 be two elements in $\mathrm{QU}_{\delta,N,r,k}(X)$ with (δ, N)-inverse v_1, v_2, respectively. If u_1 is (δ, N, r)-equivalent to u_2, then v_1 is $(4\delta, 2N, r)$-equivalent to v_2 in $\mathrm{QU}_{4\delta,2N,r,k}(X)$.*

Proof. Let $w(t')$ be the homotopy path jointing u_1 and u_2. For $\varepsilon = \frac{\delta}{N}$, there exists a partition $0 = t'_0 < t'_1 < \cdots < t'_n = 1$ such that

$$\max_{0 \le i \le n-1} \left\{ \left\| w(l) - w(l') \right\| : t'_i \le l, l' \le t'_{i+1} \right\} < \frac{\delta}{N}.$$

Assume that $s_{t'_i}$ is the (δ, N, r)-inverse of $w(t'_i)$; we require $s_0 = v_1, s_1 = v_2$. Let

$$s(t') = \frac{t' - t'_i}{t'_{i+1} - t'_i} s_{t'_{i+1}} - \frac{t' - t'_{i+1}}{t'_{i+1} - t'_i} s_{t'_i}, \quad t'_i \le t' \le t'_{i+1}.$$

We have $\|s_{t'_i} w(t') - I\| \le \|s_{t'_i}\| \cdot \|w(t') - w(t'_i)\| + \|s_{t'_i} w(t'_i) - I\| \le 2\delta$ for $t'_i \le t' \le t'_{i+1}$. Then $\|s(t')w(t') - I\| \le 4\delta$. Similarly, $\|w(t')s(t') - I\| \le 4\delta$.

Obviously, $\|s(t')\| \le 2N$ and $\mathrm{prop}(s(t')) < r$. Thus $s(t')$ is a continuous homotopy between v_1 and v_2 in $\mathrm{QU}_{4\delta,2N,r,k}(X)$. ∎

The following two lemmas can be viewed as the controlled version of the classical result in K-theory that stable homotopy equivalence of idempotents is the same as stable similarity.

Lemma 3.12. *Let $0 < \delta < 1/100$. If e is (δ, N, r)-equivalent to f in $\mathrm{QP}_{\delta,N,r,k}(X)$, then there exist a positive number m and an element u in $\mathrm{QU}_{\delta,C_1(N),C_2(N,\delta)r,k}(X)$ with $(\delta, C_1(N), C_2(N,\delta)r)$-inverse v, such that*

$$\left\| f \oplus I_m \oplus 0_m - v(e \oplus I_m \oplus 0_m)u \right\| < C_3(N)\delta,$$

where $C_1(N)$ and $C_3(N)$ depend only on N and $C_2(N,\delta)$ depends only on N and δ.

Proof. By Lemma 3.10, there exists $\alpha_N > 0, m \in \mathbb{N}$ such that $e \oplus I_m \oplus 0_m$ is $(2\delta, 3N, r)$-equivalent to $f \oplus I_m \oplus 0_m$ by an α_N-Lipschitz homotopy $e_{t'}$, i.e., $\|e_{t'} - e_{t''}\| \le \alpha_N |t' - t''|$ for any $t', t'' \in [0, 1]$. There exists a partition $0 = t'_0 < t'_1 < \cdots < t'_n = 1$ such that

$$\alpha_N |t'_{i+1} - t'_i| < \frac{1}{2N+1}.$$

Let $w_i = ((2e_{t'_i} - I)(2e_{t'_{i+1}} - I) + I)/2$. We have $I - w_i = (2e_{t'_i} - I)(e_{t'_i} - e_{t'_{i+1}}) + 2(e_{t'_i} - e_{t'_i}^2)$. Then

$$\|I - w_i\| < \|2e_{t'_i} - I\| \cdot \|e_{t'_i} - e_{t'_{i+1}}\| + 2\|e_{t'_i} - e_{t'_i}^2\| < 1/2 + 4\delta < 1.$$

Thus w_i is an invertible element and $w_i^{-1} = \Sigma_{j=0}^{\infty}(1 - w_i)^j$. Let $v_i = \Sigma_{j=0}^{l}(I - w_i)^j$ satisfying $\|v_i - w_i^{-1}\| < \delta/2((\max_i\{\|w_i\|, \|w_i^{-1}\|\} + 1)^n)$. Let

$$u = w_0 w_1 \cdots w_{n-1}, \quad v = v_{n-1} v_{n-2} \cdots v_0.$$

Then $\max\{\|u\|, \|v\|\} \leq C_1(N)$, $\max\{\text{prop}(u(t)), \text{prop}(v(t))\} \leq C_2(N, \delta)r$ for $t \in [0,1]^k$, and $\max\{\|I - uv\|, \|I - vu\|\} < \delta$, where $C_1(N)$ depends only on N and $C_2(N, \delta)$ depends only on N, δ.

By computation, we have $\|e_{t'_i} w_i - w_i e_{t'_{i+1}}\| < 26N\delta$. Then $\|ue_1 - e_0 u\| < C'N$, where C' depends only on N. Thus

$$\|e_1 - v(e_0)u\| = \|e_1 - vue_1 + v(ue_1 - e_0 u)\| < C_3(N)\delta,$$

where $C_3(N)$ depends only on N. ∎

Lemma 3.13. *Let $N \geq 1$, $0 < \delta < 1/(800N^4)$, and $0 < \varepsilon < 1/400$. For e and f in $\mathrm{QP}_{\delta,N,r,k}(B_{L,0}^p(X)^+ \otimes M_n(\mathbb{C}))$, if there exists u in $\mathrm{QU}_{\delta,N,r,k}(X)$ with (δ, N, r)-inverse v satisfying $\|uev - f\| < \varepsilon$, then $e \oplus 0_n$ is $(2\varepsilon + 4N^4\delta, 2N^3, 3r)$-equivalent to $f \oplus 0_n$ in $\mathrm{QP}_{2\varepsilon+4N^4\delta,2N^3,3r,k}(X)$.*

Proof. Let $e_{t'}$ be a homotopy connecting $f \oplus 0_n$ to $e \oplus 0_n$ obtained by combining the linear homotopy connecting $f \oplus 0_n$ to $uev \oplus 0_n$ with the following homotopy connecting $uev \oplus 0_n$ to $e \oplus 0_n$:

$$R(t')(u \oplus I_n)R^*(t')(e \oplus 0_n)R(t')(v \oplus I_n)R^*(t'),$$

where

$$R(t') = \begin{pmatrix} \cos(\pi t'/2) & \sin(\pi t'/2) \\ -\sin(\pi t'/2) & \cos(\pi t'/2) \end{pmatrix}.$$

It is not difficult to verify that $e_{t'}$ is a $(2\varepsilon + 4N^4\delta, 2N^3, 3r)$-homotopy between e and f. ∎

Definition 3.14. Let X be a proper metric space and define

$$\mathrm{GQP}_{\delta,N,r,k}(X) = \{e - f : e, f \in \mathrm{QP}_{\delta,N,r,k}(X), \pi(e) = \pi(f)\}.$$

We say that $e_1 - f_1$ is (δ, N, r)-equivalent to $e_2 - f_2$ if $e_1 \oplus f_2 \oplus I_n \oplus 0_n$ is (δ, N, r)-equivalent to $f_1 \oplus e_2 \oplus I_n \oplus 0_n$ for some n. This defines an equivalent relation on $\mathrm{GQP}_{\delta,N,r,k}$.

For any $u \in \mathrm{QU}_{\delta,N,r,k}(X)$ with a (δ, N, r)-inverse v, let $Z_t(u)$ be a homotopy connecting $I \oplus I$ to $u \oplus v$ obtained by combining the linear homotopy connecting $I \oplus I$

to $uv \oplus I$ with the homotopy $(u \oplus I)R(t)(v \oplus I)R^*(t)$ connecting $uv \oplus I$ to $u \oplus v$; let $Z'_t(u)$ be a homotopy connecting $I \oplus I$ to $v \oplus u$ obtained by combining the linear homotopy connecting $I \oplus I$ to $uv \oplus I$ with the homotopy $R(t)(u \oplus I)R^*(t)(v \oplus I)$ connecting $uv \oplus I$ to $v \oplus u$, where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Let

$$e_t(u) = Z_t(u)(I \oplus 0)Z'_t(u).$$

We have that

(1) $\|e_t^2(u) - e_t(u)\| < 8N^6\delta$;

(2) $\|e_t(u)\| \le 4N^4$ and $\|I - e_t(u)\| \le 5N^4$;

(3) $\mathrm{prop}(e_t(u)(t')) \le 2r$ for $t' \in [0,1]^k$.

Then we can define a map θ from $\mathrm{QU}_{\delta,N,r,k}(X)$ to $\mathrm{GQP}_{8N^6\delta,5N^4,2r,k+1}(X)$ by

$$\theta(u) = e_t(u) - (I \oplus 0).$$

It is not difficult to see that the definition of θ does not depend on the choice of a (δ, N, r)-inverse v of u in the sense of equivalence.

The following result can be considered as a controlled version of a classical result in the operator K-theory $K_1(A) \cong K_0(SA)$.

Lemma 3.15. $\theta : \mathrm{QU}_{\delta,N,r,k}(X) \to \mathrm{GQP}_{8N^6\delta,5N^4,2r,k+1}(X)$ *is an asymptotic isomorphism in the following sense:*

(1) *for any $0 < \delta < 1/100$, $r > 0$, $N \ge 1$, there exist $0 < \delta_1 < \delta$, $N_1 \ge N$, and $0 < r_1 < r$ such that if two elements u_1 and u_2 in $\mathrm{QU}_{\delta_1,N,r_1,k}(X)$ are (δ_1, N, r_1)-equivalent, then $\theta(u_1)$ and $\theta(u_2)$ are (δ, N_1, r)-equivalent, where δ_1 depends only on δ and N, N_1 depends only on N, and r_1 depends only on r;*

(2) *for any $0 < \delta < 1/100$, $r > 0$, $N \ge 1$, there exist $0 < \delta_2 < \delta$, $N_2 \ge N$, and $0 < r_2 < r$ such that if u' and u'' in $\mathrm{QU}_{\delta_2,N,r_2,k}(X)$ satisfy that $\theta(u')$ is (δ_2, N, r_2)-equivalent to $\theta(u'')$, then $u' \oplus I_m$ is (δ, N_2, r)-equivalent to $u'' \oplus I_m$ for some $m \in \mathbb{N}$, where δ_2 depends only on δ and N, N_2 depends only on N, and r_2 depends only on r, δ, N;*

(3) *for any $0 < \delta < 1/100$, $r > 0$, $N \ge 1$, there exist $0 < \delta_3 < \delta$, $N_3 \ge N$, and $0 < r_3 < r$ such that, for each $e - e_m \in \mathrm{GQP}_{\delta_3,N,r_3,k+1}(X)$, there exists $u \in \mathrm{QU}_{\delta,N_3,r,k}(X)$, for which $\theta(u)$ is (δ, N_3, r)-equivalent to $e - e_m$, where δ_3 depends only on δ and N, N_3 depends only on N, and r_3 depends only on r, δ, N.*

Proof. (1) Let v_i be the (δ_1, N, r_1)-inverse of u_i for $i = 1, 2$ and $w(t)$ the (δ_1, N, r_1)-homotopy between u_1 and u_2. By Lemma 3.11, there exists a $(4\delta_1, 2N, r_1)$-homotopy $s(t)$ connecting v_1 and v_2 such that $\|I - s(t)w(t)\|$ and $\|I - w(t)s(t)\|$ are less than $4\delta_1$.

Let $a(t)$ be a homotopy connecting I to v_2u_1 obtained by combining the linear homotopy connecting I to v_1u_1 with the homotopy $s(t)u_1$; let $a'(t)$ be a homotopy connecting I to v_1u_2 obtained by combining the linear homotopy connecting I to v_1u_1 with the homotopy $v_1w(t)$; let $b(t)$ be a homotopy connecting I to u_2v_1 obtained by combining the linear homotopy connecting I to u_1v_1 with the homotopy $w(t)v_1$; let $b'(t)$ be a homotopy connecting I to u_1v_2 obtained by combining the linear homotopy connecting I to u_1v_1 with the homotopy $u_1s(t)$. Define

$$x_t = Z_t(u_2)\big(a(t) \oplus b(t)\big)Z'_t(u_1),$$
$$x'_t = Z_t(u_1)\big(a'(t) \oplus b'(t)\big)Z'_t(u_2).$$

We have that

(i) $\max\{\|x_t\|, \|x'_t\|\} \leq 8N^6$;

(ii) $\max\{\|I - x_t x'_t\|, \|I - x'_t x_t\|\} < 64N^{10}\delta_1$;

(iii) $\max\{\operatorname{prop}(x_t), \operatorname{prop}(x'_t)\} < 6r_1$;

(iv) $\max\{\|x_i - I\|, \|x'_i - I\| < 3\delta_1\}$ for $i = 0, 1$.

Thus $x_t, x'_t \in \mathrm{QU}_{64N^{10}\delta_1, 8N^6, 6r_1, k+1}(X)$ and

$$\big\|x_t e_t(u_1) x'_t - e_t(u_2)\big\| < (184N^{14})\delta_1.$$

By Lemma 3.13, we can select appropriate δ_1, N_1, and r_1 satisfying Lemma 3.15 (1).

(2) Let v', v'' be (δ_2, N, r_2)-inverses of u', u'', respectively. By Lemma 3.12, there exists an element u in $\mathrm{QU}_{\delta_2, C_1(N), C_2(N,\delta_2)r_2, k+1}(X)$ with inverse v such that

$$\big\|u e_t(u' \oplus I)v - e_t(u'' \oplus I)\big\| < C_3(N)\delta_2,$$

i.e.,

$$\big\|u_t Z_t(u' \oplus I)(I \oplus 0)Z'_t(u' \oplus I)v_t - Z_t(u'' \oplus I)(I \oplus 0)Z'_t(u'' \oplus I)\big\| < C_3(N)\delta_2, \tag{3.1}$$

where $t \in [0, 1]$. Thus we have

$$\big\|Z'_t(u'' \oplus I)u_t Z_t(u' \oplus I)(I \oplus 0) - (I \oplus 0)Z'_t(u'' \oplus I)u_t Z_t(u' \oplus I)\big\| < C_4(N)\delta_2. \tag{3.2}$$

Let

$$Z'_t(u'' \oplus I)u_t Z_t(u' \oplus I) = \begin{pmatrix} b_t & g_t \\ h_t & d_t \end{pmatrix}.$$

Then by (3.2), we obtain

$$\|g_t\| < C_4(N)\delta_2, \quad \|h_t\| < C_4(N)\delta_2. \tag{3.3}$$

By (3.1), we also have

$$\big\|(I \oplus 0)Z'_t(u' \oplus I)v_t Z_t(u'' \oplus I) - Z'_t(u' \oplus I)v_t Z_t(u'' \oplus I)(I \oplus 0)\big\| < C_5(N)\delta_2. \tag{3.4}$$

Let

$$Z'_t(u' \oplus I)v_t Z_t(u'' \oplus I) = \begin{pmatrix} b'_t & g'_t \\ h'_t & d'_t \end{pmatrix}.$$

Then by (3.4), we obtain

$$\|g'_t\| < C_5(N)\delta_2, \quad \|h'_t\| < C_5(N)\delta_2. \tag{3.5}$$

Thus by (3.3) and (3.5), we know that $b_t \in \mathrm{QU}_{C_6(N)\delta_2, C_7(N), C_8(N,\delta_2)r_2, k+1}(X)$ with a $(C_6(N)\delta_2, C_7(N), C_8(N,\delta_2)r_2)$-inverse b'_t such that

$$\|c_0 - I\| \le \|u_0 - I\| < \delta_2,$$
$$\left\|c_1 - (v'' \oplus I)(u' \oplus I)\right\| < C_9(N)\delta_2.$$

Thus we can select appropriate δ_2, N_2, and r_2 satisfying Lemma 3.15 (2).

(3) $e(t)$ can be considered as a homotopy in $\mathrm{QP}_{\delta_3, N, r_3, k}(X)$, where $t \in [0, 1]$. We can assume that $e(0) = e(1) = e_m = I \oplus 0$. By the proof of Lemma 3.12, there exists a homotopy $w(t)$ in $\mathrm{QU}_{\delta_3, C_1(N), C_2(N,\delta_3)r_3, k}(X)$ with inverse $s(t)$ for which $w(0) = s(0) = I$ such that

$$\left\|w(t)(I \oplus 0 \oplus I_m \oplus 0_m)s(t) - e(t) \oplus I_m \oplus 0_m\right\| < C_3(N)\delta_3$$

for some $m \in \mathbb{N}$ and all $t \in [0, 1]$. By some minor modifications of $w(t)$ and $s(t)$, we have

$$\left\|w(1)(I \oplus 0) - (I \oplus 0)w(1)\right\| < C_4(N)\delta_3. \tag{3.6}$$

Let

$$w(1) = \begin{pmatrix} u & g \\ h & u' \end{pmatrix}, \quad s(1) = \begin{pmatrix} v & g' \\ h' & v' \end{pmatrix};$$

then by (3.6), we obtain

$$\max\left\{\|g\|, \|h\|, \|g'\|, \|h'\|\right\} < C_4(N)\delta_3.$$

Thus u and u' are two elements in $\mathrm{QU}_{C_5(N)\delta_3, C_6(N), C_7(N,\delta_3)r_3, k}(X)$ with inverse v and v', respectively.

Let a_t be a homotopy connecting $I \oplus I \oplus I$ to $v'v \oplus I \oplus I$ obtained by combining the linear homotopy connecting $I \oplus I \oplus I$ to $v'u' \oplus I \oplus I$ with the rotation homotopy connecting $(v' \oplus I \oplus I)(u' \oplus I \oplus I)$ to $(v' \oplus I \oplus I)(v \oplus u \oplus u')$ with the homotopy $(v' \oplus I \oplus I)(v \oplus w(1-t))$ connecting $(v' \oplus I \oplus I)(v \oplus u \oplus u')$ to $v'v \oplus I \oplus I$. Similarly, let b_t be a homotopy connecting $I \oplus I \oplus I$ to $uu' \oplus I \oplus I$. Define

$$y_t = \big(w(t) \oplus I \oplus I\big)\big(I \oplus a(t)\big)\big(Z'_t(u) \oplus I \oplus I\big),$$
$$y'_t = \big(Z_t(u) \oplus I \oplus I\big)\big(I \oplus b(t)\big)\big(s(t) \oplus I \oplus I\big);$$

then we have

$$y_0 = y'_0 = I, \quad \max\left\{\|y_i - I\|, \|y'_i - I\|\right\} < C_8(N)\delta_3 \tag{3.7}$$

and

$$\left\|y_t\big(e_t(u) \oplus 0\big)y'_t - (e \oplus 0)\right\| < C_9(N)\delta_3. \tag{3.8}$$

Now by Lemma 3.13, we can choose appropriate δ_3, N_3, and r_3 on the basis of (3.7) and (3.8) satisfying Lemma 3.15 (3).

Remark: we can also let

$$y_t = \big(w(t) \oplus I \oplus I\big)\big(I \oplus Z_t'(u')s(t) \oplus I\big)\big(Z_t'(u) \oplus I \oplus I\big),$$
$$y_t' = \big(Z_t(u) \oplus I \oplus I\big)\big(I \oplus w(t)Z_t(u') \oplus I\big)\big(s(t) \oplus I \oplus I\big).$$ ∎

3.2. Strongly Lipschitz homotopy invariance

Definition 3.16 (Yu [45]). Let $f, g : X \to Y$ be two proper Lipschitz maps. A continuous homotopy $F(t,x)(t \in [0,1])$ between f and g is called *strongly Lipschitz* if

(1) $F(t,x)$ is a proper map from X to Y for each t;

(2) there exists a constant C such that $d(F(t,x), F(t,y)) \leq C d(x,y)$ for all $x, y \in X$ and $t \in [0,1]$; this C is called Lipschitz constant of F;

(3) F is equicontinuous in t, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(F(t_1,x), F(t_2,x)) < \varepsilon$ for all $x \in X$ if $|t_1 - t_2| < \delta$;

(4) $F(0,x) = f(x)$, $F(1,x) = g(x)$ for all $x \in X$.

X is said to be *strongly Lipschitz homotopy equivalent* to Y if there exist proper Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that fg and gf are strongly Lipschitz homotopic to id_Y and id_X, respectively.

Lemma 3.17. *Let f and g be two Lipschitz maps from X to Y. Let $F(t,x)$ be a strongly Lipschitz homotopy connecting f to g with Lipschitz constant C. There exists $C_0 > 0$ such that, for any $u \in \mathrm{QU}_{\delta,N,r,k}(X)$, there exists a homotopy $w(t')$ in $\mathrm{QU}_{D(N)\delta,N^{100},C_0 r,k}(Y)$ for which*

$$w(0) = \mathrm{Ad}\big((V_f, V_f^+)\big)(u) \oplus I,$$
$$w(1) = \mathrm{Ad}\big((V_g, V_g^+)\big)(u) \oplus I,$$

where $D(N)$ depends only on N and C_0 depends only on C.

Proof. Choose $\{t_{i,j}\}_{i \geq 0, j \geq 0} \subseteq [0,1]$ satisfying

(1) $t_{0,j} = 0$, $t_{i,j+1} \leq t_{i,j}$, $t_{i+1,j} \geq t_{i,j}$;

(2) there exists $N_j \to \infty$ such that $t_{i,j} = 1$ for all $i \geq N_j$ and $N_{j+1} \geq N_j$;

(3) $d(F(t_{i+1,j}, x), F(t_{i,j}, x)) < \varepsilon_j = r/(j+1)$, $d(F(t_{i,j+1}, x), F(t_{i,j}, x)) < \varepsilon_j$ for all $x \in X$.

Let $f_{i,j}(x) = F(t_{i,j}, x)$. By Lemma 2.7, there exist an isometric operator $V_{f_{i,j}} : E_X^p \to E_Y^p$ and a contractive operator $V_{f_{i,j}}^+ : E_Y^p \to E_X^p$ with $V_{f_{i,j}}^+ V_{f_{i,j}} = I$ such that

$$\mathrm{supp}(V_{f_{i,j}}) \subseteq \big\{(x,y) \in X \times Y : d\big(f_{i,j}(x), y\big) < r/(1+i+j)\big\},$$
$$\mathrm{supp}(V_{f_{i,j}}^+) \subseteq \big\{(y,x) \in Y \times X : d\big(f_{i,j}(x), y\big) < r/(1+i+j)\big\}.$$

For each $i > 0$, define a family of operators $V_i(t)(t \in [0,\infty))$ from $E_X^p \oplus E_X^p$ to $E_Y^p \oplus E_Y^p$ and a family of operators $V_i^+(t)(t \in [0,\infty))$ from $E_Y^p \oplus E_Y^p$ to $E_X^p \oplus E_X^p$ by

$$V_i(t) = R(t-j)(V_{f_{i,j}} \oplus V_{f_{i,j+1}})R^*(t-j), \quad t \in [j, j+1],$$
$$V_i^+(t) = R(t-j)(V_{f_{i,j}}^+ \oplus V_{f_{i,j+1}}^+)R^*(t-j), \quad t \in [j, j+1],$$

where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Consider

$$u_0(t) = \mathrm{Ad}\big((V_f, V_f^+)\big)(u) = V_f(t)\big(u(t) \oplus I\big)V_f^+(t) + \big(I - V_f(t)V_f^+(t)\big),$$
$$u_\infty(t) = \mathrm{Ad}\big((V_g, V_g^+)\big)(u) = V_g(t)\big(u(t) \oplus I\big)V_g^+(t) + \big(I - V_g(t)V_g^+(t)\big),$$
$$u_i(t) = \mathrm{Ad}\big((V_i, V_i^+)\big)(u) = V_i(t)\big(u(t) \oplus I\big)V_i^+(t) + \big(I - V_i(t)V_i^+(t)\big).$$

Let v be the (δ, N, r)-inverse of u. Similarly, we can define

$$u_i'(t) = \mathrm{Ad}\big((V_i, V_i^+)\big)(v).$$

For each i, define n_i by

$$n_i = \begin{cases} \max\{j : i \geq N_j\}, & \{j : i \geq N_j\} \neq \emptyset; \\ 0, & \{j : i \geq N_j\} = \emptyset. \end{cases}$$

We can choose $V_{f_{i,j}}$ in such a way that $u_i(t) = u_\infty$, where $t \leq n_i$.

Define

$$w_i(t) = \begin{cases} u_i(t)\big(u_\infty'(t)\big), & t \geq n_i; \\ (n_i - t)I + (t - n_i + 1)u_i(t)u_\infty'(t), & n_i - 1 \leq t \leq n_i; \\ I, & 0 \leq t \leq n_i - 1. \end{cases}$$

Consider

$$a = \bigoplus_{i=0}^{\infty}(w_i \oplus I), \quad b = \bigoplus_{i=0}^{\infty}(w_{i+1} \oplus I), \quad c = (I \oplus I)\bigoplus_{i=1}^{\infty}(w_i \oplus I).$$

By the construction of $\{t_{i,j}\}$, we know that $a, b, c \in \mathrm{QU}_{D_1(N)\delta, N^2, C_1 r, k}(Y)$ for some constant C_1 depending only on C. Let

$$V_{i,i+1}(t') = R(t')(V_i \oplus V_{i+1})R^*(t'), \quad t' \in [0,1],$$
$$V_{i,i+1}^+(t') = R(t')(V_i^+ \oplus V_{i+1}^+)R^*(t'), \quad t' \in [0,1].$$

Define

$$u_{i,i+1}(t') = V_{i,i+1}(t')\big((u \oplus I) \oplus I\big)V^{+}_{i,i+1}(t') + \big(I - V_{i,i+1}(t')V^{+}_{i,i+1}(t')\big);$$

then

$$u_{i,i+1}(0) = \big(V_i(u \oplus I)V_i^{+} + (I - V_i V_i^{+})\big) \oplus I,$$
$$u_{i,i+1}(1) = \big(V_{i+1}(u \oplus I)V^{+}_{i+1} + (I - V_{i+1}V^{+}_{i+1})\big) \oplus I.$$

Using $u_{i,i+1}(t')$, we can construct a homotopy $s_1(t')$ in $\mathrm{QU}_{D_2(N)\delta,N^{100},C_2r,k}(Y)$ for some $C_2 \geq C_1$ depending only on C such that

$$s_1(0) = a, \quad s_1(1) = b.$$

We can also construct a homotopy $s_2(t')$ in $\mathrm{QU}_{D_3(N)\delta,N^{100},C_3r,k}(Y)$ for some $C_3 \geq C_1$ depending only on C, such that

$$s_2(0) = b \oplus I, \quad s_2(1) = c \oplus I.$$

Finally, we define $w(t')$ to be the homotopy obtained by combining the following homotopies:

(1) the linear homotopy between $(u_0 \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I)$ and $c'a((u_\infty \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I))$;

(2) $s_2'(1-t')a((u_\infty \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I))$;

(3) $s_1'(1-t')a((u_\infty \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I))$;

(4) the linear homotopy between $a'a((u_\infty \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I))$ and $(u_\infty \oplus I)\bigoplus_{i=1}^{\infty}(I \oplus I)$,

where a', b', c', s_1', s_2' are the $(D(N)\delta, N^{100}, C_0 r)$-inverses of a, b, c, s_1, s_2, respectively, in $\mathrm{QU}_{D_4(N)\delta,N^{100},C_4r,k}(Y)$ for some $C_4 \geq \max\{C_1, C_2, C_3\}$ depending only on C.

Therefore, we have that $w(t')$ is a homotopy connecting $\mathrm{Ad}((V_f, V_f^{+}))(u)\bigoplus_{i=1}^{\infty} I$ to $\mathrm{Ad}((V_g, V_g^{+}))(u)\bigoplus_{i=1}^{\infty} I$. ■

By Lemma 3.15, we have the following result.

Lemma 3.18. *Let X, Y, f, and g be as in Lemma 3.17. For any $0 < \delta < 1/100$, $N \geq 1$, $r \geq 0$, there exist $0 < \delta_1 < \delta$, $N_1 \geq N$, $0 \leq r_1 < r$ such that, for any $e \in \mathrm{QP}_{\delta_1,N,r_1,k}(X)$ $(k > 1)$, there exists a homotopy $e(t')(t' \in [0,1])$ in $\mathrm{QP}_{\delta,N_1,r,k}(Y)$ satisfying*

$$e(0) = \mathrm{Ad}\big((V_f, V_f^{+})\big)(e \oplus 0) \oplus (I \oplus 0),$$
$$e(1) = \mathrm{Ad}\big((V_g, V_g^{+})\big)(e \oplus 0) \oplus (I \oplus 0),$$

where δ_1 depends only on δ and N, N_1 depends only on N, and r_1 depends only on r, δ, N, C.

3.3. Controlled cutting and pasting

Definition 3.19 (Yu [46]). Let X be a proper metric space; let X_1 and X_2 be two subspaces. The triple $(X; X_1, X_2)$ is said to satisfy the *strong excision* condition if

(1) $X = X_1 \cup X_2$, X_i is a Borel subset, and $\mathrm{int}(X_i)$ is dense in X_i for $i = 1, 2$;

(2) there exists $r_0 > 0$, $C_0 > 0$ such that (i) for any $r' \leq r_0$, $\mathrm{bd}_{r'}(X_1) \cap \mathrm{bd}_{r'}(X_2) = \mathrm{bd}_{r'}(X_1 \cap X_2)$; (ii) for each $X' = X_1, X_2, X_1 \cap X_2$, and any $r' \leq r_0$, $\mathrm{bd}_{r'}(X')$ is strongly Lipschitz homotopy equivalent to X' with C_0 as the Lipschitz constant.

Let the triple $(X; X_1, X_2)$ be as above. Let $0 < \delta < 1/100$. For any $u \in \mathrm{QU}_{\delta,N,r,k}(X)$ with (δ, N, r)-inverse v, we take $uX_1 = \chi_{X_1} u \chi_{X_1}$ and the same for vX_1. Define

$$w_u = \begin{pmatrix} I & uX_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -vX_1 & I \end{pmatrix} \begin{pmatrix} I & uX_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix};$$

then

$$w_u^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & -uX_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ vX_1 & I \end{pmatrix} \begin{pmatrix} I & -uX_1 \\ 0 & I \end{pmatrix}.$$

We define a homomorphism

$$\partial_0 : \mathrm{QU}_{\delta,N,r,k}(X) \to \mathrm{QP}_{4N^4\delta,2N^6,6r,k}\left(\mathrm{bd}_{5r}(X_1) \cap \mathrm{bd}_{5r}(X_2)\right)$$

by

$$\partial_0(u) = \chi_{\mathrm{bd}_{5r}(X_1)\cap\mathrm{bd}_{5r}(X_2)} w_u (I \oplus 0) w_u^{-1} \chi_{\mathrm{bd}_{5r}(X_1)\cap\mathrm{bd}_{5r}(X_2)}.$$

Now we verify

$$\partial_0(u) \in \mathrm{QP}_{4N^4\delta,2N^6,6r,k}\left(\mathrm{bd}_{5r}(X_1) \cap \mathrm{bd}_{5r}(X_2)\right).$$

Firstly, $\|\partial_0(u)\|$ and $\|1 - \partial_0(u)\|$ are less than $2N^6$. Secondly, $\mathrm{prop}(\partial_0(u)) < 6r$. Finally, we estimate $\|(\partial_0(u))^2 - \partial_0(u)\|$. For convenience, we take $Y = \mathrm{bd}_{5r}(X_1) \cap \mathrm{bd}_{5r}(X_2)$:

$$\left\|\left(\partial_0(u)\right)^2 - \partial_0(u)\right\| = \left\|\chi_Y w_u (I \oplus 0) w_u^{-1} \chi_{X_1-Y} w_u (I \oplus 0) w_u^{-1} \chi_Y\right\|.$$

We now estimate $\|\chi_Y w_u (I \oplus 0) w_u^{-1} \chi_{X_1-Y}\|$. We have $\chi_{X_1} u \chi_{X_1-Y} = u\chi_{X_1-Y}$. Thus we can replace uX_1 by u in $w_u(I \oplus 0)w_u^{-1}$. Then

$$w_u(I \oplus 0)w_u^{-1} = \begin{pmatrix} (I - uv)uv + uv & (I - uv)u(I - vu) + u(I - vu) \\ (I - vu)v & (I - vu)^2, \end{pmatrix}.$$

Thus

$$\left\|\chi_Y w_u (I \oplus 0) w_u^{-1} \chi_{X_1-Y}\right\| = \left\|\chi_{Y\cap X_1}\left(\left(w_u(I \oplus 0)w_u^{-1}\right) - (I \oplus 0)\right)\chi_{X_1-Y}\right\| < 2N^2\delta.$$

Similarly,

$$\left\|\chi_{X_1-Y} w_u (I \oplus 0) w_u^{-1} \chi_Y\right\| < 2N^2\delta.$$

Assume that $r < r_0/5$, where r_0 is as in Definition 3.19. Let f be the proper Lipschitz map from $\mathrm{bd}_{5r}(X_1) \cap \mathrm{bd}_{5r}(X_2)$ to $X_1 \cap X_2$ realizing the strong Lipschitz homotopy equivalence in Definition 3.19. By Lemma 2.11, we have the pair (V_f, V_f^+) corresponding to $\{\varepsilon_m\}$, for which $\sup_m(\varepsilon_m) < r/10$.

We define the boundary map $\partial : \mathrm{QU}_{\delta,N,r,k}(X) \to \mathrm{GQP}_{4N^4\delta,2N^6,6C_0r,k}(X_1 \cap X_2)$ by

$$\partial(u) = \mathrm{Ad}\big((V_f, V_f^+)\big)\big(\partial_0(u)\big) - (I \oplus 0).$$

Then we consider the sequence

$$\mathrm{QU}_{\delta,N,r,k}(X_1) \oplus \mathrm{QU}_{\delta,N,r,k}(X_2) \xrightarrow{j} \mathrm{QU}_{\delta,N,r,k}(X) \xrightarrow{\partial} \mathrm{GQP}_{4N^4\delta,2N^6,6C_0r,k}(X_1 \cap X_2),$$

where $j(u_1 \oplus u_2) = (u_1 + \chi_{X-X_1}) \oplus (u_2 + \chi_{X-X_2})$, $r < r_0/5$.

Lemma 3.20. *Let $(X; X_1, X_2)$ be as in Definition 3.19 with r_0, C_0; then the above sequence is asymptotically exact in the following sense:*

1. *for any $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, there exist $0 < \delta_1 < \delta$, $N_1 \geq N$, $0 < r_1 < \min\{r, r_0/5\}$ such that $\partial j(u_1 \oplus u_2)$ is (δ, N_1, r)-equivalent to 0 for any $u_i \in \mathrm{QU}_{\delta_1,N,r_1,k}(X_i)$ $(i = 1, 2)$, where δ_1 depends only on δ and N, N_1 depends only on N, and r_1 depends only on δ, N, r;*
2. *for any $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, there exist $0 < \delta_2 < \delta$, $N_2 \geq N$, $0 < r_2 < \min\{r, r_0/5\}$ such that if u is an element in $\mathrm{QU}_{\delta_2,N,r_2,k}(X)$, for which $\partial(u)$ is (δ_2, N, r_2)-equivalent to 0 in $\mathrm{GQP}_{\delta_2,N,r_2,k}(X)$, then there exist $u_i \in \mathrm{QU}_{\delta,N_2,r,k}(X_i)$ $(i = 1, 2)$ such that $j(u_1 \oplus u_2)$ is (δ, N_2, r)-equivalent to u, where δ_2 depends only on δ and N, N_2 depends only on N, and r_2 depends only on δ, N, r, r_0, C_0.*

Proof. (1) follows from the definition of the boundary map and Lemma 3.13.

(2) By the strong homotopy invariance of QP, for any $0 < \delta' < \delta$, $N \geq 1$, $0 < r_2' < \min\{r, r_0/5\}$, there exist $\delta_2 < \delta'$, $N' > N$, $0 < r_2 < r_2'$ (δ_2 depends only on δ' and N, N' depends only on N, and r_2 depends only on $r_2', \delta', N, r_0, C_0$) such that, for any $u \in \mathrm{QU}_{\delta_2,N,r_2,k}(X)$ whose boundary $\partial(u)$ is (δ_2, N, r_2)-equivalent to 0, and $\partial_0(u)$ is (δ', N', r_2')-equivalent to 0. In view of Lemma 3.12, there exists an element y in $\mathrm{QU}_{\delta',C_1(N'),C_2(N',\delta')r_2',k}(\mathrm{bd}_{5r_2}(X_1) \cap \mathrm{bd}_{5r_2}(X_2))$ with $(\delta', C_1(N'), C_2(N', \delta')r_2')$-inverse y' such that

$$\big\|xw(I \oplus 0)w^{-1}x' - (I \oplus 0)\big\| < C_3(N')\delta',$$

where $x = y + \chi_{X-\mathrm{bd}_{5r_2}(X_1)\cap \mathrm{bd}_{5r_2}(X_2)}$, $x' = y' + \chi_{X-\mathrm{bd}_{5r_2}(X_1)\cap \mathrm{bd}_{5r_2}(X_2)}$, $w = w_{u\oplus I}$.

This implies that

$$\big\|xw(I \oplus 0) - (I \oplus 0)xw\big\| < C_4(N')\delta'.$$

Thus we have

$$xw = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \|b\| \leq C_4(N')\delta', \qquad \|c\| \leq C_4(N')\delta', \tag{3.9}$$

$$w^{-1}x' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \quad \|b'\| \leq C_4(N')\delta', \quad \|c'\| \leq C_4(N')\delta'. \tag{3.10}$$

Define

$$v_1 = a\chi_{\mathrm{bd}_{5r_2}}(X_1), \quad v_1' = \chi_{\mathrm{bd}_{5r_2}}(X_1)a'.$$

Equations (3.9) and (3.10) tell us that

$$v_1 \in \mathrm{QU}_{(C_4(N')+1)\delta', C_1(N')N_2^3, (C_2(N',\delta')+3)r_2', k}\left(\mathrm{bd}_{5r_2}(X_1)\right)$$

with inverse v_1'. Equations (3.9) and (3.10) together with the definition of w imply that

$$\left\| \chi_{X-\mathrm{bd}_{10r_2}(X_2)}\left(v_1'(u \oplus I - I)\right)\right\| < C_5(N')\delta', \tag{3.11}$$

$$\left\| \left(v_1'(u \oplus I - I)\right)\chi_{X-\mathrm{bd}_{10r_2}(X_2)}\right\| < C_5(N')\delta'. \tag{3.12}$$

Define

$$v_2 = \chi_{\mathrm{bd}_{10r_2}(X_2)}\left(v_1'(u \oplus I)\right)\chi_{\mathrm{bd}_{10r_2}(X_2)}, \quad v_2' = \chi_{\mathrm{bd}_{10r_2}(X_2)}\left((u' \oplus I)v_1\right)\chi_{\mathrm{bd}_{10r_2}(X_2)},$$

where u' is the (δ_2, N, r_2)-inverse of u.

Equations (3.11) and (3.12) tell us that $v_2 \in \mathrm{QU}_{C_6(N')\delta', C_7(N'), C_8(N',\delta')r_2', k}(\mathrm{bd}_{10r_2}(X_2))$ with quasi-inverse v_2'.

We require $0 < r_2 < r_0/10$. Let f_1 be the proper strong Lipschitz map from $\mathrm{bd}_{5r}(X_1)$ to X_1 realizing the strong Lipschitz homotopy equivalence. Let f_2 be the proper strong Lipschitz map from $\mathrm{bd}_{10r}(X_2)$ to X_2 realizing the strong Lipschitz homotopy equivalence. Define $u_i = \mathrm{Ad}((V_{f_i}, V_{f_i}^+))(v_i)$ for $i = 1, 2$, where the pair $(V_{f_i}, V_{f_i}^+)$ corresponds to $\{\varepsilon_k\}$, for which $\sup_k(\varepsilon_k) < r_2'$.

By (3.11) and (3.12), we have that $(v_1 + \chi_{X-\mathrm{bd}(5r_2)(X_1)}) \oplus (v_2 + \chi_{X-\mathrm{bd}(10r_2)(X_2)})$ is $(C_9(N')\delta', C_{10}(N'), C_{11}(N',\delta')r_2')$-equivalent to $u \oplus I$. Note that $C_j(N')$ depends only on N' for $j = 1, 3, 4, 5, 6, 7, 9, 10$ and $C_j(N_2, \delta')$ depends only on N', δ', C_0 for $j = 2, 8, 11$.

By Lemma 3.17, we can choose appropriate δ', N_2, and r_2' such that u_1 and u_2 satisfy the desired properties of Lemma 3.20 (2), where δ' depends only on δ and N, N_2 depends only on N, and r_2' depends only on r, δ, N, r_0, C_0. ∎

Corollary 3.21. *By Lemma 3.15 and Lemma 3.20, one has the following asymptotically exact sequence for* QU *when* $k > 1$*:*

$$\mathrm{QU}_{\delta,N,r,k}(X_1) \oplus \mathrm{QU}_{\delta,N,r,k}(X_2) \to \mathrm{QU}_{\delta,N,r,k}(X) \to \mathrm{QU}_{\delta,N,r,k-1}(X_1 \cap X_2).$$

4. Spaces with finite asymptotic dimension

In this section, we will recall some facts about spaces with finite asymptotic dimension and verify the L^p coarse Baum–Connes conjecture for spaces with finite asymptotic dimension.

Definition 4.1 (Gromov [13]). The *asymptotic dimension* of a metric space X is the smallest integer m such that, for any $r > 0$, there exists a uniformly bounded cover $C = \{U_i\}_{i \in I}$

of X, for which the r-multiplicity of C is at most $m+1$, i.e., no ball of radius r in the metric space intersects more than $m+1$ members of C. If no such m exists, we say X has infinite asymptotic dimension.

A finitely generated group can be viewed as a metric space with a left-invariant *word-length metric*. To be more precise, for a group Γ with a finite symmetric generating set S, for any $\gamma \in \Gamma$, we define its length

$$l_S(\gamma) := \min\{n : \gamma = s_1 \cdots s_n, s_i \in S\}.$$

The word-length metric d_S on Γ is defined by

$$d_S(\gamma_1, \gamma_2) := l_S(\gamma_1^{-1}\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. We remark that, for any two finite symmetric generating sets S_1, S_2 of Γ, (Γ, d_{S_1}) is quasi-isometric to (Γ, d_{S_2}).

Remark 4.2. Now we give some facts about asymptotic dimension.

(1) The concept of asymptotic dimension is a coarse geometric analogue of the covering dimension in topology.

(2) Hyperbolic groups have finite asymptotic dimension as a metric space with word-length metric [13, 37].

(3) The class of finitely generated groups with finite asymptotic dimension is hereditary [46, Proposition 6.2], i.e., if a finitely generated group Γ has finite asymptotic dimension as a metric space with word-length metric, then any finitely generated subgroup of Γ also has finite asymptotic dimension as a metric space with word-length metric.

(4) If Γ is a discrete subgroup of an almost connected Lie group, e.g., $\mathrm{SL}(n, \mathbb{Z})$, then Γ has finite asymptotic dimension.

(5) CAT(0) cube complexes have finite asymptotic dimension [43].

(6) Certain relative hyperbolic groups have finite asymptotic dimension [28].

(7) Certain Coxeter groups have finite asymptotic dimension [12].

(8) Mapping class groups have finite asymptotic dimension [4].

Construction 4.3. Let X be a proper metric space with asymptotic dimension m. By the definition of asymptotic dimension, there exists a sequence of covers C_k of X, for which there exists a sequence of positive numbers $R_k \to \infty$ such that

(1) $R_{k+1} > 4R_k$ for all k;

(2) $\text{diameter}(U) < R_k/4$ for all $U \in C_k$;

(3) the R_k-multiplicity of C_{k+1} is at most $m+1$, i.e., no ball with radius R_k intersects more than $m+1$ members of $C_k + 1$.

Let $C'_k = \{B(U, R_k) : U \in C_{k+1}\}$, where $B(U, R_k) = \{x \in X : d(x, U) < R_k\}$. Properties (1), (2), and (3) imply that $\{C'_k\}$ is an anti-Čech system for X.

Fix a positive integer n_0. For each $n > n_0$, let $r_n = \frac{R_n}{2R_{n_0+1}} - 4$. By property (1) of the sequence R_k, there exists $n_1 > n_0$ such that $r_n > 2$ if $n > n_1$ and there exists a sequence of nonnegative smooth functions $\{\chi_n\}_{n>n_1}$ on $[0, \infty)$ for which

(1) $\chi_n(t) = 1$ for all $0 \leq t \leq 2$, and $\chi_n(t) = 0$ for all $t \geq r_n$;

(2) there exists a sequence of positive numbers $\varepsilon_n \to 0$ satisfying $|\chi'_n(t)| < \varepsilon_n \leq 1$ for all $n > n_1$.

For each $U \in C_{n+1} (n > n_1)$, define

$$U' = \{V \in N_{C'_{n_0}} : V \in C'_{n_0}, U \cap V \neq \emptyset\}.$$

We define a map $G_n : N_{C'_{n_0}} \to N_{C'_n}$ by

$$G_n(x) = \sum_{U \in C_{n+1}} \frac{\chi_n\big(d(x, U')\big)}{\sum_{V \in C_{n+1}} \chi_n\big(d(x, V')\big)} B(U, R_n)$$

for all $x \in N_{C'_{n_0}}$.

Let $n > n_1$; we define a map $i_{n_0 n} : N_{C'_{n_0}} \to N_{C'_n}$ in such a way that, for each $V \in C_{n_0+1}$,

$$i_{n_0 n}\big(B(V, R_{n_0})\big) = B(U, R_n)$$

for some $U \in C_{n+1}$ satisfying $U \cap V \neq \emptyset$.

Let F_t be the linear homotopy between G_n and $i_{n_0 n}$, i.e., $F_t(x) = tG_n(x) + (1-t)i_{n_0 n}(x)$ for all $t \in [0, 1]$ and $x \in N_{C'_{n_0}}$.

By the above construction, we have the following important lemma.

Lemma 4.4 ([46, Lemma 6.3]). *Let X be a proper metric space with finite asymptotic dimension m, and let G_n, F_t, and $i_{n_0 n}$ be as above; then*

(1) *G_n is a proper Lipschitz map with a Lipschitz constant depending only on m;*

(2) *F_t is a strong Lipschitz homotopy between G_n and $i_{n_0 n}$ with a Lipschitz constant depending only on m;*

(3) *for any $\varepsilon > 0$, $R > 0$, there exists $K > 0$ such that $d(G_n(x), G_n(y)) < \varepsilon$ if $n > K$, $d(x, y) < R$.*

The following lemma plays a crucial role in the proof of Theorem 4.6. Its proof is based on the Eilenberg swindle argument and the controlled cutting and pasting exact sequence in Section 3.3.

Lemma 4.5. *Let X be a simplicial complex with finite dimension m and endowed with ℓ^1 metric. For any $k > m + 1$, $0 < \delta < 1/100$, $N \geq 1$, $r > 0$, there exist $0 < \delta_1 \leq \delta$, $N_1 \geq N$, $0 < r_1 < r$ such that every element u in $\mathrm{QU}_{\delta_1,N,r_1,k}(X)$ is (δ, N_1, r)-equivalent to I, where δ_1 depends only on δ and N, N_1 depends only on N, and r_1 depends only on r, δ, N.*

Proof. Let $X^{(n)}$ be the n-skeleton of X; we will prove our lemma for $X^{(n)}$ by induction on n.

When $n = 0$, we choose $r_1 = \min\{r, 2\}$. Let v be the (δ_1, N, r_1)-inverse of u. Then $\mathrm{prop}(u(t)) = \mathrm{prop}(v(t)) = 0$. For $t_0 \in [0, \infty)$, we define

$$u_{t_0}(t) = \begin{cases} I, & 0 \le t \le t_0; \\ u(t - t_0), & t_0 \le t < +\infty. \end{cases}$$

Similarly, we can define v_{t_0} for $t_0 \in [0, \infty)$. Thus v_{t_0} is the (δ_1, N, r_1)-inverse of u_{t_0}.

Define

$$E_X^{p,\infty} = \Big(\bigoplus_{k=0}^{\infty} E_X^p\Big) \oplus E_X^p.$$

Let $w_1(t')$ be the linear homotopy between $u \oplus (\bigoplus_{k=1}^{\infty} I) \oplus I$ and $u \oplus (\bigoplus_{k=1}^{\infty} u_k v_k) \oplus I$. Let $w_2(t') = ((\bigoplus_{k=0}^{\infty} u_k) \oplus I)(I \oplus (\bigoplus_{k=1}^{\infty} v_{k-t'}) \oplus I)$, where $t' \in [0, 1]$.

Let $T, T^* : E_X^{p,\infty} \to E_X^{p,\infty}$ be linear maps defined by

$$T\big((h_0, h_1, \ldots), h\big) = \big((0, h_0, h_1, \ldots), h\big),$$
$$T^*\big((h_0, h_1, \ldots), h\big) = \big((h_1, h_2, \ldots), h\big).$$

Thus

$$I \oplus \Big(\bigoplus_{k=1}^{\infty} v_{k-1}\Big) \oplus I = T\Big(\bigoplus_{k=0}^{\infty} (v_k - I) \oplus 0\Big) T^* + I.$$

Hence there exists a homotopy $s_1(t')(t' \in [0, 1])$ connecting $I \oplus (\bigoplus_{k=1}^{\infty} v_{k-1}) \oplus I$ and $(\bigoplus_{k=0}^{\infty} v_k) \oplus I$.

Let $s_2(t')(t' \in [0, 1])$ be the linear homotopy between $(\bigoplus_{k=0}^{\infty} u_k v_k) \oplus I$ and $(\bigoplus_{k=0}^{\infty} I) \oplus I$.

Define

$$w(t') = \begin{cases} w_1(4t'), & 0 \le t' \le 1/4; \\ w_2(4t' - 1), & 1/4 \le t' \le 1/2; \\ \big((\bigoplus_{k=0}^{\infty} u_k) \oplus I\big) s_1(4t' - 2), & 1/2 \le t' \le 3/4; \\ s_2(4t' - 3), & 3/4 \le t' \le 1. \end{cases}$$

It is not difficult to see $w(t')$ is the homotopy connecting $u \oplus I$ to I; thus we can choose appropriate δ_1 and N_1 satisfying the lemma.

Assume by induction that the lemma holds for $n = m - 1$; next we will prove the lemma holds for $n = m$. For each simplex $\triangle$ of dimension m in X, we let

$$\triangle_1 = \big\{x \in \triangle : d\big(x, c(\triangle)\big) \le 1/100\big\},$$
$$\triangle_2 = \big\{x \in \triangle : d\big(x, c(\triangle)\big) \ge 1/100\big\},$$

where $c(\triangle)$ is the center of $\triangle$.

Let

$$X_1 = \bigcup_{\triangle:\text{simplex of dimension } m \text{ in X}} \triangle_1,$$

$$X_2 = \bigcup_{\triangle:\text{simplex of dimension } m \text{ in X}} \triangle_2.$$

Notice that

(1) X_1 is strongly Lipschitz homotopy equivalent to

$$\{c(\triangle) : \triangle \text{ is } m\text{-dimensional simplex in X}\};$$

(2) X_2 is strongly Lipschitz homotopy equivalent to $X^{(m-1)}$;

(3) $X^{(m)} = X_1 \cup X_2$ and $X_1 \cap X_2$ is the disjoint union of the boundaries of all m-dimensional $\triangle_1$ in $X^{(m)}$.

Properties (1) and (2) together with strongly Lipschitz homotopy invariance of QU and the induction hypothesis imply that our lemma holds for X_1 and X_2.

By strongly Lipschitz homotopy invariance of QU and the controlled cutting and pasting exact sequence, we also know that our lemma holds for $X_1 \cap X_2$.

Obviously, $(X^{(m)}, X_1, X_2)$ satisfies the strong excision condition; thus we can complete our induction process by using the controlled cutting and pasting exact sequence and the controlled five lemma. ■

Now we are ready to prove the main theorem of this section.

Theorem 4.6. *For any $p \in [1, \infty)$, the L^p coarse Baum–Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

Proof. Let X be a proper metric space with asymptotic dimension m. By Theorem 2.14, it is enough to prove that

$$\lim_{n\to\infty} K_i\big(B^p_{L,0}(N_{C'_n})\big) = 0,$$

where C'_n is as in Construction 4.3.

Lemmas 3.8, 3.9, and 3.15 tell us that any element $[q]$ in $K_i(B^p_{L,0}(N_{C'_{n_0}}))$ can be represented as an element u in $\mathrm{QU}_{\delta_1,N,r,k}(N_{C'_{n_0}})$ for some N, r and $k > m+1$, where δ_1 is as in Lemma 4.5 for some $0 < \delta < 1/100$. Let

$$u_n = \mathrm{Ad}\big((V_{G_n}, V^+_{G_n})\big)(u),$$

where G_n is as in Lemma 4.4 and $\mathrm{Ad}((V_{G_n}, V^+_{G_n}))$ is defined by $\{\varepsilon_m\}$, for which $\sup(\varepsilon_m) < r_1/10$, where r_1 is as in Lemma 4.5.

By Lemma 4.4 (3), there exists $K > 0$ such that

$$\mathrm{prop}(u_n) < r_1, \quad \text{for } n > K.$$

Since the asymptotic dimension of X is m, we have $\dim(N_{C'_n}) \le m$ for all n. By Lemma 4.5, we have that u_n is (δ, N_1, r)-equivalent to I in $\mathrm{QU}_{\delta,N_1,r,k}(N_{C'_n})$ for $n > K$.

By Lemma 4.4(2), strongly Lipschitz homotopy invariance of QU, and Lemmas 3.8(2) and 3.9(2), we have that $\mathrm{Ad}((V_{i_{n_0}n}, V^+_{i_{n_0}n}))(u)$ and u_n correspond to the same element in $K_i(B^p_{L,0}(N_{C'_n}))$.

Thus $[q] = 0$ in $\lim_{n\to\infty} K_i(B^p_{L,0}(N_{C'_n}))$. ■

5. K-theory of L^p Roe algebras

In this section, we shall use the dual L^p K-homology as a bridge to prove that the L^p K-homology is independent of p. Combining Theorem 4.6, we obtain that the K-theory of the L^p Roe algebra does not depend on $p \in (1,\infty)$ for spaces with finite asymptotic dimension.

5.1. Dual L^p localization algebra and dual L^p K-homology

Let $p \in (1,\infty)$; let Z and Z' be countable discrete measure spaces. Then $\ell^p(Z)$ has a natural Schauder basis $\{e_i\}_{i\in Z}$, where $e_i(z) = 1$ for $i = z$ and $e_i(z) = 0$ for $i \neq z$. Similarly, $\ell^p(Z')$ has a natural Schauder basis $\{e'_i\}_{i\in Z'}$. Let T be a bounded operator from $\ell^p(Z)$ to $\ell^p(Z')$; T can be considered as a countably dimensional matrix under the Schauder bases $\{e_i\}$ and $\{e'_i\}$. We can define T^* as the transpose of the matrix of T. We call T a *dual operator*, if T^* is a bounded operator from $\ell^p(Z')$ to $\ell^p(Z)$ under the Schauder bases $\{e'_i\}$ and $\{e_i\}$. We call T a *compact dual operator* if T and T^* are compact operators from $\ell^p(Z)$ to $\ell^p(Z')$ and from $\ell^p(Z')$ to $\ell^p(Z)$, respectively. We define the *maximal norm* of dual operator T by $\|T\|_{\max} := \max\{\|T\|, \|T^*\|\}$.

For $p \in (1,\infty)$, let $\mathcal{B}^*(\ell^p(Z), \ell^p(Z'))$ be the Banach space of all dual operators from $\ell^p(Z)$ to $\ell^p(Z')$ with maximal norm. Let $\mathcal{K}^*(\ell^p(Z), \ell^p(Z'))$ be the Banach space of all compact dual operators from $\ell^p(Z)$ to $\ell^p(Z')$. It is easy to see that $\mathcal{K}^*(\ell^p(Z))$ is a closed ideal of $\mathcal{B}^*(\ell^p(Z))$.

Remark 5.1. For $p \in (1,\infty)$, let q be the dual number of p, i.e., $1/p + 1/q = 1$. If T is a dual operator acting on $\ell^p(Z)$, then T can be considered as a bounded operator acting on $\ell^q(Z)$ and $\|T\|_{\ell^q(Z)} = \|T^*\|_{\ell^p(Z)}$. This is why we call such T a dual operator. Note that $\mathcal{B}^*(\ell^p(Z)) = \mathcal{B}^*(\ell^q(Z))$ for $p, q \in (1,\infty)$ and $1/p + 1/q = 1$.

Lemma 5.2. *Let $p \in (1,\infty)$ and let Z be a countable discrete measure space. If one fixes a bijection between Z and $\mathbb{N}$, then $\ell^p(Z)$ has a natural Schauder basis $\{e_i\}_{i\in\mathbb{N}}$. For any $K \in \mathcal{K}^*(\ell^p(Z))$, one has*

$$\lim_{n\to\infty} F_n K F_n = K$$

in $\mathcal{K}^*(\ell^p(Z))$, where F_n is the coordinate projection from $\ell^p(Z)$ to the subspace generated by $e_1, \ldots, e_n$.

Proof. We just need to prove $\lim_{n\to\infty} \|F_n K F_n - K\|_{\max} = 0$, i.e.,

$$\lim_{n\to\infty} \|F_n K F_n - K\|_{l^p(Z)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|F_n^* K^* F_n^* - K^*\|_{\ell^p(Z)} = 0.$$

These are true by [33, Proposition 1.8]. ■

This lemma is false for $p=1$; N. C. Phillips constructed a rank one operator without this property in [33].

Corollary 5.3. *Let $p \in (1,\infty)$ and let Z be a countable discrete measure space; then $K_1(\mathcal{K}^*(\ell^p(Z))) = 0$ and $K_0(\mathcal{K}^*(\ell^p(Z))) = \mathbb{Z}$ generated by a rank one idempotent.*

Proof. Lemma 5.2 implies that $\mathcal{K}^*(\ell^p(Z))$ can be represented as the direct limit of matrix algebras. By the continuous property of K-group, we complete the proof. ∎

Let X be a proper metric space, $p \in (1,\infty)$. Recall that an L^p-X-module is an L^p-space $E_X^p = \ell^p(Z_X) \otimes \ell^p = \ell^p(Z_X, \ell^p)$ equipped with a natural pointwise multiplication action of $C_0(X)$ by restricting to Z_X, where Z_X is a countable dense subset in X. This action can naturally induce a morphism from $C_0(X)$ to $\mathcal{B}^*(E_X^p)$.

Definition 5.4. Let X, Y be proper metric spaces and $T \in \mathcal{B}^*(E_X^p, E_Y^p)$. The *support* of T, denoted by $\operatorname{supp}(T)$, consists of all points $(x,y) \in X \times Y$ such that $\chi_V T \chi_U \neq 0$ for all open neighborhoods U of x and V of y.

We remark that $\operatorname{supp}(T)$ has the same properties like in Remark 2.3.

Definition 5.5. Let X be a proper metric space and let T be an element in $\mathcal{B}^*(E_X^p)$.

(1) The *propagation* of T is defined to be $\operatorname{prop}(T) := \sup\{d(x,y) : (x,y) \in \operatorname{supp}(T)\}$;

(2) T is said to be a *locally compact dual operator* if $\chi_K T$ and $T\chi_K$ are in $\mathcal{K}^*(E_X^p)$ for any compact subset K in X.

Definition 5.6. The *dual L^p Roe algebra* of E_X^p, denoted by $B^{p,*}(E_X^p)$, is defined to be the maximal-norm closure of the algebra of all locally compact dual operators acting on E_X^p with finite propagations.

Let X, Y be two proper metric spaces and let f be a continuous coarse map from X to Y. Let V_f and V_f^+ be an isometric dual operator and a contractive dual operator, respectively, constructed in Lemma 2.7. Thus we have the following lemma.

Lemma 5.7. *Let f, E_X^p, and E_Y^p be as above. Then the pair (V_f, V_f^+) gives rise to a homomorphism $\operatorname{ad}((V_f, V_f^+)) : B^{p,*}(E_X^p) \to B^{p,*}(E_Y^p)$ defined by*

$$\operatorname{ad}\big((V_f, V_f^+)\big)(T) = V_f T V_f^+$$

for all $T \in B^{p,*}(E_X^p)$.

Moreover, the map $\operatorname{ad}((V_f, V_f^+))_*$ induced by $\operatorname{ad}((V_f, V_f^+))$ on K-theory depends only on f and not on the choice of the pair (V_f, V_f^+).

Proof. The proof of this lemma is the same as the proof of Lemma 2.8. ∎

Corollary 5.8. *For different L^p-X-modules E_X^p and $E_X'^p$, the algebra $B^{p,*}(E_X^p)$ is non-canonically isomorphic to $B^{p,*}(E_X'^p)$, and $K_*(B^{p,*}(E_X^p))$ is canonically isomorphic to $K_*(B^{p,*}(E_X'^p))$.*

For convenience, we replace $B^{p,*}(E_X^p)$ by $B^{p,*}(X)$ representing the dual L^p Roe algebra of X.

Definition 5.9. Let X be a proper metric space. The *dual L^p localization algebra* of X, denoted by $B_L^{p,*}(X)$, is defined to be the norm closure of the algebra of all bounded and uniformly norm-continuous functions f from $[0,\infty)$ to $B^{p,*}(X)$ such that

$$\operatorname{prop}\big(f(t)\big) \text{ is uniformly finite and } \operatorname{prop}\big(f(t)\big) \to 0 \quad \text{as } t \to \infty.$$

The *propagation* of f is defined to be $\sup\{\operatorname{prop}(f(t)) : t \in [0,\infty)\}$.

We have the following lemma for the dual L^p localization algebra just like Lemma 2.11.

Lemma 5.10. *Let X, Y be two proper metric spaces, f a uniformly continuous coarse map from X to Y, and $\{\varepsilon_k\}_k$ a sequence of positive numbers such that $\varepsilon_k \to 0$ as $k \to \infty$; then the pair $(V_f(t), V_f^+(t))$ constructed in Lemma 2.11 induces a homomorphism $\operatorname{Ad}((V_f, V_f^+))$ from $B_L^{p,*}(X)$ to $B_L^{p,*}(Y) \otimes M_2(\mathbb{C})$ defined by*

$$\operatorname{Ad}\big((V_f, V_f^+)\big)(u)(t) = V_f(t)\big(u(t) \oplus 0\big)V_f^+(t)$$

for any $u \in B_L^{p,*}(X)$ and $t \in [0,\infty)$ such that

$$\operatorname{prop}\big(\operatorname{Ad}\big((V_f, V_f^+)\big)(u)(t)\big) \leq \sup_{(x,y)\in \operatorname{supp}(u(t))} \big\{d\big(f(x), f(y)\big)\big\} + 2\varepsilon_k + 2\varepsilon_{k+1}$$

for all $t \in [k, k+1]$.

Moreover, the map $\operatorname{Ad}((V_f, V_f^+))_*$ induced by $\operatorname{Ad}((V_f, V_f^+))$ on K-theory depends only on f and not on the choice of the pairs (V_k, V_k^+) in the construction of $V_f(t)$ and $V_f^+(t)$.

Proof. The proof of this lemma is similar to the proof of Lemma 2.11. ∎

Definition 5.11. The ith *dual L^p K-homology* is defined to be $K_i(B_L^{p,*}(X))$.

5.2. Strongly Lipschitz homotopy invariance of (dual) L^p K-homology

In this section, we will prove that (dual) L^p K-homology is strongly Lipschitz homotopy invariant. In the following, we just discuss the case of dual L^p K-homology; similarly, we can obtain the same result for L^p K-homology.

Lemma 5.12. *Let f and g be two Lipschitz maps from X to Y and let $F(t,x)$ be a strongly Lipschitz homotopy connecting f and g; then*

$$\operatorname{Ad}\big((V_f, V_f^+)\big)_* = \operatorname{Ad}\big((V_g, V_g^+)\big)_* : K_*\big(B_L^{p,*}(X)\big) \to K_*\big(B_L^{p,*}(Y)\big).$$

Proof. We just prove this lemma for K_1 group, and, by suspension, we can obtain the same result for K_0 group. Choose $\{t_{i,j}\}_{i\geq 0, j\geq 0} \subseteq [0,1]$ satisfying

(1) $t_{0,j} = 0, t_{i,j+1} \le t_{i,j}, t_{i+1,j} \ge t_{i,j}$;

(2) there exists $N_j \to \infty$ such that $t_{i,j} = 1$ for all $i \ge N_j$ and $N_{j+1} \ge N_j$;

(3) $d(F(t_{i+1,j}, x), F(t_{i,j}, x)) < \varepsilon_j = 1/(j+1), d(F(t_{i,j+1}, x), F(t_{i,j}, x)) < \varepsilon_j$ for all $x \in X$.

Let $f_{i,j}(x) = F(t_{i,j}, x)$; by Lemma 2.7, there exist an isometric operator $V_{f_{i,j}} : E_X^p \to E_Y^p$ and a contractive operator $V_{f_{i,j}}^+ : E_Y^p \to E_X^p$ with $V_{f_{i,j}}^+ V_{f_{i,j}} = I$ such that

$$\operatorname{supp}(V_{f_{i,j}}) \subseteq \big\{(x, y) \in X \times Y : d\big(f_{i,j}(x), y\big) < 1/(1+i+j)\big\},$$
$$\operatorname{supp}(V_{f_{i,j}}^+) \subseteq \big\{(y, x) \in Y \times X : d\big(f_{i,j}(x), y\big) < 1/(1+i+j)\big\}.$$

For each $i > 0$, define a family of operators $V_i(t)(t \in [0,\infty))$ from $E_X^p \oplus E_X^p$ to $E_Y^p \oplus E_Y^p$ and a family of operators $V_i^+(t)(t \in [0,\infty))$ from $E_Y^p \oplus E_Y^p$ to $E_X^p \oplus E_X^p$ by

$$V_i(t) = R(t-j)(V_{f_{i,j}} \oplus V_{f_{i,j+1}})R^*(t-j), \quad t \in [j, j+1],$$
$$V_i^+(t) = R(t-j)(V_{f_{i,j}}^+ \oplus V_{f_{i,j+1}}^+)R^*(t-j), \quad t \in [j, j+1],$$

where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For any $[u] \in K_1(B_L^{p,*}(X))$, consider

$$u_0(t) = \operatorname{Ad}\big((V_f, V_f^+)\big)(u) = V_f(t)\big(u(t) \oplus I\big)V_f^+(t) + \big(I - V_f(t)V_f^+(t)\big),$$
$$u_\infty(t) = \operatorname{Ad}\big((V_g, V_g^+)\big)(u) = V_g(t)\big(u(t) \oplus I\big)V_g^+(t) + \big(I - V_g(t)V_g^+(t)\big),$$
$$u_i(t) = \operatorname{Ad}\big((V_i, V_i^+)\big)(u) = V_i(t)\big(u(t) \oplus I\big)V_i^+(t) + \big(I - V_i(t)V_i^+(t)\big).$$

For each i, define n_i by

$$n_i = \begin{cases} \max\{j : i \ge N_j\}, & \{j : i \ge N_j\} \ne \emptyset; \\ 0, & \{j : i \ge N_j\} = \emptyset. \end{cases}$$

We can choose $V_{f_{i,j}}$ in such a way that $u_i(t) = u_\infty$, where $t \le n_i$.

Define

$$w_i(t) = u_i(t)\big(u_\infty^{-1}(t)\big).$$

Consider

$$a = \bigoplus_{i=0}^{\infty}(w_i \oplus I), \quad b = \bigoplus_{i=0}^{\infty}(w_{i+1} \oplus I), \quad c = (I \oplus I)\bigoplus_{i=1}^{\infty}(w_i \oplus I).$$

By the construction of $\{t_{i,j}\}$, we know that $a, b, c \in (B_L^{p,*}(X) \otimes M_2(\mathbb{C}))^+$. It is not difficult to see that a is equivalent to b and b is equivalent to c in $K_1(B_L^{p,*}(X))$. Thus $u_0 u_\infty^{-1} \oplus_{i\ge1} I$ is equivalent to $\oplus_{i\ge0} I$ in $K_1(B_L^{p,*}(X))$. This means that $\operatorname{Ad}((V_f, V_f^+))_* = \operatorname{Ad}((V_g, V_g^+))_*$. ∎

Corollary 5.13. *If X is strongly Lipschitz homotopy equivalent to Y, then they have the same (dual) L^p K-homology.*

5.3. Cutting and pasting of the (dual) L^p K-homology

Let X be a simplicial complex endowed with the ℓ^1-metric, and let X_1 be a simplicial subcomplex of X. For $p \in (1, \infty)$, define $B_L^{p,*}(X_1; X)$ to be the closed subalgebra of $B_L^{p,*}(X)$ generated by all elements f such that there exists $c_t > 0$ satisfying $\lim_{t\to\infty} c_t = 0$ and $\mathrm{supp}(f(t)) \subset \{(x, y) \in X \times X : d((x, y), X_1 \times X_1) \leq c_t\}$ for all $t \in [0, \infty)$.

Lemma 5.14. *The inclusion homomorphism i from $B_L^{p,*}(X_1)$ to $B_L^{p,*}(X_1; X)$ induces an isomorphism from $K_*(B_L^{p,*}(X_1))$ to $K_*(B_L^{p,*}(X_1; X))$.*

Proof. For any $\varepsilon > 0$, let $B_\varepsilon(X_1) = \{x \in X : d(x, X_1) \leq \varepsilon\}$. There exists a small $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(X_1)$ is strongly Lipschitz homotopy equivalent to X_1. Any element in $K_1(B_L^{p,*}(X_1; X))$ can be represented by an invertible element $a \in (B_L^{p,*}(X_1; X))^+$ such that $a = a' + I$ and there exists $c_t > 0$ satisfying $\lim_{t\to\infty} c_t = 0$ and

$$\mathrm{supp}\left(a'(t)\right) \subset \left\{(x, y) \in X \times X : d\left((x, y), X_1 \times X_1\right) \leq c_t\right\}.$$

The uniform continuity of $a(t)$ implies that $a(t + st_0)(s \in [0, 1])$ is norm continuous in s for all $t_0 > 0$. Thus $[a(t)]$ is equivalent to $[a(t + st_0)]$ in $K_1(B_L^{p,*}(X_1; X))$ for any t_0. We can choose t_0 large enough so that $\mathrm{supp}(a'(t + t_0)) \subset B_{\varepsilon_0}(X_1) \times B_{\varepsilon_0}(X_1)$ for all t. By Lemma 5.12, we know that i_* is surjective.

A similar argument can be used to show that i_* is injective. The case for K_0 can be similarly dealt with by a suspension argument. ■

Lemma 5.15. *Let X be a simplicial complex endowed with the ℓ^1-metric and let X_1, X_2 be its two simplicial subcomplexes. One has the following six-term exact sequence:*

$$\begin{array}{ccccc}
K_0\left(B_L^{p,*}(X_1 \cap X_2)\right) & \longrightarrow & K_0\left(B_L^{p,*}(X_1)\right) \oplus K_0\left(B_L^{p,*}(X_2)\right) & \longrightarrow & K_0\left(B_L^{p,*}(X_1 \cup X_2)\right) \\
\uparrow & & & & \downarrow \\
K_1\left(B_L^{p,*}(X_1 \cup X_2)\right) & \longleftarrow & K_1\left(B_L^{p,*}(X_1)\right) \oplus K_0\left(B_L^{p,*}(X_2)\right) & \longleftarrow & K_1\left(B_L^{p,*}(X_1 \cap X_2)\right)
\end{array}$$

Proof. Let $Y = X_1 \cup X_2$; observe that $B_L^{p,*}(X_1; Y)$ and $B_L^{p,*}(X_2; Y)$ are ideals of $B_L^{p,*}(Y)$ such that $B_L^{p,*}(X_1; Y) + B_L^{p,*}(X_2; Y) = B_L^{p,*}(Y)$, and $B_L^{p,*}(X_1; Y) \cap B_L^{p,*}(X_2; Y) = B_L^{p,*}(X_1 \cap X_2; Y)$ since (X_1, X_2) is the strong excision pair of Y (Definition 3.19). Then by the Mayer–Vietoris sequence for K-theory of Banach algebras and Lemma 5.14, we can obtain this lemma. ■

Remark 5.16. By a similar argument as above, we have the following six-term exact sequence for L^p localization algebra:

$$\begin{array}{ccccc}
K_0\left(B_L^{p}(X_1 \cap X_2)\right) & \longrightarrow & K_0\left(B_L^{p}(X_1)\right) \oplus K_0\left(B_L^{p}(X_2)\right) & \longrightarrow & K_0\left(B_L^{p}(X_1 \cup X_2)\right) \\
\uparrow & & & & \downarrow \\
K_1\left(B_L^{p}(X_1 \cup X_2)\right) & \longleftarrow & K_1\left(B_L^{p}(X_1)\right) \oplus K_0\left(B_L^{p}(X_2)\right) & \longleftarrow & K_1\left(B_L^{p}(X_1 \cap X_2)\right)
\end{array}$$

5.4. Main result and proof

Let X be a finite dimensional simplicial complex endowed with ℓ^1-metric. Recall that E_X^p is the L^p-X-module and $\mathcal{B}^*(E_X^p)$ is the Banach algebra of all dual operators on E_X^p for $p \in (1,\infty)$. Every $T \in \mathcal{B}^*(E_X^p)$ can be viewed as an element in $B(E_X^p)$. Thus the inclusion map induces a contractive homomorphism

$$\phi : B_L^{p,*}(X) \to B_L^p(X).$$

Next we use the Riesz–Thorin interpolation theorem to build a connection between the dual L^p localization algebra and the localization C^*-algebra. Firstly, let us recall this interpolation theorem.

Lemma 5.17 (Riesz–Thorin). *Let (X,μ) and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}},$$

for all simple functions f on X. Then for all $0 < \theta < 1$ one has

$$\|T(f)\|_{L^{q'}} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^{p'}}$$

for all simple functions f on X, where $1/p' = (1-\theta)/p_0 + \theta/p_1$ and $1/q' = (1-\theta)/q_0 + \theta/q_1$.

By density, T has a unique extension as a bounded operator from $L^{p'}(X,\mu)$ to $L^{q'}(Y,\nu)$.

For any $p \in (1,\infty)$, let q be the dual number of p, i.e., $1/p + 1/q = 1$. Let $p_0 = q_0 = p$, $p_1 = q_1 = q$, and $\theta = 1/2$ in the above. By the Riesz–Thorin interpolation theorem, we have that each element $T \in \mathcal{B}^*(E_X^p)$ can be considered as an element in $B(E_X^2)$. This correspondence induces a contractive homomorphism

$$\psi : B_L^{p,*}(X) \to C_L^*(X),$$

where $C_L^*(X)$ is the localization C^*-algebra of X.

Proposition 5.18. *Let X be a finite dimensional simplicial complex endowed with ℓ^1-metric; then for any $p \in (1,\infty)$, ψ induces an isomorphism between $K_*(B_L^{p,*}(X))$ and $K_*(C_L^*(X))$.*

Proof. Let $X^{(n)}$ be the n-skeleton of X; we shall prove this theorem for $X^{(n)}$ by induction on n.

When $n = 0$, $K_*(B_L^{p,*}(X^{(0)}))$ equals to the direct product of $K_*(\mathcal{K}^*(\ell^p))$ and $K_*(C_L^*(X^{(0)}))$ equals to the direct product of $K_*(\mathcal{K}^*(l^2))$ using the fact that the algebra of all bounded and uniformly continuous functions from $[0,\infty)$ to a Banach algebra has the same K-theory as this Banach algebra. Then by Corollary 5.3, ψ_* is an isomorphism in this case.

Assume by induction that the theorem holds when $n = m - 1$. Next we shall prove the theorem holds when $n = m$. For each simplex $\triangle$ of dimension m in X, we let

$$\triangle_1 = \{x \in \triangle : d(x, c(\triangle)) \le 1/100\}, \quad \triangle_2 = \{x \in \triangle : d(x, c(\triangle)) \ge 1/100\},$$

where $c(\triangle)$ is the center of $\triangle$.

Let

$$X_1 = \bigcup_{\triangle:\text{simplex of dimension } m \text{ in X}} \triangle_1,$$

$$X_2 = \bigcup_{\triangle:\text{simplex of dimension } m \text{ in X}} \triangle_2.$$

Notice that

(1) X_1 is strongly Lipschitz homotopy equivalent to

$$\{c(\triangle) : \triangle \text{ is } m\text{-dimensional simplex in X}\};$$

(2) X_2 is strongly Lipschitz homotopy equivalent to $X^{(m-1)}$;

(3) $X^{(m)} = X_1 \cup X_2$ and $X_1 \cap X_2$ is the disjoint union of the boundaries of all m-dimensional $\triangle_1$ in $X^{(m)}$.

Properties (1) and (2) together with the strongly Lipschitz homotopy invariance of the dual L^p-K-homology and the induction hypothesis imply that the theorem holds for X_1 and X_2.

By the strongly Lipschitz homotopy invariance of $K_*(B_L^{p,*}(X))$, $K_*(C_L^*(X))$ and the cutting and pasting exact sequence, we also know that our lemma holds for $X_1 \cap X_2$.

Thus we can complete our induction process by using the cutting and pasting exact sequence and the five lemma. ∎

Using a similar argument for ϕ, we have the following proposition.

Proposition 5.19. *Let X be a finite dimensional simplicial complex endowed with the ℓ^1-metric; then for any $p \in (1, \infty)$, ϕ induces an isomorphism between $K_*(B_L^{p,*}(X))$ and $K_*(B_L^p(X))$.*

By Propositions 5.18 and 5.19, we obtain that the K-theory for L^p localization algebra is independent of p for a finite dimensional simplicial complex. This gives a partial answer to Question 26 in [9] proposed by Chung and Nowak.

Proposition 5.20. *Let X be a finite dimensional simplicial complex endowed with the ℓ^1-metric; $K_*(B_L^p(X))$ does not depend on $p \in (1, \infty)$.*

Furthermore, we have the following p-independency of K-theory for L^p Roe algebra.

Corollary 5.21. *Let X be a proper metric space; assume that there exists an anti-Čech system $\{C_k\}_k$ for X such that N_{C_k} is a finite dimensional simplicial complex for all k. Then if, for all $p \in (1, \infty)$, the L^p coarse Baum–Connes conjecture is true for X, one has that $K_*(B^p(X))$ does not depend on p.*

By Theorem 4.6, we have the following theorem.

Theorem 5.22. *Let X be a proper metric space. If X has finite asymptotic dimension, then $K_*(B^p(X))$ does not depend on p for $p \in (1, \infty)$.*

6. Open problems

In this last section, we list several interesting open problems.

Question 6.1. There are several versions of L^p K-homology. Are they all the same?

There are many different versions of K-homology:

(1) Kasparov's K-homology [21];

(2) K-theory of dual algebra by Paschke [32];

(3) K-theory of localization algebra of Guoliang Yu [34, 45];

(4) localization K-homology by Xiaoman Chen and Qin Wang [5];

(5) E-theory by Connes and Higson [10].

In the L^2 case, all the above concepts are the same. In this paper, we have seen that the L^p counterpart of all the above notions may be equivalent for finite dimensional simplicial complex, but we are not very optimistic that they are equivalent for general topological spaces. To prove the equivalence, we need some deep theorems, say the Voiculescu theorem [40] and the Kasparov technical lemma [22], for L^p spaces.

Question 6.2. Is it possible to prove that the K-theory of L^p Roe algebras is independent of p without using the coarse Baum–Connes conjecture?

Up to now, all the results about the p independence of the K-theory of the group algebras, crossed products, and Roe algebras rely on the Baum–Connes conjecture or the coarse Baum–Connes conjecture since the K-homology sides are easier to maneuver. A more direct approach without using the (coarse) Baum–Connes conjecture would shed some light on a Banach algebra approach to the (coarse) Baum–Connes conjecture. For example, if we know certain groups admitting proper isometric actions on L^p-spaces and the K-theory of their L^p group algebras does not depend on p, by the result of Kasparov and Yu [23], we can verify the Baum–Connes conjecture for these groups.

Question 6.3. Can we develop an L^p version of Dirac–dual-Dirac method for the L^p Baum–Connes conjecture for amenable groupoids?

In [39], Tu that showed the Baum–Connes conjecture is true for amenable groupoids or, more generally, a-T-menable groupoids. For a space with finite asymptotic dimension or, more generally, finite decomposition complexity, the coarse groupoid is amenable. For a dynamical system with finite dynamical complexity, the corresponding transformation groupoid is also amenable [17]. It would be great if we can modify Tu's method to deal with L^p groupoid algebras and give a unified proof for Chung's result on L^p crossed products and the results in this paper on L^p Roe algebras.

Question 6.4. Does the L^p coarse Baum–Connes conjecture hold for spaces coarsely embeddable into Hilbert spaces?

Recently, Shan and Wang verified the L^p coarse Novikov conjecture for spaces coarsely embeddable into simply connected nonpositively curved manifolds [38]. The key ingredient in the proof is the L^p version of Yu's twisted Roe algebra technique [47]. Shan and Wang's theorem is the first positive result on the injectivity of the L^p coarse Baum–Connes conjecture using Yu's technique. Can we also use the L^p version of Yu's technique to give a surjective argument when the space is coarsely embeddable into a Hilbert space? More generally, can we prove the L^p coarse Baum–Connes conjecture for spaces coarsely embeddable into L^p spaces?

Question 6.5. What will happen if we use $L^p(X, \nu)$ or a general L^p-space E as an L^p-X-module to define L^p Roe algebra and L^p localization algebra?

The proof of Lemma 2.7 does not work for this broader definition. Thus we need to find a new way to construct the homomorphism between L^p Roe algebras covering the map between the underlying spaces.

Question 6.6. Are there any topological and geometric implications of the L^p (coarse) Baum–Connes conjecture?

For example, does it imply the Gromov conjecture [14] that uniformly contractible manifolds with bounded geometry admit no uniform positive scalar curvature?

Question 6.7. Are there any counter-examples for the injectivity of the L^p coarse Baum–Connes conjecture?

In [18, 41], Higson–Lafforgue–Skandalis and Willett–Yu showed that Magulis-type expanders and expanders with large girth are counter-examples for the surjectivity of the coarse Baum–Connes conjecture. In [9], Chung and Nowak showed that Margulis-type expanders are still a counterexample for the L^p coarse Baum–Connes conjecture. However, the existence of an injectivity counterexample of the L^p coarse Baum–Connes conjecture is still open. In [46], Guoliang Yu gave a counterexample of the injectivity of the coarse Baum–Connes conjecture. The proof relies on a positive scalar curvature argument. Is Yu's counter-example still a counter-example for the L^p version?

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