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Induced map on K-theory for certain Γ-equivariant maps between Hilbert spaces

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Abstract. Higson–Kasparov–Trout introduced an infinite-dimensional Clifford algebra of a Hilbert space, and verified Bott periodicity on K-theory. To develop an algebraic topology of maps between Hilbert spaces, in this paper we introduce an induced Hilbert Clifford algebra and construct an induced map between K-theory of the Higson–Kasparov–Trout Clifford algebra and the induced Clifford algebra. We also compute its K-group for some concrete case.

1. Introduction

Let Γ be a discrete group and H, H' two Hilbert spaces on which Γ acts linearly and isometrically. Let $F = l + c : H' \to H$ be a Γ-equivariant map whose linear part is l, which is also Γ-equivariant. We want to construct an "induced map" of K-theory of these infinite-dimensional spaces. Of course we cannot obtain such a map in the usual sense because these spaces are locally noncompact. Thus, we introduce the infinite-dimensional Clifford C^*-algebras by Higson, Kasparov, and Trout [4].

Let E be a finite-dimensional Euclidean space, and let $\mathrm{Cl}(E)$ be the complex Clifford algebra. There is a $*$-homomorphism $\beta : C_0(\mathbb{R}) \to C_0(\mathbb{R}) \hat{\otimes} C_0(E, \mathrm{Cl}(E))$ called the Bott map, given by the functional calculus

$$f \to f(X \hat{\otimes} 1 + 1 \hat{\otimes} C),$$

where X is an unbounded multiplier of $C_0(\mathbb{R})$ by $X(f)(x) = xf(x)$, and C, which is called the Clifford operator, is also an unbounded multiplier of $C_0(E, \mathrm{Cl}(E))$ by $C(v) = v$. It turns out that β induces an isomorphism on K-theory as follows:

$$\beta_* : K_*\big(C_0(\mathbb{R})\big) \cong K_*\big(C_0(\mathbb{R}) \hat{\otimes} C_0\big(E, \mathrm{Cl}(E)\big)\big).$$

Higson–Kasparov–Trout (HKT) generalized its construction to obtain the Clifford algebra $S\mathfrak{C}(H)$ for an infinite-dimensional Hilbert space H, and verified the isomorphism

$$\beta_* : K_*\big(C_0(\mathbb{R})\big) \cong K_*\big(S\mathfrak{C}(H)\big).$$

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The idea is to use finite-dimensional approximation of the Hilbert space and inductively apply the Bott map.

K-theory plays an important role in non-commutative geometry [3]. To develop an algebraic topology of maps between Hilbert spaces, our first step is to construct an induced map in K-theory. Let $F = l + c : E' \to E$ be a proper map such that $l : E \cong E'$ gives a linear isomorphism, where l is its linear part and c is the nonlinear part between finite-dimensional Euclidean spaces. Then F induces a map

$$F^* : C_0\big(E, \mathrm{Cl}(E)\big) \to C_0\big(E', \mathrm{Cl}(E')\big)$$

given by

$$u \mapsto \big(v' \mapsto \bar{l}^*\big(u\big(F(v')\big)\big)\big)$$

where $\bar{l}$ is the unitary operator of its polar decomposition. Notice that the image $F^*(C_0(E, \mathrm{Cl}(E))) \subset C_0(E', \mathrm{Cl}(E'))$ is a C^* subalgebra.

It becomes clear why we use $\bar{l}$ rather than l to construct the infinite-dimensional version of this map. Let $F = l + c : H' \to H$ be a map between two Hilbert spaces. To extend the above "pull-back" construction to the infinite-dimensional setting we have to impose extra conditions on F. We call such special maps finitely approximable. See Definition 3.1 for more details. We then obtain the following result.

Proposition 1.1. *Suppose $F : H' \to H$ is finitely approximable. Then there is an induced Clifford C^*-algebra $S\mathfrak{C}_F(H')$.*

This C^*-algebra coincides with $F^*(C_0(E, \mathrm{Cl}(E)))$ above, in the finite-dimensional case. If a discrete group Γ acts linearly and isometrically, then it also acts on $S\mathfrak{C}_F(H)$.

The following is our main theorem.

Theorem 1.2. *Suppose $F : H' \to H$ is finitely approximable. Then it induces a $*$-homomorphism*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H').$$

In particular, it induces a homomorphism between K-groups

$$F^* : K_*\big(S\mathfrak{C}(H)\big) \to K_*\big(S\mathfrak{C}_F(H')\big).$$

If a discrete group Γ acts on both H' and H linearly and isometrically and F is Γ-finitely approximable, then F^* is a Γ-equivariant $*$-homomorphism that induces a homomorphism between K-groups

$$F^* : K_*\big(S\mathfrak{C}(H) \rtimes \Gamma\big) \to K_*\big(S\mathfrak{C}_F(H') \rtimes \Gamma\big),$$

where the crossed product is full.

Suppose $F : H' \to H$ is strongly finitely approximable. Then by approximating these Hilbert spaces by finite-dimensional linear subspaces, we can obtain its degree $\deg(F) \in \mathbb{Z}$. Then the above F^* is given by

$$F^* : \mathbb{Z} \to \mathbb{Z}$$

which sends 1 to $\deg(F)$ by choosing a suitable orientation.

We also compute the group $K(S\mathfrak{C}_F(H') \rtimes \mathbb{Z})$ for some concrete cases in Section 6.

In a successive paper [5], we will apply our construction of the K theoretic induced map to a monopole map that appears in gauge theory. Over a compact oriented four manifold, it turns out that the monopole map is strongly finitely approximable, and its degree coincides with the Bauer–Furuta degree when $b^1 = 0$ (see [2]). We will verify that the covering-monopole map on the universal covering space is Γ-finitely approximable, when its linear part gives a linear isomorphism. This produces a higher degree map of Bauer–Furuta type. The idea of the degree goes back to an old result by A. Schwarz [6].

2. Infinite-dimensional Bott periodicity

2.1. Quick review of HKT construction

We review the construction of the Hilbert space Clifford C^*-algebras by Higson, Kasparov, and Trout [4].

Let E be a finite-dimensional Euclidean space, and let $\mathrm{Cl}(E)$ be the complex Clifford algebras, where we choose a positive sign on the multiplication $e^2 = |e|^2 1$ for every $e \in E$.

This admits a natural $\mathbb{Z}_2$-grading. The embedding $C : E \to \mathrm{Cl}(E)$ gives a map which is called the Clifford operator. Let us denote $\mathfrak{C}(E) = C_0(E, \mathrm{Cl}(E))$. Let $X : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be given by $X(f)(x) = xf(x)$. Then $C_0(\mathbb{R})$ also admits a natural $\mathbb{Z}_2$-grading by even or odd functions. Both operators C and X are degree one and essentially self-adjoint unbounded multipliers on $\mathfrak{C}(E)$ and $C_0(\mathbb{R})$, respectively. In particular, $X \hat{\otimes} 1 + 1 \hat{\otimes} C$ is a degree one and essentially self-adjoint unbounded multiplier on $C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}(E)$.

Let us introduce a $*$-homomorphism

$$\beta : C_0(\mathbb{R}) \to S\mathfrak{C}(E) := C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}(E)$$

defined by

$$\beta : f \to f(X \hat{\otimes} 1 + 1 \hat{\otimes} C)$$

through functional calculus. Let E be a separable real Hilbert space, and let $E_a \subset E_b \subset E$ be a pair of finite-dimensional linear subspaces. We denote the orthogonal complement by $E_{ba} := E_b \cap E_a^{\perp}$. Then we have the canonical isomorphism $S\mathfrak{C}(E_b) \cong S\mathfrak{C}(E_{ba}) \hat{\otimes} \mathfrak{C}(E_a)$ of C^*-algebras. Let us introduce a $*$-homomorphism passing through this isomorphism

$$\beta_{ba} = \beta \hat{\otimes} 1 : S\mathfrak{C}(E_a) \to S\mathfrak{C}(E_{ba}) \hat{\otimes} \mathfrak{C}(E_a) = S\mathfrak{C}(E_b).$$

Lemma 2.1 ([4, Proposition 3.2]). *Let $E_a \subset E_b \subset E_c$. Then the composition*

$$S\mathfrak{C}(E_a) \xrightarrow{\beta_{ba}} S\mathfrak{C}(E_b) \xrightarrow{\beta_{cb}} S\mathfrak{C}(E_c)$$

coincides with the $*$-homomorphism

$$\beta_{ca} : S\mathfrak{C}(E_a) \to S\mathfrak{C}(E_c).$$

Definition 2.1. We denote the direct limit C^*-algebras by

$$S\mathfrak{C}(E) = \varinjlim_{a} S\mathfrak{C}(E_a),$$

where the direct limit is taken over all finite-dimensional linear subspaces of E.

It follows from the above construction that we can obtain a $*$-homomorphism

$$\beta : C_0(\mathbb{R}) \to S\mathfrak{C}(E).$$

Suppose a discrete group Γ acts on E linearly and isometrically. It induces the action on $S\mathfrak{C}(E)$ by

$$\gamma(f \hat{\otimes} u)(v) = f \hat{\otimes} \gamma\big(u\big(\gamma^{-1}(v)\big)\big).$$

Thus, the Bott map is Γ-equivariant. For a Γ-C^*-algebra A, let us denote

$$K^{\Gamma}(A) := K(A \rtimes \Gamma),$$

where the right-hand side C^*-algebra is given by the full crossed product of A with Γ.

Proposition 2.2 ([4, Theorem 3.5]). *β gives an equivariant asymptotic equivalence from S to $S\mathfrak{C}(E)$. In particular, it induces an isomorphism*

$$\beta_* : K^{\Gamma}_*\big(C_0(\mathbb{R})\big) \cong K^{\Gamma}_*\big(S\mathfrak{C}(E)\big).$$

2.2. Direct limit C^*-algebras

Let H be a Hilbert space on which Γ acts linearly and isometrically. Choose exhaustion by finite-dimensional linear subspaces $V_j \subset V_{j+1}$ with dense union $\bigcup_j V_j \subset H$. Let $0 < r_1 < \cdots < r_i < r_{i+1} < \cdots \to \infty$ be a divergent positive sequence with $r_{i+1} > \sqrt{2} r_i$, and let $D^j_{r_i} \subset V_j$ be the open disc with diameter r_i.

Consider the diagram of the embeddings

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \cap & & \cap & & & & \cap \\
\cdots & \subset & D^j_{r_i} & \subset & D^j_{r_{i+1}} & \subset & \cdots & \subset & V_j \\
 & & \cap & & \cap & & & & \cap \\
\cdots & \subset & D^{j+1}_{r_i} & \subset & D^{j+1}_{r_{i+1}} & \subset & \cdots & \subset & V_{j+1} \\
 & & \cap & & \cap & & & & \cap \\
 & & \vdots & & \vdots & & & & \vdots \\
\cdots & \subset & D_{r_i} & \subset & D_{r_{i+1}} & \subset & & \subset & H
\end{array}$$

Let $V_j^{\perp} \subset H$ be the orthogonal complement, and for $j' \geq j$ denote

$$V_{j,j'} := V_j^{\perp} \cap V_{j'}, \quad E^{j.j'}_{r_i} := D^j_{r_i} \times V_{j,j'} \subset V_{j'}, \quad E^j_{r_i} := D^j_{r_i} \times V_j^{\perp} \subset H.$$

Let us set

$$S\mathfrak{C}(D^j_{r_i}) \equiv C_0(\mathbb{R}) \mathbin{\hat{\otimes}} C_0\big(D^j_{r_i}, \mathrm{Cl}(V_j)\big).$$

Recall the Bott map

$$\begin{aligned}\beta : C_0(\mathbb{R}) &\to S\mathfrak{C}(V)\\ f &\to f(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} C)\end{aligned}$$

for a finite-dimensional vector space V. Then we have $*$-homomorphisms

$$\begin{aligned}\beta_{j,j'} = \beta \mathbin{\hat{\otimes}} 1 : S\mathfrak{C}(D^j_{r_i}) &\longrightarrow SC_0\big(V_{j,j'}, \mathrm{Cl}(V_{j,j'})\big) \mathbin{\hat{\otimes}} \mathfrak{C}(D^j_{r_i}) \cong S\mathfrak{C}(E^{j.j'}_{r_i})\\ &\hookrightarrow S\mathfrak{C}(V_{j'}),\end{aligned}$$

where the last embedding is the open inclusion.

Remark 2.3. Trout developed a Thom isomorphism on infinite-dimensional Euclidean bundles. One may regard $C_0(D^j_{r_i}, \mathrm{Cl}(V_j))$ as the set of continuous sections on the Clifford algebra of the tangent bundle $\mathrm{Cl}(TD^j_{r_i})$ vanishing at infinity. Then $\beta_{j,j'}$ can be described as a $*$-homomorphism

$$\beta_{j,j'} = (1 \mathbin{\hat{\otimes}} i_*) \circ \big(\beta_{E^{j,j'}_{r_i}} \mathbin{\hat{\otimes}}_{D^j_{r_i}} \mathrm{id}_{D^j_{r_i}}\big),$$

where $i : E^{j,j'}_{r_i} \hookrightarrow V'_j$ is the open inclusion; see [7, Section 2].

Let $S_r := C_0(-r, r) \subset C_0(\mathbb{R})$ and set

$$S_r\mathfrak{C}(D^j_r) \equiv C_0(-r, r) \mathbin{\hat{\otimes}} C_0\big(D^j_r, \mathrm{Cl}(V_j)\big).$$

Then the above Bott map transforms as

$$\beta_{j,j'} = \beta \mathbin{\hat{\otimes}} 1 : S_{r_i}\mathfrak{C}(D^j_{r_i}) \to S_{r_{i+1}}\mathfrak{C}(D^{j'}_{r_{i+1}}).$$

Lemma 2.4. *The direct limit C^*-algebra*

$$\lim_{i,j\to\infty} S_{r_i}\mathfrak{C}(D^j_{r_i}) = S\mathfrak{C}(H)$$

coincides with the Clifford C^*-algebra of H.

Proof. The proof consists of two steps.

Step 1. We claim that the commutativity

$$\beta_{j,j''} = \beta_{j',j''} \circ \beta_{j,j'}$$

holds. To make the notations clearer, let us denote by $\bar{\beta}_{j,j'} : S\mathfrak{C}(V_j) \to S\mathfrak{C}(V_{j'})$ the standard Bott map. Then the commutativity $\bar{\beta}_{j,j''} = \bar{\beta}_{j',j''} \circ \bar{\beta}_{j,j'}$ holds.

For $a_{i,j} \in S_{r_i}\mathfrak{C}(D^j_{r_i})$, $\overline{\beta}_{j,j''}(a_{i,j}) = \beta_{j,j''}(a_{i,j})$ holds in $S\mathfrak{C}(V_{j''})$, passing through the isometric embedding $S_{r_{i+2}}\mathfrak{C}(D^{j''}_{r_{i+2}}) \hookrightarrow S\mathfrak{C}(V_{j''})$. This implies the equalities

$$\begin{aligned}\beta_{j,j''}(a_{i,j}) &= \overline{\beta}_{j,j''}(a_{i,j}) = \overline{\beta}_{j',j''} \circ \overline{\beta}_{j,j'}(a_{i,j}) \\ &= \overline{\beta}_{j',j''} \circ \beta_{j,j'}(a_{i,j}) = \beta_{j',j''} \circ \beta_{j,j'}(a_{i,j}).\end{aligned}$$

This commutativity allows us to construct the direct limit C^*-algebra

$$\lim_{i,j\to\infty} S_{r_i}\mathfrak{C}(D^j_{r_i}).$$

There is a canonical isometric embedding

$$I : \lim_{i,j\to\infty} S_{r_i}\mathfrak{C}(D^j_{r_i}) \hookrightarrow S\mathfrak{C}(H).$$

Step 2. It remains to verify that the image of I is dense. For a linear inclusion $V \hookrightarrow H$, let $\beta : S\mathfrak{C}(V) \to S\mathfrak{C}(H)$ be the Bott $*$-homomorphism into the Clifford C^*-algebra. An element $a \in S\mathfrak{C}(H)$ is given as $\lim_j \beta(a_j)$ for some $a_j \in S\mathfrak{C}(V_j)$. Let $\chi_i \in C_c((-r_i, r_i); [0,1])$ and $\varphi_{i,j} \in C_c(D^j_{r_i}; [0,1])$ be cutoff functions with

$$\chi_i|(-r_{i-1}, r_{i-1}) \equiv 1 \quad \text{and} \quad \varphi_{i,j}|D^j_{r_{i-1}} \equiv 1.$$

Let us set $\psi_{i,j} = \chi_i \hat{\otimes} \varphi_{i,j}$. We claim that $b_{i,j} := \psi_{i,j} a_j \in S\mathfrak{C}(D^j_{r_i})$ converges to the same element:

$$\lim_{i,j\to\infty} \beta(b_{i,j}) = a \in S\mathfrak{C}(H).$$

Choose any j_0 and $\varepsilon > 0$. There exists $r > 0$ such that a_{j_0} satisfies the estimate $\|a_{j_0}\|_{C_0((D_r)^c)} < \varepsilon$. For each $f \in C_0(\mathbb{R})$, there is some $r > 0$ such that $\beta(f) \in S\mathfrak{C}(H)$ satisfies the estimate $\|\beta(f)\|_{C_0((D_r)^c)} < \varepsilon$. Thus, any $a_j \in S\mathfrak{C}(V_j)$ with $j > j_0$ also satisfies the estimate

$$\|a_j\|_{C_0((D_{\sqrt{2}r})^c)} < 2\varepsilon$$

for all large $r \gg 1$. This verifies the claim. ■

2.3. Asymptotic unitary operators

Let H' be a Hilbert space. For two finite-dimensional linear subspaces, let us set

$$d'(V'_1, V'_2) = \sup_{v_2} \inf_{v_1} \{\|v_1 - v_2\| : \|v_1\| = \|v_2\| = 1,\ v_i \in V'_i\}.$$

$d'(V'_1; V'_2) = 0$ holds if and only if V'_1 contains V'_2. Then we introduce the distance between these planes by

$$d(V'_1, V'_2) = \min\{d'(V'_1; V'_2), d'(V'_2; V'_1)\}.$$

Let $l : H' \to H$ be a linear isomorphism between Hilbert spaces, and let

$$\bar{l} = l \circ \sqrt{l^* \circ l}^{\,-1} : H' \to H$$

be the unitary corresponding to the polar decomposition of l. For any finite-dimensional linear subspace $V \subset H$, let us compare two linear subspaces

$$V' \equiv l^{-1}(V), \quad \bar{V}' \equiv \bar{l}^{-1}(V) \subset H'.$$

The following lemma will not be used later, but may be useful to understand how V' and $\bar{V}'$ differ from each other.

Lemma 2.5. *Let W_i' be a family of finite-dimensional linear subspaces with $W_i' \subset W_{i+1}'$ so that the union $\bigcup_i W_i' \subset H'$ is dense. For any finite-dimensional linear subspace $V' \subset H'$ and any small $\varepsilon > 0$, there is some i_0 such that for all $i \geq i_0$,*

$$\| (1 - \bar{\mathrm{pr}}_i) | \bar{V}' \| < \varepsilon$$

holds, where $\bar{\mathrm{pr}}_i : H' \to \bar{W}_i'$ is the orthogonal projection and $\bar{W}_i' := \bar{l}^{-1}(l(W_i'))$.

Proof. It is sufficient to verify that for any finite-dimensional linear subspace $V' \subset H'$ and any $\varepsilon > 0$, the estimate

$$d(\bar{V}', \bar{W}_i') < \varepsilon$$

holds for all large $i \gg 1$. Actually $\bar{W}' = H'$ holds when $W' = H'$ since the polar decomposition gives the unitary. Thus, for any finite-dimensional linear exhaustion W_i' such that $\bigcup_i W_i' \subset H'$ is dense, $\bigcup_i \bar{W}_i' \subset H'$ is also dense. Therefore the estimate holds. ■

Definition 2.2. Let H' and H be two Hilbert spaces and let $l : H' \to H$ be a linear isomorphism. l is asymptotically unitary if, for any $\varepsilon > 0$, there is a finite-dimensional linear subspace $V \subset H'$ such that the restriction

$$l : V^{\perp} \cong l(V^{\perp})$$

satisfies the estimate on its operator norm

$$\| (l - \bar{l}) | V^{\perp} \| < \varepsilon,$$

where $\bar{l}$ is the unitary of the polar decomposition of $l : H' \to H$.

Remark 2.6. In a subsequent paper, we will verify that a self-adjoint elliptic operator on a compact manifold is asymptotically unitary between Sobolev spaces.

Lemma 2.7. *Let $l : H' \cong H$ be asymptotically unitary. For any $\varepsilon > 0$, there is a finite-dimensional vector subspace $V_0' \subset H'$ such that the estimate*

$$d\big(V', (\bar{l}^* \circ l)(V')\big) < \varepsilon$$

holds for any $V' \supset V_0'$.

Proof. Let $V' \subset H'$ be a closed linear subspace, and let $(V')^{\perp}$ be its orthogonal complement. Let $\mathrm{pr} : H' \to V'$ be the orthogonal projection.

Step 1. Take a finite-dimensional subspace $V_0' \subset H'$ so that l satisfies the estimate $\|(l-\bar{l})|(V')^{\perp}\| < \varepsilon$ for any $V' \supset V_0'$. Then the operator norm of the restriction satisfies the estimate

$$\left\|(\bar{l}^* l - \mathrm{pr} \circ \bar{l}^* l)|V'\right\| < \varepsilon.$$

Decompose the operator $\bar{l}^* l$ with respect to $V' \oplus (V')^{\perp}$, and express $\bar{l}^* l$ by a matrix form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where both estimates $\|D - \mathrm{id}\|$ and $\|B\| < \varepsilon$ hold.

Step 2. $C = B^*$ holds since $\bar{l}^* l$ is self-adjoint. Hence the estimate $\|C\| < \varepsilon$ also holds. Then the conclusion holds because the estimate

$$d\big(V', (\bar{l}^* \circ l)(V')\big) \leq \|C\|$$

holds. ■

2.4. A variant of Clifford C^*-algebra

Let us introduce a variant of the HKT construction. Ultimately, the result of the C^*-algebra turns out to be $*$-isomorphic to the original one given by HKT. This variant naturally appears when one considers the induced Clifford C^*-algebra we introduce later.

Let $l : H' \cong H$ be an asymptotically unitary isomorphism. Let $E \subset H$ be a finite-dimensional Euclidean space, and denote

$$E' = l^{-1}(E), \quad \bar{E}' = \bar{l}^{-1}(E).$$

The map

$$C_l \equiv \bar{l}^* \circ l : E' \to \bar{E}' \hookrightarrow \mathrm{Cl}(\bar{E}')$$

is called the *induced Clifford operator*. Let us denote

$$\mathfrak{C}_l(E') = C_0\big(E', \mathrm{Cl}(\bar{E}')\big)$$

and introduce a $*$-homomorphism

$$\beta_l : C_0(\mathbb{R}) \to S\mathfrak{C}_l(E') \equiv C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}_l(E')$$

defined by $\beta_l : f \to f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_l)$ by functional calculus.

Let $E_a' \subset E_b' \subset H'$ be a pair of finite-dimensional linear subspaces, and denote the orthogonal complement as $E_{ba}' = E_b' \cap (E_a')^{\perp}$.

Lemma 2.8. *Let $l : H' \cong H$ be asymptotically unitary. Then there is a finite-dimensional vector space $V' \subset H'$ such that there is a canonical $*$-isomorphism*

$$I_{ba} : S\mathfrak{C}_l(E_{ba}') \hat{\otimes} \mathfrak{C}_l(E_a') \cong S\mathfrak{C}_l(E_b')$$

if the inclusion $V' \subset E_a'$ holds.

Proof. Let V' be the vector subspace given by Lemma 2.7. Let $\hat{E}'_{ba} \subset \bar{E}'_b$ be the orthogonal complement of $\bar{E}'_a$, and consider the orthogonal projection

$$\hat{\text{pr}} : \bar{E}'_{ba} \to \hat{E}'_{ba}.$$

By the assumption, $l^* \circ l$ is almost unitary on E'_{ba} so that the operator norm satisfies the estimate $\|(l^* \circ l) - \text{id}|E'_{ba}\| < \varepsilon$. The estimate $d(E'_a, \bar{E}'_a) < \varepsilon$ also holds from Lemma 2.7. Thus, the operator norm of the above projection satisfies the estimate

$$\|\hat{\text{pr}} - \text{id}|\bar{E}'_{ba}\| < 2\varepsilon.$$

In particular, the projection gives an isomorphism. Let $\hat{\mathbf{pr}} : \bar{E}'_{ba} \to \hat{E}'_{ba}$ be the unitary of the polar decomposition. It also satisfies the estimate $\|\hat{\mathbf{pr}} - \text{id}|\bar{E}'_{ba}\| < 4\varepsilon$, which induces a $*$-isomorphism

$$\hat{\mathbf{pr}} : \text{Cl}(\bar{E}'_{ba}) \cong \text{Cl}(\hat{E}'_{ba}).$$

It extends to the $*$-isomorphism

$$\hat{\mathbf{pr}} \hat{\otimes} 1 : \text{Cl}(\bar{E}'_{ba}) \hat{\otimes} \text{Cl}(\bar{E}'_a) \cong \text{Cl}(\hat{E}'_{ba}) \hat{\otimes} \text{Cl}(\bar{E}'_a) \cong \text{Cl}(\bar{E}'_b)$$

which induces the desired $*$-isomorphism

$$S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b). \qquad \blacksquare$$

It follows from Lemma 2.8 that there is a canonical $*$-homomorphism

$$\beta_{ba} = \beta_l \hat{\otimes} 1 : S\mathfrak{C}_l(E'_a) \to S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b).$$

Remark 2.9. Surely $\hat{\mathbf{pr}}$ induces a linear map

$$\hat{\mathbf{pr}} : \text{Cl}(\bar{E}'_{ba}) \to \text{Cl}(\hat{E}'_{ba})$$

by setting $u = u_1 + u_2 \in \text{Cl}(\hat{E}'_{ba}) \oplus \text{Cl}^0(\bar{E}'_a)$ to $u_1 \in \text{Cl}(\hat{E}'_{ba})$, where $\text{Cl}^0(E)$ is the scalarless part of $\text{Cl}(E)$. However it cannot be "almost" $*$-isomorphic in general, as $\dim E'_{ba}$ grows. To see this, let us take any $u' \in \bar{E}'_{ba}$ and set $u''' = u' - \hat{\mathbf{pr}}(u') := u' - u''$. For any orthonormal basis $\{u'_1, u'_2, \dots\}$ of $\bar{E}'_{ba}$, consider their product $u'_1 u'_2 \cdots u'_m \in \text{Cl}(\bar{E}'_{ba})$;

$$\begin{aligned} u'_1 u'_2 \cdots &= (u''_1 + u'''_1)(u''_2 + u'''_2)(u''_3 + u'''_3) \cdots (u''_m + u'''_m) \\ &= u''_1 u''_2 \cdots u''_m + \text{other terms}. \end{aligned}$$

Each norm $\|u''_i\| < 1$ is strictly less than 1 and, hence, the norm of their product in the first term above may degenerate to zero.

Let A_a be a family of C^*-algebras, and let $\beta_{ba} : A_a \to A_b$ be a family of $*$-homomorphisms, where $\{a\}$ is a semi-ordered set. The family $\{\beta_{ba}\}_{b,a}$ is *asymptotically commutative* if for any $\varepsilon > 0$, there is a_0 such that for any triplet $c \geq b \geq a \geq a_0$, the estimate

$$\|\beta_{ca} - \beta_{cb} \circ \beta_{ba}\| < \varepsilon$$

holds.

For $v_a \in A_a$, introduce the set of equivalent classes

$$\bar{v}_a := \{\beta_{ba}(v_a)\}_{b \geq a}$$

divided by all elements $\bar{v}'_a$ with

$$\lim_b \|\beta_{ba}(v'_a)\| = 0.$$

Consider the algebra generated by elements of the form $\bar{v}_a$, where the sum is given by

$$\bar{v}_a + \bar{v}_{a'} = \{\beta_{ba}(v_a) + \beta_{ba'}(v_{a'})\}_{b \geq a,a'}$$

and the multiplication is given by

$$\bar{v}_a \cdot \bar{v}_{a'} = \{\beta_{ba}(v_a) \cdot \beta_{ba'}(v_{a'})\}_{b \geq a,a'}.$$

The direct limit C^*-algebra A with respect to the family $\{\beta_{ba} : A_a \to A_b\}_{a,b}$ is defined by the closure of the above algebra with the norm

$$\|\bar{v}_a\| := \lim_b \|\beta_{ba}(v_a)\|. \tag{2.1}$$

We also denote it by

$$A := \varinjlim_a A_a.$$

Let us set

$$\beta_{ba} := I_{ba} \circ \beta_l : S\mathfrak{C}_l(E'_a) \to S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b),$$

where E'_a run over all finite-dimensional subspaces, and $b \geq a$ if and only if $E'_b \supset E'_a$ holds.

It follows from the proof of Lemma 2.8 that the following lemma holds.

Lemma 2.10. *The family $\{\beta_{ba} : S\mathfrak{C}_l(E'_a) \to S\mathfrak{C}_l(E'_b)\}_{b,a}$ is asymptotically commutative.*

Definition 2.3. Let $l : H' \cong H$ be asymptotically unitary. The direct limit C^*-algebra is given by

$$S\mathfrak{C}_l(H') = \varinjlim_a S\mathfrak{C}_l(E'_a),$$

where the norm is given in (2.1) above.

Proposition 2.11. *Assume l is asymptotically unitary. Then there is a canonical $*$-isomorphism*

$$S\mathfrak{C}_l(H') \to S\mathfrak{C}(H')$$

between the two Clifford C^*-algebras.

If a group Γ acts on H' linearly and isometrically and l is Γ-equivariant, then this $*$-isomorphism is Γ-equivariant.

Proof. The proof consists of four steps.

Step 1. It follows from Lemma 2.7 and the assumption that, for any $\varepsilon > 0$, there is a finite-dimensional vector space $V_0' \subset H'$ such that, for all $E_a' \supset V_0'$, the following two estimates hold:

$$d\big(E_a', \bar{l}^* \circ l(E_a')\big) < \varepsilon,$$
$$d\big((E_a')^\perp, \bar{l}^* \circ l\big((E_a')^\perp\big)\big) < \varepsilon.$$

Take another $E_b' \supset E_a'$ with E_{ba}', and let $\mathrm{pr}_1 : \bar{E}_a' \cong E_a'$ and $\mathrm{pr}_2 : \bar{E}_{ba}' \cong E_{ba}'$ be the orthogonal projections. Their corresponding unitaries $\mathbf{pr}_i$ satisfy the bounds

$$\|\mathbf{pr}_i - \mathrm{id}\| < 2\varepsilon.$$

They extend to $*$-isomorphisms

$$\mathbf{pr}_1 : \mathrm{Cl}(\bar{E}_a') \cong \mathrm{Cl}(E_a'),$$
$$\mathbf{pr}_2 : \mathrm{Cl}(\bar{E}_{ba}') \cong \mathrm{Cl}(E_{ba}').$$

In particular, they induce the $*$-isomorphisms

$$\mathbf{pr}_1 : C_0\big(E_a', \mathrm{Cl}(\bar{E}_a')\big) \cong C_0\big(E_a', \mathrm{Cl}(E_a')\big),$$
$$\mathbf{pr}_2 : C_0\big(E_{ba}', \mathrm{Cl}(\bar{E}_{ba}')\big) \cong C_0\big(E_{ba}', \mathrm{Cl}(E_{ba}')\big).$$

Step 2. Let us consider two Bott maps

$$\beta_1 : C_0(\mathbb{R}) \to S\mathfrak{C}_l(W'), \quad \beta_1(f) = f(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} C_l),$$
$$\beta_2 : C_0(\mathbb{R}) \to S\mathfrak{C}(W'), \quad \beta_2(f) = f(X \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} C)$$

and the diagram

$$\begin{array}{ccc} S\mathfrak{C}_l(E_a') & \xrightarrow{\beta_1} & S\mathfrak{C}_l(E_{ba}') \mathbin{\hat{\otimes}} C_0\big(E_a', \mathrm{Cl}(\bar{E}_a')\big) \\ {\scriptstyle 1\hat{\otimes}\mathbf{pr}_1}\downarrow & & \downarrow{\scriptstyle 1\hat{\otimes}\mathbf{pr}_2\hat{\otimes}\mathbf{pr}_1} \\ S\mathfrak{C}(E_a') & \xrightarrow{\beta_2} & S\mathfrak{C}(E_{ba}') \mathbin{\hat{\otimes}} C_0\big(E_a', \mathrm{Cl}(E_a')\big) \end{array}$$

Write $\mathbf{pr}_{21} := \mathbf{pr}_2 \mathbin{\hat{\otimes}} \mathbf{pr}_1$. Then this diagram satisfies the estimate

$$\|1 \mathbin{\hat{\otimes}} \mathbf{pr}_{21} \circ \beta_1 - \beta_2 \circ 1 \mathbin{\hat{\otimes}} \mathbf{pr}_1\| < 4\varepsilon.$$

ε can be arbitrarily small by choosing large E_a'.

Step 3. Let us take $x \in S\mathfrak{C}_l(H')$ and choose $x_a \in S\mathfrak{C}_l(E_a')$ with $\lim_a \|\beta_1(x_a) - x\| = 0$, where $\beta_1(x_a) \in S\mathfrak{C}_l(H')$. It follows from the above estimate on the diagram that

$$\mathbf{pr} : S\mathfrak{C}_l(H') \to S\mathfrak{C}(H'),$$
$$\mathbf{pr}(x) = \lim_a \beta_2\big(1 \mathbin{\hat{\otimes}} \mathbf{pr}_1(x_a)\big)$$

is uniquely defined and independent of the choice of x_a.

It is easy to check that this assignment gives a $*$-homomorphism. To see that it is isomorphic, we consider a converse projection, from $\bar{\text{pr}}' : E'_a \cong \bar{E}'_a$. A parallel argument gives another $*$-homomorphism $\mathbf{pr}' : S\mathfrak{C}(H') \to S\mathfrak{C}_l(H')$, and their compositions give the required identities.

Step 4. Let us consider Γ-equivariance. Suppose Γ acts on H' linearly and isometrically. We claim that $\mathbf{pr}_1 : \bar{E}'_a \to E'_a$ is Γ-equivariant.

To see this, notice that $\bar{l}$ and hence $\bar{l}^* \circ l$ are both Γ-equivariant. Then we have the equalities

$$\gamma(\bar{E}'_a) = \gamma\big(\bar{l}^* \circ l(E'_a)\big) = \bar{l}^* \circ l(\gamma E'_a) = \overline{\gamma E'_a}.$$

Therefore

$$\mathbf{pr}_1\big(\gamma(\bar{E}'_a)\big) = \mathbf{pr}_1(\overline{\gamma E'_a}) = \gamma(E'_a) = \gamma\big(\mathbf{pr}_1(\bar{E}'_a)\big).$$

As the Bott map is also Γ-equivariant, the process from Step 1 to Step 3 works equivariantly. ∎

3. Finite-dimensional approximation

Let $F : H' \to H$ be a metrically proper map between Hilbert spaces. Then there is a proper and increasing function $g : [0, \infty) \to [0, \infty)$ such that the lower bound

$$g\big(\|F(m)\|\big) \geq \|m\|$$

holds for all $m \in H'$. Later we analyze a family of maps of the form $F_i : B'_i \to W_i$, where $W_i \subset H$ is a finite-dimensional linear subspace and $B'_i \subset W'_i \subset H'$ is a closed and bounded set in a finite-dimensional linear space.

Let $D_t \subset H$ be a t ball. We say that the family of maps $\{F_i\}_i$ is *proper* if there are positive and increasing numbers $r_i, s_i \to \infty$ such that the inclusion holds:

$$F_i^{-1}(D_{s_i} \cap W_i) \subset D_{r_i} \cap W'_i.$$

Denote $F = l + c$, where l is its linear part and c is a nonlinear term.

Lemma 3.1. *Let $F = l + c : H' \to H$ be a metrically proper map. Suppose l is surjective and c is compact on each bounded set. Then there is a proper and increasing function $f : [0, \infty) \to [0, \infty)$ such that the following holds: for any $r > 0$ and $1 \geq \delta_0 > 0$, there is a finite-dimensional linear subspace $W'_0 \subset H'$ such that for any linear subspace $W'_0 \subset W' \subset H'$, the composed map*

$$\text{pr} \circ F : D_r \cap W' \to W$$

also satisfies the bound

$$f\big(\|\text{pr} \circ F(m)\|\big) \geq \|m\|$$

for any $m \in D_r \cap W'$, where $W = l(W')$ and pr *is the orthogonal projection to W.*

Moreover the estimate holds

$$\sup_{m \in D_r \cap W'} \| F(m) - \mathrm{pr} \circ F(m) \| \leq \delta_0.$$

Proof. Let $C \subset H$ be the closure of the image $c(D_r)$, which is compact. Hence there are finitely many points $w_1, \dots, w_k \in c(D_r)$ such that their δ_0 neighborhoods cover C.

Choose $w_i' \in H'$ with $l(w_i') = w_i$ for $1 \leq i \leq k$, and let W_0' be the linear span of these w_i'.

The restriction $\mathrm{pr} \circ F : D_r \cap W_0' \to W_0$ satisfies the equality

$$\mathrm{pr} \circ F = l + \mathrm{pr} \circ c,$$

where $W_0 = l(W_0')$. Then for any $m \in D_r \cap W_0'$, there is some w_i' with $\|c(m) - c(w_i')\| \leq \delta_0$, and the estimate $\|F(m) - \mathrm{pr} \circ F(m)\| \leq \delta_0$ holds.

Since g is increasing, we obtain the estimates

$$g\big(\|\mathrm{pr} \circ F(m)\| + \delta_0\big) \geq g\big(\|F(m)\|\big) \geq \|m\|.$$

The function $f(x) = g(x+1)$ satisfies the desired property.

For any other linear subspace $W_0' \subset W' \subset H'$, the same property holds for $\mathrm{pr} \circ F : D_r \cap W' \to W$ with $W = l(W')$. ■

Let $W_i' \subset H'$ and $W_i \subset H$ be two families of finite-dimensional linear subspaces. Let us say that a family of linear isomorphisms

$$l_i : W_i' \cong W_i$$

is an *asymptotic unitary family* if the following conditions hold:

(1) there exists an asymptotically unitary map $l : H' \cong H$,

(2) for each i_0, $\lim_i \|l - l_i\|_{W_{i_0}'} = 0$ holds, where $l, l_i : W_{i_0}' \to H$, and

(3) uniform bounds $C^{-1}\|l\| \leq \|l_i\| \leq C\|l\|$ hold on their norms, where C is independent of i.

Let us introduce an approximation of F as a family of maps on finite-dimensional linear subspaces. Let $D_{r_i}' \subset H'$ and $D_{s_i} \subset H$ be r_i and s_i balls, respectively.

Definition 3.1. Let $F = l + c : H' \to H$ be a metrically proper map, where l is its linear part and c is a nonlinear term. Let us say that F is finitely approximable if there is an increasing family of finite-dimensional linear subspaces

$$W_0' \subset W_1' \subset \cdots \subset W_i' \subset \cdots \subset H'$$

and a family of maps $F_i = l_i + c_i : W_i' \to W_i$, where $W_i = l_i(W_i')$, such that

(1) the union $\bigcup_{i \geq 0} W_i' \subset H'$ is dense,

(2) there are two sequences $s_0 < s_1 < \cdots \to \infty$ and $r_0 < r_1 < \cdots \to \infty$ with $r_i \geq s_i$ such that the embedding

$$F_i^{-1}(D_{s_i} \cap W_i) \subset D'_{r_i} \cap W'_i$$

holds for all i,

(3) for each i_0,

$$\lim_{i \to \infty} \sup_{m \in D'_{r_{i_0}} \cap W'_{i_0}} \| F(m) - F_i(m) \| = 0,$$

(4) $l_i : W'_i \cong W_i$ is an asymptotic unitary family with respect to l.

Let us also say that F is *strongly finitely approximable* if it is finitely approximable, $l_i = l|_{W'_i}$, and $c_i = \mathrm{pr}_i \circ c$, such that

$$\lim_{i \to \infty} \| (1 - \mathrm{pr}_i) \circ c | D'_{r_i} \| = 0,$$

where $\mathrm{pr}_i : H \to W_i$ is the orthogonal projection.

The following restates Lemma 3.1.

Corollary 3.2. *Let $F = l + c : H' \to H$ be a metrically proper map such that l is asymptotically unitary and c is compact on each bounded set. Then F is strongly finitely approximable.*

Suppose both H' and H admit linear isometric actions by a group Γ and assume that both F and l are Γ-equivariant where $F = l + c$. Then we say that F is Γ-*finitely approximable* if, moreover, the above family $\{W'_i\}_i$ satisfies that the union

$$\bigcup_i \{\gamma(W'_i) \cap W'_i\} \subset H'$$

is dense for any $\gamma \in \Gamma$.

Note that the above family $\{F_i\}_i$ satisfies convergence for any $\gamma \in \Gamma$

$$\lim_{i \to \infty} \sup_m \| \gamma F_i(m) - F_i(\gamma m) \| = 0,$$

where $m \in D'_{r_{i_0}} \cap W'_{r_{i_0}} \cap \gamma^{-1}(W'_{r_{i_0}})$ because the estimate

$$\begin{aligned} \| \gamma F_i(m) - F_i(\gamma m) \| &\leq \| \gamma F(m) - \gamma F_i(m) \| + \| \gamma F(m) - F_i(\gamma m) \| \\ &= \| F(m) - F_i(m) \| + \| F(\gamma m) - F_i(\gamma m) \| \end{aligned}$$

holds.

Let us take $\gamma \in \Gamma$, and consider the γ shift of the finite approximation data

$$\gamma(W'_i), \quad \gamma^*(F_i), \quad \gamma^*(l_i).$$

It is clear that the above shift gives another finite approximation of F.

4. Induced Clifford C^*-algebra

Let $F = l + c : H' \to H$ be a map. We aim here to construct an "induced" Clifford C^*-algebra $S\mathfrak{C}_F(H')$.

4.1. Model case

Let us start with a model case that consists of a proper and nonlinear map

$$F = l + c : E' \to E$$

between finite-dimensional Euclidean spaces, where l is a linear isomorphism. Consider a $*$-homomorphism

$$F^* : S\mathfrak{C}(E) \to S\mathfrak{C}(E') = C_0(\mathbb{R}) \hat{\otimes} C_0\big(E', \mathrm{Cl}(E')\big)$$

defined by $F^*(f \hat{\otimes} u)(v') := f \hat{\otimes} \bar{l}^{-1}(u(F(v')))$, and denote its image by

$$S\mathfrak{C}_F(E') = F^*\big(S\mathfrak{C}(E)\big)$$

which is a C^*-subalgebra in $S\mathfrak{C}(E')$, whose norm is denoted by $\| \ \|_{S\mathfrak{C}_F}$.

The induced map

$$C_F \equiv \bar{l}^{-1} \circ F : E' \to E' \hookrightarrow \mathrm{Cl}(E')$$

is called the *induced Clifford operator*. We use it to introduce a $*$-homomorphism

$$\beta_F : C_0(\mathbb{R}) \to S\mathfrak{C}_F(E')$$

defined by $\beta_F : f \to f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_F)$ by functional calculus.

Now suppose a Hilbert space H' is spanned by an infinite family of finite-dimensional Euclidean planes as

$$E_1' \oplus E_2' \oplus \cdots$$

and assume there is a family of proper maps

$$F_i = l_i + c_i : E_i' \to E_i$$

which extends to a map

$$F = (F_1, F_2, \ldots) = l + c : H' \to H,$$

where H is spanned by $E_1 \oplus E_2 \oplus \cdots$. Assume $l = (l_1, l_2, \ldots) : H' \cong H$ is asymptotically unitary.

Lemma 4.1. *Let $F = (F_1, F_2)$ be diagonal as above. Then*

$$\mathfrak{C}_F(E_1' \oplus E_2') \cong \mathfrak{C}_{F_1}(E_1') \hat{\otimes} \mathfrak{C}_{F_2}(E_2').$$

Proof. By definition $\mathfrak{C}_{F_i}(E_i') = F_i^*(\mathfrak{C}(E_i'))$ holds for $i = 1, 2$. Hence we have the isomorphisms

$$\begin{aligned}\mathfrak{C}_F(E_1' \oplus E_2') &\cong F^*\big(\mathfrak{C}(E_1' \oplus E_2')\big) \cong (F_1 \hat{\otimes} F_2)^* \mathfrak{C}(E_1') \hat{\otimes} \mathfrak{C}(E_2') \\ &\cong F_1^*\big(\mathfrak{C}(E_1')\big) \hat{\otimes} F_2^*\big(\mathfrak{C}(E_2')\big) \\ &\cong \mathfrak{C}_{F_1}(E_1') \hat{\otimes} \mathfrak{C}_{F_2}(E_2').\end{aligned}$$

■

Then the induced Bott map is given by

$$\begin{aligned}\beta_{F_{i+1}} &: S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_i') \\ &\to S\mathfrak{C}_F(E_{i+1}') \hat{\otimes} \mathfrak{C}_F(E_1' \oplus \cdots \oplus E_i') \cong S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_{i+1}')\end{aligned}$$

by the use of $C_{F_{i+1}}$.

More generally, one can induce

$$\beta_{i,j} : S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_i') \to S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_j')$$

by the use of a canonical extension

$$C_{(F_{i+1},\ldots,F_j)} : E_{i+1}' \oplus \cdots \oplus E_j' \to E_{i+1}' \cdots \oplus E_j' \subset \mathrm{Cl}(E_{i+1}' \cdots \oplus E_j').$$

Let $u \in S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_i')$ for some i. Then the limit

$$\|u\| \equiv \lim_{j \to \infty} \|\beta_{i,j}(u)\|_{S\mathfrak{C}_F}$$

exists, which gives a norm on $S\mathfrak{C}_F(E_1' \oplus E_2' \oplus \cdots)$. Then the direct limit C^*-algebra is given by

$$S\mathfrak{C}_F(H') = \lim_j S\mathfrak{C}_F(E_1' \oplus \cdots \oplus E_j')$$

whose norm is given as above.

Notice that $S\mathfrak{C}_F(H')$ is no longer a C^*-subalgebra of $S\mathfrak{C}(H')$.

Lemma 4.2. *In the case when $c_i \equiv 0$ and hence $F_i = l_i$ for all i, the induced Clifford C^*-algebra admits a canonical $*$-isomorphism*

$$S\mathfrak{C}_F(H') \cong S\mathfrak{C}(H').$$

Proof. This follows from Proposition 2.11 with the coincidence

$$S\mathfrak{C}_F(H') = S\mathfrak{C}_l(H'),$$

where the right-hand side is given in Definition 2.3. ■

4.2. Induced Clifford C^*-algebra

Assume that $F = l + c : H' \to H$ is finitely approximable as in Definition 3.1 with respect to the data $W'_0 \subset \cdots \subset W'_i \subset \cdots \subset H'$ with open disks $D'_{r_i} \subset W'_i$ and $D_{s_i} \subset W_i$, and $F_i = l_i + c_i : W'_i \to W_i$.

Let $S_r = C_0(-r, r) \subset S$ be the set of continuous functions on $(-r, r)$ vanishing at infinity, and consider the C^*-subalgebras

$$S_{r_i} \hat{\otimes} C_0\big(D'_{r_i}, \mathrm{Cl}(W'_i)\big) \equiv S_{r_i}\mathfrak{C}(D'_{r_i}).$$

Since the inclusion $F_i^{-1}(D_{s_i}) \subset D'_{r_i}$ holds, it induces a $*$-homomorphism

$$F_i^* : S_{s_i}\mathfrak{C}(D_{s_i}) \to S_{r_i}\mathfrak{C}(D'_{r_i})$$

given by $F_i^*(h)(v') := \bar{l}_i^{-1}(h(F_i(v')))$. Denote its image by

$$S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}) := F_i^*\big(S_{s_i}\mathfrak{C}(D_{s_i})\big)$$

which is a C^*-subalgebra with the norm $\| \ \|_{S_{r_i}\mathfrak{C}_{F_i}}$.

Let us consider a family of elements

$$\alpha_i \in S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}), \quad i \geq i_0$$

for some i_0. Let us say that the family is *F-compatible* if there is an element $u_{i_0} \in S_{s_{i_0}}\mathfrak{C}(D_{s_{i_0}})$ such that

$$\alpha_i = F_i^*(u_i) \in S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i})$$

holds for any $i \geq i_0$, where $u_i = \beta(u_{i_0}) \in S_{s_i}\mathfrak{C}(D_{s_i})$ with the standard Bott map β.

Remark 4.3. Consider the induced Clifford operator

$$C_{F_i} \equiv \bar{l}_i^{-1} \circ F_i : D'_{r_i} \to W_i \hookrightarrow \mathrm{Cl}(W'_i)$$

and introduce a $*$-homomorphism

$$\beta_{F_i} : S_{r_i} \to S_{r_i}\mathfrak{C}(D'_{r_i})$$

defined by $\beta_{F_i} : f \to f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{F_i})$ by functional calculus.

Then $F_i^*(\beta(f)) = \beta_{F_i}(f)$, for all $f \in S_{r_i}$.

For an element $\alpha_i \in S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i})$ and for $i_0 \leq i$, we denote its restriction by

$$\alpha_i | D'_{r_{i_0}} \in S_{r_i} \hat{\otimes} C_b(D'_{r_{i_0}}) \hat{\otimes} \mathrm{Cl}(W'_i).$$

Note that the norms satisfy the inequality

$$\|\alpha_i\|_{S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i})} \geq \big\|\alpha_i | D'_{r_{i_0}}\big\|,$$

where the right-hand side is the restriction norm.

For an F-compatible sequence $\alpha = \{\alpha_i\}_{i \geq i_0}$, the limit

$$\left\| \{\alpha_i\}_i \right\| := \lim_{j \to \infty} \lim_{i \to \infty} \|\alpha_i | D'_{r_j} \|$$

exists because both F_i and l_i converge weakly (see Definition 3.1). Moreover both F_i^* and β are $*$-homomorphisms between C^*-algebras and so are both norm-decreasing.

Definition 4.1. Let F be finitely approximable. The induced Clifford C^*-algebra is given by

$$S\mathfrak{C}_F(H') = \overline{\{\{\alpha_i\}_i;\ F\text{-compatible sequences}\}}$$

which is obtained by the norm closure of all F-compatible sequences, where the norm is the above one.

Lemma 4.4. (1) *In the model case, $S\mathfrak{C}_F(H')$ coincides with $S\mathfrak{C}_F(H')$ given in Section 4.1.*

(2) *When $F = l$ is asymptotically unitary, there is a natural $*$-isomorphism*

$$\Phi : S\mathfrak{C}_F(H') \cong S\mathfrak{C}_l(H'),$$

where the right-hand side is given in Definition 2.3.

(3) *Suppose F is Γ-finitely approximable. Then $S\mathfrak{C}_F(H')$ is independent of the choice of Γ-finite approximations.*

Proof. One can choose $W_i' = E_1 \oplus \cdots \oplus E_i'$. Then (1) follows from the equality

$$F_i^* \circ \beta = \beta_{F_i} : S_{s_i} \to S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i})$$

by Remark 4.3, with Lemma 2.4.

Let us consider (2), and set $F = l$. Recall the Bott map given in Section 2.4, and denote it by

$$\beta_l : S\mathfrak{C}_l(W_i') \to S\mathfrak{C}_l(H').$$

Let $\{l_i\}_i$ be asymptotically unitary, and denote $l_i(W_i) = \tilde{W}_i$ and $l(W_i) = W_i$. For each i_0 and $\varepsilon > 0$, there is some $i_0' \gg i_0$ such that

$$\|\mathrm{pr}_{i_0,i} - \mathrm{id}\| < \varepsilon$$

holds for any $i \geq i_0'$, where $\mathrm{pr}_{i_0,i} : W_{i_0} \to l_i(W_{i_0})$ is the orthogonal projection. Let $\mathbf{pr}_{i_0,i} : W_{i_0} \cong l_i(W_{i_0})$ be the unitary of the polar decomposition.

Take an element $\{\alpha_i\}_i \in S\mathfrak{C}_F(H')$, with $\alpha_i = l_i^*(u_i)$ and $u_i = \beta(u_{i_0}) \in S\mathfrak{C}(\tilde{W}_i)$. Note that the restriction $\beta(u_{i_0})|_{W_{i_0}} = u_{i_0}$ holds. Then by the condition of asymptotic unitarity, the restriction of their difference satisfies the estimate

$$\left\| l^*\left(\beta\left(\mathbf{pr}_{i_0,i}^*(u_{i_0})\right)\right) - \alpha_i \right\|_{W'_{i_0}} < \varepsilon,$$

where $\beta(\mathbf{pr}^*_{i_0,i}(u_{i_0})) \in S\mathfrak{C}(W_i)$. Then we set

$$\Phi(\{\alpha_i\}_i) = \lim_{i\to\infty} \beta_l\big(l^*\big(\beta\big(\mathbf{pr}^*_{i_0,i}(u_{i_0})\big)\big)\big).$$

Φ is norm-preserving, so it extends to an injective $*$-homomorphism from $S\mathfrak{C}_F(H')$.

Let us verify that it is surjective. One can follow in a converse way to the above. Take an element $\delta = \beta_l(\delta_{i_0}) \in S\mathfrak{C}_l(H')$ with $\delta_{i_0} \in S\mathfrak{C}_l(W'_{i_0})$, and set $\delta_i = l^*(\beta(\delta_{i_0}))$. Let us set $w_{i_0} = (l^*)^{-1}(\delta_{i_0}) \in S\mathfrak{C}(W_{i_0})$ by $v \to \bar{l}(\delta_{i_0}(l^{-1}(v)))$. Then we set

$$u_{i_0} = (\mathbf{pr}^{-1}_{i_0,i})^*(w_{i_0}) \in S\mathfrak{C}(\tilde{W}_{i_0})$$

and $\alpha_i = l_i^*(\beta(u_{i_0})) \in S\mathfrak{C}_{l_i}(W'_i)$. The restriction of their difference satisfies the estimate

$$\big\| l^*\big(\beta(w_{i_0})\big) - \alpha_i \big\|_{W'_{i_0}} < \varepsilon,$$

where $\beta(w_{i_0}) \in S\mathfrak{C}(W_i)$. The estimate $\|l^*(\beta(w_{i_0})) - \delta_i\|_{W'_{i_0}} < \varepsilon$ is satisfied because $\|l^*(w_{i_0}) - \delta_{i_0}\| < \varepsilon$ holds. This implies that $\Phi(\{\alpha_i\}_i) = \delta \in S\mathfrak{C}_l(H')$.

Hence Φ is an isometric $*$-homomorphism with dense image. This implies that it is surjective.

Let us verify the last property (3). Choose any subindices $j_i \geq i$ for $i = 1, 2, \ldots,$ and consider the sub-approximation given by the data $\{F_{j_i}\}_i$. If we replace the original data $\{F_i\}_i$ by this subdata, still we obtain the same C^*-algebra $S\mathfrak{C}_F(H')$ as their norms coincide as follows:

$$\lim_{i\to\infty} \big\|\alpha_i | D'_{r_{j_{i_0}}}\big\| = \lim_{i\to\infty} \big\|\alpha_{j_i} | D'_{r_{j_{i_0}}}\big\|.$$

Let us take two Γ-finite approximations and denote them by $F_i^l : (D'_{s_i})^l \to W_{i,l}$ for $l = 1, 2$.

Take an F-compatible sequence $\alpha = \{\alpha_i\}_{i\geq i_0}$ with respect to $F_i^1 : (D'_{r_i})^1 \to W_{i,1}$, where $\alpha_i = (F_i^1)^*(u_i)$ and $u_i = \beta(u_{i_0}) \in S_{s_i}\mathfrak{C}(D^1_{s_i})$. Let us take subindices $j_i \geq i$ for $i = 1, 2, \ldots$ so that $\lim_{i\to\infty} d'(W_{j_i,1}, W_{i,2}) = 0$ holds (see Section 2.3).

Let us set $\alpha'_i = (F_i^2)^*(u_i)$. Then it follows from the definition of F-compatible sequence that the convergence

$$\lim_{i\to\infty} \big\|\alpha_{j_i} | D'_{r_{j_{i_0}}} - \alpha'_i | D'_{r_{j_{i_0}}}\big\| = 0$$

holds. Combining this result with the above, we obtain the desired conclusion. ∎

Lemma 4.5. *If F is Γ-finitely approximable, then there is a canonical Γ-action on $S\mathfrak{C}_F(H')$.*

Proof. Recall that if $\{W'_i, F_i, l_i\}_i$ is a finite approximation data, then so is $\{\gamma(W'_i), \gamma^*(F_i), \gamma^*(l_i)\}_i$ (see the last sentence in Section 3).

Take an F-compatible sequence $\alpha = \{\alpha_i\}_{i \geq i_0}$ with respect to $F_i : D'_{r_i} \to W_i$, where $\alpha_i = (F_i)^*(u_i)$ and $u_i = \beta(u_{i_0}) \in S_{s_i}\mathfrak{C}(D_{s_i})$. Then $\{\gamma^*\alpha_i\}_i$ is an $\gamma^*(F)$-compatible sequence as, for $m' \in \gamma(D'_{r_i})$,

$$\begin{aligned}\gamma^*(\alpha_i)(m') &= \gamma^*\big((F_i)^*(u_i)\big)(m') = \gamma u_i\big(F_i\big(\gamma^{-1}(m')\big)\big)\\ &= \gamma\beta(u_{i_0})\big(F_i\big(\gamma^{-1}(m')\big)\big) = \beta(\gamma u_{i_0})\big(F_i\big(\gamma^{-1}(m')\big)\big)\\ &= \beta(\gamma u_{i_0}\gamma^{-1})\big(\gamma F_i\big(\gamma^{-1}(m')\big)\big) = (\gamma^* F_i)^*\big(\gamma^*(u_i)\big)(m').\end{aligned}$$

Thus, $\gamma^*(\alpha_i) = (\gamma^* F_i)^*(\beta(\gamma^*(u_{i_0})))$ holds. ■

5. Higher degree ∗-homomorphism

Let $F = l + c : H' \to H$ be a Γ-equivariant nonlinear map, whose linear part l gives an isomorphism. For a finite-dimensional linear subspace $V \subset H$, denote the orthogonal projection by $\mathrm{pr}_V : H \to V$. For $V' = l^{-1}(V)$, we have the modified map

$$F_V = l + \mathrm{pr}_V \circ c : V' \to V.$$

The restriction map $F_V \to F_U$ satisfies the formula $F_U = \mathrm{pr}_U \circ F_V|_U$, for a linear subspace $U \subset V$.

Our initial idea was to pull back $W_i = l(W'_i)$ by F_{W_i} and combine them all. For $F_i = \mathrm{pr}_i \circ F : W'_i \to W_i$, consider the induced $*$-homomorphism $F_i^* : S\mathfrak{C}(W_i) \to S\mathfrak{C}(W'_i)$. Let us explain how difficulty arises if one tries to obtain a $*$-homomorphism in this way. For simplicity, assume l is unitary and the image of c is contained in a finite-dimensional linear subspace $V \subset H$. This will be the simplest situation but already some difficulty appears when we try to construct the induced $*$-homomorphism by F.

Assume that F is metrically proper. This is equivalent to saying that the restriction $F : V' \to V$ is proper in this particular situation, where $V' = l^{-1}(V)$ is the finite-dimensional linear subspace. Let us consider the diagram

$$\begin{array}{ccc} S\mathfrak{C}(W_i) & \xrightarrow{F_i^*} & S\mathfrak{C}(W'_i) \\ {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\beta_l}\big\downarrow \\ S\mathfrak{C}(W_{i+1}) & \xrightarrow{F_{i+1}^*} & S\mathfrak{C}(W'_{i+1}). \end{array}$$

This diagram is far from commutative as the map

$$c : (W'_i)^{\perp} \cap W'_{i+1} \to V$$

can affect to control the behavior of F as $i \to \infty$. Thus the induced maps by F_i^* will not converge in $S\mathfrak{C}(H')$ in general. This is a point where we have account for the nonlinearity of F to construct the target C^*-algebra, and is the reason we have to use $S\mathfrak{C}_F(H')$ instead of $S\mathfrak{C}(H')$ below.

5.1. Degree of proper maps

Let E', E be two finite-dimensional vector spaces, and let $F = l + c : E' \to E$ be a proper smooth map whose linear part $l : E' \cong E$ gives an isomorphism.

Let us reconstruct the degree of $F \in \mathbb{Z}$ by the use of l. Let $\bar{l} : E' \to E$ be the unitary corresponding to the polar decomposition. Then $\bar{l}$ induces the algebra isomorphism $\bar{l} : \mathrm{Cl}(E') \cong \mathrm{Cl}(E)$, and we have the induced $*$-homomorphism

$$F^* : S\mathfrak{C}(E) = C_0\big(E, \mathrm{Cl}(E)\big) \to S\mathfrak{C}(E')$$
$$F^*(h)(v) = \bar{l}^{-1}\big(h\big(F(v)\big)\big).$$

Recall $S\mathfrak{C}_F(E') = F^*(S\mathfrak{C}(E))$. Then F^* can be described as a $*$-homomorphism

$$F^* : S\mathfrak{C}(E) \to S\mathfrak{C}_F(E').$$

Let us consider the induced homomorphisms between K-groups:

$$\begin{array}{ccccc} K_1\big(S\mathfrak{C}(E)\big) & \xrightarrow{F^*} & K_1\big(S\mathfrak{C}_F(E')\big) & \xrightarrow{\mathrm{inc}_*} & K_1\big(S\mathfrak{C}(E')\big), \\ \beta\big\uparrow & & & & \big\uparrow\beta \\ K_1\big(C_0(\mathbb{R})\big) & & & & K_1\big(C_0(\mathbb{R})\big) \end{array}$$

where both β give the isomorphisms by Proposition 2.2.

Let $\tilde{F}^* : K_1(C_0(\mathbb{R})) \to K_1(C_0(\mathbb{R}))$ be the homomorphism determined uniquely so that the diagram commutes. Let us equip orientations on both E' and E so that l preserves them.

Lemma 5.1. *Passing through the isomorphism $K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$,*

$$\tilde{F}^* : \mathbb{Z} \to \mathbb{Z}$$

is given by multiplication by the degree of F.

Proof. The proof consists of three steps.

Step 1. Let us consider the composition of $*$-homomorphisms

$$C_0\big(E, \mathrm{Cl}(E)\big) \to C_0\big(E', \mathrm{Cl}(E')\big) \cong C_0\big(E, \mathrm{Cl}(E)\big),$$

where the first map is F^* and the second map is given by

$$\big(\bar{l}^{-1}\big)^*(h')(v) \equiv \bar{l}\big(h'\big(\bar{l}^{-1}(v)\big)\big).$$

The latter gives an isomorphism since l is isomorphic. Thus it is sufficient to see the conclusion for the composition. The composition is given by

$$h \to \big\{v \mapsto h\big(F \circ \bar{l}^{-1}(v)\big)\big\}.$$

Step 2. Let $l_t : E' \cong E$ be another family of linear isomorphisms with $l_0 = \bar{l}$ and $l_1 = l$. It induces a family of $*$-homomorphisms

$$\begin{gathered} F_t^* : C_0\big(E, \mathrm{Cl}(E)\big) \to C_0\big(E, \mathrm{Cl}(E)\big) \\ h \to \big\{v \to h\big(F \circ (l_t)^{-1}(v)\big)\big\}. \end{gathered}$$

Since homotopic $*$-homomorphisms induce the same maps between their K-groups, it is sufficient to see the conclusion for F_1^*. Noting the equality $F \circ l^{-1} = 1 + c \circ l^{-1}$, it is enough to assume that l is the identity.

Step 3. When l is the identity, $F^* : S\mathfrak{C}(E) \to S\mathfrak{C}(E)$ is given by

$$\mathrm{id} \times F^* : S\mathfrak{C}(E) \cong \big(S \mathbin{\hat{\otimes}} \mathrm{Cl}(E)\big) \otimes C_0(E) \to \big(S \mathbin{\hat{\otimes}} \mathrm{Cl}(E)\big) \otimes C_0(E)$$

whose induced homomorphism on a K-group is given by degree F, passing through the isomorphism

$$K_1\big(S \mathbin{\hat{\otimes}} \mathrm{Cl}(E) \otimes C_0(E)\big) \cong K_1(S) \cong K_*\big(C_0(E)\big) \cong \mathbb{Z},$$

where $*$ is 0 or 1 with respect to whether $\dim E$ is even or odd. The first isomorphism comes from Proposition 2.2, and the second is the classical Bott periodicity (see [1]). ∎

5.2. Induced map for a strongly finitely approximable map

Let $F = l + c : H' \to H$ be a strongly finitely approximable map. There are finite-dimensional linear subspaces $W_i' \subset W_{i+1}' \subset \cdots \subset H'$ whose union is dense, such that the compositions with the projections $\mathrm{pr}_i \circ F : W_i' \to W_i = l(W_i)$ consist of a finitely approximable data with the constants $r_i, s_i \to \infty$.

Let us consider the restriction

$$F_{i+1} : D'_{r_i} \cap W'_{i+1} \to W_{i+1}.$$

Decompose $W_i' \oplus U_i' = W_{i+1}'$ and define $F_{i+1}^0 : W_{i+1}' \to W_{i+1}$ by

$$F_{i+1}^0(w' + u') = F_i(w') + l(u').$$

Then by definition, the estimate

$$\sup_{m \in D'_{r_i} \cap W'_{i+1}} \big\| F_{i+1}(m) - F_{i+1}^0(m) \big\| < \delta_i$$

holds, where $0 < \delta_i \to 0$.

Sublemma 5.2. *Suppose $l : H' \cong H$ is unitary. Let $\beta : S\mathfrak{C}(W_i') \to S\mathfrak{C}(W_{i+1}')$ be the Bott map. Then the equality*

$$\beta \circ F_i^* = (F_{i+1}^0)^* \circ \beta : S\mathfrak{C}(W_i) \to S\mathfrak{C}(W'_{i+1})$$

holds.

Proof. Take $f \hat{\otimes} h \in S\mathfrak{C}(W_i)$ with $(\beta \circ F_i^*)(f \hat{\otimes} h) = \beta(f) \hat{\otimes} F_i^*(h)$. By contrast, $\beta(f \hat{\otimes} h) = \beta(f) \hat{\otimes} h$ and, hence,

$$(F_{i+1}^0)^* \circ \beta(f \hat{\otimes} h) = (l \oplus F_i)^* \circ \beta(f) \hat{\otimes} h = l^*\big(\beta(f)\big) \hat{\otimes} F_i^*(h),$$

where $l^* : S\mathfrak{C}(U_i) \cong S\mathfrak{C}(U_i')$ with $U_i = l(U_i')$. Since l is unitary, the equality

$$l^*\big(\beta(f)\big) = \beta(f) \in S\mathfrak{C}(U_i')$$

holds. ∎

Proposition 5.3. *Let $F = l + c : H' \to H$ be a strongly finitely approximable map. Then the family $\{F_i^*\}_i$ induces a $*$-homomorphism*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H').$$

Proof. The proof consists of two steps.

Step 1. Let us take an element $\alpha \in S\mathfrak{C}(H)$ and its approximation $\alpha_i \in S_{r_i}\mathfrak{C}(D_{r_i})$ with $\lim_{i\to\infty} \beta(\alpha_i) = \alpha \in S\mathfrak{C}(H)$ by Lemma 2.4.

Assume $l : H' \cong H$ is unitary, and consider the following two elements:

$$\beta\big(F_i^*(\alpha_i)\big), F_{i+1}^*(\alpha_{i+1}) \in S\mathfrak{C}(W_{i+1}').$$

Then by Sublemma 5.2 we have the estimates

$$\begin{aligned} \big\|\beta\big(F_i^*(\alpha_i)\big) - F_{i+1}^*(\alpha_{i+1})\big\| &= \big\|(F_{i+1}^0)^*\big(\beta(\alpha_i)\big) - F_{i+1}^*(\alpha_{i+1})\big\|, \\ \big\|(F_{i+1}^0)^*\big(\beta(\alpha_i)\big) - F_{i+1}^*\big(\beta(\alpha_i)\big) &+ \big\|F_{i+1}^*\big(\beta(\alpha_i)\big) - F_{i+1}^*(\alpha_{i+1})\big\| \\ &\le \delta_i \big\|\beta(\alpha_i)\big\| + \big\|\beta(\alpha_i) - \alpha_{i+1}\big\|. \end{aligned}$$

The first term on the right-hand side converges to zero since $\|\beta(\alpha_i)\|$ are uniformly bounded with $\delta_i \to 0$. The second term also converges to zero. Thus the $*$-homomorphisms asymptotically commute with the Bott map. Hence the sequence $\beta(F_i^*(\alpha_i)) \in S\mathfrak{C}(H')$ converges and gives a $*$-homomorphism $F^* : \alpha \to F^*(\alpha) := \lim_i \beta(F_i^*(\alpha_i))$. Clearly this assignment is independent of the choice of approximations of α.

Step 2. Let us consider the case when l is not necessarily unitary but asymptotically unitary.

Let $\beta_l : S \to S\mathfrak{C}_l(U_i')$ be the variant of the Bott map in Section 2.4. Then the same argument as in Sublemma 5.2 verifies the equality

$$\beta_l \circ F_i^* = (F_{i+1}^0)^* \circ \beta : S\mathfrak{C}(W_i) \to S\mathfrak{C}_l(W_{i+1}').$$

Hence the parallel estimate to Step 1 above verifies that the sequence converges

$$\beta_l\big(F_i^*(\alpha_i)\big) \in S\mathfrak{C}_l(H').$$

This also gives a $*$-homomorphism $F^* : \alpha \to F^*(\alpha) := \lim_i \beta_l(F_i^*(\alpha_i))$. As $S\mathfrak{C}_l(H') \cong S\mathfrak{C}(H')$ are $*$-isomorphic by Proposition 2.11, we obtain the desired $*$-homomorphism. ∎

Remark 5.4. Suppose that $F = l + c$ satisfies the conditions to be strongly finitely approximable, except that l is not necessarily isomorphic, but the Fredholm index is zero.

We can still construct the induced $*$-homomorphism $F^* : S\mathfrak{C}(H) \to S\mathfrak{C}(H')$ as below.

There are finite-dimensional linear subspaces $V' \subset H'$ and $V \subset H$ such that the restriction gives an isomorphism $l : (V')^{\perp} \cong V^{\perp}$, where $V^{\perp} \subset H$ is the orthogonal complement. Choose any unitary $l' : V' \cong V$ and take their sum

$$L \equiv l \oplus l' : (V')^{\perp} \oplus V' \cong V^{\perp} \oplus V.$$

Let us use L to pull back the Clifford algebras and use F itself to pull back the functions. Then we can follow from Step 1 and Step 2 in the same way.

Definition 5.1. Let $F : H' \to H$ be a strongly finitely approximable map. Then the induced map

$$F^* : K_1\big(S\mathfrak{C}(H)\big) \cong \mathbb{Z} \to K_1\big(S\mathfrak{C}(H')\big) \cong \mathbb{Z}$$

is given by multiplication by an integer degree $F \in \mathbb{Z}$. We call it the K-theoretic degree of F.

5.3. Induced map for Γ-finitely approximable map

Let us start from a general property, and let H be a Hilbert space with exhaustion $W_0 \subset \cdots \subset W_i \subset \cdots \subset H$ by finite-dimensional linear subspaces. Choose divergent numbers $r_i < r_{i+1} < \cdots \to \infty$, and denote r_i balls by $D_{r_i} \subset W_i$. Let $S_r = C_c(-r, r) \subset S$ be the set of compactly supported continuous functions on $(-r, r)$.

The following restates Lemma 2.4.

Lemma 5.5. *For any $\alpha \in S\mathfrak{C}(H)$, there is a family*

$$\alpha_i \in S_{r_i} \hat{\otimes} C_0\big(D_{r_i}, \mathrm{Cl}(W_i)\big) := S_{r_i}\mathfrak{C}(D_{r_i})$$

such that their images by the Bott map converge to α:

$$\lim_{i\to\infty} \beta(\alpha_i) = \alpha \in S\mathfrak{C}(H).$$

5.3.1. Induced $*$-homomorphism. Let H', H be Hilbert spaces on which Γ act linearly and isometrically, and let $F = l + c : H' \to H$ be a Γ-equivariant map such that $l : H' \cong H$ is a linear isomorphism.

Assume that F is Γ-finitely approximable so that there is a family of finite-dimensional linear subspaces

$$W_0' \subset W_1' \subset \cdots \subset W_i' \subset \cdots \subset H'$$

with dense union, and a family of maps $F_i : W_i' \to W_i = l_i(W_i')$ with the inclusions $F_i^{-1}(D_{s_i}) \subset D'_{r_i}$. Moreover the following convergences hold for each i_0:

$$\lim_{i\to\infty} \sup_{m\in D'_{r_{i_0}}} \big\| F(m) - F_i(m) \big\| = 0,$$

$$\lim_{i\to\infty} \big\| (l - l_i)|W'_{i_0} \big\| = 0.$$

Recall the induced $*$-homomorphism

$$F_i^* : S\mathfrak{C}(D_{s_i}) \to S\mathfrak{C}_{F_i}(D'_{r_i})$$

and the induced Clifford C^*-algebra $S\mathfrak{C}_F(H')$ in Definition 4.1.

Theorem 5.6. *Let $F = l + c : H' \to H$ be Γ-finitely approximable. Then it induces the equivariant $*$-homomorphism*

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H').$$

Proof. Let us take an element $v \in S\mathfrak{C}(H)$ and its approximation $v = \lim_{i\to\infty} v_i$ with $v_i \in S_{s_i}\mathfrak{C}(D_{s_i}) = C_0(-s_i, s_i) \mathbin{\hat{\otimes}} C_0(D_{s_i}, \mathrm{Cl}(W_i))$.

Let us recall the $*$-homomorphism in Section 4.2:

$$F_i^* : S_{s_i}\mathfrak{C}(D_{s_i}) \to S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}).$$

Let us fix i_0 and let $u_i = \beta(v_{i_0}) \in S_{s_i}\mathfrak{C}(D_{s_i})$ be the image of the standard Bott map. Then the family

$$\{F_i^*(u_i)\}_{i \ge i_0}$$

determines an element in $S\mathfrak{C}_F(H')$, which gives a $*$-homomorphism

$$F^* : S_{s_{i_0}}\mathfrak{C}(D_{s_{i_0}}) \to S\mathfrak{C}_F(H')$$

since both F_i^* and β are $*$-homomorphisms. Note that the composition of two $*$-homomorphisms

$$S_{s_{i_0}}\mathfrak{C}(D_{s_{i_0}}) \xrightarrow{\beta} S_{s_{i'_0}}\mathfrak{C}(D_{s_{i'_0}}) \xrightarrow{F^*} S\mathfrak{C}_F(H')$$

coincides with $F^* : S_{s_{i_0}}\mathfrak{C}(D_{s_{i_0}}) \to S\mathfrak{C}_F(H')$.

For a small $\varepsilon > 0$, take two sufficiently large $i'_0 \ge i_0 \gg 1$ such that the estimate $\|\beta(v_{i_0}) - v_{i'_0}\| < \varepsilon$ holds, and set $u'_i = \beta(v_{i'_0}) \in S_{s_i}\mathfrak{C}(D_{s_i})$ for $i \ge i'_0$. Since F^* is norm-decreasing, the estimate $\|F_i^*(u_i) - F_i^*(u'_i)\| < \varepsilon$ holds for all $i \ge i'_0$. Hence the estimate

$$\left\| F^*(v_{i_0}) - F^*(v_{i'_0}) \right\| < \varepsilon$$

holds.

Thus we obtain the assignment $v \to \lim_{i_0\to\infty} F^*(v_{i_0})$, which gives a Γ-equivariant $*$-homomorphism

$$F^* : S\mathfrak{C}(H) \to S\mathfrak{C}_F(H'),$$

where $\{v_i\}_i$ is any approximation of v. ∎

Definition 5.2. Let $F : H' \to H$ be a Γ-finitely approximable map. Then the higher degree of F is given by the induced homomorphism

$$F^* : K_{*+1}\big(C^*(\Gamma)\big) \to K_*\big(S\mathfrak{C}_F(H') \rtimes \Gamma\big).$$

6. Computation of K-group of induced Clifford C^*-algebras

We compute the equivariant K-group of induced Clifford C^*-algebras for some particular cases. This can be a simple model case for further computation of the groups.

6.1. Basics

Let us collect some of basics which we will need. We start from some analytic aspects of Sobolev spaces. We denote by $W^{k,2}$ the Sobolev k-norm which is a linear subspace of L^2. It is a Hilbert space and, hence, complete by the norm which involves derivatives up to the kth order, and incomplete with respect to the L^2 inner product for $k \geq 1$. The following is well known.

Lemma 6.1. *Suppose $k \geq 1$. Then*

(1) *the multiplication*

$$W^{k,2}(S^1) \otimes W^{k,2}(S^1) \to W^{k,2}(S^1)$$

is compact on each bounded set;

(2) *the continuous embedding $W^{k,2}(S^1) \hookrightarrow C^0(S^1)$ holds.*

In particular, an element in $W^{k,2}(S^1)$ can be regarded as a continuous function.
Later we will consider the nonlinear map

$$F : W^{k,2}(S^1) \to W^{k,2}(S^1)$$

by $F(a) = a + a^3$.

Remark 6.2. (1) Let A be a C^*-algebra on which a finite cyclic group $\mathbb{Z}_l$ acts. Then the crossed product is defined as $A \rtimes \mathbb{Z}_l = \{(a_g)_{g \in \mathbb{Z}_l}\}$ with their product by $(a_g)(b_g) = (\sum_{g_1 g_2 = g \in \mathbb{Z}_l} a_{g_1} g_1(b_{g_2}))$. It induces the action by $\mathbb{Z}$ on A by using the natural projection $\pi_l : \mathbb{Z} \to \mathbb{Z}_l$. In such situation, there exists a six-term exact sequence between $K_*(A \rtimes \mathbb{Z})$ and $K_*(A \rtimes \mathbb{Z}_l)$. However this does not seem to contain enough information to apply to our situation. We proceed in a direct way. Recall that an element $a \in A \rtimes \mathbb{Z}$ can be approximated by $a' \in C_c(\mathbb{Z}, A)$.

(2) Let us take an element $u \in K(A \rtimes \mathbb{Z})$ and represent it by $u = [p] - [\pi(p)]$, where $\pi : \bar{A} = A \oplus \mathbb{C} \to \mathbb{C}$ is the projection. Recall that $[p] - [\pi(p)] = [q] - [\pi(q)]$ if and only if there is some $v \in M_{n,m}(A \rtimes \mathbb{Z})$ such that

$$p \oplus 1_a = v^* v, \quad v v^* = q \oplus 1_b$$

for some $a, b \geq 0$.

6.2. Computation of equivariant K-group for a toy model

6.2.1. Finite cyclic and finite-dimensional case. Consider the $\mathbb{Z}_2$-equivariant map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a + b^3 \\ b + a^3 \end{pmatrix},$$

where the involution acts by the coordinate change.

We claim that this is proper of non-zero degree. In fact, if $a + b^3 = 0$, then the equality $b + a^3 = b - b^{3^2}$ implies properness.

Consider a $\mathbb{Z}_2$-equivariant perturbation

$$F_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ta + b^3 \\ tb + a^3 \end{pmatrix}$$

for $t \in (0, 1]$. If $ta + b^3 = 0$, then $tb + a^3 = tb - t^{-3}b^{3^2}$. Thus this is a family of proper maps. At $t = 0$, $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a proper map of degree -1 since it is again $\mathbb{Z}_2$-equivariantly properly homotopic to the involution

$$I : \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} b \\ a \end{pmatrix}.$$

Note that it becomes degree zero if we replace the exponent 3 by 2.

Next we generalize slightly as follows. Consider the $\mathbb{Z}_l$-equivariant map

$$F : \mathbb{R}^l \to \mathbb{R}^l, \quad \begin{pmatrix} a_1 \\ \cdots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_1 + a_l^3 \\ a_2 + a_1^3 \\ \cdots \\ a_l + a_{l-1}^3 \end{pmatrix},$$

where the action is given by cyclic permutation of the coordinates. By the parallel argument as above, this turns out to be a proper map. To compute its degree, consider a perturbation

$$\begin{pmatrix} a_1 \\ \cdots \\ a_l \end{pmatrix} \to \begin{pmatrix} ta_1 + a_l^3 \\ ta_2 + a_1^3 \\ \cdots \\ ta_l + a_{l-1}^3 \end{pmatrix}$$

for $t \in (0, 1]$. This is a family of $\mathbb{Z}_l$-equivariant proper maps, and, at $t = 0$, $F_0 : \mathbb{R}^l \to \mathbb{R}^l$ is a proper map of degree ± 1, determined by the parity of l. In fact there is a $\mathbb{Z}_l$-equivariant proper homotopy F_t^l to the cyclic permutation

$$T_l \equiv F_0^l \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_l \\ a_1 \\ \cdots \\ a_{l-1} \end{pmatrix}.$$

Corollary 6.3. *F induces a $\mathbb{Z}_l$-equivariant isomorphism*

$$K_1^{\mathbb{Z}_l}\big(S\mathfrak{C}_F(\mathbb{R}^l)\big) \cong K_1^{\mathbb{Z}_l}\big(S\mathfrak{C}(\mathbb{R}^l)\big) \cong R(\mathbb{Z}_l)$$

on the equivariant K-theory.

Proof. F is $\mathbb{Z}_l$-equivariantly properly homotopic to T_l above, and so the isomorphism $K_1^{\mathbb{Z}_l}(S\mathfrak{C}_F(\mathbb{R}^l)) \cong K_1^{\mathbb{Z}_l}(S\mathfrak{C}_{T_l}(\mathbb{R}^l))$ holds.

T_l is a $\mathbb{Z}_l$-equivariant linear isomorphism because $\mathbb{Z}_l$ is commutative. It follows from Definition 2.2 that a linear isomorphism between finite-dimensional vector spaces is asymptotically unitary. Then, by Proposition 2.11, $S\mathfrak{C}_{T_l}(H')$ is $\mathbb{Z}_l$-equivariantly $*$-isomorphic to $S\mathfrak{C}(H')$. In particular, we have the isomorphism $K_1^{\mathbb{Z}_l}(S\mathfrak{C}_{T_l}(\mathbb{R}^l)) \cong K_1^{\mathbb{Z}_l}(S\mathfrak{C}(\mathbb{R}^l))$.

The last isomorphism comes from HKT–Bott periodicity for Euclidean space. ∎

6.2.2. Infinite cyclic case. Let $H' = H$ be the closure of $\mathbb{R}^\infty$ with the standard metric. It admits an isometric action of $\mathbb{Z}$ by the shift $T : H' \cong H'$:

$$T : (\dots, a_{-1}, a_0, a_1, \dots) \cong (\dots, a_{-2}, a_{-1}, a_0, \dots).$$

Then we consider the map $F : H' \to H$ by

$$F : \begin{pmatrix} \dots \\ a_{-1} \\ a_0 \\ a_1 \\ \dots \end{pmatrix} \to \begin{pmatrix} \dots \\ a_{-1} + a_{-2}^3 \\ a_0 + a_{-1}^3 \\ a_1 + a_0^3 \\ \dots \end{pmatrix}.$$

If we restrict on $\mathbb{R}^{2l+1} \subset H'$ by $(a_{-l}, \dots, a_l) \to (\dots, a_{-l}, \dots, a_l, 0, \dots)$, then its image is in $\mathbb{R}^{2l+2} \subset H$. In fact

$$F : \begin{pmatrix} \dots \\ 0 \\ a_{-l} \\ a_{-l+1} \\ \dots \\ a_l \\ 0 \\ \dots \\ \dots \end{pmatrix} \to \begin{pmatrix} \dots \\ 0 \\ a_{-l} \\ a_{-l+1} + a_{-l}^3 \\ \dots \\ a_l + a_{l-1}^3 \\ a_l^3 \\ 0 \\ \dots \end{pmatrix}.$$

Let us consider the map $F^l : \mathbb{R}^{2l+1} \to \mathbb{R}^{2l+1}$ by

$$F^l : \begin{pmatrix} a_{-l} \\ a_{-l+1} \\ \dots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_{-l} + a_l^3 \\ a_{-l+1} + a_{-l}^3 \\ \dots \\ a_l + a_{l-1}^3 \end{pmatrix}$$

which moves the last component to the first one. In fact F^l is still a proper map as presented in Section 6.2.1.

Let $W'_l = \mathbb{R}^{2l+1}$ be as above. Then the data (F^l, W'_l) gives the $\mathbb{Z}$-finite approximation in the sense of Definition 3.1.

First, as in the finite cyclic case, we obtain the isomorphism

$$K_1^{\mathbb{Z}_{2l+1}}\big(S\mathfrak{C}_{F^l}(W'_l)\big) \cong K_1^{\mathbb{Z}_{2l+1}}\big(S\mathfrak{C}(W'_l)\big)$$

on the equivariant K-theory. Notice that this isomorphism heavily depends on the degree being equal to ± 1.

Lemma 6.4. *The induced $*$-homomorphism*

$$(F^l)^* : S\mathfrak{C}(W_l) \to S\mathfrak{C}_{F^l}(W'_l)$$

is, in fact, an isomorphism.

Proof. Injectivity follows from surjectivity of F^l, because it has a non-zero degree. It has a closed range since it is isometric embedding. Then the conclusion follows since it has a dense range. ∎

Second, we obtain the inductive system

$$\Phi_l \equiv (F^{l+1})^* \circ \beta \circ \big((F^l)^*\big)^{-1} : S\mathfrak{C}_{F^l}(W'_l) \to S\mathfrak{C}_{F^{l+1}}(W'_{l+1}).$$

By definition, the equality

$$S\mathfrak{C}_F(H') = \lim_l S\mathfrak{C}_{F^l}(W'_l)$$

holds. Forgetting the group action, we have the isomorphisms

$$\begin{aligned} K\big(S\mathfrak{C}_F(H')\big) &= \lim_l K\big(S\mathfrak{C}_{F^l}(W'_l)\big) \cong \lim_l K\big(S\mathfrak{C}(W'_l)\big) \\ &= K\big(S\mathfrak{C}(H')\big) \cong K(S) \cong \mathbb{Z}, \end{aligned}$$

where we used the HKT–Bott periodicity.

Now consider the group action by $\mathbb{Z}$. Let F^l_t be the homotopy in Section 6.2.1, where $F^l_1 = F^l$ and $F^l_0 = T_l$.

Lemma 6.5. *There is a $*$-isomorphism*

$$I_l : S\mathfrak{C}_{F^l_0}(W'_l) \cong S\mathfrak{C}_{F^l_1}(W'_l).$$

Proof. In fact an element $u \in S\mathfrak{C}_{F^l_0}(W'_l)$ is expressed as $u = (F^l_0)^*(v_0)$ for some $v_0 \in S\mathfrak{C}(W_l)$. Because F^l_t has a non-zero degree, it follows that v_0 is uniquely determined by u. Then assign $v_1 = (F^l_1)^*(v_0)$ and denote its map by

$$I_l : S\mathfrak{C}_{F^l_0}(W'_l) \cong S\mathfrak{C}_{F^l_1}(W'_l).$$

This is a $*$-homomorphism and, in fact, is an isomorphism since if we do the same thing, replacing the role of F^l_0 and F^l_1, then we can recover u again. ∎

Proposition 6.6. *There is an isomorphism*

$$K_0(S\mathfrak{C}_F(H') \rtimes \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. Denote by $\bar{A}$ the unitization of A. Take an element $u \in K_0(A \rtimes \mathbb{Z})$ and represent it by $u = [p] - [\pi(p)]$. Approximate $p \in \mathrm{Mat}(\overline{S\mathfrak{C}_F(H') \rtimes \mathbb{Z}})$ by an element $p' = (p'_g)_{g \in B} \in \mathrm{Mat}(\overline{C_c(\mathbb{Z}, S\mathfrak{C}_F(H'))})$, where $B \subset \mathbb{Z}$ is a finite set. There is some l such that each p'_g can be approximated by $p'_{g,l} \in \overline{S\mathfrak{C}_{F^l}(W'_l)}$. Therefore p can be approximated by an element

$$p'' = (p'_{g,l}) \in C\big(\{-l, \dots, l\}, \mathrm{Mat}\,\big(\overline{S\mathfrak{C}_{F^l}(W'_l)}\big)\big).$$

Let us put

$$I_l^{-1}(p'') \in C\big(\{-l, \dots, l\}, \mathrm{Mat}\,\big(\overline{S\mathfrak{C}_{T_l}(W'_l)}\big)\big),$$

where $S_l = F_1^l$ is the cyclic permutation and I_l is in Lemma 6.5.

$$\tilde{p}'' \equiv \frac{I_l^{-1}(p'') + \big(I_l^{-1}(p'')\big)^*}{2} \in \mathrm{Mat}(\overline{S\mathfrak{C}_T(H') \rtimes \mathbb{Z}})$$

is an "almost" projection, in the sense that

$$\big\|(\tilde{p}'')^2 - \tilde{p}''\big\| < \varepsilon$$

for a small $\varepsilon > 0$. Then there is a projection $\tilde{p} \in \mathrm{Mat}(\overline{S\mathfrak{C}_T(H') \rtimes \mathbb{Z}})$ with the estimate

$$\|\tilde{p} - \tilde{p}''\| < \varepsilon'$$

for a small $\varepsilon' > 0$.

Now take another representative $u = [p] - [\pi(p)] = [q] - [\pi(q)]$. In the same way, we obtain a projection $\tilde{q} \in \mathrm{Mat}(\overline{S\mathfrak{C}_T(H') \rtimes \mathbb{Z}})$. Let us recall that there is some $v \in M_{n,m}(\overline{S\mathfrak{C}_F(H') \rtimes \mathbb{Z}})$ such that

$$p \oplus 1_a = v^* v, \quad v v^* = q \oplus 1_b$$

for some $a, b \geq 0$.

Let $v' \in C(\{-l, \dots, l\}, \mathrm{Mat}(\overline{S\mathfrak{C}_{F^l}(W'_l)}))$ be another approximation and take $\tilde{v}'' \equiv I_l^{-1}(v') \in C(\{-l, \dots, l\}, \mathrm{Mat}(\overline{S\mathfrak{C}_{T_l}(W'_l)}))$. Then we have the estimates

$$\big\|(\tilde{v}'')^* \tilde{v}'' - \tilde{p} \oplus 1_a\big\|, \quad \big\|\tilde{v}''(\tilde{v}'')^* - \tilde{q} \oplus 1_b\big\| < \varepsilon''$$

for a small $\varepsilon'' > 0$. This implies the equality

$$[\tilde{p}] - \big[\pi(\tilde{p})\big] = [\tilde{q}] - \big[\pi(\tilde{q})\big] \in K_0\big(S\mathfrak{C}_T(H') \rtimes \mathbb{Z}\big).$$

Therefore, we obtain a well-defined group homomorphism

$$K_0\big(S\mathfrak{C}_F(H') \rtimes \mathbb{Z}\big) \to K_0\big(S\mathfrak{C}_T(H') \rtimes \mathbb{Z}\big).$$

If we replace the role of F and T and proceed in the same way as above, we obtain another map in a converse direction. By construction, their compositions are both the identities. Therefore, this is an isomorphism on the K-groups.

Since the translation shift $T : H' \cong H'$ is unitary and $\mathbb{Z}$ is commutative, there is a $*$-isomorphism

$$S\mathfrak{C}_T(H') \rtimes \mathbb{Z} \cong S\mathfrak{C}(H') \rtimes \mathbb{Z}.$$

Passing through this isomorphism, we obtain the isomorphism

$$K_0\big(S\mathfrak{C}_F(H') \rtimes \mathbb{Z}\big) \to K_0\big(S\mathfrak{C}(H') \rtimes \mathbb{Z}\big).$$

The right-hand side is isomorphic to

$$K_1(C^*\mathbb{Z}) \cong K^1(S^1) \cong \mathbb{Z}$$

by HKT. ∎

6.3. Nonlinear maps between Sobolev spaces over the circle

6.3.1. Involution. Consider the space

$$S^1_2 = \mathbb{R}/2\mathbb{Z} = [0,2]/\{0 \sim 2\}$$

and $W^{k,2}(S^1_2)$ which is generated by $\sin(\pi k s)$ and $\cos(\pi k s)$ for $k \in \mathbb{Z}$.

Consider the Sobolev spaces

$$W^{k,2}(S^1_2)_0, \quad W^{k,2}(S^1_2)_1 \subset W^{k,2}(S^1_2)$$

which are generated by $W^{k,2}(0,1)_0$ and $W^{k,2}(1,2)_0$, respectively. Here, $W^{k,2}(S^1_2)_i$ is naturally isometric to $W^{k,2}(S^1_2)_{i-1}$ by the shift operator

$$T : u_1 \to u_0, \quad u_0(s) = u_1(s+1)$$

mod 2. Note that T^2 is the identity. Therefore, we can identify both Hilbert spaces by the same symbol H and, hence, the following inclusion holds:

$$H \oplus H \subset W^{k,2}(S^1_2).$$

We again consider the nonlinear map with $H' = H$

$$F : H' \oplus H' \to H \oplus H$$

by $F(a) = a + T(a)^3$, where the power is taken pointwisely. Then the map can be written as

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a + b^3 \\ b + a^3 \end{pmatrix}.$$

As we have seen, this is metrically proper.

Let $k = 1$ for simplicity of notation, consider an element $a \in W^{1,2}(S^1)$,

$$a = \sum_{k=-\infty}^{\infty} a_k \sin(2\pi ks) + b_k \cos(2\pi ks),$$

and denote

$$a^3 = \sum_{k=-\infty}^{\infty} c_k \sin(2\pi ks) + d_k \cos(2\pi ks).$$

Lemma 6.7. *Suppose $\|a\|_{W^{1,2}} \leq r$. Then for any $\varepsilon > 0$, there is $n = n(r, \varepsilon) \geq 0$ such that the estimate*

$$\Big\| \sum_{|k| \geq n+1} c_k \sin(2\pi ks) + d_k \cos(2\pi ks) \Big\|_{W^{1,2}} < \varepsilon$$

holds.

Proof. It follows from Lemma 6.1 that $W^{1,2}(S^1) \to W^{1,2}(S^1)$ by $a \to a^3$ is compact on each bounded set. ∎

Choose divergent numbers as $\lim_i n_i = \infty$. For each $i \in \mathbb{N}$, let $V_i' \subset W^{1,2}(0,1)_0$ be the finite-dimensional linear subspace spanned by $\sin(2\pi ks)$ and $\cos(2\pi ks)$ for $|k| \leq n_i$, and set

$$W_i' = V_i' \oplus T(V_i') \subset H' \oplus H'.$$

Denote $\mathrm{pr}_i : H \oplus H = H' \oplus H' \to W_i = W_i'$ as the orthogonal projection. Then the composition

$$F_i \equiv \mathrm{pr}_i \circ F : W_i' \to W_i$$

gives a strongly finitely approximable data with some s_i, r_i.

Proposition 6.8. *There is a $\mathbb{Z}_2$-equivariant $*$-isomorphism*

$$K_1^{\mathbb{Z}_2}\big(S\mathfrak{C}_F(H \oplus H)\big) \cong K_1^{\mathbb{Z}_2}\big(S\mathfrak{C}(H \oplus H)\big).$$

Proof. The proof consists of four steps.

Step 1. By the same argument as the toy case, F is metrically proper and it is $\mathbb{Z}_2$-equivariantly properly homotopic to the involution $I : H \oplus H \cong H \oplus H$ by F^t.

Sublemma 6.9. *There is a $\mathbb{Z}_2$-equivariant $*$-isomorphism*

$$S\mathfrak{C}_I(H \oplus H) \cong S\mathfrak{C}(H \oplus H).$$

Proof. By construction,

$$S\mathfrak{C}_I(H \oplus H) = \big\{\tilde{u}; u \in S\mathfrak{C}(H \oplus H)\big\},$$

where $\tilde{u}(a,b) = I^*(u(a,b))$ with $a, b \in H$. ∎

Step 2. It follows from Lemma 6.7 that $F_i^{-1}(D_{s_i} \cap W_i) \subset D'_{r_i} \cap W'_i$ holds. As in the toy case, one may assume the same property

$$(F_i^t)^{-1}(D_{s_i} \cap W_i) \subset D_{r_i} \cap W'_i \equiv D'_{r_i}$$

where $F_i^t = \mathrm{pr}_i \circ F^t$.

K-theory is stable under these continuous deformations so that the isomorphism

$$K_1^{\mathbb{Z}_2}\big(S_{r_i}\mathfrak{C}_{F_i^0}(D'_{r_i})\big) \cong K_1^{\mathbb{Z}_2}\big(S_{r_i}\mathfrak{C}_{F_i^1}(D'_{r_i})\big)$$

holds.

Step 3. Recall the induced Clifford C^*-algebra $S\mathfrak{C}_F(H)$ whose element $\{\alpha_i\}_i$ satisfies the equality $\alpha_i = F_i^*\beta(u_{i_0})$ for some $u_i = \beta(u_{i_0}) \in S_{s_i}\mathfrak{C}(D_{s_i} \cap W_i)$ and all $i \geq i_0$. Here, $S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i})$ is defined as the image of $F_i^* : S_{s_i}\mathfrak{C}(D_{s_i} \cap W_i) \to S_{r_i}\mathfrak{C}_{l_i}(D'_{r_i}) = S_{r_i}\mathfrak{C}(D'_{r_i})$ (and l_i is the identity in this particular case).

Note that $F_i|D'_{r_i}$ has non-zero degree. We claim that there is a $*$-homomorphism

$$\Phi_i : S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}) \to S_{r_{i+1}}\mathfrak{C}_{F_{i+1}}(D'_{r_{i+1}})$$

which sends α_i to α_{i+1}. In fact α_i uniquely determines u_i. Suppose the contrary, and choose two elements $u_i, u'_i \in S_{s_i}\mathfrak{C}(D_{s_i} \cap W_i)$ with $F_i^*(u_i) = F_i^*(u'_i)$. If $u_i \neq u'_i$ could hold, then there exists $m \in D_{s_i} \cap W_i$ with $u_i(m) \neq u'_i(m)$. However, since F_i has non-zero degree and is hence surjective, there exists $x \in D_{r_i}$ with $F_i(x) = m$. Then we have the equality $u_i(m) = F_i^*(u_i)(x) = F_i^*(u'_i)(x) = u'_i(m)$, which contradicts to the assumption.

Now since $F_i^* : S_{s_i}\mathfrak{C}(D_{s_i} \cap W_i) \to S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}) \subset S_{r_i}\mathfrak{C}_{l_i}(D'_{r_i})$ is an isometric $*$-embedding, it follows that the inverse

$$(F_i^*)^{-1} : S_{r_i}\mathfrak{C}_{F_i}(D'_{r_i}) \to S_{s_i}\mathfrak{C}(D_{s_i} \cap W_i)$$

is $*$-isomorphic. Then Φ_i is given by the compositions $F_{i+1}^* \circ \beta \circ (F_i^*)^{-1}$.

Step 4. Then

$$\begin{aligned} K_1^{\mathbb{Z}_2}\big(S\mathfrak{C}_F(H \oplus H)\big) &\cong \lim_i K_1^{\mathbb{Z}_2}\big(S_{r_i}\mathfrak{C}_{F_i^1}(D_{r_i})\big) \\ &\cong \lim_i K_1^{\mathbb{Z}_2}\big(S_{r_i}\mathfrak{C}_{F_i^0}(D_{r_i})\big) \\ &\cong K_1^{\mathbb{Z}_2}\big(S\mathfrak{C}_I(H \oplus H)\big). \end{aligned}$$

By Sublemma 6.9, we have the desired isomorphism. ∎

6.3.2. Finite cyclic case. Consider the space

$$S_l^1 = \mathbb{R}/l\mathbb{Z} = [0,l]/\{0 \sim l\}$$

and $W^{k,2}(S_l^1)$ which is generated by $\sin(2\pi \frac{k}{l}s)$ and $\cos(2\pi \frac{k}{l}s)$ for $k \in \mathbb{Z}$.

Consider the Sobolev spaces

$$W^{k,2}(S^1_l)_0, W^{k,2}(S^1_l)_1, \dots, W^{k,2}(S^1_l)_{l-1} \subset W^{k,2}(S^1_l)$$

which are, respectively, generated by $W^{k,2}(i, i+1)_0$. Then $W^{k,2}(S^1_l)_i$ is naturally isometric to $W^{k,2}(S^1_l)_{i+1}$ by the shift $T : u_i \to u_{i+1}$ by $u_{i+1}(s) := u_i(s-1)$, of order l. Thus we can identify these Hilbert spaces by the same symbol H and so the inclusion $H \oplus H \oplus \cdots \oplus H \subset W^{k,2}(S^1_l)$ holds.

We again consider the nonlinear map

$$F : H^l = H \oplus H \oplus \cdots \oplus H \to H \oplus H \oplus \cdots \oplus H,$$
$$F(a_1, \dots, a_l) = \big(a_1 + T(a_l)^3, a_2 + T(a_1)^3, \dots, a_l + T(a_{l-1})^3\big).$$

Then the map can be written as

$$\begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_1 + a_l^3 \\ a_2 + a_1^3 \\ \cdots \\ a_l + a_{l-1}^3 \end{pmatrix}.$$

By the same argument as the toy case, this is metrically proper, and its nonlinear part is compact on each bounded set.

By the use of F^t as above, F is $\mathbb{Z}_l$-equivariantly properly homotopic to the cyclic shift T. By a similar argument, we have the following corollary.

Corollary 6.10. *There is a $\mathbb{Z}_l$-equivariant $*$-isomorphism*

$$K_1^{\mathbb{Z}_l}\big(S\mathfrak{C}_F(H^l)\big) \cong K_1^{\mathbb{Z}_l}\big(S\mathfrak{C}(H^l)\big) \cong K_1^{\mathbb{Z}_l}(S),$$

where $\mathbb{Z}_l$ acts on H^l by the cyclic permutation of the components.

The above computation is applicable to more general situations of F and is not restricted to such a specified form of the nonlinear term.

6.3.3. Infinite cyclic case. It is not so immediate to extend the above finite cyclic case to the infinite case, following the same approach. For example, the map $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ by $\{a_i\}_i \to \{a^3_{i+1}\}_i$ is not proper.

Therefore we use a very specific approach to compute the $\mathbb{Z}$ case. Let H be the Hilbert space identified as $H = W^{k,2}(0,1)_0 \subset W^{k,2}(\mathbb{R})$, and let $\mathbf{H}$ be the closure of the sum $\oplus_{i \in \mathbb{Z}} H_i$, where H_i are the copies of the same H. Then, the T orbit of H, $\{T^n(H)\}_{n \in \mathbb{Z}}$, generates $\mathbf{H} \subset W^{k,2}(\mathbb{R})$, where $T : W^{k,2}(i, i+1)_0 \cong W^{k,2}(i+1, i+2)_0$ is the shift as before.

Consider the map $F : \mathbf{H} \to \mathbf{H}$ by

$$F : \begin{pmatrix} \cdots \\ a_{-1} \\ a_0 \\ a_1 \\ \cdots \end{pmatrix} \to \begin{pmatrix} \cdots \\ a_{-1} + a_{-2}^3 \\ a_0 + a_{-1}^3 \\ a_1 + a_0^3 \\ \cdots \end{pmatrix}.$$

Let H_l' be spanned by the vectors $(a_{-l}, \dots, a_l)$. As in the toy model case, consider the approximation $F^l : H_l' \to H_l$ by shifting the last component

$$F^l : \begin{pmatrix} a_{-l} \\ a_{-l+1} \\ \cdots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_{-l} + a_l^3 \\ a_{-l+1} + a_{-l}^3 \\ \cdots \\ a_l + a_{l-1}^3 \end{pmatrix}.$$

There is a finite-dimensional linear subspace $W_i' \subset H_i'$ with $r_i, s_i > 0$ such that (F^i, W_i', D_{r_i}') gives a $\mathbb{Z}$-finitely approximable data with $l_i = \mathrm{id}$.

An element $u \in K_0(S\mathfrak{C}_F(\mathbf{H}) \rtimes \mathbb{Z})$ has a representative as $u = [p] - [\pi(p)]$, where

$$p \in \mathrm{Mat}\left(\overline{S\mathfrak{C}_F(\mathbf{H}) \rtimes \mathbb{Z}}\right).$$

Here, p can be approximated as

$$p' \in \mathrm{Mat}\left(C\left(\{-l, \dots, l\}, \overline{S\mathfrak{C}_{F^l}(W_l')}\right)\right).$$

The rest of the process is parallel to the toy model case, and so one can proceed in the same way and then obtain an isomorphism

$$\begin{aligned} K_0\big(S\mathfrak{C}_F(\mathbf{H}) \rtimes \mathbb{Z}\big) &\cong K_0\big(S\mathfrak{C}(\mathbf{H}) \rtimes \mathbb{Z}\big) \\ &\cong K_1\big(C^*(\mathbb{Z})\big) = K^1(S^1) \\ &\cong \mathbb{Z}. \end{aligned}$$

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