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The coarse geometric ℓ^p-Novikov conjecture for subspaces of nonpositively curved manifolds

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Abstract. In this paper, we prove the coarse geometric ℓ^p-Novikov conjecture for metric spaces with bounded geometry which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature.

1. Introduction

The coarse geometric Novikov conjecture [12, 15, 27, 30] is a statement that the coarse Baum–Connes assembly map from the coarse K-homology of a metric space to the K-theory of the Roe C^*-algebra, which encodes the coarse geometry of the space, is injective. This is a geometric analogue of the strong Novikov conjecture and provides an algorithm to determine the non-vanishing problem of the higher index of the Dirac operator on a noncompact complete Riemannian manifold. It implies Gromov's conjecture that a uniformly contractible Riemannian manifold with bounded geometry cannot have a uniformly positive scalar curvature and the zero-in-the-spectrum conjecture stating that the Laplacian operator acting on the space of all L^2-forms of a uniformly contractible Riemannian manifold has zero in its spectrum.

A remarkable progress was achieved by G. Yu who proved the coarse Baum–Connes conjecture, and consequently the coarse geometric Novikov conjecture, for metric spaces with bounded geometry which admit a coarse embedding into a Hilbert space [28]. Among the main tools in [28] is the localization algebra of Yu [26] together with the twisted Roe algebra technique. A fundamental idea underlining the approach in [28] is that the index of a Dirac operator is more computable if the Dirac operator is twisted by a family of "almost flat Bott bundles." This approach inspires several later progresses on the coarse geometric Novikov conjecture for coarse embeddings into certain Banach spaces [3, 15] or nonpositively curved manifolds [22].

Recently, an ℓ^p-analog of the coarse geometric Baum–Connes assembly map for $1 < p < \infty$ was introduced in [7]; see also [31]. An important impetus behind this generalization is the unpublished work of G. Kasparov and G. Yu on the L^p-Novikov and

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Baum–Connes conjectures (cf. [14]), together with earlier works of Lafforgue's Banach KK-theory [16] and the discovery of G. Yu [29] that all Gromov hyperbolic groups, which include plenty of groups with Kazhdan's property (T), admit a proper affine isometric action on an ℓ^p-space for some $p \geq 2$. Another similar L^p-assembly map has been considered in [4] by Y. C. Chung. And closely related to these problems, rigidity and K-theory of ℓ^p-Roe-type algebras have also be studied by Y. C. Chung and K. Li [5, 6].

The ℓ^p-version of the geometric Novikov conjecture is a natural analog of the classical conjecture obtained by considering algebras of operators on ℓ^p-spaces. While applications to geometry and topology have yet to be found when $p \neq 2$, there is motivation in the coarse geometric ℓ^p-Novikov conjecture coming from comparison with the classical case and the intrinsic interest in comparing K-theories of different completions of a given algebra.

In this paper, we shall prove the following result.

Theorem 1. *Let Γ be a discrete metric space with bounded geometry. If Γ admits a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature, then the coarse geometric ℓ^p-Novikov conjecture holds for Γ, i.e., the assembly map*

$$\mu : \lim_{d \to \infty} K_*\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_*\big(B^p(\Gamma)\big)$$

is injective for all $1 < p < \infty$.

Recall that a map $f : X \to Y$ from a metric space X to another metric space Y is said to be a *coarse embedding* [11] if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}$ with $\lim_{r \to \infty} \rho_i(r) = \infty$ for $i = 1, 2$, such that

$$\rho_1\big(d(x, y)\big) \leq d\big(f(x), f(y)\big) \leq \rho_2\big(d(x, y)\big)$$

for all $x, y \in X$. The above assembly map μ is induced by the evaluation-at-zero map e from the localization ℓ^p algebra $B_L^p(P_d(\Gamma))$ of the Rips complex of Γ to the ℓ^p-Roe algebra $B^p(\Gamma)$ of Γ. The definition of the ℓ^p-assembly map is motivated by the result of G. Yu in [26] that the local index map from K-homology to the K-theory of the localization algebra is an isomorphism for a finite-dimensional simplicial complex. Due to the local nature, it can be shown (cf. [31]) that the K-theory of the ℓ^p-localization algebras $B_L^p(P_d(\Gamma))$ is independent of the choice of $1 < p < \infty$. Therefore, the left-hand side of the assembly map μ in Theorem 1 is isomorphic to the classical coarse K-homology of the space Γ.

The proof of Theorem 1 is again based on the fundamental idea and tools in [28] of G. Yu by using localization algebra technique and an ℓ^p-version of the twisted Roe algebras and ℓ^p-Bott maps. We closely follow our previous work [22] in the classical $p = 2$ case, with necessary technical adjustments.

It should be noted that techniques used in the C^*-algebraic setting often do not transfer to the L^p-setting in a straightforward manner. This is due to the more complicated geometry of L^p-spaces, including the fact that they are not reflexive unless $p = 2$. For instance, while for any closed two-sided ideals I, J in a C^*-algebra A we always have $I \cap J = IJ$ (this general fact is frequently used to establish the K-theory Mayer–Vietoris exact sequences), this equality may not hold in an arbitrary L^p-operator algebra (as a clue, consider $\mathbb{C}$ with its usual norm and the trivial product given by $xy = 0$ for all $x, y \in \mathbb{C}$). In general, an L^p-operator algebra need not have a (contractive, one-sided) approximate identity. However, we will show that closed ideals in the ℓ^p-Roe algebras or the twisted ℓ^p-Roe algebras supported on subspaces of the metric space Γ or open subsets of the manifold M do admit contractive approximate units. This allows us to establish K-theory Mayer–Vietoris sequences for the ℓ^p-Roe algebra and the twisted ℓ^p-Roe algebra.

Another subtle issue is about tensor products associated with L^p-spaces. In general, the tensor product $T \otimes S$ of a bounded operator T on $L^p(\mu)$ and a bounded operator S on a Banach space E may not extend to a bounded operator on the "natural tensor product" $L^p(\mu) \otimes_p E$, unless, for example, $E = L^p(\nu)$ is another L^p-space, in which case $\|T \otimes S\| = \|T\|\|S\|$. This suggests us to view the algebra $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ in the construction of the twisted ℓ^p-Roe algebra and Bott elements in K-theory as an L^p-operator algebra. Since the Clifford bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$ is finite-dimensional, one would naturally like to regard it as an ℓ^p-space bundle so that the algebra $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ could act on the L^p-space $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ of the ℓ^p-space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$. However, since the ℓ^p-norm on a tangent space T_xM depends on the choice of the (orthonormal) basis of T_xM, if M is not flat, we cannot end up with a consistent ℓ^p-structure on the tangent bundle TM or the Clifford bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$, which is needed for the construction of "the family of uniformly almost flat Bott elements" on M. To solve this confliction, we will view $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ acting on $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ which is the L^p-space of locally measurable sections of *the Hilbert space bundle* $\mathrm{Cliff}_{\mathbb{C}}(TM)$. It turns out that the tensor product $T \otimes S$ of a bounded operator T on ℓ^p and a bounded operator S on $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$, regarded as the L^p-space of the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$, still extends to a bounded operator on the "natural tensor product" $\ell^p \otimes_p L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ and satisfies $\|T \otimes S\| = \|T\|\|S\|$.

The paper is organized as follows. In Section 2, we recall the ℓ^p-Roe algebra, ℓ^p-localization algebras, and the coarse geometric ℓ^p-Novikov conjecture. In Section 3, we study approximate units for an ideal of the ℓ^p-Roe algebra supported on a subspace and present an ℓ^p-coarse Mayer–Vietoris principle. In Section 4, we first discuss a certain measure theory aspect of the L^p-space $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ of the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$ and the natural tensor norm $\otimes_p$, so as to view $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ as an L^p-operator algebra. Then we define the twisted ℓ^p-Roe algebra and its localization counterpart and discuss how to use ideals supported on separate open subsets of M to show that the evaluation map induces an isomorphism for twisted algebras. In Section 5, we adapt Yu's arguments about strong Lipschitz homotopy invariance to the ℓ^p-setting. In Section 6, we construct families of uniformly almost flat Bott generators to establish a Bott map β from the K-theory of the ℓ^p-Roe algebra to the K-theory of the twisted ℓ^p-Roe algebra and a Bott map β_L between the corresponding ℓ^p-localization algebras. In Section 7, we complete the proof of the main theorem of this paper.

2. The coarse geometric Novikov conjecture

In this section, we shall recall the concepts of the ℓ^p-Roe algebra [7,21], Yu's ℓ^p-localization algebras [7,26], and the coarse geometric ℓ^p-Novikov conjecture [7].

Let X be a proper metric space. Recall that the space X is called *proper* if every closed ball is compact. When X is discrete, we say that X has *bounded geometry* if, for any $R > 0$, there exists $N_R > 0$ such that for any $x \in X$ the cardinality $|B(x; R)|$ is less than or equal to N_R. For $r > 0$, an r-net in X is a discrete subset $Y \subset X$ such that, for any $y_1, y_2 \in Y$, $d(y_1, y_2) \geq r$ and for any $x \in X$ there is a $y \in Y$ such that $d(x, y) \leq r$. A general metric space X is called to have bounded geometry if X has an r-net Y for some $r > 0$ such that Y has bounded geometry.

Throughout this paper, we assume $p > 1$ and denote by $\mathcal{K}_p = \mathcal{K}(\ell^p)$ the Banach algebra of all compact operators on ℓ^p.

Definition 2 ([7,21]). Let X be a proper metric space, and fix a countable dense subset $Z \subseteq X$. Let T be a bounded operator on $\ell^p(Z, \ell^p)$, and write $T = (T(x, y))_{x,y \in Z}$ so that each $T(x, y)$ is a bounded operator on ℓ^p. The operator T is said to be *locally compact* if

- each $T(x, y)$ is a compact operator on ℓ^p;
- for every bounded subset $B \subseteq X$, the set
$$\big\{(x, y) \in (B \times B) \cap (Z \times Z) : T(x, y) \neq 0\big\}$$
is finite.

The *propagation* of T is defined to be

$$\operatorname{propagation}(T) = \inf\big\{S > 0 : T(x, y) = 0 \text{ for all } x, y \in Z \text{ with } d(x, y) > S\big\}.$$

The *algebraic ℓ^p-Roe algebra* of X, denoted by $B^p_{\mathrm{alg}}(X)$, is the subalgebra of $\mathcal{L}(\ell^p(Z, \ell^p))$ consisting of all finite propagation, locally compact operators. The *ℓ^p-Roe algebra* of X, denoted by $B^p(X)$, is the closure of $B^p_{\mathrm{alg}}(X)$ in $\mathcal{L}(\ell^p(Z, \ell^p))$.

Up to non-canonical isomorphisms, $B^p(X)$ does not depend on the choice of the dense subspace Z, while, up to canonical isomorphism, its K-theory does not depend on the choice of Z. The proof in [13] for $p = 2$ works well for general $p > 1$.

Definition 3 ([26]). The *ℓ^p-localization algebra* $B^p_L(X)$ is the norm-closure of the algebra of all bounded and uniformly norm-continuous functions $g : [0, \infty) \to B^p(X)$ such that

$$\operatorname{propagation}\big(g(t)\big) \to 0 \quad \text{as } t \to \infty.$$

The evaluation homomorphism e from $B^p_L(X)$ to $B^p(X)$ is defined by

$$e(g) = g(0)$$

for all $g \in B^p_L(X)$.

Definition 4 ([24]). Let Γ be a locally finite metric space. Let $d \geq 0$. The *Rips complex of* Γ *at scale* d, denoted by $P_d(\Gamma)$, is the simplicial complex with vertex set Γ where a subset $\{\gamma_0, \ldots, \gamma_n\}$ of Γ spans a simplex if and only if $d(\gamma_i, \gamma_j) \leq d$ for all i, j. Write points x in such a simplex $\sigma_{\{\gamma_0,\ldots,\gamma_n\}}$ of $P_d(\Gamma)$ as formal linear combinations:

$$x = \sum_{i=0}^{n} t_i \gamma_i,$$

where each coefficient t_i is in [0, 1], and $\sum_{i=0}^{n} t_i = 1$. Let $\mathrm{S}(\mathbb{R}^{n+1})$ be the sphere in the Euclidean space $\mathbb{R}^{n+1}$, and define a bijection from the simplex $\sigma_{\{\gamma_0,\ldots,\gamma_n\}}$ to $\mathrm{S}(\mathbb{R}^{n+1})$ via the map

$$\rho : x = \sum_{i=0}^{n} t_i \gamma_i \mapsto \left(\sum_{i=0}^{n} t_i^2\right)^{-\frac{1}{2}} (t_0, \ldots, t_n).$$

The spherical metric on $\sigma\{\gamma_0, \ldots, \gamma_n\}$ is the metric defined by

$$d_\sigma(x, y) := \frac{2}{\pi} \arccos\left(\langle \rho(x), \rho(y) \rangle\right),$$

i.e., the length (normalized by $2/\pi$) of the shorter arc of a great circle connecting $\rho(x)$ and $\rho(y)$.

For points $x, y \in P_d(\Gamma)$, a *simplicial path* γ (cf. [24]) between them is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of points in $P_d(\Gamma)$ together with a choice of simplices $\sigma_1, \ldots, \sigma_n$ such that each σ_i contains (x_{i-1}, x_i). The length of such a path γ is defined to be

$$l(\gamma) := \sum_{i=1}^{n} d_{\sigma_i}(x_{i-1}, x_i),$$

and the *spherical distance* between two arbitrary points $x, y \in P_d(\Gamma)$ is defined to be

$$d_S(x, y) := \inf\left\{l(\gamma) : \gamma \text{ a simplicial path between } x \text{ and } y\right\}$$

and $d_S(x, y) = \infty$ if no simplicial path exists.

A *semi-simplicial path* δ (see [24, Definition 7.2.8]) between points x and y in $P_d(\Gamma)$ consists of a sequence of the form

$$x = x_0, y_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n = y,$$

where each of the points x_i and y_i is in Γ and *some of these points may be repeated*. The *length* of such a path is defined as

$$\ell(\delta) := \sum_{i=0}^{n} d_S(x_i, y_i) + \sum_{i=0}^{n-1} d_\Gamma(y_i, x_{i+1}).$$

The *semi-spherical distance* on $P_d(\Gamma)$ is defined by

$$d_{P_d}(x, y) := \inf\left\{\ell(\delta) \mid \delta \text{ a semi-simplicial path between } x \text{ and } y\right\}.$$

Note that a semi-simplicial path between two points always exists.

The *Rips complex of* Γ is defined to be the space $P_d(\Gamma)$ equipped with the metric d_{P_d} above.

It turns out that (see [24, Proposition 7.2.11]) (1) $P_0(\Gamma)$ identifies isometrically with Γ; (2) for any $d \geq 0$, the Rips complex $P_d(\Gamma)$ is a proper, second countable metric space; (3) for each $d' \geq d \geq 0$, the canonical inclusion $i_{d'd} : P_d(\Gamma) \to P_{d'}(\Gamma)$ is a coarse equivalence and a homeomorphism onto its image.

To define the assembly map, we recall that when $p = 2$, Yu in [26] proved that the local index map from K-homology to K-theory of localization algebra is an isomorphism for a finite-dimensional simplicial complex. Y. Qiao and J. Roe in [20] later generalized this isomorphism to general locally compact metric spaces. Therefore, for $p \in (1, \infty)$, considering the analogs of ℓ^p-Roe algebra and ℓ^p-localization algebra, we define the evaluation at zero map as the assembly map, which is equivalent to the original index map when $p = 2$. The following conjecture is called *the coarse geometric* ℓ^p*-Novikov conjecture*.

Conjecture 5. *If* Γ *is a discrete metric space with bounded geometry, then the assembly map*

$$\mu := e_* : \lim_{d \to \infty} K_*\big(B_L^p\big(P_d(\Gamma)\big)\big) \to \lim_{d \to \infty} K_*\big(B^p\big(P_d(\Gamma)\big)\big) \cong K_*\big(B^p(\Gamma)\big)$$

is injective, where $1 < p < \infty$.

3. An ℓ^p coarse Mayer–Vietoris principle

In this section, we present an ℓ^p coarse Mayer–Vietoris principle similar to the argument in [13].

Definition 6 ([13]). Let X be a proper metric space, and let A and B be closed subspaces with $X = A \cup B$. We say that (A, B) is an ω-excisive pair, or that $X = A \cup B$ is an ω-excisive decomposition, if for each $R > 0$ there is some $S > 0$ such that

$$\mathrm{Pen}(A; R) \cap \mathrm{Pen}(B; R) \subset \mathrm{Pen}(A \cap B; S),$$

where $\mathrm{Pen}(A; R) = \{y \in X \mid d(y, A) \leq R\}$ is the R-neighborhood of A in X.

Definition 7 ([13]). Let A be a closed subspace of a proper metric space X. Denote by $B^p(A; X)$ the operator-norm closure of the set of all locally compact, finite propagation operators T on $\ell^p(Z, \ell^p)$ whose support is contained in $\mathrm{Pen}(A; R) \times \mathrm{Pen}(A; R)$, for some $R > 0$ depending on T.

One can see that $B^p(A; X)$ is a two-sided ideal of $B^p(X)$. For $s, t \in [0, \infty)$ with $s < t$, the inclusion $\mathrm{Pen}(A; s) \to \mathrm{Pen}(A; t)$ induces an inclusion map

$$i_{t,s} : B^p\big(\mathrm{Pen}(A; s)\big) \to B^p\big(\mathrm{Pen}(A; t)\big).$$

It follows that $B^p(A; X) = \lim_{n \to \infty} B^p(\mathrm{Pen}(A; n))$, and we get an induced map

$$i : B^p(A) \to B^p(A; X).$$

Lemma 8 ([13]). *The induced map at K-theory level*

$$i_* : K_*\big(B^p(A)\big) \to K_*\big(B^p(A;X)\big)$$

is an isomorphism.

Proof. Since the inclusions $A \subset \mathrm{Pen}(A;n)$ and $\mathrm{Pen}(A;n) \subset \mathrm{Pen}(A;n+1)$ are coarse equivalence, the induced maps on K-theory are all isomorphisms. ■

Let A be a closed subspace of S and consider the ideal $B^p(A;X)$ of $B^p(X)$. Define idempotents $Q : X \times X \to \mathcal{K}_p$ by the formula

$$Q(x,y) = 0 \quad \text{if } x \neq y,$$
$$Q(x,x) = \begin{pmatrix} I_{r(x)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $I_{r(x)}$ is the $r(x) \times r(x)$ identity matrix for some $r(x) \in \mathbb{N}$ and

$$\mathrm{Supp}\,\big(r(x)\big) \subset \mathrm{Pen}(A;R)$$

for some $R \geq 0$. We define a partial order on all such idempotents Q by the following: $Q_2 \leq Q_1$ if $\mathrm{rank}(Q_2(x,x)) \leq \mathrm{rank}(Q_1(x,x))$ for all $x \in X$.

Let $\mathcal{Q}$ be the set of all such operators Q with this order.

Proposition 9. *The collection $\mathcal{Q}$ is an approximate unit of $B^p(A;X)$.*

Proof. Let $T \in B^p(A;X)$ and $\varepsilon > 0$. For any $x, y \in X$, $T(x,y)$ is either a zero operator or a compact operator on ℓ^p. Recall that $Z \subseteq X$ is a chosen countable dense subset in the definition of the ℓ^p-Roe algebra. Enumerate $Z \times Z$ such that each pair $(x,y) \in Z \times Z$ is assigned a unique integer $n \in \mathbb{N}$.

Let $F(x,y)$ be a finite rank operator on ℓ^p such that $\|T(x,y) - F(x,y)\| < \frac{1}{2^n}\varepsilon$ when $T(x,y) \neq 0$ and n is the corresponding integer of (x,y), and $F(x,y) = 0$ when $T(x,y) = 0$. Then $F = (F(x,y))$ is a locally finite rank operator of finite propagation with $\|T - F\| < \varepsilon$.

For each fixed x, since F has finite propagation, there are only finitely many y such that $F(x,y) \neq 0$. Let

$$Q(x,x) = \begin{pmatrix} I_{r(x)} & 0 \\ 0 & 0 \end{pmatrix}$$

be a finite-rank projection for some $r(x) \in \mathbb{N}$ such that

$$Q(x,x)F(x,y) = F(x,y)$$

for all y with $F(x,y) \neq 0$. Define $Q(x,y) = 0$ if $x \neq y$. Then

$$\|QT - T\| \leq \|QT - QF\| + \|QF - F\| + \|F - T\|.$$

Here $\|F - T\| < \varepsilon$, $QF - F = 0$, and $\|QT - QF\| \leq \|Q\|\|F - T\| < \varepsilon$. So $\|QT - T\| < 2\varepsilon$ and the proof is done. ■

Proposition 10. *Let A, B be closed subspaces of X such that $X = A \cup B$. Then*

(1) $B^p(A; X) + B^p(B; X) = B^p(X)$*;*

(2) $B^p(A; X) \cap B^p(B; X) = B^p(A \cap B; X)$ *if $X = A \cup B$ is an ω-excisive decomposition.*

Proof. Obviously, $B^p(A; X) + B^p(B; X) \subseteq B^p(X)$. For the reverse inclusion, let χ_A be the characteristic function of A. For any $T \in B^p(X)$ and $\varepsilon > 0$, there exists $T_\varepsilon \in B^p_{\text{alg}}(X)$ with $\|T - T_\varepsilon\| < \varepsilon$. Then $T_\varepsilon \chi_A \in B^p(A; X)$ and $\|T\chi_A - T_\varepsilon \chi_A\| \leq \|T - T_\varepsilon\| < \varepsilon$. It follows that $T\chi_A \in B^p(A; X)$, and consequently, $T = T\chi_A + T(1 - \chi_A) \in B^p(A; X) + B^p(B; X)$. Therefore $B^p(A; X) + B^p(B; X) \supseteq B^p(X)$.

For the second part, we will show that

$$B^p(A; X) \cap B^p(B; X) = \overline{B^p(A; X) B^p(B; X)} = B^p(A \cap B; X)$$

for an ω-excisive pair (A, B) of X.

Obviously, $B^p(A \cap B; X) \subseteq B^p(A; X) \cap B^p(B; X)$ holds for any decomposition pair (A, B). On the other hand, by Proposition 9, one can easily see that $B^p(A; X) \cap B^p(B; X) \subseteq \overline{B^p(A; X) B^p(B; X)}$. Finally, for $T_A \in B^p_{\text{alg}}(A; X)$ and $T_B \in B^p_{\text{alg}}(B; X)$ with

$$\operatorname{Supp}(T_A) \subseteq \operatorname{Pen}(A; R) \times \operatorname{Pen}(A; R),$$
$$\operatorname{Supp}(T_B) \subseteq \operatorname{Pen}(B; R') \times \operatorname{Pen}(B; R'),$$

since (A, B) is w-excisive, there exists $S > 0$ such that

$$\operatorname{Supp}(T_A T_B) \subseteq \operatorname{Pen}(A \cap B; S) \times \operatorname{Pen}(A \cap B; S).$$

Hence $\overline{B^p(A; X) B^p(B; X)} \subseteq B^p(A \cap B; X)$. This completes the proof. ∎

As a general fact (cf. [24, proof of Proposition 2.7.15]), if A is a Banach algebra, and I and J are two closed two-sided ideals of A such that $I + J = A$, then standard isomorphism theorems in pure algebra give that

$$\frac{I}{I \cap J} \cong \frac{I + J}{J} = \frac{A}{J},$$

which further induces the following Mayer–Vietoris exact sequence (cf. [24, Proposition 2.7.15]).

Proposition 11 (cf. [24]). *Let A be a Banach algebra, and let I and J be two closed two-sided ideals of A such that $I + J = A$. Then there is a six term Mayer–Vietoris exact sequence on K-theory:*

$$\begin{array}{ccccc} K_0(I \cap J) & \longrightarrow & K_0(I) \oplus K_0(J) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(I) \oplus K_1(J) & \longleftarrow & K_1(I \cap J). \end{array}$$

Combining these lemmas, we obtain the following ℓ^p-version of the coarse Mayer–Vietoris principle.

Proposition 12. *Let $X = A \cup B$ be an ω-excisive decomposition of X. Then there is a six term Mayer–Vietoris exact sequence:*

$$\begin{array}{ccccc} K_0\big(B^p(A\cap B)\big) & \longrightarrow & K_0\big(B^p(A)\big)\oplus K_0\big(B^p(B)\big) & \longrightarrow & K_0\big(B^p(X)\big) \\ \uparrow & & & & \downarrow \\ K_1\big(B^p(X)\big) & \longleftarrow & K_1\big(B^p(A)\big)\oplus K_1\big(B^p(B)\big) & \longleftarrow & K_1\big(B^p(A\cap B)\big). \end{array}$$

4. Twisted ℓ^p-Roe algebras and twisted ℓ^p-localization algebras

In this section, we shall define the twisted ℓ^p-Roe algebras and the twisted ℓ^p-localization algebras for bounded geometry spaces which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature. The construction of these twisted ℓ^p-algebras follows those twisted algebras introduced in [28], with technical adjustments suitable to ℓ^p spaces.

Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature. In the following, we shall assume that the dimension of M is even. If $\dim(M)$ is odd, we can replace M by $M \times \mathbb{R}$. Indeed, the product manifold $M \times \mathbb{R}$ is also a simply connected complete Riemannian manifold with nonpositive sectional curvature. And if $f : \Gamma \to M$ is a coarse embedding, then the induced map $f' : \Gamma \to M \times \mathbb{R}$ defined by $f'(\gamma) = (f(\gamma), 0)$ is also a coarse embedding so that we can replace f by f'.

Let $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ be the C^*-algebra of continuous sections a on M which have value $a(x) \in \mathrm{Cliff}_{\mathbb{C}}(T_xM)$ at each point $x \in M$ and vanish at infinity, where $\mathrm{Cliff}_{\mathbb{C}}(T_xM)$ is the complexified Clifford algebra [1] of the tangent space T_xM at $x \in M$ with respect to the inner product on T_xM given by the Riemannian structure of M. Here $\mathrm{Cliff}_{\mathbb{C}}(TM)$ is the Clifford algebra bundle over M. Meanwhile, for any $x \in M$, $\mathrm{Cliff}_{\mathbb{C}}(T_xM)$ is also a Hilbert space, so that $\mathrm{Cliff}_{\mathbb{C}}(TM)$ is a Hilbert space bundle. Let

$$\mathfrak{B} := L^p\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big),$$

the set of all L^p sections of the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$, which is a Banach space. The C^*-algebra $\mathcal{A}$ acts on $\mathfrak{B}$ by pointwise multiplications, so that it can be regarded as an L^p-operator algebra (cf. [18, 19]).

Let us make this point of view more precise. Let ν be the Radon measure on M induced by the Riemannian metric on M (cf. [17, Chapter XVI, Theorem 4.4]). The continuous sections with compact support of the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$ generate a local ν-measurability structure $\mathcal{W}$ for the cross-sections of $\mathrm{Cliff}_{\mathbb{C}}(TM)$ (cf. [9, Chapter II, Section 15]). For $1 \le p < \infty$, the Banach space $\mathfrak{B} = L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ consists of all those locally μ-measurable cross-sections f of $\mathcal{W}$ such that

$$\|f\|_p^p = \int_M \big\|f(x)\big\|^p d\nu(x) \leq \infty,$$

and two elements of $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ are identical if they differ only on a μ-null set (cf. [9, Chapter II, Section 15.7]). Since M is a simply connected complete Riemannian manifold with nonpositive sectional curvature, by the Cartan–Hadamard theorem, for any $x \in M$, the exponential map $\exp_x : T_xM \to M$ gives rise to a diffeomorphism from $\mathbb{R}^n$ to M, so that the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$ is isomorphic to the trivial bundle $M \times \mathcal{M}_{2^k}(\mathbb{C})$, where $n = 2k = \dim(M)$ and the matrix algebra $\mathcal{M}_{2^k}(\mathbb{C})$ is endowed with a Hilbert space structure induced from the Hilbert space structure of $\mathrm{Cliff}_{\mathbb{C}}(T_xM)$. Consequently, we have

$$\mathfrak{B} := L^p\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big) \cong L^p\big(M, \nu; \mathcal{M}_{2^k}(\mathbb{C})\big) \cong L^p(M, \nu) \otimes_p \mathcal{M}_{2^k}(\mathbb{C}).$$

Let us recall some facts about the tensor norms on the spaces of p-integrable functions and tensor product operators (cf. [8, Chapter 7] and [10, Theorem 1.1 and Corollary 1.1]. For a good summary, see [4] or [18]). Let (Ω, μ) be an arbitrary measure space, $1 \leq p < \infty$, and E a Banach space. Then the space $L^p(\mu, E)$ of (classes of a.e. equal) Bochner p-integrable functions provides the algebraic tensor product $L^p(\mu) \otimes_{\mathrm{alg}} E$ with a "natural tensor norm" Δ_p via the natural injective mapping

$$\begin{aligned} L^p(\mu) \otimes_{\mathrm{alg}} E &\hookrightarrow L^p(\mu, E) \\ f \otimes x &\mapsto f(\cdot)x. \end{aligned}$$

The completion of $L^p(\mu) \otimes_{\mathrm{alg}} E$ against the tensor norm Δ_p is denoted in the following by $L^p(\mu) \otimes_p E$. Since the simple functions are dense in $L^p(\mu, E)$, the above inclusion induces an isometric isomorphism

$$L^p(\mu) \otimes_p E \cong L^p(\mu, E).$$

For the product measure $\mu \times \nu$ on $\Omega_1 \times \Omega_2$, the Fubini–Tonelli theorem shows that the inclusion $L^p(\mu) \otimes_{\mathrm{alg}} L^p(\nu) \hookrightarrow L^p(\mu \times \nu)$ induces isometric isomorphisms

$$L^p(\mu) \otimes_p L^p(\nu) \cong L^p\big(\mu, L^p(\nu)\big) \cong L^p(\mu \times \nu).$$

Moreover, the Fubini theorem for Bochner integrals (cf. [25, Appendices, Theorem B.41]; or more generally, [9, Chapter II, Section 16]) shows that we may replace the space $L^p(\nu)$ in the above identifications by $L^p(\nu, F)$ of Bochner p-integrable functions into a Banach space F. In particular, we have

$$\begin{aligned} L^p(\mu) \otimes_p \big(L^p(\nu) \otimes_p F\big) &\cong L^p(\mu) \otimes_p L^p(\nu, F) \\ &\cong L^p\big(\mu, L^p(\nu, F)\big) \\ &\cong L^p(\mu \times \nu, F), \end{aligned}$$

provided that the spaces Ω_1 and Ω_2 are locally compact and σ-compact Hausdorff spaces.

In general, for bounded linear operators $S \in \mathcal{L}(L^p(\mu))$ and $T \in \mathcal{L}(E)$, there are natural examples showing that the tensor product operator

$$S \otimes T : L^p(\mu) \otimes_{\mathrm{alg}} E \to L^p(\mu) \otimes_{\mathrm{alg}} E$$

may not extend to a bounded operator on $L^p(\mu) \otimes_p E$ (cf. [8, Chapter 7, Sections 7.5 and 7.6]). However, in the situation considered in this paper, we do not meet this difficulty. Namely, on one hand, if $S = \mathrm{id}_{L^p(\mu)}$, it is easy to verify that

$$\left\| \mathrm{id} \otimes T : L^p(\mu) \otimes_p E \to L^p(\mu) \otimes_p E \right\| = \|T\|$$

for any $T \in \mathcal{L}(E)$ (cf. [10, the proof of Theorem 1.2]). On the other hand, if $E = L^p(\nu, F)$ for a Banach space F and the same p as in $L^p(\mu)$, and if $\mathrm{id}_{L^p(\nu,F)}$ is the identity operator, then for any bounded operator $S \in \mathcal{L}(L^p(\mu))$, the operator

$$S \otimes \mathrm{id}_{L^p(\nu,F)} : L^p(\mu) \otimes_{\mathrm{alg}} L^p(\nu, F) \to L^p(\mu) \otimes_{\mathrm{alg}} L^p(\nu, F)$$

has a unique extension to a bounded linear operator

$$S \otimes \mathrm{id}_{L^p(\nu,F)} : L^p(\mu) \otimes_p L^p(\nu, F) \to L^p(\mu) \otimes_p L^p(\nu, F)$$

such that

$$\left\| S \otimes \mathrm{id}_{L^p(\nu,F)} \right\| = \|S\|.$$

This can be proved by appealing to the proof of Theorem 1.1 in [10] (replacing $L^p(\sigma)$ there by $L^p(\nu, F)$ here and confining to the case $p = q$, so that the integral version of the Minkowski inequality for $\alpha = 1$ still holds). Consequently, for any bounded linear operators $S \in \mathcal{L}(L^p(\mu))$ and $T \in \mathcal{L}(L^p(\nu, F))$, the operator $S \otimes T = (S \otimes \mathrm{id})(\mathrm{id} \otimes T)$ on $\mathcal{L}(L^p(\mu)) \otimes_{\mathrm{alg}} \mathcal{L}(L^p(\nu, F))$ extends continuously to a unique bounded linear operator

$$S \otimes T : \mathcal{L}\big(L^p(\mu)\big) \otimes_p \mathcal{L}\big(L^p(\nu, F)\big) \to \mathcal{L}\big(L^p(\mu)\big) \otimes_p \mathcal{L}\big(L^p(\nu, F)\big)$$

and $\|S \otimes T\| = \|S\|\|T\|$ (cf. [10, the proof of Corollary 1.1]).

In this paper, we will only concern ourselves with the situation where $L^p(\mu) = \ell^p$ and

$$L^p(\nu, F) = \mathfrak{B} := L^p\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big) \cong L^p\big(M, \nu; \mathcal{M}_{2^k}(\mathbb{C})\big).$$

For $a \in \mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ and $h \in \mathfrak{B}$, define

$$\|a\|_\infty = \sup\left\{ \|a(x)\| \mid x \in M \right\}.$$

Then $\|a \cdot h\| \le \|a\|_\infty \|h\|$ and $\mathcal{A} \subset \mathcal{L}(\mathfrak{B})$. For $n \in \mathbb{N}$, define

$$\mathfrak{B}_{n,p} = \mathfrak{B} \oplus_p \cdots \oplus_p \mathfrak{B},$$

the ℓ^p-direct sum of n copies of $\mathfrak{B}$. The ℓ^p-norm of $\mathfrak{B}_{n,p}$ is defined as

$$\left\| (f_1, \ldots, f_n) \right\|_p = \left(\sum_{i=1}^{n} \|f_i\|^p \right)^{1/p}, \quad \text{for } f_1, \ldots, f_n \in \mathfrak{B}.$$

Let $M_n(\mathcal{A})$ be the algebra of $n \times n$ matrices with entries in $\mathcal{A}$. Then $M_n(\mathcal{A})$ acts on $\mathfrak{B}_{n,p}$ by matrix multiplications, so that $M_n(\mathcal{A}) \subset \mathcal{L}(\mathfrak{B}_{n,p})$. Embed $M_n(\mathcal{A})$ into $M_{n+1}(\mathcal{A})$ at the top left corner, and let $M_{\infty,p}(\mathcal{A})$ be the inductive limit of $\{M_n(\mathcal{A})\}_{n=1}^{\infty}$. Define

$$\mathfrak{B}_{\infty,p} = \mathfrak{B} \oplus_p \cdots \oplus_p \mathfrak{B} \oplus_p \cdots$$

to be the ℓ^p-direct sum of infinitely many copies of $\mathfrak{B}$ with the ℓ^p-norm

$$\left\| \{f_i\}_{i=1}^{\infty} \right\|_p = \left(\sum_{i=1}^{\infty} \|f_i\|^p \right)^{1/p}, \quad \text{for } \{f_i\}_{i=1}^{\infty} \in \mathfrak{B}_{\infty,p}.$$

It follows from the above discussions that we have isometric isomorphisms

$$\mathfrak{B}_{\infty,p} \cong \ell^p(\mathbb{N}, \mathfrak{B}) \cong \ell^p \otimes_p \mathfrak{B}$$

and all $M_n(\mathcal{A})$ can be considered as subalgebras of $\mathcal{L}(\mathfrak{B}_{\infty,p})$. Denote by $\mathcal{K}_p \otimes_{\mathrm{alg}} \mathcal{A}$ the algebraic tensor product of $\mathcal{K}_p$ and $\mathcal{A}$. Naturally, $\mathcal{K}_p \otimes_{\mathrm{alg}} \mathcal{A}$ acts on $\mathfrak{B}_{\infty,p}$ and $\mathcal{K}_p \otimes_{\mathrm{alg}} \mathcal{A} \subset \mathcal{L}(\mathfrak{B}_{\infty,p})$. Let

$$\mathcal{K}_p \otimes_p \mathcal{A} = \overline{\mathcal{K}_p \otimes_{\mathrm{alg}} \mathcal{A}}^{\mathcal{L}(\mathfrak{B}_{\infty,p})}.$$

It follows that $\mathcal{K}_p \otimes_p \mathcal{A} \cong M_{\infty,p}(\mathcal{A})$.

Let Γ be a discrete metric space with bounded geometry. Let $f : \Gamma \to M$ be a coarse embedding. For each $d > 0$, we shall extend the map f to the Rips complex $P_d(\Gamma)$ in the following way. Note that f is a coarse map, i.e., there exists $R > 0$ such that, for all $\gamma_1, \gamma_2 \in \Gamma$,

$$d(\gamma_1, \gamma_2) \leq d \Rightarrow d_M\big(f(\gamma_1), f(\gamma_2)\big) \leq R.$$

For any point $x = \sum_{\gamma \in \Gamma} c_\gamma \gamma \in P_d(\Gamma)$, where $c_\gamma \geq 0$ and $\sum_{\gamma \in \Gamma} c_\gamma = 1$, we choose a point $f_x \in M$ such that

$$d\big(f_x, f(\gamma)\big) \leq R$$

for all $\gamma \in \Gamma$ with $c_\gamma \neq 0$. The correspondence $x \mapsto f_x$ gives a coarse embedding $P_d(\Gamma) \to M$, also denoted by f.

Choose a countable dense subset Γ_d of $P_d(\Gamma)$ for each $d > 0$ in such a way that $\Gamma_d \subset \Gamma_{d'}$ when $d < d'$.

Definition 13. Let $B^p_{\mathrm{alg}}(P_d(\Gamma), \mathcal{A})$ be the set of all functions

$$T : \Gamma_d \times \Gamma_d \to \mathcal{K}_p \otimes_p \mathcal{A} \subset \mathcal{L}(\mathfrak{B}_{\infty,p}) = \mathcal{L}\big(\ell^p \otimes_p L^p\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big)\big)$$

such that

(1) there exists $C > 0$ such that $\|T(x, y)\| \leq C$ for all $x, y \in \Gamma_d$;

(2) there exists $R > 0$ such that $T(x, y) = 0$ if $d(x, y) > R$;

(3) there exists $L > 0$ such that, for every $z \in P_d(\Gamma)$, the number of elements in the set

$$\big\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \leq 3R,\ d(y, z) \leq 3R,\ T(x, y) \neq 0\big\}$$

is less than L;

(4) there exists $r > 0$ such that

$$\operatorname{Supp}\big(T(x,y)\big) \subset B\big(f(x), r\big)$$

for all $x, y \in \Gamma_d$, where $B(f(x), r) = \{m \in M : d(m, f(x)) < r\}$ and, for all $x, y \in \Gamma_d$, the entry $T(x,y) \in \mathcal{K}_p \otimes_p \mathcal{A}$ is a function on M with $T(x,y)(m) \in \mathcal{K}_p \otimes_p \operatorname{Cliff}_{\mathbb{C}}(T_m M)$ for each $m \in M$ so that the *support* of $T(x,y)$ is defined by

$$\operatorname{Supp}\big(T(x,y)\big) := \big\{m \in M : T(x,y)(m) \neq 0\big\}.$$

For $f \in \ell^p(\Gamma_d, \mathfrak{B}_{\infty,p})$, we define

$$Tf(x) = \sum_{y \in \Gamma_d} T(x,y) f(y).$$

Then $T = (T(x,y)) \in \mathcal{L}(\ell^p(\Gamma_d, \mathfrak{B}_{\infty,p}))$.

Definition 14. The twisted ℓ^p-Roe algebra $B^p(P_d(\Gamma), \mathcal{A})$ is defined to be the operator norm closure of $B^p_{\mathrm{alg}}(P_d(\Gamma), \mathcal{A})$ in $\mathcal{L}(\ell^p(\Gamma_d, \mathfrak{B}_{\infty,p}))$.

The above definition of the twisted ℓ^p-Roe algebra is similar to that in [28].

Let $B^p_{L,\mathrm{alg}}(P_d(\Gamma), \mathcal{A})$ be the set of all bounded, uniformly norm-continuous functions

$$g : \mathbb{R}_+ \to B^p_{\mathrm{alg}}\big(P_d(\Gamma), \mathcal{A}\big)$$

such that

(1) there exists a bounded function $R(t) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} R(t) = 0$ such that $(g(t))(x,y) = 0$ whenever $d(x,y) > R(t)$;

(2) there exists $L > 0$ such that, for every $z \in P_d(\Gamma)$, the number of elements in the set

$$\big\{(x,y) \in \Gamma_d \times \Gamma_d : d(x,z) \leq 3R,\ d(y,z) \leq 3R,\ g(t)(x,y) \neq 0\big\}$$

is less than L for every $t \in \mathbb{R}_+$;

(3) there exists $r > 0$ such that $\operatorname{Supp}((g(t))(x,y)) \subset B(f(x), r)$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$, where $f : P_d(\Gamma) \to M$ is the extension of the coarse embedding $f : \Gamma \to M$ and $B(f(x), r) = \{m \in M : d(m, f(x)) < r\}$.

Definition 15. The twisted ℓ^p-localization algebra $B^p_L(P_d(\Gamma), \mathcal{A})$ is defined to be the norm completion of $B^p_{L,\mathrm{alg}}(P_d(\Gamma), \mathcal{A})$, where $B^p_{L,\mathrm{alg}}(P_d(\Gamma), \mathcal{A})$ is endowed with the norm

$$\|g\|_\infty = \sup_{t \in \mathbb{R}_+} \big\|g(t)\big\|_{B^p(P_d(\Gamma), \mathcal{A})}.$$

The above definition of the twisted ℓ^p-localization Roe algebra is similar to that in [28]. The evaluation homomorphism e from $B^p_L(P_d(\Gamma), \mathcal{A})$ to $B^p(P_d(\Gamma), \mathcal{A})$ defined by $e(g) = g(0)$ induces a homomorphism at K-theory level:

$$e_* : \lim_{d\to\infty} K_*\big(B^p_L(P_d(\Gamma), \mathcal{A})\big) \to \lim_{d\to\infty} K_*\big(B^p(P_d(\Gamma), \mathcal{A})\big).$$

Theorem 16. *Let Γ be a discrete metric space with bounded geometry which admits a coarse embedding $f : \Gamma \to M$ into a simply connected, complete Riemannian manifold M of nonpositive sectional curvature. Then the homomorphism*

$$e_* : \lim_{d\to\infty} K_*\big(B_L^p(P_d(\Gamma), \mathcal{A})\big) \to \lim_{d\to\infty} K_*\big(B^p(P_d(\Gamma), \mathcal{A})\big)$$

is an isomorphism.

The proof of Theorem 16 will follow the proof of Theorem 6.8 in [28]. To begin with, we need to discuss ideals of the twisted algebras associated to open subsets of the manifold M.

Definition 17. (1) The *support* of an element T in $B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})$ is defined to be

$$\begin{aligned}\operatorname{Supp}(T) &= \big\{(x, y, m) \in \Gamma_d \times \Gamma_d \times M : m \in \operatorname{Supp}\big(T(x, y)\big)\big\}\\ &= \big\{(x, y, m) \in \Gamma_d \times \Gamma_d \times M : \big(T(x, y)\big)(m) \neq 0\big\}.\end{aligned}$$

(2) The *support* of an element g in $B^p_{L,\text{alg}}(P_d(\Gamma), \mathcal{A})$ is defined to be

$$\bigcup_{t\in\mathbb{R}_+} \operatorname{Supp}\big(g(t)\big).$$

Let $O \subset M$ be an open subset of M. Define $B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_O$ to be the subalgebra of $B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})$ consisting of all elements whose supports are contained in $\Gamma_d \times \Gamma_d \times O$, i.e.,

$$B^p_{\text{alg}}\big(P_d(\Gamma), \mathcal{A}\big)_O = \big\{T \in B^p_{\text{alg}}\big(P_d(\Gamma), \mathcal{A}\big) : \operatorname{Supp}\big(T(x, y)\big) \subset O,\ \forall\, x, y \in \Gamma_d\big\}.$$

Define $B^p(P_d(\Gamma), \mathcal{A})_O$ to be the norm closure of $B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_O$. Similarly, let

$$B^p_{L,\text{alg}}\big(P_d(\Gamma), \mathcal{A}\big)_O = \big\{g \in B^p_{L,\text{alg}}\big(P_d(\Gamma), \mathcal{A}\big) : \operatorname{Supp}(g) \subset \Gamma_d \times \Gamma_d \times O\big\}$$

and define $B^p_L(P_d(\Gamma), \mathcal{A})_O$ to be the norm closure of $B^p_{L,\text{alg}}(P_d(\Gamma), \mathcal{A})_O$ under the norm $\|g\|_\infty = \sup_{t\in\mathbb{R}_+} \|g(t)\|_{B^p(P_d(\Gamma),\mathcal{A})}$.

Note that $B^p(P_d(\Gamma), \mathcal{A})_O$ and $B^p_L(P_d(\Gamma), \mathcal{A})_O$ are closed two-sided ideals of $B^p(P_d(\Gamma), \mathcal{A})$ and $B^p_L(P_d(\Gamma), \mathcal{A})$, respectively. We also have an evaluation homomorphism

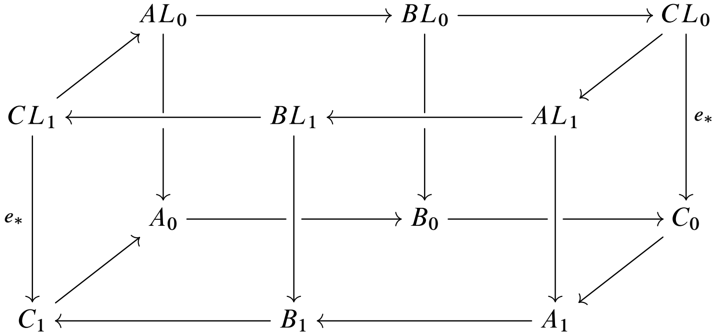
$$e : B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_O \to B^p\big(P_d(\Gamma), \mathcal{A}\big)_O$$

given by $e(g) = g(0)$.

Lemma 18. *For any two open subsets O_1, O_2 of M, one has*

$$\begin{aligned}B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} + B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2} &= B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cup O_2},\\ B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} \cap B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2} &= B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cap O_2},\\ B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} + B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_2} &= B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cup O_2},\\ B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} \cap B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_2} &= B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cap O_2}.\end{aligned}$$

Consequently, one has the following commuting diagram connecting two Mayer–Vietoris sequences at K-theory level:



where, for $* = 0, 1$*,*

$$\begin{aligned} AL_* &= K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cap O_2}\big), \quad CL_* = K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cup O_2}\big), \\ A_* &= K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cap O_2}\big), \quad C_* = K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cup O_2}\big), \\ BL_* &= K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1}\big) \oplus K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2}\big), \\ B_* &= K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1}\big) \oplus K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2}\big). \end{aligned}$$

Proof. We shall prove the first two equalities. The other two equalities can be proved similarly. Then the two Mayer–Vietoris exact sequences follow from Proposition 11.

To prove the first equality, it suffices to show that

$$B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1\cup O_2} \subseteq B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} + B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2}.$$

Now suppose $T \in B_{\mathrm{alg}}^p(P_d(\Gamma), \mathcal{A})_{O_1\cup O_2}$. Take a continuous partition of unity $\{\varphi_1, \varphi_2\}$ on $O_1 \cup O_2$ subordinate to the open over $\{O_1, O_2\}$ of $O_1 \cup O_2$. Define two functions

$$T_1, T_2 : \Gamma_d \times \Gamma_d \to \mathcal{K}_p \otimes_p \mathcal{A}$$

by

$$\begin{aligned} T_1(x, y)(m) &= \varphi_1(m)\big(T(x, y)(m)\big), \\ T_2(x, y)(m) &= \varphi_2(m)\big(T(x, y)(m)\big) \end{aligned}$$

for $x, y \in \Gamma_d$ and $m \in M$.

Then $T_1 \in B_{\mathrm{alg}}^p(P_d(\Gamma), \mathcal{A})_{O_1}$, $T_2 \in B_{\mathrm{alg}}^p(P_d(\Gamma), \mathcal{A})_{O_2}$, and

$$T = T_1 + T_2 \in B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} + B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2}$$

as desired.

For the second equality, similar to the proof of Proposition 11, it suffices to show that

$$B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} \cap B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2} \subseteq \overline{B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_1} B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_2}}.$$

Consider all (rank) functions $r : \Gamma_d \to \mathbb{N}$ and all pairs (K, ϕ), where $K \subset O_1$ is a compact subset in O_1 and $\phi \in \mathcal{A}$ is such that

$$\mathrm{Supp}(\phi) \subset O_1 \quad \text{and} \quad \phi|_K = 1.$$

For any triple $(r; K, \phi)$, define an element $Q \in B^p_{\mathrm{alg}}(P_d(\Gamma), \mathcal{A})_{O_1}$ by the formula

$$Q(x, x) = \begin{pmatrix} I_{r(x)} & 0 \\ 0 & 0 \end{pmatrix} \otimes \phi$$

and $Q(x, y) = 0$ if $x \neq y$. It is straightforward that all such elements Q constitute an approximate unit $\mathcal{Q}$ of $B^p(P_d(\Gamma), \mathcal{A})_{O_1}$. Thus the second equality follows in a similar way to the second equality in Proposition 10. This completes the proof. ■

It would be convenient to introduce the following notion associated with the coarse embedding $f : \Gamma \to M$.

Definition 19. Let $r > 0$. A family of open subsets $\{O_i\}_{i \in J}$ of M is said to be (Γ, r)-*separate* if

(1) $O_i \cap O_j = \emptyset$ if $i \neq j$;

(2) there exists $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i \in J$.

Lemma 20. *If $\{O_i\}_{i \in J}$ is a family of (Γ, r)-separate open subsets of M, then*

$$e_* : \lim_{d \to \infty} K_*\big(B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i \in J} O_i}\big) \to \lim_{d \to \infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i \in J} O_i}\big)$$

is an isomorphism, where $\bigsqcup_{i \in J} O_i$ is the (disjoint) union of $\{O_i\}_{i \in J}$.

We will prove Lemma 20 in Section 5. Granting Lemma 20 for the moment, we are able to prove Theorem 16. The proof is in much the same way as in [28].

Proof of Theorem 16. For any $r > 0$, we define $O_r \subset M$ by

$$O_r = \bigcup_{\gamma \in \Gamma} B\big(f(\gamma), r\big),$$

where $f : \Gamma \to M$ is the coarse embedding and $B(f(\gamma), r) = \{p \in M : d(p, f(\gamma)) < r\}$.

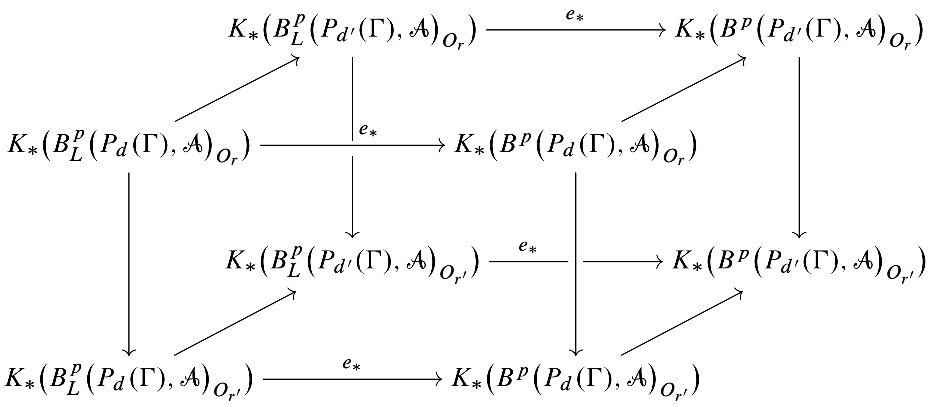
For any $d > 0$, if $r < r'$ then

$$B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_r} \subseteq B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_{r'}}, \quad B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_r} \subseteq B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_{r'}}.$$

By definition, we have

$$B^p\big(P_d(\Gamma), \mathcal{A}\big) = \lim_{r \to \infty} B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_r},$$
$$B^p_L\big(P_d(\Gamma), \mathcal{A}\big) = \lim_{r \to \infty} B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}.$$

On the other hand, for any $r > 0$, if $d < d'$ then $\Gamma_d \subseteq \Gamma_{d'}$ in $P_d(\Gamma) \subseteq P_{d'}(\Gamma)$ so that we have natural inclusions $B^p(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p(P_{d'}(\Gamma), \mathcal{A})_{O_r}$ and $B^p_L(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p_L(P_{d'}(\Gamma), \mathcal{A})_{O_r}$. These inclusions induce the commuting diagram



which allows us to change the order of limits from $\lim_{d\to\infty} \lim_{r\to\infty}$ to $\lim_{r\to\infty} \lim_{d\to\infty}$ in the second piece of the commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*\big(B^p_L\big(P_d(\Gamma), \mathcal{A}\big)\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big) \\
\cong\downarrow & & \cong\downarrow \\
\lim\limits_{d\to\infty} \lim_{r\to\infty} K_*\big(B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} \lim\limits_{r\to\infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big) \\
\cong\downarrow & & \cong\downarrow \\
\lim\limits_{r\to\infty} \lim_{d\to\infty} K_*\big(B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big) & \xrightarrow{e_*} & \lim\limits_{r\to\infty} \lim\limits_{d\to\infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big).
\end{array}$$

So, to prove Theorem 16, it suffices to show that, for any $r > 0$,

$$e_* : \lim_{d\to\infty} K_*\big(B^p_L\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big) \to \lim_{d\to\infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{O_r}\big)$$

is an isomorphism.

Let $r > 0$. Since Γ has bounded geometry and $f : \Gamma \to M$ is a coarse embedding, there exist finitely many mutually disjoint subsets of Γ, say $\Gamma_k := \{\gamma_i : i \in J_k\}$ with some index set J_k for $k = 1, 2, \ldots, k_0$, such that $\Gamma = \bigsqcup_{k=1}^{k_0} \Gamma_k$ and, for each k, $d(f(\gamma_i), f(\gamma_j)) > 2r$ for distinct elements γ_i, γ_j in Γ_k.

For each $k = 1, 2, \ldots, k_0$, let

$$O_{r,k} = \bigcup_{i\in J_k} B\big(f(\gamma_i), r\big).$$

Then $O_r = \bigcup_{k=1}^{k_0} O_{r,k}$ and each $O_{r,k}$, or an intersection of several $O_{r,k}$, is the union of a family of (Γ, r)-separate (Definition 19) open subsets of M.

Now Theorem 16 follows from Lemma 20 together with a Mayer–Vietoris sequence argument by using Lemma 18. ∎

5. Strong Lipschitz homotopy invariance

In this section, we shall present Yu's arguments about strong Lipschitz homotopy invariance for K-theory of the twisted localization algebras [28], and prove Lemma 20 of the previous section.

Let $f : \Gamma \to M$ be a coarse embedding of a bounded geometry discrete metric space Γ into a simply connected complete Riemannian manifold M of nonpositive sectional curvature, and let $r > 0$. Let $\{O_i\}_{i\in J}$ be a family of (Γ, r)-separate open subsets of M, i.e.,

(1) $O_i \cap O_j = \emptyset$ if $i \neq j$;

(2) there exists $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i \in J$.

For $d > 0$, let X_i, $i \in J$, be a family of closed subsets of $P_d(\Gamma)$ such that $\gamma_i \in X_i$ for every $i \in J$ and $\{X_i\}_{i\in J}$ is uniformly bounded in the sense that there exists $r_0 > 0$ such that diameter$(X_i) \le r_0$ for each $i \in J$. In particular, we will consider the following three cases of $\{X_i\}_{i\in J}$:

(1) $X_i = B_{P_d(\Gamma)}(\gamma_i, R) := \{x \in P_d(\Gamma) : d(x, \gamma_i) \le R\}$, for some common $R > 0$ for all $i \in J$;

(2) $X_i = \Delta_i$, a simplex in $P_d(\Gamma)$ with $\gamma_i \in \Delta_i$ for each $i \in J$;

(3) $X_i = \{\gamma_i\}$ for each $i \in J$.

For each $i \in J$, let $\mathcal{A}_{O_i}$ be the subalgebra of $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ generated by those functions whose supports are contained in O_i.

We define $A(X_i : i \in J)$ to be the closed *subalgebra* of the Banach algebra

$$\left\{ \bigoplus_{i\in J} T_i \mid T_i \in B^p(X_i) \otimes_p \mathcal{A}_{O_i},\ \sup_{i\in J} \|T_i\| < \infty \right\}$$

generated by the elements $\bigoplus_{i\in J} T_i$ for which conditions (3) and (4) from Definition 13 are satisfied by all operators T_i, $i \in J$, viewed as functions

$$T_i : (\Gamma_d \cap X_i) \times (\Gamma_d \cap X_i) \to \mathcal{K}_p \otimes_p \mathcal{A}_{O_i},$$

uniformly (cf. [23, 28]).

Similarly, we define $A_{L,\mathrm{alg}}(X_i : i \in J)$ to be the algebra of bounded, uniformly continuous maps

$$g : [0, \infty) \to A(X_i : i \in J)$$

such that if we write

$$g(t) = \bigoplus_{i\in J} g_i(t)$$

then conditions (3) and (4) from Definition 13 are satisfied by all operators $g_i(t)$, $i \in J$, $t \in [0,\infty)$, uniformly, and there exists a bounded function $c(t)$ on $\mathbb{R}_+$ with $\lim_{t\to\infty} c(t) = 0$ such that

$$\big(g_i(t)\big)(x,y) = 0$$

whenever $d(x,y) > c(t)$ for all $i \in J$, $x, y \in \Gamma_d \cap X_i$ and $t \in [0,\infty)$ (cf. [23, 28]).

Define $A_L(X_i : i \in J)$ to be the completion of $A_{L,\mathrm{alg}}(X_i : i \in J)$ for the norm

$$\|g\| = \sup_{t\in[0,\infty)} \big\|g(t)\big\|.$$

Note that there is an evaluation-at-zero map

$$e : A_L(X_i : i \in J) \to A(X_i : i \in J).$$

For each natural number $s > 0$, let $\Delta_i(s)$ be the simplex with vertices $\{\gamma \in \Gamma : d(\gamma, \gamma_i) \le s\}$ in $P_d(\Gamma)$ for $d > s$.

Lemma 21. *Let $O = \bigsqcup_{i\in J} O_i$ be the disjoint union of a family of (Γ, r)-separate open subsets $\{O_i\}_{i\in J}$ of M as above. Then*

1. $B^p(P_d(\Gamma), \mathcal{A})_O \cong \lim_{R\to\infty} A(\{x \in P_d(\Gamma) : d(x,\gamma_i) \le R\} : i \in J)$;
2. $B^p_L(P_d(\Gamma), \mathcal{A})_O \cong \lim_{R\to\infty} A_L(\{x \in P_d(\Gamma) : d(x,\gamma_i) \le R\} : i \in J)$;
3. $\lim_{d\to\infty} B^p(P_d(\Gamma), \mathcal{A})_O \cong \lim_{s\to\infty} A(\Delta_i(s) : i \in J)$;
4. $\lim_{d\to\infty} B^p_L(P_d(\Gamma), \mathcal{A})_O \cong \lim_{s\to\infty} A_L(\Delta_i(s) : i \in J)$.

Proof (cf. [28]*).* Let $\mathcal{A}_O$ be the subalgebra of $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ generated by elements whose supports are contained in O. Let $\mathfrak{B}_O = L^p(O, \mathrm{Cliff}_{\mathbb{C}}(TM))$ and let $\mathfrak{B}_{O,\infty,p}$ be the ℓ^p-direct sum of infinite copies of $\mathfrak{B}_O$ with the ℓ^p-norm

$$\big\|\{f_i\}_{i=1}^{\infty}\big\|_p = \left(\sum_{i=1}^{\infty} \|f_i\|^p\right)^{1/p}, \quad \text{for } \{f_i\}_{i=1}^{\infty} \in \mathfrak{B}_{O,\infty,p}.$$

The algebra $\mathcal{K}_p \otimes_p \mathcal{A}_O$ acts on $\mathfrak{B}_{O,\infty,p}$ and the algebra $B^p(P_d(\Gamma), \mathcal{A})_O$ acts on $\ell^p(\Gamma_d, \mathfrak{B}_{O,\infty,p})$. We have a decomposition

$$\ell^p(\Gamma_d, \mathfrak{B}_{O,\infty,p}) = \Big(\bigoplus_{i\in J} \ell^p(\Gamma_d, \mathfrak{B}_{O_i,\infty,p})\Big)_p.$$

Each $T \in B^p_{\mathrm{alg}}(P_d(\Gamma), \mathcal{A})_O$ has a corresponding decomposition

$$T = \bigoplus_{i\in J} T_i$$

such that there exists $R > 0$ for which each T_i is supported on

$$\big\{(x,y,p) : p \in O_i,\ x, y \in \Gamma_d,\ d(x,\gamma_i) \le R,\ d(y,\gamma_i) \le R\big\}.$$

On the other hand, the Banach algebra $B^p(\{x \in P_d(\Gamma) : d(x,\gamma_i) \le R\}) \otimes_p \mathcal{A}_{O_i}$ acts on

$$\ell^p\big(\{x \in \Gamma_d : d(x,\gamma_i) \le R\}, \mathfrak{B}_{O_i,\infty,p}\big),$$

so that on $\ell^p(\Gamma_d, \mathfrak{B}_{O_i,\infty,p})$, for each $R > 0$, the algebra

$$A\big(\{x \in P_d(\Gamma) : d(x,\gamma_i) \le R\} : i \in J\big)$$

can be represented as a subalgebra of $B^p(P_d(\Gamma), \mathcal{A})_O$. In this way, the decomposition $T = \bigoplus_{i\in J} T_i$ induces a Banach algebra isomorphism

$$B^p\big(P_d(\Gamma), \mathcal{A}\big)_O \cong \lim_{R\to\infty} A\big(\{x \in P_d(\Gamma) : d(x,\gamma_i) \le R\} : i \in J\big)$$

as desired in (1). Then (2), (3), and (4) follow straightforwardly from (1). ■

Now we turn to recall the notion of strong Lipschitz homotopy [26–28].

Let $\{Y_i\}_{i\in J}$ and $\{X_i\}_{i\in J}$ be two families of uniformly bounded closed subspaces of $P_d(\Gamma)$ for some $d > 0$ with $\gamma_i \in X_i$, $\gamma_i \in Y_i$ for every $i \in J$. A map $g : \bigsqcup_{i\in J} X_i \to \bigsqcup_{i\in J} Y_i$ is said to be *Lipschitz* if

(1) $g(X_i) \subseteq Y_i$ for each $i \in J$;

(2) there exists a constant c, independent of $i \in J$, such that

$$d\big(g(x), g(y)\big) \le c\,d(x,y)$$

for all $x, y \in X_i$, $i \in J$.

Let g_1, g_2 be two Lipschitz maps from $\bigsqcup_{i\in J} X_i$ to $\bigsqcup_{i\in J} Y_i$. We say g_1 is *strongly Lipschitz homotopy* equivalent to g_2 if there exists a continuous map

$$F : [0,1] \times \Big(\bigsqcup_{i\in J} X_i\Big) \to \bigsqcup_{i\in J} Y_i$$

such that

(1) $F(0,x) = g_1(x)$, $F(1,x) = g_2(x)$ for all $x \in \bigsqcup_{i\in J} X_i$;

(2) there exists a constant c for which $d(F(t,x), F(t,y)) \le c\,d(x,y)$ for all $x, y \in X_i$, $t \in [0,1]$, where i is any element in J;

(3) F is equicontinuous in t, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d\big(F(t_1,x), F(t_2,x)\big) < \varepsilon \quad \text{for all } x \in \bigsqcup_{i\in J} X_i \text{ if } |t_1 - t_2| < \delta.$$

We say $\{X_i\}_{i\in J}$ is *strongly Lipschitz homotopy* equivalent to $\{Y_i\}_{i\in J}$ if there exist Lipschitz maps $g_1 : \bigsqcup_{i\in J} X_i \to \bigsqcup_{i\in J} Y_i$ and $g_2 : \bigsqcup_{i\in J} Y_i \to \bigsqcup_{i\in J} X_i$ such that $g_1 g_2$ and $g_2 g_1$ are, respectively, strongly Lipschitz homotopy equivalent to identity maps.

Define $A_{L,0}(X_i : i \in J)$ to be the subalgebra of $A_L(X_i : i \in J)$ consisting of elements $\bigoplus_{i\in J} b_i(t)$ satisfying $b_i(0) = 0$ for all $i \in J$.

Lemma 22 ([28]). *If $\{X_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i\in J}$, then $K_*(A_{L,0}(X_i : i \in J))$ is isomorphic to $K_*(A_{L,0}(Y_i : i \in J))$.*

Let e be the evaluation homomorphism from $A_L(X_i : i \in J)$ to $A(X_i : i \in J)$ given by $\bigoplus_{i\in J} g_i(t) \mapsto \bigoplus_{i\in J} g_i(0)$.

Lemma 23 ([28]). *Let $\{\gamma_i\}_{i\in J}$ be as above, i.e., $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each i. If $\{\Delta_i\}_{i\in J}$ is a family of simplices in $P_d(\Gamma)$ for some $d > 0$ such that $\gamma_i \in \Delta_i$ for all $i \in J$, then*

$$e_* : K_*\big(A_L(\Delta_i : i \in J)\big) \to K_*\big(A(\Delta_i : i \in J)\big)$$

is an isomorphism.

Proof ([28]). Note that $\{\Delta_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{\gamma_i\}_{i\in J}$. By an argument of Eilenberg swindle, we have $K_*(A_{L,0}(\{\gamma_i\} : i \in J)) = 0$. Consequently, Lemma 23 follows from Lemma 22 and the six term exact sequence of Banach algebra K-theory. ∎

We are now ready to give a proof to Lemma 20 of the previous section.

Proof of Lemma 20 [28]. By Lemma 21, we have the commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i\in J} O_i} & \xrightarrow{\ e\ } & \lim\limits_{d\to\infty} B^p\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i\in J} O_i} \\
\cong \downarrow & & \downarrow \cong \\
\lim\limits_{s\to\infty} A_L\big(\Delta_i(s)_i : i \in J\big) & \xrightarrow{\ e\ } & \lim\limits_{s\to\infty} A\big(\Delta_i(s)_i : i \in J\big)
\end{array}$$

which induces the following commuting diagram at K-theory level:

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i\in J} O_i}\big) & \xrightarrow{\ e_*\ } & \lim\limits_{d\to\infty} K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)_{\bigsqcup_{i\in J} O_i}\big) \\
\cong \downarrow & & \downarrow \cong \\
\lim\limits_{s\to\infty} K_*\big(A_L\big(\Delta_i(s) : i \in J\big)\big) & \xrightarrow{\ e_*\ } & \lim\limits_{s\to\infty} K_*\big(A\big(\Delta_i(s) : i \in J\big)\big).
\end{array}$$

Now Lemma 20 follows from Lemma 23. ∎

6. Almost flat Bott elements and Bott maps

In this section, we shall construct uniformly almost flat Bott generators for a simply connected complete Riemannian manifold of nonpositive sectional curvature, and define a Bott map from the K-theory of the ℓ^p-Roe algebra to the K-theory of the twisted ℓ^p-Roe algebra and another Bott map between the K-theories of corresponding ℓ^p-localization algebras. We show that the Bott map from the K-theory of the ℓ^p-localization algebra to the K-theory of the twisted ℓ^p-localization algebra is an isomorphism (Theorem 27).

Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature. As remarked at the beginning of Section 4, without loss of generality, we assume in the following $\dim(M) = 2n$ for some integer $n > 0$. Recall that $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ is the C^*-algebra of continuous sections of the complex Clifford algebra bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$ of the tangent bundle of M vanishing at infinity. Let $\mathcal{B} := C_b(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ be the C^*-algebra of all continuous bounded sections of $\mathrm{Cliff}_{\mathbb{C}}(TM)$.

We can consider $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ and $\mathcal{B} = C_b(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ as L^p-operator algebras, acting on $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$, the L^p-space of the locally measurable sections of the Hilbert space bundle $\mathrm{Cliff}_{\mathbb{C}}(TM)$. Since $\mathcal{A}$ and $\mathcal{B}$ act on $L^p(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ by pointwise multiplication, both algebras have equivalent norms for different $p \in (1, \infty)$. Hence the K-theory of both $\mathcal{A}$ and $\mathcal{B}$ does not depend on p. In particular, the Bott element is still a generator for $K_0(\mathcal{A})$ if $\mathcal{A}$ is viewed as an L^p-operator algebra.

Let $x \in M$. For any $z \in M$, let $\sigma : [0, 1] \to M$ be the unique geodesic such that

$$\sigma(0) = x, \quad \sigma(1) = z.$$

Let $v_x(z) := \frac{\sigma'(1)}{\|\sigma'(1)\|} \in T_z M$. For any $c > 0$, take a continuous function $\phi_{x,c} : M \to [0, 1]$ satisfying

$$\phi_{x,c}(z) = \begin{cases} 0, & \text{if } d(x,z) \le \frac{c}{2}; \\ 1, & \text{if } d(x,z) \ge c. \end{cases} \tag{6.1}$$

For any $z \in M$, let

$$f_{x,c}(z) := \phi_{x,c}(z) \cdot v_x(z) \in T_z M.$$

Then $f_{x,c} \in \mathfrak{B}$. The following result describes certain "uniform almost flatness" of the functions $f_{x,c}$ ($x \in M$, $c > 0$).

Lemma 24. *For any $R > 0$ and $\varepsilon > 0$, there exist a constant $c > 0$ and a family of continuous functions $\{\phi_{x,c}\}_{x \in M}$ satisfying (6.1) such that, if $d(x, y) < R$, then*

$$\sup_{z \in M} \left\| f_{x,c}(z) - f_{y,c}(z) \right\|_{T_z M} < \varepsilon.$$

Proof. Let $c = \frac{2R}{\varepsilon}$. For any $x \in M$, define $\phi_{x,c} : M \to [0, 1]$ by

$$\phi_{x,c}(z) = \begin{cases} 0, & \text{if } d(x,z) \le \frac{R}{\varepsilon}; \\ \frac{\varepsilon}{R} d(x,z) - 1, & \text{if } \frac{R}{\varepsilon} \le d(x,z) \le \frac{2R}{\varepsilon}; \\ 1, & \text{if } d(x,z) \ge \frac{2R}{\varepsilon}. \end{cases}$$

Let $x, y \in M$ such that $d(x, y) < R$. Then we have several cases for the position of $z \in M$ with respect to x, y.

Consider the case where $d(x,z) > c = \frac{2R}{\varepsilon}$ and $d(y,z) > c = \frac{2R}{\varepsilon}$. Since $\phi_{x,c}(z) = \phi_{y,c}(z) = 1$, we have

$$f_{x,c}(z) - f_{y,c}(z) = v_x(z) - v_y(z).$$

Without loss of generality, assume $d(x,z) \le d(y,z)$. Then there exists a unique point y' on the unique geodesic connecting y and z such that $d(y', z) = d(x, z)$. Then $d(y', y) < R$ since $d(x, y) < R$, so that $d(x, y') < 2R$.

Let $\exp_z^{-1} : M \to T_z M$ denote the inverse of the exponential map

$$\exp_z : T_z M \to M$$

at $z \in M$. Then we have

(α) $\| \exp_z^{-1}(x)\| = d(x, z) = d(y', z) = \| \exp_z^{-1}(y')\| > c = \frac{2R}{\varepsilon}$;

(β) $\| \exp_z^{-1}(x) - \exp_z^{-1}(y')\| \leq d(x, y') < 2R$, since M has nonpositive sectional curvature;

(γ) $v_x(z) = -\frac{\exp_z^{-1}(x)}{\|\exp_z^{-1}(x)\|}$ and $v_y(z) = -\frac{\exp_z^{-1}(y')}{\|\exp_z^{-1}(y')\|}$.

Hence, for any $z \in M$, we have

$$\left\| f_{x,c}(z) - f_{y,c}(z)\right\| = \left\|v_x(z) - v_y(z)\right\| < 2R/(2R/\varepsilon) = \varepsilon$$

whenever $d(x, y) < R$. Similarly, we can check the inequality in other cases where $z \in M$ satisfies either $d(x, z) \leq c$ or $d(y, z) \leq c$. ∎

Now let us consider the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\pi} \mathcal{B}/\mathcal{A} \to 0,$$

where $\mathcal{A} = C_0(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$ and $\mathcal{B} = C_b(M, \mathrm{Cliff}_{\mathbb{C}}(TM))$. For any $f_{x,c}$ ($x \in M$, $c > 0$) constructed above, it is easy to see that $[f_{x,c}] := \pi(f_{x,c})$ is invertible in $\mathcal{B}/\mathcal{A}$ with its inverse $[-f_{x,c}]$. Thus $[f_{x,c}]$ defines an element in $K_1(\mathcal{B}/\mathcal{A})$. With the help of the index map

$$\partial : K_1(\mathcal{B}/\mathcal{A}) \to K_0(\mathcal{A}),$$

we obtain an element $\partial([f_{x,c}])$ in

$$K_0(\mathcal{A}) = K_0\big(C_0\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big)\big) \cong K_0\big(C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_{2^n}(\mathbb{C})\big) \cong \mathbb{Z}.$$

It follows from the construction of $f_{x,c}$ that, for every $x \in M$ and $c > 0$, $\partial([f_{x,c}])$ is just the Bott generator of $K_0(\mathcal{A})$.

The element $\partial([f_{x,c}])$ can be expressed explicitly as follows. Let

$$\begin{aligned} W_{x,c} &= \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{x,c} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ b_{x,c} &= W_{x,c} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_{x,c}^{-1}, \\ b_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then both $b_{x,c}$ and b_0 are idempotents in $\mathcal{M}_2(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the algebra jointing a unit to $\mathcal{A}$. It is easy to check that

$$b_{x,c} - b_0 \in C_c\big(M, \mathrm{Cliff}_{\mathbb{C}}(TM)\big) \otimes \mathcal{M}_2(\mathbb{C}),$$

the algebra of 2×2 matrices of compactly supported continuous functions, with

$$\operatorname{Supp}(b_{x,c} - b_0) \subset B_M(x,c) := \{z \in M : d(x,z) \le c\},$$

where for a matrix $a = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ of functions on M we define the support of a by

$$\operatorname{Supp}(a) = \bigcup_{i,j=1}^{2} \operatorname{Supp}(a_{i,j}).$$

Now we have the explicit expression

$$\partial([f_{x,c}]) = [b_{x,c}] - [b_0] \in K_0(\mathcal{A}).$$

Lemma 25 (Uniform almost flatness of the Bott generators). *The family of idempotents* $\{b_{x,c}\}_{x\in M, c>0}$ *in* $\mathcal{M}_2(\mathcal{A}^+) = C_0(M, \operatorname{Cliff}_{\mathbb{C}}(TM))^+ \otimes \mathcal{M}_2(\mathbb{C})$ *constructed above are uniformly almost flat in the following sense: for any* $R > 0$ *and* $\varepsilon > 0$*, there exist* $c > 0$ *and a family of continuous functions* $\{\phi_{x,c} : M \to [0,1]\}_{x\in M}$ *such that, whenever* $d(x,y) < R$*, one has*

$$\sup_{z\in M} \|b_{x,c}(z) - b_{y,c}(z)\|_{\operatorname{Cliff}_{\mathbb{C}}(T_zM)\otimes\mathcal{M}_2(\mathbb{C})} < \varepsilon,$$

where $b_{x,c}$ *is defined via* $W_{x,c}$ *and* $f_{x,c} = \phi_{x,c} v_x$ *as above, and* $\operatorname{Cliff}_{\mathbb{C}}(T_zM)$ *is the complexified Clifford algebra of the tangent space* T_zM.

Proof. Straightforward from Lemma 24. ∎

It would be convenient to introduce the following notion.

Definition 26. For $R > 0$, $\varepsilon > 0$, and $c > 0$, a family of idempotents $\{b_x\}_{x\in M}$ in $\mathcal{M}_2(\mathcal{A}^+) = C_0(M, \operatorname{Cliff}_{\mathbb{C}}(TM))^+ \otimes \mathcal{M}_2(\mathbb{C})$ is said to be $(R, \varepsilon; c)$-flat if

(1) for any $x, y \in M$ with $d(x,y) < R$ we have

$$\sup_{z\in M} \|b_x(z) - b_y(z)\|_{\operatorname{Cliff}_{\mathbb{C}}(T_zM)\otimes\mathcal{M}_2(\mathbb{C})} < \varepsilon;$$

(2) $b_x - b_0 \in C_c(M, \operatorname{Cliff}_{\mathbb{C}}(TM)) \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\operatorname{Supp}(b_x - b_0) \subset B_M(x,c) := \{z \in M : d(x,z) \le c\}.$$

Construction of the Bott map β_*

Now we shall use the above almost flat Bott generators for

$$K_0(\mathcal{A}) = K_0(C_0(M, \operatorname{Cliff}_{\mathbb{C}}(TM)))$$

to construct a “Bott map”

$$\beta_* : K_*(B^p(P_d(\Gamma))) \to K_*(B^p(P_d(\Gamma), \mathcal{A})).$$

To begin with, we give a representation of $B^p(P_d(\Gamma))$ on $\ell^p(\Gamma_d, \ell^p)$, where Γ_d is the countable dense subset of $P_d(\Gamma)$ as in the definition of $B^p(P_d(\Gamma), \mathcal{A})$.

Let $B^p_{\text{alg}}(P_d(\Gamma))$ be the algebra of functions

$$Q : \Gamma_d \times \Gamma_d \to \mathcal{K}_p$$

such that

(1) there exists $C > 0$ such that $\|Q(x, y)\| \le C$ for all $x, y \in \Gamma_d$;
(2) there exists $R > 0$ such that $Q(x, y) = 0$ whenever $d(x, y) > R$;
(3) there exists $L > 0$ such that, for every $z \in P_d(\Gamma)$, the number of elements in the set

$$\big\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \le 3R,\ d(y, z) \le 3R,\ Q(x, y) \ne 0\big\}$$

is less than L.

The product structure on $B^p_{\text{alg}}(P_d(\Gamma))$ is defined by

$$(Q_1 Q_2)(x, y) = \sum_{z \in \Gamma_d} Q_1(x, z) Q_2(z, y).$$

The algebra $B^p_{\text{alg}}(P_d(\Gamma))$ acts on $\ell^p(\Gamma_d, \ell^p)$. The operator norm completion of $B^p_{\text{alg}}(P_d(\Gamma))$ with respect to this action is isomorphic to $B^p(P_d(\Gamma))$ when Γ has bounded geometry.

Note that $B^p(P_d(\Gamma))$ is stable in the sense that $B^p(P_d(\Gamma)) \cong B^p(P_d(\Gamma)) \otimes_p \mathcal{M}_k(\mathbb{C})$ for all natural number k. Any element in $K_0(B^p(P_d(\Gamma)))$ can be expressed as the difference of the K_0-classes of two idempotents in $B^p(P_d(\Gamma))$. To define the Bott map

$$\beta_* : K_0\big(B^p\big(P_d(\Gamma)\big)\big) \to K_0\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big),$$

we need to specify the value $\beta_*([P])$ in $K_0(B^p(P_d(\Gamma), \mathcal{A}))$ for any idempotent $P \in B^p(P_d(\Gamma))$.

Now let $P \in B^p(P_d(\Gamma)) \subseteq \mathcal{B}(\ell^p(\Gamma_d, \ell^p))$ be an idempotent. Denote $\|P\| = N$. For any $0 < \varepsilon_1 < 1/100$, take an element $Q \in B^p_{\text{alg}}(P_d(\Gamma))$ such that

$$\|P - Q\| < \frac{\varepsilon_1}{2N + 2}.$$

Then $\|Q\| < \|P - Q\| + \|P\| < N + 1$, hence

$$\|Q - Q^2\| \le \|Q - P\| + \|P\|\|P - Q\| + \|P - Q\|\|Q\| < \varepsilon_1,$$

and there is $R_{\varepsilon_1} > 0$ such that $Q(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}$. For any $\varepsilon_2 > 0$, take by Lemma 25 a family of $(R_{\varepsilon_1}, \varepsilon_2; c)$-flat idempotents $\{b_x\}_{x \in M}$ in $\mathcal{M}_2(\mathcal{A}^+)$ for some $c > 0$. Define

$$\widetilde{Q}, \widetilde{Q}_0 : \Gamma_d \times \Gamma_d \to \mathcal{K}_p \otimes_p \mathcal{A}^+ \otimes_p \mathcal{M}_2(\mathbb{C})$$

by

$$\widetilde{Q}(x,y) = Q(x,y) \otimes b_x,$$
$$\widetilde{Q}_0(x,y) = Q(x,y) \otimes b_0,$$

respectively, for all $(x,y) \in \Gamma_d \times \Gamma_d$, where $b_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$\widetilde{Q}, \widetilde{Q}_0 \in B^p_{\mathrm{alg}}\big(P_d(\Gamma), \mathcal{A}^+ \otimes_p \mathcal{M}_2(\mathbb{C})\big) \cong B^p_{\mathrm{alg}}\big(P_d(\Gamma), \mathcal{A}^+\big) \otimes_p \mathcal{M}_2(\mathbb{C})$$

and

$$\widetilde{Q} - \widetilde{Q}_0 \in B^p_{\mathrm{alg}}\big(P_d(\Gamma), \mathcal{A}\big) \otimes_p \mathcal{M}_2(\mathbb{C}).$$

Since Γ has bounded geometry, by the almost flatness of the Bott generators (Lemma 25), we can choose ε_1 and ε_2 small enough to obtain $\widetilde{Q}$, $\widetilde{Q}_0$ as constructed above such that $\|\widetilde{Q}^2 - \widetilde{Q}\| < 1/5$ and $\|\widetilde{Q}_0^2 - \widetilde{Q}_0\| < 1/5$.

It follows that the spectrum of either $\widetilde{Q}$ or $\widetilde{Q}_0$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $f : S_0 \sqcup S_1 \to \mathbb{C}$ be the holomorphic function such that $f(S_0) = \{0\}$, $f(S_1) = \{1\}$. Let $\Theta = f(\widetilde{Q})$ and $\Theta_0 = f(\widetilde{Q}_0)$. Then Θ and Θ_0 are idempotents in $B^p(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$ with

$$\Theta - \Theta_0 \in B^p\big(P_d(\Gamma), \mathcal{A}\big) \otimes \mathcal{M}_2(\mathbb{C}).$$

Note that $B^p(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C})$ is a closed two-sided ideal of $B^p(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$.

At this point, we need to recall the *difference construction* in K-theory of Banach algebras introduced by Kasparov–Yu [15]. Let J be a closed two-sided ideal of a Banach algebra B. Let $p, q \in B^+$ be idempotents such that $p - q \in J$. Then a difference element $D(p,q) \in K_0(J)$ associated to the pair p, q is defined as follows. Let

$$Z(p,q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

We have

$$\big(Z(p,q)\big)^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

Define

$$D_0(p,q) = \big(Z(p,q)\big)^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p,q).$$

Let

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $D_0(p,q) \in \mathcal{M}_4(J^+)$ and $D_0(p,q) = p_1$ modulo $\mathcal{M}_4(J)$. We define the difference element

$$D(p,q) := \big[D_0(p,q)\big] - [p_1]$$

in $K_0(J)$.

Finally, for any idempotent $P \in B^p(P_d(\Gamma))$ which represents a K-theory element $[P]$ in $K_0(B^p(P_d(\Gamma)))$, we define

$$\beta_*\big([P]\big) = D(\Theta, \Theta_0) \in K_0\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

The correspondence $[P] \to \beta_*([P])$ extends to a homomorphism, the Bott map

$$\beta_* : K_0\big(B^p\big(P_d(\Gamma)\big)\big) \to K_0\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

By using suspension, we similarly define the Bott map

$$\beta_* : K_1\big(B^p\big(P_d(\Gamma)\big)\big) \to K_1\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

Construction of the Bott map $(\beta_L)_*$

Next we shall construct a Bott map for K-theory of ℓ^p-localization algebras:

$$(\beta_L)_* : K_*\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

Let $B^p_{L,\mathrm{alg}}(P_d(\Gamma))$ be the algebra of all bounded, uniformly continuous functions

$$g : \mathbb{R}_+ \to B^p_{\mathrm{alg}}\big(P_d(\Gamma)\big) \subset \mathcal{B}\big(\ell^p(\Gamma_d, \ell^p)\big)$$

with the following properties:

(1) there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} R(t) = 0$ such that $g(t)(x,y) = 0$ whenever $d(x,y) > R(t)$ for every t;

(2) there exists $L > 0$ such that, for every $z \in P_d(\Gamma)$, the number of elements in the set

$$\big\{(x,y) \in \Gamma_d \times \Gamma_d : d(x,z) \le 3R,\ d(y,z) \le 3R,\ g(t)(x,y) \ne 0\big\}$$

is less than L for every $t \in \mathbb{R}_+$.

The ℓ^p-localization algebra $B^p_L(P_d(\Gamma))$ is isomorphic to the norm completion of $B^p_{L,\mathrm{alg}}(P_d(\Gamma))$ under the norm

$$\|g\|_\infty := \sup_{t\in\mathbb{R}_+} \big\|g(t)\big\|$$

when Γ has bounded geometry. Note that $B_L^p(P_d(\Gamma))$ is stable in the sense that

$$B_L^p\big(P_d(\Gamma)\big) \cong B_L^p\big(P_d(\Gamma)\big) \otimes_p \mathcal{M}_k(\mathbb{C})$$

for all natural number k. Hence any element in $K_0(B_L^p(P_d(\Gamma)))$ can be expressed as the difference of the K_0-classes of two idempotents in $B_L^p(P_d(\Gamma))$. To define the Bott map

$$(\beta_L)_* : K_0\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_0\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big),$$

we need to specify the value $(\beta_L)_*([g])$ in $K_0(B_L^p(P_d(\Gamma), \mathcal{A}))$ for any idempotent $g \in B_L^p(P_d(\Gamma))$ representing an element $[g] \in K_0(B_L^p(P_d(\Gamma)))$.

Now let $g \in B_L^p(P_d(\Gamma))$ be an idempotent with $\|g\| = N$. For any $0 < \varepsilon_1 < 1/100$, take an element $h \in B_{L,\mathrm{alg}}^p(P_d(\Gamma))$ such that

$$\|g - h\|_\infty < \frac{\varepsilon_1}{2N + 2}.$$

Then $\|h - h^2\|_\infty < \varepsilon_1$ and there is a bounded function $R_{\varepsilon_1}(t) > 0$ with $\lim_{t\to\infty} R_{\varepsilon_1}(t) = 0$ such that $h(t)(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}(t)$ for every t. Let $\widetilde{R}_{\varepsilon_1} = \sup_{t\in\mathbb{R}_+} R(t)$. For any $\varepsilon_2 > 0$, take by Lemma 25 a family of $(\widetilde{R}_{\varepsilon_1}, \varepsilon_2; c)$-flat idempotents $\{b_x\}_{x\in M}$ in $\mathcal{M}_2(\mathcal{A}^+)$ for some $c > 0$. Define

$$\widetilde{h}, \widetilde{h}_0 : \mathbb{R}_+ \to B_{\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}^+\big) \otimes_p \mathcal{M}_2(\mathbb{C})$$

by

$$\big(\widetilde{h}(t)\big)(x, y) = \big(h(t)(x, y)\big) \otimes_p b_x \in \mathcal{K}_p \otimes_p \mathcal{A}^+ \otimes_p \mathcal{M}_2(\mathbb{C}),$$

$$\big(\widetilde{h}_0(t)\big)(x, y) = \big(h(t)(x, y)\big) \otimes_p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{K}_p \otimes_p \mathcal{A}^+ \otimes \mathcal{M}_2(\mathbb{C}),$$

for each $t \in \mathbb{R}_+$. Then we have

$$\widetilde{h}, \widetilde{h}_0 \in B_{L,\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}^+\big) \otimes_p \mathcal{M}_2(\mathbb{C})$$

and

$$\widetilde{h} - \widetilde{h}_0 \in B_{L,\mathrm{alg}}^p\big(P_d(\Gamma), \mathcal{A}\big) \otimes_p \mathcal{M}_2(\mathbb{C}).$$

Since Γ has bounded geometry, by the almost flatness of the Bott generators, we can choose ε_1 and ε_2 small enough to obtain $\widetilde{h}$, $\widetilde{h}_0$, as constructed above, such that $\|\widetilde{h}^2 - \widetilde{h}\|_\infty < 1/5$ and $\|\widetilde{h}_0^2 - \widetilde{h}_0\| < 1/5$. The spectrum of either $\widetilde{h}$ or $\widetilde{h}_0$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $f : S_0 \sqcup S_1 \to \mathbb{C}$ be the function such that $f(S_0) = \{0\}$, $f(S_1) = \{1\}$. Let $\eta = f(\widetilde{h})$ and $\eta_0 = f(\widetilde{h}_0)$. Then η and η_0 are idempotents in $B_L^p(P_d(\Gamma), \mathcal{A}^+) \otimes_p \mathcal{M}_2(\mathbb{C})$ with

$$\eta - \eta_0 \in B_L^p\big(P_d(\Gamma), \mathcal{A}\big) \otimes_p \mathcal{M}_2(\mathbb{C}).$$

Thanks to the difference construction, we define

$$(\beta_L)_*\big([g]\big) = D(\eta, \eta_0) \in K_0\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

This correspondence $[g] \mapsto (\beta_L)_*([g])$ extends to a homomorphism, the Bott map

$$(\beta_L)_* : K_0\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_0\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

By suspension, we similarly define

$$(\beta_L)_* : K_1\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_1\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big).$$

This completes the construction of the Bott map $(\beta_L)_*$.

It follows from the constructions of β_* and $(\beta_L)_*$ that we have the commuting diagram

$$\begin{array}{ccc} K_*\big(B_L^p\big(P_d(\Gamma)\big)\big) & \xrightarrow{(\beta_L)_*} & K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big) \\ {\scriptstyle e_*}\downarrow & & \downarrow{\scriptstyle e_*} \\ K_*\big(B^p\big(P_d(\Gamma)\big)\big) & \xrightarrow{\beta_*} & K_*\big(B^p\big(P_d(\Gamma), \mathcal{A}\big)\big). \end{array}$$

Theorem 27. *For any $d \geq 0$, the Bott map*

$$(\beta_L)_* : K_*\big(B_L^p\big(P_d(\Gamma)\big)\big) \to K_*\big(B_L^p\big(P_d(\Gamma), \mathcal{A}\big)\big)$$

is an isomorphism.

Proof. Note that Γ has bounded geometry, and both the ℓ^p-localization algebra and the twisted ℓ^p-localization algebra have strong Lipschitz homotopy invariance at the K-theory level. By a Mayer–Vietoris sequence argument and induction on the dimension of the skeletons [2, 26], the general case can be reduced to the 0-dimensional case; namely, if $D \subset P_d(\Gamma)$ is a δ-separated subspace (meaning $d(x, y) \geq \delta$ if $x \neq y \in D$) for some $\delta > 0$, then

$$(\beta_L)_* : K_*\big(B_L^p(D)\big) \to K_*\big(B_L^p(D, \mathcal{A})\big)$$

is an isomorphism. But this follows from the facts that

$$K_*\big(B_L^p(D)\big) \cong \prod_{\gamma \in D} K_*\big(B_L^p(\{\gamma\})\big),$$

$$K_*\big(B_L^p(D, \mathcal{A})\big) \cong \prod_{\gamma \in D} K_*\big(B_L^p(\{\gamma\}, \mathcal{A})\big)$$

and that $(\beta_L)_*$ restricts to an isomorphism from $K_*(B_L^p(\{\gamma\})) \cong K_*(\mathcal{K}_p)$ to

$$K_*\big(B_L^p(\{\gamma\}, \mathcal{A})\big) \cong K_*\big(\mathcal{K}_p \otimes \mathcal{A}\big)$$

at each $\gamma \in D$ by the classical Bott periodicity. ∎

7. Proof of the main theorem

Proof of Theorem 1. We have the commuting diagram

$$\begin{array}{ccc} \lim\limits_{d\to\infty} K_*\big(B_L^p(P_d(\Gamma))\big) & \xrightarrow[\cong]{(\beta_L)_*} & \lim\limits_{d\to\infty} K_*\big(B_L^p(P_d(\Gamma),\mathcal{A})\big) \\ {\scriptstyle e_*}\big\downarrow & & \cong\big\downarrow{\scriptstyle e_*} \\ \lim\limits_{d\to\infty} K_*\big(B^p(P_d(\Gamma))\big) & \xrightarrow{\beta_*} & \lim\limits_{d\to\infty} K_*\big(B^p(P_d(\Gamma),\mathcal{A})\big). \end{array}$$

Hence $\beta_* \circ e_* = e_* \circ (\beta_L)_*$. It follows from Theorems 16 and 27 that $\beta_* \circ e_*$ is an isomorphism. Consequently, the assembly map

$$\mu = e_* : \lim_{d\to\infty} K_*\big(B_L^p(P_d(\Gamma))\big) \to \lim_{d\to\infty} K_*\big(B^p(P_d(\Gamma))\big) \cong K_*\big(B^p(\Gamma)\big)$$

is injective. ∎

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