J. Noncommut. Geom. 15 (2021), 1485–1500
DOI 10.4171/JNCG/445

Quadratic Lie conformal superalgebras related to Novikov superalgebras

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Abstract. We study quadratic Lie conformal superalgebras associated with Novikov superalgebras. For every Novikov superalgebra $(V, \circ)$, we construct an enveloping differential Poisson superalgebra $U(V)$ with a derivation d such that $u \circ v = ud(v)$ and $\{u, v\} = u \circ v - (-1)^{|u||v|} v \circ u$ for $u, v \in V$. The latter means that the commutator Gelfand–Dorfman superalgebra of V is special. Next, we prove that every quadratic Lie conformal superalgebra constructed on a finite-dimensional special Gelfand–Dorfman superalgebra has a finite faithful conformal representation. This statement is a step towards a solution of the following open problem: whether a finite Lie conformal (super)algebra has a finite faithful conformal representation.

1. Introduction

Novikov algebras appeared in [9] as a class of algebras giving rise to Hamiltonian operators in the formal calculus of variations. Independently, these algebras were introduced in [4] as a tool for studying linear Poisson brackets of hydrodynamic type. The study of the structure theory of Novikov algebras was initiated in [23]; significant progress in this direction was obtained in [1, 7, 19, 21].

A class of more complicated structures called Gelfand–Dorfman bialgebras [20] was also introduced in [9] as a source of Hamiltonian operators. A Gelfand–Dorfman bialgebra is a linear space with two bilinear operations $(\cdot \circ \cdot)$ and $[\cdot, \cdot]$, where $(\cdot \circ \cdot)$ is a Novikov product (left symmetric, right commutative), i.e.,

$$(x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3) = (x_2 \circ x_1) \circ x_3 - x_2 \circ (x_1 \circ x_3), \tag{1.1}$$

$$(x_1 \circ x_2) \circ x_3 = (x_1 \circ x_3) \circ x_2, \tag{1.2}$$

$[\cdot, \cdot]$ is a Lie product, and the following compatibility relation holds:

$$[x_1, x_2 \circ x_3] - [x_3, x_2 \circ x_1] + [x_2, x_1] \circ x_3 - [x_2, x_3] \circ x_1 - x_2 \circ [x_1, x_3] = 0. \tag{1.3}$$

In order to avoid confusion with the well-known notion of a bialgebra as an algebra equipped with a coproduct, we will use the term *GD-algebra* [16] for a Gelfand–Dorfman bialgebra.

2020 Mathematics Subject Classification. 17B69, 17B63, 37K30.
Keywords. Poisson superalgebra, Novikov superalgebra, Gelfand–Dorfman superalgebra, conformal superalgebra.

Novikov algebras and GD-algebras play an important role in the combinatorics of differential algebras (see [16, Theorem 7]). Namely, the identities that hold for the operations

$$a \succ b = d(a) \cdot b, \quad a \prec b = a \cdot d(b),$$

$a, b \in A$, with A being an algebra in a multilinear variety Var, may be calculated by means of the Manin white product [10] of the operad Var and the operads Nov and $\mathrm{GD}^!$, the Koszul dual of GD.

Every differential Poisson algebra gives rise to a GD-algebra [22]: if P is a commutative differential algebra with a derivation d equipped with a Poisson bracket $\{\cdot, \cdot\}$ such that d acts as a derivation relative to $\{\cdot, \cdot\}$, then P is a GD-algebra with operations $x \circ y = zd(y)$, $[x, y] = \{x, y\}$, $x, y \in P$. GD-algebras that embed into Poisson differential algebras in this way are called special [16]. Not all GD-algebras are special; a series of necessary conditions for a GD-algebra to be special was found in the last paper. Below, we will state some precise examples of non-special GD-algebras.

It was noted in [9, Remark 6.3] that if we enrich a Novikov algebra $(V, \circ)$ with the operation $[x, y] = x \circ y - y \circ x$, $x, y \in V$, then we obtain a GD-algebra. It is not hard to check that such a system obtained from a Novikov algebra meets the necessary conditions of speciality found in [16]. In Section 2, we prove that all GD-superalgebras arising from Novikov superalgebras relative to (super-)commutator are indeed special, and we construct an enveloping differential Poisson superalgebra for every Novikov algebra.

It turned out (see [11, 20]) that GD-algebras are in one-to-one correspondence with quadratic Lie conformal algebras. The latter structures appeared in [12] as a tool in the study of vertex operator algebras. Conformal algebras and their generalizations (pseudoalgebras) also turn to be useful for the classification of Poisson brackets of hydrodynamic type [2]. A conformal algebra is a module C over the polynomial algebra H equipped with a "multi-valued" operation $C \otimes C \to C[\lambda]$; i.e., a product of two elements from C is a polynomial in a formal variable λ with coefficients in C. The axioms defining a conformal algebra are stated in Section 3.

One of the most intriguing questions in the theory of conformal algebras is motivated by the Ado theorem: Does a Lie conformal algebra which is a finitely generated free module over H have a faithful representation on a finitely generated free H-module? (The condition of freeness is necessary: in conformal algebras and their modules, every torsion element belongs to the corresponding annihilator.) In the case of a positive answer, we may faithfully represent every Lie conformal algebra with polynomial matrices in Cend_n; see [6]. By now, the most general result in this direction says that if a Lie conformal algebra as above has a Levi decomposition (i.e., its solvable radical splits), then it has a finite faithful representation (FFR).

Therefore, one may observe close relations between Novikov algebras, Poisson algebras, and conformal algebras. In Section 4, we prove that every quadratic Lie conformal superalgebra obtained from a special GD-algebra has a FFR. As an application, every quadratic Lie conformal superalgebra obtained from a Novikov superalgebra has a FFR.

2. Gelfand–Dorfman superalgebras

A $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$ with two bilinear operations $(\cdot \circ \cdot), [\cdot, \cdot] : V \otimes V \to V$ which respect the grading is said to be a GD-superalgebra if

- V is a Novikov superalgebra relative to $(\cdot \circ \cdot)$;
- V is a Lie superalgebra relative to $[\cdot, \cdot]$;
- for every homogeneous $a, b, c \in V$

$$[a \circ b, c] - a \circ [b, c] + [a, b] \circ c + (-1)^{|a||b|}[b, a \circ c] - (-1)^{|b||c|}[a, c] \circ b = 0. \quad (2.1)$$

A series of examples of GD-superalgebras may be obtained from differential Poisson superalgebras. Let $P = P_0 \oplus P_1$ be an associative and commutative superalgebra equipped with an operation $\{\cdot, \cdot\}$ such that

$$\{P_i, P_j\} \subseteq P_{(i+j) \pmod 2}, \quad \{a, b\} = -(-1)^{|a||b|}\{b, a\},$$
$$\{a, \{b, c\}\} - (-1)^{|a||b|}\{b, \{a, c\}\} = \{\{a, b\}, c\}, \quad (2.2)$$
$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\}, \quad (2.3)$$

for all homogeneous $a, b, c \in P$. Then P is called a Poisson superalgebra.

Suppose a Poisson superalgebra P has an (even) derivation $d : P \to P$, i.e.,

$$d(P_i) \subseteq P_i, \quad d(ab) = d(a)b + ad(b), \quad d\{a, b\} = \{d(a), b\} + \{a, d(b)\},$$

for all $a, b \in P$. Then the same space P equipped with

$$a \circ b = ad(b), \quad [a, b] = \{a, b\}$$

is a GD-superalgebra [20, Theorem 3.2] denoted by $P^{(d)}$.

For a GD-superalgebra V, we say V is *special* if there exists a Poisson superalgebra P with a derivation d such that $V \subseteq P^{(d)}$. Non-special GD-superalgebras are said to be *exceptional*.

Exceptional GD-superalgebras exist: it was shown in [16] by means of implicit computational arguments. Let us state below an explicit example of a 3-dimensional exceptional GD-algebra.

Example 1. Let V be a 3-dimensional space with a basis $\{x, y, z\}$ equipped with a Lie algebra structure

$$[x, y] = z, \quad [x, z] = [y, z] = 0$$

(Heisenberg Lie algebra). It is straightforward to check that the operation $(\cdot \circ \cdot)$ on V, given by

$$x \circ x = x - y, \quad y \circ x = -x \circ y = y,$$
$$x \circ z = z \circ x = y \circ z = z \circ y = y \circ y = z \circ z = 0,$$

turns V into a GD-algebra. If we assume V to be special, then there exists a Poisson differential algebra P such that $V \subset P^{(d)}$. Consider the expression $\{x, x'\}xx' \in P$ (here x' stands for $d(x)$). On the one hand,

$$\begin{aligned}\{x, x'\}(xx') &= \{x, x'\}(x \circ x) = \big\{x, (x \circ x)x'\big\} - \{x, x \circ x\}x' \\ &= \big[x, (x \circ x) \circ x\big] - [x, x \circ x] \circ x = -2z;\end{aligned}$$

on the other hand,

$$\{x, x'\}(xx') = \big(\{x, x'\}x\big)x' = \{x, xx'\}x' = [x, x \circ x] \circ x = 0.$$

Hence V cannot be embedded into a differential Poisson algebra.

Another series of examples was proposed in [9] for the non-graded case. In general, if V is a Novikov superalgebra, then the operation

$$[a, b] = (a \circ b) - (-1)^{|a||b|}(b \circ a), \quad a, b \in V_0 \cup V_1, \tag{2.4}$$

turns V into a GD-superalgebra denoted by $V^{(-)}$. It was conjectured in [16] that for every Novikov algebra the commutator GD-algebra $V^{(-)}$ is special. In this section, we prove the conjecture in the $\mathbb{Z}_2$-graded setting.

In the following, we will need the embedding theorem proved in [5] for Novikov algebras and then in [24] for Novikov superalgebras (in these two papers, the term "Gelfand–Dorfman–Novikov algebra" is used for what we call "Novikov algebra" following the common terminology proposed in [17]).

Theorem 1 ([5, Theorem 3], [24, Theorem 3.8]). *For every Novikov (super)algebra $(V, \circ)$, there exists an associative and (super)commutative algebra A with an even derivation d such that $V \subset A$ and $u \circ v = ud(v)$ for $u, v \in V$.*

In particular, the universal enveloping differential algebra of a given Novikov superalgebra $V = V_0 \oplus V_1$ may be constructed as follows. Let $X = X_0 \cup X_1$ be a linear basis of V, where X_i is a basis of V_i, $i = 0, 1$. Denote by $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$ the free associative supercommutative differential algebra generated by even variables X_0 and odd variables X_1 with an even derivation d. Apparently, $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle \simeq \Bbbk[d^\omega X_0] \otimes \bigwedge(\Bbbk d^\omega X_1)$, where

$$d^\omega X_i = \{d^n(x) \mid n \geq 0,\ x \in X_i\}.$$

Consider the (differential) ideal I_V of $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$ generated by $u(x, y) = xd(y) - x \circ y$, $x, y \in X$ (here $x \circ y$ stands for the linear form in $\Bbbk X$ representing the Novikov product in V). As a non-differential ideal of $\Bbbk[d^\omega X_0] \otimes \bigwedge(\Bbbk d^\omega X_1)$, I_V is generated by all derivatives of $u(x, y)$. Theorem 1 implies that $I_V \cap \Bbbk X = 0$.

In order to define a (super-)Poisson bracket $\{\cdot, \cdot\}$ on $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$, it is enough to determine polynomials $\{d^n(x), d^m(y)\}$, for $x, y \in X$, $n, m \geq 0$, and then extend the bracket in a unique way to the entire $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$ by the Leibniz rule (2.3).

If the bracket respects $\mathbb{Z}_2$-grading and is (super) anti-commutative on the generators from $d^\omega X$, then so is its extension. In order to simplify notations, let us denote $d^n(x)$ by $x^{(n)}$ for $x \in X, n \geq 0$.

Lemma 1. *Let $\{\cdot,\cdot\}$ be a bracket on $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$ obtained by expanding*

$$\{x^{(m)}, y^{(n)}\} = (n-1)x^{(m+1)}y^{(n)} - (m-1)x^{(m)}y^{(n+1)}, \quad x, y \in X,\ n, m \geq 0. \tag{2.5}$$

Then $\{\cdot,\cdot\}$ satisfies the Jacobi identity (2.2).

Proof. For $x, y, z \in X, n, m, k \geq 0$, evaluate

$$\begin{aligned}
&\{x^{(m)}\{y^{(n)}, z^{(k)}\}\} \\
&\quad = \{x^{(m)}, (k-1)y^{(n+1)}z^{(k)} - (n-1)y^{(n)}z^{(k+1)}\} \\
&\quad = (k-1)\big(\{x^{(m)}, y^{(n+1)}z^{(k)} + (-1)^{|x||y|}y^{(n+1)}\{x^{(m)}, z^{(k)}\}\}\big) \\
&\qquad - (n-1)\big(\{x^{(m)}, y^{(n)}\}z^{(k+1)} + (-1)^{|x||y|}y^{(n)}\{x^{(m)}, z^{(k+1)}\}\big) \\
&\quad = (k-1)\big(\big(nx^{(m+1)}y^{(n+1)} - (m-1)x^{(m)}y^{(n+2)}\big)z^{(k)} \\
&\qquad + y^{(n+1)}(-1)^{|x||y|}\big((k-1)x^{(m+1)}z^{(k)} - (m-1)x^{(m)}z^{(k+1)}\big)\big) \\
&\qquad - (n-1)\big(\big((n-1)x^{(m+1)}y^{(n)} - (m-1)x^{(m)}y^{(n+1)}\big)z^{(k)} \\
&\qquad - (-1)^{|x||y|}y^{(n)}\big(kx^{(m)}z^{(k+1)} - (m-1)x^{(m)}z^{(k+2)}\big)\big) \\
&\quad = (k-1)(n+k-1)x^{(m+1)}y^{(n+1)}z^{(k)} - (k-1)(m-1)x^{(m)}y^{(n+2)}z^{(k)} \\
&\qquad + (m-1)(n-k)x^{(m)}y^{(n+1)}z^{(k)} - (n-1)(k+n-1)x^{(m+1)}y^{(n)}z^{(k+1)} \\
&\qquad + (n-1)(m-1)x^{(m)}y^{(n)}z^{(k+2)}.
\end{aligned} \tag{2.6}$$

The Jacobi identity (2.2) is equivalent to

$$\begin{aligned}
&(-1)^{|x||z|}\{x^{(m)}, \{y^{(n)}, z^{(k)}\}\} + (-1)^{|y||x|}\{y^{(n)}, \{z^{(k)}, x^{(m)}\}\} \\
&\quad + (-1)^{|y||z|}\{z^{(k)}, \{x^{(m)}, y^{(n)}\}\} = 0,
\end{aligned}$$

which is easy to check by the cyclic permutation of variables in (2.6). ■

Lemma 2. *The operation d on $s\,\mathrm{Com\,Der}\langle X_0 \cup X_1, d\rangle$ is a derivation relative to the bracket from Lemma 1.*

Proof. It is enough to check that

$$\begin{aligned}
&d\{x^{(m)}, y^{(n)}\} \\
&\quad = (n-1)(x^{(m+2)}y^{(n)} + x^{(m+1)}y^{(n+1)}) - (m-1)(x^{(m)}x^{(n+2)} + x^{(m+1)}y^{(n+1)}) \\
&\quad = (n-1)x^{(m+2)}y^{(n)} - mx^{(m+1)}y^{(n+1)} + x^{(m+1)}y^{(n+1)} + nx^{(m+1)}y^{(n+1)} \\
&\qquad - (m-1)x^{(m+1)}y^{(n+1)} - x^{(m+1)}y^{(n+1)} \\
&\quad = \{x^{(m+1)}, y^{(n)}\} + \{x^{(m)}, y^{(n+1)}\} = \{dx^{(m)}, y^{(n)}\} + \{x^{(m)}, dy^{(n)}\}
\end{aligned}$$

for $x, y \in X, n, m \geq 0$. Since the bracket on the entire $s\,\mathrm{Com}\,\mathrm{Der}\langle X_0 \cup X_1, d\rangle$ is calculated via the Leibniz rule, we have

$$d\{f, g\} = \{d(f), g\} + \{f, d(g)\}$$

for all $f, g \in s\,\mathrm{Com}\,\mathrm{Der}\langle X_0 \cup X_1, d\rangle$. ■

Lemma 3. *The ideal I_V is invariant under all operations $\{f, \cdot\}$, $f \in s\,\mathrm{Com}\,\mathrm{Der}\langle X_0 \cup X_1, d\rangle$, where $\{\cdot, \cdot\}$ is the bracket from Lemma 1.*

Proof. Again, since $\{f, \cdot\}$ is defined via the Leibniz rule, it is enough to consider $f = z^{(n)}$, $z \in X, n \geq 0$. Moreover, Lemma 2 implies that

$$\{z^{(n)}, \cdot\} = d\{z^{(n-1)}, \cdot\} - \{z^{(n-1)}, d(\cdot)\},$$

so it is enough to check the invariance of I_V under $\{z, \cdot\}$, where $z = z^{(0)}$. The Leibniz rule (2.3) and Lemma 2 show that it is enough to verify $\{z, u(x, y)\} \in I_V$ for all $x, y, z \in X$. Indeed,

$$\begin{aligned}
\{z, u(x, y)\} &= \{z, xy'\} - \{z, x \circ y\} \\
&= \{z, x\}y' + (-1)^{|x||z|}x\{z, y'\} + z'(x \circ y) - zd(x \circ y) \\
&= (zx' - z'x)y' + (-1)^{|x||z|}xzy'' - z(xy')' + z'xy' \\
&= zx'y' - z'xy' + zxy'' - zxy'' - zx'y' + z'xy' \\
&= 0.
\end{aligned}$$

Hence the entire differential ideal I_V generated by $u(x, y)$ is invariant under the bracket defined by (2.5). ■

Theorem 2. *Let V be a Novikov superalgebra. Then the GD-algebra $V^{(-)}$ is special.*

Proof. Lemmas 1, 2, and 3 show that the quotient $U(V) = s\,\mathrm{Com}\,\mathrm{Der}\langle X_0 \cup X_1, d\rangle / I_V$ is a differential Poisson algebra. Theorem 1 guarantees that V embeds into $U(V)$ with $u \circ v = ud(v)$, $u, v \in V$. Moreover, (2.5) implies that

$$\{x, y\} = -x'y + xy' = x \circ y - (-1)^{|x||y|} y \circ x \quad \text{for } x, y \in X.$$

Therefore, $U(V)$ is a differential Poisson enveloping superalgebra of $V^{(-)}$. ■

3. Conformal superalgebras

In this section, we recall the main definitions concerning Lie conformal (super)algebras and their relations to GD-(super)algebras.

Let L be a Lie superalgebra over a field $\Bbbk$ of characteristic zero and let $H = \Bbbk[\partial]$ be the algebra of polynomials in one variable.

Two formal distributions $a(z), b(z) \in L[[z, z^{-1}]]$ are said to be *mutually local* if there exists $N \geq 0$ such that

$$\big[a(w), b(z)\big](w-z)^N = 0 \in L\big[[z, z^{-1}, w, w^{-1}]\big].$$

For a pair of mutually local formal distributions $a(z), b(z)$, their product $[a(w), b(z)]$ may be uniquely presented as a finite distribution

$$\big[a(w), b(z)\big] = \sum_{n=0}^{N-1} c_n(z) \frac{1}{n!} \frac{\partial^n \delta(w-z)}{\partial z^n},$$

where $c_n(z) \in L[[z, z^{-1}]]$, $\delta(w-z) = \sum_{s\in\mathbb{Z}} w^s z^{-s-1}$ is the formal delta function.

The collection of formal distributions $c_n(z)$, $n = 0, 1, \ldots, N-1$, associated to a given pair $a(z)$, $b(z)$ may be written as a polynomial in a new formal variable λ:

$$\big[a(z)\ _{(\lambda)}\ b(z)\big] = \sum_{n=0}^{N-1} \frac{\lambda^n}{n!} c_n(z) \in \big(L\big[[z, z^{-1}]\big]\big)[\lambda].$$

The latter space may be identified with $\Bbbk[\partial, \lambda] \otimes_H L[[z, z^{-1}]]$, where ∂ acts on $L[[z, z^{-1}]]$ as the ordinary derivation with respect to z.

An H-invariant subspace C of $L[[z, z^{-1}]]$ which consists of pairwise mutually local distributions and is closed with respect to the operation $[\cdot\ _{(\lambda)}\ \cdot]$ (i.e., $a, b \in C$ implies $[a\ _{(\lambda)}\ b] \in C[\lambda]$) provides an example of the following class of algebraic structures.

Definition 1 ([12, Chapter 2]). A $\mathbb{Z}_2$-graded H-module $C = C_0 \oplus C_1$ equipped with a linear map (λ-*bracket*)

$$[\cdot\ _{(\lambda)}\ \cdot] : C \otimes C \to \Bbbk[\partial, \lambda] \otimes_H C \simeq C[\lambda]$$

is called a *Lie conformal superalgebra* if $[C_i\ _{(\lambda)}\ C_j] \subseteq C_{(i+j) \pmod 2}[\lambda]$ and

$$[\partial x\ _{(\lambda)}\ y] = -\lambda[x\ _{(\lambda)}\ y], \tag{3.1}$$

$$[x\ _{(\lambda)}\ \partial y] = (\partial + \lambda)[x\ _{(\lambda)}\ y], \tag{3.2}$$

$$[x\ _{(\lambda)}\ y] = (-1)^{|x||y|}[y\ _{(-\partial-\lambda)}\ x], \tag{3.3}$$

$$\big[x\ _{(\lambda)}\ [y\ _{(\mu)}\ z]\big] - (-1)^{|x||y|}\big[y\ _{(\mu)}\ [x\ _{(\lambda)}\ z]\big] = \big[[x\ _{(\lambda)}\ y]\ _{(\lambda+\mu)}\ z\big] \tag{3.4}$$

for all $x, y \in C$. Here $|x|$ stands for the parity of a homogeneous element $x \in C_0 \cup C_1$.

An operation $[\cdot\ _{(\lambda)}\ \cdot]$ satisfying (3.1) and (3.2) is said to be *sesqui-linear*.

In particular, if C is a free H-module with a basis B, then the λ-bracket on C is uniquely determined by polynomials $p^z_{x,y} \in \Bbbk[\partial, \lambda]$, $x, y, z \in B$, so that

$$[x\ _{(\lambda)}\ y] = \sum_{z\in B} p^z_{x,y}(\partial, \lambda) z.$$

Example 2 ([20]). Let V be a GD-superalgebra. Then the space of Laurent polynomials $V[t,t^{-1}]$ is a Lie superalgebra relative to the bracket

$$[at^n, bt^m] = [a,b]t^{n+m} + n(a\circ b)t^{n+m-1} - (-1)^{|a||b|}(b\circ a)t^{n+m-1}, \quad a,b\in V.$$

For $a \in V$, let $a(z)$ stand for the formal distribution

$$a(z) = \sum_{n\in\mathbb{Z}} at^n z^{-n-1} \in \big(V[t,t^{-1}]\big)[[z,z^{-1}]].$$

The linear span of all such formal distributions and all their derivatives (with respect to z) is a Lie conformal superalgebra $L(V)$ isomorphic to $H\otimes V$ relative to the λ-bracket

$$[a\,{}_{(\lambda)}\,b] = [a,b] + (-1)^{|a||b|}(\partial+\lambda)(b\circ a) + \lambda(a\circ b), \quad a,b\in V. \tag{3.5}$$

Conformal superalgebras obtained from GD-superalgebras as in Example 2 are called *quadratic Lie conformal superalgebras*. As shown in [20, Theorem 2.2], every Lie conformal superalgebra structure on a free H-module $L = H\otimes V$ given by linear polynomials $p^z_{x,y}(\partial,\lambda)$, $x,y,z\in V$, is quadratic.

An important particular example of a quadratic Lie conformal algebra comes from the 1-dimensional GD-algebra $V = \Bbbk v$ with $v\circ v = v$:

$$[v\,{}_{(\lambda)}\,v] = \partial v + 2\lambda v.$$

In this case, $L(V)$ is the *Virasoro conformal algebra*.

Suppose that V and W are two H-modules. The space of all linear maps

$$\alpha : V \to W[\lambda], \quad v\mapsto \alpha_\lambda(v),$$

such that $\alpha_\lambda(\partial v) = (\partial+\lambda)\alpha_\lambda(v)$ for all $v\in V$, is denoted by $\operatorname{Chom}(V,W)$ (the space of *conformal homomorphisms* from V to W) [12]. If $V = W$, then $\operatorname{Chom}(V,W)$ is denoted by $\operatorname{Cend} V$.

If V is a finitely generated H-module, then the operation

$$(\cdot\lambda\cdot) : \operatorname{Cend} V\otimes \operatorname{Cend} V \to \operatorname{Cend} V[\lambda],$$

given by

$$(\alpha\,{}_{(\lambda)}\,\beta)_\mu(v) = \alpha_\lambda\big(\beta_{\mu-\lambda}(v)\big), \quad v\in V, \tag{3.6}$$

satisfies (3.1) and (3.2). Note that if V is not a finitely generated H-module, then we cannot say in general that $(\alpha\,{}_{(\lambda)}\,\beta)$ is a polynomial in λ.

Example 3. Let $U = U_0\oplus U_1$ be a finite-dimensional $\mathbb{Z}_2$-graded linear space, and let $V = H\otimes U = V_0\oplus V_1$ be the free H-module generated by U, $V_i = H\otimes U_i$. Then $\operatorname{Cend} V$ splits into the sum of even and odd components in a natural way:

$$(\operatorname{Cend} V)_0 = \operatorname{Cend}(V_0)\oplus \operatorname{Cend}(V_1), \quad (\operatorname{Cend} V)_1 = \operatorname{Chom}(V_0,V_1)\oplus \operatorname{Chom}(V_1,V_0),$$

and the bracket

$$[\alpha\,{}_{(\lambda)}\,\beta] = (\alpha\,{}_{(\lambda)}\,\beta) - (-1)^{|\alpha||\beta|}(\beta\,{}_{(-\partial-\lambda)}\,\alpha)$$

turns $\operatorname{Cend} V$ into a Lie conformal superalgebra denoted by $\operatorname{gc} V$.

If V is a free H-module of rank $n+m$, where $n = \dim U_0$, $m = \dim U_1$, then gc V is denoted by $\mathrm{gc}_{n|m}$. This conformal superalgebra may be presented as

$$H \otimes H \otimes M_{n|m}(\Bbbk) \simeq H \otimes M_{n|m}\big(\Bbbk[x]\big),$$

where $M_{n|m}$ stands for the $\mathbb{Z}_2$-graded algebra of $(n+m)$-matrices with an even component $\left(\begin{smallmatrix} M_n & 0 \\ 0 & M_m \end{smallmatrix}\right)$ and an odd component $\left(\begin{smallmatrix} 0 & M_{n,m} \\ M_{m,n} & 0 \end{smallmatrix}\right)$. The λ-bracket is given by

$$\big[A(x)\ {}_{(\lambda)}\ B(x)\big] = A(x)B(x+\lambda) - (-1)^{|A||B|}B(x)A(x-\partial-\lambda),$$

for homogeneous matrices $A, B \in M_{n|m}(\Bbbk[x])$.

A finite *representation* of a Lie conformal superalgebra L is a homomorphism of Lie conformal superalgebras

$$\rho : L \to \mathrm{gc}_{n|m}\,.$$

In order to have a FFR, a Lie conformal superalgebra has to be a torsion-free H-module since $\rho(\mathrm{tor_H}\, L) = 0$.

Every finite torsion-free Lie conformal superalgebra L has a *regular* representation on itself: $\rho(a) = \alpha$ for $a \in L$, where $\alpha_\lambda(x) = [a\ {}_{(\lambda)}\ x]$.

In general, a representation of a Lie conformal superalgebra L on a $\mathbb{Z}_2$-graded H-module $V = V_0 \oplus V_1$ is a sesqui-linear map $\rho_\lambda(\cdot,\cdot) : L \otimes V \to V[\lambda]$ which respects the gradings and

$$\rho_\lambda\big(a, \rho_\mu(b,x)\big) - (-1)^{|a||b|}\rho_\mu\big(b, \rho_\lambda(a,x)\big) = \rho_{\lambda+\mu}\big([a\ {}_{(\lambda)}\ b], x\big), \tag{3.7}$$

for $a, b \in L_0 \cup L_1$, $x \in V$.

It is unknown whether all torsion-free finite Lie conformal (super)algebras have a FFR. In the non-graded case, it is known [14] that if the solvable radical of L splits, then L has a FFR. The proof of the latter result essentially involves the representation theory of Lie conformal algebras [8]. For example, if V is the exceptional GD-algebra from Example 1, then $L(V)$ has a split solvable radical. However, a quadratic Lie conformal (super)algebra may have a non-split solvable radical. For example, the Virasoro Lie conformal algebra has a non-split extension [3, Theorem 7.2] corresponding to the 2-dimensional Novikov algebra $V = \Bbbk v + \Bbbk u$, where

$$v \circ v = v + u, \quad v \circ u = 0, \quad u \circ v = u, \quad u \circ u = 0.$$

Considered as an Abelian Lie algebra, V is a GD-algebra that gives rise to a quadratic Lie conformal algebra $L(V)$ with a non-split solvable radical $H \otimes \Bbbk u$.

In the next section, we will show that for every special GD-superalgebra V the corresponding Lie conformal superalgebra $L(V)$ has a FFR.

4. Poisson conformal superalgebras

In the study of Ado-type problems for Lie conformal algebras, Poisson structures play an important role.

Definition 2. Let P be a $\mathbb{Z}_2$-graded H-module endowed with two sesqui-linear operations

$$\big[\cdot\ {}_{(\lambda)}\ \cdot\big] : P \otimes P \to P[\lambda],$$
$$\big(\cdot\ {}_{(\lambda)}\ \cdot\big) : P \otimes P \to P[\lambda],$$

which respect the grading and satisfy the following conditions:

(1) P is a Lie conformal superalgebra relative to $[\cdot\ {}_{(\lambda)}\ \cdot]$;

(2) $(a\ {}_{(\lambda)}\ (b\ {}_{(\mu)}\ c)) = ((a\ {}_{(\lambda)}\ b)\ {}_{(\lambda+\mu)}\ c)$ for $a, b, c \in P$;

(3) $(a\ {}_{(\lambda)}\ b) = (-1)^{|a||b|}(b\ {}_{(-\partial-\lambda)}\ a)$ for homogeneous $a, b \in P$.

Then P is said to be a *Poisson conformal superalgebra* if the following conformal analogue of the Leibniz rule holds:

$$\big(a\ {}_{(\lambda)}\ [b\ {}_{(\mu)}\ c]\big) = \big(\big[a\ {}_{(\lambda)}\ b\big]\ {}_{(\lambda+\mu)}\ c\big) + (-1)^{|a||b|}\big(b\ {}_{(\mu)}\ \big[a\ {}_{(\lambda)}\ c\big]\big), \tag{4.1}$$

for $a, b \in P_0 \cup P_1$, $c \in P$.

A simplest example of a Poisson conformal (super)algebra is provided by the current functor. Namely, if $\mathfrak{p}$ is an ordinary commutative (super)algebra with a (super-)Poisson bracket $\{\cdot,\cdot\}$, then $\operatorname{Cur}\mathfrak{p} = H \otimes \mathfrak{p} \simeq \mathfrak{p}[\partial]$ equipped with

$$\big(a(\partial)\ {}_{(\lambda)}\ b(\partial)\big) = a(-\lambda)b(\partial+\lambda),$$
$$\big[a(\partial)\ {}_{(\lambda)}\ b(\partial)\big] = \big\{a(-\lambda), b(\partial+\lambda)\big\}$$

is a Poisson conformal (super)algebra.

An associative conformal algebra [12] is an H-module A equipped with a sesqui-linear operation $(\cdot\ {}_{(\lambda)}\ \cdot)$, satisfying Condition 2 of Definition 2. Assuming that A is a $\mathbb{Z}_2$-graded associative conformal algebra, the new (commutator) operation

$$\big[a\ {}_{(\lambda)}\ b\big] = \big(a\ {}_{(\lambda)}\ b\big) - (-1)^{|a||b|}\big(b\ {}_{(-\partial-\lambda)}\ a\big), \quad a, b \in A_0 \cup A_1,$$

turns the H-module A into a Lie conformal superalgebra $A^{(-)}$ (see [18, p. 323] for the non-graded case).

Given a Lie conformal superalgebra L and a $\mathbb{Z}_2$-graded associative conformal algebra A, we say that A is an *associative conformal envelope* of L if there exists a homogeneous homomorphism $\tau : L \to A^{(-)}$ of conformal algebras such that A is generated (as an associative conformal algebra) by the image of L. For a fixed L, there exists a lattice of universal associative conformal envelopes of L corresponding to different associative locality bounds on the elements of L (see [18, Section 6]). It may happen that neither of these universal envelopes contains the isomorphic image of L, i.e., there exist Lie conformal (super)algebras that cannot be embedded into associative ones.

Suppose that A is an associative conformal envelope of a Lie conformal superalgebra L. Then A has a natural filtration as an H-module:

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots,$$

where $A_1 = \tau(L)$, $A_{n+1} = A_n + H\{(\tau(L) {}_{(\lambda)} A_n)|_{\lambda=\alpha} : \alpha \in \Bbbk\}$. Then

$$(A_n {}_{(\lambda)} A_m) \subseteq A_{n+m}[\lambda], \quad [A_n {}_{(\lambda)} A_m] \subseteq A_{n+m-1}[\lambda],$$

so the associated graded H-module $\operatorname{gr} A$ has a well-defined structure of a Poisson conformal superalgebra.

For example, the *conformal Weyl algebra* $\mathrm{Cend}_{1,x}$ (see [6]) is an associative envelope of the Virasoro conformal algebra. The corresponding Poisson conformal algebra $\operatorname{gr}\mathrm{Cend}_{1,x} \simeq \Bbbk[\partial] \otimes x\Bbbk[x]$ has the following operations:

$$(x^n {}_{(\lambda)} x^m) = x^{n+m}, \quad [x^n {}_{(\lambda)} x^m] = \big(n\partial + (n+m)\lambda\big)x^{n+m-1}.$$

More examples of Poisson conformal superalgebra structures on universal associative envelopes of Lie conformal superalgebras can be found in [15].

Hereinafter, we will need the following example of a Poisson conformal superalgebra.

Lemma 4. *Let $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ be an ordinary Poisson superalgebra equipped with an even derivation $d : a \mapsto a'$. The latter means that $d(\mathfrak{p}_i) \subseteq \mathfrak{p}_i$, $i = 0, 1$, $(ab)' = ab' + a'b$, $\{a, b\}' = \{a', b\} + \{a, b'\}$ for $a, b \in \mathfrak{p}$. Then $L(\mathfrak{p}, d) = H \otimes \mathfrak{p}$ equipped with operations*

$$(a {}_{(\lambda)} b) = ab, \quad [a {}_{(\lambda)} b] = \{a, b\} + \partial a'b + \lambda(ab)',$$

for $a, b \in \mathfrak{p}_0 \cup \mathfrak{p}_1$, is a Poisson conformal superalgebra.

Proof. The operation $(\cdot {}_{(\lambda)} \cdot)$ is obviously associative and (super-)commutative. By definition, $[\cdot {}_{(\lambda)} \cdot]$ is exactly the quadratic Lie conformal bracket on the GD-superalgebra obtained from $\mathfrak{p}$ relative to d. The Leibniz rule (4.1) is straightforward to check. On the one hand,

$$\begin{aligned}
&\big[a {}_{(\lambda)} (b {}_{(\mu)} c)\big] - (-1)^{|a||b|}\big(b {}_{(\mu)} \big[a {}_{(\lambda)} c\big]\big) \\
&\quad = \{a, bc\} + \partial a'bc + \lambda(abc)' - (-1)^{|a||b|}\big(b\{a, c\} + (\partial + \mu)ba'c + \lambda b(ac)'\big) \\
&\quad = \{a, b\}c - \mu a'bc + \lambda ab'c
\end{aligned}$$

for homogeneous $a, b, c \in \mathfrak{p}$. On the other hand,

$$\big([a {}_{(\lambda)} b] {}_{(\lambda+\mu)} c\big) = \{a, b\}c - (\lambda + \mu)a'bc + \lambda(ab)'c = \{a, b\}c - \mu a'bc + \lambda ab'c.$$

Hence $L(\mathfrak{p}, d)$ is a Poisson conformal superalgebra. ■

Suppose that L is a Lie conformal superalgebra with a representation ρ on an H-module V. Then a *deformation* of ρ is a representation ρ_ε of L on the H-module $V \oplus V\varepsilon \simeq \Bbbk[\varepsilon] \otimes V/(\varepsilon^2)$, where

$$\rho_\varepsilon(a) = \alpha^{(\varepsilon)}, \quad \alpha_\lambda^{(\varepsilon)}(x) = \alpha_\lambda(x) + \varepsilon\varphi_\lambda(a, x),$$

for $a \in L$, $x \in V$, $\alpha = \rho(a)$. Here $\varphi_\lambda(\cdot,\cdot) : L \otimes V \to V[\lambda]$ is a sesqui-linear map which has to satisfy the following equation (a consequence of the Jacobi identity (3.7)):

$$\begin{aligned}\varphi_{\lambda+\mu}\big([a\,{}_{(\lambda)}\,b], x\big) &= \varphi_\lambda\big(a, \rho_\mu(b,x)\big) + \rho_\lambda\big(a, \varphi_\mu(b,x)\big)\\ &\quad - (-1)^{|a||b|}\varphi_\mu\big(b, \rho_\lambda(a,x)\big) - (-1)^{|a||b|}\rho_\mu\big(b, \varphi_\lambda(a,x)\big),\end{aligned} \tag{4.2}$$

for $a, b \in L_0 \cup L_1$, $x \in V$. In the case when V is a finitely generated H-module, the relation (4.2) exactly means that φ is a 1-cocycle in $Z^1(L, \mathrm{Cend}\, V)$ (see [3]).

Proposition 1. *Let L be a graded Lie conformal subalgebra in a Poisson conformal superalgebra P. Then L has a regular representation on P, and the sesqui-linear map $\varphi_\lambda(\cdot,\cdot) : L \otimes P \to P[\lambda]$, given by*

$$\varphi_\lambda(a, x) = \lambda(a\,{}_{(\lambda)}\,x), \quad a \in L,\ x \in P,$$

satisfies (4.2).

Proof. Obviously, the map $\rho_\lambda(a,x) = [a\,{}_{(\lambda)}\,x]$, $a \in L$, $x \in P$, is a representation of L on P. Then the right-hand side of (4.2) can be transformed by (4.1) as

$$\begin{aligned}&\lambda\big(a\,{}_{(\lambda)}\,[b\,{}_{(\mu)}\,x]\big) + \mu\big[a\,{}_{(\lambda)}\,(b\,{}_{(\mu)}\,x)\big]\\ &\quad - (-1)^{|a||b|}\mu\big(b\,{}_{(\mu)}\,[a\,{}_{(\lambda)}\,x]\big) - (-1)^{|a||b|}\lambda\big[b\,{}_{(\mu)}\,(a\,{}_{(\lambda)}\,x)\big]\\ &= \lambda\big([a\,{}_{(\lambda)}\,b]\,{}_{(\lambda+\mu)}\,x\big) + \mu\big([a\,{}_{(\lambda)}\,b]\,{}_{(\lambda+\mu)}\,x\big) = \varphi_{\lambda+\mu}\big([a\,{}_{(\lambda)}\,b], x\big).\end{aligned} \tag{4.3}$$

■

Corollary 1 ([15]). *Let L be a graded Lie conformal subalgebra of a Poisson conformal superalgebra. Then the map*

$$\hat\rho_\lambda(a, x) = [a\,{}_{(\lambda)}\,x] + \lambda(a\,{}_{(\lambda)}\,x), \quad a \in L,\ x \in P,$$

is a representation of L on P.

Proof. In order to check the Jacobi identity (3.7) for $\hat\rho$, note that the desired equation

$$\hat\rho_\lambda\big(a, \hat\rho_\mu(b,x)\big) - (-1)^{|a||b|}\hat\rho_\mu\big(b, \hat\rho_\lambda(a,x)\big) = \hat\rho_{\lambda+\mu}\big([a\,{}_{(\lambda)}\,b], x\big)$$

splits into three equations: the first one is exactly the Jacobi identity for regular representation $\rho_\lambda(a,x) = [a\,{}_{(\lambda)}\,x]$, the second one is (4.2) for $\varphi_\lambda(a,x) = \lambda(a\,{}_{(\lambda)}\,x)$, and the third one is

$$\lambda\mu\big(a\,{}_{(\lambda)}\,(b\,{}_{(\mu)}\,x)\big) - (-1)^{|a||b|}\lambda\mu\big(b\,{}_{(\mu)}\,(a\,{}_{(\lambda)}\,x)\big) = 0$$

which also holds due to conformal commutativity and associativity of $(\cdot\,{}_{(\lambda)}\,\cdot)$:

$$\begin{aligned}\big(a\,{}_{(\lambda)}\,(b\,{}_{(\mu)}\,x)\big) &= \big((a\,{}_{(\lambda)}\,b)\,{}_{(\mu+\lambda)}\,x\big) = (-1)^{|a||b|}\big((b\,{}_{(-\partial-\lambda)}\,a)\,{}_{(\lambda+\mu)}\,x\big)\\ &= (-1)^{|a||b|}\big((b\,{}_{(\mu)}\,a)\,{}_{(\lambda+\mu)}\,x\big) = (-1)^{|a||b|}\big(b\,{}_{(\mu)}\,(a\,{}_{(\lambda)}\,x)\big).\end{aligned}$$

■

Corollary 2. *If for a finite Lie conformal superalgebra L there exists an embedding $\tau : L \to P$ of L into a Poisson conformal superalgebra P in such a way that $(\tau(a)\,{}_{(\lambda)}\,\tau(L)) \neq 0$ for all $0 \neq a \in L$, then L has a FFR.*

Proof. Indeed, if we consider $V = L$ as a regular L-module,

$$M = \tau(L) + \Bbbk[\partial] \otimes \operatorname{span}\left\{\left(\tau(L)\,{}_{(\lambda)}\,\tau(L)\right)|_{\lambda=\alpha} : \alpha \in \Bbbk\right\}$$

as an L-submodule of P, and

$$\langle \cdot\,{}_{(\lambda)}\,\cdot \rangle : L \otimes V \to M[\lambda]$$

given by $\langle a\,{}_{(\lambda)}\,b\rangle = (\tau(a)\,{}_{(\lambda)}\,\tau(b))$, then all conditions of [13, Theorem 3] hold and L has a FFR. ∎

In particular, if L satisfies the Poincaré–Birkhoff–Witt condition [18], then one may choose P to be the associated graded conformal algebra of the appropriate universal associative conformal envelope of L. Therefore, for conformal algebras, the PBW theorem implies the Ado theorem immediately [13].

Theorem 3. *Let V be a finite-dimensional special GD-superalgebra. Then the Lie conformal superalgebra $L(V)$ has a finite faithful conformal representation.*

Proof. Let us fix linear bases X_0 and X_1 of V_0 and V_1, respectively, and let $X = X_0 \cup X_1$. A special GD-superalgebra embeds into its universal enveloping differential Poisson superalgebra which can be constructed as follows. Denote by $F = s\operatorname{Pois}\operatorname{Der}\langle X_0 \cup X_1, d\rangle$ the free differential Poisson superalgebra with an identity element 1 (generated by even elements X_0 and odd elements X_1) and an even derivation d. Let I_V stand for the (differential) ideal of F generated by

$$xd(y) - x \circ y, \quad x, y \in X, \tag{4.4}$$

where $x \circ y$ is a linear form in $\Bbbk X$ representing the Novikov product in V. The quotient F/I_V is the universal enveloping differential Poisson superalgebra for V denoted by $P_d(V)$. If V is a special GD-superalgebra then V embeds into $P_d(V)$.

The free algebra F may be presented as

$$F = \bigoplus_{n\in\mathbb{Z}} F_n,$$

where F_n consists of all elements of weight n. Recall that the weight in a free differential Poisson (super)algebra generated by a set X is defined as follows [16]:

$$\begin{gathered}\operatorname{wt} x = -1 \text{ for } x \in X, \quad \operatorname{wt}(1) = 0,\\ \operatorname{wt}(uv) = \operatorname{wt} u + \operatorname{wt} v, \quad \operatorname{wt}\{u, v\} = \operatorname{wt} u + \operatorname{wt} v + 1,\\ \operatorname{wt} d(u) = \operatorname{wt} u + 1.\end{gathered}$$

Since all elements in (4.4) are wt-homogeneous, the ideal I_V is wt-homogeneous and the algebra $P_d(V)$ inherits the grading:

$$P_d(V) = \bigoplus_{n\in\mathbb{Z}} U_n, \quad U_n = F_n/I_V \cap F_n.$$

Note that $V \simeq U_{-1}$. The latter was shown in [16, Theorem 10] for the non-graded case; the same reasonings work for superalgebras.

Lemma 4 states that $L(P_d(V), d) = H \otimes P_d(V)$ is a Poisson conformal superalgebra. Then by Corollary 1 the Lie conformal superalgebra $L = L(V)$ has a representation on $M = L(P_d(V), d)$ given by

$$\hat{\rho}_\lambda(a,u) = \{a,u\} + \partial d(a)u + \lambda\big(d(au) + au\big), \quad a \in V,\ u \in P_d(V). \tag{4.5}$$

Obviously, for every $m \in \mathbb{Z}$ the space $M_{\leq m} = H \otimes \bigoplus_{n \leq m} U_n$ is a conformal L-submodule of M. In particular, $\bar{M} = M_{\leq 0}/M_{\leq -2} \simeq H \otimes (U_{-1} \oplus U_0)$ is a conformal L-module corresponding to a representation $\bar{\rho}$ defined by (4.5) for $u \in U_0$ and $\bar{\rho}_\lambda(a,u) = [a\ {}_{(\lambda)}\ u]$ for $u \in U_{-1} \simeq V$. It is easy to see that the representation of L on $\bar{M}$ is faithful: $\bar{\rho}_\lambda(a,1) = (\partial + \lambda)d(a) + \lambda a \neq 0$ for $a \in V$, $a \neq 0$. However, it is not yet finite in general since $\dim U_0$ may not be finite.

Note that for every $a \in V \simeq U_{-1}$ the map $\mu_a : u \mapsto au$ maps U_0 to U_{-1}. Since $\dim U_{-1} = \dim V < \infty$, the intersection of all $\operatorname{Ker} \mu_a$, $a \in V$, is a subspace of finite codimension. So, consider $N = \{u \in U_0 \mid Vu = 0\}$. For every $u \in N$ we have

$$\bar{\rho}_\lambda(a,u) = \{a,u\} + \partial d(a)u, \quad a \in V.$$

Given $b \in V$, $b\{a,u\} = (-1)^{|a||b|}(\{a,bu\} + \{a,b\}u) = 0$, $bd(a)u = (-1)^{|a||b|}d(a)bu = 0$. Therefore, $H \otimes N$ is a conformal L-submodule of $\bar{M}$. Finally,

$$\bar{M}/N \simeq H \otimes (U_{-1} \oplus U_0/N)$$

is a finite faithful conformal L-module. ∎

Corollary 3. *If a GD-(super)algebra V is constructed from a Novikov (super)algebra with respect to the commutator, then the corresponding quadratic Lie conformal superalgebra $L(V)$ has a FFR.*

Funding. Research supported by the Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation.

References

[1] C. Bai and D. Meng, The classification of Novikov algebras in low dimensions. *J. Phys. A* **34** (2001), no. 8, 1581–1594 Zbl 1001.17002 MR 1818753

[2] B. Bakalov, A. D'Andrea, and V. G. Kac, Theory of finite pseudoalgebras. *Adv. Math.* **162** (2001), no. 1, 1–140 Zbl 1001.16021 MR 1849687

[3] B. Bakalov, V. G. Kac, and A. A. Voronov, Cohomology of conformal algebras. *Comm. Math. Phys.* **200** (1999), no. 3, 561–598 Zbl 0959.17018 MR 1675121

[4] A. A. Balinskiĭ and S. P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. *Dokl. Akad. Nauk SSSR* **283** (1985), no. 5, 1036–1039 Zbl 0606.58018 MR 802121

[5] L. A. Bokut, Y. Chen, and Z. Zhang, Gröbner–Shirshov bases method for Gelfand–Dorfman–Novikov algebras. *J. Algebra Appl.* **16** (2017), no. 1, 1750001, 22 Zbl 1405.17060 MR 3590862

[6] C. Boyallian, V. G. Kac, and J. I. Liberati, On the classification of subalgebras of Cend_N and gc_N. *J. Algebra* **260** (2003), no. 1, 32–63 Zbl 1034.17018 MR 1973575

[7] D. Burde and W. de Graaf, Classification of Novikov algebras. *Appl. Algebra Engrg. Comm. Comput.* **24** (2013), no. 1, 1–15 Zbl 1301.17023 MR 3011316

[8] S.-J. Cheng and V. G. Kac, Conformal modules. *Asian J. Math.* **1** (1997), no. 1, 181–193 Zbl 1022.17018 MR 1480993

[9] I. M. Gelfand and I. Y. Dorfman, Hamiltonian operators and algebraic structures associated with them. *Funktsional. Anal. i Prilozhen.* **13** (1979), no. 4, 13–30 Zbl 0428.58009 MR 554407

[10] V. Ginzburg and M. Kapranov, Koszul duality for operads. *Duke Math. J.* **76** (1994), no. 1, 203–272 Zbl 0855.18006 MR 1301191

[11] Y. Hong and Z. Wu, Simplicity of quadratic Lie conformal algebras. *Comm. Algebra* **45** (2017), no. 1, 141–150 Zbl 1418.17058 MR 3556562

[12] V. Kac, *Vertex Algebras for Beginners*. 2nd edn., Univ. Lecture Ser. 10, American Mathematical Society, Providence, RI, 1998 Zbl 0924.17023 MR 1651389

[13] P. Kolesnikov, On finite representations of conformal algebras. *J. Algebra* **331** (2011), 169–193 Zbl 1231.81047 MR 2774653

[14] P. Kolesnikov, The Ado theorem for finite Lie conformal algebras with Levi decomposition. *J. Algebra Appl.* **15** (2016), no. 7, 1650130 Zbl 1395.17068 MR 3528558

[15] P. S. Kolesnikov, Universal enveloping Poisson conformal algebras. *Internat. J. Algebra Comput.* **30** (2020), no. 5, 1015–1034 Zbl 07252755 MR 4135018

[16] P. S. Kolesnikov, B. Sartayev, and A. Orazgaliev, Gelfand–Dorfman algebras, derived identities, and the Manin product of operads. *J. Algebra* **539** (2019), 260–284 Zbl 1448.17023 MR 3995778

[17] J. M. Osborn, Novikov algebras. *Nova J. Algebra Geom.* **1** (1992), no. 1, 1–13 Zbl 0876.17005 MR 1163779

[18] M. Roitman, Universal enveloping conformal algebras. *Selecta Math. (N.S.)* **6** (2000), no. 3, 319–345 Zbl 1048.17017 MR 1817616

[19] X. Xu, On simple Novikov algebras and their irreducible modules. *J. Algebra* **185** (1996), no. 3, 905–934 Zbl 0863.17003 MR 1419729

[20] X. Xu, Quadratic conformal superalgebras. *J. Algebra* **231** (2000), no. 1, 1–38 Zbl 1001.17024 MR 1779590

[21] X. Xu, Classification of simple Novikov algebras and their irreducible modules of characteristic 0. *J. Algebra* **246** (2001), no. 2, 673–707 Zbl 1003.17003 MR 1872120

[22] X. Xu, Gel'fand-Dorfman bialgebras. *Southeast Asian Bull. Math.* **27** (2003), no. 3, 561–574 Zbl 1160.17301 MR 2045568

[23] E. I. Zel'manov, On a class of local translation invariant Lie algebras. *Soviet Math. Dokl.* **35** (1987), 216–218 Zbl 0629.17002

[24] Z. Zhang, Y. Chen, and L. A. Bokut, Free Gelfand–Dorfman–Novikov superalgebras and a Poincaré–Birkhoff–Witt type theorem. *Internat. J. Algebra Comput.* **29** (2019), no. 3, 481–505 Zbl 1444.17001 MR 3955819

Received 3 January 2020; revised 27 November 2020.

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