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Measured quantum groupoids on a finite basis and equivariant Kasparov theory

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Abstract. In this article, we generalize to the case of measured quantum groupoids on a finite basis some important results concerning the equivariant Kasparov theory for actions of locally compact quantum groups, see Baaj and Skandalis (1989, 1993). To every pair (A, B) of C*-algebras continuously acted upon by a regular measured quantum groupoid on a finite basis $\mathcal{G}$, we associate a $\mathcal{G}$-equivariant Kasparov theory group $\mathsf{KK}_{\mathcal{G}}(A, B)$. The Kasparov product generalizes to this setting. By applying recent results by Baaj and Crespo (2017, 2018) concerning actions of regular measured quantum groupoids on a finite basis, we obtain two canonical homomorphisms $J_{\mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(A, B) \to \mathsf{KK}_{\widehat{\mathcal{G}}}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$ and $J_{\widehat{\mathcal{G}}} : \mathsf{KK}_{\widehat{\mathcal{G}}}(A, B) \to \mathsf{KK}_{\mathcal{G}}(A \rtimes \widehat{\mathcal{G}}, B \rtimes \widehat{\mathcal{G}})$ inverse of each other through the Morita equivalence coming from a version of the Takesaki–Takai duality theorem. We investigate in detail the case of colinking measured quantum groupoids. In particular, if $\mathbb{G}_1$ and $\mathbb{G}_2$ are two monoidally equivalent regular locally compact quantum groups, we obtain a new proof of the canonical equivalence of the associated equivariant Kasparov categories, see Baaj and Crespo (2017).

Contents

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1. Introduction

The notion of monoidal equivalence of compact quantum groups has been introduced by Bichon, De Rijdt, and Vaes in [6]. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are said to be monoidally equivalent if their categories of representations are equivalent as monoidal C*-categories. They have proved that $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent if and only if there exists a unital C*-algebra equipped with commuting continuous ergodic actions of full multiplicity of $\mathbb{G}_1$ on the left and of $\mathbb{G}_2$ on the right. Among the applications of monoidal equivalence to the geometric theory of free discrete quantum groups, we mention the contributions to random walks and their associated boundaries [14, 30], CCAP property and Haagerup property [13], the Baum-Connes conjecture, and K-amenability [32, 33].

In his Ph.D. thesis [11], De Commer has extended the notion of monoidal equivalence to the locally compact case. Two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ (in the sense of Kustermans and Vaes [20]) are said to be monoidally equivalent if there exists a von Neumann algebra equipped with a left Galois action of $\mathbb{G}_1$ and a right Galois action of $\mathbb{G}_2$ that commute. He proved that this notion is completely encoded by a measured quantum groupoid (in the sense of Enock and Lesieur [16]) on the basis $\mathbb{C}^2$. Such a groupoid is called a colinking measured quantum groupoid.

The measured quantum groupoids have been introduced and studied by Lesieur and Enock (see [16, 22]). Roughly speaking, a measured quantum groupoid (in the sense of Enock–Lesieur) is an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$, where N and M are von Neumann algebras (the basis N and M are the algebras of the groupoid corresponding respectively to the space of units and the total space for a classical groupoid), α and β are faithful normal *-homomorphisms from N and N° (the opposite algebra) to M (corresponding to the source and target maps for a classical groupoid) with commuting ranges, Γ is a coproduct taking its values in a certain fiber product, ν is a normal semi-finite weight on N, and T and T' are operator-valued weights satisfying some axioms.

In the case of a finite-dimensional basis N, the definition has been greatly simplified by De Commer [10–12] and we will use this point of view in this article. Broadly speaking, we can take for ν the non-normalized Markov trace on the C*-algebra $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$. The relative tensor product of Hilbert spaces (resp. the fiber product of von Neumann algebras) is replaced by the ordinary tensor product of Hilbert spaces (resp. von Neumann algebras). The coproduct takes its values in $M \otimes M$ but is no longer unital.

In [2], the authors introduce a notion of (strongly) continuous actions on C*-algebras of measured quantum groupoids on a finite basis. They extend the construction of the crossed product and the dual action and give a version of the Takesaki–Takai duality generalizing the Baaj–Skandalis duality theorem [4] in this setting.

If a colinking measured quantum groupoid $\mathcal{G}$, associated with a monoidal equivalence of two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, acts (strongly) continuously on a C*-algebra A, then A splits up as a direct sum $A = A_1 \oplus A_2$ of C*-algebras and the action of $\mathcal{G}$ on A restricts to an action of $\mathbb{G}_1$ (resp. $\mathbb{G}_2$) on A_1 (resp. A_2).

They also extend the induction procedure to the case of monoidally equivalent regular locally compact quantum groups. To any continuous action of $\mathbb{G}_1$ on a C*-algebra A_1, they associate canonically a C*-algebra A_2 endowed with a continuous action of $\mathbb{G}_2$. As important consequences of this construction, we mention the following:

- a one-to-one functorial correspondence between the continuous actions of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, which generalizes the compact case [14] and the case of deformations by a 2-cocycle [24];
- a complete description of the continuous actions of colinking measured quantum groupoids;
- the equivalence of the categories $\mathsf{KK}_{\mathbb{G}_1}$ and $\mathsf{KK}_{\mathbb{G}_2}$, which generalizes to the regular locally compact case a result of Voigt [33].

The proofs of the above results rely crucially on the regularity of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. They prove that the regularity of $\mathbb{G}_1$ and $\mathbb{G}_2$ is equivalent to the regularity of the associated colinking measured quantum groupoid in the sense of [15] (see also [27, 28]).

In [9], the author generalizes to the case of (semi-)regular measured quantum groupoid on a finite basis some important properties of (semi-)regular locally compact quantum groups [1, 4], which then allow him to generalize some crucial results of [5] concerning actions of (semi-)regular locally compact quantum groups. More precisely, if $\mathcal{G}$ is a regular measured quantum groupoid on a finite basis, then any weakly continuous action of $\mathcal{G}$ is necessarily continuous in the strong sense.

Let $\mathcal{G}$ be a measured quantum groupoid on a finite basis. The author provides a notion of action of $\mathcal{G}$ on Hilbert C*-modules in line with the corresponding notion for quantum groups [3]. By using the previous result, if $\mathcal{G}$ is regular, then any action of $\mathcal{G}$ on a Hilbert C*-module is necessarily continuous. The author defines the notion of $\mathcal{G}$-equivariant Morita equivalence between $\mathcal{G}$-C*-algebras. By applying the version of the Takesaki–Takai duality theorem obtained in [2], the author finally obtains that any $\mathcal{G}$-C*-algebra A is $\mathcal{G}$-equivariantly Morita equivalent to its double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ in a canonical way.

In this article, we generalize to the setting of measured quantum groupoid on a finite basis some crucial results concerning equivariant Kasparov theory for actions of quantum groups [3]. More precisely, we define the equivariant Kasparov groups $\mathsf{KK}_{\mathcal{G}}(A, B)$ for any pair of $\mathcal{G}$-C*-algebras (A, B) and extend the functorial properties and the Kasparov product in this framework. For all pair of $\mathcal{G}$-C*-algebras (resp. $\widehat{\mathcal{G}}$-C*-algebras), we build a homomorphism

$$J_{\mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(A, B) \to \mathsf{KK}_{\widehat{\mathcal{G}}}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G}) \quad \big(\text{resp. } J_{\widehat{\mathcal{G}}} : \mathsf{KK}_{\widehat{\mathcal{G}}}(A, B) \to \mathsf{KK}_{\mathcal{G}}(A \rtimes \widehat{\mathcal{G}}, B \rtimes \widehat{\mathcal{G}})\big).$$

We also prove that $J_{\mathcal{G}}$ and $J_{\widehat{\mathcal{G}}}$ are inverse of each other through the Morita equivalences obtained in [9]. The rest of the paper is dedicated to the applications of the above theory to monoidal equivalence. In particular, we provide a new proof of the equivalence of the equivariant Kasparov categories $\mathsf{KK}_{\mathbb{G}_1}$ and $\mathsf{KK}_{\mathbb{G}_2}$ when $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent regular locally compact quantum groups [2] (see also [33] for the compact case). It should be mentioned that the equivariant Kasparov theory for actions of locally compact topological groupoid has been studied by Le Gall in [21].

For the notions and notations used in this paper, we invite the reader to find them in [9] and the references therein where a comprehensive and detailed study is done. For more information on locally compact quantum groups and measured quantum groupoids, we refer to the classical literature on these subjects (see [4, 16, 20, 22, 29]).

This article is organized as follows.

- *Chapter 1*. In the first section, we recall the notion of action of measured quantum groupoids on a finite basis on Hilbert C*-modules (cf. [9]). In the second section, we study the crossed product construction in this setting and we state a version of the Takesaki–Takai duality theorem. The last section begins with a reminder of the case of a colinking measured quantum groupoid (cf. [2, 9]). The structure of the double crossed product is investigated at the end of this section.
- *Chapter 2*. In this chapter, we give the definition and some properties of equivariant Kasparov groups by a regular measured quantum groupoid on a finite basis. We generalize to our setting the Kasparov technical theorem, which allows us to build the Kasparov product. In the last section, we build the so-called “descent morphisms” $J_{\mathcal{G}}$ and $J_{\widehat{\mathcal{G}}}$ and prove that they are inverse of each other up to Morita equivalences.
- *Chapter 3*. We apply the previous results to the case of a colinking measured quantum groupoid $\mathcal{G}$ associated with two monoidally equivalent regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. We obtain canonical equivalences between the equivariant Kasparov categories of $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathcal{G}$. In particular, we provide a new proof of the isomorphism obtained in [2, §4.5].

2. Hilbert C*-modules acted upon by measured quantum groupoids

2.1. Notion of actions of measured quantum groupoids on a finite basis on Hilbert C*-modules

In this paragraph, we recall the notion of $\mathcal{G}$-equivariant Hilbert C*-module for a measured quantum groupoid $\mathcal{G}$ on a finite basis in the spirit of [3] (cf. [9, §6.1]). We fix a measured quantum groupoid $\mathcal{G}$ on a finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ endowed with the non-normalized Markov trace $\epsilon = \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l$. We use all the notations introduced in [9, §3.1 and §3.2] concerning the objects associated with $\mathcal{G}$. For example, (S, δ) denotes the weak Hopf C*-algebra associated to $\mathcal{G}$ represented on its standard Hilbert space $\mathcal{H}$ and the morphisms $\alpha : N \to \mathcal{M}(S)$ and $\beta : N^{\circ} \to \mathcal{M}(S)$ are the base maps. Let us fix a $\mathcal{G}$-C*-algebra A. We denote by $\delta_A : A \to \mathcal{M}(A \otimes S)$ and $\beta_A : N^{\circ} \to \mathcal{M}(A)$ the morphisms which define the continuous coaction on the C*-algebra A.

Following [3, §2], an action of $\mathcal{G}$ on a Hilbert A-module $\mathcal{E}$ is defined in [9] by three equivalent data:

- a pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ consisting of a *-homomorphism $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ and a linear map $\delta_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ (cf. Definition 2.1.1),

- a pair $(\beta_{\mathcal{E}}, \mathcal{V})$ consisting of a *-homomorphism $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ and an isometry $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ (cf. Definition 2.1.4),
- an action (β_J, δ_J) of $\mathcal{G}$ on $J := \mathcal{K}(\mathcal{E} \oplus A)$ (cf. Definition 2.1.8),

satisfying some conditions.

We have the following unitary equivalences of Hilbert modules:

$$\begin{aligned} A \otimes_{\delta_A} (A \otimes S) &\to q_{\beta_A \alpha}(A \otimes S) \\ a \otimes_{\delta_A} x &\mapsto \delta_A(a)x; \end{aligned} \tag{2.1}$$

$$\begin{aligned} (A \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S) &\to q_{\beta_A \alpha, 12}(A \otimes S \otimes S) \\ x \otimes_{\delta_A \otimes \mathrm{id}_S} y &\mapsto (\delta_A \otimes \mathrm{id}_S)(x)y; \end{aligned} \tag{2.2}$$

$$\begin{aligned} (A \otimes S) \otimes_{\mathrm{id}_A \otimes \delta} (A \otimes S \otimes S) &\to q_{\beta \alpha, 23}(A \otimes S \otimes S) \\ x \otimes_{\mathrm{id}_A \otimes \delta} y &\mapsto (\mathrm{id}_A \otimes \delta)(x)y. \end{aligned} \tag{2.3}$$

In the following, we fix a Hilbert A-module $\mathcal{E}$. We will apply the usual identifications $\mathcal{M}(A \otimes S) = \mathcal{L}(A \otimes S)$, $\mathcal{K}(\mathcal{E}) \otimes S = \mathcal{K}(\mathcal{E} \otimes S)$ and $\mathcal{M}(\mathcal{K}(\mathcal{E}) \otimes S) = \mathcal{L}(\mathcal{E} \otimes S)$.

Definition 2.1.1. An action of $\mathcal{G}$ on the Hilbert A-module $\mathcal{E}$ is a pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$, where $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ is a non-degenerate *-homomorphism and $\delta_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ is a linear map such that

(1) for all $a \in A$ and $\xi, \eta \in \mathcal{E}$, we have

$$\delta_{\mathcal{E}}(\xi a) = \delta_{\mathcal{E}}(\xi)\delta_A(a) \quad \text{and} \quad \langle \delta_{\mathcal{E}}(\xi), \delta_{\mathcal{E}}(\eta) \rangle = \delta_A(\langle \xi, \eta \rangle),$$

(2) $[\delta_{\mathcal{E}}(\mathcal{E})(A \otimes S)] = q_{\beta_{\mathcal{E}} \alpha}(\mathcal{E} \otimes S)$,

(3) for all $\xi \in \mathcal{E}$ and $n \in N$, we have $\delta_{\mathcal{E}}(\beta_{\mathcal{E}}(n^{\mathrm{o}})\xi) = (1_{\mathcal{E}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathcal{E}}(\xi)$,

(4) the linear maps $\delta_{\mathcal{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathcal{E}} \otimes \delta$ extend to linear maps from $\mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$ to $\mathcal{L}(A \otimes S \otimes S, \mathcal{E} \otimes S \otimes S)$ and we have

$$(\delta_{\mathcal{E}} \otimes \mathrm{id}_S)\delta_{\mathcal{E}}(\xi) = (\mathrm{id}_{\mathcal{E}} \otimes \delta)\delta_{\mathcal{E}}(\xi) \in \mathcal{L}(A \otimes S \otimes S, \mathcal{E} \otimes S \otimes S), \text{ for all } \xi \in \mathcal{E}.$$

Remarks 2.1.2. (1) If the second formula of condition (1) holds, then $\delta_{\mathcal{E}}$ is isometric (cf. [3] and [9, Remarks A.3.2]).

(2) If condition (1) holds, then condition (2) is equivalent to

$$[\delta_{\mathcal{E}}(\mathcal{E})(1_A \otimes S)] = q_{\beta_{\mathcal{E}} \alpha}(\mathcal{E} \otimes S).$$

Indeed, if $(u_\lambda)_\lambda$ is an approximate unit of A we have

$$\delta_{\mathcal{E}}(\xi) = \lim_\lambda \delta_{\mathcal{E}}(\xi u_\lambda) = \lim_\lambda \delta_{\mathcal{E}}(\xi)\delta_A(u_\lambda) = \delta_{\mathcal{E}}(\xi) q_{\beta_A \alpha} \quad \text{for all } \xi \in \mathcal{E}.$$

By strong continuity of the action (β_A, δ_A), condition (1) of Definition 2.1.1, and the equality $\mathcal{E}A = \mathcal{E}$, we then have $[\delta_{\mathcal{E}}(\mathcal{E})(A \otimes S)] = [\delta_{\mathcal{E}}(\mathcal{E})(1_A \otimes S)]$ and the equivalence follows.

(3) Note that we have $q_{\beta_{\mathcal{E}} \alpha}\delta_{\mathcal{E}}(\xi) = \delta_{\mathcal{E}}(\xi) = \delta_{\mathcal{E}}(\xi) q_{\beta_A \alpha}$ for all $\xi \in \mathcal{E}$.

(4) We will prove (cf. Remarks 2.1.7) that if $\delta_{\mathcal{E}}$ satisfies conditions (1) and (2) of Definition 2.1.1, then the extensions of $\delta_{\mathcal{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathcal{E}} \otimes \delta$ always exist and satisfy the formulas

$$(\mathrm{id}_{\mathcal{E}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) = (\mathrm{id}_{\mathcal{E}} \otimes \delta)(Tx);$$
$$(\delta_{\mathcal{E}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) = (\delta_{\mathcal{E}} \otimes \mathrm{id}_S)(Tx)$$

for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$.

Notation 2.1.3. For $\xi \in \mathcal{E}$, let us denote by $T_\xi \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes_{\delta_A} (A \otimes S))$ the operator defined by

$$T_\xi(x) := \xi \otimes_{\delta_A} x \quad \text{for all } x \in A \otimes S.$$

We have $T_\xi^*(\eta \otimes_{\delta_A} y) = \delta_A(\langle \xi, \eta \rangle)y$ for all $\eta \in \mathcal{E}$ and $y \in A \otimes S$. In particular, we have $T_\xi^* T_\eta = \delta_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathcal{E}$.

Definition 2.1.4. Let $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ be an isometry and $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ a non-degenerate $*$-homomorphism such that

(1) $\mathcal{V}\mathcal{V}^* = q_{\beta_{\mathcal{E}}\alpha}$,

(2) $\mathcal{V}(\beta_{\mathcal{E}}(n^{\mathrm{o}}) \otimes_{\delta_A} 1) = (1_{\mathcal{E}} \otimes \beta(n^{\mathrm{o}}))\mathcal{V}$ for all $n \in N$.

Then, $\mathcal{V}$ is said to be admissible if we further have

(3) $\mathcal{V}T_\xi \in \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ for all $\xi \in \mathcal{E}$,

(4) $(\mathcal{V} \otimes_{\mathbb{C}} \mathrm{id}_S)(\mathcal{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1) = \mathcal{V} \otimes_{\mathrm{id}_A \otimes \delta} 1 \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A^2} (A \otimes S \otimes S), \mathcal{E} \otimes S \otimes S)$.

The fourth statement in the previous definition makes sense since we have used the canonical identifications thereafter. By combining the associativity of the internal tensor product with the unitary equivalences (2.2) and (2.3), we obtain the following unitary equivalences of Hilbert $A \otimes S$-modules:

$$\big(\mathcal{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S) \to \mathcal{E} \otimes_{\delta_A^2} (A \otimes S \otimes S)$$
$$(\xi \otimes_{\delta_A} x) \otimes_{\delta_A \otimes \mathrm{id}_S} y \mapsto \xi \otimes_{\delta_A^2} (\delta_A \otimes \mathrm{id}_S)(x)y; \tag{2.4}$$
$$\big(\mathcal{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes_{\mathrm{id}_A \otimes \delta} (A \otimes S \otimes S) \to \mathcal{E} \otimes_{\delta_A^2} (A \otimes S \otimes S)$$
$$(\xi \otimes_{\delta_A} x) \otimes_{\mathrm{id}_A \otimes \delta} y \mapsto \xi \otimes_{\delta_A^2} (\mathrm{id}_A \otimes \delta)(x)y. \tag{2.5}$$

We also have the following:

$$(\mathcal{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S) \to \big(\mathcal{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes S$$
$$(\xi \otimes s) \otimes_{\delta_A \otimes \mathrm{id}_S} (x \otimes t) \mapsto (\xi \otimes_{\delta_A} x) \otimes st; \tag{2.6}$$
$$(\mathcal{E} \otimes S) \otimes_{\mathrm{id}_A \otimes \delta} (A \otimes S \otimes S) \to q_{\beta\alpha,23}(\mathcal{E} \otimes S \otimes S) \subset \mathcal{E} \otimes S \otimes S$$
$$\xi \otimes_{\mathrm{id}_A \otimes \delta} y \mapsto (\mathrm{id}_{\mathcal{E}} \otimes \delta)(\xi)y. \tag{2.7}$$

In particular, $\mathcal{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1 \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A^2} (A \otimes S \otimes S), (\mathcal{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S))$ (2.4), and $\mathcal{V} \otimes_{\mathbb{C}} \mathrm{id}_S \in \mathcal{L}((\mathcal{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S), \mathcal{E} \otimes S \otimes S)$ (2.6).

The next result provides an equivalence of Definitions 2.1.1 and 2.1.4.

Proposition 2.1.5. (a) *Let* $\delta_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ *be a linear map and* $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ *a non-degenerate* **-homomorphism which satisfy conditions* (1)*,* (2)*, and* (3) *of Definition* 2.1.1*. Then, there exists a unique isometry* $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ *such that* $\delta_{\mathcal{E}}(\xi) = \mathcal{V} T_\xi$ *for all* $\xi \in \mathcal{E}$*. Moreover, the pair* $(\beta_{\mathcal{E}}, \mathcal{V})$ *satisfies conditions* (1)*,* (2)*, and* (3) *of Definition* 2.1.4.

(b) *Conversely, let* $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ *be an isometry and* $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ *a non-degenerate* **-homomorphism which satisfy conditions* (1)*,* (2)*, and* (3) *of Definition* 2.1.4*. Let us consider the map* $\delta_{\mathcal{E}} : \mathcal{E} \to \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$ *given by* $\delta_{\mathcal{E}}(\xi) := \mathcal{V} T_\xi$ *for all* $\xi \in \mathcal{E}$*. Then, the pair* $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ *satisfies conditions* (1)*,* (2)*, and* (3) *of Definition* 2.1.1.

(c) *Let us assume that the above statements hold. Then, the pair* $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ *is an action of* $\mathcal{G}$ *on* $\mathcal{E}$ *if and only if* $\mathcal{V}$ *is admissible.*

Notation 2.1.6. Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert C*-modules. Let $q \in \mathcal{L}(\mathscr{E})$ be a self-adjoint projection and $T \in \mathcal{L}(q\mathscr{E}, \mathscr{F})$. Let $\widetilde{T} : \mathscr{E} \to \mathscr{F}$ be the map defined by $\widetilde{T}\xi := Tq\xi$ for all $\xi \in \mathscr{E}$. Therefore, $\widetilde{T} \in \mathcal{L}(\mathscr{E}, \mathscr{F})$ and $\widetilde{T}^* = qT^*$. By abuse of notation, we will still denote by T the adjointable operator $\widetilde{T}$.

Remarks 2.1.7. As a consequence of Proposition 2.1.5, we have the statements below.

- By applying Notation 2.1.6 and the identifications (2.3), (2.7), we have obtained a linear map $\mathrm{id}_{\mathcal{E}} \otimes \delta : \mathcal{L}(A \otimes S, \mathcal{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathcal{E} \otimes S \otimes S)$ given by

$$(\mathrm{id}_{\mathcal{E}} \otimes \delta)(T) := T \otimes_{\mathrm{id}_A \otimes \delta} 1 \quad \text{for all } T \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S).$$

- If $\delta_{\mathcal{E}}$ satisfies conditions (1) and (2) of Definition 2.1.1, let $\mathcal{V}$ be the isometry associated with $\delta_{\mathcal{E}}$ (cf. Proposition 2.1.5 (a)). By applying Notation 2.1.6 and the identifications (2.2), (2.6), the linear map

$$\delta_{\mathcal{E}} \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathcal{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathcal{E} \otimes S \otimes S)$$

  is defined by

$$(\delta_{\mathcal{E}} \otimes \mathrm{id}_S)(T) := (\mathcal{V} \otimes_{\mathbb{C}} 1_S)(T \otimes_{\delta_A \otimes \mathrm{id}_S} 1) \quad \text{for all } T \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S).$$

Note that the extensions $\mathrm{id}_{\mathcal{E}} \otimes \delta$ and $\delta_{\mathcal{E}} \otimes \mathrm{id}_S$ satisfy the following formulas:

$$\begin{aligned} (\mathrm{id}_{\mathcal{E}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) &= (\mathrm{id}_{\mathcal{E}} \otimes \delta)(Tx); \\ (\delta_{\mathcal{E}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) &= (\delta_{\mathcal{E}} \otimes \mathrm{id}_S)(Tx) \end{aligned} \tag{2.8}$$

for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$.

Let us denote by $J := \mathcal{K}(\mathcal{E} \oplus A)$ the linking C*-algebra associated with the Hilbert A-module $\mathcal{E}$. In the following, we apply the usual identifications $\mathcal{M}(J) = \mathcal{L}(\mathcal{E} \oplus A)$ and $\mathcal{M}(J \otimes S) = \mathcal{L}((\mathcal{E} \otimes S) \oplus (A \otimes S))$.

Definition 2.1.8. An action (β_J, δ_J) of $\mathcal{G}$ on J is said to be compatible with the action (β_A, δ_A) if

(1) $\delta_J : J \to \mathcal{M}(J \otimes S)$ is compatible with δ_A; i.e., $\iota_{A\otimes S} \circ \delta_A = \delta_J \circ \iota_A$,

(2) $\beta_J : N^{\mathrm{o}} \to \mathcal{M}(J)$ is compatible with β_A; i.e., $\iota_A(\beta_A(n^{\mathrm{o}})a) = \beta_J(n^{\mathrm{o}})\iota_A(a)$ for all $n \in N$ and $a \in A$.

Proposition 2.1.9. *Let (β_J, δ_J) be a compatible action of $\mathcal{G}$ on J. There exists a unique non-degenerate *-homomorphism $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ such that*

$$\beta_J(n^{\mathrm{o}}) = \begin{pmatrix} \beta_{\mathcal{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix} \quad \text{for all } n \in N.$$

Moreover, we have $q_{\beta_J\alpha} = \begin{pmatrix} q_{\beta_{\mathcal{E}}\alpha} & 0 \\ 0 & q_{\beta_A\alpha} \end{pmatrix}$.

Proposition 2.1.10. (a) *Let us assume that the* C**-algebra J is endowed with a compatible action (β_J, δ_J) of $\mathcal{G}$ such that $\delta_J(J) \subset \widetilde{\mathcal{M}}(J \otimes S)$. Then, we have the following statements:*

- *there exists a unique linear map $\delta_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ such that $\iota_{\mathcal{E}\otimes S} \circ \delta_{\mathcal{E}} = \delta_J \circ \iota_{\mathcal{E}}$; furthermore, the pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ is an action of $\mathcal{G}$ on $\mathcal{E}$, where $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ is the *-homomorphism defined in Proposition 2.1.9;*
- *there exists a unique faithful *-homomorphism $\delta_{\mathcal{K}(\mathcal{E})} : \mathcal{K}(\mathcal{E}) \to \widetilde{\mathcal{M}}(\mathcal{K}(\mathcal{E}) \otimes S)$ such that $\iota_{\mathcal{K}(\mathcal{E}\otimes S)} \circ \delta_{\mathcal{K}(\mathcal{E})} = \delta_J \circ \iota_{\mathcal{K}(\mathcal{E})}$; moreover, the pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{K}(\mathcal{E})})$ is an action of $\mathcal{G}$ on $\mathcal{K}(\mathcal{E})$.*

(b) *Conversely, let $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ be an action of $\mathcal{G}$ on the Hilbert A-module $\mathcal{E}$. Then, there exists a faithful *-homomorphism $\delta_J : J \to \widetilde{\mathcal{M}}(J \otimes S)$ such that $\iota_{\mathcal{E}\otimes S} \circ \delta_{\mathcal{E}} = \delta_J \circ \iota_{\mathcal{E}}$. Moreover, we define a unique action (β_J, δ_J) of $\mathcal{G}$ on J compatible with (β_A, δ_A) by setting*

$$\beta_J(n^{\mathrm{o}}) = \begin{pmatrix} \beta_{\mathcal{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix} \quad \text{for all } n \in N.$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert A-modules acted upon by $\mathcal{G}$, then so is their direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ in a canonical way.

Proposition-Definition 2.1.11. *For $i = 1, 2$, let $\mathcal{E}_i$ be a Hilbert A-module acted upon by $\mathcal{G}$. Let $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$. For $i = 1, 2$, let $j_{\mathcal{E}_i} : \mathcal{L}(A \otimes S, \mathcal{E}_i \otimes S) \to \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$ be the linear extension of the canonical injection $\mathcal{E}_i \otimes S \to \mathcal{E} \otimes S$. Let $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ and $\delta_{\mathcal{E}} : \mathcal{E} \to \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$ be the maps defined by*

$$\beta_{\mathcal{E}}(n^{\mathrm{o}}) := \begin{pmatrix} \beta_{\mathcal{E}_1}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_{\mathcal{E}_2}(n^{\mathrm{o}}) \end{pmatrix}, \quad n \in N,$$

$$\delta_{\mathcal{E}}(\xi) := \sum_{i=1,2} j_{\mathcal{E}_i} \circ \delta_{\mathcal{E}_i}(\xi_i), \quad \xi = (\xi_1, \xi_2) \in \mathcal{E}.$$

Then, the pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ is an action of $\mathcal{G}$ on $\mathcal{E}$.

Remarks 2.1.12. Let $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ be an action of $\mathcal{G}$ on the Hilbert A-module $\mathcal{E}$.

(1) The map $\delta_{\mathcal{K}(\mathcal{E})} : \mathcal{K}(\mathcal{E}) \to \widetilde{\mathcal{M}}(\mathcal{K}(\mathcal{E}) \otimes S)$ defined in Proposition 2.1.10 (a) is the unique *-homomorphism satisfying the relation $\delta_{\mathcal{K}(\mathcal{E})}(\theta_{\xi,\eta}) = \delta_{\mathcal{E}}(\xi) \circ \delta_{\mathcal{E}}(\eta)^*$ for all $\xi, \eta \in \mathcal{E}$.

(2) For all $F \in \mathcal{L}(\mathcal{E})$ and $\zeta \in \mathcal{E}$, $\delta_{\mathcal{E}}(F\zeta) = \delta_{\mathcal{K}(\mathcal{E})}(F)\delta_{\mathcal{E}}(\zeta)$. Indeed, for all $\xi, \eta, \zeta \in \mathcal{E}$ we have $\delta_{\mathcal{E}}(\theta_{\xi,\eta}\zeta) = \delta_{\mathcal{E}}(\xi)\delta_A(\langle \eta, \zeta\rangle) = \delta_{\mathcal{E}}(\xi)\langle \delta_{\mathcal{E}}(\eta), \delta_{\mathcal{E}}(\zeta)\rangle = \delta_{\mathcal{K}(\mathcal{E})}(\theta_{\xi,\eta})\delta_{\mathcal{E}}(\zeta)$. Hence, $\delta_{\mathcal{E}}(k\zeta) = \delta_{\mathcal{K}(\mathcal{E})}(k)\delta_{\mathcal{E}}(\zeta)$ for all $k \in \mathcal{K}(\mathcal{E})$ and $\zeta \in \mathcal{E}$. The claim is then proved by strict continuity of $\delta_{\mathcal{K}(\mathcal{E})}$.

(3) For all $k \in \mathcal{K}(\mathcal{E})$, $\delta_{\mathcal{K}(\mathcal{E})}(k) = \mathcal{V}(k \otimes_{\delta_A} 1)\mathcal{V}^*$, where $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ is the isometry associated with the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ (cf. Definition 2.1.4).

Proposition-Definition 2.1.13. *Let $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ be an action of $\mathcal{G}$ on the Hilbert A-module $\mathcal{E}$. Let $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ be the isometry associated with $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ (cf. Proposition* 2.1.5 (a)*). Let us endow the* C*-algebras J and $\mathcal{K}(\mathcal{E})$ with the actions defined in Proposition* 2.1.10*. Let $F \in \mathcal{L}(\mathcal{E})$. The following statements are equivalent:*

(i) $\delta_{\mathcal{E}}(F\xi) = (F \otimes 1_S)\delta_{\mathcal{E}}(\xi)$ *for all* $\xi \in \mathcal{E}$*;*

(ii) F *is* $\delta_{\mathcal{K}(\mathcal{E})}$*-invariant;*

(iii) $\mathcal{V}(F \otimes_{\delta_A} 1)\mathcal{V}^* = q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)$*;*

(iv) $\iota_{\mathcal{K}(\mathcal{E})}(F)$ *is* δ_J*-invariant.*

In that case, F is said to be ($\delta_{\mathcal{E}}$-)invariant.

Proof. (ii)⇒(i) For all $\xi \in \mathcal{E}$, we have (cf. [9, Remarks 6.1.23], Remark 2.1.2 (3))

$$\delta_{\mathcal{E}}(F\xi) = \delta_{\mathcal{K}(\mathcal{E})}(F)\delta_{\mathcal{E}}(\xi) = (F \otimes 1_S)q_{\beta_{\mathcal{E}}\alpha}\delta_{\mathcal{E}}(\xi) = (F \otimes 1_S)\delta_{\mathcal{E}}(\xi).$$

(i)⇒(ii) For all $\xi, \eta \in \mathcal{E}$, we have (cf. Remark 2.1.12 (1))

$$\begin{aligned}\delta_{\mathcal{K}(\mathcal{E})}(F\theta_{\xi,\eta}) = \delta_{\mathcal{K}(\mathcal{E})}(\theta_{F\xi,\eta}) &= \delta_{\mathcal{E}}(F\xi)\delta_{\mathcal{E}}(\eta)^* = (F \otimes 1_S)\delta_{\mathcal{E}}(\xi)\delta_{\mathcal{E}}(\eta)^* \\ &= (F \otimes 1_S)\delta_{\mathcal{K}(\mathcal{E})}(\theta_{\xi,\eta}).\end{aligned}$$

Hence, $\delta_{\mathcal{K}(\mathcal{E})}(Fk) = (F \otimes 1_S)\delta_{\mathcal{K}(\mathcal{E})}(k)$ for all $k \in \mathcal{K}(\mathcal{E})$. Hence,

$$\delta_{\mathcal{K}(\mathcal{E})}(F) = (F \otimes 1_S)q_{\beta_{\mathcal{E}}\alpha}.$$

(ii)⇔(iii) See Remark 2.1.12 (3).

(iii)⇔(iv) This is a direct consequence of the relation

$$\delta_J \circ \iota_{\mathcal{K}(\mathcal{E})} = \iota_{\mathcal{K}(\mathcal{E} \otimes S)} \circ \delta_{\mathcal{K}(\mathcal{E})}. \qquad \blacksquare$$

Let us recall the notion of equivariant unitary equivalence between Hilbert C*-modules over possibly different C*-algebras acted upon by $\mathcal{G}$.

Definition 2.1.14. Let A and B be two $\mathcal{G}$-C*-algebras and $\phi : A \to B$ a $\mathcal{G}$-equivariant *-isomorphism. Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert modules over, respectively, A and B acted upon by $\mathcal{G}$. A ϕ-compatible unitary operator $\Phi : \mathcal{E} \to \mathcal{F}$ (cf. [9, Definition A.3.1]) is said to be $\mathcal{G}$-equivariant if we have

$$\delta_{\mathcal{F}}(\Phi\xi) = (\Phi \otimes \mathrm{id}_S)\delta_{\mathcal{E}}(\xi) \quad \text{for all } \xi \in \mathcal{E}.$$

We recall that the linear map $\Phi \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathcal{E} \otimes S) \to \mathcal{L}(B \otimes S, \mathcal{F} \otimes S)$ (cf. [9, Notation A.3.6]) is the extension of the $\phi \otimes \mathrm{id}_S$-compatible unitary operator $\Phi \otimes \mathrm{id}_S : \mathcal{E} \otimes S \to \mathcal{F} \otimes S$ (cf. [9, Proposition-Definition A.3.4]). Note that we have $\Phi \circ \beta_{\mathcal{E}}(n^{\mathrm{o}}) = \beta_{\mathcal{F}}(n^{\mathrm{o}}) \circ \Phi$ for all $n \in N$ (cf. [9, Proposition 6.1.13]).

Definition 2.1.15. Two Hilbert C*-modules $\mathcal{E}$ and $\mathcal{F}$ acted upon by $\mathcal{G}$ are said to be $\mathcal{G}$-equivariantly unitarily equivalent if there exists a $\mathcal{G}$-equivariant unitary operator from $\mathcal{E}$ onto $\mathcal{F}$.

It is clear that the $\mathcal{G}$-equivariant unitary equivalence defines an equivalence relation on the class consisting of the Hilbert C*-modules acted upon by $\mathcal{G}$. For equivalent definitions of the $\mathcal{G}$-equivariant unitary equivalence in the two other pictures, we refer to [9, §6.1].

Remark 2.1.16. Let B be a $\widehat{\mathcal{G}}$-C*-algebra. An action of the dual measured quantum groupoid $\widehat{\mathcal{G}}$ on a Hilbert B-module $\mathcal{F}$ is defined by three equivalent data:

- a pair $(\alpha_{\mathcal{F}}, \delta_{\mathcal{F}})$ consisting of a *-homomorphism $\alpha_{\mathcal{F}} : N \to \mathcal{L}(\mathcal{F})$ and a linear map $\delta_{\mathcal{F}} : \mathcal{F} \to \widetilde{\mathcal{M}}(\mathcal{F} \otimes \widehat{S})$,
- a pair $(\alpha_{\mathcal{E}}, \mathcal{V})$ consisting of a *-homomorphism $\alpha_{\mathcal{F}} : N \to \mathcal{L}(\mathcal{F})$ and an isometry $\mathcal{V} \in \mathcal{L}(\mathcal{F} \otimes_{\delta_B} (B \otimes \widehat{S}), \mathcal{F} \otimes \widehat{S})$,
- an action (α_K, δ_K) of $\widehat{\mathcal{G}}$ on $K := \mathcal{K}(\mathcal{F} \oplus B)$,

satisfying some conditions. The details are left to the reader's attention.

2.2. Equivariant Hilbert modules and bimodules

In this paragraph, we recall the notion of continuity for actions of the quantum groupoid $\mathcal{G}$ on Hilbert C*-modules and the notion of equivariant representation of a $\mathcal{G}$-C*-algebra on a Hilbert C*-module acted upon by $\mathcal{G}$ (cf. [9, §7]). Let A be a $\mathcal{G}$-C*-algebra.

Definition 2.2.1. An action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ of $\mathcal{G}$ on a Hilbert A-module $\mathcal{E}$ is said to be continuous if we have $[(1_{\mathcal{E}} \otimes S)\delta_{\mathcal{E}}(\mathcal{E})] = (\mathcal{E} \otimes S)q_{\beta_A \alpha}$. A $\mathcal{G}$-equivariant Hilbert A-module is a Hilbert A-module $\mathcal{E}$ endowed with a continuous action of $\mathcal{G}$.

Proposition 2.2.2. *Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant Hilbert A-module. Let $B := \mathcal{K}(\mathcal{E})$. We have the following statements:*

1. *the action (β_B, δ_B) of $\mathcal{G}$ on B defined in Proposition 2.1.10 is strongly continuous;*
2. *we define a continuous action of $\mathcal{G}$ on the Hilbert B-module $\mathcal{E}^*$ by setting*
   - *$\beta_{\mathcal{E}^*}(n^{\mathrm{o}})T := \beta_A(n^{\mathrm{o}}) \circ T$ for all $n \in N$ and $T \in \mathcal{E}^*$,*
   - *$\delta_{\mathcal{E}^*}(T)x := \delta_{\mathcal{E}}(T^*)^* \circ x$ for all $T \in \mathcal{E}^*$ and $x \in B \otimes S$,*

   where we have applied the usual identifications $B \otimes S = \mathcal{K}(\mathcal{E} \otimes S)$ and $\mathcal{E} = \mathcal{K}(A, \mathcal{E})$.

Proposition 2.2.3. *Let $\mathcal{E}$ be a Hilbert A-module endowed with an action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ of $\mathcal{G}$ on $\mathcal{E}$. Let $J := \mathcal{K}(\mathcal{E} \oplus A)$ be the associated linking C*-algebra. Let (β_J, δ_J) be the action defined in Proposition 2.1.10. Then, the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ is continuous if and only if the action (β_J, δ_J) is strongly continuous.*

Notations 2.2.4. There is a one-to-one correspondence between $\mathcal{G}$-equivariant Hilbert C*-modules (cf. Definition 2.2.1) and linking $\mathcal{G}$-C*-algebras (cf. [9, Definition 6.1.22]).

- Let $(J, \beta_J, \delta_J, e_1, e_2)$ be a linking $\mathcal{G}$-C*-algebra. By restriction of the action (β_J, δ_J), the corner $e_2 J e_2$ (resp. $e_1 J e_2$) turns into a $\mathcal{G}$-C*-algebra (resp. $\mathcal{G}$-equivariant Hilbert C*-module over $e_2 J e_2$). We also have the identification of $\mathcal{G}$-C*-algebras $\mathcal{K}(e_1 J e_2) = e_1 J e_1$.
- Conversely, if $(\mathcal{E}, \beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ is a full $\mathcal{G}$-equivariant Hilbert A-module, then the C*-algebra $J := \mathcal{K}(\mathcal{E} \oplus A)$ endowed with the continuous action (β_J, δ_J) (cf. Propositions 2.1.10 and 2.2.3) and the projections $e_1 := \iota_{\mathcal{E}}(1_{\mathcal{E}})$ and $e_2 := \iota_A(1_A)$ is a linking $\mathcal{G}$-C*-algebra.

Theorem 2.2.5. *Let $\mathcal{E}$ be a Hilbert A-module. If the quantum groupoid $\mathcal{G}$ is regular, then any action of $\mathcal{G}$ on $\mathcal{E}$ is continuous.*

Notation 2.2.6. Let A and B be two C*-algebras and $\mathcal{E}$ a Hilbert B-module. If $\gamma : A \to \mathcal{L}(\mathcal{E})$ is a *-homomorphism, then we extend $\gamma \otimes \mathrm{id}_S$ to a *-homomorphism $\gamma \otimes \mathrm{id}_S : \widetilde{\mathcal{M}}(A \otimes S) \to \mathcal{L}(\mathcal{E} \otimes S)$ up to the identification $\mathcal{M}(\mathcal{K}(\mathcal{E}) \otimes S) = \mathcal{L}(\mathcal{E} \otimes S)$.

Definition 2.2.7. Let A and B be two $\mathcal{G}$-C*-algebras, $\mathcal{E}$ a Hilbert B-module, $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ an action of $\mathcal{G}$ on $\mathcal{E}$, and $\gamma : A \to \mathcal{L}(\mathcal{E})$ a *-representation. We say that γ is $\mathcal{G}$-equivariant if we have

(1) $\delta_{\mathcal{E}}(\gamma(a)\xi) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathcal{E}}(\xi)$ for all $a \in A$ and $\xi \in \mathcal{E}$,

(2) $\beta_{\mathcal{E}}(n^{\mathrm{o}}) \circ \gamma(a) = \gamma(\beta_A(n^{\mathrm{o}})a)$ for all $n \in N$ and $a \in A$.

A $\mathcal{G}$-equivariant Hilbert A-B-bimodule is a countably generated $\mathcal{G}$-equivariant Hilbert B-module endowed with a $\mathcal{G}$-equivariant *-representation of A.

Remarks 2.2.8. (1) Provided that the second condition in the above definition is verified, the first condition is equivalent to

$$\mathcal{V}\big(\gamma(a) \otimes_{\delta_B} 1\big)\mathcal{V}^* = (\gamma \otimes \mathrm{id}_S)\delta_A(a) \quad \text{for all } a \in A, \tag{2.9}$$

where $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_B} (B \otimes S), \mathcal{E} \otimes S)$ denotes the isometry defined in Proposition 2.1.5 (a).

(2) We recall that the action $\delta_{\mathcal{K}(\mathcal{E})}$ of $\mathcal{G}$ on $\mathcal{K}(\mathcal{E})$ is defined by

$$\delta_{\mathcal{K}(\mathcal{E})}(k) := \mathcal{V}(k \otimes_{\delta_B} 1)\mathcal{V}^* \quad \text{for all } k \in \mathcal{K}(\mathcal{E}).$$

Hence, (2.9) can be restated as follows: $\delta_{\mathcal{K}(\mathcal{E})}(\gamma(a)) = (\gamma \otimes \mathrm{id}_S)\delta_A(a)$ for all $a \in A$. In particular, if γ is non-degenerate, then Definition 2.2.7 simply means that the *-homomorphism $\gamma : A \to \mathcal{M}(\mathcal{K}(\mathcal{E}))$ is $\mathcal{G}$-equivariant (cf. [9, Definition 5.1.10]).

(3) If $\gamma : A \to \mathcal{L}(\mathcal{E})$ is a non-degenerate *-representation such that

$$\delta_{\mathcal{E}}\big(\gamma(a)\xi\big) = (\gamma \otimes \mathrm{id}_S)\big(\delta_A(a)\big) \circ \delta_{\mathcal{E}}(\xi) \quad \text{for all } a \in A \text{ and } \xi \in \mathcal{E},$$

then the second condition of Definition 2.2.7 is satisfied.

We recall below the tensor product construction.

Proposition 2.2.9. *Let C (resp. B) be a $\mathcal{G}$-C*-algebra. Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be a Hilbert module over C (resp. B) endowed with an action $(\beta_{\mathcal{E}_1}, \delta_{\mathcal{E}_1})$ (resp. $(\beta_{\mathcal{E}_2}, \delta_{\mathcal{E}_2})$) of $\mathcal{G}$. Let $\gamma_2 : C \to \mathcal{L}(\mathcal{E}_2)$ be a $\mathcal{G}$-equivariant *-representation. Consider the Hilbert B-module $\mathcal{E} := \mathcal{E}_1 \otimes_{\gamma_2} \mathcal{E}_2$. Denote*

$$\Delta(\xi_1, \xi_2) := \big(\delta_{\mathcal{E}_1}(\xi_1) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} 1\big) \circ \delta_{\mathcal{E}_2}(\xi_2) \quad \text{for } \xi_1 \in \mathcal{E}_1 \text{ and } \xi_2 \in \mathcal{E}_2.$$

We have $\Delta(\xi_1, \xi_2) \in \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ for all $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$. Let $\beta_{\mathcal{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E})$ be the *-homomorphism defined by

$$\beta_{\mathcal{E}}(n^{\mathrm{o}}) := \beta_{\mathcal{E}_1}(n^{\mathrm{o}}) \otimes_{\gamma_2} 1 \quad \text{for all } n \in N.$$

There exists a unique map $\delta_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{M}}(\mathcal{E} \otimes S)$ defined by the formula

$$\delta_{\mathcal{E}}(\xi_1 \otimes_{\gamma_2} \xi_2) := \Delta(\xi_1, \xi_2) \quad \text{for } \xi_1 \in \mathcal{E}_1 \text{ and } \xi_2 \in \mathcal{E}_2$$

such that the pair $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ is an action of $\mathcal{G}$ on $\mathcal{E}$.

The operator $\delta_{\mathcal{E}_1}(\xi_1)$ is considered here as an element of $\mathcal{L}(\widetilde{C} \otimes S, \mathcal{E}_1 \otimes S) \supset \widetilde{\mathcal{M}}(\mathcal{E}_1 \otimes S)$. In particular, we have $\delta_{\mathcal{E}_1}(\xi_1) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} 1 \in \mathcal{L}(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes S)$ up to the identifications

$$\begin{aligned} (\widetilde{C} \otimes S) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\mathcal{E}_2 \otimes S) &\to \mathcal{E}_2 \otimes S \\ x \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} \eta &\mapsto (\widetilde{\gamma}_2 \otimes \mathrm{id}_S)(x)\eta; \end{aligned} \tag{2.10}$$

$$\begin{aligned} (\mathcal{E}_1 \otimes S) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\mathcal{E}_2 \otimes S) &\to \mathcal{E} \otimes S \\ (\xi_1 \otimes s) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\xi_2 \otimes t) &\mapsto (\xi_1 \otimes_{\gamma_2} \xi_2) \otimes st. \end{aligned} \tag{2.11}$$

Remark 2.2.10. We recall the definition of the isometry $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_B} (B \otimes S), \mathcal{E} \otimes S)$ associated with the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ (cf. Definition 2.1.4). We refer to the proof of Proposition 7.9 [9] for more details. For $i = 1, 2$, let $\mathcal{V}_i$ be the isometry associated with the actions $(\beta_{\mathcal{E}_i}, \delta_{\mathcal{E}_i})$. Let $\widetilde{\mathcal{V}}_2 \in \mathcal{L}(\mathcal{E} \otimes_{\delta_B} (B \otimes S), \mathcal{E}_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} (\mathcal{E}_2 \otimes S))$ be the unitary defined for all $\xi_1 \in \mathcal{E}_1$, $\xi_2 \in \mathcal{E}_2$ and $x \in B \otimes S$ by

$$\widetilde{\mathcal{V}}_2\big((\xi_1 \otimes_{\gamma_2} \xi_2) \otimes_{\delta_B} x\big) := \xi_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} \mathcal{V}_2(\xi_2 \otimes_{\delta_B} x).$$

Up to the identifications

$$\begin{aligned} \big(\mathcal{E}_1 \otimes_{\delta_C} (C \otimes S)\big) \otimes_{\gamma_2 \otimes \mathrm{id}_S} (\mathcal{E}_2 \otimes S) &\to \mathcal{E}_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} (\mathcal{E}_2 \otimes S) \\ (\xi_1 \otimes_{\delta_C} x) \otimes_{\gamma_2 \otimes \mathrm{id}_S} \eta &\mapsto \xi_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} (\gamma_2 \otimes \mathrm{id}_S)(x)\eta; \end{aligned} \tag{2.12}$$

$$\begin{aligned} (\mathcal{E}_1 \otimes S) \otimes_{\gamma_2 \otimes \mathrm{id}_S} (\mathcal{E}_2 \otimes S) &\to \mathcal{E} \otimes S \\ (\xi_1 \otimes s) \otimes_{\gamma_2 \otimes \mathrm{id}_S} (\xi_2 \otimes t) &\mapsto (\xi_1 \otimes_{\gamma_2} \xi_2) \otimes st \end{aligned} \tag{2.13}$$

we have $\mathcal{V} = (\mathcal{V}_1 \otimes_{\gamma_2 \otimes \mathrm{id}_S} 1)\widetilde{\mathcal{V}}_2$.

The following result is straightforward.

Proposition 2.2.11. *We use all the notations and hypotheses of Proposition 2.2.9. If A is a $\mathcal{G}$-C*-algebra and $\gamma_1 : A \to \mathcal{L}(\mathcal{E}_1)$ is a $\mathcal{G}$-equivariant *-representation, then $\gamma : A \to \mathcal{L}(\mathcal{E})$, the *-representation defined by $\gamma(a) := \gamma_1(a) \otimes_{\gamma_2} 1$ for all $a \in A$, is $\mathcal{G}$-equivariant.*

If $(\mathcal{E}_1, \gamma_1)$ *is a* $\mathcal{G}$*-equivariant* A*-*C*-bimodule and* $\mathcal{E}_2$ *is a* $\mathcal{G}$*-equivariant* C*-*B*-bimodule, then the pair* $(\mathcal{E}, \gamma)$ *is a* $\mathcal{G}$*-equivariant* A*-*B*-bimodule.*

The $\mathcal{G}$-equivariance of the internal tensor product associativity map is straightforward and left to the reader's discretion.

Lemma 2.2.12. *We use all the notations and hypotheses of Proposition 2.2.9. If* $F \in \mathcal{L}(\mathcal{E}_1)$ *is invariant, then so is* $F \otimes_{\gamma_2} 1 \in \mathcal{L}(\mathcal{E})$.

Proof. For all $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$, $\Delta(F\xi_1, \xi_2) = ((F \otimes 1_S) \otimes_{\tilde{\gamma}_2 \otimes \mathrm{id}_S} 1)\Delta(\xi_1, \xi_2)$. However, $(F \otimes 1_S) \otimes_{\tilde{\gamma}_2 \otimes \mathrm{id}_S} 1$ is identified to $(F \otimes_{\gamma_2} 1) \otimes 1_S$ through the identification (2.11). Hence, $\delta_{\mathcal{E}}(F\xi_1 \otimes_{\gamma_2} \xi_2) = ((F \otimes_{\gamma_2} 1) \otimes 1_S)\delta_{\mathcal{E}}(\xi_1 \otimes_{\gamma_2} \xi_2)$ for all $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$. Hence, $F \otimes_{\gamma_2} 1 \in \mathcal{L}(\mathcal{E})$ is invariant (cf. Proposition-Definition 2.1.13). ■

2.3. Biduality and equivariant Morita equivalence

In this paragraph, we recall the notion of equivariant Morita equivalence between $\mathcal{G}$-C*-algebras ([9, §7]).

Definition 2.3.1 (cf. [25, §6]). Let A and B be two C*-algebras. An imprimitivity A-B-bimodule is an A-B-bimodule $\mathcal{E}$, which is a full left Hilbert A-module for an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ and a full right Hilbert B-module for a B-valued inner product $\langle\cdot,\cdot\rangle_B$ such that ${}_A\langle\xi,\eta\rangle\zeta = \xi\langle\eta,\zeta\rangle_B$ for all $\xi, \eta, \zeta \in \mathcal{E}$.

Remarks 2.3.2. Let A and B be two C*-algebras and $\mathcal{E}$ an imprimitivity A-B-bimodule. We recall that the norms defined by the inner products ${}_A\langle\cdot,\cdot\rangle$ on ${}_A\mathcal{E}$ and $\langle\cdot,\cdot\rangle_B$ on $\mathcal{E}_B$ coincide. We also recall that the left (resp. right) action of A (resp. B) on $\mathcal{E}$ defines a non-degenerate *-homomorphism $\gamma : A \to \mathcal{L}(\mathcal{E}_B)$ (resp. $\rho : B \to \mathcal{L}({}_A\mathcal{E})$).

Definition 2.3.3. Let A and B be two $\mathcal{G}$-C*-algebras. A $\mathcal{G}$-equivariant imprimitivity A-B-bimodule is an imprimitivity A-B-bimodule $\mathcal{E}$ endowed with a continuous action of $\mathcal{G}$ on $\mathcal{E}_B$ such that the left action $\gamma : A \to \mathcal{L}(\mathcal{E}_B)$ is $\mathcal{G}$-equivariant. In that case, we say that A and B are $\mathcal{G}$-equivariantly Morita equivalent.

If the quantum groupoid $\mathcal{G}$ is regular, then the $\mathcal{G}$-equivariant Morita equivalence is a reflexive, symmetric, and transitive relation on the class of $\mathcal{G}$-C*-algebras (cf. [9, Definition 7.13]).

In what follows, we recall the canonical equivariant Morita equivalence of the double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ (resp. $(B \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G}$) with the initial $\mathcal{G}$-C*-algebra (resp. $\widehat{\mathcal{G}}$-C*-algebra) A (resp. B) (cf. [9, Theorem 7.22]).

Let (A, β_A, δ_A) (resp. (B, α_B, δ_B)) be a $\mathcal{G}$-C*-algebra (resp. $\widehat{\mathcal{G}}$-C*-algebra).

Notations 2.3.4. The *-representation of A (resp. B) on the Hilbert A-module $A \otimes \mathcal{H}$ (resp. the Hilbert B-module $B \otimes \mathcal{H}$)

$$\pi_R := (\mathrm{id}_A \otimes R) \circ \delta_A : A \to \mathcal{L}(A \otimes \mathcal{H})$$
$$\big(\text{resp. } \pi_\rho := (\mathrm{id}_B \otimes \rho) \circ \delta_B : B \to \mathcal{L}(B \otimes \mathcal{H})\big)$$

extends uniquely to a strictly/*-strongly continuous faithful *-representation $\pi_R : \mathcal{M}(A) \to \mathcal{L}(A \otimes \mathcal{H})$ (resp. $\pi_\rho : \mathcal{M}(B) \to \mathcal{L}(B \otimes \mathcal{H})$) satisfying $\pi_R(m) = (\mathrm{id}_A \otimes R)\delta_A(m)$ for all $m \in \mathcal{M}(A)$ and $\pi_R(1_A) = q_{\beta_A\widehat{\alpha}}$ (resp. $\pi_\rho(m) = (\mathrm{id}_B \otimes \rho)\delta_B(m)$ for all $m \in \mathcal{M}(B)$ and $\pi_\rho(1_B) = q_{\alpha_B\beta}$). Consider the Hilbert A-module (resp. the Hilbert B-module)

$$\mathcal{E}_{A,R} := q_{\beta_A\widehat{\alpha}}(A \otimes \mathcal{H}) \quad \big(\text{resp. } \mathcal{E}_{B,\rho} := q_{\alpha_B\beta}(B \otimes \mathcal{H})\big).$$

We recall that the Banach space

$$D := \big[\pi_R(a)\big(1_A \otimes \lambda(x)L(y)\big);\ a \in A,\ x \in \widehat{S},\ y \in S\big]$$
$$\big(\text{resp. } E := \big[\pi_\rho(b)\big(1_B \otimes R(y)\lambda(x)\big);\ b \in B,\ y \in S,\ x \in \widehat{S}\big]\big)$$

is a C*-subalgebra of $\mathcal{L}(A \otimes \mathcal{H})$ (resp. $\mathcal{L}(B \otimes \mathcal{H})$) such that $uq_{\beta_A\widehat{\alpha}} = u = uq_{\beta_A\widehat{\alpha}}$ for all $u \in D$ (resp. $vq_{\alpha_B\beta} = v = q_{\alpha_B\beta}v$ for all $v \in E$). Moreover, we have $D(A \otimes \mathcal{H}) = \mathcal{E}_{A,R}$ (resp. $E(B \otimes \mathcal{H}) = \mathcal{E}_{B,\rho}$). We also recall that there exists a unique strictly/*-strongly continuous faithful *-representation $j_D : \mathcal{M}(D) \to \mathcal{L}(A \otimes \mathcal{H})$ (resp. $j_E : \mathcal{M}(E) \to \mathcal{L}(B \otimes \mathcal{H})$) extending the inclusion map $D \subset \mathcal{L}(A \otimes \mathcal{H})$ (resp. $E \subset \mathcal{L}(B \otimes \mathcal{H})$) such that $j_D(1_D) = q_{\beta_A\widehat{\alpha}}$ (resp. $j_E(1_E) = q_{\alpha_B\beta}$).

Proposition 2.3.5. *There exists a unique *-isomorphism $\phi : (A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ (resp. $\psi : (B \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G} \to E$) such that $\phi(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y))$ for all $a \in A$, $x \in \widehat{S}$, and $y \in S$ (resp. $\psi(\pi(\widehat{\pi}(b)\theta(y))\widehat{\theta}(x)) = \pi_\rho(b)(1_B \otimes R(y)\rho(x))$ for all $b \in B$, $y \in S$, and $x \in \widehat{S}$).*

Notations 2.3.6. We denote $\mathcal{K} := \mathcal{K}(\mathcal{H})$ for short. Let $\delta_0 : A \otimes \mathcal{K} \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ (resp. $\delta_0 : B \otimes \mathcal{K} \to \mathcal{M}(B \otimes \mathcal{K} \otimes \widehat{S})$) be the *-homomorphism defined by

$$\delta_0(a \otimes k) = \delta_A(a)_{13}(1_A \otimes k \otimes 1_S) \quad \big(\text{resp. } \delta_0(b \otimes k) = \delta_B(b)_{13}(1_B \otimes k \otimes 1_{\widehat{S}})\big)$$

for all $a \in A$ (resp. $b \in B$) and $k \in \mathcal{K}$. The morphism δ_0 extends uniquely to a strictly continuous *-homomorphism still denoted by $\delta_0 : \mathcal{M}(A \otimes \mathcal{K}) \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ (resp. $\delta_0 : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{M}(B \otimes \mathcal{K} \otimes \widehat{S})$) such that $\delta_0(1_{A\otimes\mathcal{K}}) = q_{\beta_A\alpha,13}$ (resp. $\delta_0(1_{B\otimes\mathcal{K}}) = q_{\alpha_B\beta,13}$). Let us denote by $\mathcal{V} \in \mathcal{L}(\mathcal{H} \otimes S)$ (resp. $\widetilde{\mathcal{V}} \in \mathcal{L}(\mathcal{H} \otimes \widehat{S})$) the unique partial isometry such that $(\mathrm{id}_\mathcal{K} \otimes L)(\mathcal{V}) = V$ (resp. $(\mathrm{id}_\mathcal{K} \otimes \rho)(\widetilde{\mathcal{V}}) = \widetilde{V}$).

Theorem 2.3.7. *There exists a unique strongly continuous action (β_D, δ_D) (resp. (α_E, δ_E)) of $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$) on the C*-algebra $D := [\pi_R(a)(1_A \otimes \lambda(x)L(y));\ a \in A,\ x \in \widehat{S},\ y \in S]$ (resp. $E := [\pi_\rho(b)(1_B \otimes R(y)\lambda(x));\ b \in B,\ y \in S,\ x \in \widehat{S}]$) defined by the relations*

$$(j_D \otimes \mathrm{id}_S)\delta_D(u) = \mathcal{V}_{23}\delta_0(u)\mathcal{V}_{23}^*,\ u \in D;\ j_D\big(\beta_D(n^{\mathrm{o}})\big) = q_{\beta_A\widehat{\alpha}}\big(1_A \otimes \beta(n^{\mathrm{o}})\big),\ n \in N$$
$$\big(\textit{resp. } (j_E \otimes \mathrm{id}_{\widehat{S}})\delta_E(v) = \widetilde{\mathcal{V}}_{23}\delta_0(v)\widetilde{\mathcal{V}}_{23}^*,\ v \in E;\ j_E\big(\alpha_E(n)\big) = q_{\alpha_B\beta}\big(1_B \otimes \widehat{\alpha}(n)\big),\ n \in N\big).$$

*Moreover, the canonical *-isomorphism $\phi : (A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ (resp. $\psi : (B \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G} \to E$) (cf. Proposition 2.3.5) is $\mathcal{G}$-equivariant (resp. $\widehat{\mathcal{G}}$-equivariant). If the groupoid $\mathcal{G}$ is regular, then we have $D = q_{\beta_A\widehat{\alpha}}(A \otimes \mathcal{K})q_{\beta_A\widehat{\alpha}}$ (resp. $E = q_{\alpha_B\beta}(B \otimes \mathcal{K})q_{\alpha_B\beta}$).*

The $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra D (resp. E) defined above will be referred to as the bidual $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra of A (resp. B). We investigate below the case of a linking $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra.

Lemma 2.3.8. *Let (A, β_A, δ_A) be a $\mathcal{G}$-C*-algebra (resp. $\widehat{\mathcal{G}}$-C*-algebra). For all $m \in \mathcal{M}(A)$, $\pi(m) \in \mathcal{M}(A \rtimes \mathcal{G})$ (resp. $\widehat{\pi}(m) \in \mathcal{M}(A \rtimes \widehat{\mathcal{G}})$) is $\delta_{A\rtimes\mathcal{G}}$-invariant (resp. $\delta_{A\rtimes\widehat{\mathcal{G}}}$-invariant).*

Proof. We have $(\widehat{\theta} \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(1_{\widehat{S}}) = (\widehat{\theta} \otimes \mathrm{id}_{\widehat{S}})(q_{\widehat{\alpha}\beta}) = q_{\alpha_{A\rtimes\mathcal{G}}\beta}$. By strict continuity of $\widehat{\theta}$ and $\delta_{A\rtimes\mathcal{G}}$, it follows from [9, Proposition-Definition 5.1.15 (1)] that $\delta_{A\rtimes\mathcal{G}}(\pi(a)) = (\pi(a) \otimes 1_{\widehat{S}})q_{\alpha_{A\rtimes\mathcal{G}}\beta}$ for all $a \in A$. Hence, $\delta_{A\rtimes\mathcal{G}}(\pi(m)) = q_{\alpha_{A\rtimes\mathcal{G}}\beta}(\pi(m) \otimes 1_{\widehat{S}})$ by strict continuity of π and $\delta_{A\rtimes\mathcal{G}}$, i.e., $\pi(m)$ is $\delta_{A\rtimes\mathcal{G}}$-invariant. ■

Proposition 2.3.9. *If the quintuple $(J, \beta_J, \delta_J, e_1, e_2)$ (resp. $(K, \alpha_K, \delta_K, f_1, f_2)$) is a linking $\mathcal{G}$-C*-algebra (resp. linking $\widehat{\mathcal{G}}$-C*-algebra), then the quintuple $(J \rtimes \mathcal{G}, \alpha_{J\rtimes\mathcal{G}}, \delta_{J\rtimes\mathcal{G}}, \pi(e_1), \pi(e_2))$ (resp. $(K \rtimes \widehat{\mathcal{G}}, \beta_{K\rtimes\widehat{\mathcal{G}}}, \delta_{K\rtimes\widehat{\mathcal{G}}}, \widehat{\pi}(f_1), \widehat{\pi}(f_2))$) is a linking $\widehat{\mathcal{G}}$-C*-algebra (resp. linking $\mathcal{G}$-C*-algebra).*

Proof. Let $(J, \beta_J, \delta_J, e_1, e_2)$ be a linking $\mathcal{G}$-C*-algebra. Let $K := J \rtimes \mathcal{G}$ and let (α_K, δ_K) be the dual action of (β_J, δ_J). Since the canonical morphism $\pi : J \to \mathcal{M}(K)$ is non-degenerate, we have $\pi(e_1) + \pi(e_2) = 1_K$. Let $j = 1, 2$. Since $\pi(e_j) \in \mathcal{M}(K)$, we have $[K\pi(e_j)K] \subset K$. Any element of K is the norm limit of finite sums of the form $\sum_\lambda \widehat{\theta}(x_\lambda)\pi(a_\lambda)\widehat{\theta}(x'_\lambda)$ with $x_\lambda, x'_\lambda \in \widehat{S}$ and $a_\lambda \in J$. Since $J = [Je_jJ]$, any element of K is the norm limit of finite sums of the form $\sum_\lambda \widehat{\theta}(x_\lambda)\pi(a_\lambda)\pi(e_j)\pi(b_\lambda)\widehat{\theta}(x'_\lambda)$ with $x_\lambda, x'_\lambda \in \widehat{S}$ and $a_\lambda, b_\lambda \in J$. Hence, $K \subset [K\pi(e_j)K]$. Hence, $K = [K\pi(e_j)K]$. Thus, the quintuple $(J \rtimes \mathcal{G}, \alpha_{J\rtimes\mathcal{G}}, \delta_{J\rtimes\mathcal{G}}, \pi(e_1), \pi(e_2))$ is a linking $\widehat{\mathcal{G}}$-C*-algebra (cf. Lemma 2.3.8). ■

Remark 2.3.10. Let $(J, \beta_J, \delta_J, e_1, e_2)$ (resp. $(K, \alpha_K, \delta_K, f_1, f_2)$) be linking $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra. We have a bidual linking $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra $(D, \beta_D, \delta_D, \pi_R(e_1), \pi_R(e_2))$ (resp. $(E, \alpha_E, \delta_E, \pi_\rho(f_1), \pi_\rho(f_2))$) and $\phi : (J \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ (resp. $\psi : (K \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G} \to E$) is an isomorphism of linking $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebras.

In the following result, we assume the quantum groupoid $\mathcal{G}$ to be regular.

Theorem 2.3.11. *Let (A, β_A, δ_A) (resp. (B, α_B, δ_B)) be a $\mathcal{G}$-C*-algebra (resp. $\widehat{\mathcal{G}}$-C*-algebra).*

(1) *There exists a unique continuous action $(\beta_{\mathcal{E}_{A,R}}, \delta_{\mathcal{E}_{A,R}})$ (resp. $(\alpha_{\mathcal{E}_{B,\rho}}, \delta_{\mathcal{E}_{B,\rho}})$) of $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$) on the Hilbert A-module $\mathcal{E}_{A,R}$ (resp. the Hilbert B-module $\mathcal{E}_{B,\rho}$) given for all $a \in A$ (resp. $b \in B$), $\zeta \in \mathcal{H}$ and $n \in N$ by the formulas*

$$\delta_{\mathcal{E}_{A,R}}\big(q_{\beta_A\widehat{\alpha}}(a \otimes \zeta)\big) = \mathcal{V}_{23}\delta_A(a)_{13}(1_A \otimes \zeta \otimes 1_S);$$

$$\beta_{\mathcal{E}_{A,R}}(n^{\mathrm{o}}) := \big(1_A \otimes \beta(n^{\mathrm{o}})\big)\restriction_{\mathcal{E}_{A,R}};$$

$$\big(\textit{resp. } \delta_{\mathcal{E}_{B,\rho}}\big(q_{\alpha_B\beta}(b \otimes \zeta)\big) = \widetilde{\mathcal{V}}_{23}\delta_B(b)_{13}(1_B \otimes \zeta \otimes 1_{\widehat{S}});$$

$$\alpha_{\mathcal{E}_{B,\rho}}(n) := \big(1_B \otimes \widehat{\alpha}(n)\big)\restriction_{\mathcal{E}_{B,\rho}}\big).$$

(2) *Endowed with the* **-representation*

$$D \to \mathcal{L}(\mathcal{E}_{A,R});\ u \mapsto u\restriction_{\mathcal{E}_{A,R}} \quad \big(\textit{resp. } E \to \mathcal{L}(\mathcal{E}_{B,\rho});\ v \mapsto v\restriction_{\mathcal{E}_{B,\rho}}\big),$$

the $\mathcal{G}$*-equivariant Hilbert* A*-module* $\mathcal{E}_{A,R}$ *(resp. the* $\widehat{\mathcal{G}}$*-equivariant Hilbert* B*-module* $\mathcal{E}_{B,\rho}$*) is a* $\mathcal{G}$*-equivariant Hilbert* D*-*A*-bimodule (resp.* $\widehat{\mathcal{G}}$*-equivariant Hilbert* E*-*B*-bimodule).*

(3) *The* $\mathcal{G}$*-*C**-algebras (resp.* $\widehat{\mathcal{G}}$*-*C**-algebras)* A *and* D *(resp.* B *and* E*) are Morita equivalent via the* $\mathcal{G}$*-equivariant (resp.* $\widehat{\mathcal{G}}$*-equivariant) imprimitivity* D*-*A*-bimodule* $\mathcal{E}_{A,R}$ *(resp.* E*-*B*-bimodule* $\mathcal{E}_{B,\rho}$*).*

2.4. Crossed product, dual action, and biduality

2.4.1. Crossed product. In this paragraph, we define and investigate the crossed product of a Hilbert module acted upon by a measured quantum groupoid on a finite-dimensional basis. Let us specify some notations.

Let (A,β_A,δ_A) be a $\mathcal{G}$-C*-algebra. Denote by $B := A \rtimes \mathcal{G}$ the crossed product endowed with the dual action (α_B,δ_B). Let $\pi : A \to \mathcal{M}(B)$ and $\widehat{\theta} : \widehat{S} \to \mathcal{M}(B)$ be the canonical morphisms (cf. [9, Proposition-Definition 5.1.14]). Let $\mathcal{E}$ be a Hilbert A-module and $(\beta_{\mathcal{E}},\delta_{\mathcal{E}})$ an action of $\mathcal{G}$ on $\mathcal{E}$.

Definition 2.4.1. We call the crossed product of $\mathcal{E}$ by the action $(\beta_{\mathcal{E}},\delta_{\mathcal{E}})$ the Hilbert B-module $\mathcal{E} \otimes_\pi B$ denoted by $\mathcal{E} \rtimes \mathcal{G}$.

Notation 2.4.2. For $\xi \in \mathcal{E}$, we denote by $\Pi(\xi) \in \mathcal{L}(B, \mathcal{E} \rtimes \mathcal{G})$ the adjointable operator defined by $\Pi(\xi)b := \xi \otimes_\pi b$ for all $b \in B$. We have $\Pi(\xi)^*(\eta \otimes_\pi b) = \pi(\langle \xi,\eta\rangle)b$ for all $\eta \in \mathcal{E}$ and $b \in B$. We then have a linear map $\Pi : \mathcal{E} \to \mathcal{L}(B, \mathcal{E} \rtimes \mathcal{G})$ (also denoted by $\Pi_{\mathcal{E}}$ for emphasis).

Proposition 2.4.3. *We have*

(1) Π *is non-degenerate; i.e.,* $[\Pi(\mathcal{E})B] = \mathcal{E} \rtimes \mathcal{G}$,

(2) $\Pi(\xi a) = \Pi(\xi)\pi(a)$ *for all* $\xi \in \mathcal{E}$ *and* $a \in A$,

(3) $\Pi(\xi)^*\Pi(\eta) = \pi(\langle \xi,\eta\rangle)$ *for all* $\xi,\eta \in \mathcal{E}$,

(4) $\Pi(\xi)\widehat{\theta}(x) \in \mathcal{E} \rtimes \mathcal{G}$ *for all* $\xi \in \mathcal{E}$ *and* $x \in \widehat{S}$ *and* $\mathcal{E} \rtimes \mathcal{G} = [\Pi(\xi)\widehat{\theta}(x);\ \xi \in \mathcal{E},\ x \in \widehat{S}]$.

Proof. Statements (1), (2), and (3) are direct consequences of the definitions. For all $\xi \in \mathcal{E}$, $a \in A$, and $x \in \widehat{S}$, we have $\Pi(\xi a)\widehat{\theta}(x) = \Pi(\xi)(\pi(a)\widehat{\theta}(x)) \in \mathcal{E} \rtimes \mathcal{G}$. Hence, $\Pi(\xi)\widehat{\theta}(x) \in \mathcal{E} \rtimes \mathcal{G}$ for all $\xi \in \mathcal{E}$ and $x \in \widehat{S}$ since $\mathcal{E}A = \mathcal{E}$. The formula $\mathcal{E} \rtimes \mathcal{G} = [\Pi(\xi)\widehat{\theta}(x);\ \xi \in \mathcal{E},\ x \in \widehat{S}]$ follows from the relations $[\Pi(\mathcal{E})B] = \mathcal{E} \rtimes \mathcal{G}$ and $B = [\pi(a)\widehat{\theta}(x);\ a \in A,\ x \in \widehat{S}]$. ■

Proposition 2.4.4. *Let* $\alpha_{\mathcal{E}\rtimes\mathcal{G}} : N \to \mathcal{L}(\mathcal{E} \rtimes \mathcal{G})$ *and* $\delta_{\mathcal{E}\rtimes\mathcal{G}} : \mathcal{E} \rtimes \mathcal{G} \to \mathcal{L}(B \otimes \widehat{S}, (\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S})$ *be the linear maps defined by*

$$\alpha_{\mathcal{E}\rtimes\mathcal{G}}(n) := 1_{\mathcal{E}} \otimes_\pi \alpha_B(n), \quad n \in N;$$
$$\delta_{\mathcal{E}\rtimes\mathcal{G}}(\xi \otimes_\pi b) := \big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)\delta_B(b), \quad \xi \in \mathcal{E},\ b \in B.$$

Then, the pair $(\alpha_{\mathcal{E}\rtimes\mathcal{G}},\delta_{\mathcal{E}\rtimes\mathcal{G}})$ *is a continuous action of* $\widehat{\mathcal{G}}$ *on the crossed product* $\mathcal{E} \rtimes \mathcal{G}$.

Proof. Since $\delta_B(B) \in \widetilde{\mathcal{M}}(B \otimes \widehat{S})$, it is clear that $(\Pi(\xi) \otimes 1_{\widehat{S}})\delta_B(b) \in \widetilde{\mathcal{M}}((\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S})$ for all $\xi \in \mathcal{E}$ and $b \in B$. We have $\delta_B(\pi(a)b) = (\pi(a) \otimes 1_{\widehat{S}})\delta_B(b)$ for all $a \in A$ and $b \in B$ (cf. [9, Proposition-Definition 5.1.15 (1)]). Therefore, we have a well-defined linear map

$$\mathcal{E} \odot_B B \to \widetilde{\mathcal{M}}(B \otimes \widehat{S}) \subset \mathcal{L}\big(B \otimes \widehat{S}, (\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S}\big); \quad \xi \otimes_\pi b \mapsto \big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)\delta_B(b).$$

Let $\xi, \eta \in \mathcal{E}$. For all $b, c \in B$ and $x, y \in \widehat{S}$, we have

$$\begin{aligned}\big\langle\big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)(b \otimes x), \big(\Pi(\eta) \otimes 1_{\widehat{S}}\big)(c \otimes y)\big\rangle &= \big\langle(\xi \otimes_\pi b) \otimes x, (\eta \otimes_\pi c) \otimes y\big\rangle \\ &= b^*\pi\big(\langle\xi, \eta\rangle\big)c \otimes x^*y \\ &= (b \otimes x)^*\big(\pi\big(\langle\xi, \eta\rangle\big) \otimes 1_{\widehat{S}}\big)(c \otimes y).\end{aligned}$$

Hence, $(\Pi(\xi) \otimes 1_{\widehat{S}})^*(\Pi(\eta) \otimes 1_{\widehat{S}}) = \pi(\langle\xi, \eta\rangle) \otimes 1_{\widehat{S}}$. Therefore, for all $b, c \in B$ we have

$$\begin{aligned}\big\langle\big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)\delta_B(b), \big(\Pi(\eta) \otimes 1_{\widehat{S}}\big)\delta_B(c)\big\rangle &= \delta_B(b)^*\big(\pi\big(\langle\xi, \eta\rangle\big) \otimes 1_{\widehat{S}}\big)\delta_B(c) \\ &= \delta_B\big(b^*\pi\big(\langle\xi, \eta\rangle\big)c\big) \\ &= \delta_B\big(\langle\xi \otimes_\pi b, \eta \otimes_\pi c\rangle\big).\end{aligned}$$

Hence, there exists a unique bounded linear map $\delta_{\mathcal{E} \rtimes \mathcal{G}} : \mathcal{E} \rtimes \mathcal{G} \to \widetilde{\mathcal{M}}((\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S})$ such that $\delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi \otimes_B b) = (\Pi(\xi) \otimes 1_{\widehat{S}})\delta_B(b)$ for all $\xi \in \mathcal{E}$ and $b \in B$. Moreover, we have also proved that $\langle\delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi), \delta_{\mathcal{E} \rtimes \mathcal{G}}(\eta)\rangle = \delta_B(\langle\xi, \eta\rangle)$ for all $\xi, \eta \in \mathcal{E} \rtimes \mathcal{G}$. It is clear that $\delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi)\delta_B(b) = \delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi b)$ for all $\xi \in \mathcal{E} \rtimes \mathcal{G}$ and $b \in B$.

Let us fix $n \in N$. We recall that $\alpha_B(n) := \widehat{\theta}(\widehat{\alpha}(n))$. It follows from the inclusion $\widehat{\alpha}(N) \subset M'$ that $[1_A \otimes \rho(\widehat{\alpha}(n)), \pi_L(a)] = 0$ for all $a \in A$. Hence, $[\alpha_B(n), \pi(a)] = 0$ for all $a \in A$. Thus, the map $1_{\mathcal{E}} \otimes_\pi \alpha_B(n) \in \mathcal{L}(\mathcal{E} \rtimes \mathcal{G})$ is well defined. It is clear that $\alpha_{\mathcal{E} \rtimes \mathcal{G}} : N \to \mathcal{L}(\mathcal{E} \rtimes \mathcal{G})$ is a non-degenerate *-homomorphism.

We have $[1_{\mathcal{E} \rtimes \mathcal{G}} \otimes \widehat{\alpha}(n), \Pi(\xi) \otimes 1_{\widehat{S}}] = 0$ and $(1_B \otimes \widehat{\alpha}(n))\delta_B(b) = \delta_B(\alpha_B(n)b)$ for all $n \in N$, $\xi \in \mathcal{E}$, and $b \in B$. It then follows that $\delta_{\mathcal{E} \rtimes \mathcal{G}}(\alpha_{\mathcal{E} \rtimes \mathcal{G}}(n)\xi) = (1_{\mathcal{E} \rtimes \mathcal{G}} \otimes \widehat{\alpha}(n))\delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi)$ for all $\xi \in \mathcal{E} \rtimes \mathcal{G}$ and $n \in N$.

By continuity of the dual action (α_B, δ_B), we have

$$\big[\delta_{\mathcal{E} \rtimes \mathcal{G}}(\mathcal{E} \rtimes \mathcal{G})(B \otimes \widehat{S})\big] = \big[\big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)q_{\alpha_B\beta}(b \otimes x);\ \xi \in \mathcal{E},\ b \in B,\ x \in \widehat{S}\big].$$

Let $n, n' \in N$, $b \in B$, $x \in \widehat{S}$, and $\xi \in \mathcal{E}$. We have

$$\begin{aligned}\big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)\big(\alpha_B(n') \otimes \beta(n^{\mathrm{o}})\big)(b \otimes x) &= \big(\xi \otimes_\pi \alpha_B(n')b\big) \otimes \beta(n^{\mathrm{o}})x \\ &= \big(\alpha_{\mathcal{E} \rtimes \mathcal{G}}(n') \otimes \beta(n^{\mathrm{o}})\big)\big((\xi \otimes_\pi b) \otimes x\big).\end{aligned}$$

Hence, $(\Pi(\xi) \otimes 1_{\widehat{S}})q_{\alpha_B\beta}(b \otimes x) = q_{\alpha_{\mathcal{E} \rtimes \mathcal{G}}\beta}((\xi \otimes_\pi b) \otimes x)$. Therefore, we have

$$\big[\delta_{\mathcal{E} \rtimes \mathcal{G}}(\mathcal{E} \rtimes \mathcal{G})(B \otimes \widehat{S})\big] = q_{\alpha_{\mathcal{E} \rtimes \mathcal{G}}\beta}\big((\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S}\big).$$

The maps $\delta_{\mathcal{E} \rtimes \mathcal{G}} \otimes \mathrm{id}_{\widehat{S}}$ and $\mathrm{id}_{\mathcal{E} \rtimes \mathcal{G}} \otimes \widehat{\delta}$ extend to linear maps from $\mathcal{L}(B \otimes \widehat{S}, (\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S})$ to $\mathcal{L}(B \otimes \widehat{S} \otimes \widehat{S}, (\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S} \otimes \widehat{S})$ (cf. Remarks 2.1.7). For all $\xi \in \mathcal{E}$ and $b \in B$, we have

$$\begin{aligned}(\mathrm{id}_{\mathcal{E} \rtimes \mathcal{G}} \otimes \widehat{\delta})\delta_{\mathcal{E} \rtimes \mathcal{G}}(\xi \otimes_\pi b) &= (\mathrm{id}_{\mathcal{E} \rtimes \mathcal{G}} \otimes \widehat{\delta})\big(\Pi(\xi) \otimes 1_{\widehat{S}}\big)\delta_B(b) \\ &= (\Pi(\xi) \otimes 1_{\widehat{S}} \otimes 1_{\widehat{S}})(\mathrm{id}_B \otimes \widehat{\delta})\delta_B(b)\end{aligned}$$

$$
\begin{aligned}
&= \big((\Pi(\xi) \otimes 1_{\widehat{S}})\delta_B \otimes \mathrm{id}_{\widehat{S}}\big)\delta_B(b) \\
&= \big(\delta_{\mathcal{E}\rtimes\mathcal{G}} \circ \Pi(\xi) \otimes \mathrm{id}_{\widehat{S}}\big)\delta_B(x) \\
&= (\delta_{\mathcal{E}\rtimes\mathcal{G}} \otimes \mathrm{id}_{\widehat{S}})\delta_{\mathcal{E}\rtimes\mathcal{G}}(\xi \otimes_\pi b).
\end{aligned}
$$

Hence, $(\delta_{\mathcal{E}\rtimes\mathcal{G}} \otimes \mathrm{id}_{\widehat{S}})\delta_{\mathcal{E}\rtimes\mathcal{G}} = (\mathrm{id}_{\mathcal{E}\rtimes\mathcal{G}} \otimes \widehat{\delta})\delta_{\mathcal{E}\rtimes\mathcal{G}}$. It follows from the above that the pair $(\alpha_{\mathcal{E}\rtimes\mathcal{G}}, \delta_{\mathcal{E}\rtimes\mathcal{G}})$ is an action of $\widehat{\mathcal{G}}$ on the Hilbert B-module $\mathcal{E} \rtimes \mathcal{G}$. By continuity of (α_B, δ_B), we have

$$
\big[(1_{\mathcal{E}\rtimes\mathcal{G}} \otimes \widehat{S})\delta_{\mathcal{E}\rtimes\mathcal{G}}(\mathcal{E} \rtimes \mathcal{G})\big] = \big((\mathcal{E} \rtimes \mathcal{G}) \otimes \widehat{S}\big)q_{\alpha_B\beta}
$$

and the triple $(\mathcal{E} \rtimes \mathcal{G}, \alpha_{\mathcal{E}\rtimes\mathcal{G}}, \delta_{\mathcal{E}\rtimes\mathcal{G}})$ is actually a $\widehat{\mathcal{G}}$-equivariant Hilbert B-module. ■

Definition 2.4.5. The action $(\alpha_{\mathcal{E}\rtimes\mathcal{G}}, \delta_{\mathcal{E}\rtimes\mathcal{G}})$ of the measured quantum groupoid $\widehat{\mathcal{G}}$ on the crossed product $\mathcal{E} \rtimes \mathcal{G}$ is called the dual action of $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$.

Lemma 2.4.6. *For all $F \in \mathcal{L}(\mathcal{E})$, the operator $F \otimes_\pi 1_B \in \mathcal{L}(\mathcal{E} \rtimes \mathcal{G})$ is invariant.*

Proof. This is an immediate consequence of the definition of the action of $\widehat{\mathcal{G}}$ on the crossed product $\mathcal{E} \rtimes \mathcal{G}$ and the fact that $\Pi(F\xi) = (F \otimes_\pi 1_B)\Pi(\xi)$ for all $\xi \in \mathcal{E}$. ■

Proposition 2.4.7. *Let A_1 and A_2 be two $\mathcal{G}$-C*-algebras, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert C*-modules over, respectively, A_1 and A_2 acted upon by $\mathcal{G}$. Let $\phi : A_1 \to A_2$ be a $\mathcal{G}$-equivariant *-isomorphism and $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ a $\mathcal{G}$-equivariant unitary equivalence of Hilbert modules over the isomorphism ϕ. There exists a unique $\widehat{\mathcal{G}}$-equivariant unitary equivalence of Hilbert modules $\Phi_* : \mathcal{E}_1 \rtimes \mathcal{G} \to \mathcal{E}_2 \rtimes \mathcal{G}$ over the $\widehat{\mathcal{G}}$-equivariant *-isomorphism $\phi_* : A_1 \rtimes \mathcal{G} \to A_2 \rtimes \mathcal{G}$ such that*

$$
\Phi_*(\xi \otimes_{\pi_{A_1}} b) = \Phi\xi \otimes_{\pi_{A_2}} \phi_*(b) \quad \textit{for all } b \in A_1 \rtimes \mathcal{G} \textit{ and } \xi \in \mathcal{E}_1.
$$

Proof. We have $\Phi(\xi a) = \Phi(\xi)\phi(a)$ and $\phi_*(\pi_{A_1}(a)) = \pi_{A_2}(\phi(a))$ for all $\xi \in \mathcal{E}_1$ and $a \in A_1$. Hence, $\Phi(\xi a) \otimes_{\pi_{A_2}} \phi_*(b) = \Phi\xi \otimes_{\pi_{A_2}} \phi_*(\pi_{A_1}(a)b)$ for all $a \in A_1$ and $\xi \in \mathcal{E}_1$. Therefore, we have a linear map

$$
\Phi_* : \mathcal{E}_1 \odot_{\pi_{A_1}} (A_1 \rtimes \mathcal{G}) \to \mathcal{E}_2 \rtimes \mathcal{G}; \quad \xi \otimes_{\pi_{A_1}} b \mapsto \Phi\xi \otimes_{\pi_{A_2}} \phi_*(b).
$$

For all $\xi, \eta \in \mathcal{E}_1$, we have $\pi_{A_2}(\langle \Phi\xi, \Phi\eta \rangle) = \pi_{A_2}(\phi(\langle \xi, \eta \rangle)) = \phi_*(\pi_{A_1}(\langle \xi, \eta \rangle))$. Hence,

$$
\big\langle \Phi\xi \otimes_{\pi_{A_1}} \phi_*(b), \Phi\eta \otimes_{\pi_{A_2}} \phi_*(c) \big\rangle = \phi_*\big(\langle \xi \otimes_{\pi_{A_1}} b, \eta \otimes_{\pi_{A_1}} c \rangle\big)
$$

for all $\xi, \eta \in \mathcal{E}_1$ and $b, c \in A_1 \rtimes \mathcal{G}$. Therefore, Φ_* extends to a unitary equivalence $\Phi_* : \mathcal{E}_1 \rtimes \mathcal{G} \to \mathcal{E}_2 \rtimes \mathcal{G}$ over ϕ_*. Since ϕ_* is $\widehat{\mathcal{G}}$-equivariant and $\Pi_{\mathcal{E}_2}(\Phi\xi) \circ \phi_* = \Phi_* \circ \Pi_{\mathcal{E}_1}(\xi)$ for all $\xi \in \mathcal{E}_1$, we have $\delta_{\mathcal{E}_1\rtimes\mathcal{G}}(\Phi_*(\xi \otimes_{\pi_{A_1}} b)) = (\Phi_* \otimes \mathrm{id}_{\widehat{S}})\delta_{\mathcal{E}_1\rtimes\mathcal{G}}(\xi \otimes_{\pi_{A_1}} b)$ for all $\xi \in \mathcal{E}_1$ and $b \in A_1 \rtimes \mathcal{G}$. Hence, $\delta_{\mathcal{E}_1\rtimes\mathcal{G}} \circ \Phi_* = (\Phi_* \otimes \mathrm{id}_{\widehat{S}})\delta_{\mathcal{E}_1\rtimes\mathcal{G}}$. Hence, Φ_* is equivariant. ■

Let $(J, \beta_J, \delta_J, e_1, e_2)$ be a linking $\mathcal{G}$-C*-algebra. Let us denote $A := e_2 J e_2$ and $\mathcal{E} := e_1 J e_2$ with their structure of $\mathcal{G}$-C*-algebra and $\mathcal{G}$-equivariant Hilbert A-module (cf. Notations 2.2.4). We consider the crossed products $A \rtimes \mathcal{G}$ (resp. $K := J \rtimes \mathcal{G}$) endowed with the dual action $(\alpha_{A\rtimes\mathcal{G}},\delta_{A\rtimes\mathcal{G}})$ (resp. (α_K,δ_K)) and the canonical morphisms $\pi_A : A\to\mathcal{M}(A\rtimes\mathcal{G})$ and $\widehat{\theta}_A:\widehat{S}\to\mathcal{M}(A\rtimes\mathcal{G})$ (resp. $\pi_J : J\to\mathcal{M}(K)$ and $\widehat{\theta}_J:\widehat{S}\to\mathcal{M}(K)$).

We know that the quintuple $(K,\alpha_K,\delta_K,\pi_J(e_1),\pi_J(e_2))$ is a linking $\widehat{\mathcal{G}}$-C*-algebra (cf. Proposition 2.3.9). Let $B:=\pi_J(e_2)K\pi_J(e_2)$ and $\mathcal{F}:=\pi_J(e_1)K\pi_J(e_2)$, respectively, endowed with their structure of $\widehat{\mathcal{G}}$-C*-algebra and $\widehat{\mathcal{G}}$-equivariant Hilbert B-module (cf. Notations 2.2.4). We show that we have a $\widehat{\mathcal{G}}$-equivariant unitary equivalence between $\mathcal{E}\rtimes\mathcal{G}$ and $\mathcal{F}$. More precisely, we have the proposition below.

Proposition 2.4.8. *With the above notations and hypotheses, there exists a unique $\widehat{\mathcal{G}}$-equivariant *-isomorphism $\chi : A\rtimes\mathcal{G}\to B$ such that $\chi(\pi_A(a)\widehat{\theta}_A(x))=\pi_J(a)\widehat{\theta}_J(x)$ for all $a\in A$ and $x\in\widehat{S}$. Moreover, the map $X:\mathcal{E}\rtimes\mathcal{G}\to\mathcal{F}$; $\xi\otimes_{\pi_A}u\mapsto\pi_J(\xi)\chi(u)$ is a χ-compatible unitary operator.*

Proof. Since $[e_2Je_2]=A$, the inclusion map $A\otimes\mathcal{K}\subset J\otimes\mathcal{K}$ extends uniquely to a *-strong continuous *-homomorphism $\tau_A:\mathcal{L}(A\otimes\mathcal{H})\to\mathcal{L}(J\otimes\mathcal{H})$ such that $\tau_A(1_{A\otimes\mathcal{K}})=e_2\otimes 1_{\mathcal{K}}$ up to the identifications $\mathcal{M}(A\otimes\mathcal{K})=\mathcal{L}(A\otimes\mathcal{H})$ and $\mathcal{M}(J\otimes\mathcal{K})=\mathcal{L}(J\otimes\mathcal{H})$. Now we recall that we have the identifications

$$\mathcal{L}(\mathcal{E}_{A,L})=\{T\in\mathcal{L}(A\otimes\mathcal{H});\ Tq_{\beta_A\alpha}=T=q_{\beta_A\alpha}T\};$$
$$\mathcal{L}(\mathcal{E}_{J,L})=\{T\in\mathcal{L}(J\otimes\mathcal{H});\ Tq_{\beta_J\alpha}=T=q_{\beta_J\alpha}T\}.$$

We also recall that for $n\in N$, we have $\beta_A(n^{\mathrm{o}}):=\beta_J(n^{\mathrm{o}})\restriction_A$ (with the identification $\mathcal{M}(A)=\mathcal{L}(A)$) since $[\beta_J(n^{\mathrm{o}}),e_2]=0$. As a result, τ_A induces by restriction to $\mathcal{L}(\mathcal{E}_{A,L})$ a *-strong *-homomorphism still denoted by $\tau_A:\mathcal{L}(\mathcal{E}_{A,L})\to\mathcal{L}(\mathcal{E}_{J,L})$. We have the following formulas:

$$\tau_A\big(\widehat{\theta}_A(x)\big)=\widehat{\theta}_J(x),\ x\in\widehat{S};\quad \tau_A(\pi_A(a))=\pi_J(a),\ a\in A.$$

Hence, $\chi:=\tau_A\restriction_{A\rtimes\mathcal{G}}: A\rtimes\mathcal{G}\to K$ is the unique *-homomorphism such that

$$\chi\big(\pi_A(a)\widehat{\theta}_A(x)\big)=\pi_J(a)\widehat{\theta}_J(x)\quad\text{for all } a\in A \text{ and } x\in\widehat{S}.$$

Note that since τ_A is faithful so is χ. It follows from $K=[\pi_J(A)\widehat{\theta}_J(\widehat{S})]$ and the fact that $[\pi_J(e_2),\widehat{\theta}_J(x)]=0$ for all $x\in\widehat{S}$ that the image of χ is $B:=\pi_J(e_2)K\pi_J(e_2)$. Let us prove that χ is $\widehat{\mathcal{G}}$-equivariant. We recall that $\delta_{A\rtimes\mathcal{G}}(\pi_A(a)\widehat{\theta}_A(x))=(\pi_A(a)\otimes 1_{\widehat{S}})(\widehat{\theta}_A\otimes\mathrm{id}_{\widehat{S}})\widehat{\delta}(x)$ for all $a\in A$ and $x\in\widehat{S}$. It then follows from $\chi\circ\pi_A=\pi_J$ and $\chi\circ\widehat{\theta}_A=\widehat{\theta}_J$ that for all $a\in A$ and $x\in\widehat{S}$ we have

$$(\chi\otimes\mathrm{id}_{\widehat{S}})\delta_{A\rtimes\mathcal{G}}\big(\pi_A(a)\widehat{\theta}_A(x)\big)=\delta_K\big(\pi_J(a)\widehat{\theta}_J(x)\big)=\delta_K\big(\chi\big(\pi_A(a)\widehat{\theta}_A(x)\big)\big).$$

Since $\pi_J(xa)=\pi_J(x)\chi(\pi_A(a))$ for $x\in J$ and $a\in A$, we have

$$\pi_J(xa)\chi(b)=\pi_J(x)\chi\big(\pi_A(a)b\big)\quad\text{for } x\in J,\ a\in A,\ \text{and } b\in A\rtimes\mathcal{G}.$$

Therefore, we have a well-defined linear map

$$X:\mathcal{E}\odot_{\pi_A}(A\rtimes\mathcal{G})\to K;\quad \xi\otimes_{\pi_A}u\mapsto\pi_J(\xi)\chi(u).$$

For $\xi,\eta\in\mathcal{E}$, we have $\pi_J(\xi)^*\pi_J(\eta)=\pi_J(\langle\xi,\eta\rangle)=\chi(\pi_A(\langle\xi,\eta\rangle))$. Therefore, for all $\xi,\eta\in\mathcal{E}$ and $u,v\in A\rtimes\mathcal{G}$ we have $(\pi_J(\xi)\chi(u))^*\pi_J(\eta)\chi(v)=\chi(\langle\xi\otimes_{\pi_A}u,\eta\otimes_{\pi_A}v\rangle)$. As a result, X extends uniquely to a bounded linear map $X:\mathcal{E}\rtimes\mathcal{G}\to K$ such that $X(\langle\zeta_1,\zeta_2\rangle)=\chi(\langle\zeta_1,\zeta_2\rangle)$ for all $\zeta_1,\zeta_2\in\mathcal{E}\rtimes\mathcal{G}$. It is clear that $X(\zeta u)=X(\zeta)\chi(u)$ for all $\zeta\in\mathcal{E}\rtimes\mathcal{G}$ and $u\in A\rtimes\mathcal{G}$.

For all $\xi\in\mathcal{E}$, we have $(X\otimes\mathrm{id}_{\widehat{S}})\circ(T_\xi\otimes\mathrm{id}_{\widehat{S}})=(\pi_J(\xi)\otimes\mathrm{id}_{\widehat{S}})\circ(\chi\otimes\mathrm{id}_{\widehat{S}})$. Hence

$$(X\otimes\mathrm{id}_{\widehat{S}})\delta_{\mathcal{E}\rtimes\mathcal{G}}(\xi\otimes_{\pi_A}u)=(\pi_J(\xi)\otimes 1_{\widehat{S}})(\chi\otimes\mathrm{id}_{\widehat{S}})\delta_{A\rtimes\mathcal{G}}(u)=\delta_K\big(\pi_J(\xi)\chi(u)\big)$$

for all $\xi\in\mathcal{E}$ and $u\in A\rtimes\mathcal{G}$, which proves that X is $\widehat{\mathcal{G}}$-equivariant. Finally, X induces a $\widehat{\mathcal{G}}$-equivariant unitary equivalence of Hilbert modules from $\mathcal{E}\rtimes\mathcal{G}$ onto $\mathcal{F}:=\pi_J(e_1)K\pi_J(e_2)$ over the isomorphism of $\widehat{\mathcal{G}}$-C*-algebras $\chi:A\rtimes\mathcal{G}\to\pi_J(e_2)K\pi_J(e_2)$. ∎

The continuous action (β_J,δ_J) (resp. (α_K,δ_K)) also endows the C*-algebra e_1Je_1 (resp. $\pi_J(e_1)K\pi_J(e_1)$) identified with $\mathcal{K}(\mathcal{E})$ (resp. $\mathcal{K}(\mathcal{F})$) with a continuous action $(\beta_{\mathcal{K}(\mathcal{E})},\delta_{\mathcal{K}(\mathcal{E})})$ (resp. $(\alpha_{\mathcal{K}(\mathcal{F})},\delta_{\mathcal{K}(\mathcal{F})})$) of $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$).

Proposition 2.4.9. *The map $\mathcal{K}(\mathcal{E})\rtimes\mathcal{G}\to\mathcal{K}(\mathcal{F})$; $\pi_{\mathcal{K}(\mathcal{E})}(k)\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\mapsto\pi_J(k)\widehat{\theta}_J(x)$ is a $\widehat{\mathcal{G}}$-equivariant *-isomorphism.*

Proof. The proof is the same as that of the above proposition (by exchanging the projections e_1 and e_2). ∎

Corollary 2.4.10. *Let A be a $\mathcal{G}$-C*-algebra and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert A-module. We have a canonical $\widehat{\mathcal{G}}$-equivariant *-isomorphism $\mathcal{K}(\mathcal{E})\rtimes\mathcal{G}\simeq\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$. Moreover, if $F\in\mathcal{L}(\mathcal{E})$, then the operator $F\otimes_{\pi_A}1\in\mathcal{L}(\mathcal{E}\rtimes\mathcal{G})$ is identified with $\pi_{\mathcal{K}(\mathcal{E})}(F)$ through the identification $\mathcal{L}(\mathcal{E}\rtimes\mathcal{G})\simeq\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})$.*

Proof. It suffices to apply Propositions 2.4.8 and 2.4.9 to $J:=\mathcal{K}(\mathcal{E}\oplus A)$ equipped with its structure of linking $\mathcal{G}$-C*-algebra (cf. Notations 2.2.4). ∎

Corollary 2.4.11. *Let A be a $\mathcal{G}$-C*-algebra and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert A-module. Let $\widehat{\theta}:\widehat{S}\to\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})$ and $\Pi:\mathcal{E}\to\mathcal{L}(A\rtimes\mathcal{G},\mathcal{E}\rtimes\mathcal{G})$ be the canonical morphisms. With the identification $\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})=\mathcal{L}(\mathcal{E}\rtimes\mathcal{G})$ (cf. Corollary 2.4.10), we have $\widehat{\theta}(x)\Pi(\xi)\in\mathcal{E}\rtimes\mathcal{G}$ for all $x\in\widehat{S}$ and $\xi\in\mathcal{E}$. Moreover, we have $\mathcal{E}\rtimes\mathcal{G}=[\widehat{\theta}(x)\Pi(\xi);\ \xi\in\mathcal{E},\ x\in\widehat{S}]$.*

Proof. Let us equip $J:=\mathcal{K}(\mathcal{E}\oplus A)$ with its structure of linking $\mathcal{G}$-C*-algebra (cf. Notations 2.2.4). Let $\xi\in\mathcal{E}$, $b\in A\rtimes\mathcal{G}$, and $k\in\mathcal{K}(\mathcal{E})$, then $\Pi(\xi)b$ (resp. $\pi_{\mathcal{K}(\mathcal{E})}(k)$) is identified to $\pi_J(\iota_{\mathcal{E}}(\xi))\chi(b)$ (resp. $\pi_J(\iota_{\mathcal{K}(\mathcal{E})}(k))$) through the identification of Proposition 2.4.8 (resp. Proposition 2.4.9). Hence, $\pi_{\mathcal{K}(\mathcal{E})}(k)\Pi(\xi)b$ is identified to $\pi_J(\iota_{\mathcal{E}}(k\xi))\chi(b)$. Thus, we have $\pi_{\mathcal{K}(\mathcal{E})}(k)\Pi(\xi)b=\Pi(k\xi)\chi(b)$. As a result, we have $\pi_{\mathcal{K}(\mathcal{E})}(k)\Pi(\xi)=\Pi(k\xi)$ for all $k\in\mathcal{K}(\mathcal{E})$ and $\xi\in\mathcal{E}$.

If $\xi\in\mathcal{E}$, $x\in\widehat{S}$, and $k\in\mathcal{K}(\mathcal{E})$, we have $\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\Pi(k\xi)=\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\pi_{\mathcal{K}(\mathcal{E})}(k)\Pi(\xi)\in\mathcal{E}\rtimes\mathcal{G}$ since $\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\pi_{\mathcal{K}(\mathcal{E})}(k)\in\mathcal{K}(\mathcal{E})\rtimes\mathcal{G}=\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$. Hence, $\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\Pi(\xi)\in\mathcal{E}\rtimes\mathcal{G}$ for all $x\in\widehat{S}$ and $\xi\in\mathcal{E}$ since $\mathcal{K}(\mathcal{E})\mathcal{E}=\mathcal{E}$. Let $\xi\in\mathcal{E}$ and $x\in\widehat{S}$, then $\Pi(\xi)\widehat{\theta}_A(x)$ is identified to $\pi_J(\iota_{\mathcal{E}}(\xi))\widehat{\theta}_J(x)$. Moreover, $\pi_J(\iota_{\mathcal{E}}(\xi))\widehat{\theta}_J(x)$ is the norm limit of finite sums of the form

$\sum_i \widehat{\theta}_J(x_i)\pi_J(u_i)$ with $x_i \in \widehat{S}$ and $u_i \in J$. However, we have $[\pi_J(e_j), \widehat{\theta}_J(y)] = 0$ for all $y \in \widehat{S}$ and $j = 1, 2$. Hence, $\Pi(\xi)\widehat{\theta}_A(x)$ is the norm limit of finite sums of elements of the form $\sum_i \widehat{\theta}_J(x_i)\pi_J(e_1 u_i e_2)$. Write $e_1 u_i e_2 = \iota_{\mathcal{E}}(\xi_i)$ with $\xi_i \in \mathcal{E}$, then $\Pi(\xi)\widehat{\theta}_A(x)$ is the norm limit of finite sums of the form $\sum_i \widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_i)\Pi(\xi_i)$. ■

In the following result, we investigate the functoriality of the crossed product construction.

Proposition 2.4.12. *For $i = 1, 2$, let A_i be a $\mathcal{G}$-C*-algebra. Let $f : A_1 \to \mathcal{M}(A_2)$ be a $\mathcal{G}$-equivariant *-homomorphism.*

(1) *There exists a unique *-homomorphism $f_* : A_1 \rtimes \mathcal{G} \to \mathcal{M}(A_2 \rtimes \mathcal{G})$ such that for all $a \in A_1$ and $x \in \widehat{S}$, $f(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)) = \pi_{A_2}(f(a))\widehat{\theta}_{A_2}(x)$. Moreover, f_* is $\widehat{\mathcal{G}}$-equivariant. Note that if $f(A_1) \subset A_2$, then $f_*(A_1 \rtimes \mathcal{G}) \subset A_2 \rtimes \mathcal{G}$.*

(2) *The correspondence $- \rtimes \mathcal{G} : \mathsf{Alg}_{\mathcal{G}} \to \mathsf{Alg}_{\widehat{\mathcal{G}}}$ is functorial.*

Proof. (1) Let $i = 1, 2$. Let us denote $\pi_{A_i,L} : A_i \to \mathcal{L}(A_i \otimes \mathcal{H}); a \mapsto (\mathrm{id}_{A_i} \otimes L)\delta_{A_i}(a)$ and $B_i := A_i \rtimes \mathcal{G}$. In this proof, we make the identifications

$$\widetilde{\mathcal{M}}(A_i \otimes \mathcal{K}) \subset \mathcal{M}(A_i \otimes \mathcal{K}) = \mathcal{L}(A_i \otimes \mathcal{H}).$$

We also identify

$$\mathcal{L}(\mathcal{E}_{A_i,L}) = \big\{T \in \mathcal{L}(A_i \otimes \mathcal{H});\ T q_{\beta_{A_i}\alpha} = T = q_{\beta_{A_i}\alpha} T\big\}.$$

We then have $B_i = [\pi_{A_i,L}(a)(1_{A_i} \otimes \rho(x));\ a \in A_i,\ x \in \widehat{S}]$. It follows from $L(S)\mathcal{K} = \mathcal{K}$ and $\delta_{A_i}(A_i) \subset \widetilde{\mathcal{M}}(A_i \otimes S)$ that $B_i \subset \widetilde{\mathcal{M}}(A_i \otimes \mathcal{K})$. Let $f \otimes \mathrm{id}_{\mathcal{K}} : \widetilde{\mathcal{M}}(A_1 \otimes \mathcal{K}) \to \mathcal{L}(A_2 \otimes \mathcal{H})$. By a straightforward computation, we have

$$(f \otimes \mathrm{id}_{\mathcal{K}})\big(\pi_{A_1,L}(a)\big(1_{A_1} \otimes \rho(x)\big)\big) = \pi_{A_2,L}\big(f(a)\big)\big(1_{A_2} \otimes \rho(x)\big) \quad \text{for all } a \in A_1 \text{ and } x \in \widehat{S}.$$

Hence, $(f \otimes \mathrm{id}_{\mathcal{K}})(B_1) \subset \mathcal{M}(B_2)$. Let $f_* := (f \otimes \mathrm{id}_{\mathcal{K}})\restriction_{B_1} : B_1 \to \mathcal{M}(B_2)$. We have proved that the *-homomorphism f_* satisfies $f(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)) = \pi_{A_2}(f(a))\widehat{\theta}_{A_2}(x)$ for all $a \in A_1$ and $x \in \widehat{S}$. In particular, for all $a \in A_1$ and $u \in \widehat{S} \otimes \widehat{S}$ we have

$$(f_* \otimes \mathrm{id}_{\widehat{S}})\big((\pi_{A_1}(a) \otimes 1_{\widehat{S}})(\widehat{\theta}_{A_1} \otimes \mathrm{id}_{\widehat{S}})(u)\big) = \big(\pi_{A_2}\big(f(a)\big) \otimes 1_{\widehat{S}}\big)(\widehat{\theta}_{A_2} \otimes \mathrm{id}_{\widehat{S}})(u).$$

Let $a \in A_1$ and $x, x' \in \widehat{S}$. By a straightforward computation, it follows from [9, Proposition-Definition 5.1.15 (1)], the previous formula, and the relation $\widehat{\delta}(x)(1_{\widehat{S}} \otimes x') \in \widehat{S} \otimes \widehat{S}$ that

$$(f_* \otimes \mathrm{id}_{\widehat{S}})\big(\delta_{B_1}\big(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)\big)(1_{B_1} \otimes x')\big) = \delta_{B_2}\big(f_*\big(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)\big)\big)(1_{B_2} \otimes x').$$

Hence, $(f_* \otimes \mathrm{id}_{\widehat{S}})\delta_{B_1}(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)) = \delta_{B_2}(f_*(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x)))$ for all $a \in A_1$ and $x \in \widehat{S}$. Moreover, it is easy to see that $f_*(\alpha_{A_1}(n)b) = \alpha_{A_2}(n)f_*(b)$ for all $b \in B_1$. Hence, f_* is $\widehat{\mathcal{G}}$-equivariant.

(2) If f is non-degenerate, then so is f_*. Indeed, we have $A_2 = f(A_1)A_2$. Let $a \in A_1$, $b \in A_2$, and $x \in \widehat{S}$. By [9, Proposition-Definition 5.1.14],

$$\pi_{A_2}\big(f(a)b\big)\widehat{\theta}_{A_2}(x) = \pi_{A_2}\big(f(a)\big)\pi_{A_2}(b)\widehat{\theta}_{A_2}(x)$$

is the norm limit of finite sums of the form

$$\sum_i \pi_{A_2}\big(f(a)\big)\widehat{\theta}_{A_2}(x_i x_i')\pi_{A_2}(b_i) = \sum_i f_*\big(\pi_{A_1}(a)\widehat{\theta}_{A_1}(x_i)\big)\widehat{\theta}_{A_2}(x_i')\pi_{A_2}(b_i)$$

with $x_i, x_i' \in \widehat{S}$ and $b_i \in A_2$. Hence, $\pi_{A_2}(f(a)b)\widehat{\theta}_{A_2}(x) \in [f_*(B_1)B_2]$. Hence, f_* is non-degenerate. The functoriality of the correspondence $- \rtimes \mathcal{G} : \mathsf{Alg}_{\mathcal{G}} \to \mathsf{Alg}_{\widehat{\mathcal{G}}}$ follows. ∎

We are now able to define the bimodule structure on the crossed product $\mathcal{E} \rtimes \mathcal{G}$.

Proposition-Definition 2.4.13. *Let B be a $\mathcal{G}$-C*-algebra and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert B-module. Let $\gamma : A \to \mathcal{L}(\mathcal{E})$ be a $\mathcal{G}$-equivariant *-representation. By applying Proposition 2.4.12 and Corollary 2.4.10, we have a canonical $\widehat{\mathcal{G}}$-equivariant *-representation $\gamma_* : A \rtimes \mathcal{G} \to \mathcal{L}(\mathcal{E} \rtimes \mathcal{G})$. Moreover, if $\mathcal{E}$ is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule, then $\mathcal{E} \rtimes \mathcal{G}$ is a $\widehat{\mathcal{G}}$-equivariant Hilbert $A \rtimes \mathcal{G}$-$B \rtimes \mathcal{G}$-bimodule.*

Proof. We only have to prove that if $\mathcal{E}$ is countably generated as a Hilbert B-module, then $\mathcal{E} \rtimes \mathcal{G}$ is countably generated as a Hilbert $B \rtimes \mathcal{G}$-module. Let $\{\xi_i;\ i \in \mathbb{N}\}$ be a generating set for the Hilbert B-module $\mathcal{E}$. We have $\widehat{S} = \{(\mathrm{id} \otimes \omega_{\xi,\eta})(V);\ \xi, \eta \in \mathcal{H}\}$. Moreover, $\mathcal{H}$ is separable then so is $\widehat{S}$. Let $\{x_i;\ i \in \mathbb{N}\}$ be a total subset of $\widehat{S}$. We claim that $\{\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_i)\Pi(\xi_j);\ i \in \mathbb{N},\ j \in \mathbb{N}\} \subset \mathcal{E} \rtimes \mathcal{G}$ is a generating set for the Hilbert $B \rtimes \mathcal{G}$-module $\mathcal{E} \rtimes \mathcal{G}$. Indeed, this follows from the relation $\mathcal{E} \rtimes \mathcal{G} = [\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x)\Pi(\xi)\widehat{\theta}_B(x');\ \xi \in \mathcal{E}, x, x' \in \widehat{S}]$ (cf. Proposition 2.4.3, Corollary 2.4.11 and the fact that any element of $\widehat{S}$ can be written as a product of two elements of $\widehat{S}$) and Proposition 2.4.3 (2). ∎

Proposition 2.4.14. *Let A, B, and C be three $\mathcal{G}$-C*-algebras. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert modules over C and B, respectively. Let $(\beta_{\mathcal{E}_1}, \delta_{\mathcal{E}_1})$ and $(\beta_{\mathcal{E}_2}, \delta_{\mathcal{E}_2})$ be actions of $\mathcal{G}$ on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Let $\gamma_2 : C \to \mathcal{L}(\mathcal{E}_2)$ be a $\mathcal{G}$-equivariant *-representation. Let $\mathcal{E} := \mathcal{E}_1 \otimes_{\gamma_2} \mathcal{E}_2$ be the Hilbert B-module acted upon by $\mathcal{G}$ defined in Proposition 2.2.9. Let $\gamma_{2*} : C \rtimes \mathcal{G} \to \mathcal{L}(\mathcal{E}_2 \rtimes \mathcal{G})$ be the $\widehat{\mathcal{G}}$-equivariant *-representation defined in Proposition-Definition 2.4.13.*

(1) *There exists a unique $\mathcal{G}$-equivariant unitary $\Xi : (\mathcal{E}_1 \rtimes \mathcal{G}) \otimes_{\gamma_{2*}} (\mathcal{E}_1 \rtimes \mathcal{G}) \to \mathcal{E} \rtimes \mathcal{G}$ such that*

$$\Xi\big(\widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)}(x_1)\Pi_{\mathcal{E}_1}(\xi_1) \otimes_{\gamma_{2*}} \Pi_{\mathcal{E}_2}(\xi_2)\widehat{\theta}_B(x_2)\big) = \widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_1)\Pi_{\mathcal{E}}(\xi_1 \otimes_{\gamma_2} \xi_2)\widehat{\theta}_B(x_2)$$

for all $x_1, x_2 \in \widehat{S}$, $\xi_1 \in \mathcal{E}_1$, and $\xi_2 \in \mathcal{E}_2$.

(2) *Let $\gamma_1 : A \to \mathcal{L}(\mathcal{E}_1)$ be a $\mathcal{G}$-equivariant *-representation. Denote by $\gamma : A \to \mathcal{L}(\mathcal{E})$ the $\mathcal{G}$-equivariant *-representation defined by $\gamma(a) := \gamma_1(a) \otimes_{\gamma_2} 1$ for all $a \in A$ (cf. Proposition 2.2.11) and $\gamma_{1*} : A \rtimes \mathcal{G} \to \mathcal{L}(\mathcal{E}_2 \rtimes \mathcal{G})$ the $\widehat{\mathcal{G}}$-equivariant *-representation defined in Proposition-Definition 2.4.13. Let*

$$\kappa : A \to \mathcal{L}\big((\mathcal{E}_1 \rtimes \mathcal{G}) \otimes_{\gamma_{2*}} (\mathcal{E}_2 \rtimes \mathcal{G})\big)$$

*be the $\mathcal{G}$-equivariant *-representation defined by $\kappa(a) := \gamma_{1*}(a) \otimes_{\gamma_{2*}} 1$ for all $a \in A$ (cf. Proposition 2.2.11). We then have $\Xi \circ \kappa(a) = \gamma_*(a) \circ \Xi$ for all $a \in A$.*

For the proof, we will need to use a concrete interpretation of the crossed product.

Notations 2.4.15. Let A be a $\mathcal{G}$-C*-algebra. Consider the $\widehat{\mathcal{G}}$-C*-algebra $B := A \rtimes \mathcal{G}$. Let $\mathcal{E}$ be a Hilbert A-module acted upon by $\mathcal{G}$.

(1) Let $\kappa : S \to \mathcal{B}(\mathcal{H})$ be a non-degenerate *-homomorphism. We have the following unitary equivalences of Hilbert A-modules:

$$\begin{aligned}(A \otimes S) \otimes_{\mathrm{id}_A \otimes \kappa} (A \otimes \mathcal{H}) &\to A \otimes \mathcal{H}\\ (a \otimes s) \otimes_{\mathrm{id}_A \otimes \kappa} (b \otimes \eta) &\mapsto ab \otimes \kappa(s)\eta;\\ (\mathcal{E} \otimes S) \otimes_{\mathrm{id}_A \otimes \kappa} (A \otimes \mathcal{H}) &\to \mathcal{E} \otimes \mathcal{H}\\ (\xi \otimes s) \otimes_{\mathrm{id}_A \otimes \kappa} (a \otimes \eta) &\mapsto \xi a \otimes \kappa(s)\eta.\end{aligned}$$

By using the above identifications, the map $\mathrm{id}_{\mathcal{E}} \otimes \kappa$ extends to a linear map

$$\begin{aligned}\mathrm{id}_{\mathcal{E}} \otimes \kappa : \mathcal{L}(A \otimes S, \mathcal{E} \otimes S) &\to \mathcal{L}(A \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H})\\ T &\mapsto (\mathrm{id}_{\mathcal{E}} \otimes \kappa)(T) := T \otimes_{\mathrm{id}_A \otimes \kappa} 1.\end{aligned}$$

The extension is uniquely determined by the formula

$$\begin{aligned}&(\mathrm{id}_{\mathcal{E}} \otimes \kappa)(Tx)\\ &\quad = (\mathrm{id}_{\mathcal{E}} \otimes \kappa)(T)(\mathrm{id}_A \otimes \kappa)(x) \quad \text{for } T \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S) \text{ and } x \in A \otimes S.\end{aligned}$$

For all $T, S \in \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$ we have

$$(\mathrm{id}_{\mathcal{E}} \otimes \kappa)(T)^*(\mathrm{id}_{\mathcal{E}} \otimes \kappa)(S) = (\mathrm{id}_A \otimes \kappa)(T^*S)$$

with the identification $\mathcal{L}(A \otimes S) = \mathcal{M}(A \otimes S)$.

(2) By using the above notation, we consider the linear map

$$\Pi_L : \mathcal{E} \to \mathcal{L}(A \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H})$$

defined by $\Pi_L(\xi) := (\mathrm{id}_{\mathcal{E}} \otimes L)\delta_{\mathcal{E}}(\xi)$ for all $\xi \in \mathcal{E}$. Note that we have

- $\Pi_L(\xi a) = \Pi_L(\xi)\pi_L(a)$ for all $a \in A$ and $\xi \in \mathcal{E}$,
- $\Pi_L(\xi)^*\Pi_L(\eta) = \pi_L(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathcal{E}$,
- $q_{\beta_{\mathcal{E}}\alpha}\Pi_L(\xi) = \Pi_L(\xi) = \Pi_L(\xi)q_{\beta_A\alpha}$ for all $\xi \in \mathcal{E}$.

(3) There exists a unique isometric linear map $\Psi_{L,\rho} : \mathcal{E} \rtimes \mathcal{G} \to \mathcal{L}(A \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H})$ such that $\Psi_{L,\rho}(\Pi(\xi)\widehat{\theta}(x)) = \Pi_L(\xi)(1_A \otimes \rho(x))$ for all $\xi \in \mathcal{E}$ and $x \in \widehat{S}$. Indeed, let us denote by $\psi_{L,\rho} : B \to \mathcal{L}(A \otimes \mathcal{H})$ the unique faithful strictly/*-strongly continuous *-representation such that $\psi_{L,\rho}(\pi(a)\widehat{\theta}(x)) = \pi_L(a)(1_A \otimes \rho(x))$ for all $a \in A$ and $x \in \widehat{S}$ (cf. [8, Proposition 4.2.3]). Since $\psi_{L,\rho}(\pi(a)) = \pi_L(a)$ for all $a \in A$, there exists a unique isometric linear map $\Psi_{L,\rho} : \mathcal{E} \rtimes \mathcal{G} \to \mathcal{L}(A \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H})$ such that $\Psi_{L,\rho}(\xi \otimes_\pi b) = \Pi_L(\xi)\psi_{L,\rho}(b)$ for all $\xi \in \mathcal{E}$ and $b \in B$. It is clear that $\Psi_{L,\rho}(\Pi(\xi)\widehat{\theta}(x)) = \Pi_L(\xi)(1_A \otimes \rho(x))$ for all $\xi \in \mathcal{E}$ and $x \in \widehat{S}$.

Proof of Proposition 2.4.14. (1) According to Notation 2.4.15 (3), we identify $\mathcal{E}_1 \rtimes \mathcal{G}$ (resp. $\mathcal{E}_2 \rtimes \mathcal{G}$) with a subspace of $\mathcal{L}(C \otimes \mathcal{H}, \mathcal{E}_1 \otimes \mathcal{H})$ (resp. $\mathcal{L}(B \otimes \mathcal{H}, \mathcal{E}_2 \otimes \mathcal{H})$). Since $\delta_C(C)(1_C \otimes S) \subset C \otimes S$, we even have $\mathcal{E}_1 \rtimes \mathcal{G} \subset \mathcal{L}(\widetilde{C} \otimes \mathcal{H}, \mathcal{E}_1 \otimes \mathcal{H})$. Let $\widetilde{\gamma}_2 \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{L}(\widetilde{C} \otimes \mathcal{H}) \to \mathcal{L}(\mathcal{E}_2 \otimes \mathcal{H})$. With the identification $(\widetilde{C} \otimes \mathcal{H}) \otimes_{\widetilde{\gamma}_2} \mathcal{E}_2 = \mathcal{E}_2 \otimes \mathcal{H}$, we have

$$(\widetilde{\gamma}_2 \otimes \mathrm{id}_{\mathcal{K}})(T) = T \otimes_{\widetilde{\gamma}_2} 1 \quad \text{for all } T \in \mathcal{L}(\widetilde{C} \otimes \mathcal{H}).$$

By using the identification $(\mathcal{E}_1 \otimes \mathcal{H}) \otimes_{\widetilde{\gamma}_2} \mathcal{E}_2 = \mathcal{E} \otimes \mathcal{H}$, we then obtain an isometric adjointable operator

$$\begin{aligned} \Xi : \mathcal{L}(\widetilde{C} \otimes \mathcal{H}, \mathcal{E}_1 \otimes \mathcal{H}) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_{\mathcal{K}}} \mathcal{L}(B \otimes \mathcal{H}, \mathcal{E}_2 \otimes \mathcal{H}) &\to \mathcal{L}(B \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H}) \\ T_1 \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_{\mathcal{K}}} T_2 &\mapsto (T_1 \otimes_{\widetilde{\gamma}_2} 1) T_2. \end{aligned}$$

We prove that

$$\Xi\big(\widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)}(x_1)\Pi_{\mathcal{E}_1}(\xi_1) \otimes_{\gamma_{2*}} \Pi_{\mathcal{E}_2}(\xi_2)\widehat{\theta}_B(x_2)\big) = \widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_1)\Pi_{\mathcal{E}}(\xi_1 \otimes_{\gamma_2} \xi_2)\widehat{\theta}_B(x_2)$$

for all $x_1, x_2 \in \widehat{S}$, $\xi_1 \in \mathcal{E}_1$, and $\xi_2 \in \mathcal{E}_2$ by a straightforward computation. Hence, Ξ induces by restriction a unitary $\Xi \in \mathcal{L}((\mathcal{E}_1 \rtimes \mathcal{G}) \otimes_{\gamma_{2*}} (\mathcal{E}_1 \rtimes \mathcal{G}), \mathcal{E} \rtimes \mathcal{G})$. The equivariance of Ξ will follow from the definitions and the formulas

$$\begin{aligned} \delta_{\mathcal{E} \rtimes \mathcal{G}}\big(\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_1)\Pi_{\mathcal{E}}(\xi)\widehat{\theta}_B(x_2)\big) &= (\widehat{\theta}_{\mathcal{K}(\mathcal{E})} \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(x_1)\big(\Pi_{\mathcal{E}}(\xi) \otimes 1_{\widehat{S}}\big)(\widehat{\theta}_B \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(x_2); \\ \delta_{\mathcal{E}_1 \rtimes \mathcal{G}}\big(\widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)}(x_1)\Pi_{\mathcal{E}_1}(\xi_1)\big) &= (\widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)} \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(x_1)\big(\Pi_{\mathcal{E}_1}(\xi_1) \otimes 1_{\widehat{S}}\big); \\ \delta_{\mathcal{E}_2 \rtimes \mathcal{G}}\big(\Pi_{\mathcal{E}_2}(\xi_2)\widehat{\theta}_B(x_2)\big) &= \big(\Pi_{\mathcal{E}_2}(\xi_2) \otimes 1_{\widehat{S}}\big)(\widehat{\theta}_B \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(x_2) \end{aligned}$$

for all $\xi \in \mathcal{E}$, $\xi_1 \in \mathcal{E}_1$, $\xi_2 \in \mathcal{E}_2$, and $x_1, x_2 \in \widehat{S}$.

(2) This will follow from the formulas

$$\begin{aligned} \gamma_{1*}(a)\widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)}(x_1)\Pi_{\mathcal{E}_1}(\xi_1) &= \widehat{\theta}_{\mathcal{K}(\mathcal{E}_1)}(x_1)\Pi_{\mathcal{E}_1}(\gamma_1(a)\xi_1); \\ \kappa(a)\widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_1)\Pi_{\mathcal{E}}(\xi)\widehat{\theta}_B(x_2) &= \widehat{\theta}_{\mathcal{K}(\mathcal{E})}(x_1)\Pi_{\mathcal{E}}(\gamma(a)\xi)\widehat{\theta}_B(x_2) \end{aligned}$$

for all $\xi_1 \in \mathcal{E}_1$, $\xi \in \mathcal{E}$, and $x_1, x_2 \in \widehat{S}$ (cf. Proposition 2.4.12 (1)). ■

In a similar way, we define the crossed product of a Hilbert C*-module by an action of the dual measured quantum groupoid $\widehat{\mathcal{G}}$. The details are left to the reader's attention.

Let (B, α_B, δ_B) be a $\widehat{\mathcal{G}}$-C*-algebra. Let us denote by $C := B \rtimes \widehat{\mathcal{G}}$ the crossed product endowed with the dual action (β_C, δ_C). Let $\widehat{\pi} : B \to \mathcal{M}(C)$ and $\theta : S \to \mathcal{M}(C)$ be the canonical morphisms. Let $\mathcal{F}$ be a Hilbert B-module and $(\alpha_{\mathcal{F}}, \delta_{\mathcal{F}})$ an action of $\widehat{\mathcal{G}}$ on $\mathcal{F}$.

Definition 2.4.16. We call the crossed product of $\mathcal{F}$ by the action $(\alpha_{\mathcal{F}}, \delta_{\mathcal{F}})$ the Hilbert C-module $\mathcal{F} \otimes_{\widehat{\pi}} C$ denoted by $\mathcal{F} \rtimes \widehat{\mathcal{G}}$.

Notation 2.4.17. For $\xi \in \mathcal{F}$, we denote by $\widehat{\Pi}(\xi) \in \mathcal{L}(B, \mathcal{F} \rtimes \widehat{\mathcal{G}})$ the adjointable operator defined by $\widehat{\Pi}(\xi)c := \xi \otimes_{\widehat{\pi}} c$ for all $c \in C$. We have $\widehat{\Pi}(\xi)^*(\eta \otimes_{\widehat{\pi}} c) = \widehat{\pi}(\langle \xi, \eta \rangle)c$ for all $\eta \in \mathcal{F}$ and $c \in C$. We then have a linear map $\widehat{\Pi} : \mathcal{F} \to \mathcal{L}(B, \mathcal{F} \rtimes \widehat{\mathcal{G}})$.

Proposition-Definition 2.4.18. *Let us denote by* $\delta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}} : \mathcal{F}\rtimes\widehat{\mathcal{G}} \to \mathcal{L}(C\otimes S, (\mathcal{F}\rtimes\widehat{\mathcal{G}})\otimes S)$ *and* $\beta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{F}\rtimes\widehat{\mathcal{G}})$ *the linear maps defined by*

$$\beta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}}(n^{\mathrm{o}}) := 1_{\mathcal{F}} \otimes_{\widehat{\pi}} \beta_C(n^{\mathrm{o}}), \quad n\in N;$$
$$\delta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}}(\xi\otimes_{\widehat{\pi}} c) := \big(\widehat{\Pi}(\xi)\otimes 1_S\big)\delta_C(c), \quad \xi\in\mathcal{F},\ c\in C.$$

Then, the pair $(\beta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}}, \delta_{\mathcal{F}\rtimes\widehat{\mathcal{G}}})$ *is a continuous action of* $\mathcal{G}$ *on the crossed product* $\mathcal{F}\rtimes\widehat{\mathcal{G}}$ *called the dual action of* $(\alpha_{\mathcal{F}}, \delta_{\mathcal{F}})$.

Let $(K, \alpha_K, \delta_K, f_1, f_2)$ be a linking $\widehat{\mathcal{G}}$-C*-algebra. Let us consider the $\widehat{\mathcal{G}}$-C*-algebra $B := f_2Kf_2$ and the $\widehat{\mathcal{G}}$-equivariant Hilbert B-module $\mathcal{F} := f_1Kf_2$. We consider the $\mathcal{G}$-C*-algebras $B\rtimes\widehat{\mathcal{G}}$ (resp. $L := K\rtimes\widehat{\mathcal{G}}$) endowed with the canonical morphisms $\widehat{\pi}_B : B\to \mathcal{M}(B\rtimes\widehat{\mathcal{G}})$ and $\theta_B : S\to\mathcal{M}(B\rtimes\widehat{\mathcal{G}})$ (resp. $\widehat{\pi}_L : L\to\mathcal{M}(K)$ and $\theta_L : S\to\mathcal{M}(K)$). We know that $(L, \beta_L, \delta_L, \widehat{\pi}_L(f_1), \widehat{\pi}_L(f_2))$ is a linking $\mathcal{G}$-C*-algebra. Let us consider the $\mathcal{G}$-C*-algebra $C := \widehat{\pi}_K(f_2)L\widehat{\pi}_K(f_2)$ and $\mathcal{G}$-equivariant Hilbert C-module $\mathcal{G} := \widehat{\pi}_K(f_1)L\widehat{\pi}_K(f_2)$.

Proposition 2.4.19. *With the above notations and hypotheses, we have the following statements.*

(1) *There exists a unique* $\mathcal{G}$*-equivariant* **-isomorphism* $\psi : B\rtimes\widehat{\mathcal{G}}\to C$ *such that for all* $b\in B$ *and* $y\in S$ *we have* $\psi(\widehat{\pi}_B(b)\theta_B(y)) = \widehat{\pi}_K(b)\theta_K(y)$. *Moreover, we have* $\mathcal{G}$*-equivariant unitary equivalence* $\Psi : \mathcal{F}\rtimes\widehat{\mathcal{G}}\to\mathcal{G}$; $\eta\otimes_{\widehat{\pi}_B} u\mapsto\widehat{\pi}_K(\eta)\psi(u)$ *over* $\psi : B\rtimes\widehat{\mathcal{G}}\to C$.

(2) *The map*

$$\mathcal{K}(\mathcal{E})\rtimes\mathcal{G}\to\mathcal{K}(\mathcal{F});\quad \widehat{\pi}_{\mathcal{K}(\mathcal{F})}(k)\theta_{\mathcal{K}(\mathcal{F})}(y)\mapsto\widehat{\pi}_K(k)\widehat{\theta}_K(y)$$

is a $\mathcal{G}$*-equivariant* **-isomorphism.*

Corollary 2.4.20. *Let* B *be a* $\widehat{\mathcal{G}}$-C*-*algebra and* $\mathcal{F}$ *a* $\widehat{\mathcal{G}}$*-equivariant Hilbert* B*-module. We have a canonical* $\mathcal{G}$*-equivariant* **-isomorphism* $\mathcal{K}(\mathcal{F})\rtimes\widehat{\mathcal{G}}\simeq\mathcal{K}(\mathcal{F}\rtimes\widehat{\mathcal{G}})$. *Moreover, if* $F\in\mathcal{L}(\mathcal{F})$, *then the operator* $F\otimes_{\widehat{\pi}_B} 1\in\mathcal{L}(\mathcal{F}\rtimes\widehat{\mathcal{G}})$ *is identified with* $\widehat{\pi}_{\mathcal{K}(\mathcal{F})}(F)$ *through the identification* $\mathcal{L}(\mathcal{F}\rtimes\widehat{\mathcal{G}})\simeq\mathcal{M}(\mathcal{K}(\mathcal{F})\rtimes\widehat{\mathcal{G}})$.

2.4.2. Takesaki–Takai duality. In the following paragraph, we investigate the double crossed product. Let A be a $\mathcal{G}$-C*-algebra and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert A-module. Let D be the bidual $\mathcal{G}$-C*-algebra (cf. Notations 2.3.4).

Proposition-Definition 2.4.21. *Let* $\Pi_R : \mathcal{E}\to\mathcal{L}(A\otimes\mathcal{H}, \mathcal{E}\otimes\mathcal{H})$ *be the linear map defined by* $\Pi_R(\xi) := (\mathrm{id}_{\mathcal{E}}\otimes R)\delta_{\mathcal{E}}(\xi)$ *for all* $\xi\in\mathcal{E}$ *(cf. Notations 2.4.15). Let*

$$\mathcal{D} := \big[\Pi_R(\xi)\big(1_A\otimes\lambda(x)L(y)\big);\ \xi\in\mathcal{E},\ x\in\widehat{S},\ y\in S\big]\subset\mathcal{L}(A\otimes\mathcal{H}, \mathcal{E}\otimes\mathcal{H}).$$

For the natural right action of D *by composition of operators and the* D*-valued inner product given by* $\langle\zeta_1, \zeta_2\rangle := \zeta_1^*\circ\zeta_2$ *for* $\zeta_1, \zeta_2\in\mathcal{D}$, *we turn* $\mathcal{D}$ *into a Hilbert* D*-module. Let*

$$\mathcal{E}_{\mathcal{E},R} := q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\mathcal{E}\otimes\mathcal{H})\subset\mathcal{E}\otimes\mathcal{H}.$$

Then, $\mathcal{E}_{\mathcal{E},R}$ *is a Hilbert sub-*A*-module of* $\mathcal{E}\otimes\mathcal{H}$ *and* $[\mathcal{D}\mathcal{E}_{A,R}] = \mathcal{E}_{\mathcal{E},R}$.

Proof. By combining the facts that $[\lambda(\widehat{S})L(S)]$ and D are C*-algebras with the formula $\Pi_R(\xi)\pi_R(a) = \Pi_R(\xi a)$ for $\xi \in \mathcal{E}$ and $a \in A$, we obtain the inclusion $\mathcal{D}D \subset \mathcal{D}$. Moreover, we have $\Pi_R(\xi)^*\Pi_R(\eta) = \pi_R(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathcal{E}$. It then follows that $\mathcal{D}^*\mathcal{D} \subset D$. Since $q_{\beta_{\mathcal{E}}\widehat{\alpha}}\Pi_R(\xi) = \Pi_R(\xi) = \Pi_R(\xi)q_{\beta_A\widehat{\alpha}}$ for all $\xi \in \mathcal{E}$ (cf. Remark 2.1.2 (3)), we have the inclusion $[\Pi_R(\mathcal{E})\mathcal{E}_{A,R}] \subset \mathcal{E}_{\mathcal{E},R}$. The converse inclusion follows from $[\delta_{\mathcal{E}}(\mathcal{E})(1_A \otimes S)] = q_{\beta_{\mathcal{E}}\alpha}(\mathcal{E} \otimes S)$. Hence, $[\Pi_R(\mathcal{E})\mathcal{E}_{A,R}] = \mathcal{E}_{\mathcal{E},R}$. The relation $[\mathcal{D}\mathcal{E}_{A,R}] = \mathcal{E}_{\mathcal{E},R}$ follows from $\mathcal{E}_{A,R} = [D\mathcal{E}_{A,R}]$ and $[\Pi_R(\mathcal{E})D] = \mathcal{D}$. ■

We will endow $\mathcal{D}$ with a structure of $\mathcal{G}$-equivariant Hilbert D-module. Actually, the action $(\beta_{\mathcal{D}}, \delta_{\mathcal{D}})$ defined in Theorem 2.4.22 will be obtained by transport of structure through the identification $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \simeq \mathcal{D}$ of Theorem 2.4.23.

Let us denote by $\sigma : S \otimes \mathcal{K} \to \mathcal{K} \otimes S$ the flip *-homomorphism. As in Remarks 2.1.7, we define the linear extensions

$$\begin{aligned} \mathrm{id}_{\mathcal{E}} \otimes \sigma &: \mathcal{L}(A \otimes S \otimes \mathcal{K}, \mathcal{E} \otimes S \otimes \mathcal{K}) \to \mathcal{L}(A \otimes \mathcal{K} \otimes S, \mathcal{E} \otimes \mathcal{K} \otimes S); \\ \delta_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{K}} &: \mathcal{L}(A \otimes \mathcal{K}, \mathcal{E} \otimes \mathcal{K}) \to \mathcal{L}(A \otimes S \otimes \mathcal{K}, \mathcal{E} \otimes S \otimes \mathcal{K}). \end{aligned}$$

We state below the main results of this paragraph.

Theorem 2.4.22. *Let $\delta_{\mathcal{D}} : \mathcal{D} \to \mathcal{L}(D \otimes S, \mathcal{D} \otimes S)$ and $\beta_{\mathcal{D}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{D})$ be the maps defined by the formulas*

$$\begin{aligned} \beta_{\mathcal{D}}(n^{\mathrm{o}}) &:= q_{\beta_{\mathcal{E}}\widehat{\alpha}}\big(1_{\mathcal{E}} \otimes \beta(n^{\mathrm{o}})\big), \quad n \in N; \\ \delta_{\mathcal{D}}(\zeta) &:= (1_{\mathcal{E}} \otimes \mathcal{V})(\mathrm{id}_{\mathcal{E}} \otimes \sigma)(\delta_{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{K}})(\zeta)(1_A \otimes \mathcal{V})^*, \quad \zeta \in \mathcal{D}. \end{aligned}$$

Then, the pair $(\beta_{\mathcal{D}}, \delta_{\mathcal{D}})$ is a continuous action of $\mathcal{G}$ on the Hilbert D-module $\mathcal{D}$. Moreover, for all $\xi \in \mathcal{E}$, $x \in \widehat{S}$, and $y \in S$ we have

$$\delta_{\mathcal{D}}\big(\Pi_R(\xi)\big(1_A \otimes \lambda(x)L(y)\big)\big) = \big(\Pi_R(\xi) \otimes 1_S\big)(1_A \otimes \lambda(x) \otimes 1_S)\big(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)\big).$$

If $\mathcal{G}$ is regular, then we have $\mathcal{D} = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\mathcal{E} \otimes \mathcal{K})q_{\beta_A\widehat{\alpha}}$.

If $\mathcal{G}$ is regular, we have $\mathcal{D} \subset \mathcal{E} \otimes \mathcal{K}$ up to the identification $\mathcal{E} \subset \mathcal{L}(A, \mathcal{E})$.

Theorem 2.4.23. *There exists a unique unitary equivalence $\Phi : (\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to \mathcal{D}$ over the canonical *-isomorphism $\phi : (A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ (cf. Proposition 2.3.5) such that*

$$\Phi\big(\widehat{\Pi}\big(\Pi(\xi)\widehat{\theta}(x)\big)\theta(y)\big) = \Pi_R(\xi)\big(1_A \otimes \lambda(x)L(y)\big) \quad \textit{for all } \xi \in \mathcal{E},\ x \in \widehat{S},\ \textit{and } y \in S.$$

Moreover, Φ is $\mathcal{G}$-equivariant.

Proofs of Theorems 2.4.22 and 2.4.23. At the risk of considering $\mathcal{K}(\mathcal{E} \oplus A)$, we can assume that $\mathcal{E}$ is a top right-hand corner in some linking $\mathcal{G}$-C*-algebra $(J, \beta_J, \delta_J, e_1, e_2)$. By combining Proposition 2.4.19 (1) and Proposition 2.4.9, we can identify $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ with the top right-hand corner of the linking $\mathcal{G}$-C*-algebra

$$\big((J \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}, \beta_{(J \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}}, \delta_{(J \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}}, \widehat{\pi}\big(\pi(e_1)\big), \widehat{\pi}\big(\pi(e_2)\big)\big).$$

Let us denote by $D_J \subset \mathcal{L}(J \otimes \mathcal{H})$ the bidual $\mathcal{G}$-C*-algebra of J. By applying the biduality theorem (cf. Theorem 2.3.7 and Remark 2.3.10), we can identify $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ with the top right-hand corner of the linking $\mathcal{G}$-C*-algebra $(D_J, \beta_{D_J}, \delta_{D_J}, \pi_R(e_1), \pi_R(e_2))$. Since $\pi_R(e_j) = q_{\beta_J \widehat{\alpha}}(e_j \otimes 1_{\mathcal{K}})$ for $j = 1, 2$ and $\widehat{\alpha}(N) = M' \cap \widehat{M}'$, we have

$$\left[\pi_R(e_j),\ 1_J \otimes \lambda(x)L(y)\right] = 0 \quad \text{for all } x \in \widehat{S} \text{ and } y \in S.$$

Hence, we can identify $\pi_R(e_1) D_J \pi_R(e_2)$ with $\mathcal{D}$. The action $(\beta_{\mathcal{D}}, \delta_{\mathcal{D}})$ is then obtained by transport of structure. ■

Corollary 2.4.24. *Assume that $\mathcal{G}$ is regular. The formulas*

$$\delta_{\mathcal{E}_{\mathcal{E},R}}\big(q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\xi \otimes \eta)\big) := \mathcal{V}_{23}\delta_{\mathcal{E}}(\xi)_{13}(1_A \otimes \eta \otimes 1_S), \quad \xi \in \mathcal{E},\ \eta \in \mathcal{H};$$
$$\beta_{\mathcal{E}_{\mathcal{E},R}}(n^{\mathrm{o}}) := \big(1_{\mathcal{E}} \otimes \beta(n^{\mathrm{o}})\big)\restriction_{\mathcal{E}_{\mathcal{E},R}}, \quad n \in N,$$

define an action of $\mathcal{G}$ on the Hilbert A-module $\mathcal{E}_{\mathcal{E},R}$. Moreover, we have a canonical identification of $\mathcal{G}$-equivariant Hilbert A-modules

$$\big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}\big) \otimes_{(A \otimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} \mathcal{E}_{A,R} = \mathcal{E}_{\mathcal{E},R}$$

up to the identification of $\mathcal{G}$-C-algebras $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} = D$.*

Proof. It is clear that the formula $\mathcal{D} \otimes_D \mathcal{E}_{A,R} \to \mathcal{E}_{\mathcal{E},R}$; $\zeta \otimes_D \xi \mapsto \zeta(\xi)$ defines a unitary equivalence of Hilbert A-modules. Let $(\beta_{\mathcal{E}_{\mathcal{E},R}}, \delta_{\mathcal{E}_{\mathcal{E},R}})$ be the action of $\mathcal{G}$ on $\mathcal{E}_{\mathcal{E},R}$ obtained from the action of $\mathcal{G}$ on $\mathcal{D} \otimes_D \mathcal{E}_{A,R}$ by transport of structure. By a straightforward computation, we prove that $(\beta_{\mathcal{E}_{\mathcal{E},R}}, \delta_{\mathcal{E}_{\mathcal{E},R}})$ satisfies the formulas stated above. By Theorem 2.4.23, we have a unitary equivalence of $\mathcal{G}$-equivariant Hilbert D-modules

$$\big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}\big) \otimes_{(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} D = \mathcal{D}.$$

By taking the internal tensor product by $\mathcal{E}_{A,R}$ and using the associativity, we obtain

$$\begin{aligned}\Big(\big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}\big) \otimes_{(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} D\Big) \otimes_D \mathcal{E}_{A,R} &= \big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}\big) \otimes_{(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} (D \otimes_D \mathcal{E}_{A,R}) \\ &= \big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}\big) \otimes_{(A \otimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} \mathcal{E}_{A,R}.\end{aligned}$$

Hence, $((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}) \otimes_{(A \otimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}} \mathcal{E}_{A,R} = \mathcal{E}_{\mathcal{E},R}$. ■

Lemma 2.4.25. *Assume that $\mathcal{G}$ is regular. For all $F \in \mathcal{L}(\mathcal{E})$, $(\zeta \mapsto \pi_R(F) \circ \zeta) \in \mathcal{L}(\mathcal{D})$ and $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ are invariant.*

In order to keep the notations simple, we will sometimes denote by $\pi_R(F)$ the operators defined above since no ambiguity will arise.

Proof. Let $T \in \mathcal{L}(\mathcal{D})$ be the operator defined by $T(\zeta) := \pi_R(F) \circ \zeta$ for all $\zeta \in \mathcal{D}$. The operator $(F \otimes_\pi 1) \otimes_{\widehat{\pi}} 1 \in \mathcal{L}((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}})$ is invariant (cf. Lemma 2.4.6). However, the operator $(F \otimes_\pi 1) \otimes_{\widehat{\pi}} 1$ is identified to $T \in \mathcal{L}(\mathcal{D})$ through the identification $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} = \mathcal{D}$

(cf. Theorem 2.4.23). Hence, T is invariant. The operator $T \otimes_D 1 \in \mathcal{L}(\mathcal{D} \otimes_D \mathcal{E}_{A,R})$ is identified to $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ through the identification

$$\mathcal{D} \otimes_D \mathcal{E}_{A,R} \to \mathcal{E}_{\mathcal{E},R}; \quad \zeta \otimes_D \xi \mapsto \zeta(\xi).$$

Hence, the operator $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ is invariant. ■

Proposition-Definition 2.4.26. *Assume that $\mathcal{G}$ is regular. Let B be a $\mathcal{G}$-C*-algebra, $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert B-module, and $\gamma : A \to \mathcal{L}(\mathcal{E})$ a $\mathcal{G}$-equivariant *-representation of A on $\mathcal{E}$. Then, for all $d \in D$ we have*

$$(\gamma \otimes \mathrm{id}_{\mathcal{K}})(d)q_{\beta_{\mathcal{E}}\widehat{\alpha}} = (\gamma \otimes \mathrm{id}_{\mathcal{K}})(d) = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\gamma \otimes \mathrm{id}_{\mathcal{K}})(d) \quad \text{in } \mathcal{L}(\mathcal{E} \otimes \mathcal{H}).$$

*Moreover, the map $d \in D \mapsto (\gamma \otimes \mathrm{id}_{\mathcal{K}})(d)\restriction_{\mathcal{E}_{\mathcal{E},R}}$ is a $\mathcal{G}$-equivariant *-representation of D on $\mathcal{E}_{\mathcal{E},R}$. If $\mathcal{E}$ is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule, then $\mathcal{E}_{\mathcal{E},R}$ is a $\mathcal{G}$-equivariant Hilbert D-B-bimodule.*

Proof. We have $\gamma(\beta_A(n^{\mathrm{o}})a) = \beta_{\mathcal{E}}(n^{\mathrm{o}})\gamma(a)$ for all $a \in A$ and $n \in N$. It then follows that $(\gamma \otimes \mathrm{id}_{\mathcal{K}})(q_{\beta_A\widehat{\alpha}}xq_{\beta_A\widehat{\alpha}}) = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\gamma \otimes \mathrm{id}_{\mathcal{K}})(x)q_{\beta_{\mathcal{E}}\widehat{\alpha}}$ for all $x \in A \otimes \mathcal{K}$. In particular, we have $(\gamma \otimes \mathrm{id}_{\mathcal{K}})(d)q_{\beta_{\mathcal{E}}\widehat{\alpha}} = (\gamma \otimes \mathrm{id}_{\mathcal{K}})(d) = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\gamma \otimes \mathrm{id}_{\mathcal{K}})(d)$ for all $d \in D$. As a result, the *-representation $\gamma \otimes \mathrm{id}_{\mathcal{K}} : A \otimes \mathcal{K} \to \mathcal{L}(\mathcal{E} \otimes \mathcal{H})$ induces by restriction a *-representation $\gamma_0 : D \to \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$. Let us prove that γ_0 is $\mathcal{G}$-equivariant. Let us fix $\xi \in \mathcal{E}$, $\eta \in \mathcal{H}$, $a \in A$, and $k \in \mathcal{K}$. We have $\delta_{\mathcal{E}}(\gamma(a)\xi) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathcal{E}}(\xi)$. By a straightforward computation, we have $\delta_{\mathcal{E}_{\mathcal{E},R}}(q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\gamma(a)\xi \otimes k\eta)) = (\gamma \otimes \mathrm{id}_{\mathcal{K}} \otimes \mathrm{id}_S)(\mathcal{V}_{23}\delta_0(a \otimes k))\delta_{\mathcal{E}}(\xi)_{13}(1_A \otimes \eta \otimes 1_S)$. For all $x \in A \otimes \mathcal{K}$, we have $\delta_0(xq_{\beta_A\widehat{\alpha}}) = \delta_0(x)q_{\widehat{\alpha}\beta,23}$. Hence,

$$\mathcal{V}_{23}\delta_0(xq_{\beta_A\widehat{\alpha}}) = \delta_{A\otimes\mathcal{K}}(x)\mathcal{V}_{23} \quad \text{for all } x \in A \otimes \mathcal{K}.$$

In particular, we have $\mathcal{V}_{23}\delta_0(d) = \delta_D(d)\mathcal{V}_{23}$ for all $d \in D$. Hence,

$$\delta_{\mathcal{E}_{\mathcal{E},R}}\big(\gamma_0(d)\zeta\big) = (\gamma_0 \otimes \mathrm{id}_D)\big(\delta_D(d)\big) \circ \delta_{\mathcal{E}_{\mathcal{E},R}}(\zeta) \quad \text{for all } \zeta \in \mathcal{E}_{\mathcal{E},R} \text{ and } d \in D.$$

It is easily seen that $\gamma_0(\beta_D(n^{\mathrm{o}})d) = \beta_{\mathcal{E}_{\mathcal{E},R}}(n^{\mathrm{o}})\gamma_0(d)$ for all $n \in N$ and $d \in D$. If $\mathcal{E}$ is countably generated as a Hilbert B-module, then so is $\mathcal{E} \otimes \mathcal{H}$ since $\mathcal{H}$ is separable. Hence, the submodule $\mathcal{E}_{\mathcal{E},R}$ of $\mathcal{E} \otimes \mathcal{H}$ is countably generated. ■

2.5. Case of a colinking measured quantum groupoid

Let us fix a colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$.

2.5.1. Hilbert C*-modules acted upon by a colinking measured quantum groupoid. In the following, we recall the description of Hilbert C*-modules acted upon by $\mathcal{G}$ in terms of Hilbert C*-modules acted upon by $\mathbb{G}_1$ and $\mathbb{G}_2$ (cf. [9, §6.2]). Let A be a $\mathcal{G}$-C*-algebra. We follow the notations of [9, §3.3] (resp. [9, Notation 5.2.1 and Proposition 5.2.2]) concerning the objects associated with $\mathcal{G}$ (resp. A). Let $\mathcal{E}$ be a Hilbert A-module endowed with an action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ of $\mathcal{G}$.

Notations 2.5.1. We introduce some useful notations to describe the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$.

- Let $q_{\mathcal{E},j} := \beta_{\mathcal{E}}(\varepsilon_j)$ for $j = 1, 2$. Note that $q_{\mathcal{E},1}$ and $q_{\mathcal{E},2}$ are orthogonal self-adjoint projections of $\mathcal{L}(\mathcal{E})$ and $q_{\mathcal{E},1} + q_{\mathcal{E},2} = 1_{\mathcal{E}}$.
- Let $J := \mathcal{K}(\mathcal{E} \oplus A)$ be the linking C*-algebra associated with $\mathcal{E}$ endowed with the action (β_J, δ_J) of $\mathcal{G}$ (cf. Proposition 2.1.10 (b)). Since $\beta_J(\mathbb{C}^2) \subset \mathcal{Z}(\mathcal{M}(J))$ (cf. [2, §3.2.3]), we have $\beta_{\mathcal{E}}(n)\xi = \xi\beta_A(n)$ in $\mathcal{L}(A, \mathcal{E})$ for all $n \in \mathbb{C}^2$ and $\xi \in \mathcal{E}$; i.e.,
$$\big(\beta_{\mathcal{E}}(n)\xi\big)a = \xi\big(\beta_A(n)a\big) \quad \text{for all } n \in \mathbb{C}^2,\ \xi \in \mathcal{E},\ \text{and } a \in A.$$
Hence,
$$(q_{\mathcal{E},j}\xi)a = \xi(q_{A,j}a) \quad \text{for all } \xi \in \mathcal{E},\ a \in A,\ j = 1, 2. \tag{2.14}$$
In particular, we have
$$\langle q_{\mathcal{E},j}\xi, q_{\mathcal{E},j}\eta\rangle = q_{A,j}\langle \xi, \eta\rangle \quad \text{for all } \xi, \eta \in \mathcal{E}.$$
For $j = 1, 2$, we then define the following Hilbert A_j-module $\mathcal{E}_j := q_{\mathcal{E},j}\mathcal{E}$. Note that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$.
- For $j, k = 1, 2$, let $\Pi_j^k : \mathcal{E}_k \otimes S_{kj} \to \mathcal{E} \otimes S$ be the inclusion map. It is clear that the map Π_j^k is a π_j^k-compatible operator. Then we can consider its canonical linear extension $\Pi_j^k : \mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj}) \to \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$, up to the canonical injective maps $\mathcal{E}_k \otimes S_{kj} \to \mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ and $\mathcal{E} \otimes S \to \mathcal{L}(A \otimes S, \mathcal{E} \otimes S)$, defined by $\Pi_j^k(T)(x) := \Pi_j^k \circ T((q_{A,k} \otimes p_{kj})x)$ for all $T \in \mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ and $x \in A \otimes S$.

Lemma 2.5.2. *With the above notations and hypotheses, we have a canonical unitary equivalence of Hilbert $A \otimes S$-modules $\mathcal{E} \otimes_{\delta_A} (A \otimes S) = \bigoplus_{j,k=1,2} \mathcal{E}_j \otimes_{\delta_{A_j}^k} (A_k \otimes S_{kj})$.*

Proposition-Definition 2.5.3. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_A} (A \otimes S), \mathcal{E} \otimes S)$ be the isometry associated with the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ (cf. Proposition 2.1.5 (a)). For all $j, k = 1, 2$, there exists a unique unitary*
$$\mathcal{V}_j^k \in \mathcal{L}\big(\mathcal{E}_j \otimes_{\delta_{A_j}^k} (A_k \otimes S_{kj}), \mathcal{E}_k \otimes S_{kj}\big)$$
such that
$$\mathcal{V}(\xi \otimes_{\delta_A} x) = \sum_{j,k=1,2} \mathcal{V}_j^k\big(q_{\mathcal{E},j}\xi \otimes_{\delta_{A_j}^k} (q_{A,k} \otimes p_{kj})x\big) \quad \textit{for all } \xi \in \mathcal{E} \textit{ and } x \in A \otimes S.$$

For $j, k, l = 1, 2$ we have the following set of unitary equivalences of Hilbert modules:
$$\begin{aligned} A_j \otimes_{\delta_{A_j}^k} (A_k \otimes S_{kj}) &\to A_k \otimes S_{kj} \\ a \otimes_{\delta_{A_j}^k} x &\mapsto \delta_{A_j}^k(a)x; \end{aligned} \tag{2.15}$$
$$\begin{aligned} (A_k \otimes S_{kj}) \otimes_{\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}) &\to A_l \otimes S_{lk} \otimes S_{kj} \\ x \otimes_{\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}}} y &\mapsto (\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})(x)y; \end{aligned} \tag{2.16}$$

$$(A_l \otimes S_{lj}) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj}) \to A_l \otimes S_{lk} \otimes S_{kj}$$
$$x \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto (\mathrm{id}_{A_l} \otimes \delta^k_{lj})(x)y; \tag{2.17}$$
$$\big(\mathcal{E}_j \otimes_{\delta^k_{A_j}} (A_k \otimes S_{kj})\big) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \mathcal{E}_j \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj})$$
$$(\xi \otimes_{\delta^k_{A_j}} x) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} y \mapsto \xi \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} (\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})(x)y; \tag{2.18}$$
$$\big(\mathcal{E}_j \otimes_{\delta^l_{A_j}} (A_l \otimes S_{lj})\big) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \mathcal{E}_j \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj})$$
$$(\xi \otimes_{\delta^l_{A_j}} x) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto \xi \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} (\mathrm{id}_{A_l} \otimes \delta^k_{lj})(x)y; \tag{2.19}$$
$$(\mathcal{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \big(\mathcal{E}_k \otimes_{\delta^l_{A_k}} (A_l \otimes S_{lk})\big) \otimes S_{kj}$$
$$(\xi \otimes s) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (x \otimes t) \mapsto (\xi \otimes_{\delta^l_{A_k}} x) \otimes st; \tag{2.20}$$
$$(\mathcal{E}_l \otimes S_{lj}) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \mathcal{E}_l \otimes S_{lk} \otimes S_{kj}$$
$$\xi \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto (\mathrm{id}_{\mathcal{E}_l} \otimes \delta^k_{lj})(\xi)y. \tag{2.21}$$

Proposition 2.5.4. *For all $j,k,l = 1,2$, we have*

$$(\mathcal{V}^l_k \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}})(\mathcal{V}^k_j \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1) = \mathcal{V}^l_j \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} 1.$$

For $j,k,l = 1,2$,

$$\mathcal{V}^l_k \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}} \in \mathcal{L}\big((\mathcal{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}), \mathcal{E}_l \otimes S_{lk} \otimes S_{kj}\big) \quad (2.20);$$
$$\mathcal{V}^k_j \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1 \in \mathcal{L}\big(\mathcal{E}_j \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), (\mathcal{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj})\big) \quad (2.18);$$
$$\mathcal{V}^l_j \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} 1 \in \mathcal{L}\big(\mathcal{E}_j \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), \mathcal{E}_l \otimes S_{lk} \otimes S_{kj}\big) \quad (2.21).$$

Moreover, the composition $(\mathcal{V}^l_k \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}})(\mathcal{V}^k_j \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1)$ does make sense since $(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j} = (\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}$.

Proposition-Definition 2.5.5. *For $j,k = 1,2$, let $\delta^k_{\mathcal{E}_j} : \mathcal{E}_j \to \mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ be the linear map defined by*

$$\delta^k_{\mathcal{E}_j}(\xi)x := \mathcal{V}^k_j(\xi \otimes_{\delta^k_{A_j}} x) \quad \textit{for all } \xi \in \mathcal{E}_j \textit{ and } x \in A_k \otimes S_{kj}.$$

For all $j,k,l = 1,2$, we have the following statements.

(i) $\delta_{\mathcal{E}}(\xi) = \sum_{k,j=1,2} \Pi^k_j \circ \delta^k_{\mathcal{E}_j}(q_{\mathcal{E},j}\xi)$ *for all* $\xi \in \mathcal{E}$.

(ii) $\delta^k_{\mathcal{E}_j}(\mathcal{E}_j) \subset \widetilde{\mathcal{M}}(\mathcal{E}_k \otimes S_{kj})$.

(iii) $\delta^k_{\mathcal{E}_j}(\xi a) = \delta^k_{\mathcal{E}_j}(\xi)\delta^k_{A_j}(a)$ *and* $\langle \delta^k_{\mathcal{E}_j}(\xi), \delta^k_{\mathcal{E}_j}(\eta)\rangle = \delta^k_{A_j}(\langle \xi, \eta\rangle)$ *for all* $\xi, \eta \in \mathcal{E}_j$ *and* $a \in A_j$.

(iv) $[\delta^k_{\mathcal{E}_j}(\mathcal{E}_j)(1_{A_k} \otimes S_{kj})] = \mathcal{E}_k \otimes S_{kj}$*; in particular, we have*

$$\mathcal{E}_k = \left[(\mathrm{id}_{\mathcal{E}_k} \otimes \omega)\delta^k_{\mathcal{E}_j}(\xi);\ \omega \in \mathcal{B}(\mathcal{H}_{kj})_*,\ \xi \in \mathcal{E}_j\right]$$

cf. [9, Proposition-Definition 2.3.6].

(v) $\delta^l_{\mathcal{E}_k} \otimes \mathrm{id}_{S_{kj}}$ *(resp.* $\mathrm{id}_{\mathcal{E}_l} \otimes \delta^k_{lj}$*) extends to a linear map from* $\mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ *(resp.* $\mathcal{L}(A_l \otimes S_{lj}, \mathcal{E}_l \otimes S_{lj})$*) to* $\mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathcal{E}_l \otimes S_{lk} \otimes S_{kj})$ *and for all* $\xi \in \mathcal{E}_j$ *we have*

$$(\delta^l_{\mathcal{E}_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathcal{E}_j}(\xi) = (\mathrm{id}_{\mathcal{E}_l} \otimes \delta^k_{lj})\delta^l_{\mathcal{E}_j}(\xi) \in \mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathcal{E}_l \otimes S_{lk} \otimes S_{kj}).$$

(vi) *If* $\mathcal{E}$ *is a* $\mathcal{G}$*-equivariant Hilbert* A*-module, then we have* $[(1_{\mathcal{E}_k} \otimes S_{kj})\delta^k_{\mathcal{E}_j}(\mathcal{E}_j)] = \mathcal{E}_k \otimes S_{kj}$.

If $\mathcal{E}$ *is a* $\mathcal{G}$*-equivariant Hilbert* A*-module, then* $(\mathcal{E}_j, \delta^j_{\mathcal{E}_j})$ *is a* $\mathbb{G}_j$*-equivariant Hilbert* A_j*-module and* $\mathcal{V}^j_j$ *is the associated unitary.*

According to this concrete description of $\mathcal{G}$-equivariant Hilbert C*-modules, we have a description of the $\mathcal{G}$-equivariant unitary equivalences in terms of $\mathbb{G}_j$-equivariant unitary equivalences for $j = 1, 2$.

Lemma 2.5.6. *Let* A *and* B *be* $\mathcal{G}$-C**-algebras. Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be Hilbert* C**-modules over* A *and* B*, respectively, acted upon by* $\mathcal{G}$.

(1) *Let* $\Phi : \mathcal{E} \to \mathcal{F}$ *be a* $\mathcal{G}$*-equivariant unitary equivalence over a* $\mathcal{G}$*-equivariant* **-isomorphism* $\phi : A \to B$*. For* $j = 1, 2$*, there exists a unique map* $\Phi_j : \mathcal{E}_j \to \mathcal{F}_j$ *satisfying the formula* $\Phi(\xi) = \Phi_1(q_{\mathcal{E},1}\xi) + \Phi_2(q_{\mathcal{E},2}\xi)$ *for all* $\xi \in \mathcal{E}$*. Moreover, we have*

  (i) *for* $j = 1, 2$*, the map* Φ_j *is a unitary equivalence over the* **-isomorphism* $\phi_j : A_j \to B_j$ *(cf.* [9, Proposition 5.2.3 (1)]*);*

  (ii) *for all* $j, k = 1, 2$*,* $(\Phi_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{\mathcal{E}_j} = \delta^k_{\mathcal{F}_j} \circ \Phi_j$.

  In particular, Φ_j *is a* $\mathbb{G}_j$*-equivariant* ϕ_j*-compatible unitary operator.*

(2) *Conversely, for* $j = 1, 2$ *let* $\Phi_j : \mathcal{E}_j \to \mathcal{F}_j$ *be a* $\mathbb{G}_j$*-equivariant unitary equivalence over a* $\mathbb{G}_j$*-equivariant* **-isomorphism*

$$\phi_j : A_j \to B_j$$

such that $(\phi_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{A_j} = \delta^k_{B_j} \circ \phi_j$ *and* $(\Phi_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{\mathcal{E}_j} = \delta^k_{\mathcal{F}_j} \circ \Phi_j$ *for all* $j, k = 1, 2$*. Then, the map* $\Phi : \mathcal{E} \to \mathcal{F}$*, defined by* $\Phi(\xi) := \Phi_1(q_{\mathcal{E},1}\xi) + \Phi_2(q_{\mathcal{E},2}\xi)$ *for all* $\xi \in \mathcal{E}$*, is a* $\mathcal{G}$*-equivariant unitary equivalence over the* $\mathcal{G}$*-equivariant* **-isomorphism* $\phi : A \to B$ *(cf.* [9, Proposition 5.2.3 (2)]*).*

We also have a description of the $\mathcal{G}$-equivariant *-representations in terms of $\mathbb{G}_j$-equivariant *-representations for $j = 1, 2$.

Lemma 2.5.7. *Let A, B be two $\mathcal{G}$-C*-algebras and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert B-module. We follow* [9, Notations 5.2.1–5.2.2] *and Notations* 2.5.1 *concerning these objects.*

(1) *Let $\gamma : A \to \mathcal{L}(\mathcal{E})$ be a $\mathcal{G}$-equivariant *-representation. We have*

$$\gamma(a)q_{\mathcal{E},j} = \gamma(q_{A,j}a) = q_{\mathcal{E},j}\gamma(a) \quad \text{for all } a \in A \text{ and } j = 1,2.$$

There exist unique *-representations $\gamma_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ for $j = 1,2$ such that for all $a \in A$, $\gamma(a) = \gamma_1(q_{A,1}a) + \gamma_2(q_{A,2}a)$. Furthermore, for $j,k = 1,2$ we have

$$\delta^k_{\mathcal{E}_j}\big(\gamma_j(a)\xi\big) = (\gamma_k \otimes \mathrm{id}_{S_{kj}})\big(\delta^k_{A_j}(a)\big) \circ \delta^k_{\mathcal{E}_j}(\xi) \quad \text{for all } a \in A_j \text{ and } \xi \in \mathcal{E}_j. \tag{2.22}$$

In particular, the *-representation $\gamma_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ is $\mathbb{G}_j$-equivariant.

(2) *Conversely, let $\gamma_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ be a $\mathbb{G}_j$-equivariant *-representation for $j = 1,2$. Let $\gamma : A \to \mathcal{L}(\mathcal{E})$ be the *-representation defined by $\gamma(a) := \gamma_1(q_{A,1}a) + \gamma_2(q_{A,2}a)$ for all $a \in A$. Assume further that* (2.22) *holds for all $j,k = 1,2$. Then, the *-representation $\gamma : A \to \mathcal{L}(\mathcal{E})$ is $\mathcal{G}$-equivariant.*

Moreover, the pair $(\mathcal{E},\gamma)$ is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule if and only if the pair $(\mathcal{E}_j,\gamma_j)$ is a $\mathbb{G}_j$-equivariant Hilbert A_j-B_j-bimodule for $j = 1,2$.

Proof. Since β_A is central and γ is $\mathcal{G}$-equivariant, we have

$$\big[\gamma(a), \beta_{\mathcal{E}}(n)\big] = \gamma\big(\big[a, \beta_A(n)\big]\big) = 0 \quad \text{for all } n \in \mathbb{C}^2.$$

Hence, $\gamma(a)q_{\mathcal{E},j} = \gamma(q_{A,j}a) = q_{\mathcal{E},j}\gamma(a)$ for all $a \in A$ and $j = 1,2$. For $j = 1,2$, we denote by $\gamma_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ the *-representation defined by $\gamma_j(a) := \gamma(a){\upharpoonright}_{\mathcal{E}_j}$ for all $a \in A_j$. We have $\gamma(a) = \gamma_1(q_{A,1}a) + \gamma_2(q_{A,2}a)$ for all $a \in A$ and (2.22) is a straightforward restatement of the fact that $\delta_{\mathcal{E}}(\gamma(a)\xi) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathcal{E}}(\xi)$. The converse and the last statement are obvious. ∎

Note that (2.22) can be restated in the following ways:

$$\mathcal{V}^k_j(\gamma_j(a) \otimes_{\delta^k_{B_j}} 1)(\mathcal{V}^k_j)^* = (\gamma_k \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}(a), \quad a \in A;$$
$$\delta^k_{\mathcal{K}(\mathcal{E}_j)}(\gamma_j(a)) = (\gamma_k \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}(a), \quad a \in A.$$

The following lemma is straightforward.

Lemma 2.5.8. *Let A and B be two $\mathcal{G}$-C*-algebras. Let $\mathcal{E}$ and $\mathcal{F}$ be two $\mathcal{G}$-equivariant Hilbert B-modules.*

(1) *Let $\pi : A \to \mathcal{L}(\mathcal{E})$ and $\gamma : A \to \mathcal{L}(\mathcal{F})$ be $\mathcal{G}$-equivariant *-representations. Let $\Phi \in \mathcal{L}(\mathcal{E}, F)$ be a $\mathcal{G}$-equivariant unitary such that $\Phi \circ \pi(a) = \gamma(a) \circ \Phi$ for all $a \in A$. Then, for $j = 1,2$ the $\mathbb{G}_j$-equivariant unitary $\Phi_j \in \mathcal{L}(\mathcal{E}_j, \mathcal{F}_j)$ (cf. Lemma* 2.5.6 (1)*) satisfies for all $a \in A_j$ the relation $\Phi_j \circ \pi_j(a) = \gamma_j(a) \circ \Phi_j$ (cf. Lemma* 2.5.7 (1)*).*

(2) *Conversely, for* $j = 1, 2$ *let us fix* $\mathbb{G}_j$*-equivariant* **-representations* $\pi_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ *and* $\gamma_j : A_j \to \mathcal{L}(\mathcal{F}_j)$ *and a* $\mathbb{G}_j$*-equivariant unitary* $\Phi_j \in \mathcal{L}(\mathcal{E}_j, \mathcal{F}_j)$ *satisfying the relation* $\Phi_j \circ \pi_j(a) = \gamma_j(a) \circ \Phi_j$ *for all* $a \in A_j$*. Then, the* $\mathcal{G}$*-equivariant unitary* $\Phi \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ *(cf. Lemma 2.5.6 (2)) satisfies the relation* $\Phi \circ \pi(a) = \gamma(a) \circ \Phi$ *for all* $a \in A$ *(cf. Lemma 2.5.7 (2)).*

2.5.2. Induction of equivariant Hilbert C*-modules. In the following, we recall the induction procedure for equivariant Hilbert C*-modules (cf. [2, Proposition 4.3], [9, §6.3]). We assume that the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are regular. Fix a $\mathbb{G}_1$-C*-algebra (A_1, δ_{A_1}) and a $\mathbb{G}_1$-equivariant Hilbert A_1-module $(\mathcal{E}_1, \delta_{\mathcal{E}_1})$. We denote by $J_1 := \mathcal{K}(\mathcal{E}_1 \oplus A_1)$ the associated linking C*-algebra endowed with the continuous action δ_{J_1} of $\mathbb{G}_1$.

Notations 2.5.9. Let us fix some notations.

- Let $\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{11}^2 : \mathcal{L}(A_1 \otimes S_{11}, \mathcal{E}_1 \otimes S_{11}) \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{21}, \mathcal{E}_1 \otimes S_{12} \otimes S_{21})$ be the unique linear extension of $\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{11}^2 : \mathcal{E}_1 \otimes S_{11} \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{21}, \mathcal{E}_1 \otimes S_{12} \otimes S_{21})$ such that $(\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{11}^2)(T)(\mathrm{id}_{A_1} \otimes \delta_{11}^2)(x) = (\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{11}^2)(Tx)$ for all $x \in \mathcal{M}(A_1 \otimes S_{11})$ and $T \in \mathcal{L}(A_1 \otimes S_{11}, \mathcal{E}_1 \otimes S_{11})$.
- Let $\delta_{\mathcal{E}_1}^{(2)} : \mathcal{E}_1 \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{21}, \mathcal{E}_1 \otimes S_{12} \otimes S_{21})$ be the linear map defined by $\delta_{\mathcal{E}_1}^{(2)}(\xi) := (\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{11}^2)\delta_{\mathcal{E}_1}(\xi)$ for all $\xi \in \mathcal{E}_1$.
- Consider the Banach subspace of $\mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12})$ defined by (cf. [9, Proposition-Definition 2.3.6])

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1) := \left[(\mathrm{id}_{\mathcal{E}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathcal{E}_1}^{(2)}(\xi);\ \xi \in \mathcal{E}_1,\ \omega \in \mathcal{B}(\mathcal{H}_{21})_*\right].$$

Proposition 2.5.10. *Let us denote by* $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ *the induced* $\mathbb{G}_2$*-C*-algebra of* A_1*. Let* $\mathcal{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1)$*.*

(1) *We have* $[\mathcal{E}_2(1_{A_1} \otimes S_{12})] = \mathcal{E}_1 \otimes S_{12} = [(1_{\mathcal{E}_1} \otimes S_{11})\mathcal{E}_2]$*. In particular,* $\mathcal{E}_2 \subset \widetilde{\mathcal{M}}(\mathcal{E}_1 \otimes S_{12})$*.*

(2) $\mathcal{E}_2$ *is a Hilbert* A_2*-module for the right action by composition and the* A_2*-valued inner product given by* $\langle \xi, \eta \rangle := \xi^* \circ \eta$ *for* $\xi, \eta \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1)$*.*

Let us denote by $(A_2, \delta_{A_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1, \delta_{A_1})$ and $(J_2, \delta_{J_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1, \delta_{J_1})$ the induced $\mathbb{G}_2$-C*-algebra of (A_1, δ_{A_1}) and (J_1, δ_{J_1}), respectively. We also denote by $\mathcal{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1)$ the induced Hilbert A_2-module as defined above.

Notation 2.5.11. Let

$$\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{12}^2 : \mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12}) \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{22}, \mathcal{E}_1 \otimes S_{12} \otimes S_{22})$$

be the unique linear extension of

$$\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{12}^2 : \mathcal{E}_1 \otimes S_{12} \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{22}, \mathcal{E}_1 \otimes S_{12} \otimes S_{22})$$

such that $(\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{12}^2)(T)(\mathrm{id}_{A_1} \otimes \delta_{12}^2)(x) = (\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{12}^2)(Tx)$ for all $x \in \mathcal{M}(A_1 \otimes S_{12})$ and $T \in \mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12})$.

Proposition-Definition 2.5.12. *There exists a unique linear map*

$$\delta_{\mathcal{E}_2} : \mathcal{E}_2 \to \mathcal{L}(A_2 \otimes S_{22}, \mathcal{E}_2 \otimes S_{22})$$

satisfying the relation $[\delta_{\mathcal{E}_2}(\xi)a]b = (\mathrm{id}_{\mathcal{E}_1} \otimes \delta^2_{12})(\xi)(ab)$ *for all* $\xi \in \mathcal{E}_2$, $a \in A_2 \otimes S_{22}$, *and* $b \in A_1 \otimes S_{12} \otimes S_{22}$. *Moreover, the pair* $(\mathcal{E}_2, \delta_{\mathcal{E}_2})$ *is a* $\mathbb{G}_2$*-equivariant Hilbert* A_2*-module.*

In the proposition below, we state that the above induction procedure for equivariant Hilbert C*-modules is equivalent to that of [2, §4.3].

Notations 2.5.13. Let $e_{1,1} := \iota_{\mathcal{K}(\mathcal{E}_1)}(1_{\mathcal{E}_1}) \in \mathcal{M}(J_1)$ and $e_{2,1} := \iota_{A_1}(1_{A_1}) \in \mathcal{M}(J_1)$, where we identify $\mathcal{M}(J_1) = \mathcal{L}(\mathcal{E}_1 \oplus A_1)$. Let $(J_2, \delta_{J_2}, e_{1,2}, e_{2,2})$ be the induced linking $\mathbb{G}_2$-C*-algebra, with $e_{l,2} := e_{l,1} \otimes 1_{S_{12}} \in \mathcal{M}(J_2)$ for $l = 1, 2$ (cf. [2, Proposition 2.14]). Consider $e_{2,2}J_2e_{2,2}$ and $e_{1,2}J_2e_{2,2}$ endowed with their structure of $\mathbb{G}_2$-C*-algebra and $\mathbb{G}_2$-equivariant Hilbert $e_{2,2}J_2e_{2,2}$-module (cf. [3]). Recall that $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1} : A_2 \to J_2$; $x \mapsto (\iota_{A_1} \otimes \mathrm{id}_{S_{12}})(x)$ induces a $\mathbb{G}_2$-equivariant *-isomormorphism $A_2 \to e_{2,2}J_2e_{2,2}$ (cf. [2, Propositions 2.17 and 2.18]).

Proposition 2.5.14. *We use the above notations.*

(i) *There exists a unique bounded linear map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1} : \mathcal{E}_2 \to J_2$ *such that*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1}\big((\mathrm{id}_{\mathcal{E}_1 \otimes S_{12}} \otimes \omega)\delta^{(2)}_{\mathcal{E}_1}(\xi)\big) = (\mathrm{id}_{J_1 \otimes S_{12}} \otimes \omega)\delta^{(2)}_{J_1}\big(\iota_{\mathcal{E}_1}(\xi)\big),$$

for all $\xi \in \mathcal{E}_1$ *and* $\omega \in \mathcal{B}(\mathcal{H}_{21})_*$. *Moreover, we have* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1}(\mathcal{E}_2) = e_{1,2}J_2e_{2,2}$ *and* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1}$ *induces a* $\mathbb{G}_2$*-equivariant unitary equivalence* $\mathcal{E}_2 \to e_{1,2}J_2e_{2,2}$; $\xi \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1}(\xi)$ *over the* $\mathbb{G}_2$*-equivariant *-isomorphism* $A_2 \to e_{2,2}J_2e_{2,2}$; $a \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}(a)$.

(ii) *There exists a unique *-homomorphism* $\tau : \mathcal{K}(\mathcal{E}_2 \oplus A_2) \to J_2$ *such that* $\tau \circ \iota_{\mathcal{E}_2} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathcal{E}_1}$ *and* $\tau \circ \iota_{A_2} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}$. *Moreover,* τ *is an isomorphism of linking* $\mathbb{G}_2$*-C*-algebras.*

(iii) *If* $T \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathcal{E}_1)) \subset \mathcal{L}(\mathcal{E}_1 \otimes S_{12})$ *and* $\eta \in \mathcal{E}_2 \subset \mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12})$, *then we have* $T \circ \eta \in \mathcal{E}_2$. *Moreover, for all* $T \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathcal{E}_1))$, *we have* $[\eta \mapsto T \circ \eta] \in \mathcal{K}(\mathcal{E}_2)$. *More precisely, the map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathcal{E}_1)) \to \mathcal{K}(\mathcal{E}_2)$; $T \mapsto [\eta \mapsto T \circ \eta]$ *is a* $\mathbb{G}_2$*-equivariant *-isomorphism.*

In the result below, we recall how to induce $\mathbb{G}_1$-equivariant unitary equivalence.

Proposition-Definition 2.5.15. *Let us fix some notations. Consider*

- *two* $\mathbb{G}_1$*-C*-algebras* A_1 *and* B_1,
- *two* $\mathbb{G}_1$*-equivariant Hilbert modules* $\mathcal{E}_1$ *and* $\mathcal{F}_1$ *over* A_1 *and* B_1, *respectively,*
- *a* $\mathbb{G}_1$*-equivariant unitary equivalence* $\Phi_1 : \mathcal{E}_1 \to \mathcal{F}_1$ *over a* $\mathbb{G}_1$*-equivariant *-isomorphism* $\phi_1 : A_1 \to B_1$.

Denote by

- $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ *and* $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ *the induced* $\mathbb{G}_2$*-C*-algebras,*

- $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\phi_1) : A_2 \to B_2$ *the induced* $\mathbb{G}_2$*-equivariant* **-isomorphism,*
- $\mathcal{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1)$ *and* $\mathcal{F}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{F}_1)$ *the induced* $\mathbb{G}_2$*-equivariant Hilbert modules over* A_2 *and* B_2*, respectively,*
- $\Phi_1 \otimes \mathrm{id}_{S_{12}} : \mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12}) \to \mathcal{L}(B_1 \otimes S_{12}, \mathcal{F}_1 \otimes S_{12})$ *the unique linear map such that*

$$(\Phi_1 \otimes \mathrm{id}_{S_{12}})(T)(\phi_1 \otimes \mathrm{id}_{S_{12}})(x) = (\Phi_1 \otimes \mathrm{id}_{S_{12}})(Tx)$$

for all $\mathcal{L}(A_1 \otimes S_{12}, \mathcal{E}_1 \otimes S_{12})$ *and* $x \in A_1 \otimes S_{12}$ *(cf.* [9, Notation A.3.6]*).*

Then, $(\Phi_1 \otimes \mathrm{id}_{S_{12}})(\mathcal{E}_2) \subset \mathcal{F}_2$ *and the map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\Phi_1) := (\Phi_1 \otimes \mathrm{id}_{S_{12}})\restriction_{\mathcal{E}_2} : \mathcal{E}_2 \to \mathcal{F}_2$ *is a* $\mathbb{G}_2$*-equivariant unitary equivalence over* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\phi_1) : A_2 \to B_2$*. Moreover, for all* $\xi \in \mathcal{E}_1$ *and* $\omega \in \mathcal{B}(\mathcal{H}_{21})_*$ *we have* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\Phi_1)((\mathrm{id}_{\mathcal{E}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathcal{E}_1}^{(2)}(\xi)) = (\mathrm{id}_{\mathcal{F}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathcal{F}_1}^{(2)}(\Phi_1 \xi)$*.*

We can also induce $\mathbb{G}_1$-equivariant *-representations. Let us state a preliminary result.

Lemma 2.5.16. *Let* A_1 *be a* $\mathbb{G}_1$*-*C**-algebra. If* A_1 *is* σ*-unital (resp. separable), then so is the induced* $\mathbb{G}_2$*-*C**-algebra* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$*.*

Proof. Let us assume that A_1 is σ-unital. Let $(u_n)_{n \geqslant 1}$ be a countable approximate unit of A_1. Let $\omega \in \mathcal{B}(\mathcal{H}_{21})_*$ such that $\omega(1) = 1$. Then, the sequence $((\mathrm{id}_{A_1 \otimes S_{12}} \otimes \omega)\delta_{A_1}^{(2)}(u_n))_{n \geqslant 1}$ is an approximate unit of $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$. Hence, $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ is σ-unital. Suppose now that A_1 is separable. Let X (resp. Y) be a countable total subset of A_1 (resp. $\mathcal{H}_{21}$). Hence, the subset $\{(\mathrm{id}_{A_1 \otimes S_{12}} \otimes \omega_{\xi,\eta})\delta_{A_1}^{(2)}(a);\ a \in X,\ \xi, \eta \in Y\}$ of $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ is countable and spans a dense subspace of $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$. Hence, the C*-algebra $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ is separable. ∎

Proposition-Definition 2.5.17. *Let* A_1 *and* B_1 *be* $\mathbb{G}_1$*-*C**-algebras and* $\mathcal{E}_1$ *a* $\mathbb{G}_1$*-equivariant Hilbert* A_1*-*B_1*-bimodule. Let* $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ *and* $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ *be the induced* $\mathbb{G}_2$*-*C**-algebras. Let* $\mathcal{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1)$ *be the induced* $\mathbb{G}_2$*-equivariant Hilbert* B_2*-module. Let us consider a* $\mathbb{G}_1$*-equivariant* **-representation* $\gamma_1 : A_1 \to \mathcal{L}(\mathcal{E}_1)$*. Up to the identifications* $\mathcal{L}(\mathcal{E}_1) = \mathcal{M}(\mathcal{K}(\mathcal{E}_1))$ *and* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \mathcal{K}(\mathcal{E}_1) = \mathcal{K}(\mathcal{E}_2)$ *(cf. Proposition* 2.5.14 (iii)*) and by functoriality of the induction (cf.* [2, Proposition 4.3 (c)]*), we have a* $\mathbb{G}_2$*-equivariant* **-representation*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \gamma_1 : A_2 \to \mathcal{L}(\mathcal{E}_2)$$

called the induced $\mathbb{G}_2$*-equivariant* **-representation of* γ_1*. If* $\mathcal{E}_1$ *is a* $\mathbb{G}_1$*-equivariant Hilbert* A_1*-*B_1*-bimodule, then* $\mathcal{E}_2$ *is a* $\mathbb{G}_2$*-equivariant Hilbert* A_2*-*B_2*-bimodule called the induced* $\mathbb{G}_2$*-equivariant bimodule of* $\mathcal{E}_1$*.*

Proof. The fact that we have a well-defined induced $\mathbb{G}_2$-equivariant *-representation $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \gamma_1 : A_2 \to \mathcal{L}(\mathcal{E}_2)$ is just a restatement of [2, Proposition 4.3 (c)] and Proposition 2.5.14 (iii). Let us assume that $\mathcal{E}_1$ is countably generated as a Hilbert B_1-module; i.e., the C*-algebra $\mathcal{K}(\mathcal{E}_1)$ is σ-unital. Hence, $\mathcal{K}(\mathcal{E}_2)$ is σ-unital (cf. Proposition 2.5.14 (iii) and Lemma 2.5.16); i.e., $\mathcal{E}_2$ is a countably generated Hilbert B_2-module. ∎

By exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$, we define as above an induction procedure for $\mathbb{G}_2$-equivariant Hilbert modules.

In the following, we investigate the composition of $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}$ and $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}$. Let $j,k=1,2$. Let A_j be a $\mathbb{G}_j$-C*-algebra and $\mathcal{E}_j$ a $\mathbb{G}_j$-equivariant Hilbert A_j-module. Denote by

- $A_k := \mathrm{Ind}_{\mathbb{G}_j}^{\mathbb{G}_k}(A_j)$ and $\mathcal{E}_k = \mathrm{Ind}_{\mathbb{G}_j}^{\mathbb{G}_k}(\mathcal{E}_j) \subset \mathcal{L}(A_j \otimes S_{jk}, \mathcal{E}_j \otimes S_{jk})$ the induced $\mathbb{G}_k$-C*-algebra and the induced $\mathbb{G}_k$-equivariant Hilbert A_k-module,
- $C = \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k)$ and $\mathcal{F} := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{E}_k) \subset \mathcal{L}(A_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ the induced $\mathbb{G}_j$-C*-algebra and the induced $\mathbb{G}_j$-equivariant Hilbert C-module.

Proposition 2.5.18. *With the above notations and hypotheses, we have the following statements:*

(1) *there exists a unique map $\Pi_j : \mathcal{E}_j \to \mathcal{F}$ such that*

$$\big(\Pi_j(\xi)x\big)a = \delta_{\mathcal{E}_j}^{(k)}(\xi)(xa) \quad \text{for all } \xi \in \mathcal{E}_j,\ x \in A_k \otimes S_{kj},\ \text{and } a \in A_j \otimes S_{jk} \otimes S_{kj};$$

*moreover, Π_j is a $\mathbb{G}_j$-equivariant unitary equivalence over the $\mathbb{G}_j$-equivariant *-isomorphism $\pi_j : A_j \to C;\ a \mapsto \delta_{A_j}^{(k)}(a)$;*

(2) *$\delta_{\mathcal{E}_j}^{k} : \mathcal{E}_j \to \widetilde{\mathcal{M}}(\mathcal{E}_k \otimes S_{kj});\ \xi \mapsto \Pi_j(\xi)$ is a well-defined linear map such that*

(i) *$\delta_{\mathcal{E}_j}^{k}(\xi a) = \delta_{\mathcal{E}_j}^{k}(\xi)\delta_{A_j}^{k}(a)$ and $\langle \delta_{\mathcal{E}_j}^{k}(\xi), \delta_{\mathcal{E}_j}^{k}(\eta)\rangle = \delta_{A_j}^{k}(\langle \xi, \eta\rangle)$ for all $\xi, \eta \in \mathcal{E}_j$ and $a \in A_j$,*

(ii) *$[\delta_{\mathcal{E}_j}^{k}(\mathcal{E}_j)(1_{A_k} \otimes S_{kj})] = \mathcal{E}_2 \otimes S_{kj} = [(1_{\mathcal{E}_k} \otimes S_{kj})\delta_{\mathcal{E}_j}^{k}(\mathcal{E}_j)]$.*

Theorem 2.5.19. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent regular locally compact quantum groups. The map*

$$\begin{aligned}&\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} : (\mathcal{E}_1, \delta_{\mathcal{E}_1})\\ &\quad\mapsto \big(\mathcal{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1),\ \delta_{\mathcal{E}_2} : \xi \in \mathcal{E}_2 \mapsto \big[x \in A_2 \otimes S_{22} \mapsto (\mathrm{id}_{\mathcal{E}_1} \otimes \delta_{12}^2)(\xi)x\big]\big),\end{aligned}$$

where $\mathcal{E}_1$ is a Hilbert module over the $\mathbb{G}_1$-C-algebra A_1 and $A_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ denotes the induced $\mathbb{G}_2$-C*-algebra, is a one-to-one correspondence up to unitary equivalence. The inverse map, up to unitary equivalence, is*

$$\begin{aligned}&\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1} : (\mathcal{F}_2, \delta_{\mathcal{F}_2})\\ &\quad\mapsto \big(\mathcal{F}_1 := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathcal{F}_2),\ \delta_{\mathcal{F}_1} : \xi \in \mathcal{F}_1 \mapsto \big[x \in B_1 \otimes S_{11} \mapsto (\mathrm{id}_{\mathcal{F}_2} \otimes \delta_{21}^1)(\xi)x\big]\big),\end{aligned}$$

where $\mathcal{F}_2$ is a Hilbert module over the $\mathbb{G}_2$-C-algebra B_2 and $B_1 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_2)$ denotes the induced $\mathbb{G}_1$-C*-algebra.*

Let B_1 be a $\mathbb{G}_1$-C*-algebra. Let us denote by $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ the induced $\mathbb{G}_2$-C*-algebra. Let $\delta_{B_j}^{k} : B_j \to \mathcal{M}(B_k \otimes S_{kj})$, for $j,k=1,2$, be the *-homomorphisms defined in [9, Notation 5.2.7].

Notations 2.5.20. Let $\mathcal{F}_1$ be a $\mathbb{G}_1$-equivariant Hilbert B_1-module. Let us denote by $\mathcal{F}_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{F}_1)$ the induced $\mathbb{G}_2$-equivariant Hilbert B_2-module. We have four linear maps

$$\delta^k_{\mathcal{F}_j} : \mathcal{F}_j \to \mathcal{L}(B_k \otimes S_{kj}, \mathcal{F}_k \otimes S_{kj}) \quad \text{for } j,k = 1,2,$$

defined as follows:

- $\delta^1_{\mathcal{F}_1} := \delta_{\mathcal{F}_1}$ and $\delta^2_{\mathcal{F}_2} := \delta_{\mathcal{F}_2}$;
- $\delta^2_{\mathcal{F}_1} : \mathcal{F}_1 \to \mathcal{L}(B_2 \otimes S_{21}, \mathcal{F}_2 \otimes S_{21})$ is the unique linear map such that

$$\big(\delta^2_{\mathcal{F}_1}(\xi)x\big)b = \delta^{(2)}_{\mathcal{F}_1}(\xi)(xb)$$

  for all $\xi \in \mathcal{F}_1$, $x \in B_2 \otimes S_{21}$, and $b \in B_1 \otimes S_{12} \otimes S_{22}$, where $\delta^{(2)}_{\mathcal{F}_1}(\xi) := (\mathrm{id}_{\mathcal{E}_1} \otimes \delta^2_{11})\delta_{\mathcal{F}_1}(\xi)$ (cf. Proposition 2.5.18);
- $\delta^1_{\mathcal{F}_2} : \mathcal{F}_2 \to \mathcal{L}(B_1 \otimes S_{12}, \mathcal{F}_1 \otimes S_{12})$ is the unique linear map such that for all $\xi \in \mathcal{F}_2$, $x \in \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2) \otimes S_{12}$ and $y \in B_2 \otimes S_{21} \otimes S_{12}$, we have

$$\big[(\Pi_1 \otimes \mathrm{id}_{S_{12}})\big(\delta^1_{\mathcal{F}_2}(\xi)\big)x\big]y = \delta^{(1)}_{\mathcal{F}_2}(\xi)(xy),$$

  where $\delta^{(1)}_{\mathcal{F}_2}(\xi) := (\mathrm{id}_{\mathcal{F}_1} \otimes \delta^1_{22})\delta_{\mathcal{F}_2}(\xi)$ and $\Pi_1 : \mathcal{F}_1 \to \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathcal{F}_2)$ (cf. Proposition 2.5.18 (1)).

Lemma 2.5.21. *For all $j,k,l = 1,2$, we have the following statements:*

(1) $\delta^k_{\mathcal{F}_j}(\mathcal{F}_j) \subset \widetilde{\mathcal{M}}(\mathcal{F}_k \otimes S_{kj})$;

(2) $\delta^k_{\mathcal{F}_j}(\xi b) = \delta^k_{\mathcal{F}_j}(\xi)\delta^k_{B_j}(b)$ *and* $\langle \delta^k_{\mathcal{F}_j}(\xi), \delta^k_{\mathcal{F}_j}(\eta)\rangle = \delta^k_{B_j}(\langle \xi, \eta\rangle)$ *for all* $\xi, \eta \in \mathcal{F}_j$ *and* $b \in B_j$;

(3) $[\delta^k_{\mathcal{F}_j}(\mathcal{F}_j)(1_{B_k} \otimes S_{kj})] = \mathcal{F}_k \otimes S_{kj} = [(1_{\mathcal{F}_k} \otimes S_{kj})\delta^k_{\mathcal{F}_j}(\mathcal{F}_j)]$;

(4) $\delta^l_{\mathcal{F}_k} \otimes \mathrm{id}_{S_{kj}}$ *(resp.* $\mathrm{id}_{\mathcal{F}_l} \otimes \delta^k_{lj}$*) extends uniquely to a linear map from* $\mathcal{L}(B_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ *to* $\mathcal{L}(B_l \otimes S_{lk} \otimes S_{kj}, \mathcal{E}_l \otimes S_{lk} \otimes S_{kj})$ *such that*

$$(\delta^l_{\mathcal{F}_k} \otimes \mathrm{id}_{S_{kj}})(T)(\delta^l_{B_k} \otimes \mathrm{id}_{S_{kj}})(x) = (\delta^l_{\mathcal{F}_k} \otimes \mathrm{id}_{S_{kj}})(Tx)$$
$$\big(\textit{resp. } (\mathrm{id}_{\mathcal{F}_l} \otimes \delta^k_{lj})(T)(\mathrm{id}_{B_l} \otimes \delta^k_{lj})(x) = (\mathrm{id}_{\mathcal{F}_l} \otimes \delta^k_{lj})(Tx)\big)$$

for all $T \in \mathcal{L}(B_k \otimes S_{kj}, \mathcal{E}_k \otimes S_{kj})$ *and* $x \in B_k \otimes S_{kj}$;

(5) $(\delta^l_{\mathcal{F}_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathcal{F}_j} = (\mathrm{id}_{\mathcal{F}_l} \otimes \delta^k_{lj})\delta^l_{\mathcal{F}_j}$.

Let us consider the C*-algebra $B := B_1 \oplus B_2$ endowed with the continuous action (β_B, δ_B) (cf. [9, Proposition 5.2.9]).

Proposition 2.5.22. *Let $\mathcal{F}_1$ be a $\mathbb{G}_1$-equivariant Hilbert B_1-module. Let $\mathcal{F}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{F}_1)$ be the induced $\mathbb{G}_2$-equivariant Hilbert B_2-module. Consider the Hilbert B-module $\mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2$. Denote by $\Pi^k_j : \mathcal{L}(B_k \otimes S_{kj}, \mathcal{F}_k \otimes S_{kj}) \to \mathcal{L}(B \otimes S, \mathcal{F} \otimes S)$ the linear extension of the canonical injection $\mathcal{F}_k \otimes S_{kj} \to \mathcal{F} \otimes S$. Let us consider the linear maps*

$\delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{L}(B \otimes S, \mathcal{F} \otimes S)$ and $\beta_{\mathcal{F}} : \mathbb{C}^2 \to \mathcal{L}(\mathcal{F})$ defined by

$$\delta_{\mathcal{F}}(\xi) := \sum_{k,j=1,2} \Pi_j^k \circ \delta_{\mathcal{F}_j}^k(\xi_j), \quad \xi = (\xi_1, \xi_2) \in \mathcal{F};$$

$$\beta_{\mathcal{F}}(\lambda, \mu) := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad (\lambda, \mu) \in \mathbb{C}^2.$$

Then, the triple $(\mathcal{F}, \beta_{\mathcal{F}}, \delta_{\mathcal{F}})$ is a $\mathcal{G}$-equivariant Hilbert B-module.

Proposition 2.5.23. *Let $(\mathcal{E}, \beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ be a $\mathcal{G}$-equivariant Hilbert A-module. In the following, we use the notations of Proposition-Definition 2.5.5. Let $j, k = 1, 2$ with $j \neq k$. Let*

$$\widetilde{A}_j := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k, \delta_{A_k}^k) \quad \text{and} \quad \widetilde{\mathcal{E}}_j := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{E}_k, \delta_{\mathcal{E}_k}^k).$$

If $\xi \in \mathcal{E}_j$, then we have $\delta_{\mathcal{E}_j}^k(\xi) \in \widetilde{\mathcal{E}}_j \subset \widetilde{\mathcal{M}}(\mathcal{E}_k \otimes S_{kj})$ and the map $\widetilde{\Pi}_j : \mathcal{E}_j \to \widetilde{\mathcal{E}}_j ; \xi \mapsto \delta_{\mathcal{E}_j}^k(\xi)$ is a $\mathbb{G}_j$-equivariant unitary equivalence over $\widetilde{\pi}_j : A_j \to \widetilde{A}_j$ (cf. [9, Proposition 5.2.8]*).*

Theorem 2.5.24. *Let $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid between two regular monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let $j = 1, 2$. The map $(\mathcal{E}, \beta_{\mathcal{E}}, \delta_{\mathcal{E}}) \mapsto (\mathcal{E}_j, \delta_{\mathcal{E}_j}^j)$ is a one-to-one correspondence up to unitary equivalence between $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$-equivariant modules and $\mathbb{G}_j$-equivariant modules (cf. Proposition-Definition 2.5.5 and Lemma 2.5.6* (1)*). The inverse map, up to unitary equivalence, is $(\mathcal{F}_j, \delta_{\mathcal{F}_j}) \mapsto (\mathcal{F}, \beta_{\mathcal{F}}, \delta_{\mathcal{F}})$ (cf. Proposition 2.5.22, Proposition-Definition 2.5.15, and Lemma 2.5.6* (2)*).*

Proposition 2.5.25. *We follow the hypotheses and notations of Proposition 2.5.22. Let $\gamma_1 : A_1 \to \mathcal{L}(\mathcal{E}_1)$ be a $\mathbb{G}_1$-equivariant *-representation of a $\mathbb{G}_1$-C*-algebra A_1. Let A_2 be the induced $\mathbb{G}_2$-C*-algebra of A_1 and let $\gamma_2 : A_2 \to \mathcal{L}(\mathcal{F}_2)$ be the induced $\mathbb{G}_2$-equivariant *-representation of γ_1 (cf. Proposition-Definition 2.5.17). Let us endow the C*-algebra $A := A_1 \oplus A_2$ with the continuous action (β_A, δ_A) (cf.* [9, Proposition 5.2.9]*). The map*

$$\gamma : A \to \mathcal{L}(\mathcal{F}); \quad (a_1, a_2) \mapsto \begin{pmatrix} \gamma_1(a_1) & 0 \\ 0 & \gamma_2(a_2) \end{pmatrix}$$

is a $\mathcal{G}$-equivariant *-representation. Moreover, if $\mathcal{F}_1$ is a $\mathbb{G}_1$-equivariant Hilbert A_1-B_1-bimodule, then $\mathcal{F}$ is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule.

Proof. This is a straightforward consequence of Lemma 2.5.7 (2) and Proposition-Definition 2.5.17. ∎

2.5.3. Structure of the double crossed product. For further usage, let us introduce a writing convention concerning bimodule structures.

Convention 2.5.26. Let A and B be two $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebras. When dealing with a A-B-bimodule structure and in order to avoid any confusion between similar objects associated with A and B, we will sometimes specify those associated with A (resp. B) by using the lower index "g" (resp. "d")[1]. For example, we will denote by D_{g} and D_{d} (resp. E_{g} and E_{d}) the bidual $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra of A and B, respectively (cf. Notations 2.3.4).

In this paragraph, we assume the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ to be regular. We recall that the colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ is regular.

Let A, B be $\mathcal{G}$-C*-algebras and $\mathcal{E}$ a $\mathcal{G}$-equivariant Hilbert A-B-module. In this paragraph, we restate the main results of [2, §4.4] in order to describe the structure of the double crossed product $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$. Let D_{g} (resp. D_{d}) be the bidual $\mathcal{G}$-C*-algebra of A (resp. B) (cf. Notations 2.3.4 and Convention 2.5.26). Let $\mathcal{D}$ be the bidual $\mathcal{G}$-equivariant Hilbert D_{g}-D_{d}-module of $\mathcal{E}$ (cf. Section 2.4.2). In the following, we use all the notations and results of [9, Notation 5.2.1] and Section 2.5.1.

Let us recall the notations of [2, Notations 3.48] concerning the structure of the bidual $\mathcal{G}$-C*-algebra D_{d} (and similarly for D_{g}).

- We have $D_{\mathrm{d}} = q_{\beta_B\widehat{\alpha}}(B \otimes \mathcal{K})q_{\beta_B\widehat{\alpha}} = \oplus_{j=1,2} B_j \otimes \mathcal{K}(\mathcal{H}_{1j} \oplus \mathcal{H}_{2j})$. For all $j = 1,2$, we will identify $D_{j,\mathrm{d}} = B_j \otimes \mathcal{K}(\mathcal{H}_{1j} \oplus \mathcal{H}_{2j})$. Let $\mathcal{B}_{ll',j,\mathrm{d}} := B_j \otimes \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj})$ for $l, l', j = 1,2$. Let $\mathcal{B}_{l,j,\mathrm{d}} := \mathcal{B}_{ll,j,\mathrm{d}} = B_j \otimes \mathcal{K}(\mathcal{H}_{lj})$ for $l, j = 1,2$. For $l, l', j = 1,2$, $\mathcal{B}_{l,j,\mathrm{d}}$ and $\mathcal{B}_{l',j,\mathrm{d}}$ are C*-algebras and $\mathcal{B}_{ll',j,\mathrm{d}}$ turns into a Hilbert $\mathcal{B}_{l,j,\mathrm{d}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule.
- For $l, l', j, k = 1,2$, let
$$\delta^k_{\mathcal{B}_{ll',j,\mathrm{d}},0} : \mathcal{B}_{ll',j,\mathrm{d}} \to \mathcal{L}\big(B_k \otimes \mathcal{K}(\mathcal{H}_{l'j}) \otimes S_{kj}, B_k \otimes \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj}) \otimes S_{kj}\big)$$
be the linear map defined by
$$\delta^k_{\mathcal{B}_{ll',j,\mathrm{d}},0}(b \otimes T) := \delta^k_{B_j}(b)_{13}(1_{B_k} \otimes T \otimes 1_{S_{kj}}) \quad \text{for all } b \in B_j \text{ and } T \in \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj}).$$
- For $l, l', j, k = 1,2$, let $\delta^k_{\mathcal{B}_{ll',j,\mathrm{d}}} : \mathcal{B}_{ll',j,\mathrm{d}} \to \mathcal{L}(\mathcal{B}_{l',k,\mathrm{d}} \otimes S_{kj}, \mathcal{B}_{ll',k,\mathrm{d}} \otimes S_{kj})$ be the linear map defined by
$$\delta^k_{\mathcal{B}_{ll',j,\mathrm{d}}}(x) := (V^l_{kj})_{23}\delta^k_{\mathcal{B}_{ll',j,\mathrm{d}},0}(x)(V^{l'}_{kj})^*_{23} \quad \text{for all } x \in \mathcal{B}_{ll',j,\mathrm{d}}.$$
Note that $\delta^k_{\mathcal{B}_{l,j,\mathrm{d}}} : \mathcal{B}_{l,j,\mathrm{d}} \to \mathcal{L}(\mathcal{B}_{l,k,\mathrm{d}} \otimes S_{kj})$ is a *-homomorphism.

We identify $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} = \mathcal{D}$ (cf. Theorem 2.4.23). We have $\mathcal{D} = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(\mathcal{E} \otimes \mathcal{K})q_{\beta_B\widehat{\alpha}} \subset \mathcal{E} \otimes \mathcal{K}$. In the following, we investigate the precise structure of $\mathcal{D}$.

- We have $q_{\mathcal{D},j} := \beta_{\mathcal{D}}(\varepsilon_j) = q_{\beta_{\mathcal{E}}\widehat{\alpha}}(1_{\mathcal{E}} \otimes \beta(\varepsilon_j)) = \beta_{\mathcal{E}}(\varepsilon_j) \otimes \beta(\varepsilon_j) = \sum_{l=1,2} q_{\mathcal{E},j} \otimes p_{lj}$ for all $j = 1,2$ (cf. [2, Lemme 2.21 and Notations 2.22]).
- For $j = 1,2$, let us consider the Hilbert $D_{j,\mathrm{g}}$-$D_{j,\mathrm{d}}$-module $\mathcal{D}_j := q_{\mathcal{D},j}\mathcal{D}$ (cf. Section 2.5.1). We have
$$\mathcal{D}_j = \mathcal{E}_j \otimes \mathcal{K}(\mathcal{H}_{1j} \oplus \mathcal{H}_{2j}) = \bigoplus_{l,l'=1,2} \mathcal{E}_j \otimes \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj}).$$

[1] The letter "g" (resp. "d") stands for "*gauche*" (resp. "*droite*"), the french word for "left" (resp. "right"). This choice was not motivated by chauvinism but only by the fact that the letters "l" and "r" would have been confusing.

For $l, l', j = 1, 2$, let us consider the Hilbert $\mathcal{B}_{l,j,\mathrm{g}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule $\mathcal{E}_{ll',j} := \mathcal{E}_j \otimes \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj})$. For $l, j = 1, 2$, let $\mathcal{E}_{l,j} := \mathcal{E}_{ll,j} = \mathcal{E}_j \otimes \mathcal{K}(\mathcal{H}_{lj})$.

- For $j, k = 1, 2$, let us denote by $\Pi_j^k : \mathcal{L}(D_{k,\mathrm{d}} \otimes S_{kj}, \mathcal{D}_k \otimes S_{kj}) \to \mathcal{L}(D_\mathrm{d} \otimes S, \mathcal{D} \otimes S)$ the linear extension of the inclusion map $\mathcal{D}_k \otimes S_{kj} \to \mathcal{D} \otimes S$. For $j, k = 1, 2$, let us denote by $\delta_{\mathcal{D}_j}^k : \mathcal{D}_j \to \mathcal{L}(D_{k,\mathrm{d}} \otimes S_{kj}, \mathcal{D}_k \otimes S_{kj})$ the linear map defined in Proposition-Definition 2.5.5. For all $\zeta \in \mathcal{D}$, we have

$$\delta_{\mathcal{D}}(\zeta) = \sum_{j,k=1,2} \Pi_j^k \circ \delta_{\mathcal{D}_j}^k (q_{\mathcal{D},j} \zeta).$$

We recall that for $j = 1, 2$ the pair $(\mathcal{D}_j, \delta_{\mathcal{D}_j}^j)$ is a $\mathbb{G}_j$-equivariant Hilbert $D_{j,\mathrm{g}}$-$D_{j,\mathrm{d}}$-bimodule.

- For $l, l', j, k = 1, 2$, let

$$\delta_{\mathcal{E}_{ll',j},0}^k : \mathcal{E}_{ll',j} \to \mathcal{L}(B_k \otimes \mathcal{K}(\mathcal{H}_{l'j}) \otimes S_{kj}, \mathcal{E}_k \otimes \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj}) \otimes S_{kj})$$

be the linear map defined by

$$\delta_{\mathcal{E}_{ll',j},0}^k (\xi \otimes T) := \delta_{\mathcal{E}_j}^k (\xi)_{13} (1_{B_k} \otimes T \otimes 1_{S_{kj}}) \quad \text{for all } \xi \in \mathcal{E}_j \text{ and } T \in \mathcal{K}(\mathcal{H}_{l'j}, \mathcal{H}_{lj}).$$

- Let $\delta_{\mathcal{E}_{ll',j}}^k : \mathcal{E}_{ll',j} \to \mathcal{L}(\mathcal{B}_{l',k,\mathrm{d}} \otimes S_{kj}, \mathcal{E}_{ll',k} \otimes S_{kj})$ be the linear defined by

$$\delta_{\mathcal{E}_{ll',j}}^k (\zeta) := (V_{kj}^l)_{23} \delta_{\mathcal{E}_{ll',j},0}^k (\zeta) (V_{kj}^{l'})_{23}^* \quad \text{for all } \zeta \in \mathcal{E}_{ll',j}.$$

- Let $j, k, l, l' = 1, 2$. We denote by

$$\Pi_{ll',j}^k : \mathcal{L}(\mathcal{B}_{l',k,\mathrm{d}} \otimes S_{kj}, \mathcal{E}_{ll',k}) \to \mathcal{L}(D_{k,\mathrm{d}} \otimes S_{kj}, \mathcal{D}_k \otimes S_{kj})$$

the linear extension of the inclusion map $\mathcal{E}_{ll',k} \otimes S_{kj} \to \mathcal{D}_k \otimes S_{kj}$. For $\zeta \in \mathcal{D}_j$, let us denote by $\zeta_{ll'}$ the element of $\mathcal{E}_{ll',j}$ defined by $\zeta_{ll'} := (q_{\mathcal{E},j} \otimes p_{lj}) \zeta (q_{B,j} \otimes p_{l'j})$. For all $j, k = 1, 2$, we have

$$\delta_{\mathcal{D}_j}^k (\zeta) = \sum_{l,l'=1,2} \Pi_{ll',j}^k \circ \delta_{\mathcal{E}_{ll',j}}^k (\zeta_{ll'}).$$

For all $l, l', j = 1, 2$, the pair $(\mathcal{E}_{ll',j}, \delta_{\mathcal{E}_{ll',j}}^j)$ is a $\mathbb{G}_j$-equivariant Hilbert $\mathcal{B}_{l,j,\mathrm{g}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule.

3. Equivariant Kasparov theory

In this chapter, we fix a regular measured quantum groupoid $\mathcal{G}$ on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ endowed with the non-normalized Markov trace. We use all the notations introduced in [9, §3.1 and §3.2] concerning the objects associated with $\mathcal{G}$.

3.1. Equivariant Kasparov groups

Let us recall a definition.

Definition 3.1.1 (cf. [18]). Let A and B be C*-algebras. A Kasparov A-B-bimodule is a triple $(\mathcal{E}, \gamma, F)$ consisting of a countably generated Hilbert B-module $\mathcal{E}$, a *-homomorphism $\gamma : A \to \mathcal{L}(\mathcal{E})$, and an operator $F \in \mathcal{L}(\mathcal{E})$ such that

$$\big[\gamma(a), F\big] \in \mathcal{K}(\mathcal{E}), \quad \gamma(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E}), \quad \gamma(a)(F - F^*) \in \mathcal{K}(\mathcal{E}) \quad \text{for all } a \in A. \tag{3.1}$$

If we have

$$\big[\gamma(a), F\big] = \gamma(a)(F^2 - 1) = \gamma(a)(F - F^*) = 0 \quad \text{for all } a \in A, \tag{3.2}$$

then the Kasparov A-B-bimodule $(\mathcal{E}, \gamma, F)$ is said to be degenerate.

Let A and B be $\mathcal{G}$-C*-algebras.

Definition 3.1.2. A $\mathcal{G}$-equivariant Kasparov A-B-bimodule is a triple $(\mathcal{E}, \gamma, F)$, consisting of $\mathcal{G}$-equivariant A-B-bimodule $(\mathcal{E}, \gamma)$ (cf. Definition 2.2.7) and an operator $F \in \mathcal{L}(\mathcal{E})$ such that

(1) the triple $(\mathcal{E}, \gamma, F)$ is a Kasparov A-B-bimodule,

(2) $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$,

(3) $(\gamma \otimes \mathrm{id}_S)(x)(\delta_{\mathcal{K}(\mathcal{E})}(F) - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)) \in \mathcal{K}(\mathcal{E} \otimes S)$ for all $x \in A \otimes S$.

The $\mathcal{G}$-equivariant Kasparov A-B-bimodule will sometimes be simply denoted by $(\mathcal{E}, F)$ when the representation γ is clear from the context.

Remarks 3.1.3. Let us make some comments concerning the previous definition.

(1) Since $\beta_{\mathcal{E}}(n^{\mathrm{o}})\gamma(a) = \gamma(\beta_A(n^{\mathrm{o}})a)$ for all $a \in A$ and $n \in N$, we have

$$(\gamma \otimes \mathrm{id}_S)(x q_{\beta_A \alpha}) = (\gamma \otimes \mathrm{id}_S)(x) q_{\beta_{\mathcal{E}}\alpha} = (\gamma \otimes \mathrm{id}_S)(x)\mathcal{V}\mathcal{V}^* \quad \text{for all } x \in A \otimes S,$$

where $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_B} (B \otimes S), \mathcal{E} \otimes S)$ is the isometry defined in Proposition 2.1.5 (a). The following statements are then equivalent to condition (3):

- $(\gamma \otimes \mathrm{id}_S)(x)(\delta_{\mathcal{K}(\mathcal{E})}(F) - F \otimes 1_S) \in \mathcal{K}(\mathcal{E} \otimes S)$ for all $x \in (A \otimes S)q_{\beta_A \alpha}$;
- $(\gamma \otimes \mathrm{id}_S)(x)(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)) \in \mathcal{K}(\mathcal{E} \otimes S)$ for all $x \in A \otimes S$;
- $(\gamma \otimes \mathrm{id}_S)(x)(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - F \otimes 1_S) \in \mathcal{K}(\mathcal{E} \otimes S)$ for all $x \in (A \otimes S)q_{\beta_A \alpha}$.

Note that it follows from condition (2) that $[F \otimes 1_S, q_{\beta_{\mathcal{E}}\alpha}] = 0$.

(2) As in [3, Remarque 3.4 (2)], we prove that

$$\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - F \otimes 1_S\big)(\gamma \otimes \mathrm{id}_S)(x) \in \mathcal{K}(\mathcal{E} \otimes S) \quad \text{for all } x \in q_{\beta_A \alpha}(A \otimes S).$$

Since $(\gamma \otimes \mathrm{id}_S)(q_{\beta_A \alpha} x) = q_{\beta_{\mathcal{E}}\alpha}(\gamma \otimes \mathrm{id}_S)(x)$ for all $x \in A \otimes S$ and $[F \otimes 1_S, q_{\beta_{\mathcal{E}}\alpha}] = 0$, this is also equivalent to

$$\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)\big)(\gamma \otimes \mathrm{id}_S)(x) \in \mathcal{K}(\mathcal{E} \otimes S) \quad \text{for all } x \in A \otimes S.$$

Note that the converse is also true; i.e., condition (3) is equivalent to these assertions.

(3) If F is invariant (cf. Proposition-Definition 2.1.13), then conditions (2) and (3) of Definition 3.1.2 are satisfied.

Definition 3.1.4. (1) Two $\mathcal{G}$-equivariant Kasparov A-B-bimodules $(\mathcal{E}_1, \gamma_1, F_1)$ and $(\mathcal{E}_2, \gamma_2, F_2)$ are said to be unitarily equivalent if there exists a unitary $\Phi \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ such that

(i) Φ is $\mathcal{G}$-equivariant (cf. Definition 2.1.14),

(ii) $\Phi \circ \gamma_1(a) = \gamma_2(a) \circ \Phi$ for all $a \in A$,

(iii) $F_2 \circ \Phi = \Phi \circ F_1$.

(2) Let $\mathsf{E}_{\mathcal{G}}(A, B)$ be the set of unitary equivalence classes of $\mathcal{G}$-equivariant Kasparov A-B-bimodules.

(3) A $\mathcal{G}$-equivariant Kasparov A-B-bimodule $(\mathcal{E}, \gamma, F)$ is said to be degenerate, if $(\mathcal{E}, \gamma, F)$ is a degenerate Kasparov A-B-bimodule such that

$$(\gamma \otimes \mathrm{id}_S)(x)\big(\delta_{\mathcal{K}(\mathcal{E})}(F) - (F \otimes 1_S)q_{\beta_{\mathcal{E}}\alpha}\big) = 0 \quad \text{for all } x \in A \otimes S.$$

We denote by $\mathsf{D}_{\mathcal{G}}(A, B) \subset \mathsf{E}_{\mathcal{G}}(A, B)$ the set of unitary equivalence classes of degenerate $\mathcal{G}$-equivariant Kasparov A-B-bimodules.

(4) Let us consider the C*-algebra $B[0,1] := \mathrm{C}([0,1], B)$ of the B-valued norm continuous functions on $[0,1]$. We make the identification $B[0,1] = \mathrm{C}([0,1]) \otimes B$ in the following way:

$$(f \otimes b)(t) := f(t)b \quad \text{for all } f \in \mathrm{C}([0,1]),\ b \in B,\ \text{and } t \in [0,1].$$

Similarly, we make the identification $B[0,1] \otimes S = (B \otimes S)[0,1]$. In particular, $\mathcal{M}(B[0,1] \otimes S)$ (resp. $\mathcal{M}(B[0,1])$) consists of the *strictly* continuous $\mathcal{M}(B \otimes S)$-valued (resp. $\mathcal{M}(B)$-valued) functions on $[0,1]$. We then warn the reader that in our context we will abusively denote $\mathcal{M}(B[0,1] \otimes S)$ (resp. $\mathcal{M}(B[0,1])$) by $\mathcal{M}(B \otimes S)[0,1]$ (resp. $\mathcal{M}(B)[0,1]$). Let $\delta_{B[0,1]} : B[0,1] \to \mathcal{M}(B[0,1] \otimes S)$ and $\beta_{B[0,1]} : N^{\mathrm{o}} \to \mathcal{M}(B[0,1])$ be the maps defined by $\delta_{B[0,1]}(f)(t) := \delta_B(f(t))$ and $[\beta_{B[0,1]}(n^{\mathrm{o}})f](t) = \beta_B(n^{\mathrm{o}})f(t)$ for all $f \in B[0,1]$, $n \in N$, and $t \in [0,1]$. Then, the pair $(\beta_{B[0,1]}, \delta_{B[0,1]})$ is a continuous action of $\mathcal{G}$ on $B[0,1]$. For $t \in [0,1]$, let $\mathrm{e}_t : B[0,1] \to B$ be the evaluation at point t, i.e., the surjective *-homomorphism defined for all $f \in B[0,1]$ by $\mathrm{e}_t(f) := f(t)$. Note that e_t is $\mathcal{G}$-equivariant by definition of the action of $\mathcal{G}$ on $B[0,1]$.

(5) Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant Hilbert B-module. Let us consider the Hilbert $B[0,1]$-module $\mathcal{E}[0,1] := \mathrm{C}([0,1], \mathcal{E})$ of $\mathcal{E}$-valued continuous functions on $[0,1]$. We make the identification $\mathcal{E}[0,1] = \mathrm{C}[0,1] \otimes \mathcal{E}$ as above. We equip the Hilbert B-module $\mathcal{E}[0,1]$ with the action of $\mathcal{G}$ obtained by transport of structure through the identification

$$\mathcal{K}\big(\mathcal{E}[0,1] \oplus B[0,1]\big) = \mathcal{K}(\mathcal{E} \oplus B)[0,1].$$

We have $\beta_{\mathcal{E}[0,1]} = \beta_{\mathcal{K}(\mathcal{E})[0,1]}$ up to the identification $\mathcal{L}(\mathcal{E}[0,1]) = \mathcal{M}(\mathcal{K}(\mathcal{E})[0,1])$. For all $x \in B[0,1] \otimes S$ and $\xi \in \mathcal{E}[0,1] \otimes S$, we have $(\delta_{\mathcal{E}[0,1]}(\xi)x)(t) = \delta_{\mathcal{E}}(\xi(t))x(t)$ up to the identifications $B[0,1] \otimes S = (B \otimes S)[0,1]$ and $\mathcal{E}[0,1] \otimes S = (\mathcal{E} \otimes S)[0,1]$.

Proposition 3.1.5. *Let A_1, A_2, A, B_1, B_2, and B be $\mathcal{G}$-C*-algebras.*

(1) *Let $f : A_1 \to A_2$ be a $\mathcal{G}$-equivariant *-homomorphism. Let $(\mathcal{E}, \gamma, F)$ be a $\mathcal{G}$-equivariant Kasparov A_2-B-bimodule. Let $\gamma^* := \gamma \circ f : A_1 \to \mathcal{L}(\mathcal{E})$. Then the triple $(\mathcal{E}, \gamma^*, F)$ is a $\mathcal{G}$-equivariant Kasparov A_1-B-bimodule. Moreover, we have the well-defined map*

$$f^* : \mathsf{E}_{\mathcal{G}}(A_2, B) \to \mathsf{E}_{\mathcal{G}}(A_1, B); \quad (\mathcal{E}, \gamma, F) \mapsto (\mathcal{E}, \gamma^*, F).$$

(2) *Let $g : B_1 \to B_2$ be a $\mathcal{G}$-equivariant *-homomorphism. Let $(\mathcal{E}, \gamma, F)$ be a $\mathcal{G}$-equivariant Kasparov A-B_1-bimodule. Let $\gamma_* : A \to \mathcal{L}(\mathcal{E} \otimes_g B_2)$ be the *-representation defined by $\gamma_*(a) := \gamma(a) \otimes_g 1_{B_2}$ for all $a \in A$. Then the triple $(\mathcal{E} \otimes_g B_2, \gamma_*, F \otimes_g 1_{B_2})$ is a $\mathcal{G}$-equivariant Kasparov A-B_1-bimodule. Moreover, we have the following well-defined map*

$$g_* : \mathsf{E}_{\mathcal{G}}(A, B_1) \to \mathsf{E}_{\mathcal{G}}(A, B_2); \quad (\mathcal{E}, \gamma, F) \mapsto (\mathcal{E} \otimes_g B_2, \gamma_*, F \otimes_g 1_{B_2}).$$

Proof. Straightforward verifications. ∎

Definition 3.1.6. Let $(\mathcal{E}_0, F_0), (\mathcal{E}_1, F_1) \in \mathsf{E}_{\mathcal{G}}(A, B)$. A homotopy between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ is an element $x \in \mathsf{E}_{\mathcal{G}}(A, B[0, 1])$ such that $\mathrm{e}_{0*}(x) = (\mathcal{E}_0, F_0)$ and $\mathrm{e}_{1*}(x) = (\mathcal{E}_1, F_1)$. In that case, we say that $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ are homotopic. The homotopy relation is an equivalence relation on the class $\mathsf{E}_{\mathcal{G}}(A, B)$. We denote by $\mathsf{KK}_{\mathcal{G}}(A, B)$ the quotient set of $\mathsf{E}_{\mathcal{G}}(A, B)$ by the homotopy relation. We also denote by $[(\mathcal{E}, F)]$ the class of $(\mathcal{E}, F) \in \mathsf{E}_{\mathcal{G}}(A, B)$ in $\mathsf{KK}_{\mathcal{G}}(A, B)$.

Examples 3.1.7 (cf. [3, 18, 26]). We can build homotopies of $\mathsf{E}_{\mathcal{G}}(A, B[0, 1])$ in the following ways.

(1) An operator homotopy is a triple $(\mathcal{E}, \gamma, (F_t)_{t\in[0,1]})$ consisting of a $\mathcal{G}$-equivariant Hilbert A-B-bimodule $(\mathcal{E}, \gamma)$ and a family of adjointable operators $(F_t)_{t\in[0,1]}$ on $\mathcal{E}$ such that

- the triple $(\mathcal{E}, \gamma, F_t)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule for all $t \in [0, 1]$,
- the map $(t \mapsto F_t)$ is norm continuous.

The family of operators $(F_t)_{t\in[0,1]}$ defines an element F of $\mathcal{L}(\mathcal{E}[0, 1])$ (up to the identification $\mathcal{L}(\mathcal{E})[0, 1] = \mathcal{L}(\mathcal{E}[0, 1])$; cf. Definition 3.1.4 (4), (5)) and the triple $(\mathcal{E}[0, 1], \gamma \otimes 1, F)$ is a homotopy between $(\mathcal{E}, \gamma, F_0)$ and $(\mathcal{E}, \gamma, F_1)$.

(2) An important example of operator homotopy can be obtained in the following case. Let $(\mathcal{E}, \gamma, F)$ be a $\mathcal{G}$-equivariant Kasparov A-B-bimodule. We call an operator $G \in \mathcal{L}(\mathcal{E})$ a compact perturbation of F if for all $a \in A$ we have $\gamma(a)(F - G) \in \mathcal{K}(\mathcal{E})$ and $(F - G)\gamma(a) \in \mathcal{K}(\mathcal{E})$. In that case, the triple $(\mathcal{E}, \gamma, G)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule. Moreover, the triples $(\mathcal{E}, \gamma, F)$ and $(\mathcal{E}, \gamma, G)$ are operator homotopic via the obvious continuous path defined by $F_t := (1 - t)F + tG$ for $t \in [0, 1]$.

(3) Let $(\mathcal{E}, (\gamma_t)_{t\in[0,1]}, F)$ be a triple where $\mathcal{E}$ is a $\mathcal{G}$-equivariant Hilbert B-module, $(\gamma_t)_{t\in[0,1]}$ is a family of $\mathcal{G}$-equivariant *-representations of A on $\mathcal{E}$ and $F \in \mathcal{L}(\mathcal{E})$ such that the triple $(\mathcal{E}, \gamma_t, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule for all $t \in [0, 1]$ and the map $(t \mapsto \gamma_t(a))$ is norm continuous for all $a \in A$. Up to the identification $\mathcal{L}(\mathcal{E})[0, 1] = \mathcal{L}(\mathcal{E}[0, 1])$, the family $(\gamma_t)_{t\in[0,1]}$ defines a $\mathcal{G}$-equivariant *-representation

$\gamma : A \to \mathcal{L}(\mathcal{E}[0,1])$. Moreover, the triple $(\mathcal{E}[0,1], \gamma, 1 \otimes F)$ is a homotopy between $(\mathcal{E}, \gamma_0, F)$ and $(\mathcal{E}, \gamma_1, F)$.

As for actions of quantum groups (cf. [3, Proposition 3.3]), we have the following proposition.

Proposition 3.1.8. *Endowed with the binary operation induced by the direct sum operation* $([(\mathcal{E}_1, F_1)], [(\mathcal{E}_2, F_2)]) \mapsto [(\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2)]$*, the quotient set* $\mathsf{KK}_{\mathcal{G}}(A, B)$ *is an abelian group and the class of the identity element* $0 \in \mathsf{KK}_{\mathcal{G}}(A, B)$ *is represented by any element of* $\mathsf{D}_{\mathcal{G}}(A, B)$.

It follows from Definition 3.1.6 that the maps defined in Proposition 3.1.5 factorize over the quotient maps so that we obtain homomorphisms of abelian groups $f^* : \mathsf{KK}_{\mathcal{G}}(A_2, B) \to \mathsf{KK}_{\mathcal{G}}(A_1, B)$ and $g_* : \mathsf{KK}_{\mathcal{G}}(A, B_1) \to \mathsf{KK}_{\mathcal{G}}(A, B_2)$.

3.2. Kasparov's technical theorem

Notation 3.2.1. Let A be a C*-algebra. We denote by

$$\mathrm{Der}(A) := \{d \in \mathcal{B}(A);\ \forall x, y \in A,\ d(xy) = d(x)y + xd(y)\}$$

the Banach subspace of $\mathcal{B}(A)$ consisting of the continuous derivations of A. Any $d \in \mathrm{Der}(A)$ extends uniquely to a strictly continuous linear map $d : \mathcal{M}(A) \to \mathcal{M}(A)$ defined by

$$d(m)a := d(ma) - md(a) \quad \text{and} \quad ad(m) := d(am) - d(a)m \quad \text{for all } m \in \mathcal{M}(A),\ a \in A.$$

Note that $d(1) = 0$ and $d \in \mathrm{Der}(\mathcal{M}(A))$. For all $x \in A$, we have $\mathrm{Ad}(x) \in \mathrm{Der}(A)$ (inner derivations of A).

Remarks 3.2.2. (1) If $m \in \mathcal{M}(A)$, we define an inner derivation $\mathrm{Ad}(m) \in \mathrm{Der}(A)$ by setting $\mathrm{Ad}(m)(x) := [m, x] = mx - xm \in A$ for all $x \in A$.

(2) Let J be a closed two-sided ideal of A and $d \in \mathrm{Der}(A)$. We have $d(J) \subset J$. Indeed, let (u_λ) be an approximate unit of J and $x \in J$. We have $d(x) = \lim_\lambda d(xu_\lambda)$ with respect to the norm topology since d is norm continuous. Moreover, for all λ we have $d(xu_\lambda) = d(x)u_\lambda + xd(u_\lambda) \in J$ since J is an ideal of A. Hence, $d(x) \in J$. In particular, d induces an element of $\mathrm{Der}(J)$ by restriction and we have a continuous linear map $\mathrm{Der}(A) \to \mathrm{Der}(J)$.

Lemma 3.2.3 (cf. [3, Lemme 4.1]). *Let* J *be a* C*-*algebra and* (β_J, δ_J) *an action of* $\mathcal{G}$ *on* J*. For all* $h \in J$, $k \in S$, $z \in J \otimes S$*, for all compact* $K \subset \mathrm{Der}(J)$*, and for all real number* $\varepsilon > 0$*, there exists* $u \in J$*, such that* $0 \leqslant u \leqslant 1$ *and* $[u, \beta_J(n^{\mathrm{o}})] = 0$ *for all* $n \in N$*, which further satisfies*

(a) $\|uh - h\| \leqslant \varepsilon$,

(b) *for all* $d \in K$, $\|d(u)\| < \varepsilon$,

(c) $\|(\delta_J(u) - q_{\beta_J\alpha}(u \otimes 1))(1 \otimes k)\| < \varepsilon$,

(d) $\|\delta_J(u)z - q_{\beta_J\alpha}z\| < \varepsilon$.

Proof. Denote by $J(K)$ the C*-algebra consisting of the J-valued continuous functions on K. We adapt the proof of Lemme 4.1 [3] by considering the affine map

$$\Phi : J \to A := J \oplus J(K) \oplus (J \otimes S) \oplus (J \otimes S)$$
$$x \mapsto \Phi(x) := \big(xh - h, [d \mapsto d(x)], \big(\delta_J(x) - q_{\beta_J\alpha}(x \otimes 1)\big)(1 \otimes k), \delta_J(x)z - q_{\beta_J\alpha}z\big),$$

which admits a unique strictly continuous extension, still denoted by $\Phi : \mathcal{M}(J) \to \mathcal{M}(A)$, such that $\Phi(1) = 0$. By applying the Hahn–Banach theorem, we then conclude as in [3] that 0 is an adherent point of $\Phi(C)$, where C is the nonempty closed convex subset $\{u \in J;\ 0 \leqslant u \leqslant 1,\ \forall n \in N,\ [u, \beta_J(n^{\mathrm{o}})] = 0\}$ of J, which is just a restatement of the above lemma. ∎

If J is a closed two-sided ideal of A, we recall that $\mathcal{M}(A; J)$ denotes the strictly closed C*-subalgebra of $\mathcal{M}(A)$ consisting of the elements $m \in \mathcal{M}(A)$ such that the relations $mA \subset J$ and $Am \subset J$ hold. Note that the restriction homomorphism from $\mathcal{M}(A)$ to $\mathcal{M}(J)$ identifies $\mathcal{M}(A; J)$ to a C*-subalgebra of $\mathcal{M}(J)$ and if B is another C*-algebra, we have $\widetilde{\mathcal{M}}(A \otimes B) = \mathcal{M}(\widetilde{A} \otimes B; A \otimes B)$.

Definition 3.2.4. Let A be a C*-algebra endowed with an action (β_A, δ_A) of $\mathcal{G}$ such that $\delta_A(A) \subset \widetilde{\mathcal{M}}(A \otimes S)$. A closed two-sided ideal J of A is said to be invariant by (β_A, δ_A) (or (β_A, δ_A)-invariant) if $\delta_A(J) \subset \mathcal{M}(\widetilde{A} \otimes S; J \otimes S)$. We denote by $\delta_J : J \to \widetilde{\mathcal{M}}(J \otimes S)$ the map obtained by restricting δ_A. We also denote by $\beta_J : N^{\mathrm{o}} \to \mathcal{M}(J)$ the map β_A composed with the projection map $\mathcal{M}(A) \to \mathcal{M}(J)$. Then, we obtain an action of $\mathcal{G}$ on J.

Before stating the generalization of Kasparov's technical theorem, we first state an easy lemma that will be used several times.

Lemma 3.2.5. *Let B be a* C**-algebra and J a closed two-sided ideal of B. Let $x \in B$ and $b \in B_+$ such that $[x, b] \in J$. Then, we have $[x, b^{1/2}] \in J$. In particular, if b commutes with x so does $b^{1/2}$.*

Proof. Let us denote $A := \{b \in B;\ [b, x] \in J\}$. It is easily seen that A is a closed subalgebra of B. If $b \in B$ is a self-adjoint element commuting with x, then b is an element of the C*-subalgebra $A \cap A^*$ of B. In particular, if b is a positive element of B which also belongs to A, then so does $b^{1/2}$. ∎

Theorem 3.2.6 (cf. [3, Théorème 4.3]). *Let us consider*

- *J_1 a σ-unital* C**-algebra endowed with an action $(\beta_{J_1}, \delta_{J_1})$ of $\mathcal{G}$ such that $\delta_{J_1}(J_1) \subset \widetilde{\mathcal{M}}(J_1 \otimes S)$,*
- *J a $(\beta_{J_1}, \delta_{J_1})$-invariant ideal of J_1,*
- *J_2 a σ-unital subalgebra of $\mathcal{M}(J_1; J)$,*
- *$\mathcal{F}$ a separable subspace of* $\mathrm{Der}(J_1)$,
- *J_2' a σ-unital subalgebra of $\mathcal{M}(J \otimes S)$ such that $\delta_{J_1}(x)y \in J \otimes S$ for all $x \in J_1$ and $y \in J_2'$.*

There exists $M \in \mathcal{M}(J_1; J)$ *such that* $0 \leqslant M \leqslant 1$ *and* $[M, \beta_{J_1}(n^{\mathrm{o}})] = 0$ *for all* $n \in N$*, which satisfies the following statements:*

- $(1 - M)J_2 \subset J$;
- *for all* $d \in \mathcal{F}$, $d(M) \in J$;
- $\delta_{J_1}(M) - q_{\beta_{J_1}\alpha}(M \otimes 1_S) \in \widetilde{\mathcal{M}}(J \otimes S)$;
- $(q_{\beta_{J_1}\alpha} - \delta_{J_1}(M))J_2' \subset J \otimes S$.

Proof. In essence, the proof is that of [3, Théorème 4.3]. We denote $q := q_{\beta_{J_1}\alpha}$ for short. Let $h_1 \in J_1, h_2 \in J_2, h_2' \in J_2'$, and $k \in S$ be strictly positive elements. Indeed, we have $S = [(\omega_{\xi,\eta} \otimes \mathrm{id})(V);\ \xi, \eta \in \mathcal{H}]$ and $\mathcal{H}$ is separable. Hence, S is σ-unital. Let K be a compact of $\mathcal{F}$ such that $\mathcal{F} = [K]$. By Lemma 3.2.3, there exists an increasing sequence $(u_l)_{l \in \mathbb{N}}$ of elements of J_1 with $u_0 = 0$, which satisfies for all integer $l \geqslant 1$ the following statements:

(i) $0 \leqslant u_l \leqslant 1$; $[u_l, \beta_{J_1}(n^{\mathrm{o}})] = 0$ for all $n \in N$;

(ii) $\|u_l h_1 - h_1\| \leqslant 2^{-l}$;

(iii) for all $d \in K$, $\|d(u_l)\| \leqslant 2^{-l}$;

(iv) $\|(\delta_{J_1}(u_l) - q(u_l \otimes 1))(1 \otimes k)\| \leqslant 2^{-l}$.

Let us recall that any derivation of J_1 induces a derivation of J by restriction (cf. Remark 3.2.2 (2)). It follows from Lemma 3.2.3 that there exists a sequence $(v_l)_{l \in \mathbb{N}^*}$ of elements of J such that $0 \leqslant v_l \leqslant 1$ and $[v_l, \beta_J(n^{\mathrm{o}})] = 0$ for all $n \in N$ and $l \in \mathbb{N}^*$, which satisfies for all integer $l \geqslant 1$ the following statements:

(a) $\|v_l(u_l - u_{l-1})^{1/2}h_2 - (u_l - u_{l-1})^{1/2}h_2\| \leqslant 2^{-l}$;

(b) for all $d \in K$, $\|d(v_l)\| \leqslant 2^{-l}$ and $\|[(u_l - u_{l-1})^{1/2}, v_l]\| \leqslant 2^{-l}$;

(c) $\|(\delta_{J_1}(v_l) - q(v_l \otimes 1))(1 \otimes k)\| \leqslant 2^{-l}$;

(d) $\|(\delta_{J_1}(v_l) - q)\delta_{J_1}(u_l - u_{l-1})^{1/2}h_2'\| \leqslant 2^{-l}$.

More precisely, for each fixed integer $l \geqslant 1$, we have applied Lemma 3.2.3 with $h := (u_l - u_{l-1})^{1/2}h_2 \in J$, $z := \delta_{J_1}(u_l - u_{l-1})^{1/2}h_2' \in J \otimes S$, $\varepsilon := 2^{-l}$ and the compact subset of $\mathrm{Der}(J)$ consists of the derivation $\mathrm{Ad}((u_l - u_{l-1})^{1/2})$ and the image of the compact subset $K \subset \mathrm{Der}(J_1)$ by the continuous map $\mathrm{Der}(J_1) \to \mathrm{Der}(J)$. For all integer $l \geqslant 1$, let us denote

- $M_l := \sum_{i=1}^{l}(u_i - u_{i-1})^{1/2}v_i(u_i - u_{i-1})^{1/2}$,
- $M_l' := \sum_{i=1}^{l} v_i(u_i - u_{i-1})$,
- $N_l := \sum_{i=1}^{l}(u_i - u_{i-1})^{1/2}(1 - v_i)(u_i - u_{i-1})^{1/2}$.

Let us notice the following statements.

(A) For all $l \geqslant 1$, we have $M_l \in J$, $M_l' \in J$, and $N_l \in J_1$.

(B) For all $l \in \mathbb{N}^*$, we have

$$M_l - M_l' = \sum_{i=1}^{l} \big[(u_i - u_{i-1})^{1/2}, v_i\big](u_i - u_{i-1})^{1/2}$$

for all $l \in \mathbb{N}^*$ and $1 \leqslant i \leqslant l$, we have $\|[(u_i - u_{i-1})^{1/2}, v_i](u_i - u_{i-1})^{1/2}\| \leqslant 2^{-(i-1)}$ by (b); hence, the sequence $(M_l - M'_l)_{l\geqslant 1}$ is norm convergent.

(C) $M_l + N_l = \sum_{i=1}^{l}(u_i - u_{i-1}) = u_l \xrightarrow[l\to\infty]{} 1$ in $\mathcal{M}(J_1)$ with respect to the strict topology.

Let us prove that $(M_l)_{l\geqslant 1}$ converges strictly in $\mathcal{M}(J_1)$. Since h_1 is strictly positive, it suffices to prove that $(M_l h_1)_{l\geqslant 1}$ and $(h_1 M_l)_{l\geqslant 1}$ are norm convergent in J_1. For all integer $l \geqslant 1$, we have

$$M'_l h_1 = \sum_{i=1}^{l} v_i(u_i h_1 - h_1) - \sum_{i=1}^{l} v_i(u_{i-1} h_1 - h_1),$$

with $\|v_i(u_i h_1 - h_1)\| \leqslant 2^{-i}$ and $\|v_i(u_{i-1}h_1 - h_1)\| \leqslant 2^{-(i-1)}$ for all $1 \leqslant i \leqslant l$. Hence, $(M'_l h_1)_{l\geqslant 1}$ is norm convergent. The norm convergence of $(M_l h_1)_{l\geqslant 1}$ follows from (B). Since $h_1 M_l = (M_l h_1)^*$, the norm convergence of $(h_1 M_l)_{l\geqslant 1}$ is proved. Let $M \in \mathcal{M}(J_1)$ (resp. $M' \in \mathcal{M}(J_1)$) be the strict limit of $(M_l)_{l\geqslant 1}$ (resp. $(M'_l)_{l\geqslant 1}$). We have $M, M' \in \mathcal{M}(J_1; J)$ by (A) and $M - M' \in J$ by (B). Since M_l (resp. N_l) is positive for all integer $l \geqslant 1$, so is M (resp. $1 - M$). Hence, $0 \leqslant M \leqslant 1$. Let $n \in N$. Since $[\beta_{J_1}(n^{\mathrm{o}}), u_l] = 0$ and $[\beta_{J_1}(n^{\mathrm{o}}), v_l] = 0$ for all integer $l \geqslant 1$, we have $[\beta_{J_1}(n^{\mathrm{o}}), M_l] = 0$ (cf. Lemma 3.2.5). Hence, $[M, \beta_{J_1}(n^{\mathrm{o}})] = 0$ for all $n \in N$. In particular, we have $[M \otimes 1_S, q] = 0$.

For all $d \in K$, the sequence $(d(M'_l))_{l\geqslant 1}$ is norm convergent in $\mathcal{M}(J_1)$. Indeed, we have

$$\begin{aligned} d(M'_l) &= \sum_{i=1}^{l} d(v_i)(u_i - u_{i-1}) + \sum_{i=1}^{l} v_i\big(d(u_i) - d(u_{i-1})\big) \\ &= \sum_{i=1}^{l} d(v_i)(u_i - u_{i-1}) + \sum_{i=1}^{l} (v_i - v_{i-1}) d(u_i) \end{aligned}$$

(recall that $u_0 = 0$), which is norm convergent by (iii) and (b). Since d is strictly continuous, the norm limit of $(d(M'_l))_{l\geqslant 1}$ is $d(M')$. It follows from (A) and Remark 3.2.2 (2) that $d(M'_l) \in J$ for all $l \geqslant 1$. Hence, $d(M') \in J$. Since $M - M' \in J$ and $d(M') \in J$, it then follows that $d(M) \in J$ for all $d \in K$. Hence, $d(M) \in J$ for all $d \in \mathcal{F}$. Let us prove that

$$\big(\delta_{J_1}(M'_l) - q(M'_l \otimes 1_S)\big)(1_{J_1} \otimes k) \xrightarrow[l\to\infty]{} \big(\delta_{J_1}(M') - q(M' \otimes 1_S)\big)(1_{J_1} \otimes k) \tag{3.3}$$

with respect to the norm topology. It suffices to see that

$$\Big(\big(\delta_{J_1}(M'_l) - q(M'_l \otimes 1_S)\big)(1_{J_1} \otimes k)\Big)_{l\geqslant 1}$$

is norm convergent since this sequence is already convergent with respect to the strict topology towards $(\delta_{J_1}(M') - q(M' \otimes 1_S))(1_{J_1} \otimes k)$. For all integer $l \geqslant 1$, we have

$$\delta_{J_1}(M'_l) - q(M'_l \otimes 1_S) = \sum_{i=1}^{l} \delta_{J_1}(v_i)\big(\delta_{J_1}(u_i) - \delta_{J_1}(u_{i-1})\big) - q \sum_{i=1}^{l} v_i(u_i - u_{i-1}) \otimes 1_S$$

$$= \sum_{i=1}^{l} \big(\delta_{J_1}(v_i)\delta_{J_1}(u_i) - q(v_i u_i \otimes 1_S)\big)$$
$$- \sum_{i=1}^{l} \big(\delta_{J_1}(v_i)\delta_{J_1}(u_{i-1}) - q(v_i u_{i-1} \otimes 1_S)\big).$$

We have $q(v_i u_i \otimes 1_S) = q(v_i \otimes 1_S)q(u_i \otimes 1_S)$ for all integer $i \geqslant 1$. Hence,

$$\delta_{J_1}(v_i)\delta_{J_1}(u_i) - q(v_i u_i \otimes 1_S) = \delta_{J_1}(v_i)\big(\delta_{J_1}(u_i) - q(u_i \otimes 1_S)\big)$$
$$+ \big(\delta_{J_1}(v_i) - q(v_i \otimes 1_S)\big)q(u_i \otimes 1_S).$$

Hence,

$$\sum_i \delta_{J_i}(v_i)\big(\delta_{J_1}(u_i) - q(u_i \otimes 1_S)\big)(1_{J_1} \otimes k);\ \sum_i \big(\delta_{J_1}(v_i) - q(v_i \otimes 1_S)\big)q(u_i \otimes 1)(1_{J_1} \otimes k)$$

are convergent by application of (iv) and (c) (and the fact that $\|u_l\| \leqslant 1$ and $\|v_l\| \leqslant 1$ for all integer $l \geqslant 1$); hence, so is $\sum_i (\delta_{J_1}(v_i)\delta_{J_1}(u_i) - q(v_i u_i \otimes 1))(1_S \otimes k)$. We prove that the series $\sum_i (\delta_{J_1}(v_i)\delta_{J_1}(u_{i-1}) - q(v_i u_{i-1} \otimes 1))(1_S \otimes k)$ is convergent in a similar way and (3.3) is proved.

Since k is strictly positive, the sequence $((\delta_{J_1}(M'_l) - q(M'_l \otimes 1_S))(1_{J_1} \otimes s))_{l\geqslant 1}$ is norm convergent towards $(\delta_{J_1}(M') - q(M' \otimes 1_S))(1_{J_1} \otimes s)$ for all $s \in S$. However, since for all integer $l \geqslant 1$ and $s \in S$ we have $(\delta_{J_1}(M'_n) - q(M'_n \otimes 1))(1_{J_1} \otimes s) \in J \otimes S$ ($M'_l \in J$ and J is invariant), it follows that $(\delta_{J_1}(M') - q(M' \otimes 1_S))(1_{J_1} \otimes s) \in J \otimes S$ for all $s \in S$. Hence, $\delta_{J_1}(M') - q(M' \otimes 1) \in \widetilde{\mathcal{M}}(J \otimes S)$ since M' is self-adjoint and $[M \otimes 1_S, q] = 0$. Moreover, we have $M = (M - M') + M'$ and $M - M' \in J$. Hence,

$$\delta_{J_1}(M) - q(M \otimes 1_S) = \delta_{J_1}(M - M') - q\big((M - M') \otimes 1_S\big)$$
$$+ \delta_{J_1}(M') - q(M' \otimes 1_S) \in \widetilde{\mathcal{M}}(J \otimes S).$$

By (C), the sequence $(N_l)_{l\geqslant 1}$ converges strictly towards $1 - M$. It follows from (a), the fact that $\|(1 - v_i)(u_i - u_{i-1})^{1/2}h_2\| \leqslant 2^{-i}$ for all integer $i \geqslant 1$, (i), and the previous statement that the sequence $(N_l h_2)_{l\geqslant 1}$ converges in norm towards $(1 - M)h_2$. However, we have $h_2 \in J_2$ and $N_l \in J_1$ for all integer $l \geqslant 1$. Hence, $N_l h_2 \in J$ for all integer $l \geqslant 1$. We then have $(1 - M)h_2 \in J$. Hence, $(1 - M)J_2 \subset J$ since h_2 is strictly positive.

By combining (d) with the fact that $\|\delta_{J_1}(1 - v_i)\delta_{J_1}(u_i - u_{i-1})^{1/2}h'_2\| \leqslant 2^{-i}$ for all integer $i \geqslant 1$, we prove in a similar way that the sequence $(\delta_{J_1}(N_l)h'_2)_{l\geqslant 1}$ converges in norm towards $\delta_{J_1}(1 - M)h'_2$ and we prove that $\delta_{J_1}(1 - M)J'_2 \subset J \otimes S$. ∎

3.3. Kasparov's product

In this paragraph, we define the Kasparov product in the equivariant framework for actions of measured quantum groupoids on a finite basis.

Let C and B be two $\mathcal{G}$-C*-algebras. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\mathcal{G}$-equivariant Hilbert C*-modules over C and B, respectively. Let $\gamma_2 : C \to \mathcal{L}(\mathcal{E}_2)$ be a $\mathcal{G}$-equivariant *-representation.

Let us also consider the $\mathcal{G}$-equivariant Hilbert B-module $\mathcal{E} := \mathcal{E}_1 \otimes_{\gamma_2} \mathcal{E}_2$ (cf. Proposition 2.2.9). For $\xi \in \mathcal{E}_1$, we denote by $T_\xi \in \mathcal{L}(\mathcal{E}_2, \mathcal{E})$ the operator defined by $T_\xi(\eta) := \xi \otimes_{\gamma_2} \eta$ for all $\eta \in \mathcal{E}_2$.

Let us recall the notion of connection.

Definition 3.3.1 (cf. [7, Definition A.1] and [26, Definition 8]). Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$. We say that $F \in \mathcal{L}(\mathcal{E})$ is an F_2-connection for $\mathcal{E}_1$ if for all $\xi \in \mathcal{E}_1$, we have $T_\xi F_2 - F T_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E})$ and $F_2 T_\xi^* - T_\xi^* F \in \mathcal{K}(\mathcal{E}, \mathcal{E}_2)$.

In the lemmas below, we assume that the Hilbert A-module $\mathcal{E}_1$ is countably generated.

Lemma 3.3.2 (cf. [7, Proposition A.2 a)]). *Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$ such that $[F_2, \gamma_2(a)] \in \mathcal{K}(\mathcal{E}_2)$ for all $a \in A$. Then there exist F_2-connections F for $\mathcal{E}_1$ such that $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$.*

Proof. By Kasparov's stabilization theorem (cf. [17, Theorem 2]), we can assume that $\mathcal{E}_1$ is a submodule of $\mathcal{H}_{\widetilde{C}} = \mathcal{H} \otimes_{\mathbb{C}} \widetilde{C}$ and $\mathcal{E}_1 = P(\mathcal{H}_{\widetilde{C}})$, where $P \in \mathcal{L}(\mathcal{H}_{\widetilde{C}})$ is a projection. Let

$$F := (P \otimes_{\gamma_2} 1_{\mathcal{E}_2})(1_{\mathcal{H}_{\widetilde{C}}} \otimes_{\mathbb{C}} F_2)(P \otimes_{\gamma_2} 1_{\mathcal{E}_2})$$

be the Grassmann connection (cf. [7, A.2 a)]). But, since $\beta_{\mathcal{E}}(n^{\mathrm{o}}) = \beta_{\mathcal{E}_1}(n^{\mathrm{o}}) \otimes_\gamma 1$, we have $[1_{\mathcal{H}_{\widetilde{C}}} \otimes_{\mathbb{C}} F_2, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$. Moreover, if $T \in \mathcal{L}(\mathcal{E}_1)$, we have $PT\xi = T\xi = TP\xi$ for all $\xi \in \mathcal{E}_1$. Hence, $[P \otimes_\gamma 1_{\mathcal{E}_2}, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$ and the result is proved. ∎

Lemma 3.3.3 (cf. [3, Lemme 5.1]). *Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$ such that $(\mathcal{E}_2, \gamma_2, F_2) \in \mathsf{E}_{\mathcal{G}}(C, B)$. Let $F \in \mathcal{L}(\mathcal{E})$ be an F_2-connection for $\mathcal{E}_1$ such that $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$ (cf. Lemma 3.3.2). Then, we have $(\mathcal{E}, \gamma, F) \in \mathsf{E}_{\mathcal{G}}(\mathcal{K}(\mathcal{E}_1), B)$, where the left action $\gamma : \mathcal{K}(\mathcal{E}_1) \to \mathcal{L}(\mathcal{E})$ of $\mathcal{K}(\mathcal{E}_1)$ on $\mathcal{E}$ is defined by $\gamma(k) := k \otimes_{\gamma_2} 1$ for all $k \in \mathcal{K}(\mathcal{E}_1)$.*

In the following proof, we use all the notations of Remark 2.2.10.

Proof. The pair $(\mathcal{E}, \gamma)$ is a $\mathcal{G}$-equivariant Hilbert $\mathcal{K}(\mathcal{E}_1)$-$B$-bimodule (cf. Propositions 2.2.9 and 2.2.11 where $A := \mathcal{K}(\mathcal{E}_1)$ and γ_1 is the inclusion map $\mathcal{K}(\mathcal{E}_1) \subset \mathcal{L}(\mathcal{E}_1)$). By [26, Proposition 9 (h)], it then remains to prove that

$$\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)\big)(\gamma \otimes \mathrm{id}_S)(x) \in \mathcal{K}(\mathcal{E} \otimes S) \quad \text{for all } x \in \mathcal{K}(\mathcal{E}_1) \otimes S.$$

It suffices to prove that

$$\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)\big)T_\xi \in \mathcal{K}(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes S) \quad \text{for all } \xi \in \mathcal{E}_1 \otimes S,$$

where $T_\xi \in \mathcal{L}(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes S)$ is defined for all $\eta \in \mathcal{E}_2 \otimes S$ by $T_\xi(\eta) := \xi \otimes_{\gamma_2 \otimes \mathrm{id}_S} \eta$ up to (2.13). Let $\xi \in \mathcal{E}_1 \otimes S$ and $\xi' := q_{\beta_{\mathcal{E}_1}\alpha}\xi$, we have $q_{\beta_{\mathcal{E}}\alpha} T_\xi = T_{\xi'}$. Since $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$, we have $q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)T_\xi = (F \otimes 1_S)T_{\xi'}$. Moreover, since $\mathcal{V}^* = \mathcal{V}^* q_{\beta_{\mathcal{E}}\alpha}$, we have $\mathcal{V}^* T_\xi = \mathcal{V}^* T_{\xi'}$. Hence, $\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* T_\xi = \mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* T_{\xi'}$. Thus, we have

$$\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)\big)T_\xi = \big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - F \otimes 1_S\big)T_{\xi'}.$$

Therefore, we have to prove that $(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - F \otimes 1_S)T_\xi \in \mathcal{K}(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes S)$ for all $\xi \in q_{\beta_{\mathcal{E}_1}\alpha}(\mathcal{E}_1 \otimes S)$. Since $\{\delta_{\mathcal{E}_1}(\xi_1)x;\ \xi_1 \in \mathcal{E}_1, x \in C \otimes S\}$ is a total subset of $q_{\beta_{\mathcal{E}_1}\alpha}(\mathcal{E}_1 \otimes S)$, it suffices to consider the case where $\xi = \delta_{\mathcal{E}_1}(\xi_1)x$ with $\xi_1 \in \mathcal{E}_1$ and $x \in q_{\beta_C\alpha}(C \otimes S)$ (cf. Definition 2.1.1 (2) and Remark 2.1.2 (3)). Let $y := (\gamma_2 \otimes \mathrm{id}_S)(x) \in \mathcal{L}(\mathcal{E}_2 \otimes S)$. We have $\xi = \mathcal{V}_1 T_{\xi_1}(x)$. Hence, we have

$$T_\xi = (\mathcal{V}_1 \otimes_{\gamma_2 \otimes \mathrm{id}_S} 1)T_{\xi_1} y \quad \text{and then} \quad \mathcal{V}^* T_\xi = \widetilde{\mathcal{V}}_2^* T_{\xi_1} y.$$

By a direct computation, we have

$$\widetilde{\mathcal{V}}_2^*(\xi_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} \mathcal{V}_2 \eta) = (T_{\xi_1} \otimes_{\delta_B} 1)\eta \quad \text{for all } \eta \in \mathcal{E}_2 \otimes_{\delta_B} (B \otimes S).$$

Since $\mathcal{V}_2\mathcal{V}_2^* = q_{\beta_{\mathcal{E}_2}\alpha}$ and $\mathcal{V}_2^*\mathcal{V}_2 = 1$, we have

$$\widetilde{\mathcal{V}}_2^* T_{\xi_1}\eta = (T_{\xi_1} \otimes_{\delta_B} 1)\mathcal{V}_2^*\eta \quad \text{for all } \eta \in q_{\beta_{\mathcal{E}_2}\alpha}(\mathcal{E}_2 \otimes S).$$

In particular, we have $\widetilde{\mathcal{V}}_2^* T_{\xi_1} y = (T_{\xi_1} \otimes_{\delta_B} 1)\mathcal{V}_2^* y$ (indeed, since $x \in q_{\beta_C\alpha}(C \otimes S)$ we have $q_{\beta_{\mathcal{E}_2}\alpha} y = y$). Hence, $\mathcal{V}^* T_\xi = (T_{\xi_1} \otimes_{\delta_B} 1)\mathcal{V}_2^* y$. In particular, we then obtain the relation $(F \otimes_{\delta_B} 1)\mathcal{V}^* T_\xi = (F T_{\xi_1} \otimes_{\delta_B} 1)\mathcal{V}_2^* y$. For all $\xi_2 \in \mathcal{E}_2$, $\zeta \in \mathcal{E}$ and $s \in S$, we have

$$\theta_{\zeta,\xi_2} \otimes_{\delta_B} 1 = T_\zeta T_{\xi_2}^*;$$
$$(T_{\xi_2}^* \otimes_{\delta_B} 1)\mathcal{V}_2^*(1_{\mathcal{E}_2} \otimes s) = \big((1_{\mathcal{E}_2} \otimes s^*)\mathcal{V}_2 T_{\xi_2}\big)^* \in \mathcal{K}(\mathcal{E}_2 \otimes S, B \otimes S).$$

Hence, $(k \otimes_{\delta_B} 1)\mathcal{V}_2^*(1_{\mathcal{E}_2} \otimes s) \in \mathcal{K}(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes_{\delta_B} (B \otimes S))$ for all $k \in \mathcal{K}(\mathcal{E}_2, \mathcal{E})$ and $s \in S$. In particular, since F is an F_2-connection for $\mathcal{E}_1$ and $y \in \mathcal{L}(\mathcal{E}_2) \otimes S$, we have

$$\big((F T_{\xi_1} - T_{\xi_1} F_2) \otimes_{\delta_B} 1\big)\mathcal{V}_2^* y \in \mathcal{K}\big(\mathcal{E}_2 \otimes S, \mathcal{E} \otimes_{\delta_B} (B \otimes S)\big).$$

However, $(\mathcal{V}_2(F_2 \otimes_{\delta_B} 1)\mathcal{V}_2^* - F_2 \otimes 1_S)y \in \mathcal{K}(\mathcal{E}_2 \otimes S)$ (cf. Remark 3.1.3 (2)), $[F_2 \otimes 1_S, y] \in \mathcal{K}(\mathcal{E}_2 \otimes S)$ (since $y \in \gamma_2(C) \otimes S$) and $\mathcal{V}(T_{\xi_1} \otimes_{\delta_B} 1)\mathcal{V}_2^* y = T_\xi$ (since $\mathcal{V}\mathcal{V}^* = q_{\beta_{\mathcal{E}}\alpha}$ and $q_{\beta_{\mathcal{E}}\alpha} T_\xi = T_\xi$). This completes the proof. ∎

Definition 3.3.4 (cf. [3, Définition 5.2]). Let A, C, and B be three $\mathcal{G}$-C*-algebras. Let $(\mathcal{E}_1, \gamma_1, F_1) \in \mathsf{E}_{\mathcal{G}}(A, C)$ and $(\mathcal{E}_2, \gamma_2, F_2) \in \mathsf{E}_{\mathcal{G}}(C, B)$. Let $\mathcal{E} := \mathcal{E}_1 \otimes_{\gamma_2} \mathcal{E}_2$ be the $\mathcal{G}$-equivariant Hilbert A-B-bimodule defined in Propositions 2.2.9 and 2.2.11, where $\gamma : A \to \mathcal{L}(\mathcal{E});\ a \mapsto \gamma_1(a) \otimes_{\gamma_2} 1$ denotes the left action of A on $\mathcal{E}$. We denote by $F_1 \#_{\mathcal{G}} F_2$ the set of operators $F \in \mathcal{L}(\mathcal{E})$ satisfying the following conditions:

(a) $(\mathcal{E}, F) \in \mathsf{E}_{\mathcal{G}}(A, B)$;

(b) F is an F_2-connection;

(c) for all $a \in A$, the image of $\gamma(a)[F_1 \otimes_{\gamma_2} 1, F]\gamma(a^*)$ in $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ is positive.

With the notations and hypothesis of the above definition, we have the following result.

Theorem 3.3.5 (cf. [3, Théorème 5.3]). *We assume that the* C*-*algebra A is separable. Then, the set $F_1 \#_{\mathcal{G}} F_2$ is nonempty and the class of $(\mathcal{E}, F)$ in $\mathsf{KK}_{\mathcal{G}}(A, B)$ is independent of $F \in F_1 \#_{\mathcal{G}} F_2$ and only depends on the class of $(\mathcal{E}_1, F_1)$ in $\mathsf{KK}_{\mathcal{G}}(A, C)$ and that of $(\mathcal{E}_2, F_2)$ in $\mathsf{KK}_{\mathcal{G}}(C, B)$.*

Proof. The proof is basically identical to that of the non-equivariant case (cf. [18, 26]) or the equivariant case for actions of quantum groups (cf. [3]). Let us prove that $F_1 \#_{\mathcal{G}} F_2$ is nonempty. Let us denote $q := q_{\beta_{\mathcal{E}}\alpha}$ for short. Let $\mathcal{V} \in \mathcal{L}(\mathcal{E} \otimes_{\delta_B} (B \otimes S), \mathcal{E} \otimes S)$ be the isometry associated with the action of $\mathcal{G}$ on $\mathcal{E}$. Let T be an F_2-connection for $\mathcal{E}_1$ such that $[T, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$ (cf. Lemma 3.3.2). Let us fix a strictly positive element $k \in S$. Let us define

- $J_1 := \mathcal{K}(\mathcal{E}_1) \otimes_{\gamma_2} 1 + \mathcal{K}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E}) = \mathcal{M}(\mathcal{K}(\mathcal{E}))$, $J := \mathcal{K}(\mathcal{E})$,
- $J_2 := \mathrm{C}^*(\{T - T^*,\ 1 - T^2,\ [T, F_1 \otimes_{\gamma_2} 1]\} \cup \{[T, \gamma(a)];\ a \in A\}) \subset \mathcal{L}(\mathcal{E})$,
- $\mathcal{F} := [\{\mathrm{Ad}(F_1 \otimes_{\gamma_2} 1),\ \mathrm{Ad}(T)\} \cup \{\mathrm{Ad}(\gamma(a));\ a \in A\}] \subset \mathrm{Der}(J_1)$,
- $J_2' := \mathrm{C}^*(\{q(1 \otimes k)(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S)),$ $(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S))(1 \otimes k)q\}) \subset \mathcal{L}(\mathcal{E} \otimes S)$.

Then, we have the following facts.

- J is an invariant closed two-sided ideal of J_1.
- J_2 is a C^*-subalgebra of $\mathcal{M}(J_1; J)$; by assumption A is separable, then so is J_2; hence, J_2 is σ-unital.
- $\mathcal{F}$ is a separable (since A separable).
- J_2' is a σ-unital C^*-subalgebra of $\mathcal{M}(J \otimes S)$ (separable).

Let $x \in \mathcal{K}(\mathcal{E}_1) \otimes_{\gamma_2} 1$. We have $\delta_{J_1}(x) = \mathcal{V}(x \otimes_{\delta_B} 1)\mathcal{V}^*$. Since $\mathcal{V}^*\mathcal{V} = 1$, we have

$$[\delta_{J_1}(x), \mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*] = \mathcal{V}([x, T] \otimes_{\delta_B} 1)\mathcal{V}^*;$$
$$\mathcal{V}([x, T] \otimes_{\delta_B} 1)\mathcal{V}^* = \delta_J([x, T]) \in \tilde{\mathcal{M}}(\mathcal{K}(\mathcal{E}) \otimes S) \quad (\text{cf. [26, Proposition 9 (e)]}).$$

Hence,

$$[\delta_{J_1}(x), \mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*] = \mathcal{V}([x, T] \otimes_{\delta_B} 1)\mathcal{V}^* \in \tilde{\mathcal{M}}(\mathcal{K}(\mathcal{E}) \otimes S). \tag{3.4}$$

Moreover, by Lemma 3.3.3 we have $\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S)) \in \mathcal{K}(\mathcal{E} \otimes S)$ and $(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S))\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k) \in \mathcal{K}(\mathcal{E} \otimes S)$. Hence,

$$[\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k), \mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*] = [\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k), q(T \otimes 1_S)] \mod \mathcal{K}(\mathcal{E} \otimes S).$$

By combining the fact that $\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k) \in q(J_1 \otimes S)$ with the fact that $[T, y] \in \mathcal{K}(\mathcal{E} \otimes S)$ for all $y \in J_1$ (cf. [26, Proposition 9 (h)]), we obtain $[\delta_{J_1}(x)(1 \otimes k), q(T \otimes 1_S)] \in \mathcal{K}(\mathcal{E} \otimes S)$. Hence,

$$[\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k), \mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*] \in \mathcal{K}(\mathcal{E} \otimes S). \tag{3.5}$$

We also have

$$\begin{aligned}
&\delta_{J_1}(x)[\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S), 1_{\mathcal{E}} \otimes k] \\
&\quad = \delta_{J_1}(x)(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1))(1_{\mathcal{E}} \otimes k) \\
&\qquad - \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S)) \\
&\quad = \delta_{J_1}(x)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*(1_{\mathcal{E}} \otimes k) - \delta_{J_1}(x)(T \otimes 1_S)(1_{\mathcal{E}} \otimes k) \\
&\qquad - \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* + \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)q(T \otimes 1_S)
\end{aligned}$$

$$= \delta_{J_1}(x)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*(1_{\mathcal{E}} \otimes k) - \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*$$
$$- \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)(T \otimes 1_S)(1-q).$$

Hence,

$$\begin{aligned}
&\delta_{J_1}(x)\big[\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S), (1_{\mathcal{E}} \otimes k)q\big] \\
&\quad = \delta_{J_1}(x)\big[\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S), 1_{\mathcal{E}} \otimes k\big]q \\
&\quad = \delta_{J_1}(x)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*(1_{\mathcal{E}} \otimes k)q - \delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*.
\end{aligned}$$

By applying (3.5), we have

$$\begin{aligned}
&\delta_{J_1}(x)\big[\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S), (1_{\mathcal{E}} \otimes k)q\big] \\
&\quad = \big[\delta_{J_1}(x), \mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^*\big](1_{\mathcal{E}} \otimes k)q \quad \operatorname{mod} \mathcal{K}(\mathcal{E} \otimes S).
\end{aligned}$$

By applying (3.4), we finally obtain

$$\delta_{J_1}(x)\big[\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S), (1_{\mathcal{E}} \otimes k)q\big] \in \mathcal{K}(\mathcal{E} \otimes S). \tag{1}$$

By combining the previous relation with

$$\delta_{J_1}(x)(1_{\mathcal{E}} \otimes k)\big(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S)\big) \in \mathcal{K}(\mathcal{E} \otimes S) \quad \text{(cf. Lemma 3.3.3)}, \tag{2}$$

we have

$$\delta_{J_1}(x)\big(\mathcal{V}(T \otimes_{\delta_B} 1)\mathcal{V}^* - q(T \otimes 1_S)\big)(1_{\mathcal{E}} \otimes k)q \in \mathcal{K}(\mathcal{E} \otimes S).$$

Since the above holds true when replacing T by T^* (cf. Remark 3.1.3 (2) and [26, Proposition 9 (b)]), it then follows that $\delta_{J_1}(x)y \in J \otimes S$ for all $x \in J_1$ and $y \in J_2'$. We can apply Theorem 3.2.6. Let us consider M as in the theorem. Let

$$F := M^{1/2}(F_1 \otimes_{\gamma_2} 1) - (1-M)^{1/2}T.$$

For all $n \in N$, we have $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$. Indeed, since $(\mathcal{E}_1, F_1) \in \mathsf{E}_{\mathcal{G}}(A, C)$ we have $[F_1, \beta_{\mathcal{E}_1}(n^{\mathrm{o}})] = 0$. Hence, $[F_1 \otimes_{\gamma_2} 1, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$. We also have $[T, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ by assumption. By Lemma 3.2.5 ($[M, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ and M is positive), we also have $[M^{1/2}, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$. Similarly, we have $[(1-M)^{1/2}, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$. According to the non-equivariant case, it only remains to prove that

$$x\big(\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* - q(F \otimes 1_S)\big) \in \mathcal{K}(\mathcal{E} \otimes S) \quad \text{for all } x \in (\gamma \otimes \mathrm{id}_S)(A \otimes S).$$

Let us fix $x \in (\gamma \otimes \mathrm{id}_S)(A \otimes S)$. We have $\mathcal{V}(F \otimes_{\delta_B} 1)\mathcal{V}^* = \delta_J(F)$. By combining the formula $\delta_J(F_1 \otimes_{\gamma_2} 1) = \delta_{\mathcal{K}(\mathcal{E}_1)}(F_1) \otimes_{\gamma_2 \otimes \mathrm{id}_S} 1$ with the fact that the pair $(\mathcal{E}_1, F_1)$ is a $\mathcal{G}$-equivariant Kasparov A-C-bimodule, we obtain

$$\begin{aligned}
x\big(\delta_J(F_1 \otimes_{\gamma_2} 1) - q\big((F_1 \otimes_{\gamma_2} 1) \otimes 1_S\big)\big) &= x\big(\delta_{\mathcal{K}(\mathcal{E}_1)}(F_1) - q_{\beta_{\mathcal{E}_1}\alpha}(F_1 \otimes 1_S)\big) \otimes_{\gamma_2 \otimes \mathrm{id}_S} 1 \\
&\in \big(\mathcal{K}(\mathcal{E}_1) \otimes S\big) \otimes_{\gamma_2 \otimes \mathrm{id}_S} 1 \subset J_1 \otimes S.
\end{aligned}$$

Since M is an element of the C*-subalgebra $\mathcal{M}(J_1; J)$ of $\mathcal{M}(J_1)$, we have $M^{1/2} \in \mathcal{M}(J_1; J)$. Hence,

$$(M^{1/2} \otimes 1_S)x\big(\delta_J(F_1 \otimes_{\gamma_2} 1) - q\big((F_1 \otimes_{\gamma_2} 1) \otimes 1_S\big)\big) \in J \otimes S = \mathcal{K}(\mathcal{E} \otimes S). \quad (3.6)$$

Let $a \in A$ and $s \in S$, then we have

$$\begin{aligned}(\gamma \otimes \mathrm{id}_S)(a \otimes s)&\big(\delta_J(M^{1/2}) - q(M^{1/2} \otimes 1_S)\big)\delta_J(F_1 \otimes_{\gamma_2} 1) \\ &= \big(\gamma(a)M^{1/2} \otimes s\big)\delta_J(M^{1/2}) - \big(\gamma(a)M^{1/2} \otimes s\big)\delta_J(F_1 \otimes_{\gamma_2} 1)\end{aligned}$$

(since $[M \otimes 1_S, q] = 0$ we have $[M^{1/2} \otimes 1_S, q] = 0$; cf. Lemma 3.2.5) and

$$(1_J \otimes s)\delta_J(F_1 \otimes_{\gamma_2} 1), \quad (1_J \otimes s)\delta_J(M^{1/2}) \in J \otimes S.$$

Hence,

$$x\big(\delta_J(M^{1/2}) - q(M^{1/2} \otimes 1_S)\big)\delta_J(F_1 \otimes_{\gamma_2} 1) \in \mathcal{K}(\mathcal{E} \otimes S). \quad (3.7)$$

For all $a \in A$ and $s \in S$, since $[\gamma(a), M] \in J$ (recall that $\mathrm{Ad}(\gamma(a)) \in \mathcal{F}$), we have

$$\big[(\gamma \otimes \mathrm{id}_S)(a \otimes s), M^{1/2} \otimes 1_S\big] = [\gamma(a), M^{1/2}] \otimes s \in J \otimes S \quad \text{(cf. Lemma 3.2.5)}.$$

Thus, we have $[x, M^{1/2} \otimes 1_S] \in \mathcal{K}(\mathcal{E} \otimes S)$. Hence,

$$[x, M^{1/2} \otimes 1_S]\big(\delta_J(F_1 \otimes_{\gamma_2} 1) - q\big((F_1 \otimes_{\gamma_2} 1) \otimes 1_S\big)\big) \in \mathcal{K}(\mathcal{E} \otimes S). \quad (3.8)$$

By summing up (3.6), (3.7), and (3.8), we have proved that

$$x\big(\delta_J\big(M^{1/2}(F_1 \otimes_{\gamma_2} 1)\big) - q\big(M^{1/2}(F_1 \otimes_{\gamma_2} 1) \otimes 1_S\big)\big) \in \mathcal{K}(\mathcal{E} \otimes S)$$

(recall that $[M^{1/2} \otimes 1_S, q] = 0$). Let

$$E := \big\{u \in \mathcal{M}(J_1; J);\ \delta_J(u) - q(u \otimes 1_S) \in \widetilde{\mathcal{M}}(J \otimes S) \text{ and } [q, u \otimes 1_S] = 0\big\}.$$

Then E is a closed subalgebra of $\mathcal{M}(J_1; J)$. Indeed, E is clearly a closed subspace of $\mathcal{M}(J_1; J)$. Moreover, for all $u, v \in E$ we have

$$\begin{aligned}\delta_J(uv) - q(uv \otimes 1_S) &= \delta_J(u)\delta_J(v) - q(u \otimes 1_S)q(v \otimes 1_S) \\ &= \delta_J(u)\big(\delta_J(v) - q(v \otimes 1_S)\big) + \big(\delta_J(u) - q(u \otimes 1_S)\big)q(v \otimes 1_S) \\ &= \big(\delta_J(u) - q(u \otimes 1_S)\big)\big(\delta_J(v) - q(v \otimes 1_S)\big) \\ &\quad + (u \otimes 1_S)(\delta_J(v) - q(v \otimes 1_S)) \\ &\quad + \big(\delta_J(u) - q(u \otimes 1_S)\big)(v \otimes 1_S) \in \widetilde{\mathcal{M}}(J \otimes S)\end{aligned}$$

and $[q, uv \otimes 1_S] = 0$. Hence, we have

$$\delta_J\big((1 - M)^{1/2}\big) - q\big((1 - M)^{1/2} \otimes 1_S\big) \in \widetilde{\mathcal{M}}(J \otimes S).$$

Therefore, we have

$$x\big(\delta_J\big((1-M)^{1/2}\big)-q\big((1-M)^{1/2}\otimes 1_S\big)\big)\in\mathcal{K}(\mathcal{E}\otimes S). \tag{3.9}$$

We also have $(\delta_J((1-M)^{1/2})-q((1-M)^{1/2}\otimes 1_S))x\in\mathcal{K}(\mathcal{E}\otimes S)$ by taking the adjoint in (3.9). In particular, we have $[x,\delta_J((1-M)^{1/2})]=[x,q((1-M)^{1/2}\otimes 1_S)]$ mod $\mathcal{K}(\mathcal{E}\otimes S)$. Moreover, we have $[x,(1-M)\otimes 1_S]=-[x,M\otimes 1_S]\in\mathcal{K}(\mathcal{E}\otimes S)$. It follows from Lemma 4.2.9 that $[x,(1-M)^{1/2}\otimes 1_S]\in\mathcal{K}(\mathcal{E}\otimes S)$. Since q is a projection such that $[q,(1-M)^{1/2}\otimes 1_S]=0$, we have $[x,q((1-M)^{1/2}\otimes 1_S)]=[qxq,(1-M)^{1/2}\otimes 1_S]\in\mathcal{K}(\mathcal{E}\otimes S)$. Hence,

$$\big[x,\delta_J\big((1-M)^{1/2}\big)\big]\in\mathcal{K}(\mathcal{E}\otimes S). \tag{3.10}$$

We have

$$\delta_J\big((1-M)^{1/2}\big)x\big(\delta_J(T)-q(T\otimes 1_S)\big)\in\mathcal{K}(\mathcal{E}\otimes S). \tag{3.11}$$

Indeed, since k is a strictly positive element of S, we can assume that $x=x'(1_{\mathcal{E}}\otimes k)$ with $x'\in(\gamma\otimes\mathrm{id}_S)(A\otimes S)$. In virtue of (3.10), we have

$$\begin{aligned}&\delta_J\big((1-M)^{1/2}\big)x\big(\delta_J(T)-q(T\otimes 1_S)\big)\\&\quad=x'\delta_J\big((1-M)^{1/2}\big)(1\otimes k)\big(\delta_J(T)-q(T\otimes 1)\big)\quad\text{mod }\mathcal{K}(\mathcal{E}\otimes S).\end{aligned}$$

Note that $F:=\{u\in\mathcal{M}(J_1);\ \delta_J(u)J_2'\subset J\otimes S\text{ and }J_2'\delta_J(u)\in J\otimes S\}$ is a C*-algebra and $1-M\in F$. Hence, $(1-M)^{1/2}\in F$ and (3.11) is proved. By using again (3.10), we prove that

$$x\delta_J\big((1-M)^{1/2}\big)\big(\delta_J(T)-q(T\otimes 1_S)\big)\in\mathcal{K}(\mathcal{E}\otimes S). \tag{3.12}$$

Therefore, we have

$$x\big(\delta_J\big((1-M)^{1/2}T\big)-q\big((1-M)^{1/2}T\otimes 1_S\big)\big)\in\mathcal{K}(\mathcal{E}\otimes S).$$

Indeed, we have (recall that $[q,(1-M)^{1/2}\otimes 1_S]=0$)

$$\begin{aligned}&x\big(\delta_J\big((1-M)^{1/2}T\big)-q\big((1-M)^{1/2}T\otimes 1_S\big)\big)\\&\quad=x\delta_J\big((1-M)^{1/2}\big)\delta_J(T)-xq\big((1-M)^{1/2}\otimes 1\big)q(T\otimes 1_S)\\&\quad=x\delta_J\big((1-M)^{1/2}\big)\big(\delta_J(T)-q(T\otimes 1_S)\big)+x\big(\delta_J\big((1-M)^{1/2}\big)\\&\qquad-q\big((1-M)^{1/2}\otimes 1_S\big)\big)(T\otimes 1_S).\end{aligned}$$

By (3.9) and (3.12), we obtain

$$x\big(\delta_J\big((1-M)^{1/2}T\big)-q\big((1-M)^{1/2}T\otimes 1_S\big)\big)\in\mathcal{K}(\mathcal{E}\otimes S).$$ ∎

Definition 3.3.6. Under the notations and hypotheses of Theorem 3.3.5, the class x in $\mathsf{KK}_{\mathcal{G}}(A,B)$ of $(\mathcal{E},F)$, where $F\in F_1\#_{\mathcal{G}}F_2$, is called the Kasparov product of the class x_1 of $(\mathcal{E}_1,F_1)$ in $\mathsf{KK}_{\mathcal{G}}(A,C)$ and the class x_2 of $(\mathcal{E}_2,F_2)$ in $\mathsf{KK}_{\mathcal{G}}(C,B)$. We denote $x=x_1\otimes_C x_2$.

As in the non-equivariant case [18,19] and the equivariant case for actions of quantum groups [3], we have the following theorem.

Theorem 3.3.7. *The Kasparov product*

$$\mathsf{KK}_{\mathcal{G}}(A,C)\times\mathsf{KK}_{\mathcal{G}}(C,B)\to\mathsf{KK}_{\mathcal{G}}(A,B);\quad (x_1,x_2)\mapsto x_1\otimes_C x_2$$

is bilinear, contravariant in A, covariant in B, functorial in C, and associative.

Definition 3.3.8. Let A and B be $\mathcal{G}$-C*-algebras.

- Let $\phi : A\to B$ be a $\mathcal{G}$-equivariant *-homomorphism. If the C*-algebra B is σ-unital, then the triple $(B,\phi,0)$ is an equivariant Kasparov A-B-bimodule and we define $[\phi]:=[(B,\phi,0)]\in\mathsf{KK}_{\mathcal{G}}(A,B)$.
- If the C*-algebra A is σ-unital, we define $1_A:=[\mathrm{id}_A]=[(A,0)]\in\mathsf{KK}_{\mathcal{G}}(A,A)$.

The Kasparov product generalizes the composition of equivariant *-homomorphisms. More precisely, we have the following result.

Proposition 3.3.9. *Let A, B, and C be $\mathcal{G}$-C*-algebras with B and C σ-unital. Let $\phi : A\to C$ and $\psi : C\to B$ be $\mathcal{G}$-equivariant *-homomorphisms.*

(1) *We have $\phi^*([\psi])=[\psi\circ\phi]=\psi_*([\phi])$ in $\mathsf{KK}_{\mathcal{G}}(A,B)$.*

(2) *If A is separable, we have $[\psi\circ\phi]=[\phi]\otimes_C[\psi]$ in $\mathsf{KK}_{\mathcal{G}}(A,B)$.*

Proposition 3.3.10. *Let A and B be two $\mathcal{G}$-C*-algebras. We have that*

(1) *if B is σ-unital and A separable, then $x\otimes_B 1_B=x$ for all $x\in\mathsf{KK}_{\mathcal{G}}(A,B)$,*

(2) *if A is separable, then $1_A\otimes_A x=x$ for all $x\in\mathsf{KK}_{\mathcal{G}}(A,B)$.*

Only statement (2) is not obvious. For the proof, we will need the following easy lemma.

Lemma 3.3.11. *Let A be a $\mathcal{G}$-C*-algebra and $p\geqslant 2$ an integer. Denote by $\mathrm{M}_p(A)$ the C*-algebra of matrices of size p with entries in A. Let $\delta_{\mathrm{M}_p(A)}:\mathrm{M}_p(A)\to\mathcal{M}(\mathrm{M}_p(A)\otimes S)$ and $\beta_{\mathrm{M}_p(A)}:N^{\mathrm{o}}\to\mathcal{M}(\mathrm{M}_p(A))$ be the maps defined by*

- $\delta_{\mathrm{M}_p(A)}:=\mathrm{id}_{\mathrm{M}_p(\mathbb{C})}\otimes\delta_A : (a_{ij})\mapsto(\delta_A(a_{ij}))$, *up to the identifications*
$$\mathrm{M}_p(A)=\mathrm{M}_p(\mathbb{C})\otimes A;$$
$$\mathrm{M}_p(\mathbb{C})\otimes\mathcal{M}(A\otimes S)=\mathrm{M}_p(\mathcal{M}(A\otimes S))\subset\mathcal{M}(\mathrm{M}_p(A\otimes S))=\mathcal{M}(\mathrm{M}_p(A)\otimes S),$$
- $\beta_{\mathrm{M}_p(A)}(n^{\mathrm{o}})(a_{ij})=(\beta_A(n^{\mathrm{o}})a_{ij})$ *and* $(a_{ij})\beta_{\mathrm{M}_p(A)}(n^{\mathrm{o}})=(a_{ij}\beta_A(n^{\mathrm{o}}))$ *for all $n\in N$ and $(a_{ij})\in\mathrm{M}_p(A)$.*

Then, the pair $(\beta_{\mathrm{M}_p(A)},\delta_{\mathrm{M}_p(A)})$ is a continuous action of $\mathcal{G}$ on $\mathrm{M}_p(A)$.

Proof of Proposition 3.3.10 (2). The idea of the proof is the same as that of [23, Lemma 3.3] (see also [26, Proposition 17]). Let $(\mathcal{E},\gamma,F)$ be an equivariant Kasparov A-B-bimodule. Consider the Hilbert B-submodule $\mathcal{E}_2:=[\gamma(A)\mathcal{E}]$ of $\mathcal{E}$. Let $\mathcal{E}_1:=\mathcal{E}$ and $F_1:=F\in\mathcal{L}(\mathcal{E}_1)$. Define maps $\gamma_{ij}:A\to\mathcal{L}(\mathcal{E}_j,\mathcal{E}_i)$ for $i,j=1,2$ obtained by range/domain restriction of $\gamma(a)$ (for $a\in A$ fixed) and denote $\gamma_i:=\gamma_{ii}$ for $i=1,2$. Note that $\gamma_1=\gamma$

and $\gamma_2 : A \to \mathcal{L}(\mathcal{E}_2)$ is a non-degenerate *-representation of A on $\mathcal{E}_2$. By equivariance of γ, the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ induces by restriction an action $(\beta_{\mathcal{E}_2}, \delta_{\mathcal{E}_2})$ of $\mathcal{G}$ on $\mathcal{E}_2$ and it is clear that $(\mathcal{E}_2, \gamma_2)$ is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule. It is easily seen that we have an identification of equivariant Hilbert bimodules $A \otimes_\gamma \mathcal{E} \to \mathcal{E}_2;\, a \otimes_\gamma \xi \mapsto \gamma(a)\xi$. Let $F_2 \in 0 \,\#_{\mathcal{G}} F_1 \subset \mathcal{L}(\mathcal{E}_2)$. By combining the maps γ_{ij}, we obtain an equivariant *-representation $\phi : \mathrm{M}_2(A) \to \mathcal{L}(\mathcal{E}_1 \oplus \mathcal{E}_2);\, (a_{ij}) \mapsto (\gamma_{ij}(a_{ij}))$ of $\mathrm{M}_2(A)$ on $\mathcal{E}_1 \oplus \mathcal{E}_2$ (cf. Lemma 3.3.11 and Proposition-Definition 2.1.11). Hence, the pair $(\mathcal{E}_1 \oplus \mathcal{E}_2, \phi)$ is an equivariant Hilbert $\mathrm{M}_2(A)$-B-bimodule. We claim that the triple $(\mathcal{E}_1 \oplus \mathcal{E}_2, \phi, F_1 \oplus F_2)$ is an equivariant Kasparov bimodule. For $a \in A$, the operator $T_a \in \mathcal{L}(\mathcal{E}, A \otimes_\gamma \mathcal{E})$ is identified to $\gamma_{21}(a)$ through the identification $A \otimes_\gamma \mathcal{E} = \mathcal{E}_2$. Hence, $F_1\gamma_{12}(a) - \gamma_{12}(a)F_2 \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_1)$ and $F_2\gamma_{21}(a) - \gamma_{21}(a)F_1 \in \mathcal{K}(\mathcal{E}_1, \mathcal{E}_2)$ since F_2 is an F_1-connection (for A). In particular, if $x \in \mathrm{M}_2(A)$ is an off-diagonal matrix (i.e., the diagonal entries equal zero), then $[F_1 \oplus F_2, \phi(x)] \in \mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)$. Since any element of A is a product of two elements of A, any diagonal matrix of $\mathrm{M}_2(A)$ is a product of two off-diagonal matrices of $\mathrm{M}_2(A)$. Moreover, if $x, y \in \mathrm{M}_2(A)$ are off-diagonal, we have $[F_1 \oplus F_2, \phi(xy)] = [F_1 \oplus F_2, \phi(x)]\phi(y) + \phi(x)[F_1 \oplus F_2, \phi(y)] \in \mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)$. Hence, $[F_1 \oplus F_2, \phi(x)] \in \mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ if $x \in \mathrm{M}_2(A)$ is diagonal. This relation extend by linearity to all $x \in \mathrm{M}_2(A)$. The relation $\phi(x)(1 - (F_1 \oplus F_2)^2) \in \mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ holds if $x \in \mathrm{M}_2(A)$ is diagonal. Since we can factorize a finite number of elements of A by a common element of A, any matrix of $\mathrm{M}_p(A)$ factorizes on the right by a diagonal matrix. Hence, this relation extends to all $x \in \mathrm{M}_p(A)$. The remaning relation of Definition 3.1.1 is proved in a same way. With the identifications $\mathcal{M}(\mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2) \otimes S) = \mathcal{L}((\mathcal{E}_1 \oplus \mathcal{E}_2) \otimes S) = \mathcal{L}((\mathcal{E}_1 \otimes S) \oplus (\mathcal{E}_2 \otimes S))$, we have

$$\delta_{\mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)}(F_1 \oplus F_2) = \delta_{\mathcal{K}(\mathcal{E}_1)}(F_1) \oplus \delta_{\mathcal{K}(\mathcal{E}_2)}(F_2) \quad \text{and} \quad q_{\beta_{\mathcal{E}_1 \oplus \mathcal{E}_2}\alpha} = q_{\beta_{\mathcal{E}_1}\alpha} \oplus q_{\beta_{\mathcal{E}_2}\alpha}.$$

By using the above trick and the identification $\mathrm{M}_2(A) \otimes S = \mathrm{M}_2(A \otimes S)$, we prove that

$$(\phi \otimes \mathrm{id}_S)(x)\big(\delta_{\mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2)}(F_1 \oplus F_2) - q_{\beta_{\mathcal{E}_1 \oplus \mathcal{E}_2}\alpha}\big((F_1 \oplus F_2) \otimes 1_S\big)\big) \in \mathcal{K}\big((\mathcal{E}_1 \oplus \mathcal{E}_2) \otimes S\big)$$

for all $x \in \mathrm{M}_2(A) \otimes S$. Let $\tau := (\mathcal{E}_1 \oplus \mathcal{E}_2, \phi, F_1 \oplus F_2) \in \mathsf{E}_{\mathcal{G}}(\mathrm{M}_2(A), B)$. For $t \in [0, 1]$, let $\iota_t : A \to \mathrm{M}_2(A)$ be the $\mathcal{G}$-equivariant *-homomorphism defined for all $a \in A$ by

$$\iota_t(a) := \begin{pmatrix} (1-t^2)a & t\sqrt{1-t^2}a \\ t\sqrt{1-t^2}a & t^2 a \end{pmatrix}.$$

We have $\iota_0^*(\tau) = (\mathcal{E}_1, \gamma_1, F_1) \oplus (\mathcal{E}_2, 0, F_2)$ and $\iota_1^*(\tau) = (\mathcal{E}_1, 0, F_1) \oplus (\mathcal{E}_2, \gamma_2, F_2)$ in $\mathsf{E}_{\mathcal{G}}(A, B)$. Moreover, $(\mathcal{E}_2, 0, F_2)$ and $(\mathcal{E}_1, 0, F_1)$ are degenerate $\mathcal{G}$-equivariant Kasparov bimodules. Hence, the triple $(\mathcal{E}_1 \oplus \mathcal{E}_2, (\phi \circ \iota_t)_{t \in [0,1]}, F_1 \oplus F_2)$ defines a homotopy between $(\mathcal{E}_1, \gamma_1, F_1)$ and $(\mathcal{E}_2, \gamma_2, F_2)$ (cf. Proposition 3.1.5 (1) and Example 3.1.7 (3)). This completes the proof. ■

Remark 3.3.12. If $x := [(\mathcal{E}, F)] \in \mathsf{KK}_{\mathcal{G}}(A, B)$ (with A separable), we can always require the left action $A \to \mathcal{L}(\mathcal{E})$ of A on $\mathcal{E}$ to be non-degenerate (cf. Proposition 3.3.10 (2)).

The result below is a generalization of Proposition 3.3.9 (2) and follows straightforwardly from the above remark.

Proposition-Definition 3.3.13. *Let A, B, and C be $\mathcal{G}$-C*-algebras with B σ-unital. Let $\phi : A \to B$ be a $\mathcal{G}$-equivariant *-homomorphism.*

(1) *If C is separable, we have $x \otimes_A [\phi] = \phi_*(x)$ in $\mathsf{KK}_{\mathcal{G}}(C, B)$ for all $x \in \mathsf{KK}_{\mathcal{G}}(C, A)$.*

(2) *If A is separable, we have $[\phi] \otimes_B x = \phi^*(x)$ in $\mathsf{KK}_{\mathcal{G}}(A, C)$ for all $x \in \mathsf{KK}_{\mathcal{G}}(B, C)$.*

In particular, if A is a separable $\mathcal{G}$-C-algebra, then the abelian group $\mathsf{KK}_{\mathcal{G}}(A, A)$ endowed with the Kasparov product is a unital ring called the equivariant Kasparov ring of A.*

3.4. Descent morphisms

Lemma 3.4.1 (cf. [3, Lemme 6.13]). *Let $(\mathcal{E}, \gamma)$ be a $\mathcal{G}$-equivariant A-B-bimodule with a non-degenerate left action $\gamma : A \to \mathcal{L}(\mathcal{E})$ and $F \in \mathcal{L}(\mathcal{E})$. We assume that $[\gamma(a), F] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$, $[F, \beta_{\mathcal{E}}(n^{\mathrm{o}})] = 0$ for all $n \in N$, and $(\delta_{\mathcal{K}(\mathcal{E})}(F) - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S))(\gamma \otimes \mathrm{id}_S)(x) \subset \mathcal{K}(\mathcal{E} \otimes S)$ for all $x \in A \otimes S$. Let D be the bidual $\mathcal{G}$-C*-algebra of A (cf. Notations 2.3.4).*

(1) *Through the identification of Hilbert D-B-bimodules (cf. Theorem 2.3.11, Corollary 2.4.24, and Proposition-Definition 2.4.26 for the definitions and notations)*

$$\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E} \to \mathcal{E}_{\mathcal{E},R}; \quad q_{\beta_A\alpha}(a \otimes \xi) \otimes_\gamma \eta \mapsto q_{\beta_{\mathcal{E}}\alpha}\big(\gamma(a)\eta \otimes \xi\big), \tag{3.13}$$

the operator $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ is identified to an F-connection for $\mathcal{E}_{A,R}$.

(2) *The operator $F \in \mathcal{L}(\mathcal{E})$ is identified to a $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}}$-connection through the identification of Hilbert A-B-bimodules $\mathcal{E}^*_{A,R} \otimes_D \mathcal{E}_{\mathcal{E},R} = \mathcal{E}$.*

Proof. (1) It is clear that the formula

$$(A \otimes \mathcal{H}) \otimes_\gamma \mathcal{E} \to \mathcal{E} \otimes \mathcal{H}; \quad (a \otimes \xi) \otimes_\gamma \eta \mapsto \gamma(a)\eta \otimes \xi \tag{3.14}$$

defines an adjointable unitary of Hilbert B-modules, which intertwines the left actions of $A \otimes \mathcal{K}$. However, we have $\gamma(\beta_A(n^{\mathrm{o}})a) = \beta_{\mathcal{E}}(n^{\mathrm{o}})\gamma(a)$ for all $n \in N$ and $a \in A$. Hence, the above unitary induces by restriction the identification of Hilbert D-B-bimodules $\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E} = \mathcal{E}_{\mathcal{E},R}$. By compactness of the commutators $[\gamma(a), F]$ for all $a \in A$, the operator $F \otimes 1_{\mathcal{H}} \in \mathcal{L}(\mathcal{E} \otimes \mathcal{H})$ is identified to an F-connection through (3.14). Hence, the operator $(F \otimes 1_{\mathcal{H}})\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ is identified to an F-connection through (3.13). Moreover, since $(\delta_{\mathcal{K}(\mathcal{E})}(F) - F \otimes 1_S)(\gamma \otimes \mathrm{id}_S)(q_{\beta_A\alpha}(A \otimes S)) \subset \mathcal{K}(\mathcal{E} \otimes S)$, the operator $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} - (F \otimes 1_{\mathcal{H}})\restriction_{\mathcal{E}_{\mathcal{E},R}}$ is identified to a 0-connection (cf. [26, Proposition 9 (d)]) through (3.14). Hence, the operator $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R})$ is identified to an F-connection through (3.13) (cf. [26, Proposition 9 (c)]).

(2) By associativity, we have $\mathcal{E}^*_{A,R} \otimes_D \mathcal{E}_{\mathcal{E},R} = (\mathcal{E}^*_{A,R} \otimes_D \mathcal{E}_{A,R}) \otimes_A \mathcal{E} = A \otimes_A \mathcal{E}$ (cf. [9, Proposition 7.12]). By non-degeneracy of γ, we have $A \otimes_A \mathcal{E} = \mathcal{E}$. We then obtain a canonical identification of Hilbert A-B-bimodules $\mathcal{E}^*_{A,R} \otimes_D \mathcal{E}_{\mathcal{E},R} = \mathcal{E}$. For $\xi \in \mathcal{E}_{A,R}$, the operator $T_{\xi^*} \in \mathcal{L}(\mathcal{E}_{\mathcal{E},R}, \mathcal{E}^*_{A,R} \otimes_D \mathcal{E}_{\mathcal{E},R})$ is identified to $T^*_\xi \in \mathcal{L}(\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E}, \mathcal{E})$ up to

the identifications $\mathcal{E}_{A,R}^* \otimes_D \mathcal{E}_{\mathcal{E},R} = \mathcal{E}$ and $\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E} = \mathcal{E}_{\mathcal{E},R}$. Therefore, the fact that $F \in \mathcal{L}(\mathcal{E})$ is a $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}}$-connection is just a restatement of (1). ∎

The result below will be used in the proof of Lemma 3.4.3.

Lemma 3.4.2. *If $m \in \mathcal{M}(A)$ is δ_A-invariant, then $[\pi(m), \widehat{\theta}(x)] = 0$ for all $x \in \widehat{S}$.*

Proof. Let us fix a δ_A-invariant element m of $\mathcal{M}(A)$ and $x \in \widehat{S}$. We have to show that $[\pi_L(m), 1_A \otimes \rho(x)] = 0$. This follows from the formula $\pi_L(m) = q_{\beta_A \alpha}(m \otimes 1_{\mathcal{H}})$ and the commutation relation $[q_{\beta_A \alpha}, 1_A \otimes \rho(x)] = 0$ ensuing from $\widehat{S} \subset \widehat{M}' \subset \alpha(N)'$. ∎

Lemma 3.4.3 (cf. [3, Lemme 6.14] and [31, Proposition 5.3]). *We follow the notations and hypotheses of Lemma 3.4.1. Let $\mathcal{E}_0 := \mathcal{E}_{\mathcal{E},R}$ be the $\mathcal{G}$-equivariant Hilbert D-B-bimodule defined in Corollary 2.4.24 and Proposition-Definition 2.4.26. Let $F_0 := \pi_R(F)\restriction_{\mathcal{E}_0} \in \mathcal{L}(\mathcal{E}_0)$. Let $\gamma_0 : D \to \mathcal{L}(\mathcal{E}_0)$; $d \mapsto (\gamma \otimes \mathrm{id}_{\mathcal{K}})(d)\restriction_{\mathcal{E}_0}$ be the left action of D on $\mathcal{E}_0$. Then, we have (cf. Proposition-Definition 2.4.13)*

$$\left[\gamma_{0*}(D \rtimes \mathcal{G}), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)\right] \subset \mathcal{K}(\mathcal{E}_0 \rtimes \mathcal{G}).$$

If $(\mathcal{E}, \gamma, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule, then the triple $(\mathcal{E}_0, \gamma_0, F_0)$ is a $\mathcal{G}$-equivariant Kasparov D-B-bimodule and the triple $(\mathcal{E}_0 \rtimes \mathcal{G}, \gamma_{0*}, \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0))$ is a $\widehat{\mathcal{G}}$-equivariant Kasparov $D \rtimes \mathcal{G}$-$B \rtimes \mathcal{G}$-bimodule.

Proof. From the relations $((F \otimes 1_S)q_{\beta_{\mathcal{E}}\alpha} - \delta_{\mathcal{K}(\mathcal{E})}(F))(\gamma \otimes \mathrm{id}_S)(A \otimes S) \subset \mathcal{K}(\mathcal{E}) \otimes S$ and $R(S)\mathcal{K} = \mathcal{K}$, we infer that $((F \otimes 1)q_{\beta_{\mathcal{E}}\widehat{\alpha}} - \pi_R(F))(\gamma \otimes \mathrm{id}_{\mathcal{K}})(A \otimes \mathcal{K}) \subset \mathcal{K}(\mathcal{E} \otimes \mathcal{H})$. Hence, $((F \otimes 1)\restriction_{\mathcal{E}_0} - F_0)\gamma_0(D) \subset \mathcal{K}(\mathcal{E}_0)$. We also have $\gamma_0(D)(F_0 - (F \otimes 1_{\mathcal{H}})\restriction_{\mathcal{E}_0}) \subset \mathcal{K}(\mathcal{E}_0)$. Hence, $[\gamma_0(d), F_0] = [\gamma_0(d), (F_0 \otimes 1_{\mathcal{H}})\restriction_{\mathcal{E}_0}] \bmod \mathcal{K}(\mathcal{E}_0)$. By compactness of the commutators $[\gamma(a), F]$ for all $a \in A$, we have $[(\gamma \otimes \mathrm{id}_{\mathcal{K}})(x), F \otimes 1_{\mathcal{H}}] \in \mathcal{K}(\mathcal{E} \otimes \mathcal{H})$ for all $x \in A \otimes \mathcal{K}$. In particular, $[\gamma_0(d), (F_0 \otimes 1_{\mathcal{H}})\restriction_{\mathcal{E}_0}] \in \mathcal{K}(\mathcal{E}_0)$ for all $d \in D$. Hence, $[\gamma_0(d), F_0] \in \mathcal{K}(\mathcal{E}_0)$ for all $d \in D$. Let $d \in D$ and $x \in \widehat{S}$. We have

$$\begin{aligned}\left[\gamma_{0*}\big(\pi_D(d)\widehat{\theta}_D(x)\big), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)\right] &= \pi_{\mathcal{K}(\mathcal{E}_0)}\big([\gamma_0(d), F_0]\big)\widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(x) \\ &\quad + \pi_{\mathcal{K}(\mathcal{E}_0)}\big(\gamma_0(d)\big)\big[\widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(x), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)\big].\end{aligned}$$

The first term of the right-hand side belongs to $\mathcal{K}(\mathcal{E}_0 \rtimes \mathcal{G})$ since $[\gamma_0(d), F_0] \in \mathcal{K}(\mathcal{E}_0)$ (cf. Corollary 2.4.10) and the second one is zero since $[\widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(x), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)] = 0$ (cf. Lemmas 2.4.25 and 3.4.2). Therefore, we have

$$\left[\gamma_{0*}\big(\pi_D(d)\widehat{\theta}_D(x)\big), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)\right] \in \mathcal{K}(\mathcal{E}_0 \rtimes \mathcal{G}) \quad \text{for all } d \in D \text{ and } x \in \widehat{S}.$$

Hence, $[\gamma_{0*}(D \rtimes \mathcal{G}), \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)] \subset \mathcal{K}(\mathcal{E}_0 \rtimes \mathcal{G})$. Assume that $(\mathcal{E}, \gamma, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule. By arguing as at the beginning of the proof, we prove the remaining relations of Definition 3.1.1 so that the triple $(\mathcal{E}_0, \gamma_0, F_0)$ is a Kasparov D-B-bimodule. By invariance of F_0, the triple $(\mathcal{E}_0, \gamma_0, F_0)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule (cf. Remark 3.1.3 (3)). We prove the remaining relations of Definition 3.1.1 so that the triple $(\mathcal{E}_0 \rtimes \mathcal{G}, \gamma_{0*}, \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0))$ is a Kasparov D-B-bimodule. For instance, we have

$$\gamma_{0*}\big(\widehat{\theta}_D(x)\pi_D(d)\big)\big(\pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)^* - \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0)\big)$$
$$= \widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(x)\pi_{\mathcal{K}(\mathcal{E}_0)}\big(\gamma_0(d)(F_0^* - F_0)\big) \in \mathcal{K}(\mathcal{E}_0) \rtimes \mathcal{G} \text{ (cf. Proposition-Definition 2.4.13)},$$

for all $d \in D$ and $x \in \widehat{S}$. Moreover, the triple $(\mathcal{E}_0 \rtimes \mathcal{G}, \gamma_{0*}, \pi_{\mathcal{K}(\mathcal{E}_0)}(F_0))$ is a $\mathcal{G}$-equivariant Kasparov D-B-bimodule (cf. Lemma 2.4.6 and Remark 3.1.3 (3)). ∎

Now, we are in position to define the descent morphism.

Theorem 3.4.4 (cf. [3, Proposition 6.18 and Théorème 6.19], [4, Remarque 7.7 (b)], and [31, Proposition 5.3 and Lemme 5.4]). *Let A, B, and C be $\mathcal{G}$-C*-algebras.*

(1) *If $(\mathcal{E}, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule (with a non-degenerate left action), then $(\mathcal{E} \rtimes \mathcal{G}, F \otimes_{\pi_B} 1)$ is a $\widehat{\mathcal{G}}$-equivariant Kasparov $A \rtimes \mathcal{G}$-$B \rtimes \mathcal{G}$-bimodule. Moreover, if $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ are unitarily equivalent (resp. homotopic) $\mathcal{G}$-equivariant Kasparov A-B-bimodules, then so are $(\mathcal{E}_1 \rtimes \mathcal{G}, F_1 \otimes_{\pi_B} 1)$ and $(\mathcal{E}_2 \rtimes \mathcal{G}, F_2 \otimes_{\pi_B} 1)$.*

Let $J_{\mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(A, B) \to \mathsf{KK}_{\widehat{\mathcal{G}}}(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$ be the homomorphism of abelian groups defined for all $[(\mathcal{E}, F)] \in \mathsf{KK}_{\mathcal{G}}(A, B)$ (with a non-degenerate left action) by

$$J_{\mathcal{G}}\big([(\mathcal{E}, F)]\big) := \big[(\mathcal{E} \rtimes \mathcal{G}, F \otimes_{\pi_B} 1)\big].$$

(2) *Let $\phi : A \to B$ be a $\mathcal{G}$-equivariant *-homomorphism. We recall that the equivariance of ϕ allows us to define a $\widehat{\mathcal{G}}$-equivariant *-homomorphism $\phi_* : A \rtimes \mathcal{G} \to B \rtimes \mathcal{G}$ (cf. Proposition 2.4.12 (1)). We have $J_{\mathcal{G}}([\phi]) = [\phi_*]$. In particular, we have $J_{\mathcal{G}}(1_A) = 1_{A \rtimes \mathcal{G}}$.*

(3) *Assume the C*-algebra A to be separable. For all $x_1 \in \mathsf{KK}_{\mathcal{G}}(A, C)$ and $x_2 \in \mathsf{KK}_{\mathcal{G}}(C, B)$, we have*

$$J_{\mathcal{G}}(x_1 \otimes_C x_2) = J_{\mathcal{G}}(x_1) \otimes_{C \rtimes \mathcal{G}} J_{\mathcal{G}}(x_2).$$

The following proof is broadly inspired by those of [31, Proposition 5.3 and Lemma 5.4], of which we take some of the notations. In the proof, we will also follow the notations introduced in Lemmas 3.4.1 and 3.4.3.

Proof. (1) Let $(\mathcal{E}, \gamma, F)$ be a $\mathcal{G}$-equivariant Kasparov A-B-bimodule (with a non-degenerate left action). Let $(\mathcal{E}_0, \gamma_0, F_0)$ be the $\mathcal{G}$-equivariant Kasparov D-B-bimodule defined in Lemma 3.4.3. Note that $\mathcal{E} \oplus \mathcal{E}_0$ is a $\mathcal{G}$-equivariant Hilbert B-module. We can consider the canonical morphism $\pi_{\mathcal{K}(\mathcal{E} \oplus \mathcal{E}_0)} : \mathcal{L}(\mathcal{E} \oplus \mathcal{E}_0) \to \mathcal{M}(\mathcal{K}(\mathcal{E} \oplus \mathcal{E}_0) \rtimes \mathcal{G})$. Up to the canonical identifications $\mathcal{K}(\mathcal{E} \oplus \mathcal{E}_0) \rtimes \mathcal{G} = \mathcal{K}((\mathcal{E} \oplus \mathcal{E}_0) \rtimes \mathcal{G})$ (cf. Corollary 2.4.10) and $(\mathcal{E} \oplus \mathcal{E}_0) \rtimes \mathcal{G} = (\mathcal{E} \rtimes \mathcal{G}) \oplus (\mathcal{E}_0 \rtimes \mathcal{G})$, we can consider the following restrictions of $\pi_{\mathcal{K}(\mathcal{E} \oplus \mathcal{E}_0)}$:

$$\begin{aligned} i &: \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{E} \rtimes \mathcal{G}); \\ i_0 &: \mathcal{L}(\mathcal{E}_0, \mathcal{E}) \to \mathcal{L}(\mathcal{E}_0 \rtimes \mathcal{G}, \mathcal{E} \rtimes \mathcal{G}); \\ i^0 &: \mathcal{L}(\mathcal{E}, \mathcal{E}_0) \to \mathcal{L}(\mathcal{E} \rtimes \mathcal{G}, \mathcal{E}_0 \rtimes \mathcal{G}); \\ i_0^0 &: \mathcal{L}(\mathcal{E}_0) \to \mathcal{L}(\mathcal{E}_0 \rtimes \mathcal{G}). \end{aligned}$$

Note that, up to the identifications $\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})=\mathcal{K}(\mathcal{E})\rtimes\mathcal{G}$ and $\mathcal{K}(\mathcal{E}_0\rtimes\mathcal{G})=\mathcal{K}(\mathcal{E}_0)\rtimes\mathcal{G}$, we have $i=\pi_{\mathcal{K}(\mathcal{E})}$ and $i_0^0=\pi_{\mathcal{K}(\mathcal{E}_0)}$. By using the identification $\mathcal{E}_{A,R}^*\otimes_{\gamma_0}\mathcal{E}_0=\mathcal{E}$ (cf. Lemma 3.4.1 (2)), for $\zeta\in\mathcal{E}_{A,R}^*$ let $T_\zeta\in\mathcal{L}(\mathcal{E}_0,\mathcal{E})$ be the operator defined by $T_\zeta(\eta)=\zeta\otimes_{\gamma_0}\eta$ for all $\eta\in\mathcal{E}_{\mathcal{E},R}$. For $\eta\in\mathcal{H}$, we denote by $\tau_\eta\in\mathcal{L}(\mathcal{E},\mathcal{E}\otimes\mathcal{H})$ the operator defined by $\tau_\xi(\eta):=\xi\otimes\eta$ for all $\xi\in\mathcal{E}$. It should be noted that $\tau_\eta^*(\xi'\otimes\eta')=\langle\eta,\eta'\rangle\xi'$ for all $\xi'\in\mathcal{E}$ and $\eta'\in\mathcal{H}$. It follows from $[T_\zeta\gamma_0(x)T_\xi^*;\ \zeta,\xi\in\mathcal{E}_{A,R},\ x\in D]=\gamma(A)$ that

$$\left[i_0(T_\zeta)\gamma_{0*}(x)i^0(T_\xi^*);\ x\in D\rtimes\mathcal{G},\zeta,\xi\in\mathcal{E}_{A,R}^*\right]=\gamma_*(A\rtimes\mathcal{G}).$$

By combining the non-degeneracy of the canonical morphism $\pi_D:D\to\mathcal{M}(D\rtimes\mathcal{G})$ with the fact that $\gamma_{0*}(\pi_D(d))=i_0^0(\gamma_0(d))$ for all $d\in D$, we have that $\gamma_*(A\rtimes\mathcal{G})$ is the closed linear span of the elements of the form

$$i_0\big(T_\zeta\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big),\quad \text{with } x\in D\rtimes\mathcal{G},\ b,c\in D,\ \text{and } \zeta,\ \xi\in\mathcal{E}_{A,R}^*.$$

Let us prove that $i(F)$ commutes with these elements modulo $\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$. We will carry out the computations modulo $\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$ by using the inclusions

$$i_0\big(\mathcal{K}(\mathcal{E}_0,\mathcal{E})\big)\gamma_{0*}(A\rtimes\mathcal{G})\subset\mathcal{K}(\mathcal{E}_0\rtimes\mathcal{G},\mathcal{E}\rtimes\mathcal{G});\tag{3.15}$$

$$\gamma_{0*}(A\rtimes\mathcal{G})i^0\big(\mathcal{K}(\mathcal{E},\mathcal{E}_0)\big)\subset\mathcal{K}(\mathcal{E}\rtimes\mathcal{G},\mathcal{E}_0\rtimes\mathcal{G}).\tag{3.16}$$

Let us prove (3.15) since (3.16) will follow by taking the adjoint in (3.15). By the relation $\mathcal{K}(\mathcal{E}_0,\mathcal{E})=[\mathcal{K}(\mathcal{E}_0,\mathcal{E})\mathcal{K}(\mathcal{E}_0)]$, it suffices to prove that $i_0^0(\mathcal{K}(\mathcal{E}_0))\gamma_{0*}(A\rtimes\mathcal{G})\subset\mathcal{K}(\mathcal{E}_0\rtimes\mathcal{G})$. Let $k\in\mathcal{K}(\mathcal{E}_0)$ and $x\in A\rtimes\mathcal{G}$. In virtue of the non-degeneracy of the canonical morphism $\widehat{\theta}_A:\widehat{S}\to\mathcal{M}(A\rtimes\mathcal{G})$, we can assume that $x=\widehat{\theta}_A(y)x'$ with $y\in\widehat{S}$ and $x'\in A\rtimes\mathcal{G}$. By the equivariance of γ_{0*}, we have $\gamma_{0*}(x)=\widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(y)\gamma_{0*}(x')$. Hence, $i_0^0(k)\gamma_{0*}(x)\in\mathcal{K}(\mathcal{E}_0\rtimes\mathcal{G})$ since $i_0^0(k)\widehat{\theta}_{\mathcal{K}(\mathcal{E}_0)}(y)\in\mathcal{K}(\mathcal{E}_0)\rtimes\mathcal{G}$ and $\gamma_{0*}(x)\in\mathcal{M}(\mathcal{K}(\mathcal{E}_0)\rtimes\mathcal{G})$.

Let us fix $x\in D\rtimes\mathcal{G}$, $b,c\in D$, and $\zeta,\xi\in\mathcal{E}_{A,R}^*$. We have

$$i(F)i_0\big(T_\zeta\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big)=i_0\big(FT_\zeta\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big).$$

Let $\zeta=(a\otimes\zeta'^*)q_{\beta_A\widehat{\alpha}}$ with $a\in A$ and $\zeta'\in\mathcal{H}$. Let $\eta_0=q_{\beta_\mathcal{E}\widehat{\alpha}}(\eta\otimes\chi)\in\mathcal{E}_0$ with $\eta\in\mathcal{E}$ and $\chi\in\mathcal{H}$. Let $(e_{ij}^{(l)})_{1\leqslant l\leqslant k,\,1\leqslant i,j\leqslant n_l}$ be a system of matrix units of N. We have (cf. [9, Proposition-Definition A.2.19])

$$T_\zeta(\eta_0)=\sum_{l=1}^k n_l^{-1}\sum_{i,j=1}^{n_l}\langle\zeta',\widehat{\alpha}\big(e_{ji}^{(l)}\big)\chi\rangle\gamma(a)\beta_\mathcal{E}\big(e_{ij}^{(l)\mathrm{o}}\big)\eta.$$

In particular, we have

$$FT_\zeta(\eta_0)=\sum_{l=1}^k n_l^{-1}\sum_{i,j=1}^{n_l}\langle\zeta',\widehat{\alpha}\big(e_{ji}^{(l)}\big)\chi\rangle F\gamma(a)\beta_\mathcal{E}\big(e_{ij}^{(l)\mathrm{o}}\big)\eta=F\gamma(a)\tau_{\zeta'}^*(\eta_0)$$

and since $[F,\beta_\mathcal{E}(n^\mathrm{o})]=0$ for all $n\in N$, we also have

$$T_\zeta(F\otimes 1)(\eta_0)=\sum_{l=1}^k n_l^{-1}\sum_{i,j=1}^{n_l}\langle\zeta',\widehat{\alpha}\big(e_{ji}^{(l)}\big)\chi\rangle\gamma(a)F\beta_\mathcal{E}\big(e_{ij}^{(l)\mathrm{o}}\big)\eta=\gamma(a)F\tau_{\zeta'}^*(\eta_0).$$

Hence, we have $FT_\xi - T_\xi(F\otimes 1)\restriction_{\mathcal{E}_0} = [F,\gamma(a)]\tau_{\xi'}^*\restriction_{\mathcal{E}_0} \in \mathcal{K}(\mathcal{E}_0,\mathcal{E})$. Thus, we have (cf. (3.15))

$$i_0\big(FT_\xi\gamma_0(b)\big)\gamma_{0*}(x) = i_0\big(T_\xi(F\otimes 1)\gamma_0(b)\big)\gamma_{0*}(x) \mod \mathcal{K}(\mathcal{E}_0\rtimes\mathcal{G},\mathcal{E}\rtimes\mathcal{G}). \quad (3.17)$$

We recall (cf. proof of Lemma 3.4.3) that

$$\big((F\otimes 1)\restriction_{\mathcal{E}_0} - F_0\big)\gamma_0(D)\subset\mathcal{K}(\mathcal{E}_0). \quad (3.18)$$

We have

$$\begin{aligned} & i(F)i_0\big(T_\xi\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big) \\ &\quad = i_0\big(T_\xi(F\otimes 1)\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G}) \quad (3.17)\\ &\quad = i_0\big(T_\xi F_0\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G}) \quad (3.18), (3.15)\\ &\quad = i_0(T_\xi)i_0^0(F_0)\gamma_{0*}\big(\pi_D(b)x\pi_D(c)\big)i^0(T_\xi^*) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G})\\ &\quad = i_0(T_\xi)\gamma_{0*}\big(\pi_D(b)x\pi_D(c)\big)i_0^0(F_0)i^0(T_\xi^*) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G}) \quad (\text{Lemma } 3.4.3)\\ &\quad = i_0\big(T_\xi\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)F_0T_\xi^*\big) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G}). \end{aligned}$$

By using (3.16) and Definition 3.1.2 (3), we prove in a similar way that

$$\begin{aligned} &\gamma_{0*}(x)i^0\big(\gamma_0(c)(F\otimes 1)T_\xi^*\big) = \gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*F\big) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G},\mathcal{E}_0\rtimes\mathcal{G});\\ &\gamma_0(D)\big((F\otimes 1)\restriction_{\mathcal{E}_0} - F_0\big)\subset\mathcal{K}(\mathcal{E}_0), \end{aligned}$$

which allows us to conclude the above computation by stating that

$$\begin{aligned} & i(F)i_0\big(T_\xi\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big)\\ &\quad = i_0\big(T_\xi\gamma_0(b)\big)\gamma_{0*}(x)i^0\big(\gamma_0(c)T_\xi^*\big)i(F) \mod \mathcal{K}(\mathcal{E}\rtimes\mathcal{G}). \end{aligned}$$

The other statements of (3.1) are obtained by a direct computation. For instance, for all $x\in A\rtimes\mathcal{G}$ we have $\gamma_*(x)(i(F)^*-i(F))\in\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$. Indeed, this follows from the fact that $\{\widehat{\theta}(y)i(\gamma(a));\ y\in\widehat{S},\ a\in A\}$ is a total subset of $\gamma_*(A\rtimes\mathcal{G})$ and the fact that $\widehat{\theta}(y)i(\gamma(a))(i(F)^*-i(F)) = \widehat{\theta}(y)i(\gamma(a)(F^*-F))\in\mathcal{K}(\mathcal{E}\rtimes\mathcal{G})$ for all $y\in\widehat{S}$ and $a\in A$.

It follows from the definition of the dual action (2.4.4) and the fact that $T_{F\xi} = i(F)T_\xi$ for all $\xi\in\mathcal{E}$ that $i(F)$ is $\delta_{\mathcal{E}\rtimes\mathcal{G}}$-invariant. It is also straightforward that $[i(F),\alpha_{\mathcal{E}\rtimes\mathcal{G}}(n)] = 0$ for all $n\in N$. Hence, $(\mathcal{E}\rtimes\mathcal{G},\gamma_*,i(F))$ is an equivariant Kasparov bimodule (Remark 3.1.3 (3)).

It is clear that $(\mathcal{E},\gamma,F)\in\mathsf{E}_{\mathcal{G}}(A,B)$ defines a unique $(\mathcal{E}\rtimes\mathcal{G},\gamma_*,i(F))\in\mathsf{E}_{\widehat{\mathcal{G}}}(A\rtimes\mathcal{G},B\rtimes\mathcal{G})$ (cf. Proposition 2.4.7). If $(\mathcal{E},F)\in\mathsf{E}_{\mathcal{G}}(A,B[0,1])$ is a homotopy between $(\mathcal{E}_1,F_1)$ and $(\mathcal{E}_2,F_2)$, then $(\mathcal{E}\rtimes\mathcal{G},i(F))$ is a homotopy between $(\mathcal{E}_1\rtimes\mathcal{G},i(F_1))$ and $(\mathcal{E}_2\rtimes\mathcal{G},i(F_2))$. This statement makes sense in virtue of the following result.

Lemma 3.4.5. *There exists a unique equivariant* **-isomorphism*

$$B[0,1]\rtimes\mathcal{G}\to B\rtimes\mathcal{G}[0,1];\quad \pi_{B[0,1]}(f)\widehat{\theta}_{B[0,1]}(x)\mapsto\big[t\mapsto\pi_B\big(f(t)\big)\widehat{\theta}_B(x)\big].$$

Proof of Lemma 3.4.5. We have the identifications (cf. Definition 3.1.4 (4))

$$\mathcal{L}\big(B[0,1]\otimes\mathcal{H}\big)=\mathcal{M}\big(B[0,1]\otimes\mathcal{K}\big)=\mathcal{M}\big((B\otimes\mathcal{K})[0,1]\big)$$
$$=\mathcal{M}(B\otimes\mathcal{K})[0,1]=\mathcal{L}(B\otimes\mathcal{H})[0,1].$$

For $f\in B[0,1]$ and $x\in\widehat{S}$, the operator $\pi_{B[0,1],L}(f)(1_{B[0,1]}\otimes\rho(x))\in\mathcal{L}(B[0,1]\otimes\mathcal{H})$ is identified to the continuous function $[t\mapsto\pi_{B,L}(f(t))(1_B\otimes\rho(x))]\in\mathcal{L}(B\otimes\mathcal{H})[0,1]$. Furthermore, we also have the identifications

$$\mathcal{L}(\mathcal{E}_{B[0,1],L})=\big\{T\in\mathcal{L}\big(B[0,1]\otimes\mathcal{H}\big);\ Tq_{\beta_{B[0,1]}\alpha}=T=q_{\beta_{B[0,1]}\alpha}T\big\};$$
$$\mathcal{L}(\mathcal{E}_{B,L})=\big\{T\in\mathcal{L}(B\otimes\mathcal{H});\ Tq_{\beta_B\alpha}=T=q_{\beta_B\alpha}T\big\}.$$

We then obtain an identification between the C*-algebras $\mathcal{L}(\mathcal{E}_{B[0,1],L})$ and $\mathcal{L}(\mathcal{E}_{B,L})[0,1]$, which identifies $\pi_{B[0,1]}(f)\widehat{\theta}_{B[0,1]}(x)$ with $[t\mapsto\pi_B(f(t))\widehat{\theta}_B(x)]$. By restriction, we obtain an injective *-homomorphism $\phi:B[0,1]\rtimes\mathcal{G}\to B\rtimes\mathcal{G}[0,1]$. Moreover, for all $b\in B$, $x\in\widehat{S}$, and $f\in\mathrm{C}([0,1])$ we have $f\otimes\pi_B(b)\widehat{\theta}_B(y)=\phi(\pi_{B[0,1]}(f\otimes b)\widehat{\theta}_B(x))$ (cf. $B[0,1]=\mathrm{C}([0,1])\otimes B$ and $B\rtimes\mathcal{G}[0,1]=\mathrm{C}([0,1])\otimes B\rtimes\mathcal{G}$), which proves that the range of ϕ is dense. The surjectivity of ϕ is then proved. The $\mathcal{G}$-equivariance of ϕ is a direct consequence of the definition. ∎

End of the proof of Theorem 3.4.4. (2) Straightforward.

(3) Let $x_1\in\mathsf{KK}_{\mathcal{G}}(A,C)$ and $x_2\in\mathsf{KK}_{\mathcal{G}}(C,B)$. For $i=1,2$, we consider an equivariant Kasparov bimodule $(\mathcal{E}_i,\gamma_i,F_i)$ such that $x_i=[(\mathcal{E}_i,\gamma_i,F_i)]$. Let us consider the $\mathcal{G}$-equivariant Hilbert B-module $\mathcal{E}:=\mathcal{E}_1\otimes_{\gamma_2}\mathcal{E}_2$, the $\mathcal{G}$-equivariant *-representation $\gamma:A\to\mathcal{L}(\mathcal{E}_2)$ defined by $\gamma(a):=\gamma_1(a)\otimes_{\gamma_2}1$ for all $a\in A$ and an operator $F\in F_1\#_{\mathcal{G}}F_2\subset\mathcal{L}(\mathcal{E})$ (cf. Definition 3.3.4). Let $y:=x_1\otimes_C x_2=[(\mathcal{E},\gamma,F)]$ (cf. Theorem 3.3.5 and Definition 3.3.6). For $i=1,2$, denote by $(\mathcal{E}'_i,\gamma'_i,F'_i)$ (resp. $(\mathcal{E}',\gamma',F')$) the equivariant Kasparov bimodules obtained from $(\mathcal{E}_i,\gamma_i,F_i)$ (resp. $(\mathcal{E},\gamma,F)$) by the crossed product construction. By definition, we have $J_{\mathcal{G}}(x_i)=[(\mathcal{E}'_i,\gamma'_i,F'_i)]$ for $i=1,2$ and $J_{\mathcal{G}}(y)=[(\mathcal{E}',\gamma',F')]$. We have a canonical identification $\mathcal{E}'=\mathcal{E}'_1\otimes_{\gamma'_2}\mathcal{E}'_2$, which intertwines the left actions (cf. Proposition 2.4.14). Let us denote by $\pi:\mathcal{K}(\mathcal{E})\to\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})$ and $\widehat{\theta}:\widehat{S}\to\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})$ the canonical morphisms. We recall that $F'=\pi(F)$ up to the identification $\mathcal{L}(\mathcal{E}')=\mathcal{M}(\mathcal{K}(\mathcal{E})\rtimes\mathcal{G})$ (cf. Proposition 2.4.9). We also have $F'_1\otimes_{\gamma'_2}1=\pi(F_1\otimes_{\gamma_2}1)$ up to the identification $\mathcal{E}'=\mathcal{E}'_1\otimes_{\gamma'_2}\mathcal{E}'_2$. We have $\gamma(a)[F_1\otimes_{\gamma_2}1,F]\gamma(a)^*\in\mathcal{L}(\mathcal{E})_++\mathcal{K}(\mathcal{E})$ for all $a\in A$ by assumption. For all $a\in A$ and $u\in A\rtimes\mathcal{G}$, we have

$$\gamma'(u)\pi\big(\gamma(a)\big)[F'_1\otimes_{\gamma'_2}1,F']\pi\big(\gamma(a)^*\big)\gamma'(u)^*$$
$$=\gamma'(u)\pi\big(\gamma(a)[F_1\otimes_{\gamma_2}1,F]\gamma(a)^*\big)\gamma'(u)^*\in\mathcal{L}(\mathcal{E}')_++\mathcal{K}(\mathcal{E}')$$

since $\gamma'(u)\pi(\mathcal{L}(\mathcal{E})_+)\gamma'(u)^*\subset\mathcal{L}(\mathcal{E}')_+$ and $\gamma'(u)\pi(\mathcal{K}(\mathcal{E}))\gamma'(u)^*\subset\mathcal{K}(\mathcal{E}')$.

The positivity condition follows from the non-degeneracy of the canonical morphism $\pi_A:A\to\mathcal{M}(A\rtimes\mathcal{G})$. Indeed, for any $v\in A\rtimes\mathcal{G}$ there exist $a\in A$ and $u\in A\rtimes\mathcal{G}$ such that $v=u\pi_A(a)$. Then $\gamma'(v)=\gamma'(u)\gamma'(\pi_A(a))=\gamma'(u)\pi(\gamma(a))$. The compatibility with the direct sum is straightforward. ∎

In a similar way, we prove the following theorem.

Theorem 3.4.6. *Let A, B, and C be $\widehat{\mathcal{G}}$-C^*-algebras.*

(1) *If $(\mathcal{F}, G)$ is a $\widehat{\mathcal{G}}$-equivariant Kasparov A-B-bimodule (with a non-degenerate left action), then $(\mathcal{F} \rtimes \widehat{\mathcal{G}}, G \otimes_{\widehat{\pi}_B} 1)$ is a $\mathcal{G}$-equivariant Kasparov $A \rtimes \widehat{\mathcal{G}}$-$B \rtimes \widehat{\mathcal{G}}$-bimodule. Moreover, if $(\mathcal{F}_1, G_1)$ and $(\mathcal{F}_2, G_2)$ are unitarily equivalent (resp. homotopic) $\widehat{\mathcal{G}}$-equivariant Kasparov A-B-bimodules, then so are $(\mathcal{F}_1 \rtimes \widehat{\mathcal{G}}, G_1 \otimes_{\widehat{\pi}_B} 1)$ and $(\mathcal{F}_2 \rtimes \widehat{\mathcal{G}}, G_2 \otimes_{\widehat{\pi}_B} 1)$.*

Let $J_{\widehat{\mathcal{G}}} : \mathsf{KK}_{\widehat{\mathcal{G}}}(A, B) \to \mathsf{KK}_{\mathcal{G}}(A \rtimes \widehat{\mathcal{G}}, B \rtimes \widehat{\mathcal{G}})$ be the homomorphism of groups defined for all $[(\mathcal{F}, G)] \in \mathsf{KK}_{\widehat{\mathcal{G}}}(A, B)$ (with a non-degenerate left action) by

$$J_{\widehat{\mathcal{G}}}\big([(\mathcal{F}, G)]\big) = \big[(\mathcal{F} \rtimes \widehat{\mathcal{G}}, G \otimes_{\widehat{\pi}_B} 1)\big].$$

(2) *Let $\phi : A \to B$ be a $\widehat{\mathcal{G}}$-equivariant $*$-homomorphism. We recall that the equivariance of ϕ allows us to define a $\mathcal{G}$-equivariant $*$-homomorphism $\phi_* : A \rtimes \widehat{\mathcal{G}} \to B \rtimes \widehat{\mathcal{G}}$. We have $J_{\widehat{\mathcal{G}}}([\phi]) = [\phi_*]$. In particular, we have $J_{\widehat{\mathcal{G}}}(1_A) = 1_{A \rtimes \widehat{\mathcal{G}}}$.*

(3) *Assume the C^*-algebra A to be separable. For all $x_1 \in \mathsf{KK}_{\widehat{\mathcal{G}}}(A, C)$ and $x_2 \in \mathsf{KK}_{\widehat{\mathcal{G}}}(C, B)$, we have*

$$J_{\widehat{\mathcal{G}}}(x_1 \otimes_C x_2) = J_{\widehat{\mathcal{G}}}(x_1) \otimes_{C \rtimes \widehat{\mathcal{G}}} J_{\widehat{\mathcal{G}}}(x_2).$$

Notations 3.4.7. Before stating the main theorem of this article, we need to specify some further notations. Let A (resp. B) be a $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C^*-algebra. Let D (resp. E) be the bidual $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C^*-algebra defined in Notations 2.3.4. We recall that $\mathcal{E}_{A,R}$ (resp. $\mathcal{E}_{B,\rho}$) defines a $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-equivariant Morita equivalence between A (resp. B) and D (resp. E) (cf. Theorem 2.3.11). Let us define

$$\mathfrak{b}_A := \big[(\mathcal{E}_{A,R}, 0)\big] \in \mathsf{KK}_{\mathcal{G}}(D, A) \quad \text{and} \quad \mathfrak{a}_A := \big[(\mathcal{E}_{A,R}^*, 0)\big] \in \mathsf{KK}_{\mathcal{G}}(A, D)$$

$$\big(\text{resp. } \widehat{\mathfrak{b}}_B := \big[(\mathcal{E}_{B,\rho}, 0)\big] \in \mathsf{KK}_{\widehat{\mathcal{G}}}(E, B) \quad \text{and} \quad \widehat{\mathfrak{a}}_B := \big[(\mathcal{E}_{B,\rho}^*, 0)\big] \in \mathsf{KK}_{\widehat{\mathcal{G}}}(B, E)\big).$$

Lemma 3.4.8. *Let A be a separable $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C^*-algebra. Let D (resp. E) be the bidual $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C^*-algebra of A. We have*

$$\mathfrak{b}_A \otimes_A \mathfrak{a}_A = 1_D \quad \textit{and} \quad \mathfrak{a}_A \otimes_D \mathfrak{b}_A = 1_A$$

$$\big(\textit{resp. } \widehat{\mathfrak{b}}_A \otimes_A \widehat{\mathfrak{a}}_A = 1_E \quad \textit{and} \quad \widehat{\mathfrak{a}}_A \otimes_E \widehat{\mathfrak{b}}_A = 1_A\big).$$

In particular, if A and B are separable $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C^*-algebras, then the map

$$\mathsf{KK}_{\mathcal{G}}(A, B) \to \mathsf{KK}_{\mathcal{G}}(D_{\mathrm{g}}, D_{\mathrm{d}}); \quad x \mapsto \mathfrak{b}_A \otimes_A x \otimes_B \mathfrak{a}_B$$

$$\big(\textit{resp. } \mathsf{KK}_{\widehat{\mathcal{G}}}(A, B) \to \mathsf{KK}_{\widehat{\mathcal{G}}}(E_{\mathrm{g}}, E_{\mathrm{d}}); \quad x \mapsto \widehat{\mathfrak{b}}_A \otimes_A x \otimes_B \widehat{\mathfrak{a}}_B\big)$$

are isomorphisms of abelian groups (cf. Convention 2.5.26 for the writing conventions).

Proof. This is a consequence of [9, Proposition 7.12] and Theorem 2.3.11. ∎

We can state the main result of this paragraph. We refer the reader to [3, Théorème 6.20], [3, Remarque 7.7 (b)], and [31, §5.1] for the corresponding statement in the quantum group framework.

Theorem 3.4.9. *Let A and B be $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebras. If A is separable and B is σ-unital, then for all $x \in \mathsf{KK}_{\mathcal{G}}(A,B)$ (resp. $x \in \mathsf{KK}_{\widehat{\mathcal{G}}}(A,B)$), we have*

$$J_{\widehat{\mathcal{G}}} \circ J_{\mathcal{G}}(x) = \mathfrak{b}_A \otimes_A x \otimes_B \mathfrak{a}_B \quad \big(\text{resp. } J_{\mathcal{G}} \circ J_{\widehat{\mathcal{G}}}(x) = \widehat{\mathfrak{b}}_A \otimes_A x \otimes_B \widehat{\mathfrak{a}}_B\big)$$

up to the identifications $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} = D_{\mathrm{g}}$ and $(B \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} = D_{\mathrm{d}}$ (resp. $(A \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G} = E_{\mathrm{g}}$ and $(B \rtimes \widehat{\mathcal{G}}) \rtimes \mathcal{G} = E_{\mathrm{d}}$) (cf. Theorem 2.3.7).

Proof. Let $x \in \mathsf{KK}_{\mathcal{G}}(A,B)$. It suffices to prove that $J_{\widehat{\mathcal{G}}}(J_{\mathcal{G}}(x)) \otimes_{D_{\mathrm{d}}} \mathfrak{b}_B = \mathfrak{b}_A \otimes_A x$. Let $(\mathcal{E}, \gamma, F) \in \mathsf{E}_{\mathcal{G}}(A,B)$ such that $x = [(\mathcal{E}, \gamma, F)]$. With no loss of generality, we can assume that the *-representation γ is non-degenerate. Let us consider the canonical morphisms $\pi : \mathcal{K}(\mathcal{E}) \to \mathcal{M}(\mathcal{K}(\mathcal{E}) \rtimes \mathcal{G})$ and $\widehat{\pi} : \mathcal{K}(\mathcal{E}) \rtimes \mathcal{G} \to \mathcal{M}((\mathcal{K}(\mathcal{E}) \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}})$. We make the identifications $\mathcal{M}((\mathcal{K}(\mathcal{E}) \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}) = \mathcal{L}((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}})$ (cf. Proposition 2.4.9 and Corollary 2.4.20). We have

$$J_{\widehat{\mathcal{G}}} \circ J_{\mathcal{G}}(x) = \big[\big((\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}, \widehat{\pi} \circ \pi(F)\big)\big]$$

(recall that $(F \otimes_{\widehat{\pi}_B} 1) \otimes_{\pi_B} 1$ is identified with $\widehat{\pi} \circ \pi(F)$). Let us compute the Kasparov product $J_{\widehat{\mathcal{G}}}(J_{\mathcal{G}}(x)) \otimes_{D_{\mathrm{d}}} \mathfrak{b}_B$. Denote by $\pi_{D_{\mathrm{d}}} : D_{\mathrm{d}} \to \mathcal{L}(\mathcal{E}_{B,R})$ the equivariant *-representation given by $\pi_{D_{\mathrm{d}}}(d) := d\restriction_{\mathcal{E}_{B,R}}$ for all $d \in D_{\mathrm{d}}$. Recall that we have the identification of equivariant Hilbert D_{g}-B-bimodules $(\mathcal{E} \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \otimes_{\pi_{D_{\mathrm{d}}}} \mathcal{E}_{B,R} = \mathcal{E}_{\mathcal{E},R}$ (cf. Corollary 2.4.24) and the operator $\widehat{\pi}(\pi(F))$ is identified to $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}}$. Hence,

$$J_{\widehat{\mathcal{G}}}\big(J_{\mathcal{G}}(x)\big) \otimes_{D_{\mathrm{d}}} \mathfrak{b}_B = \big[\big(\mathcal{E}_{\mathcal{E},R}, \pi_R(F)\big)\big].$$

Let us compute the Kasparov product $\mathfrak{b}_A \otimes_A x$. We have an identification of Hilbert D_{g}-B-bimodules $\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E} = \mathcal{E}_{\mathcal{E},R}$ (cf. Lemma 3.4.1 (1)). It is easily seen that this identification is $\mathcal{G}$-equivariant. By Lemma 3.4.1, the operator $\pi_R(F)\restriction_{\mathcal{E}_{\mathcal{E},R}} \in \mathcal{L}(\mathcal{E}_{A,R} \otimes_\gamma \mathcal{E})$ is an F-connection. Since the positivity condition is trivial, we have proved that $\mathfrak{b}_A \otimes_A x = [(\mathcal{E}_{\mathcal{E},R}, \pi_R(F))]$. ∎

Corollary 3.4.10. *The homomorphisms $J_{\mathcal{G}}$ and $J_{\widehat{\mathcal{G}}}$ are isomorphisms of abelian groups.*

Proof. This follows from Lemma 3.4.8 and Theorem 3.4.9. ∎

Corollary 3.4.11. *If A is a separable $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$)-C*-algebra, then the descent morphism $J_{\mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(A,A) \to \mathsf{KK}_{\widehat{\mathcal{G}}}(A \rtimes \mathcal{G}, A \rtimes \mathcal{G})$ (resp. $J_{\widehat{\mathcal{G}}} : \mathsf{KK}_{\widehat{\mathcal{G}}}(A,A) \to \mathsf{KK}_{\mathcal{G}}(A \rtimes \widehat{\mathcal{G}}, A \rtimes \widehat{\mathcal{G}})$) is an isomorphism of rings.*

Proof. This is a straightforward consequence of Theorem 3.4.9 and Corollary 3.4.10. ∎

4. Monoidal equivalence and equivariant KK-theory

In this chapter, we fix a colinking measured quantum groupoid $\mathscr{G} := \mathscr{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between two monoidally equivalent regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$.

4.1. Description of the $\mathscr{G}_{\mathbb{G}_1,\mathbb{G}_2}$-equivariant Kasparov bimodules

In this paragraph, we fix two $\mathscr{G}$-C*-algebras A and B. We also fix a $\mathscr{G}$-equivariant Hilbert A-B-bimodule $(\mathcal{E}, \gamma)$. We use all the notations and results of [9, §5.2] and Section 2.5.1 concerning these objects. We assume the C*-algebra A to be separable. In particular, for $j = 1, 2$ the C*-algebra A_j is separable.

Lemma 4.1.1. *Let $F \in \mathcal{L}(\mathcal{E})$. There exist unique operators $F_1 \in \mathcal{L}(\mathcal{E}_1)$ and $F_2 \in \mathcal{L}(\mathcal{E}_2)$ such that $F = F_1 \oplus F_2$. We have the following statements:*

(1) *the pair $(\mathcal{E}, \gamma, F)$ is a Kasparov A-B-bimodule if and only if for $j = 1, 2$ the pair $(\mathcal{E}_j, \gamma_j, F_j)$ is a Kasparov A_j-B_j-bimodule,*

(2) *the conditions below are equivalent:*

 (i) $(\gamma \otimes \mathrm{id}_S)(x)(\delta_{\mathcal{K}(\mathcal{E})}(F) - q_{\beta_{\mathcal{E}}\alpha}(F \otimes 1_S)) \in \mathcal{K}(\mathcal{E} \otimes S)$ *for all* $x \in A \otimes S$,

 (ii) $(\gamma_k \otimes \mathrm{id}_{S_{kj}})(x)(\delta^k_{\mathcal{K}(\mathcal{E}_j)}(F_j) - F_k \otimes 1_{S_{kj}}) \in \mathcal{K}(\mathcal{E}_k \otimes S_{kj})$ *for all* $x \in A_k \otimes S_{kj}$ *and* $j, k = 1, 2$,

(3) *if the triple $(\mathcal{E}, \gamma, F)$ is a $\mathscr{G}$-equivariant Kasparov A-B-bimodule, then the triple $(\mathcal{E}_j, \gamma_j, F_j)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule for $j = 1, 2$.*

Proof. We recall that $\beta_{\mathcal{E}}(\mathbb{C}^2) \subset \mathcal{Z}(\mathcal{L}(\mathcal{E}))$ (cf. [2, Equation (3.11)]). Hence, we have $[F, q_{\mathcal{E},j}] = 0$ for $j = 1, 2$. Let $F_j := F \restriction_{\mathcal{E}_j} \in \mathcal{L}(\mathcal{E}_j)$ for $j = 1, 2$. We have $F = F_1 \oplus F_2$. The equivalence of the first statement follows from the relation $\mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}_1) \oplus \mathcal{K}(\mathcal{E}_2)$ and the definitions; for instance we have $[\gamma(a), F]\xi = \sum_{j=1,2} [\gamma_j(q_{A,j}a), F_j] q_{\mathcal{E},j}\xi$ for all $\xi \in \mathcal{E}$. Condition (ii) is just a straightforward restatement of condition (i). Statement (3) follows by taking $k = j$ in (ii) and by using statement (1). ∎

Proposition-Definition 4.1.2. *With the notations and hypotheses of Lemma 4.1.1, for $j = 1, 2$ the map*

$$J_{\mathbb{G}_j,\mathscr{G}} : \mathsf{KK}_{\mathscr{G}}(A, B) \to \mathsf{KK}_{\mathbb{G}_j}(A_j, B_j); \quad \big[(\mathcal{E}, \gamma, F)\big] \mapsto \big[(\mathcal{E}_j, \gamma_j, F_j)\big]$$

is a homomorphism of abelian groups.

Proof. We first prove that $J_{\mathbb{G}_j,\mathscr{G}}$ is well defined. By Lemmas 2.5.6 (1) and 2.5.8 (1), we have well-defined maps:

$$J_{\mathbb{G}_j,\mathscr{G}} : \mathsf{E}_{\mathscr{G}}(A, B) \to \mathsf{E}_{\mathbb{G}_j}(A_j, B_j); \quad (\mathcal{E}, \gamma, F) \mapsto (\mathcal{E}_j, \gamma_j, F_j) \quad \text{for } j = 1, 2.$$

The fact that the map $J_{\mathbb{G}_j,\mathscr{G}}$ factorizes over the quotient map $\mathsf{E}_{\mathscr{G}}(A, B) \to \mathsf{KK}_{\mathscr{G}}(A, B)$ will follow from the following result.

Lemma 4.1.3. *Let C be a third $\mathcal{G}$-C*-algebra. Let $g : B \to C$ be a $\mathcal{G}$-equivariant *-homomorphism. We recall that g induces a $\mathbb{G}_j$-equivariant *-homomorphism $g_j : B_j \to C_j$ for $j = 1, 2$ (cf.* [9, Proposition 5.2.3 (1)]*). Then, the diagram*

$$\begin{array}{ccc} \mathsf{E}_{\mathcal{G}}(A, B) & \xrightarrow{g_*} & \mathsf{E}_{\mathcal{G}}(A, C) \\ {\scriptstyle J_{\mathbb{G}_j,\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle J_{\mathbb{G}_j,\mathcal{G}}} \\ \mathsf{E}_{\mathbb{G}_j}(A_j, B_j) & \xrightarrow{(g_j)_*} & \mathsf{E}_{\mathbb{G}_j}(A_j, C_j) \end{array}$$

commutes for all $j = 1, 2$.

The proof of the above lemma is effortless and the details are left to the reader. We recall that $B[0, 1]$ is a $\mathcal{G}$-C*-algebra (cf. Definition 3.1.4 (4)). Then, we apply the notations of [9, §5.2] as follows:

- $B[0, 1]_j := \beta_{B[0,1]}(\varepsilon_j)B[0, 1] = B_j[0, 1]$ for $j = 1, 2$;
- $\delta^k_{B_j[0,1]} : B_j[0, 1] \to \mathcal{M}(B_k[0, 1] \otimes S_{kj}) = \mathcal{M}(B_k \otimes S_{kj})[0, 1]$ for $j, k = 1, 2$, the *-homomorphism defined by $\delta^k_{B_j[0,1]}(f)(t) := \delta^k_{B_j}(f(t))$ for all $f \in B_j[0, 1]$ and $t \in [0, 1]$.

We recall that for $t \in [0, 1]$ the evaluation map $\mathrm{e}_t : B[0, 1] \to B$ is $\mathcal{G}$-equivariant. Moreover, for $j = 1, 2$, it is clear that the *-homomorphism $(\mathrm{e}_t)_j : B_j[0, 1] \to B_j$ is the evaluation at t. It then follows from the above lemma that the image of a homotopy of $\mathsf{E}_{\mathcal{G}}(A, B[0, 1])$ by $\mathsf{E}_{\mathcal{G}}(A, B[0, 1]) \to \mathsf{E}_{\mathbb{G}_j}(A_j, B_j[0, 1])$ is a homotopy, which finally proves that the map $J_{\mathbb{G}_j,\mathcal{G}}$ is well defined on $\mathsf{KK}_{\mathcal{G}}(A, B)$. The compatibility with the direct sum is straightforward. ■

Proposition 4.1.4. *Let C be a third $\mathcal{G}$-C*-algebra. For $j = 1, 2$, we have*

(1) $J_{\mathbb{G}_j,\mathcal{G}}(1_A) = 1_{A_j}$,

(2) *for all $x \in \mathsf{KK}_{\mathcal{G}}(A, C)$ and $y \in \mathsf{KK}_{\mathcal{G}}(C, B)$,*
$J_{\mathbb{G}_j,\mathcal{G}}(x \otimes_C y) = J_{\mathbb{G}_j,\mathcal{G}}(x) \otimes_{C_j} J_{\mathbb{G}_j,\mathcal{G}}(y)$ *in* $\mathsf{KK}_{\mathbb{G}_j}(A_j, B_j)$.

Proof. The first statement is straightforward. Let us write $x := [(\mathcal{E}, F)] \in \mathsf{KK}_{\mathcal{G}}(A, C)$ and $y := [(\mathcal{E}', F')] \in \mathsf{KK}_{\mathcal{G}}(C, B)$. Let $\gamma : C \to \mathcal{L}(\mathcal{E}')$ be the left action of C on $\mathcal{E}'$. Let $\mathcal{F} := \mathcal{E} \otimes_\gamma \mathcal{E}'$. We have $x \otimes_C y = [(\mathcal{F}, T)]$ for some $T \in F\#_{\mathcal{G}} F' \subset \mathcal{L}(\mathcal{F})$. In the following, we use the notations of Section 2.5.1 and Lemma 4.1.1 for the Kasparov bimodules $(\mathcal{E}, F)$, $(\mathcal{E}', F')$, and $(\mathcal{F}, T)$. We have a well-defined B_j-linear isometric map $\Phi : \mathcal{E}_j \otimes_{\gamma_j} \mathcal{E}'_j \to \mathcal{F}_j$; $\xi \otimes_{\gamma_j} \eta \mapsto \xi \otimes_\gamma \eta$. Let $\xi \in \mathcal{E}$ and $\eta \in \mathcal{E}'$. Let us write $\xi = \zeta c$ with $\zeta \in \mathcal{E}$ and $c \in C$. We have

$$\begin{aligned} q_{\mathcal{F},j}(\xi \otimes_\gamma \eta) &= q_{\mathcal{E},j}\xi \otimes_\gamma \eta = (q_{\mathcal{E},j}\zeta)c \otimes_\gamma \eta = (q_{\mathcal{E},j}\zeta)q_{C,j}c \otimes_\gamma \eta \\ &= q_{\mathcal{E},j}\zeta \otimes_\gamma \gamma(q_{C,j}c)\eta = q_{\mathcal{E},j}\zeta \otimes_\gamma \gamma(c)q_{\mathcal{E}',j}\eta = q_{\mathcal{E},j}\xi \otimes_\gamma q_{\mathcal{E}',j}\eta. \end{aligned}$$

Therefore, the range of Φ contains the total subset $\{q_{\mathcal{F},j}(\xi \otimes_\gamma \eta);\ \xi \in \mathcal{E},\ \eta \in \mathcal{E}'\}$ of $\mathcal{F}_j$. Hence, $\Phi \in \mathcal{L}(\mathcal{E}_j \otimes_{\gamma_j} \mathcal{E}'_j, \mathcal{F}_j)$ and Φ is unitary. We have $\Phi^*(\xi \otimes_\gamma \eta) = q_{\mathcal{E},j}\xi \otimes_{\gamma_j} q_{\mathcal{E}',j}\eta$ for all $\xi \otimes_\gamma \eta \in \mathcal{F}_j$. It is clear that Φ is $\mathbb{G}_j$-equivariant and intertwines the left actions of A_j.

Therefore, $(\mathcal{E}_j \otimes_{\gamma_j} \mathcal{E}'_j, \Phi^* T_j \Phi)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule unitarily equivalent to $(\mathcal{F}_j, T_j)$. Hence, $[(\mathcal{E}_j \otimes_{\gamma_j} \mathcal{E}'_j, \Phi^* T_j \Phi)] = [(\mathcal{F}_j, T_j)]$ in $\mathsf{KK}_{\mathbb{G}_j}(A_j, B_j)$.

Since T is an F'-connection for $\mathcal{E}$, the operator $\Phi^* T_j \Phi$ is an F'_j-connection for $\mathcal{E}_j$. Indeed, for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{E}'$, we have

$$q_{\mathcal{F},j} T_\xi(\eta) = \Phi(q_{\mathcal{E},j}\xi \otimes_{\gamma_j} q_{\mathcal{E}',j}\eta) = \Phi T_{q_{\mathcal{E},j}\xi} q_{\mathcal{E}',j}(\eta).$$

Hence, $q_{\mathcal{F},j} T_\xi F' = \Phi T_{q_{\mathcal{E},j}\xi} F'_j q_{\mathcal{E}',j}$. We also have $q_{\mathcal{F},j} T T_\xi = T_j \Phi T_{q_{\mathcal{E},j}\xi} q_{\mathcal{E}',j}$. Hence,

$$T_\xi F'_j - \Phi^* T_j \Phi T_\xi = \Phi^* q_{\mathcal{F},j}(T_\xi F' - T T_\xi)\restriction_{\mathcal{E}'_j} \in \mathcal{K}(\mathcal{E}'_j, \mathcal{F}_j) \quad \text{for all } \xi \in \mathcal{E}_j.$$

Let $\phi : A \to \mathcal{L}(\mathcal{E})$ and $\pi : A \to \mathcal{L}(\mathcal{F})$; $a \mapsto \phi(a) \otimes_\gamma 1$ be the left actions of A. For $j = 1, 2$, we have $\phi_j : A_j \to \mathcal{L}(\mathcal{E}_j)$ and $\pi_j : A_j \to \mathcal{L}(\mathcal{F}_j)$. We have $q_{\mathcal{F},j}(F \otimes_\gamma 1) = \Phi(F_j \otimes_{\gamma_j} 1)\Phi^* q_{\mathcal{F},j}$. Hence, we have

$$\begin{aligned} &q_{\mathcal{F},j}(F \otimes_\gamma 1)T = \Phi(F_j \otimes_{\gamma_j} 1)\Phi^* T_j q_{\mathcal{F},j}; \\ &q_{\mathcal{F},j} T(F \otimes_\gamma 1) = T_j q_{\mathcal{F},j}(F \otimes_\gamma 1) = T_j \Phi(F_j \otimes_{\gamma_j} 1)\Phi^* q_{\mathcal{F},j}. \end{aligned}$$

Hence,

$$\begin{aligned} q_{\mathcal{F},j}[F \otimes_\gamma 1, T] &= \big(\Phi(F_j \otimes_{\gamma_j} 1)\Phi^* T_j - T_j \Phi(F_j \otimes_{\gamma_j} 1)\Phi^*\big) q_{\mathcal{F},j} \\ &= \Phi[F_j \otimes_{\gamma_j} 1, \Phi^* T_j \Phi]\Phi^* q_{\mathcal{F},j}. \end{aligned}$$

Therefore, for all $a \in A_j$ we have

$$\begin{aligned} \pi(a)[F \otimes_\gamma 1, T]\pi(a)\restriction_{\mathcal{E}_j} &= \pi_j(a)\Phi[F_j \otimes_{\gamma_j} 1, \Phi^* T_j \Phi]\Phi^* \pi_j(a) \\ &= \Phi(\phi_j(a) \otimes_{\gamma_j} 1)[F_j \otimes_{\gamma_j} 1, \Phi^* T_j \Phi](\phi_j(a) \otimes_{\gamma_j} 1)\Phi^*. \end{aligned}$$

Hence, the image of $(\phi_j(a) \otimes_{\gamma_j} 1)[F_j \otimes_{\gamma_j} 1, \Phi^* T_j \Phi](\phi_j(a) \otimes_{\gamma_j} 1)$ in $\mathcal{L}(\mathcal{E}_j)/\mathcal{K}(\mathcal{E}_j)$ is positive. Hence, $\Phi^* T_j \Phi \in F_j \#_{\mathbb{G}_j} F'_j$ and

$$\big[(\mathcal{E}_j \otimes_{\gamma_j} \mathcal{E}'_j, \Phi^* T_j \Phi)\big] = \big[(\mathcal{E}_j, F_j)\big] \otimes_{C_j} \big[(\mathcal{E}'_j, F'_j)\big] = J_{\mathbb{G}_j,\mathcal{G}}(x) \otimes_{C_j} J_{\mathbb{G}_j,\mathcal{G}}(y). \quad \blacksquare$$

4.2. Induction of equivariant Kasparov bimodules

We begin this paragraph with two technical lemmas that will be used in the proof of Proposition 4.2.11.

Lemma 4.2.1. *Let $\mathbb{G}$ be a locally compact quantum group. Let A, A', B, and B' be $\mathbb{G}$-C*-algebras. Let $(\mathcal{E}, \phi)$ (resp. $(\mathcal{E}', \phi')$) be a $\mathbb{G}$-equivariant Hilbert A-B (resp. A'-B')-bimodule. Denote by A_0 (resp. B_0) the $\mathbb{G}$-C*-algebras $A \oplus A'$ (resp. $B \oplus B'$). Denote by $\mathcal{E}_0$ the Hilbert B_0-module $\mathcal{E} \oplus \mathcal{E}'$. Denote by ϕ_0 the *-representation of A_0 on $\mathcal{E}_0$ defined by $\phi_0(a \oplus a') := \phi(a) \oplus \phi'(a')$ for all $a \in A$ and $a' \in A'$. Fix $T \in \mathcal{L}(\mathcal{E})$ and $T' \in \mathcal{L}(\mathcal{E}')$ and denote $T_0 := T \oplus T' \in \mathcal{L}(\mathcal{E}_0)$.*

(1) *The triple $(\mathcal{E}_0, \phi_0, T_0)$ is a $\mathbb{G}$-equivariant Kasparov A_0-B_0-bimodule if and only if the triples $(\mathcal{E}, \phi, T)$ and $(\mathcal{E}', \phi', T')$ are $\mathbb{G}$-equivariant Kasparov bimodules.*

(2) *Denote by* $p_{\mathrm{g}} : A_0 \to A$, $p'_{\mathrm{g}} : A_0 \to A'$, $p_{\mathrm{d}} : B_0 \to B$, *and* $p'_{\mathrm{d}} : B_0 \to B'$ *the canonical surjections. Denote also by* $i_{\mathrm{g}} : A \to A_0$, $i'_{\mathrm{g}} : A' \to A_0$, $i_{\mathrm{d}} : B \to B_0$, *and* $i'_{\mathrm{d}} : B \to B_0$ *the canonical injections. Assume the* C*-*algebras* A *and* A' *to be separable and* B *and* B' *to be* σ-*unital. If the conditions above hold true, then we have*

$$[i_{\mathrm{g}}] \otimes_{A_0} \big[(\mathcal{E}_0, F_0)\big] \otimes_{B_0} [p_{\mathrm{d}}] = \big[(\mathcal{E}, F)\big];$$
$$[i'_{\mathrm{g}}] \otimes_{A_0} \big[(\mathcal{E}_0, F_0)\big] \otimes_{B_0} [p'_{\mathrm{d}}] = \big[(\mathcal{E}', F')\big];$$
$$[p_{\mathrm{g}}] \otimes_{A} \big[(\mathcal{E}, F)\big] \otimes_{B} [i_{\mathrm{d}}] = \big[(\mathcal{E}_0, F_0)\big] = [p'_{\mathrm{g}}] \otimes_{A'} \big[(\mathcal{E}', F')\big] \otimes_{B'} [i'_{\mathrm{d}}]$$

in $\mathsf{KK}_{\mathbb{G}}(A, B)$, $\mathsf{KK}_{\mathbb{G}}(A', B')$, *and* $\mathsf{KK}_{\mathbb{G}}(A_0, B_0)$, *respectively.*

Proof. (1), (3) The result is a straightforward consequence of the canonical identifications of C*-algebras $\mathcal{K}(\mathcal{E}) \oplus \mathcal{K}(\mathcal{E}') = \mathcal{K}(\mathcal{E}_0)$, $(\mathcal{K}(\mathcal{E}) \otimes \mathrm{C}_0(\mathbb{G})) \oplus (\mathcal{K}(\mathcal{E}') \otimes \mathrm{C}_0(\mathbb{G})) = \mathcal{K}(\mathcal{E}_0) \otimes \mathrm{C}_0(\mathbb{G})$.

(2) By definition of the structures of $\mathbb{G}$-C*-algebra on A_0 and B_0, the maps defined above are $\mathbb{G}$-equivariant *-homomorphisms. We have

$$[i_{\mathrm{g}}] \otimes_{A_0} \big[(\mathcal{E}_0, \phi_0, F_0)\big] \otimes_{B_0} [p_{\mathrm{d}}] = \big[(\mathcal{E}_0 \otimes_{p_{\mathrm{d}}} B, \big(a \mapsto \phi_0(i_{\mathrm{g}}(a)) \otimes_{p_{\mathrm{d}}} 1\big), F_0 \otimes_{p_{\mathrm{d}}} 1)\big].$$

However, the triples $(\mathcal{E}_0 \otimes_{p_{\mathrm{d}}} B, (a \mapsto \phi_0(i_{\mathrm{g}}(a)) \otimes_{p_{\mathrm{d}}} 1), F_0 \otimes_{p_{\mathrm{d}}} 1)$ and $(\mathcal{E}, \phi, F)$ are unitarily equivalent via the map $\mathcal{E}_0 \otimes_{p_{\mathrm{d}}} B \to \mathcal{E}$; $(\xi \oplus \xi') \otimes_{p_{\mathrm{d}}} b \mapsto \xi b$. Therefore, we obtain the relation $[i_{\mathrm{g}}] \otimes_{A_0} [(\mathcal{E}_0, \phi_0, F_0)] \otimes_{B_0} [p_{\mathrm{d}}] = [(\mathcal{E}, \phi, F)]$. With a similar argument, we also prove that $[i'_{\mathrm{g}}] \otimes_{A_0} [(\mathcal{E}_0, \phi_0, F_0)] \otimes_{B_0} [p'_{\mathrm{d}}] = [(\mathcal{E}', \phi', F')]$. The last formula follows from the first two ones since we have $p_{\mathrm{g}} \circ i_{\mathrm{g}} = \mathrm{id}_A$, $p'_{\mathrm{g}} \circ i'_{\mathrm{g}} = \mathrm{id}_{A'}$, $p_{\mathrm{d}} \circ i_{\mathrm{d}} = \mathrm{id}_B$, and $p'_{\mathrm{d}} \circ i'_{\mathrm{d}} = \mathrm{id}_{B'}$ (cf. Proposition 3.3.9 (2)). ∎

Before stating the second technical lemma, we need to fix the notion of operators acting by factorization.

Proposition-Definition 4.2.2. *Let* B *be a* C*-*algebra. Let* H *and* K *be two Hilbert spaces. Let* $\mathcal{E}$ *be a Hilbert* B-*module. We consider the Hilbert* $B \otimes \mathcal{K}(K)$-*module* $\mathcal{E} \otimes \mathcal{K}(K)$ *and the Hilbert* $B \otimes \mathcal{K}(H)$-*module* $\mathcal{E} \otimes \mathcal{K}(H, K)$. *If* $F \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(K))$, *then there exists a unique operator* $\widetilde{F} \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(H, K))$ *such that*

$$\widetilde{F}(\xi \otimes kT) = F(\xi \otimes k)(1_B \otimes T) \quad \textit{for all } \xi \in \mathcal{E},\ k \in \mathcal{K}(K),\ \textit{and } T \in \mathcal{K}(H, K).$$

The operator $\widetilde{F}$ *will be referred to as the operator* F *acting on* $\mathcal{E} \otimes \mathcal{K}(H, K)$ *by factorization and will sometimes be simply denoted by* F. *Furthermore, for all* $F \in \mathcal{K}(\mathcal{E} \otimes \mathcal{K}(K))$ *we have* $\widetilde{F} \in \mathcal{K}(\mathcal{E} \otimes \mathcal{K}(H, K))$.

Proof. We have $\mathcal{K}(H, K) = \mathcal{K}(K)\mathcal{K}(H, K)$. Indeed, $\mathcal{K}(H, K)$ is Hilbert $\mathcal{K}(K)$-module under the natural left action of $\mathcal{K}(K)$ by composition and the $\mathcal{K}(K)$-valued inner product defined by $\langle T, S\rangle := T \circ S^*$ for $T, S \in \mathcal{K}(H, K)$. Let (u_i) be an approximate unit of the C*-algebra $\mathcal{K}(K)$. Let us fix $\xi \in \mathcal{E}$ and write $\xi = \eta b$ with $\eta \in \mathcal{E}$ and $b \in B$. For all $k \in \mathcal{K}(K)$ and $T \in \mathcal{K}(H, K)$, we have

$$F(\xi \otimes k)(1_B \otimes T) = \lim_i \ F(\eta b \otimes u_i k)(1_B \otimes T) = \lim_i \ F(\eta \otimes u_i)(b \otimes kT).$$

Hence, $F(\xi \otimes k)(1_B \otimes T) = F(\xi \otimes k')(1_B \otimes T')$ for all $k, k' \in \mathcal{K}(K)$ and $T, T' \in \mathcal{K}(H, K)$ such that $kT = k'T'$. Thus, the map $\widetilde{F}$ is well defined. Moreover, it is easily seen that $\widetilde{F} \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(H, K))$ with $\widetilde{F}^* = \widetilde{F^*}$. Note also that the map $F \mapsto \widetilde{F}$ is a *-homomorphism. If the Hilbert space H is nonzero, we have $\mathcal{K}(K) = [\mathcal{K}(H, K)\mathcal{K}(K, H)]$. However, for all $\xi, \eta \in \mathcal{E}$ and $T_1, T_2 \in \mathcal{K}(H, K)$ the image of $\theta_{\xi \otimes k, \eta \otimes T_1 T_2^*}$ by the *-homomorphism $F \mapsto \widetilde{F}$ is the operator $\theta_{\xi \otimes kT_2, \eta \otimes T_1}$. ∎

Lemma 4.2.3. *Let A and B be two* C*-*algebras. Let H and K be two Hilbert spaces. Let $\mathcal{E}$ be a Hilbert A-B-bimodule. We consider the Hilbert $A \otimes \mathcal{K}(K)$-$B \otimes \mathcal{K}(K)$-bimodule $\mathcal{E} \otimes \mathcal{K}(K)$ and the Hilbert $A \otimes \mathcal{K}(K)$-$B \otimes \mathcal{K}(H)$-bimodule $\mathcal{E} \otimes \mathcal{K}(H, K)$. Let $F \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(K))$ such that the pair $(\mathcal{E} \otimes \mathcal{K}(K), F)$ is a Kasparov bimodule. If $\widetilde{F} \in \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(H, K))$ denotes the operator F acting on $\mathcal{E} \otimes \mathcal{K}(H, K)$ by factorization, then the pair $(\mathcal{E} \otimes \mathcal{K}(H, K), \widetilde{F})$ is a Kasparov bimodule.*

Proof. Let $\gamma : A \to \mathcal{L}(\mathcal{E})$ be the left action of A on $\mathcal{E}$. The left action of $A \otimes \mathcal{K}(K)$ on the Hilbert $B \otimes \mathcal{K}(K)$-module $\mathcal{E} \otimes \mathcal{K}(K)$ is

$$\phi := \gamma \otimes \mathrm{id}_{\mathcal{K}(K)} : A \otimes \mathcal{K}(K) \to \mathcal{L}\big(\mathcal{E} \otimes \mathcal{K}(K)\big).$$

It is clear that the left action of $A \otimes \mathcal{K}(K)$ on the Hilbert $B \otimes \mathcal{K}(H)$-module $\mathcal{E} \otimes \mathcal{K}(H, K)$ is the *-representation $\widetilde{\phi} : A \otimes \mathcal{K}(K) \to \mathcal{L}(\mathcal{E} \otimes \mathcal{K}(H, K))$, where for $x \in A \otimes \mathcal{K}(K)$ the operator $\widetilde{\phi}(x)$ is the operator $\phi(x)$ acting on $\mathcal{E} \otimes \mathcal{K}(H, K)$ by factorization. The proof follows from the last statement of Proposition-Definition 4.2.2 and the fact that the factorization map $F \mapsto \widetilde{F}$ is a *-homomorphism. ∎

In the following, we fix two $\mathbb{G}_1$-C*-algebras A_1 and B_1. Let us denote by $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ and $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ the induced $\mathbb{G}_2$-C*-algebras. We also fix a $\mathbb{G}_1$-equivariant Hilbert A_1-B_1-bimodule $(\mathcal{E}_1, \gamma_1)$ (with a non-degenerate left action) and denote by

$$(\mathcal{E}_2, \gamma_2) := \big(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{E}_1), \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \gamma_1\big)$$

the induced $\mathbb{G}_2$-equivariant Hilbert A_2-B_2-bimodule. Let us consider the $\mathcal{G}$-C*-algebras $A := A_1 \oplus A_2$ and $B := B_1 \oplus B_2$. We also equip the Hilbert C*-module $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$ with the structure of $\mathcal{G}$-equivariant Hilbert A-B-bimodule defined by the action $(\beta_{\mathcal{E}}, \delta_{\mathcal{E}})$ of $\mathcal{G}$ (cf. Proposition 2.5.22) and the equivariant *-representation $\gamma : A \to \mathcal{L}(\mathcal{E})$ (cf. Proposition 2.5.25). In what follows, we will make some obvious identifications without always mentioning them explicitly; e.g., $A_1 = A_1 \oplus \{0\}$. We will also use the notations and results of Section 2.5.3 concerning the objects associated with A, B, and $\mathcal{E}$.

Before recalling the definition of the homomorphisms

$$J_{\mathbb{G}_k, \mathbb{G}_j} : \mathsf{KK}_{\mathbb{G}_j}(A_j, B_j) \to \mathsf{KK}_{\mathbb{G}_k}(A_k, B_k)$$

for $j, k = 1, 2$ with $j \neq k$ (cf. [2, §4.5]) we first have to fix some notations.

Notations 4.2.4. Let $\mathbb{G}$ be a regular locally compact quantum group. Let $\mathcal{A}$ be a $\mathbb{G}$-C*-algebra. By the Baaj–Skandalis duality theorem (cf. [4]), we identify the $\mathbb{G}$-C*-algebras $(\mathcal{A} \rtimes \mathbb{G}) \rtimes \widehat{\mathbb{G}}$ and $\mathcal{A} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$. Let $\mathfrak{b}_{\mathcal{A}} := [(\mathcal{A} \otimes \mathrm{L}^2(\mathbb{G}), 0)] \in \mathsf{KK}_{\mathbb{G}}(\mathcal{A} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})), \mathcal{A})$ and $\mathfrak{a}_{\mathcal{A}} := [(\mathcal{A} \otimes \mathrm{L}^2(\mathbb{G})^*, 0)] \in \mathsf{KK}_{\mathcal{G}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})))$ (cf. [3]).

Notations 4.2.5 (cf. [2, Corollaire 3.50 (c) and Notations 3.51]). For all $j, l, l' = 1, 2$, the Hilbert C*-module $\mathcal{B}_{ll',j,\mathrm{g}}$ (resp. $\mathcal{B}_{ll',j,\mathrm{d}}$) is a $\mathbb{G}_j$-equivariant imprimitivity $\mathcal{B}_{l,j,\mathrm{g}}$-$\mathcal{B}_{l',j,\mathrm{g}}$-bimodule (resp. $\mathbb{G}_j$-equivariant imprimitivity $\mathcal{B}_{l,j,\mathrm{d}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule) and we denote by $\mathfrak{c}_{ll',j,\mathrm{g}}$ (resp. $\mathfrak{c}_{ll',j,\mathrm{d}}$) the class of $(\mathcal{B}_{ll',j,\mathrm{g}}, 0)$ (resp. $(\mathcal{B}_{ll',j,\mathrm{d}}, 0)$) in $\mathsf{KK}_{\mathbb{G}_j}(\mathcal{B}_{l,j,\mathrm{g}}, \mathcal{B}_{l',j,\mathrm{g}})$ (resp. $\mathsf{KK}_{\mathbb{G}_j}(\mathcal{B}_{l,j,\mathrm{d}}, \mathcal{B}_{l',j,\mathrm{d}})$).

For $i, j = 1, 2$, we have the faithful non-degenerate *-representations of the C*-algebra S_{ij}, $L_{ij} : S_{ij} \to \mathcal{B}(\mathcal{H}_{ij})$, $x \mapsto L(x)\restriction_{\mathcal{H}_{ij}}$ and $R_{ij} : S_{ij} \to \mathcal{B}(\mathcal{H}_{ji})$, $x \mapsto U_{ij} L_{ij}(x) U_{ji}$, where $U_{kl} = p_{lk} U p_{kl} \in \mathcal{B}(\mathcal{H}_{kl}, \mathcal{H}_{lk})$.

Proposition 4.2.6 (cf. [2, Proposition 4.30 and Corollaire 4.33]). *Let $F_1 \in \mathcal{L}(\mathcal{E}_1)$ such that the pair $(\mathcal{E}_1, F_1)$ is $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule. We have that*

(1) *the pair $(\mathcal{E}_{1,2}, (\mathrm{id}_{\mathcal{K}(\mathcal{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathcal{E}_1)}(F_1))$ is a $\mathbb{G}_1$-equivariant $\mathcal{B}_{1,2,\mathrm{g}}$-$\mathcal{B}_{1,2,\mathrm{d}}$-bimodule,*

(2) *there exists an operator $F_2 \in \mathcal{L}(\mathcal{E}_2)$ such that*

  (a) *$(\mathcal{E}_2, F_2)$ is a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule,*

  (b) *in $\mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$, we have*

$$\begin{aligned}&\mathfrak{b}_{A_2} \otimes_{A_2} \left[(\mathcal{E}_2, F_2)\right] \otimes_{B_2} \mathfrak{a}_{B_2} \\ &\qquad = \mathfrak{c}_{21,2,\mathrm{g}} \otimes_{\mathcal{B}_{1,2,\mathrm{g}}} \left[\left(\mathcal{E}_{1,2}, \left(\mathrm{id}_{\mathcal{K}(\mathcal{E}_2)} \otimes R_{21}\right)\delta^2_{\mathcal{K}(\mathcal{E}_1)}(F_1)\right)\right] \otimes_{\mathcal{B}_{1,2,\mathrm{d}}} \mathfrak{c}_{12,2,\mathrm{d}},\end{aligned}$$

(3) *if $F_2, F_2' \in \mathcal{L}(\mathcal{E}_2)$ satisfy conditions (a) and (b) above, then $[(\mathcal{E}_2, F_2)] = [(\mathcal{E}_2, F_2')]$ in $\mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$.*

If $x := [(\mathcal{E}_1, F_1)] \in \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1)$, let us denote by $J_{\mathbb{G}_2,\mathbb{G}_1}(x)$ the unique element $y \in \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$ satisfying the relation

$$\mathfrak{b}_{A_2} \otimes_{A_2} y \otimes_{B_2} \mathfrak{a}_{B_2} = \mathfrak{c}_{21,2,\mathrm{g}} \otimes_{\mathcal{B}_{1,2,\mathrm{g}}} \left[\left(\mathcal{E}_{1,2}, \left(\mathrm{id}_{\mathcal{K}(\mathcal{E}_2)} \otimes R_{21}\right)\delta^2_{\mathcal{K}(\mathcal{E}_1)}(F_1)\right)\right] \otimes_{\mathcal{B}_{1,2,\mathrm{d}}} \mathfrak{c}_{12,2,\mathrm{d}}.$$

Then, the map $J_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1) \to \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$ is a homomorphism of abelian groups.

In order to define the homomorphism $J_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1)$, we first need to fix further objects.

We consider the induced $\mathbb{G}_1$-C*-algebras

$$A_1' := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2) \quad \text{and} \quad B_1' := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2).$$

We denote by $\mathcal{E}_1' := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathcal{E}_2)$ the induced $\mathbb{G}_1$-equivariant Hilbert A_1'-B_1'-bimodule. Let us consider the $\mathcal{G}$-C*-algebras $A' := A_1' \oplus A_2$ and $B' := B_1' \oplus B_2$. We also equip the Hilbert C*-module $\mathcal{E}' := \mathcal{E}_1' \oplus \mathcal{E}_2$ with the structure of $\mathcal{G}$-equivariant Hilbert A'-B'-bimodule defined by the action $(\beta_{\mathcal{E}'}, \delta_{\mathcal{E}'})$ of $\mathcal{G}$ (cf. Proposition 2.5.22) and the left action

$\gamma' : A' \to \mathcal{L}(\mathcal{E}')$ (cf. Proposition 2.5.25). We will use the notations of Section 2.5.3 decorated with a prime concerning the objects associated with A', B', and $\mathcal{E}'$.

Notations 4.2.7. Let $\pi_{1,\mathrm{g}} : A_1 \to A_1'$ and $\pi_{1,\mathrm{d}} : B_1 \to B_1'$ be the $\mathbb{G}_1$-equivariant *-isomorphisms defined in [9, Proposition 5.2.6]. We recall that we have $\mathcal{G}$-equivariant *-isomorphisms $A \to A'$, $(a_1, a_2) \mapsto (\pi_{1,\mathrm{g}}(a_1), a_2)$ and $B \to B'$, $(b_1, b_2) \mapsto (\pi_{1,\mathrm{d}}(b_1), b_2)$ (cf. [2, §4.1]), which then induce a $\mathcal{G}$-equivariant *-isomorphism between the bidual $\mathcal{G}$-C*-algebras associated with A and A' (resp. B and B') by applying the functoriality of the crossed product and the biduality theorem (cf. Propositions 2.4.12 and 2.3.5 and Theorem 2.3.7). By restriction, for all $j, l = 1, 2$ we have two $\mathbb{G}_j$-equivariant *-isomorphisms $f_{l,j,\mathrm{g}} : \mathcal{B}_{l,j,\mathrm{g}} \to \mathcal{B}'_{l,j,\mathrm{g}}$ and $f_{l,j,\mathrm{d}} : \mathcal{B}_{l,j,\mathrm{d}} \to \mathcal{B}'_{l,j,\mathrm{g}}$.

Proposition-Definition 4.2.8. *Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$ such that the pair $(\mathcal{E}_2, F_2)$ is a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule. Let $y := [(\mathcal{E}_2, F_2)] \in \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$. Let $J_{\mathbb{G}_1,\mathbb{G}_2}(y)$ be the unique element $x \in \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1)$ satisfying the relation*

$$\begin{aligned} &\mathfrak{b}_{A_1} \otimes_{A_1} x \otimes_{B_1} \mathfrak{a}_{B_1} \\ &= \mathfrak{c}_{12,1,\mathrm{g}} \otimes_{\mathcal{B}_{2,1,\mathrm{g}}} [f_{2,1,\mathrm{g}}] \otimes_{\mathcal{B}'_{2,1,\mathrm{g}}} \big[\big(\mathcal{E}'_{2,1}, \big(\mathrm{id}_{\mathcal{K}(\mathcal{E}'_1)} \otimes R_{12}\big)\delta^1_{\mathcal{K}(\mathcal{E}_2)}(F_2)\big)\big] \otimes_{\mathcal{B}'_{2,1,\mathrm{d}}} [f^{-1}_{2,1,\mathrm{d}}]. \end{aligned}$$

Then, the map $J_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1)$ is a homomorphism of abelian groups.

Lemma 4.2.9. *For $j = 1, 2$, let $F_j \in \mathcal{L}(\mathcal{E}_j)$. Let $F := F_1 \oplus F_2 \in \mathcal{L}(\mathcal{E})$. The pair $(\mathcal{E}, \gamma)$ is a Kasparov A-B-bimodule if and only if the pair $(\mathcal{E}_j, F_j)$ is a Kasparov A_j-B_j-bimodule for $j = 1, 2$.*

Proof. For all $a = (a_1, a_2) \in A$, we have the relations $[\gamma(a), F] = \oplus_{j=1,2}[\gamma_j(a_j), F_j]$, $\gamma(a)(F^2 - 1) = \oplus_{j=1,2}\gamma_j(a_j)(F_j^2 - 1)$, and $\gamma(a)(F - F^*) = \oplus_{j=1,2}\gamma_j(a_j)(F_j - F_j^*)$. Therefore, the equivalence follows directly from $\mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}_1) \oplus \mathcal{K}(\mathcal{E}_2)$. ∎

Lemma 4.2.10. *For all $j, l, l' = 1, 2$, the pair $(\mathcal{E}_{ll',j}, (\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(F_l))$ is a Kasparov $\mathcal{B}_{l,j,\mathrm{g}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule.*

Proof. If $l' = l$, we refer the reader to [3] for $l = j$ and [2, Propositions 4.30 and 4.34] for $l \neq j$. By applying Lemma 4.2.3, the general case follows from the case where $l' = l$. ∎

Actually, we can prove that the pair $(\mathcal{E}_{ll',j}, (\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(F_l))$ is a $\mathbb{G}_j$-equivariant Kasparov $\mathcal{B}_{l,j,\mathrm{g}}$-$\mathcal{B}_{l',j,\mathrm{d}}$-bimodule. Indeed, as above the case where $l' = l$ is already known (cf. [2,3]). Moreover, the operator $(\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(F_l) \in \mathcal{L}(\mathcal{E}_{l,j})$ is invariant (cf. [2, 4.29]). By a direct computation, we show that the operator

$$(\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(F_l) \in \mathcal{L}(\mathcal{E}_{ll',j})$$

is invariant (cf. Section 2.5.3 for the definitions).

In the following, we assume the C*-algebra A_1 to be separable. Hence, the C*-algebras A_2 and A are separable (cf. Lemma 2.5.16).

Proposition 4.2.11. *For $j = 1, 2$, let $F_j \in \mathcal{L}(\mathcal{E}_j)$ such that $(\mathcal{E}_j, F_j)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule. Let $F := F_1 \oplus F_2 \in \mathcal{L}(\mathcal{E})$. We have that*

(1) *the pair $(\mathcal{D}, \pi_R(F))$ is a $\mathcal{G}$-equivariant Kasparov D_{g}-D_{d}-bimodule,*

(2) *there exists $T \in \mathcal{L}(\mathcal{E})$ such that*
   (a) *the pair $(\mathcal{E}, T)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule,*
   (b) $\mathfrak{b}_A \otimes_A [(\mathcal{E}, T)] \otimes_B \mathfrak{a}_B = [(\mathcal{D}, \pi_R(F))]$.

Moreover, we have that

(3) *if $T, T' \in \mathcal{L}(\mathcal{E})$ satisfy conditions* (a) *and* (b)*, then $[(\mathcal{E}, T)] = [(\mathcal{E}, T')]$ in $\mathsf{KK}_{\mathcal{G}}(A, B)$,*

(4) *if $T \in \mathcal{L}(\mathcal{E})$ satisfies conditions (a) and (b), then the class of $(\mathcal{E}, T)$ in $\mathsf{KK}_{\mathcal{G}}(A, B)$ only depends on those of $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ in $\mathsf{KK}_{\mathbb{G}_1}(A_1, B_1)$ and $\mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$, respectively,*

(5) *for $j = 1, 2$ and $T \in \mathcal{L}(\mathcal{E})$ satisfying conditions (a) and (b), let $T_j \in \mathcal{L}(\mathcal{E}_j)$ such that $T = T_1 \oplus T_2$ (cf. Lemma 4.1.1), then the pair $(\mathcal{E}_j, T_j)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule and we have $[(\mathcal{E}_j, T_j)] = [(\mathcal{E}_j, F_j)]$ in $\mathsf{KK}_{\mathbb{G}_j}(A_j, B_j)$.*

Proof. (1) Let $X \in \mathcal{L}(\mathcal{D})$ be the operator defined by $X(\zeta) := \pi_R(F) \circ \zeta$ for all $\zeta \in \mathcal{D}$. It suffices to prove that $(\mathcal{D}, X)$ is a Kasparov D_{g}-D_{d}-bimodule (cf. Lemma 2.4.25 and Remark 3.1.3 (3)). This amounts again to proving that $(\mathcal{D}_j, X_j)$ is a Kasparov $D_{j,\mathrm{g}}$-$D_{j,\mathrm{d}}$-bimodule for $j = 1, 2$ (cf. Lemma 4.2.9 (1)). However, this follows straightforwardly from Lemmas 4.2.10 and 4.2.1 (1).

(2), (3) These statements are direct consequences of Lemma 3.4.8.

(4) For $j = 1, 2$, let $F_j, F'_j \in \mathcal{L}(\mathcal{E}_j)$ such that $(\mathcal{E}_j, F_j)$ and $(\mathcal{E}_j, F'_j)$ are $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodules satisfying $[(\mathcal{E}_j, F_j)] = [(\mathcal{E}_j, F'_j)]$ in $\mathsf{KK}_{\mathbb{G}_j}(A_j, B_j)$. Let $F := F_1 \oplus F_2 \in \mathcal{L}(\mathcal{E})$ and $F' := F'_1 \oplus F'_2 \in \mathcal{L}(\mathcal{E})$. Let $T \in \mathcal{L}(\mathcal{E})$ (resp. $T' \in \mathcal{L}(\mathcal{E})$) be an operator satisfying conditions (a) and (b) for F (resp. F'). Let us prove that $[(\mathcal{E}, T)] = [(\mathcal{E}, T')]$ in $\mathsf{KK}_{\mathcal{G}}(A, B)$. For $j = 1, 2$, there exists a degenerate $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule $(\mathcal{F}_j, X_j)$ such that $(\mathcal{E}_j \oplus \mathcal{F}_j, F_j \oplus X_j)$ and $(\mathcal{E}_j \oplus \mathcal{F}_j, F'_j \oplus X_j)$ are operator homotopic (cf. [3, Remarques 5.11 (2)]). In particular, for $j = 1, 2$ there exists an operator homotopy $(\mathcal{E}_j, F_{j,t})_{t \in [0,1]}$ between $(\mathcal{E}_j, F_j)$ and $(\mathcal{E}_j, F'_j)$. For $t \in [0, 1]$, let $F_t := F_{1,t} \oplus F_{2,t} \in \mathcal{L}(\mathcal{E})$. For $t \in [0, 1]$, let $T_t \in \mathcal{L}(\mathcal{E})$ be an operator satisfying conditions (a) and (b) for F_t. Then, $(\mathcal{D}, \pi_R(T_t))_{t \in [0,1]}$ is an operator homotopy between $(\mathcal{D}, \pi_R(T))$ and $(\mathcal{D}, \pi_R(T'))$. Hence, $[(\mathcal{D}, \pi_R(T))] = [(\mathcal{D}, \pi_R(T'))]$. It then follows that

$$\mathfrak{b}_A \otimes_A \left[(\mathcal{E}, T)\right] \otimes_B \mathfrak{a}_B = \mathfrak{b}_A \otimes_A \left[(\mathcal{E}, T')\right] \otimes_B \mathfrak{a}_B.$$

Hence, $[(\mathcal{E}, T)] = [(\mathcal{E}, T')]$ (cf. Lemma 3.4.8).

(5) For the fact that $(\mathcal{E}_j, T_j)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule for $j = 1, 2$, we refer to Section 4.1. Let $X, Y \in \mathcal{L}(\mathcal{D})$ be the operators defined by $X(\zeta) := \pi_R(F) \circ \zeta$ and $Y(\zeta) := \pi_R(T) \circ \zeta$ for all $\zeta \in \mathcal{D}$. It follows from (b) and Theorem 3.4.9 that $[(\mathcal{D}, X)] = [(\mathcal{D}, Y)]$ in $\mathsf{KK}_{\mathcal{G}}(D_{\mathrm{g}}, D_{\mathrm{d}})$. By composing by $J_{\mathbb{G}_j, \mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(D_{\mathrm{g}}, D_{\mathrm{d}}) \to \mathsf{KK}_{\mathbb{G}_j}(D_{j,\mathrm{g}}, D_{j,\mathrm{d}})$, we have $[(\mathcal{D}_j, X_j)] = [(\mathcal{D}_j, Y_j)]$ for all $j = 1, 2$ (cf. Section 4.1). Hence,

we have (cf. Lemma 4.2.1 (2))

$$\begin{aligned}&\big[\big(\mathcal{E}_{ll',j}, (\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(F_l)\big)\big]\\ &\quad= \big[\big(\mathcal{E}_{ll',j}, (\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathcal{E}_l)}(T_l)\big)\big] \quad \text{for all } j,l,l' = 1,2.\end{aligned}$$

In particular, we have

$$\begin{aligned}&\big[\big(\mathcal{E}_{j,j}, \big(\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jj}\big)\delta^j_{\mathcal{K}(\mathcal{E}_j)}(F_j)\big)\big]\\ &\quad= \big[\big(\mathcal{E}_{j,j}, \big(\mathrm{id}_{\mathcal{K}(\mathcal{E}_j)} \otimes R_{jj}\big)\delta^j_{\mathcal{K}(\mathcal{E}_j)}(T_j)\big)\big] \quad \text{for } j = 1,2.\end{aligned}$$

By composing by the isomorphism $x \mapsto \mathfrak{a}_{A_j} \otimes_{D_{j,\mathrm{g}}} x \otimes_{D_{j,\mathrm{d}}} \mathfrak{b}_{B_j}$, we obtain $[(\mathcal{E}_j, F_j)] = [(\mathcal{E}_j, T_j)]$ (cf. [3]). ∎

In virtue of Proposition 4.2.11 (1)–(4), the following definition makes sense.

Definition 4.2.12. Let $j,k = 1,2$ such that $j \neq k$. Let $F_j \in \mathcal{L}(\mathcal{E}_j)$ such that the pair $(\mathcal{E}_j, F_j)$ is a $\mathbb{G}_j$-equivariant Kasparov A_j-B_j-bimodule. Let $x := [(\mathcal{E}_j, F_j)] \in \mathsf{KK}_{\mathbb{G}_j}(A_j, B_j)$ and $J_{\mathbb{G}_k,\mathbb{G}_j}(x) = [(\mathcal{E}_k, F_k)]$. Let $F := F_1 \oplus F_2 \in \mathcal{L}(\mathcal{E})$. We denote by $J_{\mathcal{G},\mathbb{G}_j}(x)$ the unique element $y \in \mathsf{KK}_{\mathcal{G}}(A, B)$ such that $\mathfrak{b}_A \otimes_A y \otimes_B \mathfrak{a}_B = [(\mathcal{D}, \pi_R(F))]$. For $j = 1,2$, we have a well-defined homomorphism of abelian groups $J_{\mathcal{G},\mathbb{G}_j} : \mathsf{KK}_{\mathbb{G}_j}(A_j, B_j) \to \mathsf{KK}_{\mathcal{G}}(A, B)$.

Lemma 4.2.13. *With the notations and hypotheses of Definition 4.2.12, if the pair $(\mathcal{E}, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule, then we have $J_{\mathcal{G},\mathbb{G}_j}(x) = [(\mathcal{E}, F)]$.*

Proof. This is a straightforward consequence of Theorem 3.4.9. ∎

Proposition 4.2.14. *Let $j,k = 1,2$ with $j \neq k$. We have*

(1) $J_{\mathbb{G}_j,\mathcal{G}} \circ J_{\mathcal{G},\mathbb{G}_j} = \mathrm{id}_{\mathsf{KK}_{\mathbb{G}_j}(A_j,B_j)}$,

(2) $J_{\mathbb{G}_k,\mathcal{G}} \circ J_{\mathcal{G},\mathbb{G}_j} = J_{\mathbb{G}_k,\mathbb{G}_j}$,

(3) $J_{\mathbb{G}_k,\mathbb{G}_j} \circ J_{\mathbb{G}_j,\mathcal{G}} = J_{\mathbb{G}_k,\mathcal{G}}$.

Proof. Formulas (1) and (2) are immediate consequences of Proposition 4.2.11 (5). The last statement follows by plugging the second formula in the left-hand side and by simplifying with the first one. ∎

We can state the main results of this paragraph.

Theorem 4.2.15. *Let $j = 1,2$. The maps*

$$J_{\mathbb{G}_j,\mathcal{G}} : \mathsf{KK}_{\mathcal{G}}(A, B) \to \mathsf{KK}_{\mathbb{G}_j}(A_j, B_j) \quad \text{and} \quad J_{\mathcal{G},\mathbb{G}_j} : \mathsf{KK}_{\mathbb{G}_i}(A_j, B_j) \to \mathsf{KK}_{\mathcal{G}}(A, B)$$

are isomorphisms of abelian groups inverse of each other.

Proof. Let $j,k = 1,2$ with $j \neq k$. It remains to prove that $J_{\mathcal{G},\mathbb{G}_j} \circ J_{\mathbb{G}_j,\mathcal{G}} = \mathrm{id}_{\mathsf{KK}_{\mathcal{G}}(A,B)}$ (cf. Proposition 4.2.14 (1)). Let $F \in \mathcal{L}(\mathcal{E})$ such that the pair $(\mathcal{E}, F)$ is a $\mathcal{G}$-equivariant Kasparov A-B-bimodule. Let $x := [(\mathcal{E}, F)] \in \mathsf{KK}_{\mathcal{G}}(A, B)$. We have $F = F_1 \oplus F_2$ with $F_1 \in$

$\mathcal{L}(\mathcal{E}_1)$ and $F_2 \in \mathcal{L}(\mathcal{E}_2)$. It follows from Proposition 4.2.14 (3) that $J_{\mathbb{G}_k,\mathbb{G}_j}([(\mathcal{E}_j, F_j)]) = [(\mathcal{E}_k, F_k)]$. By applying Lemma 4.2.13, we then obtain

$$J_{\mathcal{G},\mathbb{G}_j}\big(J_{\mathbb{G}_j,\mathcal{G}}(x)\big) = J_{\mathcal{G},\mathbb{G}_j}\big(\big[(\mathcal{E}_j, F_j)\big]\big) = \big[(\mathcal{E}, F)\big] = x. \qquad \blacksquare$$

We then obtain another proof of Théorème 4.36 [2].

Corollary 4.2.16. *The map $J_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}_{\mathbb{G}_1}(A_1, B_1) \to \mathsf{KK}_{\mathbb{G}_2}(A_2, B_2)$ is an isomorphism of abelian groups and $(J_{\mathbb{G}_2,\mathbb{G}_1})^{-1} = J_{\mathbb{G}_1,\mathbb{G}_2}$.*

Proof. This is an immediate consequence of Proposition 4.2.14 (2) and Theorem 4.2.15. ■

Let us fix a third $\mathbb{G}_1$-C*-algebra C_1. Consider the induced $\mathbb{G}_2$-C*-algebra $C_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(C_1)$ and the $\mathcal{G}$-C*-algebra $C := C_1 \oplus C_2$.

Proposition 4.2.17. *For $j = 1, 2$, we have*

1. $J_{\mathcal{G},\mathbb{G}_j}(1_{A_j}) = 1_A$,
2. *for all $x \in \mathsf{KK}_{\mathbb{G}_j}(A_j, C_j)$ and $y \in \mathsf{KK}_{\mathbb{G}_j}(C_j, B_j)$,*

$$J_{\mathcal{G},\mathbb{G}_j}(x \otimes_{C_j} y) = J_{\mathcal{G},\mathbb{G}_j}(x) \otimes_C J_{\mathcal{G},\mathbb{G}_j}(y) \quad \text{in } \mathsf{KK}_{\mathcal{G}}(A, B).$$

Proof. This follows from Theorem 4.2.15 and Proposition 4.1.4. ■

Proposition 4.2.18. *For $j, k = 1, 2$ with $j \neq k$, we have*

1. $J_{\mathbb{G}_k,\mathbb{G}_j}(1_{A_j}) = 1_{A_k}$,
2. *for all $x \in \mathsf{KK}_{\mathbb{G}_j}(A_j, C_j)$ and $y \in \mathsf{KK}_{\mathbb{G}_j}(C_j, B_j)$,*

$$J_{\mathbb{G}_k,\mathbb{G}_j}(x \otimes_{C_j} y) = J_{\mathbb{G}_k,\mathbb{G}_j}(x) \otimes_{C_k} J_{\mathbb{G}_k,\mathbb{G}_j}(y) \quad \text{in } \mathsf{KK}_{\mathbb{G}_k}(A_k, B_k).$$

Proof. This is a direct consequence of Propositions 4.2.14 (2), 4.1.4, and 4.2.17. ■

Notations 4.2.19. We denote by $\mathsf{KK}_{\mathcal{G}}$ (resp. $\mathsf{KK}_{\mathbb{G}_j}$ for $j = 1, 2$) the category of separable $\mathcal{G}$ (resp. $\mathbb{G}_j$)-C*-algebras whose set of arrows between two $\mathcal{G}$ (resp. $\mathbb{G}_j$)-C*-algebras A and B is the equivariant Kasparov group $\mathsf{KK}_{\mathcal{G}}(A, B)$ (resp. $\mathsf{KK}_{\mathbb{G}_j}(A, B)$).

Theorem 4.2.20. *We have that*

1. *for $j = 1, 2$, the correspondences $J_{\mathbb{G}_j,\mathcal{G}} : \mathsf{KK}_{\mathcal{G}} \to \mathsf{KK}_{\mathbb{G}_j}$ and $J_{\mathcal{G},\mathbb{G}_j} : \mathsf{KK}_{\mathbb{G}_j} \to \mathsf{KK}_{\mathcal{G}}$ are equivalences of categories inverse of each other,*
2. *the correspondences $J_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}_{\mathbb{G}_1} \to \mathsf{KK}_{\mathbb{G}_2}$ and $J_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}_{\mathbb{G}_2} \to \mathsf{KK}_{\mathbb{G}_1}$ are equivalences of categories inverse of each other.*

Proof. The first (resp. second) statement is just a restatement of Theorem 2.5.24, Proposition 4.2.17, and Theorem 4.2.15 (resp. Theorem 2.5.19, Proposition 4.2.18, and Corollary 4.2.16). ■

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