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Quantum Euclidean spaces with noncommutative derivatives

Li Gao, Marius Junge, and Edward McDonald

Abstract. Quantum Euclidean spaces, as Moyal deformations of Euclidean spaces, are the model examples of noncompact noncommutative manifold. In this paper, we study the quantum Euclidean space equipped with partial derivatives satisfying canonical commutation relation (CCR). This gives an example of semifinite spectral triple with nonflat geometric structure. We develop an abstract symbol calculus for the pseudo-differential operators with noncommuting derivatives. We also obtain a local index formula in our setting via the computation of the Connes–Chern character of the corresponding spectral triple.

1. Introduction

The theory of pseudo-differential operators (ΨDOs) plays an influential role in the index theory of elliptic operators. This approach also prevails in noncommutative geometry. In [13], Connes and Moscovici established the local index formula for spectral triples, which gives an analytic expression for the index pairing between K-theory of noncommutative algebras and the K-homology class induced by a Dirac-type operator. This local index formula was extended to the locally compact (i.e., nonunital) setting by Carey, Gayral, Rennie, and Sukochev [9]. In both proofs of the local index formula [9, 13], an abstract theory of ΨDOs is crucial to the analysis. On the prototypical example of a noncommutative geometry–quantum tori, pseudo-differential operators have been widely used in studying curvatures and other geometric structures (see, e.g., [2, 14–16, 28]). Recently several works [22, 23, 30, 43] give detailed accounts of the symbol calculus for ΨDOs on quantum tori.

Quantum Euclidean spaces are model examples of noncommutative spaces in the locally compact setting, and can be viewed as locally compact counterparts of quantum tori. They are noncommutative deformations of Euclidean spaces which originate from the Heisenberg relation and Moyal products in quantum mechanics. Let $\theta = (\theta_{jk})_{j,k=1}^{d}$ be a skew-symmetric $d \times d$ matrix. Roughly speaking, a d-dimensional quantum Euclidean space is given by the von Neumann algebra $\mathbb{R}_\theta$ generated by the spectral projections of d self-adjoint operators $x_1, \dots, x_d$ satisfying the canonical commutation relation (CCR)

$$[x_j, x_k] = -i\theta_{jk}.$$

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We will review a rigorous definition of $\mathbb{R}_\theta$ in Section 2. Despite having a relatively simple algebraic structure (a type I von Neumann algebra) the connection to Euclidean spaces and quantum physics make them indispensable in various scenarios. For example, from the perspective of harmonic and functional analysis, Calderón–Zygmund theory and pseudo-differential operator theory on quantum Euclidean spaces were established in the recent article [19] and the theory of distributions goes back to [20, 44]. In noncommutative geometry, quantum Euclidean spaces serve as model examples for nonunital spectral triples [18]. In mathematical physics, noncommutative Euclidean spaces have been heavily studied under the name of CCR algebras [5, Section 5.2.2.2] and in the context of Weyl quantization [24, Chapter 14], [42, Chapter 2, Section 3]. Also, the discovery of instantons on noncommutative $\mathbb{R}^4$ makes an influential connection to string theory [11, 36, 40].

In this paper, we revisit the connection between ΨDOs and the local index formula for quantum Euclidean spaces. Both topics have been considered for $\mathbb{R}_\theta$, with its standard geometric structure. Recall that $\mathbb{R}_\theta$ is associated with a Weyl quantization map, defined for functions in the Schwartz class $S(\mathbb{R}^d)$ as

$$\lambda_\theta : f \in S(\mathbb{R}^d) \mapsto \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi})\, d\xi \in \mathbb{R}_\theta,$$

where $\lambda_\theta(\boldsymbol{\xi}) = e^{\boldsymbol{\xi}_1 x_1 + \cdots + \boldsymbol{\xi}_d x_d}$, $\boldsymbol{\xi} \in \mathbb{R}^d$ is a projective unitary representation of $\mathbb{R}^d$,

$$\lambda_\theta(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\eta}) = e^{\frac{i}{2}\boldsymbol{\xi}\cdot\theta\boldsymbol{\eta}}\lambda_\theta(\boldsymbol{\xi} + \boldsymbol{\eta})$$

(see Section 2 for further details). The canonical trace associated to $\mathbb{R}_\theta$ is defined on the image of $S(\mathbb{R}^d)$ under λ_θ as $\tau_\theta(\lambda_\theta(f)) = \int f$. Differentiation operators $\frac{\partial}{\partial \mathbf{x}_j}$ admit a canonical extension to $\mathbb{R}_\theta$, defined on $\lambda_\theta(S(\mathbb{R}^d))$ by $D_j\lambda_\theta(f) = \lambda_\theta(-i\frac{\partial}{\partial \mathbf{x}_j} f)$. The operators D_j have self-adjoint extensions to the Hilbert–Schmidt space $L_2(\mathbb{R}_\theta, \tau_\theta)$. Since partial differentiation operators on $S(\mathbb{R}^d)$ commute, it follows immediately that $[D_j, D_k] = 0$ for $1 \le j,k \le d$. The fact that these partial derivatives mutually commute reflects a "flat" geometry of $\mathbb{R}_\theta$.

The scope of this paper is to consider a more general but still computable differential structure on $\mathbb{R}_\theta$. More precisely, we shall equip $\mathbb{R}_\theta$ with "covariant derivatives" $\xi_1, \ldots, \xi_d$ satisfying (another) CCR relation. Unlike the standard case

$$[x_j, x_k] = -i\theta_{j,k}, \quad [D_j, x_k] = -i\delta_{j,k}, \quad [D_j, D_k] = 0, \tag{1.1}$$

we consider that x_j's and ξ_k's together have the commutation relations

$$[x_j, x_k] = -i\theta_{j,k}, \quad [\xi_j, x_k] = -i\delta_{jk}, \quad [\xi_j, \xi_k] = -i\theta'_{jk}, \tag{1.2}$$

where δ is the Kronecker delta notation and θ' is an arbitrary but fixed skew-symmetric matrix. In the classical case, when $\theta = 0$ and $\mathbb{R}_0 = L_\infty(\mathbb{R}^d)$, such ξ_j's are covariant derivatives of connections with a constant curvature form (see Section 4.1). From this perspective, (1.2) can be viewed as a natural deformation of (1.1) by adding a nonzero curvature form. From the perspective of quantum physics, noncommuting derivatives occur in the presence of a magnetic field [1]. One can view the matrix θ' as representing a constant magnetic field on $\mathbb{R}_\theta$. The noncommutativity of the covariant derivatives ξ_j adds an essential difficulty in developing the theory of ΨDOs. When $\theta' = 0$, the commutativity of D_j's makes the phase space (or the Fourier transform side) a commutative space, and then the symbol of a ΨDO is an operator-valued function $a : \mathbb{R}^d \to \mathbb{R}_\theta$. In our setting for noncommuting ξ_j's, the symbol will become purely abstract as operators affiliated to $\mathbb{R}_\theta \otimes \mathbb{R}'_\theta$. Moreover, due to the unbounded nature of symbol functions, we have to inevitably deal with unbounded but smooth elements. The idea of incorporating noncommuting derivatives into pseudo-differential calculus has also appeared in the related context of magnetic pseudo-differential calculus [34, 35].

We now briefly explain our setting and illustrate the main results. Let $\mathbb{R}_\theta \mathbin{\bar{\otimes}} \mathbb{R}'_\theta$ be the $2d$-dimensional quantum Euclidean space generated by the relations

$$[x_j, x_k] = -i\theta_{j,k}, \quad [\xi_j, \xi_k] = -i\theta'_{j,k}, \quad [x_j, \xi_k] = 0$$

and let $\mathbb{R}_\Theta$ be the $2d$-dimensional space generated by (1.2) with parameter matrix $\Theta = \left[\begin{smallmatrix} \theta & I_d \\ -I_d & \theta' \end{smallmatrix}\right]$. We will consider a pseudo-differential calculus defined with symbols as operators affiliated to $\mathbb{R}_\theta \mathbin{\bar{\otimes}} \mathbb{R}'_\theta$ and the ΨDOs themselves are operators affiliated to $\mathbb{R}_\Theta$. The operator or quantization map "Op" sending symbols to ΨDOs is simple: for $a \in \mathbb{R}_\theta$, $b \in \mathbb{R}_{\theta'}$

$$\mathrm{Op}(a \otimes b) = ab \in \mathbb{R}_\Theta, \tag{1.3}$$

where $\mathbb{R}_\theta, \mathbb{R}'_\theta$ are viewed as subalgebras of $\mathbb{R}_\Theta$. The domain of Op can be extended to the following abstract symbol class.

- We say an operator a affiliated to $\mathbb{R}_\theta \mathbin{\bar{\otimes}} \mathbb{R}_{\theta'}$ is a symbol of order m (write as $a \in \Sigma^m$) if for any multi-indices α and β, $D_x^\alpha D_\xi^\beta(a)(1 + \sum_j \xi_j^2)^{-\frac{m+|\beta|}{2}}$ extends to a bounded operator in $\mathbb{R}_\theta \mathbin{\bar{\otimes}} \mathbb{R}_{\theta'}$.

Here D_x are the canonical (commuting) differentiation operators acting on the first component $\mathbb{R}_\theta$ and D_ξ are the same for $\mathbb{R}_{\theta'}$. *A priori* it is not clear that this definition is closed under multiplication, and adjoint, or whether we have the expected properties $\Sigma^m \cdot \Sigma^n = \Sigma^{m+n}$ and $(\Sigma^m)^* = \Sigma^m$, which are important components for the development of a symbol calculus. To resolve that, we introduce in Section 3 a notation of "asymptotic degree" to measure the unboundedness of operators affiliated to $\mathbb{R}_\theta$. This is a notion directly inspired by the abstract pseudo-differential calculus developed by Connes and Moscovici [13, Appendix B] and Higson [25]. With this definition of symbol class, we establish in Section 4 the two core parts of ΨDOs calculus—the L_2-boundedness theorem for 0-order ΨDOs and the composition formula.

Theorem 1.1 (cf. Theorem 4.12). *Let a be a symbol of order* 0 *(i.e., $a \in \Sigma^0$). Then* $\mathrm{Op}(a)$*, initially defined on $\lambda_\Theta(S(\mathbb{R}^{2d}))$, has unique extension to a bounded operator on the Hilbert space $L_2(\mathbb{R}_\Theta)$.*

Theorem 1.2 (cf. Theorem 4.14). *Let a be a symbol of order m and b a symbol of order n. Then* $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c)$ *for some symbol c of order $m+n$. Moreover,*

$$c \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_{\xi}^{\alpha}(a) D_{x}^{\alpha}(b)$$

in the sense that for any positive integer N, $c - \sum_{|\alpha|\le N} \frac{i^{|\alpha|}}{\alpha!} D_{\xi}^{\alpha}(a) D_{x}^{\alpha}(b)$ is a symbol of order $m+n-N-1$.

The proofs of the above theorems use the idea of co-representation maps. The co-representation maps enable us to convert the operator map Op to an operator-valued classical operator map on the $\mathbb{R}^d$. In particular, this gives an alternative approach to some parts of symbol calculus in [19] for $\theta' = 0$.

In Section 5, we apply the ΨDO calculus proving that

$$\Big(W^{1,\infty}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, D = \sum_j \xi_j \otimes c_j\Big) \tag{1.4}$$

forms a semifinite nonunital spectral triple (in the sense of [9, Definition 2.1]). Here, c_j are generators of the Clifford algebra Cl^d and $W^{\infty,1}(\mathbb{R}_\theta) = \{a | D^\alpha(a) \in L_1(\mathbb{R}_\theta)\ \forall \alpha\}$ is the noncommutative Sobolev spaces. We denote by $W^{\infty,1}(\mathbb{R}_\theta)^{\sim} = W^{\infty,1}(\mathbb{R}_\theta) + \mathbb{C}$ the minimal unitalization. The triple in (1.4) forms a smoothly summable semifinite spectral triple with isolated spectrum dimension (see Section 5 for further details). We are able to apply the even case of the local index formula in [9, Theorem 3.33], yielding the following.

Theorem 1.3 (cf. Corollary 5.10). *Let d be even and $\mathbb{R}_\theta$ a d-dimensional quantum Euclidean space. Then $(A, H, D) := (W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes M_N, \sum_j \xi_j \otimes c_j)$ is an even, smoothly summable, semifinite spectral triple with isolated spectrum dimension. Moreover, for a projection $e \in M_n(W^{\infty,1}(\mathbb{R}_\theta)^{\sim})$, the index pairing is given by*

$$\begin{aligned}&\big\langle [e] - [1_e], (A, H, D)\big\rangle \\ &= \pi^{\frac{d}{2}} \Bigg(\tau_\theta \otimes \mathrm{tr}\Big(\gamma(e - 1_e)\frac{\omega^{\frac{d}{2}}}{\frac{d}{2}!}\Big) + \sum_{m=1}^{\frac{d}{2}} \frac{1}{2m!} \tau_\theta \otimes \mathrm{tr}\Big(\gamma e (de)^{2m} \frac{\omega^{\frac{d}{2}-m}}{(\frac{d}{2}-m)!}\Big)\Bigg),\end{aligned}$$

where $\omega = \frac{i}{2}\sum_{j,k} \theta'_{j,k} c_j c_k$.

Note that the Dirac Laplacian has square given by

$$D^2 = \Big(\sum_j \xi_j \otimes c_j\Big)^2 = \sum_j \xi_j^2 - \omega,$$

where ω plays the role of a curvature form in the index pairing. One direct application of the above index formula is Theorem 5.12, in which we prove that for $d = 2$, the noncommutative analog of Bott projection is a generator of the K_0-group of $\mathbb{R}_\theta$ for all θ.

The general local index formula in [9, 13] contains residue cocycles which involve higher-order residues at $z = 0$ for zeta functions

$$\zeta_k(z) = \operatorname{tr}\big(\gamma a_0 da_1^{(k_1)} \cdots da_m^{(k_m)} (1 + D^2)^{-\frac{m}{2}-k-z}\big),$$

where

$$a_j \in A, \quad da = [D, a], \quad da^{(k)} := \underbrace{\big[D^2, \big[D^2, \ldots, [D^2}_{k\text{-times}}, da]\big]\big].$$

For compact spin Riemannian manifolds, it was observed in [13] and fully proved by Ponge [37] that the above zeta functions only has nonzero residue for $|k| = 0$ and the poles are simple, which recovers the Atiyah–Singer index theorem for spin Dirac operators. Theorem 1.3 shows that $\mathbb{R}_\theta$, as a noncommutative deformation of Euclidean space $\mathbb{R}_d$, enjoys the same simplified index formula as a manifold. This result suggests that there should be a class of "mild" noncommutative spectral triples whose index pairing behaves as classical cases. It would be interesting to find a criterion for such "mild" noncommutative manifolds as well as the K-theory meaning behind it. Nevertheless, this is beyond the scope of this paper, and our work can be viewed as an invitation to such a study.

The paper is organized as follows: we first review some preliminary facts about quantum Euclidean spaces in Section 2. Section 3 introduces and discusses the notation "asymptotic degree," which is a key tool in the subsequent discussions. In Section 4, we discuss the symbol calculus of ΨDOs and prove Theorems 1.1 and 1.2. Section 5 is devoted to the local index formula and Theorem 1.3.

2. Preliminaries on quantum Euclidean spaces

In this section, we review the basic structures of Quantum Euclidean spaces. Quantum Euclidean spaces in the literature have been studied under several different names: Moyal planes [18, 20, 44], CCR algebras [4, Section 5.2.2.2], noncommutative Euclidean Spaces [17, 32] and quantum Euclidean spaces [19]. In particular, [4] gives a detailed account from the operator theoretic perspective. The distribution theory was studied in [20, 44]. More recently, González-Pérez, Junge, and Parcet [19] studied harmonic analysis on quantum Euclidean spaces. From the noncommutative geometric perspective, an early exposition is in [18].

2.1. Definitions and notations

Throughout the paper, we use the usual letters $x_1, x_2, \ldots$ and $\xi_1, \xi_2, \ldots$ for operators and the boldface letters $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d)$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_d)$ for vectors and scalars. Let $d \geq 2$ and let $\theta = (\theta_{jk})_{j,k=1}^d$ be a real skew-symmetric $d \times d$ matrix. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of complex Schwartz functions (smooth, rapidly decreasing) on $\mathbb{R}^d$. The Moyal product $\star_\theta$ associated to θ is defined as (see [38])

$$f \star_\theta g(\mathbf{x}) := (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\Big(\mathbf{x} + \frac{\boldsymbol{\theta}}{2}\mathbf{v}\Big) g(\mathbf{x} - \mathbf{w}) e^{i\mathbf{v}\cdot\mathbf{w}}\, d\mathbf{v}\, d\mathbf{w}, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

The Moyal product is bilinear, associative, and reversed under complex conjugation $\bar{f} \star_\theta \bar{g} = \overline{g \star_\theta f}$, which makes $(\mathcal{S}(\mathbb{R}^d), \star_\theta)$ a $*$-algebra. The left Moyal multiplication gives the following *-homomorphism $\lambda_\theta : (\mathcal{S}(\mathbb{R}^d), \star_\theta) \to B(L_2(\mathbb{R}^d))$,

$$\lambda_\theta(f)g = f \star_\theta g, \quad \lambda_\theta(f)\lambda_\theta(g) = \lambda_\theta(f \star_\theta g). \tag{2.1}$$

Definition 2.1. The quantum Euclidean space associated to θ is given by the following objects in $B(L_2(\mathbb{R}^d))$:

(i) $\mathcal{S}_\theta := \lambda_\theta(\mathcal{S}(\mathbb{R}^d))$ as the quantized Schwartz class;

(ii) $\mathbb{E}_\theta := \overline{\mathcal{S}_\theta}^{\|\cdot\|}$ as the C^*-algebra generated by $\mathcal{S}_\theta$;

(iii) $\mathbb{R}_\theta := (\mathcal{S}_\theta)''$ as the von Neumann algebra generated by $\mathcal{S}_\theta$.

When $\theta = 0$, $\star_0$ is the usual pointwise multiplication, $\mathbb{E}_0 = C_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ which vanish at infinity, and $\mathbb{R}_0 = L_\infty(\mathbb{R}^d)$ is the space of essentially bounded functions on $\mathbb{R}^d$. The algebra $\mathbb{E}_\theta$ is identical to a deformation quantization of $C_0(\mathbb{R}^d)$ as defined by Rieffel [38]. An equivalent approach to defining $\mathbb{R}_\theta$ is to consider the θ-twisted regular representation of the group $\mathbb{R}^d$ on $L_2(\mathbb{R}^d)$. For each vector $\boldsymbol{\xi} \in \mathbb{R}^d$, we define the unitary operator $\lambda_\theta(\boldsymbol{\xi})$ on $L_2(\mathbb{R}^d)$,

$$\big(\lambda_\theta(\boldsymbol{\xi})g\big)(\mathbf{x}) = e^{i\boldsymbol{\xi}\cdot\mathbf{x}} g\Big(\mathbf{x} - \frac{\theta}{2}\boldsymbol{\xi}\Big). \tag{2.2}$$

The family of operators $\{\lambda_\theta(\boldsymbol{\xi})\}_{\boldsymbol{\xi}\in\mathbb{R}^d}$ satisfies the commutation relation

$$\lambda_\theta(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\eta}) = e^{\frac{i}{2}\boldsymbol{\xi}\cdot\theta\boldsymbol{\eta}}\lambda_\theta(\boldsymbol{\xi}+\boldsymbol{\eta}) = e^{i\boldsymbol{\xi}\cdot\theta\boldsymbol{\eta}}\lambda_\theta(\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\xi})$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$. The map $\lambda_\theta : \mathbb{R}^d \to B(L_2(\mathbb{R}^d))$ is a projective unitary representation of $\mathbb{R}^d$ called the twisted left regular representation. The Moyal multiplication (2.1) for $(S(\mathbb{R}^d), \star_\theta)$ can also be formulated via quantized Fourier transform

$$\lambda_\theta(f) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi})d\boldsymbol{\xi}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Here $\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}$ is the Fourier transform of f and the integral converges in operator norm. Let $u_j(t) = \lambda_\theta(0, 0, \ldots, t, \ldots, 0)$ be the one parameter unitary group associated to the jth coordinate. The generator x_j of $u_j(t)$ satisfying $u_j(t) = e^{ix_j t}$ is the self-adjoint operator given by

$$(x_j g)(\mathbf{x}) = \mathbf{x}_j g(\mathbf{x}) + \frac{i}{2}\sum_k \theta_{jk}\frac{\partial g}{\partial \mathbf{x}_k}(\mathbf{x}).$$

The d-tuple $(x_1, \ldots, x_d)$ consists of d self-adjoint operators on $L_2(\mathbb{R}^d)$ having a common core $\mathcal{S}(\mathbb{R}^d)$ which satisfy the CCR relation $[x_j, x_k] = -i\theta_{jk}$ on $\mathcal{S}(\mathbb{R}^d)$. The operators $\{x_j\}_{j=1}^d$ are affiliated with the von Neumann algebra $\mathbb{R}_\theta$ in the usual sense that their spectral measures consist of projections in $\mathbb{R}_\theta$. The projective unitary representation $\boldsymbol{\xi} \to$

$\lambda_\theta(\boldsymbol{\xi})$ can be recovered formally from $(x_1, \ldots, x_d)$ using the Baker–Campbell–Hausdorff formula; i.e.,

$$\lambda_\theta(\boldsymbol{\xi}) := e^{i(\xi_1 x_1 + \cdots + \xi_d x_d)} = e^{-\frac{i}{2}\sum_{j<k}\theta_{jk}\xi_j\xi_k} e^{i\xi_1 x_1} \ldots e^{i\xi_d x_d}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

The family of generators $(x_1, \ldots, x_d)$, unitaries $\lambda_\theta(\boldsymbol{\xi})$, and the quantized Schwartz class $\{\lambda_\theta(f) : f \in \mathcal{S}(\mathbb{R}^d)\}$ all represent equivalent formulations of quantum Euclidean spaces. We will use them interchangeably in the paper.

2.2. The Stone–von Neumann theorem

We say that two self-adjoint operators P, Q satisfy the Heisenberg relation $[P, Q] = -iI$ if for any $s, t \in \mathbb{R}$,

$$e^{isP} e^{itQ} = e^{ist} e^{itQ} e^{isP}.$$

The well-known Stone–von Neumann theorem states that any irreducible representations of $[P, Q] = -iI$ are unitarily equivalent to the 1-dimensional Schrödinger representation given by

$$Pf = -i\frac{df}{d\mathbf{x}}, \quad (Qf)(\mathbf{x}) = \mathbf{x}f(\mathbf{x}), \quad f \in \mathcal{S}(\mathbb{R}).$$

Here P, Q are unbounded self-adjoint operators on $L_2(\mathbb{R})$ and the one-parameter unitary groups are

$$(e^{itP} f)(\mathbf{x}) = f(\mathbf{x} + t), \quad (e^{isQ} f)(\mathbf{x}) = e^{is\mathbf{x}} f(\mathbf{x}); \tag{2.3}$$

see, e.g., [24, Chapter 14]. The Stone–von Neumann theorem extends to n pairs of Heisenberg relations that mutually commute; i.e.,

$$[P_j, Q_k] = \begin{cases} -iI, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \quad [P_j, P_k] = [Q_j, Q_k] = 0, \quad \forall j, k. \tag{2.4}$$

The following is Theorem 14.8 in [24].

Theorem 2.2 (Stone–von Neumann theorem). *Suppose that $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ are self-adjoint operators on H satisfying the CCR relations (2.4). Then H can be decomposed as an orthogonal direct sum of closed subspaces $\{H_j\}$ satisfying that*

(i) *each H_l is invariant under e^{itP_j} and e^{itQ_j} for all j and t,*

(ii) *there exist unitary operators $U_l : H_l \to L_2(\mathbb{R}^d)$ such that*

$$U_l P_j U_l^* f = -i\frac{\partial}{\partial \mathbf{x}_j} f, \quad (U_l Q_j U_l^* f)(\mathbf{x}) = \mathbf{x}_j f(\mathbf{x}). \tag{2.5}$$

The above theorem says that any representation of (2.4) is a finite or infinite direct sum of the n-dimensional Schrödinger representation on $L_2(\mathbb{R}^n)$. When $d = 2n$ is even-dimensional, this gives the standard noncommutative case for $\mathbb{R}_\theta$ that $\theta = \left[\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right]$, where I_n is the n-dimensional identity matrix. In this case, $\mathbb{E}_\theta \cong K(L_2(\mathbb{R}^n))$ the compact operators and $\mathbb{R}_\theta \cong B(L_2(\mathbb{R}^n))$. The following proposition gives a change of variables between $\mathbb{R}_\theta$'s with different θ.

Proposition 2.3. *Let $T = (T_{jk})_{j,k=1}^d$ be a real invertible matrix and T^t its transpose. Let θ and $\widetilde{\theta}$ be two skew-symmetric matrices such that $\widetilde{\theta} = T\theta T^t$. Then the map Φ_T:*

$$\Phi_T\big(\lambda_{\widetilde{\theta}}(\boldsymbol{\xi})\big) = \lambda_\theta(T^t\boldsymbol{\xi}), \Phi_T\big(\lambda_{\widetilde{\theta}}(f)\big) = \lambda_\theta(f \circ T),$$

extends to a $*$-isomorphism from $\mathbb{E}_{\widetilde{\theta}}$ to $\mathbb{E}_\theta$ and a normal $*$-isomorphism from $\mathbb{R}_{\widetilde{\theta}}$ to $\mathbb{R}_\theta$.

Proof. Define the operator U_T on $L_2(\mathbb{R}^d)$ as follows:

$$(U_T f)(\mathbf{x}) = f(T^{-1}\mathbf{x}),$$

where U_T is bounded and invertible with $\|U_T\| = |\det(T)|^{\frac{1}{2}}$ and $(U_T)^{-1} = U_{T^{-1}}$. For any Schwartz function f, one verifies that

$$\begin{aligned}\big(U_T^{-1}\lambda_{\widetilde{\theta}}(\boldsymbol{\xi})U_T f\big)(\mathbf{x}) &= e^{i\boldsymbol{\xi}\cdot T\mathbf{x}} f\left(T^{-1}\left(T\mathbf{x} + \frac{1}{2}\widetilde{\theta}\boldsymbol{\xi}\right)\right) = e^{i(T^t\boldsymbol{\xi})\cdot\mathbf{x}} f\left(\mathbf{x} + \frac{1}{2}\theta T^t\boldsymbol{\xi}\right) \\ &= \lambda_\theta(T^t\boldsymbol{\xi})f(\mathbf{x}).\end{aligned}$$

Then it is clear that $U_T^{-1}\mathcal{S}_{\widetilde{\theta}}U_T = \mathcal{S}_\theta$. Since U_T is a bounded invertible operator on $L_2(\mathbb{R}^d)$, then $\Phi_T(\cdot) = U_T^{-1}(\cdot)U_T$ extends to a $*$-isomorphism from $\mathbb{E}_{\widetilde{\theta}}$ to $\mathbb{E}_\theta$ and a normal $*$-isomorphism from $\mathbb{R}_{\widetilde{\theta}}$ to $\mathbb{R}_\theta$. ∎

In general, let θ be a skew-symmetric matrix of rank $2n \le d$. There exists an invertible matrix T such that $\widetilde{\theta} = T\theta T^t$ is the following standard form:

$$\begin{bmatrix} 0 & -I_n & \\ I_n & 0 & \\ & & 0_{d-2n} \end{bmatrix}, \tag{2.6}$$

where 0_{d-2n} is a $(d-2n) \times (d-2n)$ zero matrix. Recall that $x_1, \dots, x_d$ denote the generators of the unitary semigroups $t_j \mapsto \lambda_\theta(0, \dots, t_j, \dots)$. Then $x_1, \dots, x_{2n}$, by the Stone–von Neumann theorem, are unitarily equivalent to (a multiple of) the derivatives and position operators $-i\frac{\partial}{\partial \mathbf{x}_1}, \dots, -i\frac{\partial}{\partial \mathbf{x}_n}, \mathbf{x}_1, \dots, \mathbf{x}_n$ on $L_2(\mathbb{R}^n)$, and $x_{2n+1}, \dots, x_d$ are $d-2n$ position operators $\mathbf{x}_{n+1}, \dots, \mathbf{x}_{d-n}$ on $L_2(\mathbb{R}^{d-2n})$. Hence if θ is of rank $2n < d$, then we have up to multiplicity [38, Proposition 5.2]

$$\mathbb{E}_\theta \cong \mathcal{K}\big(L_2(\mathbb{R}^n)\big) \otimes C_0(\mathbb{R}^{d-2n}), \quad \mathbb{R}_\theta \cong B\big(L_2(\mathbb{R}^n)\big) \mathbin{\overline{\otimes}} L_\infty(\mathbb{R}^{d-2n}).$$

In particular, the C^*-algebra $\mathbb{E}_\theta$ is simple if and only if the matrix θ is of full rank.

2.3. Integrals and derivatives

We start with the noncommutative integrals.

Proposition 2.4. *The linear functional*

$$\tau_\theta\big(\lambda_\theta(f)\big) = \int_{\mathbb{R}^d} f, \quad f \in \mathcal{S}(\mathbb{R}^d)$$

extends to a normal faithful semifinite trace on $\mathbb{R}_\theta$.

(i) *Let T be a real invertible matrix and let θ and $\widetilde{\theta}$ be two skew-symmetric matrices such that $\widetilde{\theta} = T\theta T^t$. Then the normal $*$-isomorphism*

$$\Phi_T : \mathbb{R}_{\widetilde{\theta}} \to \mathbb{R}_\theta, \quad \Phi_T\big(\lambda_{\widetilde{\theta}}(f)\big) = \lambda_\theta(f \circ T) \tag{2.7}$$

satisfies $\tau_\theta \circ \Phi_T = |\det T|^{-1}\tau_{\widetilde{\theta}}$.

(ii) *Let $\mathbf{x} \in \mathbb{R}^d$ and let $\alpha_{\mathbf{x}}$ be the translation action $\alpha_{\mathbf{x}}(f)(\cdot) = f(\cdot + \mathbf{x})$. Define the map*

$$\alpha_{\mathbf{x}}\big(\lambda_\theta(\boldsymbol{\xi})\big) = e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\lambda_\theta(\boldsymbol{\xi}), \quad \alpha_{\mathbf{x}}\big(\lambda_\theta(f)\big) = \lambda_\theta\big(\alpha_{\mathbf{x}}(f)\big).$$

Then $\alpha_{\mathbf{x}}$ is a τ_θ-preserving automorphism on $\mathbb{R}_\theta$.

Proof. The fact that τ_θ extends to a normal faithful trace on $\mathbb{R}_\theta$ was proved in [19] by writing $\mathbb{R}_\theta$ as an iterated crossed product $L_\infty(\mathbb{R}) \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$. Here we present a proof using a change of variables, which is useful for our later discussion. A similar discussion can be found in [29]. Denote the multiplier and translation unitary groups on $L_2(\mathbb{R}^n)$ as follows:

$$\big(u(\boldsymbol{\xi})f\big)(\mathbf{x}) = f(\mathbf{x}+\boldsymbol{\xi}), \quad \big(v(\boldsymbol{\eta})f\big)(\mathbf{x}) = e^{i\boldsymbol{\eta}\cdot\mathbf{x}}f(\mathbf{x}).$$

We first consider the case $d = 2n$ and $\theta = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. By the Stone–von Neumann theorem, there exists some Hilbert space H and a unitary $W : L_2(\mathbb{R}_\theta) \to L_2(\mathbb{R}^n) \otimes I_H$ such that

$$W\lambda_\theta(\boldsymbol{\xi}, \mathbf{0})W^* = u(\boldsymbol{\xi}) \otimes I_H, \quad W\lambda_\theta(\mathbf{0}, \boldsymbol{\eta})W^* = v(\boldsymbol{\eta}) \otimes I_H,$$

where $\boldsymbol{\xi} \in \mathbb{R}^n$ are the first n coordinates and $\boldsymbol{\eta} \in \mathbb{R}^n$ are the last n coordinates. For $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, the quantization $\lambda_\theta(f_1 \otimes f_2)$ is unitarily equivalent to (a multiple of) the following operator T_{f_1,f_2}. For $h \in L_2(\mathbb{R}^n)$;

$$\begin{aligned}(T_{f_1,f_2}h)(\mathbf{y}) &= (2\pi)^{-2n} \iint \hat{f}_1(\boldsymbol{\xi})\hat{f}_2(\boldsymbol{\eta})e^{-\frac{i}{2}\boldsymbol{\xi}\cdot\boldsymbol{\eta}}e^{i\boldsymbol{\eta}\cdot(\mathbf{y}+\boldsymbol{\xi})}h(\mathbf{y}+\boldsymbol{\xi})\,d\xi\,d\boldsymbol{\eta} \\ &= (2\pi)^{-2n} \iint \hat{f}_1(\mathbf{x}-\mathbf{y})\hat{f}_2(\boldsymbol{\eta})e^{-\frac{i}{2}(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\eta}}e^{i\mathbf{x}\cdot\boldsymbol{\eta}}h(\mathbf{x})\,d\mathbf{x}\,d\eta \\ &= (2\pi)^{-n} \int \hat{f}_1(\mathbf{x}-\mathbf{y})f_2\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big)h(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

Because $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, it follows from [6, Proposition 1.1 and Theorem 3.1] that T_{f_1,f_2} is a trace class operator on $L_2(\mathbb{R}^n)$ and

$$\begin{aligned}\operatorname{tr}(T_{f_1,f_2}) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}_1(\mathbf{y}-\mathbf{y})f_2\Big(\frac{\mathbf{y}+\mathbf{y}}{2}\Big)\,d\mathbf{y} \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}_1(0)f_2(\mathbf{y})\,d\mathbf{y} = (2\pi)^{-n} \int_{\mathbb{R}^n} f_1 \cdot \int_{\mathbb{R}^n} f_2,\end{aligned}$$

which coincides with τ_θ on $\mathbb{R}_\theta$ up to a normalization constant $(2\pi)^{-n}$. Now we consider the case where θ is a singular standard form: $\theta = \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $\theta_1 = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ be the nonsingular part. $\mathbb{R}_{\theta_1} \cong B(L_2(\mathbb{R}^n))$ is a Type I factor and the degenerated part gives the left regular representation $\lambda_0 : \mathbb{R}^{d-2n} \to B(L_2(\mathbb{R}^{d-2n}))$. Then

$$\mathbb{R}_\theta \cong \mathbb{R}_{\theta_1} \bar{\otimes} \mathbb{R}_0 \cong B\big(L_2(\mathbb{R}^n)\big) \bar{\otimes} L_\infty(\mathbb{R}^{d-2n})$$

as von Neumann algebras, where the tensor product $\bar{\otimes}$ is the von Neumann algebra tensor product. The trace τ_θ on $\mathbb{R}_\theta$ is the product trace $\tau_{\theta_1} \otimes \tau_0$, where τ_0 on $L_\infty(\mathbb{R}^{d-2n})$ is the Lebesgue integral and τ_{θ_1} is up to a constant the standard trace tr on $B(L_2(\mathbb{R}^n))$. Then τ_θ is normal faithful semifinite and the case for general θ follows from (i). Recall that the $*$-isomorphism Φ_T is implemented by the bounded invertible operator

$$U_T : L_2(\mathbb{R}_{\tilde{\theta}}) \to L_2(\mathbb{R}_\theta), \quad U_T \lambda_{\tilde{\theta}}(f) = \lambda_\theta(f \circ T^{-1}).$$

For $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}\tau_\theta \circ \Phi_T\big(\lambda_{\tilde{\theta}}(f)\big) &= \tau_\theta\bigg(\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\lambda_\theta(T\boldsymbol{\xi})\,d\boldsymbol{\xi}\bigg) = |\det T|^{-1}\tau_\theta\bigg(\int_{\mathbb{R}^d} \hat{f}(T^{-1}\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\eta})\,d\boldsymbol{\eta}\bigg)\\ &= |\det T|^{-1}\hat{f}(0) = |\det T|^{-1}\tau_{\tilde{\theta}}\big(\lambda_{\tilde{\theta}}(f)\big).\end{aligned}$$

For (ii), $\alpha_{\mathbf{x}}$ is implemented by the shifting unitary $U_{\mathbf{x}}$ on $L_2(\mathbb{R}^d)$ that

$$\alpha_{\mathbf{x}}\big(\lambda_\theta(f)\big) = U_{\mathbf{x}}\lambda_\theta(f)U_{\mathbf{x}}^*, \quad U_{\mathbf{x}}f(\mathbf{y}) = f(\mathbf{y}+\mathbf{x}).$$

Hence $\alpha_{\mathbf{x}}$ extends to an automorphism on $\mathbb{R}_\theta$. ∎

The automorphism $\alpha_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^d$, is called the *transference action* on $\mathbb{R}_\theta$. For $1 \le p \le \infty$, we write $L_p(\mathbb{R}_\theta)$ for the noncommutative L_p-space with respect to τ_θ and identify $L_\infty(\mathbb{R}_\theta) = \mathbb{R}_\theta$. Following the standard definition of the L_p-spaces for a von Neumann algebra, for $1 \le p < \infty$, the L_p-norm is defined as

$$\|x\|_p = \tau_\theta\big(|x|^p\big)^{1/p}$$

for those $x \in \mathbb{R}_\theta$ such that the norm is finite. The L_p-space is then defined as the completion of $\{x \in \mathbb{R}_\theta : \|x\|_p < \infty\}$ with respect to the L_p-norm.

For all θ, $L_2(\mathbb{R}_\theta) \cong L_2(\mathbb{R}^d)$ and λ_θ is exactly the left regular representation of $\mathbb{R}_\theta$ on $L_2(\mathbb{R}_\theta)$. The density of $\mathcal{S}_\theta$ in $\mathbb{E}_\theta$ is by definition, and the density of $\mathcal{S}_\theta$ in $L_2(\mathbb{R}_\theta)$ follows from the unitarity of the map λ_θ and the density of $\mathcal{S}(\mathbb{R}^d)$ in $L_2(\mathbb{R}^d)$.

The following lemma proves that $\mathcal{S}_\theta$ is also dense in the noncommutative L_1-space $L_1(\mathbb{R}_\theta)$. A similar result with a different proof is in [31, Proposition 3.14].

Lemma 2.5. *$\mathcal{S}_\theta$ is dense in $L_1(\mathbb{R}_\theta)$.*

Proof. If $a \in L_1(\mathbb{R}_\theta)$, then it follows from the general theory of semifinite von Neumann algebras that there exists a factorization $a = a_1 a_2$ for some $a_1, a_2 \in L_2(\mathbb{R}_\theta)$ and $\|a_1\|_2 = \|a_2\|_2 = \|a\|_1^{\frac{1}{2}}$; indeed, one can define $a_2 = |a|^{1/2}$ and $a_1 = u|a|^{1/2}$, where $a = u|a|$ is a polar decomposition. Then we can find $f_1, f_2 \in S(\mathbb{R}^d)$ such that $\|\lambda_\theta(f_j) - a_j\|_2 \le \varepsilon$,

$j = 1, 2$. Then

$$\left\|a - \lambda_\theta(f_1)\lambda_\theta(f_2)\right\|_1 \le \left\|a_1 a_2 - a_1\lambda_\theta(f_2)\right\|_1 + \left\|a_1\lambda_\theta(f_2) - \lambda_\theta(f_1)\lambda_\theta(f_2)\right\|_1$$
$$\le \|a_1\|_2\varepsilon + \|f_2\|_2\varepsilon \le \big(2\|a\|_1^{\frac{1}{2}} + \varepsilon\big)\varepsilon. \qquad \blacksquare$$

The noncommutative Lorentz space $L_{p,\infty}(\mathbb{R}_\theta)$ is the space of measurable operators a affiliated to $\mathbb{R}_\theta$ such that the following quasinorm is finite:

$$\|a\|_{L_{p,\infty}}^p = \sup_{t>0} t^p \tau_\theta(1_{|a|>t}),$$

where $1_{|a|>t}$ denotes the spectral projection of $|a|$. In other words, $a \in L_{p,\infty}(\mathbb{R}_\theta)$ if $\tau_\theta(1_{|a|>t})$ is asymptotically at most $O(t^{-p})$. This is a special case of the general notion of weak L_p-space on a semifinite von Neumann algebra. For $\det(\theta) \neq 0$, the above (weak) L_p-spaces are nothing but the (weak) Schatten p-spaces.

Proposition 2.6. *Denote* $|x| := (\sum_j x_j^2)^{\frac{1}{2}}$ *and* $\langle x\rangle := (1 + \sum_j x_j^2)^{\frac{1}{2}}$. *For all* θ, *the following holds.*

(i) $\tau_\theta(e^{-t|x|^2}) = t^{-\frac{d}{2}} \det(\frac{\pi i t\theta}{\sinh(it\theta)})^{1/2}$ *for* $t > 0$.

Here the function $\mu \mapsto \frac{\pi\mu}{\sinh\mu}$ *is a real function continuously extended to* $\mu = 0$ *and* $\frac{\pi i\theta}{\sinh(i\theta)}$ *is the functional calculus for self-adjoint matrix* $i\theta$.

(ii) $\langle x\rangle^{-1} \in L_{d,\infty}(\mathbb{R}_\theta)$.

Proof. Let us first consider that θ is the standard form (2.6) of rank $2n$. We have shown in Proposition 2.4 that there is (up to a factor $(2\pi)^n$) a trace preserving $*$-isomorphism $\pi : \mathbb{R}_\theta \to B(L_2(\mathbb{R}^n)) \bar{\otimes} L_\infty(\mathbb{R}^{d-2n})$ on $L_2(\mathbb{R}^{d-n})$ such that for $1 \le j \le n$, $1 \le k \le d - 2n$

$$x_j \mapsto D_{\mathbf{y}_j}, \quad x_{j+n} \mapsto \mathbf{y}_j, \quad x_{2n+k} \mapsto \mathbf{y}_{n+k},$$

where $D_{\mathbf{y}_j}$ and $\mathbf{y}_j$ are the self-adjoint derivative and position operators on $L_2(\mathbb{R}^{d-n})$

$$D_{\mathbf{y}_j} g = -i\frac{\partial g}{\partial \mathbf{y}_j}, \quad (y_j g)(\mathbf{y}) = \mathbf{y}_j g(\mathbf{y}).$$

Then $\langle x\rangle^2$ is unitary equivalent to (a multiple) of the following operator on $L_2(\mathbb{R}^{d-n})$:

$$H := \left(\sum_{j=1}^n D_{\mathbf{y}_j}^2 + \mathbf{y}_j^2\right) \otimes \mathrm{id}_{L_2(\mathbb{R}^{d-2n})} + \mathrm{id}_{L_2(\mathbb{R}^n)} \otimes \left(1 + \sum_{l=n+1}^{d-n} \mathbf{y}_l^2\right).$$

The first part is the Hamiltonian of n-dimensional quantum harmonic oscillator and the second part is a multiplier on $L_2(\mathbb{R}^{d-2n})$. It is known (see [24, Chapter 11]) that

$$H_1 := \left(\sum_{j=1}^n D_{\mathbf{y}_j}^2 + \mathbf{y}_j^2\right)$$

has a discrete spectrum $\{n + 2k\}_{k=0}^\infty$ and the multiplicity of $n + 2k$ is $\binom{k+n-1}{k}$.

Moreover, if T is a real invertible matrix such that $T\theta T^t$ is the standard form (2.6), then $\det(T) = (\mu_1\mu_2\cdots\mu_n)^{-1}$, where $\mu_1, \mu_2, \dots, \mu_n$ are imaginary parts of eigenvalues of θ. Thus, by the isomorphism in (2.7), we have

$$\begin{aligned}
\tau_\theta(e^{-t|x|^2}) &= \mu_1\mu_2\cdots\mu_n(2\pi)^n \cdot \operatorname{tr}\big(e^{-t\sum_{j=1}^n \mu_j (D_{\mathbf{y}_j}^2 + \mathbf{y}_j^2)}\big) \\
&\quad \cdot \int_{\mathbb{R}^{d-2n}} e^{-t\sum_{j=n+1}^{d-n} \mathbf{y}_j^2}\, d\mathbf{y}_{n+1}\cdots d\mathbf{y}_{d-n} \\
&= \mu_1\mu_2\cdots\mu_n(2\pi)^n \cdot \Big(\prod_{j=1}^n \sum_{k=0} e^{-t\mu_j(1+2k)}\Big)\cdot\Big(\frac{\pi}{t}\Big)^{\frac{d-2n}{2}} \\
&= \Big(\prod_{j=1}^n \frac{2\pi t\mu_j}{e^{t\mu_j} - e^{-t\mu_j}}\Big)(\pi)^{\frac{d-2n}{2}} t^{-\frac{d}{2}} \\
&= t^{-\frac{d}{2}}\Big(\prod_{j=1}^n \frac{\pi t\mu_j}{\sinh t\mu_j}\Big)(\pi)^{\frac{d-2n}{2}} \\
&= t^{-\frac{d}{2}} \det\Big(\frac{\pi i t\theta}{\sinh(it\theta)}\Big)^{1/2}.
\end{aligned}$$

The last equality follows from $\lim_{\mu\to 0} \frac{\pi\mu}{\sinh(\mu)} = \pi$. We now explain how (i) implies (ii). We observe that if F denotes the function

$$F(\lambda) = \tau(1_{|x|^2} < \lambda), \quad \lambda > 0,$$

then $\tau_\theta(e^{-t|x|^2})$ is the Laplace–Stieltjes transform of F. It follows from a Tauberian theorem of Hardy and Littlewood [26, Chapter 1, Theorem 15.3] that the existence of the limit

$$\lim_{t\to 0} t^{\frac{d}{2}} \tau_\theta(e^{-t|x|^2})$$

implies that there exists a limit as $\lambda \to \infty$ of $F(\lambda)\lambda^{-\frac{d}{2}}$. In particular, $F(\lambda) = O(\lambda^{\frac{d}{2}})$ as $\lambda \to \infty$, and hence for $t < 1$ we have

$$\tau(1_{\langle x\rangle^{-1} > t}) = \tau(1_{\langle x\rangle < t^{-1}}) = \tau(1_{|x|^2 < t^{-2}-1}) = F(t^{-2}-1) \le Ct^{-d}.$$

Since $\langle x\rangle \ge 1$, we also have $\tau(1_{\langle x\rangle^{-1}>t}) = 0$ when $t > 1$, and hence $\langle x\rangle^{-1} \in L_{d,\infty}(\mathbb{R}_\theta)$. ∎

Let $D_{\mathbf{x}_1}, \dots, D_{\mathbf{x}_d}$ be the partial derivative operators

$$D_{\mathbf{x}_j} f = -i\frac{\partial}{\partial \mathbf{x}_j} f,$$

which are unbounded self-adjoint operators on $L_2(\mathbb{R}^d)$ with a common core $\mathcal{S}(\mathbb{R}^d)$. On $\mathbb{R}_\theta$, we define for $\lambda_\theta(f)$ in $\mathcal{S}_\theta \subset B(L_2(\mathbb{R}^d))$ the partial derivatives

$$D_j\lambda_\theta(f) := \big[D_{\mathbf{x}_j}, \lambda_\theta(f)\big] = \lambda_\theta(D_{\mathbf{x}_j} f).$$

Since $D_{\mathbf{x}_j}$ is the same as D_j for $\theta = 0$, we will often write $D_{\mathbf{x}_j}$ simply as D_j. Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distribution on $\mathbb{R}^d$. In [20, 44] (see also [18]), Moyal product and the Weyl quantization are weakly extended to $\mathcal{S}'(\mathbb{R}^d)$ as follows:

$$\langle T \star_\theta f, g\rangle = \langle T, f \star_\theta g\rangle, \quad \langle f \star_\theta T, g\rangle = \langle T, g \star_\theta f\rangle,$$

where the bracket is the pairing between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. For $T \in S'(\mathbb{R}^d)$, $\lambda_\theta(T)$ is the quantized operator $\lambda_\theta(T)f = T \star_\theta f$ and satisfies

$$\lambda_\theta(T)\lambda_\theta(f) = \lambda_\theta(T \star_\theta f), \quad \lambda_\theta(f)\lambda_\theta(T) = \lambda_\theta(f \star_\theta T).$$

For all $T \in \mathcal{S}'(\mathbb{R}^d)$, $\lambda_\theta(T)$ commutes with the right Moyal multiplication hence affiliated to $\mathbb{R}_\theta$. We will use the multiplier algebra introduced in [44],

$$\mathcal{M}_\theta = \{\lambda_\theta(T) \mid T \in \mathcal{S}'(\mathbb{R}^d), \lambda_\theta(T)\mathcal{S}_\theta \subset \mathcal{S}_\theta, \mathcal{S}_\theta\lambda_\theta(T) \subset \mathcal{S}_\theta\}.$$

The pairing between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ coincides with the τ_θ-trace duality for the quantization. Namely, for $\lambda_\theta(T) \in \mathcal{M}_\theta, \lambda_\theta(f) \in \mathcal{S}_\theta$,

$$\tau_\theta\big(\lambda_\theta(T)\lambda_\theta(f)\big) = \tau_\theta\big(\lambda_\theta(T \star_\theta f)\big) = \int T \star_\theta f = \langle T, f\rangle.$$

In particular, $\mathcal{M}_\theta$ contains the noncommutative polynomials of $x_1, \dots, x_d$ as the quantized coordinate function $\mathbf{x}_j$,

$$\lambda_\theta(\mathbf{x}_j) = x_j, \quad x_j\lambda_\theta(f) = \lambda_\theta(\mathbf{x}_j f) + \frac{1}{2}\sum_k \theta_{jk} D_k \lambda_\theta(f).$$

The transference automorphism $\alpha_{\mathbf{x}}$ and the partial derivatives D_j weakly extend to $\mathcal{M}_\theta$

$$\langle\alpha_{\mathbf{x}}(a), \lambda_\theta(f)\rangle := \langle a, \alpha_{-\mathbf{x}}\lambda_\theta(f)\rangle, \quad \langle D_j(a), \lambda_\theta(f)\rangle = \langle a, D_j\lambda_\theta(f)\rangle.$$

Viewing $a \in \mathcal{M}_\theta$ as an unbounded operator densely defined on $S(\mathbb{R}^d) \subset L_2(\mathbb{R}^d)$, the weak derivatives satisfy $D_j(a) = [D_j, a]$.

3. Asymptotic degrees

In this section, we introduce a notion of "asymptotic degree" which measures the "growth" of unbounded elements in $\mathcal{M}_\theta$, and which serves as a key technical tool for later discussions. The idea is inspired by the abstract ΨDOs introduced by Connes and Moscovici in [12, 13] and the abstract Weyl algebras of Guillemin [21]. We briefly recall the basic setting here. Let D be a (possibly unbounded) self-adjoint operator on a Hilbert space H such that $|D|$ is strictly positive. For each $s \in \mathbb{R}$, put $H^s = \mathrm{Dom}(|D|^s)$ with inner product

$$\langle v_1, v_2\rangle_{H^s} := \langle |D|^s v_1, |D|^s v_2\rangle_H, \quad v_1, v_2 \in \mathrm{Dom}\,(|D|^s).$$

Let $H^\infty = \bigcap_{s\in\mathbb{Z}} H^s$. Because $\mathrm{Dom}(e^{|D|^2}) \subset H^\infty$, H^∞ is a dense subspace of H. Let F be a closed operator on H such that $H^\infty \subset \mathrm{Dom}(F)$, $F(H^\infty) \subset H^\infty$. Because $|D|^{-s} : H^0 \to H^s$ is an isometric isomorphism, one sees that

$$\|F : H^s \to H^{s-r}\| = \big\||D|^{s-r} F |D|^{-s}\big\|.$$

For a fixed $r \in \mathbb{R}$, F extends to a bounded operator from H^s to H^{s-r} for any s if and only if $|D|^{s-r}F|D|^{-s}$ are bounded on H. Such an F is considered as an abstract ΨDO of order r.

We use the above idea to characterize the asymptotic degree (we use the word "degree" to distinguish with the notation "order" for ΨDOs) of elements in $\mathcal{M}_\theta$. We choose the strictly positive operator D as $\langle x\rangle := (1+\sum_j x_j^2)^{\frac{1}{2}}$.

Definition 3.1. We say that an operator $a \in \mathcal{M}_\theta$ is of *asymptotic degree r* if for any $s \in \mathbb{R}$,

$$\langle x\rangle^s a \langle x\rangle^{-s-r}$$

extends to a bounded operator in $B(L_2(\mathbb{R}_\theta))$ (hence also in $\mathbb{R}_\theta \subset B(L_2(\mathbb{R}_\theta))$). We denote by O^r the set of all elements of asymptotic degree r and write $O^{-\infty} = \bigcap_{r\in\mathbb{Z}} O^r$.

Let $L_2^s(\mathbb{R}_\theta)$ be the Hilbert space completion of $\mathcal{S}_\theta$ with respect to the inner product

$$\big\langle \lambda_\theta(f), \lambda_\theta(g)\big\rangle_s = \tau_\theta\big(\lambda_\theta(f)^*\langle x\rangle^{2s}\lambda_\theta(g)\big).$$

It is clear that $a \in O^r$ if and only if for any $s \in \mathbb{R}$, the left multiplication operator $\lambda_\theta(f) \mapsto a\lambda_\theta(f)$ extends continuously from $L_2^s(\mathbb{R}_\theta)$ to $L_2^{s-r}(\mathbb{R}_\theta)$. The following theorem estimates the degrees of some common elements. We introduce the standard notation of multi-indices that for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$,

$$x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d}, \quad D^\alpha := D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_d^{\alpha_d}.$$

Note that the product x^α is ordered because x_j's are noncommutative.

Theorem 3.2. *For all multi-indices α and r ∈ ℝ,*

$$x^\alpha \in O^{|\alpha|}, \quad \big[x^\alpha, \langle x\rangle^r\big] \in O^{r+|\alpha|-2}, \quad D^\alpha\big(\langle x\rangle^r\big) \in O^{r-|\alpha|}.$$

Proof. We divide the proof into several steps.

Step 1. $[D_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$, $[x_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$ are bounded for $0 < r < 2$.

We use the fractional power for a positive operator A:

$$A^{-s} = C_s\int_0^\infty (t+A)^{-1}t^{-s}\,dt, \quad 0 < s < 1,$$

where C_s is a nonzero constant depending on s. Since the constant does not affect the boundedness, we suppress all constant C_s's. Denote $\Delta := \langle x\rangle^2 = 1 + \sum_j x_j^2$. For $0 < r < 2$,

$$\begin{aligned}
&\big[D_j, \langle x\rangle^{-r}\big] \\
&= \int_0^\infty \big[D_j, (t+\Delta)^{-1}\big]t^{-\frac{r}{2}}\,dt \\
&= \int_0^\infty (t+\Delta)^{-1}\big[(t+\Delta), D_j\big](t+\Delta)^{-1}t^{-\frac{r}{2}}\,dt \\
&= 2i\int_0^\infty (t+\Delta)^{-1}x_j(t+\Delta)^{-1}t^{-\frac{r}{2}}\,dt
\end{aligned}$$

$$= 2i \int_0^\infty x_j (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt + 2i \int_0^\infty \big[(t+\Delta)^{-1}, x_j\big](t+\Delta)^{-1} t^{-\frac{r}{2}}\,dt$$
$$= 2i \int_0^\infty x_j (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt + 2i \int_0^\infty (t+\Delta)^{-1}\big[x_j, (t+\Delta)\big](t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt$$
$$= 2i x_j \int_0^\infty (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt + 2\sum_k \theta_{jk} \int_0^\infty (t+\Delta)^{-1} x_k (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt.$$

For the first integral,

$$2i x_j \int_0^\infty (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt \cdot \Delta^{\frac{1+r}{2}} = 2i x_j \Delta^{-1-\frac{r}{2}} \Delta^{\frac{1+r}{2}} = 2i x_j \Delta^{-\frac{1}{2}}$$

is bounded. For the second integral,

$$\left\| \int_0^\infty (t+\Delta)^{-1} x_k (t+\Delta)^{-2} t^{-\frac{r}{2}}\,dt \langle x\rangle^{1+r} \right\| \le \int_0^\infty \big\|(t+\Delta)^{-2+\frac{r}{2}}\big\| t^{-\frac{r}{2}}\,dt$$
$$\le \int_0^\infty (t+1)^{-2+\frac{r}{2}} t^{-\frac{r}{2}}\,dt < \infty$$

converges absolutely. For the commutator with x_j, we have

$$\big[x_j, \langle x\rangle^{-r}\big] = \int (t+\Delta)^{-1}\big[(t+\Delta), x_j\big](t+\Delta)^{-1} t^{-\frac{r}{2}}\,dt$$
$$= 2i \sum_k \theta_{jk} \int (t+\Delta)^{-1} x_k (t+\Delta)^{-1} t^{-\frac{r}{2}}\,dt = 2i \sum_k \theta_{jk} \big[D_j, \langle x\rangle^{-r}\big].$$

Then $[x_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$ for $0 < r < 2$ which is bounded by the previous case. In particular, we also obtained that

$$\langle x\rangle^{-r} x_j \langle x\rangle^{r+1} = \big[\langle x\rangle^{-r}, x_j\big]\langle x\rangle^{r+1} + x_j \langle x\rangle^{-1}$$

is bounded for $0 < r < 2$.

Step 2. $[x_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$, $[D_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$ are bounded for all r.

First for $-2 < r < 0$, the boundedness follows from

$$\big[x_j, \langle x\rangle^{-r}\big]\langle x\rangle^{r+1} = \big[x_j, \langle x\rangle^{-r-2}\big]\langle x\rangle^{r+3} + 2i \sum_k \theta_{jk} \langle x\rangle^{-r-2} x_k \langle x\rangle^{r+1}.$$

Then we have the initial case for $-2 < r < 2$ and use the following induction steps $r \to -r+1$ for $r < 0$ and $r \to -r-1$ for $r > 0$:

$$\begin{aligned}
\big[x_j, \langle x\rangle^{r}\big]\langle x\rangle^{-r+1} &= \langle x\rangle\big[x_j, \langle x\rangle^{r-1}\big]\langle x\rangle^{-r+1} + \big[x_j, \langle x\rangle\big] \\
&= \langle x\rangle^{r}\big[\langle x\rangle^{-r+1}, x_j\big] + \big[x_j, \langle x\rangle\big], \\
\big[x_j, \langle x\rangle^{r}\big]\langle x\rangle^{-r+1} &= \langle x\rangle^{-1}\big[x_j, \langle x\rangle^{r+1}\big]\langle x\rangle^{-r+1} + \big[x_j, \langle x\rangle^{-1}\big]\langle x\rangle^{2} \\
&= \langle x\rangle^{r}\big[\langle x\rangle^{-r-1}, x_j\big]\langle x\rangle^{2} + \big[x_j, \langle x\rangle^{-1}\big]\langle x\rangle^{2} \\
&= \langle x\rangle^{r}\big[\langle x\rangle^{-r-1}, x_j\big] - \langle x\rangle^{-1}\big[\langle x\rangle^{2}, x_j\big] + \big[x_j, \langle x\rangle^{-1}\big]\langle x\rangle^{2}.
\end{aligned}$$

The argument for $[D_j, \langle x\rangle^{-r}]\langle x\rangle^{r+1}$ is similar.

Step 3. $x^\alpha \in O^{|\alpha|}$ and $[x^\alpha, \langle x\rangle^r] \in O^{|\alpha|+r-2}$ for all α and r.

First, by Step 2 we have that for all s

$$\begin{aligned} \langle x\rangle^s x_j \langle x\rangle^{-s-1} &= \big[\langle x\rangle^s, x_j\big]\langle x\rangle^{-s-1} + x_j\langle x\rangle^{-1}, \\ \langle x\rangle^{-s}\big[x_j, \langle x\rangle^r\big]\langle x\rangle^{-r+s+1} &= \big[x_j, \langle x\rangle^{r-s}\big]\langle x\rangle^{-r+s+1} + \big[x_j, \langle x\rangle^{-s}\big]\langle x\rangle^{s+1}, \\ \langle x\rangle^{-s}\big[D_j, \langle x\rangle^r\big]\langle x\rangle^{-r+s+1} &= \big[D_j, \langle x\rangle^{r-s}\big]\langle x\rangle^{-r+s+1} + \big[D_j, \langle x\rangle^{-s}\big]\langle x\rangle^{s+1} \end{aligned}$$

are all bounded. This implies that

$$x_j \in O^1, \quad \big[x_j, \langle x\rangle^r\big] \in O^{r-1}, \quad \big[D_j, \langle x\rangle^r\big] \in O^{r-1}.$$

Thus $x^\alpha \in O^{|\alpha|}$ by product. For $[x^\alpha, \langle x\rangle^r]$, we use the induction step that by the Leibniz rule

$$\big[x_j x^\alpha, \langle x\rangle^r\big] = x_j\big[x^\alpha, \langle x\rangle^r\big] + \big[x_j, \langle x\rangle^r\big]x^\alpha,$$

and $[x_j, x^\alpha]$ is a polynomial of order less than $|\alpha|$.

Step 4. $D^\alpha(\langle x\rangle^r) \in O^{r-|\alpha|}$ for all $r \in \mathbb{R}$.

We first do induction on $|\alpha|$ for $-2 < r = -2s < 0$. For $0 < s < 1$, we introduce the notation

$$\begin{aligned} &I_s(a_1, a_2, \dots, a_l) \\ &\quad := \int_0^\infty t^{-s}(t+\Delta)^{-1}a_1(t+\Delta)^{-1}a_2(t+\Delta)^{-1}\cdots(t+\Delta)^{-1}a_l(t+\Delta)^{-1}\,dt. \end{aligned}$$

For $|\alpha| = 1$, $[D_j, \langle x\rangle^{-2s}] = 2iI_s(x_j)$. Note that by the Leibniz rule, we have

$$\begin{aligned} \big[D_j, I_\alpha(a_1, \dots, a_l)\big] = &\sum_{1\le k\le l} I_\alpha\big(a_1, \dots, \underbrace{[D_j, a_k]}_{k\text{th}}, \dots, a_l\big) \\ &+ \sum_{1\le k\le l+1} I_\alpha\big(a_1, \dots, \underbrace{[\Delta, D_j]}_{k\text{th}}, a_k, \dots, a_l\big). \end{aligned} \tag{3.1}$$

Then all higher-order derivatives of $\langle x\rangle^{-2s}$ are the sum of $I_s(a_1, a_2, \dots, a_l)$ terms with $a_1, \dots, a_l \in \{1, x_1, \dots, x_n\}$. Moreover, their degree can be tracked inductively. Let s_k be the degree of a_k. We show in the next lemma that $I_s(a_1, \dots, a_l)$ is at most of degree $-2l - 2s + \sum_k s_k$. Now assume that for $|\alpha| \le N$, $D^\alpha(\langle x\rangle^r)$ is a sum of the terms $I_s(a_1, a_2, \dots, a_l)$ with $-2l - 2s + \sum_k s_k \le r - |\alpha|$. Then $[D_j, D^\alpha(\langle x\rangle^r)]$ is a sum of commutators as (3.1). The degree of the first part in (3.1) is lowered by 1 because $[D_j, x_j] = -i$ and $[D_j, 1] = 0$, and the second part has degree at most

$$-2(l+1) - 2s + \Big(1 + \sum_k s_k\Big) = -2l - 2s - 1 + \sum_k s_k$$

because $[\Delta, D_j] = 2ix_j$ and the length l is increased by 1. Thus by induction on $|\alpha|$ we prove the case $-2 < r < 0$. For general r, one can always write $r = r_1 + r_2 + \cdots + r_l$ as a finite sum of $r_k \in (-2, 0] \cup 2\mathbb{N}$. Then by the Leibniz rule

$$D_\alpha(\langle x\rangle^r) = \sum_{\alpha_1+\cdots+\alpha_l=\alpha} \binom{\alpha}{\alpha_1, \ldots, \alpha_n} D_{\alpha_1}(\langle x\rangle^{r_1}) \cdots D_{\alpha_l}(\langle x\rangle^{r_l}),$$

where $\binom{\alpha}{\alpha_1,\ldots,\alpha_n} = \alpha!(\alpha_1!)^{-1}\cdots(\alpha_d!)^{-1}$ is the multinomial coefficient. For positive integer m, $D_\alpha(x^{2m})$ is a polynomial of degree $2m - |\alpha|$ and the term $D_\alpha(\langle x\rangle^{r_k}), -2 < r_k < 0$ has degree at most $r_k - |\alpha|$ as proved above. Therefore, $D_\alpha(\langle x\rangle^r)$ is of degree at most $\sum_k r_k - |\alpha_k| = r - |\alpha|$. ∎

The following lemma is inspired from the abstract ΨDO calculus in [25].

Lemma 3.3. *Let $0 < s < 1$ and let I_s be the notation*

$$I_s(a_1, a_2, \ldots, a_l) := \int_0^\infty t^{-s}(t+\Delta)^{-1}a_1(t+\Delta)^{-1}a_2(t+\Delta)^{-1}\cdots(t+\Delta)^{-1}a_l(t+\Delta)^{-1}\,dt.$$

Then

(i) *if $a_k \in O^{s_k}$, then $I_s(a_1, a_2, \ldots, a_l) \in O^{-2l-2s+\sum_k s_k+\varepsilon}$ for any $\varepsilon > 0$,*

(ii) *if $a_k \in \{1, x_1, x_2, \ldots, x_n\}$, then $I_\alpha(a_1, a_2, \ldots, a_l) \in O^{-2l-2s+\sum_k s_k}$.*

Proof. Let $q, r \in \mathbb{R}$ with $-q + r = -2l - 2s + \sum_k s_k + \varepsilon$.

$$\begin{aligned}&\langle x\rangle^q \int_0^\infty t^{-s}(t+\Delta)^{-1}a_1(t+\Delta)^{-1}a_2(t+\Delta)^{-1}\cdots(t+\Delta)^{-1}a_l(t+\Delta)^{-1}dt\,\langle x\rangle^{-r}\\ &\quad= \int_0^\infty t^{-s}(t+\Delta)^{-1+\alpha-\varepsilon/2}\langle x\rangle^q(t+\Delta)^{-s+\varepsilon/2}a_1(t+\Delta)^{-1}\\ &\qquad\cdots(t+\Delta)^{-1}a_l(t+\Delta)^{-1}\langle x\rangle^{-r}\,dt.\end{aligned}$$

Note that

$$\begin{aligned}&\big\|\langle x\rangle^q(t+\Delta)^{-s+\varepsilon/2}a_1(t+\Delta)^{-1}a_2(t+\Delta)^{-1}\cdots(t+\Delta)^{-1}a_l(t+\Delta)^{-1}\langle x\rangle^{-r}\big\|\\ &\le \big\|\langle x\rangle^{2q-\varepsilon}(t+\Delta)^{-q+\varepsilon/2}\big\|\,\big\|\langle x\rangle^{q-2s+\varepsilon}a_1\langle x\rangle^{-q+2s-\varepsilon-s_1}\big\|\,\big\|\langle x\rangle^2(t+\Delta)^{-1}\big\|\\ &\quad\cdots\big\|\langle x\rangle^2(t+\Delta)^{-1}\big\|\,\big\|\langle x\rangle^{q+\sum_{k\le l-1}s_k-2(n-1)-2s+\varepsilon}a_l\langle x\rangle^{-q-\sum_{k\le l}s_k+2s+2(n-1)-\varepsilon}\big\|\\ &\quad\times\big\|\langle x\rangle^2(t+\Delta)^{-1}\big\|\\ &\le \big\|\langle x\rangle^{q-2s+\varepsilon}a_1\langle x\rangle^{-q+2s-\varepsilon-s_1}\big\|\\ &\quad\cdots\big\|\langle x\rangle^{q+\sum_{k\le l-1}s_k-2(l-1)-2s+\varepsilon}a_l\langle x\rangle^{-q-\sum_{k\le l}s_k+2s+2(l-1)-\varepsilon}\big\|\end{aligned}$$

which is uniformly bounded. Thus,

$$\begin{aligned}
&\left\| \langle x\rangle^q \int_0^\infty t^{-s}(t+\Delta)^{-1}a_1(t+\Delta)^{-1}a_2(t+\Delta)^{-1}\cdots(t+\Delta)^{-1}a_n(t+\Delta)^{-1}dt\langle x\rangle^{-r}\right\| \\
&\quad\lesssim \int_0^\infty \left\|t^{-q}(t+\Delta)^{-1+s-\varepsilon/2}\right\| dt \\
&\quad\le \int_0^\infty t^{-s}(t+1)^{-1+s-\varepsilon/2}\,dt \\
&\quad< \infty.
\end{aligned}$$

For (ii), note that

$$I_s(\underbrace{1,\dots,1}_{l}) = \int_0^\infty (t+\Delta)^{-l}t^{-s}\,dt = C_s\langle x\rangle^{-2(l-1)-2s}.$$

Let k be the last position in $I_s(a_1,\dots,a_l)$ such that a_k is nonscalar. That is, for all $n\le k$, $a_n = x_{j_n}$ for some $1\le j_n\le d$ and $a_m = 1$ for all $k<m\le l$. We have that

$$\begin{aligned}
&I_s\Big(\underbrace{a_1,\dots,a_{k-1},x_j,1,\dots,1}_{l}\Big) \\
&\quad= I_s\Big(\underbrace{a_1,\dots,a_{k-1},1,x_j,1,\dots,1}_{l}\Big) + I_s\Big(\underbrace{a_1,\dots,a_{k-1},1,[\Delta,x_j],1,\dots,1}_{l+1}\Big) \\
&\quad= I_s\Big(\underbrace{a_1,\dots,a_{k-1},1,\dots,1}_{l}\Big)x_j + \sum_{k+1\le m\le l+1} I_s\big(a_1,\dots,a_{k-1},1,\dots,\underbrace{[\Delta,x_j]}_{m\text{th}},\dots,1\big).
\end{aligned}$$

Note that $[\Delta,x_j] = -2i\sum_k \theta_{kj}x_k$. Then by (i), the second part belongs to

$$O^{-2l-2+\sum_k s_k-2s+\varepsilon}\subseteq O^{-2l+\sum_k s_k-2s}.$$

We then finish the proof by the induction on the last nonscalar position. ∎

Proposition 3.4. (i) *Let $s\in\mathbb{R}$. If $D^\alpha(a)\langle x\rangle^{-s}$ is bounded for all α, then $a\in O^s$.*

(ii) *$\mathcal{S}_\theta = \{a\in\mathbb{R}_\theta \mid D^\alpha(a)\in O^{-\infty}$ for all $\alpha\}$. Moreover, the map $f\mapsto\lambda_\theta(f)$ is bi-continuous from $\mathcal{S}(\mathbb{R}^d)$ equipped with the standard seminorms to $\mathcal{S}_\theta$ with the seminorms $\|D^\alpha(\cdot)\langle x\rangle^{2n}\|$ for all α and n. In particular, $\langle x\rangle^r\mathcal{S}_\theta\subset\mathcal{S}_\theta$ for any r.*

Proof. (i) Define the notation

$$\begin{aligned}
a^{(1)} &:= [\Delta,a] = i\sum_l \theta_{jl}\big(x_jD_l(a)+D_l(a)x_j\big); \\
a^{(2)} &:= \big[\Delta,[\Delta,a]\big] = -2\sum_l\sum_m \theta_{jl}\theta_{mj}\big(x_mD_l(a)+D_l(a)x_m\big) \\
&\quad -\sum_{l,m}\theta_{jl}\theta_{km}\big(x_jx_kD_lD_m(a)+x_jD_lD_m(a)x_k+x_kD_lD_m(a)x_j+D_lD_m(a)x_kx_j\big).
\end{aligned}$$

We first give the proof for $s = 0$. Assume that $D^\alpha(a)$ is bounded for all α. Then $a^{(1)}\langle x\rangle^{-1}$ is bounded because

$$\begin{aligned} x_j D_l(a)\langle x\rangle^{-1} &= D_l(a)x_j\langle x\rangle^{-1} + \big[x_j, D_l(a)\big]\langle x\rangle^{-1} \\ &= D_l(a)x_j\langle x\rangle^{-1} - \sum_k \theta_{jk} D_k D_l(a)\langle x\rangle^{-1}, \end{aligned}$$

and similarly one can verify that $a^{(2)}\langle x\rangle^{-2}$ is bounded. Then for $0 < r < 2$,

$$\begin{aligned} \big[a, \langle x\rangle^{-r}\big]\langle x\rangle^r &= I_{\frac{r}{2}}\big([\Delta, a]\big)\langle x\rangle^r = I_{\frac{r}{2}}(a^{(1)})\langle x\rangle^r \\ &= a^{(1)} I_{\frac{r}{2}}(1)\langle x\rangle^r + I_{\frac{r}{2}}(a^{(2)}, 1)\langle x\rangle^r \\ &= a^{(1)}\langle x\rangle^{-1} + I_{\frac{r}{2}}(a^{(2)}, 1)\langle x\rangle^r. \end{aligned}$$

The second part is bounded because

$$\begin{aligned} \big\| I_{\frac{r}{2}}(a^{(2)}, 1)\langle x\rangle^r \big\| &\le \int_0^\infty t^{-\frac{r}{2}} \big\|(\Delta + t)^{-1}\big\| \big\|a^{(2)}(t+\Delta)^{-1}\big\| \big\|\langle x\rangle^r (t+\Delta)^{-1}\big\|\, dt \\ &\lesssim \int_0^\infty t^{-\frac{r}{2}} \big\|\langle x\rangle^r (t+\Delta)^{-2}\big\|\, dt \le \int_0^\infty t^{-\frac{r}{2}} (t+1)^{-2+\frac{r}{2}}\, dt < \infty. \end{aligned}$$

Thus we have that $\langle x\rangle^{-r} a \langle x\rangle^r$ is bounded for $0 \le r \le 2$ and for $-2 \le r \le 0$ by taking the adjoint. Moreover, the same argument applies to $D^\beta(a)$ for all β. Consider that $b = \langle x\rangle^{-r} a \langle x\rangle^r$; then

$$D^\alpha(b) = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \binom{\alpha}{\alpha_1, \alpha_2, \alpha_3} D^{\alpha_1}\big(\langle x\rangle^{-r}\big) D^{\alpha_2}(a) D^{\alpha_3}\big(\langle x\rangle^r\big)$$

is bounded for all α by the Leibniz rule and Theorem 3.2. Thus we have shown that $\langle x\rangle^{-r} a \langle x\rangle^r$ is bounded for $-4 \le r \le 4$. By induction, this can be extended for all $r \in \mathbb{R}$ which proves the case $s = 0$. For general s, we have

$$D^\alpha\big(a\langle x\rangle^{-s}\big) = \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1, \alpha_2} D^{\alpha_1}(a) D^{\alpha_2}\big(\langle x\rangle^{-s}\big),$$

where the assumption $D^\alpha(a)\langle x\rangle^{-s}$ is bounded and $D^{\alpha_2}(\langle x\rangle^{-s}) \in O^{s-|\alpha|}$ by Theorem 3.2. Thus by the case of $s = 0$, we know that $a\langle x\rangle^{-s} \in O^0$, which implies that $a \in O^s$.

For (ii), we first show that for $f \in \mathcal{S}(\mathbb{R}^d)$, $\lambda_\theta(f)\langle x\rangle^{2m}$ is bounded for all positive integers m. Note that $\langle x\rangle^{2m}$ is a polynomial of x with degree $2m$. And

$$\begin{aligned} x_j \lambda_\theta(f) &= \lambda_\theta\Big(x_j f + \frac{i}{2}\sum_k \theta_{jk}\partial_j f\Big), \\ \lambda_\theta(f) x_j &= \big(x_j \lambda_\theta(\bar f)\big)^* = \Big(\lambda_\theta\Big(x_j \bar f + \frac{i}{2}\sum_k \theta_{jk}\overline{\partial_j f}\Big)\Big)^* \\ &= \lambda_\theta(x_j f) - \frac{i}{2}\sum_k \theta_{jk}\lambda_\theta(\partial_j f). \end{aligned}$$

Then $\lambda_\theta(f)\langle x\rangle^{2m}$ are again in $\mathcal{S}_\theta$; hence bounded. Therefore, for any $r>0$, $\lambda_\theta(f)\langle x\rangle^r$ is bounded and similarly for the derivatives $D^\alpha(\lambda_\theta(f))$. Thus by (i), $D^\alpha(\lambda_\theta(f))\in O^{-\infty}$ for all α. For the other direction, $a\in O^r$ for $r<-\frac{d}{2}$ implies that

$$\|a\|_2\le \big\|\langle x\rangle^r\big\|_2\big\|\langle x\rangle^{-r}a\big\|_\infty<\infty.$$

Thus $a=\lambda_\theta(f)$ for some $f\in L_2(\mathbb{R}^d)$ and $D^\alpha(a)=\lambda_\theta(D_\alpha(f))$ in the distribution sense. Then all the derivatives of f belongs to $L_2(\mathbb{R}^d)$ and hence f is in the Sobolev space $H^s(\mathbb{R}^d)=\{f\mid(1+\Delta)^{\frac{s}{2}}f\in L_2(\mathbb{R}^d)\}$ for all s. Using the Sobolev embedding theorem, $f\in C_0^\infty(\mathbb{R}^d)$ with all derivatives bounded. To see that $\mathbf{x}^\beta f$ are bounded functions for β, we use induction on $|\beta|$ and

$$\lambda_\theta(\mathbf{x}_j f)=x_j\lambda_\theta(f)-\frac{i}{2}\sum_k\theta_{jk}\lambda_\theta(D_j f). \tag{3.2}$$

Similarly, we know that $D_\alpha(f)\mathbf{x}^\beta$ are bounded for all α,β. To show that seminorms are equivalent, let $f\in\mathcal{S}(\mathbb{R}^d)$ and denote $\hat{f}$ as its Fourier transform. Let n be the smallest even integer greater than $\frac{d}{2}$,

$$\big\|D^\beta(f)\langle\mathbf{x}\rangle^{2m}\big\|_\infty\le\big\|\widehat{D^\beta(f)\langle\mathbf{x}\rangle^{2m}}\big\|_1\le\big\|\langle\boldsymbol{\xi}\rangle^n\widehat{D^\beta(f)\langle\mathbf{x}\rangle^{2m}}\big\|_2\big\|\langle\boldsymbol{\xi}\rangle^{-n}\big\|_2.$$

Let $\langle\boldsymbol{\xi}\rangle^n\widehat{D^\beta(f)\langle\mathbf{x}\rangle^{2m}}\in\mathcal{S}(\mathbb{R}^d)$ be the Fourier transform of g. g can be expressed as a linear combination of $\mathbf{x}^\beta D^\alpha(f)$ with $|\alpha|$ up to n and β up to $2m$. Therefore,

$$\begin{aligned}\big\|D^\beta(f)\langle\mathbf{x}\rangle^{2m}\big\|_\infty&\lesssim\big\|\lambda_\theta(g)\big\|_2\lesssim\big\|\lambda_\theta(g)\langle x\rangle^n\big\|_\infty\\&\lesssim\sup\big\{\big\|D^\alpha\lambda_\theta(f)x^\beta\big\|_\infty\mid|\alpha|\le n,|\beta|\le n+2m\big\}.\end{aligned}$$

Finally, we note that $D^\alpha\lambda_\theta(f)\in\mathcal{S}_\theta\subset O^{-\infty}$ and by Theorem 3.2 $D^\alpha\langle x\rangle^r\in O^{r-|\alpha|}$. By the product rule, $D^\alpha(\langle x\rangle^r\lambda_\theta(f))\in O^{-\infty}$ for all α. Then $\langle x\rangle^r\mathcal{S}_\theta\subset\mathcal{S}_\theta$. ■

Lemma 3.5. *Let* $\mathbf{y}\in\mathbb{R}^d$. *Denote* $\langle x+\mathbf{y}\rangle:=(1+\sum_j(x_j+\mathbf{y}_j)^2)^{\frac{1}{2}}$. *Then*

(i) $\alpha_{\mathbf{y}}(\langle x\rangle^r)=\langle x+\mathbf{y}\rangle^r$;

(ii) *for any* $0<r\le 2n$ *with* n *integer, there exists a constant* $c_{r,n}$ *such that*

$$\big\|\langle x+\mathbf{y}\rangle^r\langle x\rangle^{-r}\big\|_\infty\le c_{r,n}\langle\mathbf{y}\rangle^{2n},\quad\big\|\langle x\rangle^r\langle x+\mathbf{y}\rangle^{-r}\big\|_\infty\le c_{r,n}\langle\mathbf{y}\rangle^{2n}.$$

Proof. It is clear that $\langle\alpha_{\mathbf{y}}(x)\rangle^2=1+\sum_j(x_j+\mathbf{y}_j)^2=\alpha_{\mathbf{y}}(\langle x\rangle^2)$. Then by the fact that $\alpha_{\mathbf{y}}$ is a $*$-isomorphism on $\mathcal{M}_\theta$, $\alpha_{\mathbf{y}}(\langle x\rangle^{-2})=\langle\alpha_{\mathbf{y}}(x)\rangle^{-2}$. Then we apply the operator integral for $0<s<2$,

$$\langle x\rangle^{-s}=C_r\int_0^\infty(t+\langle x\rangle^2)^{-1}t^{-\frac{s}{2}}\,dt.$$

Then the general case follows from writing $r=2n-s$. For (ii), for $r=2$,

$$\big\|\langle x+\mathbf{y}\rangle^2\langle x\rangle^{-2}\big\|\le\Big\|1+\sum_j 2\mathbf{y}_j x_j\langle x\rangle^{-2}+\sum_j\mathbf{y}_j^2\langle x\rangle^{-2}\Big\|\lesssim\langle\mathbf{y}\rangle^2$$

$$\big\|\big(\langle x\rangle^{-2}-\langle x+\mathbf{y}\rangle^{2}\big)\big(t+\langle x\rangle^{2}\big)^{-1}\big\| \leq \Big\|\sum_j 2\mathbf{y}_j x_j\big(t+\langle x\rangle^2\big)^{-1}+\sum_j \mathbf{y}_j^2\big(t+\langle x\rangle^2\big)^{-1}\Big\| \lesssim t^{-\frac{1}{2}}\langle \mathbf{y}\rangle^2.$$

For $r=2n$, $\langle x\rangle^{2n}$ is a $2n$-degree polynomial of x_j whose largest coefficient is the constant term $\langle \mathbf{y}\rangle^2 n$. By a similar argument for $\langle x\rangle^{2n}$, we have

$$\big\|\langle x+\mathbf{y}\rangle^{2n}\langle x\rangle^{-2n}\big\| \lesssim \langle \mathbf{y}\rangle^{2n},$$
$$\big\|\big(\langle x\rangle^{-2n}-\langle x+\mathbf{y}\rangle^{2n}\big)\big(t+\langle x\rangle^{2n}\big)^{-1}\big\| \lesssim t^{-\frac{1}{2n}}\langle \mathbf{y}\rangle^2.$$

Using the transference,

$$\big\|\langle x\rangle^{2n}\langle x+\mathbf{y}\rangle^{-2n}\big\| = \big\|\alpha_{\mathbf{y}}\big(\langle x-\mathbf{y}\rangle^{2n}\langle x\rangle^{-2n}\big)\big\| = \big\|\langle x\rangle^{2n}\langle x+\mathbf{y}\rangle^{-2n}\big\| \lesssim \langle \mathbf{y}\rangle^{2n}.$$

This proves the inequality for $r=2n$ even integers. For general positive r, choose an integer n such that $0<r<2n-1$ and consider that

$$1-\langle x\rangle^{r}\langle x+\mathbf{y}\rangle^{-r} = \langle x\rangle^{r}\big(\langle x\rangle^{-r}-\langle x+\mathbf{y}\rangle^{-r}\big).$$

Take $s=\frac{r}{2n}<1-\frac{1}{2n}$, then we have

$$\begin{aligned}
&\langle x\rangle^{r}\big(\langle x\rangle^{-r}-\langle x+\mathbf{y}\rangle^{-r}\big)\\
&= C_s\langle x\rangle^{r}\int_0^\infty \big(\big(t+\langle x\rangle^{2n}\big)^{-1}-\big(t+\langle x+\mathbf{y}\rangle^{2n}\big)^{-1}\big)t^{-s}\,dt.\\
&= C_s\int_0^\infty \big(\langle x\rangle^{r}\big(t+\langle x\rangle^{2n}\big)^{-1}\big)\big(\big(\langle x+\mathbf{y}\rangle^{2n}-\langle x\rangle^{2n}\big)\big(t+\langle x+\mathbf{y}\rangle^{2n}\big)^{-1}\big)t^{-s}\,dt. \quad (3.3)
\end{aligned}$$

Note that $\|\langle x\rangle^{r}(t+\langle x\rangle^{2n})^{-1}\|\leq (t+1)^{s-1}$ and

$$\big\|\big(\langle x+\mathbf{y}\rangle^{2n}-\langle x\rangle^{2n}\big)\big(t+\langle x\rangle^{2n}\big)\big\| \lesssim t^{-\frac{1}{2n}}\langle \mathbf{y}\rangle^{2n}.$$

Therefore,

$$\big\|\langle x\rangle^{r}\big(\langle x\rangle^{-r}-\langle x+\mathbf{y}\rangle^{-r}\big)\big\| \lesssim \int_0^\infty (1+t)^{s-1}t^{-\frac{1}{2n}-s}\langle \mathbf{y}\rangle^{2n}\,dt \lesssim \langle \mathbf{y}\rangle^{2n}.$$

This proves the inequality for $\langle x\rangle^{r}\langle x+\mathbf{y}\rangle^{-r}$ and the other case follows from transference. ■

Using Lemma 3.5, we show that quantized partial derivatives defined in Section 2.3 are indeed the vector derivatives of transference action.

Proposition 3.6. *Let* $\mathbf{e}_j=(0,\dots,1,\dots,0)$ *be the* j*th basis vector.*

(i) *For* $\lambda_\theta(f)\in\mathcal{S}_\theta$, $D_j\lambda_\theta(f)=-i\lim_{h\to 0}\frac{1}{h}(\alpha_{h\mathbf{e}_j}(\lambda_\theta(f))-\lambda_\theta(f))$ *in* $\mathcal{S}_\theta$.

(ii) *Let* $m\in\mathbb{R}$. *If* $a\in\mathcal{M}_\theta$ *and* $D^\alpha(a)\langle x\rangle^m\in\mathbb{R}_\theta$ *for all* $|\alpha|\leq 2$, *then*

$$\lim_{h\to 0}\frac{1}{h}\big\|\big(\alpha_{h\mathbf{e}_j}(a)-a-hD_j(a)\big)\langle x\rangle^{m}\big\|_\infty = 0.$$

Proof. For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) = \sum_j \int_0^1 \mathbf{y}_j (\partial_j f)(\mathbf{x}+t\mathbf{y}) dt.$$

In terms of the function f, we have

$$\alpha_{\mathbf{y}}(f) - f = \sum_j \int_0^1 \mathbf{y}_j \alpha_{t\mathbf{y}}(iD_j f) dt.$$

Since $\{\alpha_{t\mathbf{y}}(iD_j f) \mid 0 \le t \le 1\}$ is uniformly bounded for every seminorm of $\mathcal{S}(\mathbb{R}^d)$, we have that $\mathbf{y} \to \alpha_{\mathbf{y}}(f)$ is continuous in $\mathcal{S}(\mathbb{R}^d)$. Because $\mathcal{S}_\theta$ and $\mathcal{S}(\mathbb{R}^d)$ have equivalent seminorms, we have that $\mathbf{y} \mapsto \alpha_{\mathbf{y}}(\lambda_\theta(f)) = \lambda_\theta(\alpha_{\mathbf{y}} f)$ is also continuous. It holds that

$$\begin{aligned} &\frac{1}{h}\big(\alpha_{h\mathbf{e}_j}(\lambda_\theta(f)) - \lambda_\theta(f) - h\lambda_\theta(iD_j f)\big) \\ &\quad = \int_0^1 \alpha_{t\mathbf{h}_j}\lambda_\theta(iD_j f) - \lambda_\theta(iD_j f) dt \\ &\quad = \int_0^1 \big(\alpha_{th\mathbf{e}_j}\lambda_\theta(iD_j f) - \lambda_\theta(iD_j f)\big) dt \end{aligned}$$

which goes to 0 in $\mathcal{S}_\theta$ for $h \to 0$ because of the continuity of $\mathbf{y} \to \alpha_{\mathbf{y}}(\lambda_\theta(D_j f))$. For (ii), we have the integral

$$\alpha_{\mathbf{y}}(a)\langle x\rangle^m - a\langle x\rangle^m = \sum_j \mathbf{y}_j \int_0^1 \alpha_{t\mathbf{y}}(iD_j a)\langle x\rangle^m \, dt, \tag{3.4}$$

which holds weakly. Suppose that $a\langle x\rangle^m$ and $D_j(a)\langle x\rangle^m$ are bounded. Then

$$\|\alpha_{\mathbf{y}}(D_j a)\langle x\rangle^m\| \le \|\alpha_{\mathbf{y}}(D_j a\langle x\rangle^m)\| \, \|\langle x+\mathbf{y}\rangle^{-m}\langle x\rangle^m\| \le \|D_j a\langle x\rangle^m\| \langle \mathbf{y}\rangle^{2n},$$

for some $2n > |m|$. So $\alpha_{\mathbf{y}}(D_j a)\langle x\rangle^m$ is uniformly bounded for small $\mathbf{y}$, which by the integral (3.4) implies that $\mathbf{y} \mapsto \alpha_{\mathbf{y}}(a)\langle x\rangle^m$ is continuous in norm. Now if $D^\alpha(a)\langle x\rangle^m$ is bounded for all $|\alpha| \le 2$, then

$$\left\|\frac{1}{h}\big(\alpha_{h\mathbf{e}_j}(a) - a - hD_j(a)\big)\langle x\rangle^m\right\|_\infty \le \int_0^1 \left\|\big(\alpha_{th\mathbf{e}_j}(iD_j a) - iD_j a\big)\langle x\rangle^m\right\|_\infty dt.$$

This goes 0 in norm as $h \to 0$ because $\mathbf{y} \to \alpha_{\mathbf{y}}(D_j a)\langle x\rangle^m$ is continuous. ∎

The next proposition gives an approximation of identity for $L_p(\mathbb{R}_\theta)$.

Proposition 3.7. *There exists a sequence $f_n \in \mathcal{S}(\mathbb{R}^d)$ independent of θ such that (i) for any $a \in \mathbb{E}_\theta$ and $p = \infty$, and (ii) for any $a \in L_p(\mathbb{R}_\theta)$ and $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|a\lambda_\theta(f_n) - a\|_p = \lim_{n\to\infty} \|\lambda_\theta(f_n)a - a\|_p = 0.$$

Proof. The case $a \in \mathbb{E}_\theta$ and $p = \infty$ is a special case of [38, Proposition 4.13]. Here we provide an argument for all $1 \le p \le \infty$. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a smooth positive function such that ϕ is supported on $|\mathbf{x}| \le 1$ and $\int \phi = (2\pi)^d$. Take $\phi_n = n^d \phi(n\mathbf{x})$ and the inverse Fourier transform $\check{\phi}_n$. We first show that for any $\lambda_\theta(g) \in \mathcal{S}_\theta$,

$$\left\|\lambda_\theta(g)\lambda_\theta(\check{\phi}_n) - \lambda_\theta(g)\right\|_\infty \to 0.$$

Indeed,

$$\begin{aligned}
\lambda_\theta(g)\lambda_\theta(\check{\phi}_n) &= \left(\frac{1}{2\pi^d}\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi})d\boldsymbol{\xi}\right)\left(\frac{1}{2\pi^d}\int_{\mathbb{R}^d} \phi_n(\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\eta})d\boldsymbol{\eta}\right) \\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta})e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}}\lambda_\theta(\boldsymbol{\xi}+\boldsymbol{\eta})d\boldsymbol{\xi}\,d\boldsymbol{\eta} \\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta}-\boldsymbol{\xi})e^{\frac{i}{2}\boldsymbol{\xi}\theta(\boldsymbol{\eta}-\boldsymbol{\xi})}\,d\boldsymbol{\xi}\right)\lambda_\theta(\boldsymbol{\eta})d\boldsymbol{\eta} := \lambda_\theta(g_n),
\end{aligned}$$

where

$$\hat{g}_n(\boldsymbol{\eta}) = \frac{1}{2\pi^d}\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta}-\boldsymbol{\xi})e^{\frac{i}{2}\boldsymbol{\xi}\theta(\boldsymbol{\eta}-\boldsymbol{\xi})}\,d\boldsymbol{\xi}.$$

Then, for some $R > 0$,

$$\begin{aligned}
\|\hat{g} - \hat{g}_n\|_1 &= \int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\eta}) - \frac{1}{2\pi^d}\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta}-\boldsymbol{\xi})e^{\frac{i}{2}\boldsymbol{\xi}\theta(\boldsymbol{\eta}-\boldsymbol{\xi})}\,d\boldsymbol{\xi}\right|d\boldsymbol{\eta} \\
&= \int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\eta}) - \frac{1}{2\pi^d}\int_{\mathbb{R}^d} \hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta}-\boldsymbol{\xi})d\boldsymbol{\xi}\right|d\boldsymbol{\eta} \\
&\quad + \frac{1}{2\pi^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta}-\boldsymbol{\xi})(1-e^{\frac{i}{2}\boldsymbol{\xi}\theta(\boldsymbol{\eta}-\boldsymbol{\xi})})\right|d\boldsymbol{\xi}\,d\boldsymbol{\eta} \\
&\le \left\|\hat{g} - \hat{g} * \left(\frac{1}{2\pi}\phi_n\right)\right\|_1 + \frac{1}{2\pi^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta})(1-e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}})\right|d\boldsymbol{\eta}\,d\boldsymbol{\xi},
\end{aligned}$$

where $\hat{g} * (\frac{1}{2\pi}\phi_n)$ is the convolution and for the second term we used a change of variable $\boldsymbol{\eta} \to \boldsymbol{\eta} + \boldsymbol{\xi}$. It is clear that $\|\hat{g} - \hat{g} * (\frac{1}{2\pi}\phi_n)\|_1 \to 0$ as $n \to \infty$ because $\frac{1}{2\pi}\phi_n$ is an approximation identity. For the second term, for any ε we can find R large enough such that $\int_{|\boldsymbol{\xi}|>R}|\hat{g}(\boldsymbol{\xi})| < \frac{\varepsilon}{3}$ and then n is large enough such that $|1 - e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}}| < \frac{\varepsilon}{3\|\hat{g}\|_1}$ for all $|\boldsymbol{\xi}| < R$ and $|\boldsymbol{\xi}| \le \frac{1}{n}$. Since ϕ_n is supported on $|\boldsymbol{\xi}| \le \frac{1}{n}$,

$$\begin{aligned}
&\iint \left|\hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta})(1-e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}})\right|d\boldsymbol{\eta}\,d\boldsymbol{\xi} \\
&\le \int_{|\boldsymbol{\xi}|>R}\int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta})(1-e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}})\right|d\boldsymbol{\eta}\,d\boldsymbol{\xi} \\
&\quad + \int_{|\boldsymbol{\xi}|<R}\int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\xi})\phi_n(\boldsymbol{\eta})(1-e^{\frac{i}{2}\boldsymbol{\xi}\theta\boldsymbol{\eta}})\right|d\boldsymbol{\eta}\,d\boldsymbol{\xi} \\
&\le \int_{|\boldsymbol{\xi}|>R}\int_{\mathbb{R}^d} 2\left|\hat{g}(\boldsymbol{\xi})\right|\phi_n(\boldsymbol{\eta})d\boldsymbol{\eta}\,d\boldsymbol{\xi} + \frac{\varepsilon}{3\|\hat{g}\|_1}\int_{|\boldsymbol{\xi}|<R}\int_{\mathbb{R}^d}\left|\hat{g}(\boldsymbol{\xi})\right|\phi_n(\boldsymbol{\eta})d\boldsymbol{\eta}\,d\boldsymbol{\xi} \\
&\le (2\pi)^d\left(\frac{2\varepsilon}{3} + \frac{\varepsilon}{3}\right) = (2\pi)^d\,\varepsilon.
\end{aligned}$$

Thus, we obtained

$$\left\|\lambda_\theta(g_n) - \lambda_\theta(g)\right\|_\infty \le \|\hat{g}_n - \hat{g}\|_1 \to 0.$$

For $1 \le p < \infty$, we apply the argument for $\langle x\rangle^d \lambda_\theta(g)$. Note that $\langle x\rangle^{d+1}\lambda_\theta(g) \in \mathcal{S}_\theta$ by Proposition 3.4. Thus, we have

$$\left\|\lambda_\theta(g)\lambda_\theta(f_n) - \lambda_\theta(g)\right\|_p \le \left\|\langle x\rangle^{d+1}\big(\lambda_\theta(g)\lambda_\theta(f_n) - \lambda_\theta(g)\big)\right\|_\infty \left\|\langle x\rangle^{-d-1}\right\|_p \to 0.$$

Given $a \in L_1(\mathbb{R}_\theta)$, we choose $g \in \mathcal{S}_\theta$ so that $\|\lambda_\theta(g) - a\|_1 \le \varepsilon/3$. Note that for all n,

$$\left\|\lambda_\theta(\check{\phi}_n)\right\|_\infty \le \|\phi_n\|_1 = 1.$$

Then, for n large enough,

$$\begin{aligned}
&\left\|a - a\lambda_\theta(\check{\phi}_n)\right\|_1 \\
&\le \left\|a - \lambda_\theta(g)\right\|_1 + \left\|\lambda_\theta(g) - \lambda_\theta(g)\lambda_\theta(\check{\phi}_n)\right\|_1 + \left\|\lambda_\theta(g)\lambda_\theta(\check{\phi}_n) - a\lambda_\theta(\check{\phi}_n)\right\|_1 \\
&\le \left\|a - \lambda_\theta(g)\right\|_1 + \left\|\lambda_\theta(g) - \lambda_\theta(g)\lambda_\theta(\check{\phi}_n)\right\|_1 + \left\|\lambda_\theta(g) - a\right\|_1 \left\|\lambda_\theta(\check{\phi}_n)\right\|_\infty \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{3.5}$$

The argument for ∞-norm and $a \in \mathbb{E}_\theta$ is similar. For $1 < p < \infty$, we use interpolation inequality that

$$\left\|a - a\lambda_\theta(\check{\phi}_n)\right\|_p \le \left\|a - a\lambda_\theta(\check{\phi}_n)\right\|_1^{\frac{1}{p}} \left\|a - a\lambda_\theta(\check{\phi}_n)\right\|_\infty^{1-\frac{1}{p}} \to 0,$$

for any $a \in L_1(\mathbb{R}_\theta) \cap L_\infty(\mathbb{R}_\theta)$. Since $L_1 \cap L_\infty$ is dense in L_p, the argument for general $a \in L_p$ is similar to (3.5). ∎

4. Pseudo-differential calculus for noncommutative derivatives

On $\mathbb{R}^d$ the CCR relation for covariant derivatives corresponds to a constant curvature form. Consider the connection

$$\nabla : \mathbb{C}^\infty(\mathbb{R}^d) \to \Omega^1(\mathbb{R}^d), \quad \nabla f = df + \frac{i}{2}\sum_{j,k} \theta'_{j,k} f \mathbf{x}_j d\mathbf{x}_k \tag{4.1}$$

with curvature form $d\omega = \frac{i}{2}\sum_{j,k} \theta_{jk} d\mathbf{x}_j \wedge d\mathbf{x}_k$. The self-adjoint covariant derivatives $\nabla_j = \nabla_{-\frac{\partial}{\partial_j}}$ satisfy that

$$\nabla_j f = -i\frac{\partial}{\partial \mathbf{x}_j}(f) - \sum_k \frac{1}{2}\theta'_{jk} f \mathbf{x}_k, \quad [\nabla_j, \nabla_k] = -i\theta'_{jk}.$$

The physical meaning behind this is a constant magnetic field perpendicular to the space $\mathbb{R}^d$. In this section, we develop the symbol calculus of ΨDOs of the above structure for a noncommutative $\mathbb{R}_\theta$. Let $\mathbb{R}_\theta$ be the quantum Euclidean space generated by $[x_j, x_k] = -i\theta_{jk}$. We equipped $\mathbb{R}_\theta$ with noncommuting covariant derivatives ξ_j satisfying

$$[\xi_j, x_k] = -i\delta_{jk}, \quad [\xi_j, \xi_k] = -i\theta'_{jk}, \tag{4.2}$$

where δ is the Kronecker delta notation. For $\theta' = 0$, González-Pérez, Junge, and Parcet [19] established the ΨDOs as operators on $L_2(\mathbb{R}_\theta)$ via $\xi_j = D_j$. For general θ and θ', x_j's and ξ_k's satisfying above commutation relations together generate a $2d$-dimensional quantum Euclidean space $\mathbb{R}_\Theta$ with parameter $\Theta = \left[\begin{smallmatrix} \theta & -I \\ I & \theta' \end{smallmatrix}\right]$. In general, x_j's and ξ_k's do not admit a canonical representation on $L_2(\mathbb{R}_\theta)$ because Θ can be singular. Hence we consider the ΨDOs as operators (densely) defined on $L_2(\mathbb{R}_\Theta) \cong L_2(\mathbb{R}_\theta) \otimes_2 L_2(\mathbb{R}_{\theta'})$ affiliated to $\mathbb{R}_\Theta$. Here $\otimes_2$ is the Hilbert space tensor product.

4.1. Abstract symbols

In the classical case for $\mathbb{R}^d$, a standard symbol of order m is a smooth bi-variable function $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that the

$$\left|D_x^\alpha D_\xi^\beta (a)(\mathbf{x}, \boldsymbol{\xi})\right| \le C_{\alpha,\beta}\left(1 + |\boldsymbol{\xi}|^2\right)^{(m-|\beta|)/2} \tag{4.3}$$

(see, e.g., [41, Chapter 6].) In our setting, the symbols are operators affiliated to the von Neumann algebra tensor product $\mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}$. Let us denote $\mathbb{R}_{\theta,\theta'} := \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}$, $\mathcal{M}_{\theta,\theta'}$ for the multiplier algebra of $\mathbb{R}_{\theta,\theta'}$, and $\mathcal{S}_{\theta,\theta'}$ for the Schwartz class. $\mathbb{R}_{\theta,\theta'}$ is a $2d$-dimensional quantum Euclidean space with parameter matrix $\left[\begin{smallmatrix} \theta & 0 \\ 0 & \theta' \end{smallmatrix}\right]$, in which x and ξ variables are mutually commuting; i.e., $[x_j, \xi_k] = 0$ for all j, k. We specify the canonical partial derivatives for x variables by $D_{x_1}, \dots, D_{x_d}$ and for ξ variables by $D_{\xi_1}, \dots, D_{\xi_d}$. That is, for $a \in \mathcal{M}_{\theta,\theta'}$

$$D_{x_j}(a) = [D_j \otimes 1, a], \quad D_{\xi_j}(a) = [1 \otimes D_j, a].$$

We index the transference action by the position $\alpha_{\mathbf{y}} \otimes \alpha_{\boldsymbol{\eta}}(a) = \alpha^1_{\boldsymbol{\eta}} \alpha^2_{\mathbf{y}}(a)$. We use the standard multi-derivative notation that for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$,

$$D_x^\alpha(a) = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_d}^{\alpha_d}(a), \quad D_\xi^\alpha(a) = D_{\xi_1}^{\alpha_1} D_{\xi_2}^{\alpha_2} \cdots D_{\xi_d}^{\alpha_d}(a).$$

Write $\langle \xi \rangle := (1 + \sum_{j=1^d} \xi_j^2)^{\frac{1}{2}}$, where ξ_j's are the noncommuting generators for $\mathbb{R}_{\theta'}$. We start with the abstract reformulation of the definition (4.3).

Definition 4.1. For a real number m, define Σ^m as the set of all operators $a \in \mathcal{M}_{\theta,\theta'}$ such that for all α, β,

$$D_x^\alpha D_\xi^\beta (a) \langle \xi \rangle^{|\beta| - r}$$

extends to be a bounded operator in $\mathbb{R}_{\theta,\theta'}$. We call Σ^m the space of symbols of order m and write $\Sigma^{-\infty} = \bigcap_m \Sigma^m$ and $\Sigma^\infty = \bigcup_m \Sigma^m$.

A priori it is not clear that the above definition satisfies the properties that $\Sigma^m \cdot \Sigma^n = \Sigma^{m+n}$ and $(\Sigma^m)^* = \Sigma^m$. To resolve these questions, we refine the notion of asymptotic degree introduced in Section 3.

Definition 4.2. Given two real numbers s and r, we say that an operator $a \in \mathcal{M}_{\theta,\theta'}$ is of bi-degree (s,r) if for all $s', r' \in \mathbb{R}$

$$\langle x\rangle^{s'}\langle\xi\rangle^{r'}a\langle x\rangle^{-s'-s}\langle\xi\rangle^{-r'-r}$$

extends to a bounded element in $\mathbb{R}_{\theta,\theta'}$. We denote by $O^{s,r}$ the set of all elements of bi-degree (s,r) and write $O^{-\infty,r} = \bigcap_{s\in\mathbb{R}} O^{s,r}$ and $O^{-\infty,-\infty} = \bigcap_{s,r\in\mathbb{R}} O^{s,r}$.

Note that on $\mathbb{R}_{\theta,\theta'}$, $\langle x\rangle$ and $\langle\xi\rangle$ commute so the order of the product $\langle x\rangle^s\langle\xi\rangle^r$ does not matter. It follows that

$$O^{k,l}O^{m,n} \subset O^{k+m,l+n}, \quad \forall k,l,m,n \in \mathbb{R}. \tag{4.4}$$

Indeed, for any $a \in O^{k,l}$, $b \in O^{m,n}$, and $s, r \in \mathbb{R}$,

$$\begin{aligned}&\langle x\rangle^s\langle\xi\rangle^r ab\langle x\rangle^{-k-m-s}\langle\xi\rangle^{-l-n-r}\\ &\quad= \big(\langle x\rangle^s\langle\xi\rangle^r a\langle x\rangle^{-k-s}\langle\xi\rangle^{-l-r}\big)\big(\langle x\rangle^{k+s}\langle\xi\rangle^{l+r} b\langle x\rangle^{-k-m-s}\langle\xi\rangle^{-l-n-r}\big)\end{aligned}$$

is bounded. The "bi-degree" gives an alternative characterization of the abstract symbol classes.

Theorem 4.3. *Let m be a real number and $a \in \mathcal{M}_{\theta,\theta'}$. Then $a \in \Sigma^m$ if and only if for all α, β,*

$$D_x^\alpha D_\xi^\beta(a) \in O^{0,m-|\beta|}.$$

Proof. The sufficiency is clear by the definition. Let $a \in \Sigma^m$. It follows from Lemma 3.4 that for all α, β, $D_x^\alpha D_\xi^\beta(a)$ is of degree 0 for x and degree $m - |\beta|$ for ξ. Because $\langle x\rangle$ and $\langle\xi\rangle$ commute, we have that $D_x^\alpha D_\xi^\beta(a) \in O^{0,m-|\beta|}$. ∎

Recall that we write the transference action on $\mathbb{R}_\theta$ as $\alpha = \alpha^1 \otimes \alpha^2$, so that for $a \in \Sigma^m$, the notation $\alpha^1(a)$ is the function on $\mathbb{R}^d$ given by the restriction of $\alpha(a)$ – which is a function on $\mathbb{R}^d$ – to the first d-coordinates and $\alpha^2(a)$ is the restriction of $\alpha(a)$ to the final d coordinates.

Proposition 4.4. *Σ^m equipped with the seminorms $\|\cdot\|_{\alpha,\beta} := \|D_x^\alpha D_\xi^\beta(\cdot)\langle\xi\rangle^{|\beta|-m}\|$ is a Fréchet space. In particular, for $a \in \Sigma^m$, $D_{x_j}(a)$ and $D_{\xi_j}(a)$ are the vector derivatives*

$$D_{x_j}(a) = i\lim_{h\to 0}\frac{1}{h}\big(\alpha^1_{h\mathbf{e}_j}(a) - a\big),$$
$$D_{\xi_j}(a) = i\lim_{h\to 0}\frac{1}{h}\big(\alpha^2_{h\mathbf{e}_j}(a) - a\big),$$

where the limits converge in the Fréchet topology of Σ^m.

Proof. Let $\{a_n\}_{n=0}^\infty \subset \Sigma^m$ be a Cauchy sequence in Σ^m with respect to all the seminorms $\|\cdot\|_{\alpha,\beta}$. In particular, a_n is Cauchy in the norm of $\mathbb{R}_{\theta,\theta'}$ and hence there exists $b_{\alpha,\beta} \in \mathbb{R}_{\theta,\theta'}$ such that

$$\big\|D_x^\alpha D_\xi^\beta(a_n)\langle\xi\rangle^{|\beta|-m} - b_{\alpha,\beta}\big\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Denote $c_{\alpha,\beta} = b_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}$ and $c_{0,0} = b_{0,0}\langle\xi\rangle^{m}$. Let $\lambda_{\theta,\theta'}(f) \in \mathcal{S}_{\theta,\theta'}$, then

$$\begin{aligned}\big\langle c_{\alpha,\beta}, \langle\xi\rangle^{|\beta|-m}\lambda_{\theta,\theta'}(f)\big\rangle &= \big\langle b_{\alpha,\beta}\langle\xi\rangle^{|\beta|-m}, \lambda_{\theta,\theta'}(f)\big\rangle = \big\langle b_{\alpha,\beta}, \lambda_{\theta,\theta'}(f)\big\rangle \\ &= \lim_{n\to\infty}\big\langle D_x^\alpha D_\xi^\beta(a_n)\langle\xi\rangle^{|\beta|-m}, \lambda_{\theta,\theta'}(f)\big\rangle \\ &= \lim_{n\to\infty}\big\langle a_n\langle\xi\rangle^{-m}, \langle\xi\rangle^{m} D_x^\alpha D_\xi^\beta\big(\langle\xi\rangle^{|\beta|-m}\lambda_{\theta,\theta'}(f)\big)\big\rangle \\ &= \big\langle b_{0,0}, \langle\xi\rangle^{m} D_x^\alpha D_\xi^\beta\big(\langle\xi\rangle^{|\beta|-m}\lambda_{\theta,\theta'}(f)\big)\big\rangle \\ &= \big\langle D_x^\alpha D_\xi^\beta(c_{0,0}), \langle\xi\rangle^{|\beta|-m}\lambda_{\theta,\theta'}(f)\big\rangle.\end{aligned}$$

Because we have an identity of sets $\langle\xi\rangle^{|\beta|-m}\mathcal{S}_{\theta,\theta'} = \mathcal{S}_{\theta,\theta'}$ by Proposition 3.4, we have $D_x^\alpha D_\xi^\beta(c_{0,0}) = c_{\alpha,\beta}$ weakly. To see that $c_{0,0}$ is again in the multiplier algebra $\mathcal{M}_{\theta,\theta'}$, it suffices to show that for any $\lambda_{\theta,\theta'}(f) \in \mathcal{S}_{\theta,\theta'}$,

$$\Big\| D_x^\alpha D_\xi^\beta\big(c_{0,0}\lambda_{\theta,\theta'}(f)\big)\Big(1 + \sum_j x_j^2 + \xi_j^2\Big)^\gamma \Big\|$$

is bounded for any α, β, γ. Using the Leibniz rule,

$$\begin{aligned}&D_x^\alpha D_\xi^\beta\big(c_{0,0}\lambda_{\theta,\theta'}(f)\big) \\ &\quad= \sum_{\alpha_1+\alpha_2=\alpha}\ \sum_{\beta_1+\beta_2=\beta}\binom{\alpha}{\alpha_1,\alpha_2}\binom{\beta}{\beta_1,\beta_2} D_x^{\alpha_1}D_\xi^{\beta_1}(c_{0,0}) D_x^{\alpha_2}D_\xi^{\beta_2}\big(\lambda_{\theta',\theta}(f)\big) \\ &\quad= \sum_{\alpha_1+\alpha_2=\alpha}\ \sum_{\beta_1+\beta_2=\beta}\binom{\alpha}{\alpha_1,\alpha_2}\binom{\beta}{\beta_1,\beta_2} c_{\alpha_1,\beta_1}\lambda_{\theta',\theta}\big(D_x^{\alpha_2}D_\xi^{\beta_2}(f)\big).\end{aligned}$$

Note that for each α_1, β_1, $c_{\alpha_1,\beta_1} = b_{\alpha_1,\beta_1}\langle\xi\rangle^{m-|\beta|}$ and b_{α_1,β_1} are bounded. Then, for any γ,

$$\begin{aligned}&c_{\alpha_1,\beta_1}\lambda_{\theta',\theta}\big(D_x^{\alpha_2}D_\xi^{\beta_2}(f)\big)\Big(1 + \sum_j x_j^2 + \xi_j^2\Big)^\gamma \\ &\qquad= b_{\alpha_1,\beta_1}\langle\xi\rangle^{m-|\beta|}\lambda_{\theta',\theta}\big(D_x^{\alpha_2}D_\xi^{\beta_2}(f)\big)\Big(1 + \sum_j x_j^2 + \xi_j^2\Big)^\gamma\end{aligned}$$

is bounded since $\lambda_{\theta,\theta'}(D_x^{\alpha_2}D_\xi^{\beta_2} f)$ are in $O^{-\infty,-\infty}$. By again Proposition 3.4, this implies that $c_{0,0}\lambda_{\theta,\theta'}(f) \in S_{\theta,\theta'}$ and $c_{0,0} \in \mathcal{M}_{\theta,\theta'}$ is a multiplier. The convergence of the vector derivatives is a consequence of applying Proposition 3.6 to $\mathbb{R}_{\theta,\theta'}$. ■

Corollary 4.5. *For all multi-indices α and real numbers m, n,*

(i) $\xi^\alpha \in \Sigma^{|\alpha|}$, $\langle\xi\rangle^m \in \Sigma^m$;

(ii) *if $a \in \Sigma^m$, then $a^* \in \Sigma^m$;*

(iii) *if $a \in \Sigma^m, b \in \Sigma^n$, then $ab \in \Sigma^{m+n}$.*

Proof. (i) is a direct consequence of Theorem 3.2. (ii) follows from the fact that

$$D_x^\alpha D_\xi^\beta(a^*) = (-1)^{|\alpha|+|\beta|}\big(D_x^\alpha D_\xi^\beta(a)\big)^*.$$

For (iii), by the Leibniz rule

$$D_x^\alpha D_\xi^\beta(ab) = \sum_{\alpha_1+\alpha_2=\alpha,\beta_1+\beta_2=\beta} \binom{\alpha}{\alpha_1,\alpha_2}\binom{\beta}{\beta_1,\beta_2} D_x^{\alpha_1} D_\xi^{\beta_1}(a) D_x^{\alpha_2} D_\xi^{\beta_2}(b). \quad (4.5)$$

Using Theorem 4.3,

$$D_x^{\alpha_1} D_\xi^{\beta_1}(a) \in O^{0,m-|\beta_1|}, \quad D_x^{\alpha_2} D_\xi^{\beta_2}(b) \in O^{0,n-|\beta_2|}.$$

Using the property (4.4), all summands in (4.5) belong to $O^{0,m+n-|\beta_1|-|\beta_2|} = O^{0,m+n-|\beta|}$. Again by Theorem 4.3, $ab \in \Sigma^{n+m}$. ■

4.2. Comultiplications

One key tool that will be used in the proof of our symbol calculus is the comultiplication maps of $\mathbb{R}_\theta$ and $\mathbb{R}_{\theta,\theta'}$. The comultiplication map of $\mathbb{R}^d$ as an abelian group is

$$\sigma : L_\infty(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d \times \mathbb{R}^d) \cong L_\infty(\mathbb{R}^d) \mathbin{\overline{\otimes}} L_\infty(\mathbb{R}^d), \quad \sigma(f)(\mathbf{x},\mathbf{y}) = f(\mathbf{x}+\mathbf{y}).$$

Algebraically, $\sigma(u(\boldsymbol{\xi})) = u(\boldsymbol{\xi}) \otimes u(\boldsymbol{\xi})$, where $u(\boldsymbol{\xi})$ is the unitary function $u(\boldsymbol{\xi})(\mathbf{x}) = e^{i\boldsymbol{\xi}\cdot\mathbf{x}}$. For $\mathbb{R}_\theta$, we consider the deformed comultiplication map

$$\sigma_\theta : \mathbb{R}_\theta \to L_\infty(\mathbb{R}^n) \mathbin{\overline{\otimes}} \mathbb{R}_\theta, \quad \sigma_\theta\big(\lambda_\theta(\boldsymbol{\xi})\big) = u(\boldsymbol{\xi}) \otimes \lambda_\theta(\boldsymbol{\xi}),$$

where $\overline{\otimes}$ is the von Neumann algebra tensor product. $L_\infty(\mathbb{R}^n) \mathbin{\overline{\otimes}} \mathbb{R}_\theta$ can be identified with $\mathbb{R}_\theta$-valued functions $L_\infty(\mathbb{R}^d, \mathbb{R}_\theta)$, and at a point $\mathbf{x} \in \mathbb{R}^d$,

$$\sigma_\theta\big(\lambda_\theta(\boldsymbol{\xi})\big)(\mathbf{x}) = e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\lambda_\theta(\boldsymbol{\xi}) = \alpha_{\mathbf{x}}\big(\lambda_\theta(\boldsymbol{\xi})\big).$$

The same co-representation map is used in [19, Corollary 1.4] in the study of ΨDOs of $\mathbb{R}_\theta$ with commuting derivatives.

Proposition 4.6. *The map*

$$\sigma_\theta : \mathcal{S}_\theta \to L_\infty(\mathbb{R}^d, \mathbb{R}_\theta), \quad \sigma_\theta\big(\lambda_\theta(f)\big)(\mathbf{x}) = \alpha_{\mathbf{x}}\big(\lambda_\theta(f)\big)$$

(i) *extends to an injective normal $*$-homomorphism from $\mathbb{R}_\theta$ to $L_\infty(\mathbb{R}^d, \mathbb{R}_\theta)$;*

(ii) *extends to an injective algebraic $*$-homomorphism from $\mathcal{M}_\theta$ to $L_\infty(\mathbb{R}^d, \mathcal{M}_\theta)$. Moreover, for all $a \in \mathcal{M}_\theta$, $\sigma_\theta(D_j a) = D_{x_j}(\sigma_\theta(a)) = D_{\mathbf{x}_j}(\sigma_\theta(a))$;*

(iii) *extends to a complete isometry V_θ right from $L_2(\mathbb{R}_\theta)^c$ to $L_2^c(\mathbb{R}^d) \otimes_{wh} \mathbb{R}_\theta$. Here $\otimes_{wh}$ denotes the W^*-Haagerup tensor product (see [3]) and $L_2^c(\mathbb{R}^d)$ is the column space.*

Proof. (i) follows from the fact that at each point $\mathbf{x} \in \mathbb{R}^d$, $\alpha_{\mathbf{x}}$ is a $*$-automorphism of $\mathbb{R}_\theta$. The normality was proved in [19, Corollary 1.4]. (ii) is similar to (i). For the derivatives, let $D_{\mathbf{x}_j}$ denote the jth partial derivatives for $\mathbb{R}_d$ and let D_{x_j} denote the partial derivatives on $\mathbb{R}_\theta$. For all $\mathbf{x} \in \mathbb{R}^d$ and $a \in \mathcal{M}_\theta$,

$$D_{\mathbf{x}_j}\big(\sigma_\theta(a)\big)(\mathbf{x}) = \lim_{h\to 0} -\frac{i}{h}\big(\alpha_{\mathbf{x}+h\mathbf{e}_j}(a) - \alpha_{\mathbf{x}}(a)\big) = D_{x_j}\big(\alpha_{\mathbf{x}}(a)\big) = \alpha_{\mathbf{x}}(D_{x_j}a).$$

For (iii), let $b = \sum_k b_k \lambda_\theta(f_k)$ with $b_k \in \mathbb{C}$ and $\lambda_\theta(f_k)$ being an orthonormal set in $L_2(\mathbb{R}_\theta)$. Then $\|b\|^2_{L_2(\mathbb{R}_\theta)} = \sum_k |b_k|^2$. The norm of $L_2^c(\mathbb{R}^d) \otimes_{wh} \mathbb{R}_\theta$ is given by the $\mathbb{R}_\theta$-valued inner product that for $f, g \in L_2(\mathbb{R}^d)$ and $a, c \in \mathbb{R}_\theta$

$$\langle f \otimes a, g \otimes c\rangle_{\mathbb{R}_\theta} = \langle f, g\rangle_{L_2(\mathbb{R}^d)} a^* c, \quad \|B\|_{L_2^c(\mathbb{R}^d)\otimes_{wh}\mathbb{R}_\theta} = \big\|\langle B, B\rangle_{\mathbb{R}_\theta}\big\|_{\mathbb{R}_\theta}.$$

Note that on the Fourier transform side,

$$V_\theta\big(\lambda_\theta(f)\big)(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi}).$$

Therefore,

$$\begin{aligned}\Big\|V_\theta\Big(\sum_k b_k\lambda_\theta(f_k)\Big)\Big\|_{L_2^c(\mathbb{R}^d)\otimes_{wh}\mathbb{R}_\theta} &= \Big\|\sum_{k,k'} b_k\bar{b}_{k'} \int \hat{f}_k(\boldsymbol{\xi})\bar{\hat{f}}_{k'}(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi})\lambda_\theta(\boldsymbol{\xi})^*\, d\boldsymbol{\xi}\Big\|_{\mathbb{R}_\theta} \\ &= \Big\|\Big(\sum_k |b_k|^2\Big)\mathbf{1}\Big\|_{\mathbb{R}_\theta} = \sum_k |b_k|^2.\end{aligned}$$

Replacing $b_k \in \mathbb{C}$ with matrices $b_k \in M_n$ in the above argument gives the complete isometry. ∎

Let us write $\lambda_{\theta,\theta'}(\boldsymbol{\eta}, \mathbf{y}) := \lambda_\theta(\boldsymbol{\eta}) \otimes \lambda_{\theta'}(\mathbf{y})$ for the generators of $\mathbb{R}_{\theta,\theta'} := \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}$. The quantization map for $\mathbb{R}_{\theta,\theta'}$ is

$$\lambda_{\theta,\theta'}(F) = (2\pi)^{-2d}\int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta}, \mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta}, \mathbf{y})d\eta\, d\mathbf{y},$$

where

$$\widehat{F}(\boldsymbol{\eta}, \mathbf{y}) = \int_{\mathbb{R}^{2d}} F(\mathbf{x}, \boldsymbol{\xi})e^{-i(\mathbf{x}\boldsymbol{\eta}+\boldsymbol{\xi}\mathbf{y})}d\mathbf{x}\, d\boldsymbol{\xi}$$

is the Fourier transform. By Proposition 4.6, we can dilate the symbols affiliated to $\mathbb{R}_{\theta,\theta'}$ to operator-valued symbols,

$$\begin{gathered}\sigma_\theta \otimes \sigma_{\theta'} : \mathbb{R}_{\theta,\theta'} \to L_\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}), \\ (\sigma_\theta \otimes \sigma'_\theta)\lambda_{\theta,\theta'}(F)(\mathbf{x}, \mathbf{y}) = \alpha^1_{\mathbf{x}}\alpha^2_{\mathbf{y}}\big(\lambda_{\theta,\theta'}(F)\big),\end{gathered}$$

where α^1 (resp. α^2) is the transference action on $\mathbb{R}_\theta$ (resp. $\mathbb{R}_{\theta'}$). For the ΨDOs, we consider the comultiplication maps for $\mathbb{R}_\Theta$ with $\Theta = \left[\begin{smallmatrix}\theta & -I_n \\ I_n & \theta'\end{smallmatrix}\right]$. Note that $\mathbb{R}_\theta$ and $\mathbb{R}_{\theta'}$ embed into $\mathbb{R}_\Theta$ as the subalgebras generated respectively by

$$\mathbb{R}_\theta \cong \operatorname{span}\big\{\lambda_\Theta(\boldsymbol{\eta}, \mathbf{0})\big\}'', \quad \mathbb{R}_{\theta'} \cong \operatorname{span}\big\{\lambda_\Theta(\mathbf{0}, \mathbf{y})\big\}'',$$

where $''$ denotes the double commutant, or equivalently the closure in the weak operator topology. For the ease of notation, we identify $\mathbb{R}_\theta, \mathbb{R}'_\theta$ with their embedding in $\mathbb{R}_\Theta$ and write

$$\lambda_\theta(\boldsymbol{\eta}) := \lambda_\Theta(\boldsymbol{\eta}, \mathbf{0}), \quad \lambda_{\theta'}(\mathbf{y}) := \lambda_\Theta(\mathbf{0}, \mathbf{y}).$$

In $\mathbb{R}_\Theta$, these two families of unitary generators satisfy the commutation relation

$$\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y}) = e^{i\boldsymbol{\eta}\mathbf{y}}\lambda_{\theta'}(\mathbf{y})\lambda_\theta(\boldsymbol{\eta}).$$

We define the following quantization for $\mathbb{R}_\Theta$:

$$\lambda_\Theta(F) = (2\pi)^{-2d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{F}(\boldsymbol{\eta}, \mathbf{y})\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y}) d\eta\, d\mathbf{y}, \quad F \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d).$$

We have the Hilbert space isometry between two quantizations:

$$W : L_2(\mathbb{R}_\Theta) \to L_2(\mathbb{R}_{\theta,\theta'}), \quad W|\lambda_\Theta(F)\rangle = |\lambda_{\theta,\theta'}(F)\rangle.$$

Here and in the following, we will use the "ket" notation $|\cdot\rangle$ to emphasize membership to the corresponding L_2-space.

Proposition 4.7. *Define the unitary*

$$u_\theta(\mathbf{y}) : L_2(\mathbb{R}_\theta) \to L_2(\mathbb{R}_\theta), \quad v_\theta(\mathbf{y})|\lambda_\theta(f)\rangle = \big|\lambda_\theta(\alpha_y f)\big\rangle.$$

The map

$$\begin{aligned} &\sigma_\Theta : \mathcal{S}_\Theta \to B\big(L_2(\mathbb{R}_\theta)\big) \mathbin{\overline{\otimes}} \mathbb{R}_{\theta'} \\ &\lambda_\Theta(F) \mapsto (2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta}, \mathbf{y})\lambda_\theta(\boldsymbol{\eta}) v_\theta(\mathbf{y}) \otimes \lambda_{\theta'}(\mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y} \end{aligned}$$

(i) *satisfies* $\sigma_\Theta(\lambda_\Theta(F)) = W\lambda_\Theta(F)W^*$, *by viewing*

$$\mathcal{S}_\Theta \subset B\big(L_2(\mathbb{R}_\Theta)\big), \quad B\big(L_2(\mathbb{R}_\theta)\big) \mathbin{\overline{\otimes}} \mathbb{R}_{\theta'} \subset B\big(L_2(\mathbb{R}_\theta) \otimes_2 L_2(\mathbb{R}_{\theta'})\big);$$

(ii) *extends to an injective normal $*$-homomorphism from $\mathbb{R}_\Theta$ to $B(L_2(\mathbb{R}_\theta)) \mathbin{\overline{\otimes}} \mathbb{R}_{\theta'}$.*

Proof. By linearity, it suffices to verify that

$$W\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)W^* = \lambda_\theta(\boldsymbol{\eta}_0)v_\theta(\mathbf{y}_0) \otimes \lambda_{\theta'}(\mathbf{y}_0).$$

Indeed, for $\lambda_{\theta,\theta'}(G) \in \mathcal{S}_{\theta,\theta'}$,

$$W\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)W^*\big|\lambda_{\theta,\theta'}(G)\big\rangle = W\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)\big|\lambda_\Theta(G)\big\rangle = W\big|\lambda_\Theta(G_1)\big\rangle,$$

where

$$\begin{aligned} \lambda_\Theta(G_1) &= \int_{\mathbb{R}^{2d}} \widehat{G}(\boldsymbol{\eta}, \mathbf{y})\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y}) d\mathbf{y}\, d\boldsymbol{\eta} \\ &= \int_{\mathbb{R}^{2d}} \widehat{G}(\boldsymbol{\eta} - \boldsymbol{\eta}_0, \mathbf{y} - \mathbf{y}_0) e^{i\boldsymbol{\eta}\mathbf{y}_0} e^{\frac{i}{2}(\boldsymbol{\eta}\theta\boldsymbol{\eta}_0 + \mathbf{y}\theta'\mathbf{y}_0)} \lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y}) d\mathbf{y}\, d\boldsymbol{\eta}. \end{aligned}$$

Then

$$W|\lambda_\Theta(G_1)\rangle = \big|\lambda_{\theta,\theta'}(G_1)\big\rangle = \big(\lambda_\theta(\boldsymbol{\eta}_0)v_\theta(\mathbf{y}_0) \otimes \lambda_{\theta'}(\mathbf{y}_0)\big)\big|\lambda_{\theta,\theta'}(G)\big\rangle. \qquad \blacksquare$$

Now let us consider the Gelfand–Naimark–Segal (GNS) construction of $B(L_2(\mathbb{R}_\theta))$ with respect to its standard trace. Define for a Schwartz function F on $\mathbb{R}^{2d}$ the operator

$$T_F = (2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) \lambda_\theta(\boldsymbol{\eta}) v_\theta(\mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y}.$$

For $|\lambda_\theta(f)\rangle \in L_2(\mathbb{R}_\theta)$,

$$T_F \big|\lambda_\theta(f)\big\rangle = (2\pi)^{-2d} \int \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) \lambda_\theta(\boldsymbol{\eta}) v_\theta(\mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y} \big|\lambda_\theta(f)\big\rangle =: \big|\lambda_\theta(g)\big\rangle,$$

where T_F has the kernel representation

$$\widehat{g}(\boldsymbol{\eta}) = (2\pi)^{-2d} \int \widehat{F}(\boldsymbol{\eta} - \boldsymbol{\xi}, \mathbf{y}) e^{i\mathbf{y}\boldsymbol{\eta}} e^{\frac{i}{2}\boldsymbol{\eta}\theta\boldsymbol{\xi}}\, d\mathbf{y} \hat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

Since $F \in \mathcal{S}(\mathbb{R}^{2d})$, T_F is trace class and

$$\operatorname{tr}(T_F) = (2\pi)^{-2d} \int \widehat{F}(0, \mathbf{y}) e^{i\mathbf{y}\boldsymbol{\eta}}\, d\mathbf{y}\, d\boldsymbol{\eta} = (2\pi)^{-d} \int F.$$

One calculates that

$$T_F^* T_F = (2\pi)^{-4d} \int_{\mathbb{R}^{2d}} \bigg(\int_{\mathbb{R}^{2d}} \overline{\widehat{F}}(\boldsymbol{\eta}_1, \mathbf{y}_1) \widehat{F}(\boldsymbol{\eta} + \boldsymbol{\eta}_1, \mathbf{y} + \mathbf{y}_1) e^{-\frac{i}{2}\boldsymbol{\eta}\theta\boldsymbol{\eta}_1} \times e^{-i\boldsymbol{\eta}_1\mathbf{y}}\, d\boldsymbol{\eta}_1\, d\mathbf{y}_1 \bigg) \lambda_\theta(\boldsymbol{\eta}) v_\theta(\mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y}.$$

Hence

$$\operatorname{tr}(T_F^* T_F) = (2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \overline{\widehat{F}}(\boldsymbol{\eta}_1, \mathbf{y}_1) \widehat{F}(\boldsymbol{\eta}_1, \mathbf{y}_1) d\boldsymbol{\eta}_1\, d\mathbf{y}_1 = (2\pi)^{-2d} \|F\|_2^2.$$

Up to a scalar we have a Hilbert space isometry

$$V : L_2\big(B\big(L_2(\mathbb{R}_\theta)\big), \operatorname{tr}\big) \to L_2\big(\mathbb{R}^d, L_2(\mathbb{R}_\theta)\big), \quad V(T_F)(\mathbf{x}) = \lambda_\theta\big(F(\mathbf{x}, \cdot)\big).$$

Write $\widetilde{\pi}$ as the GNS construction of $B(L_2(\mathbb{R}_\theta))$ on $L_2(B(L_2(\mathbb{R}_\theta)), \operatorname{tr})$. Then $\pi(\cdot) = V\widetilde{\pi}(\cdot)V^*$ gives a normal faithful $*$-homomorphism form $B(L_2(\mathbb{R}_\theta))$ to $B(L_2(\mathbb{R}^d)) \overline{\otimes} \mathbb{R}_\theta$ as follows:

$$\begin{aligned} \pi(T_F) &:= V\widetilde{\pi}(T_F)V^* \\ &= (2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) v(\boldsymbol{\eta}) u(\mathbf{y}) \otimes \lambda_\theta(\boldsymbol{\eta}) d\boldsymbol{\eta}\, d\mathbf{y} \in B\big(L_2(\mathbb{R}^d)\big) \overline{\otimes} \mathbb{R}_\theta, \end{aligned}$$

where $v(\boldsymbol{\eta})$ is translation unitary on $L_2(\mathbb{R}^d)$. Combining π with the co-representation σ_Θ, we obtain another representation of $\mathbb{R}_\Theta$.

Proposition 4.8. *The map*

$$\begin{aligned} \widetilde{\sigma}_\Theta : \mathcal{S}_\Theta &\to B\big(L_2(\mathbb{R}^d)\big) \overline{\otimes} \mathbb{R}_{\theta,\theta'} \\ \lambda_\Theta(F) &\mapsto (2\pi)^{-2d} \int \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) \big(u(\boldsymbol{\eta}) v(\mathbf{y}) \otimes \lambda_{\theta,\theta'}(\boldsymbol{\eta}, \mathbf{y})\big) d\boldsymbol{\eta}\, d\mathbf{y} \end{aligned}$$

(i) *extends to a normal injective $*$-homomorphism from $\mathbb{R}_\Theta$ to $B(L_2(\mathbb{R}^d)) \overline{\otimes} \mathbb{R}_{\theta,\theta'}$;*

(ii) *satisfies the intertwining relation $(V_\theta \otimes \mathrm{id}_{\mathbb{R}_{\theta'}})\widetilde{\sigma}_\Theta(\cdot) = \sigma_\Theta(\cdot)(V_\theta \otimes \mathrm{id}_{\mathbb{R}_{\theta'}})$ for the isometry*

$$V_\theta \otimes \mathrm{id}_{\mathbb{R}_{\theta'}} : L_2^c(\mathbb{R}_\theta) \otimes_{wh} \mathbb{R}_{\theta'} \to L_2^c(\mathbb{R}^d) \otimes_{wh} (\mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'}).$$

Proof. (i) We verify that $\widetilde{\sigma}_\Theta = (\pi \otimes \mathrm{id}_{\mathbb{R}_{\theta'}}) \circ \sigma_\Theta$. Indeed,

$$\begin{aligned}
&(\pi \otimes \mathrm{id}_{\mathbb{R}_{\theta'}}) \circ \sigma_\Theta(\lambda_\Theta(F)) \\
&\quad = \pi \otimes \mathrm{id}_{\mathbb{R}_{\theta'}} \Big((2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) \lambda_\theta(\boldsymbol{\eta}) v_\theta(\mathbf{y}) \otimes \lambda_{\theta'}(\mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y} \Big) \\
&\quad = (2\pi)^{-2d} \int \widehat{F}(\boldsymbol{\eta}, \mathbf{y}) \big(u(\boldsymbol{\eta}) v(\mathbf{y}) \otimes \lambda_\theta(\boldsymbol{\eta}) \otimes \lambda_{\theta'}(\mathbf{y}) \big) d\boldsymbol{\eta}\, d\mathbf{y} = \widetilde{\sigma}_\Theta\big(\lambda_\Theta(F)\big).
\end{aligned}$$

For (ii), recall that $B(L_2(\mathbb{R}_\theta)) \overline{\otimes} \mathbb{R}_{\theta'}$ is canonically isomorphic to the adjointable $\mathbb{R}'_\theta$-module map $\mathcal{L}(L_2^c(\mathbb{R}_\theta) \otimes_{wh} \mathbb{R}_{\theta'})$ and similarly

$$B\big(L_2(\mathbb{R}^d)\big) \overline{\otimes} \mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'} \cong \mathcal{L}\big(L_2^c(\mathbb{R}_\theta) \otimes_{wh} \mathbb{R}_{\theta,\theta'}\big)$$

as an $\mathbb{R}_{\theta,\theta'}$-module map (see [27]). The complete isometry V_θ in Proposition 4.6 gives an isometry

$$V_\theta \otimes \mathrm{id}_{\theta'} : L_2^c(\mathbb{R}_\theta) \otimes_{wh} \mathbb{R}_{\theta'} \to L_2^c(\mathbb{R}^d) \otimes_{wh} (\mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'}).$$

We verify the intertwining relation $(V_\theta \otimes \mathrm{id})\sigma_\Theta(\cdot) = \widetilde{\sigma}_\Theta(\cdot)(V_\theta \otimes \mathrm{id})$. For any $\lambda_\Theta(F) \in \mathcal{S}_\Theta$ and $\lambda_{\theta,\theta'}(G) \in \mathcal{S}_{\theta,\theta'}$, we have $\sigma_\Theta(\lambda_\Theta(F))|\lambda_{\theta,\theta'}(G)\rangle = |\lambda_{\theta,\theta'}(G_1)\rangle$, where

$$\widehat{G}_1(\boldsymbol{\eta}, \mathbf{y}) = (2\pi)^{-2d} \int \widehat{F}(\boldsymbol{\eta} - \boldsymbol{\eta}_1, \mathbf{y} - \mathbf{y}_1) \widehat{G}(\boldsymbol{\eta}_1, \mathbf{y}_1) e^{i\boldsymbol{\eta}_1(\mathbf{y}-\mathbf{y}_1)} e^{\frac{i}{2}\boldsymbol{\eta}\theta\boldsymbol{\eta}_1} e^{\frac{i}{2}\mathbf{y}\theta\mathbf{y}_1}\, d\boldsymbol{\eta}_1\, d\mathbf{y}_1.$$

On the other hand, one verifies that

$$\begin{aligned}
&\widetilde{\sigma}_\Theta \otimes \mathrm{id}\, \big(\lambda_\Theta(F)\big) V_\theta \big|\lambda_{\theta,\theta'}(G)\big\rangle \\
&\quad = \Big| \int \widehat{G}_1(\boldsymbol{\eta}, \mathbf{y}) u(\boldsymbol{\eta}) \otimes \lambda_{\theta,\theta'}(\boldsymbol{\eta}, \mathbf{y}) d\boldsymbol{\eta}\, d\mathbf{y} \Big\rangle = V_\theta \otimes \mathrm{id}\, \big(\sigma_\Theta(\lambda_\Theta(F)) \big|\lambda_{\theta,\theta'}(G)\big\rangle\big).
\end{aligned}$$

We see that the representation $(V_\theta \otimes \mathrm{id})^* \sigma_\Theta(\cdot)(V_\theta \otimes \mathrm{id})$ is a restriction of $\widetilde{\sigma}_\Theta$. ∎

4.3. Pseudo-differential operator calculus

Recall that on $\mathbb{R}^d$ the pseudo-differential operator of a symbol $a(\mathbf{x}, \boldsymbol{\xi})$ is given by the singular integral form

$$\mathrm{op}_0(a)(f)(\mathbf{x}) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{4.6}$$

In [19], the ΨDOs on $\mathbb{R}_\theta$ are defined as those operators of the form

$$\mathrm{op}_\theta(a)\big(\lambda_\theta(f)\big) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(\boldsymbol{\xi}) \lambda_\theta(\boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{4.7}$$

where $a : \mathbb{R}^d \to \mathbb{R}_\theta$ is the symbol as an $\mathbb{R}_\theta$-valued function. The ΨDOs in our setting are operators densely defined on $L_2(\mathbb{R}_{\theta,\theta'}) \cong L_2(\mathbb{R}_\theta) \otimes_2 L_2(\mathbb{R}_{\theta'})$. The main idea to define the operator map "Op" is that for a symbol $a_1 \otimes a_2$ with $a_1 \in \mathbb{R}_\theta, a_2 \in \mathbb{R}_{\theta'}$,

$$\mathrm{Op}(a_1 \otimes a_2) = \sigma_\Theta(a_1 a_2) \in B\big(L_2(\mathbb{R}_{\theta,\theta'})\big), \tag{4.8}$$

where $a_1 a_2$ is the product in $\mathbb{R}_\Theta$ by viewing $\mathbb{R}_\theta, \mathbb{R}'_\theta \subset \mathbb{R}_\Theta$ as subalgebras, and σ_Θ is the $*$-representation of $\mathbb{R}_\Theta$ on $L_2(\mathbb{R}_{\theta,\theta'})$ defined in Proposition 4.7. The definition for general symbol $a \in \Sigma^m$ is given as follows.

Definition 4.9. For a symbol $a \in \Sigma^m$, we define the operator $\mathrm{Op}(a) : \mathcal{S}_{\theta,\theta'} \to \mathcal{S}_{\theta,\theta'}$ as follows:

$$\mathrm{Op}(a)\lambda_{\theta,\theta'}(F) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\, d\mathbf{y}.$$

An operator of this form will be called a pseudo-differential operator (ΨDO). We denote by op^m the set of all ΨDOs of order at most m.

We justify the above definition and verify the property (4.8).

Proposition 4.10. *For a symbol $a \in \Sigma^m$,* Op(a) *is a continuous map from $\mathcal{S}_{\theta,\theta'}$ to $\mathcal{S}_{\theta,\theta'}$ and* Op(a) *is an operator affiliated to $\sigma_\Theta(\mathbb{R}_\Theta) \subset B(L_2(\mathbb{R}_{\theta,\theta'}))$. In particular, if $a_1 \in \mathbb{R}_\theta$ and $a_2 \in \mathbb{R}_{\theta'}$,* $\mathrm{Op}(a_1 \otimes a_2) = \sigma_\Theta(a_1 a_2)$.

Proof. In the calculation below, the normalization constant $(2\pi)^{-d}$ will be omitted. Recall from Proposition 4.7 that

$$W : L_2(\mathbb{R}_\Theta) \to L_2(\mathbb{R}_{\theta,\theta'}), \quad W\big|\lambda_\Theta(F)\big\rangle = \big|\lambda_{\theta,\theta'}(F)\big\rangle$$

is the isometry such that $W^*\sigma_\Theta(\cdot)W$ is the left regular representation of $\mathbb{R}_\Theta$ on $L_2(\mathbb{R}_\Theta)$. To verify that Op(a) is affiliated to $\sigma_\Theta(\mathbb{R}_\Theta)$, it suffices to show that $W\,\mathrm{Op}(a)W^*$ commutes with right multiplication of $\mathbb{R}_\Theta$. For any $\boldsymbol{\eta}_0, \mathbf{y}_0 \in \mathbb{R}^d$,

$$\begin{aligned}\lambda_\Theta(F)\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0) &= \bigg(\int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y})d\boldsymbol{\eta}\, d\mathbf{y}\bigg)\lambda_\theta(\boldsymbol{\eta}_0)\lambda'_\theta(\mathbf{y}_0)\\ &= \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})e^{i\mathbf{y}\boldsymbol{\eta}_0}\lambda_\theta(\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y})\lambda_{\theta'}(\mathbf{y}_0)d\boldsymbol{\eta}\, d\mathbf{y}.\end{aligned}$$

Then $W(\lambda_\Theta(F)\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)) = \alpha^2_{\boldsymbol{\eta}_0}(\lambda_{\theta,\theta'}(F))\lambda_{\theta,\theta'}(\boldsymbol{\eta}_0,\mathbf{y}_0)$. We verify that

$$\begin{aligned}&\mathrm{Op}(a)W\big(\lambda_\Theta(F)\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)\big)\\ &\quad= \mathrm{Op}(a)\big(\alpha^2_{\boldsymbol{\eta}_0}\big(\lambda_{\theta,\theta'}(F)\big)\lambda_{\theta,\theta'}(\boldsymbol{\eta}_0,\mathbf{y}_0)\big)\\ &\quad= \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}+\boldsymbol{\eta}_0}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})e^{i\mathbf{y}\boldsymbol{\eta}_0}e^{\frac{i}{2}\boldsymbol{\eta}\theta\boldsymbol{\eta}_0}e^{\frac{i}{2}\mathbf{y}\theta'\mathbf{y}_0}\lambda_{\theta,\theta'}(\boldsymbol{\eta}+\boldsymbol{\eta}_0,\mathbf{y}+\mathbf{y}_0)d\boldsymbol{\eta}\, d\mathbf{y}\\ &\quad= \bigg(\int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}+\boldsymbol{\eta}_0}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\alpha^2_{\boldsymbol{\eta}_0}\big(\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)d\boldsymbol{\eta}\, d\mathbf{y}\bigg)\lambda_{\theta,\theta'}(\boldsymbol{\eta}_0,\mathbf{y}_0)\end{aligned}$$

$$= \alpha^2_{\boldsymbol{\eta}_0}\Big(\int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\Big)\lambda_{\theta,\theta'}(\boldsymbol{\eta}_0,\mathbf{y}_0)$$
$$= \alpha^2_{\boldsymbol{\eta}_0}\big(\mathrm{Op}(a)\lambda_{\theta,\theta'}(F)\big)\lambda_{\theta,\theta'}(\boldsymbol{\eta}_0,\mathbf{y}_0).$$

Hence

$$W^*\,\mathrm{Op}(a)W\big(\lambda_\Theta(F)\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0)\big) = \big(W^*\,\mathrm{Op}(a)W\lambda_\Theta(F)\big)\lambda_\theta(\boldsymbol{\eta}_0)\lambda_{\theta'}(\mathbf{y}_0),$$

which implies that $\mathrm{Op}(a)$ is affiliated to the representation on

$$\sigma(\mathbb{R}_\Theta) \subset B\big(L_2(\mathbb{R}_\theta)\otimes_2 L_2(\mathbb{R}_{\theta'})\big).$$

Now we show that $\mathrm{Op}(a): \mathcal{S}_{\theta,\theta'} \to \mathcal{S}_{\theta,\theta'}$ is continuous. Let us first assume that $a \in \Sigma^0$ is a 0-order symbol. Then a is bounded in $\mathbb{R}_{\theta,\theta'}$ and $\|a\|_\infty = \|\alpha^2_{\boldsymbol{\eta}}(a)\|_\infty$ for all $\boldsymbol{\eta}$. Thus the singular integral

$$\Big\|\int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\Big\|_\infty \le \|\widehat{F}\|_1\|a\|_\infty$$

converges in $\mathbb{R}_{\theta,\theta'}$. Write the set $\Omega := \{\mathrm{Op}(a)\lambda_\Theta(F) \mid F \in \mathcal{S}(\mathbb{R}^{2d}),\ a \in \Sigma^0\} \subset \mathbb{R}_{\theta,\theta'}$. For derivatives, we know that $D_{x_j}(\lambda_\theta(\boldsymbol{\eta})) = \boldsymbol{\eta}_j\lambda_\theta(\boldsymbol{\eta})$, $D_{\xi_j}(\lambda_{\theta'}(\mathbf{y})) = \mathbf{y}_j\lambda_{\theta'}(\mathbf{y})$, and $D^\beta_x D^\gamma_\xi(a) \in \Sigma^{-|\gamma|}$. Using product rules in the integral,

$$\begin{aligned} D_{\xi_j}\big(\mathrm{Op}(a)\lambda_{\theta,\theta'}(F)\big) &= D_{\xi_j}\Big(\int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_\theta(\boldsymbol{\eta})\otimes\lambda_{\theta'}(\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\Big) \\ &= \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(D_{\xi_j}a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\mathbf{y}_j\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y} \\ &= \mathrm{Op}(D_{\xi_j}a)\lambda_{\theta,\theta'}(F) + \mathrm{Op}(a)\lambda_{\theta,\theta'}(D_{\boldsymbol{\xi}_j}F), \end{aligned}$$

which is again in the set Ω hence bounded in $\mathbb{R}_{\theta,\theta'}$. By induction, $D^\beta_x D^\gamma_\xi(\mathrm{Op}(a)\lambda_{\theta,\theta'}(F))$ is in Ω for any β,γ. On the other hand, let $h \in \mathbb{R}$ and $\mathbf{e}_j = (0,\dots,1,\dots,0)$, then

$$\lambda_\theta(\boldsymbol{\eta})e^{ix_jh} = e^{-\frac{i}{2}\sum_k h\theta_{jk}\boldsymbol{\eta}_k}\lambda_\theta(\boldsymbol{\eta}+h\mathbf{e}_j), \quad \lambda_{\theta'}(\mathbf{y})e^{i\xi_jh} = e^{-\frac{i}{2}\sum_k h\theta'_{jk}\mathbf{y}_k}\lambda_\theta(\mathbf{y}+h\mathbf{e}_j).$$

Taking derivatives at $h = 0$,

$$\begin{aligned} \lambda_\theta(\boldsymbol{\eta})x_j &= D_{\boldsymbol{\eta}_j}\big(\lambda_\theta(\boldsymbol{\eta})\big) - \frac{1}{2}\sum_k \theta_{jk}\boldsymbol{\eta}_k\lambda_\theta(\boldsymbol{\eta}), \\ \lambda_\theta(\mathbf{y})\xi_j &= D_{\mathbf{y}_j}\big(\lambda_{\theta'}(\mathbf{y})\big) - \frac{1}{2}\sum_k \theta'_{jk}\boldsymbol{\eta}_k\lambda_{\theta'}(\mathbf{y}) \end{aligned}$$

holds weakly. Then

$$\begin{aligned} \big(\mathrm{Op}(a)\lambda_{\theta,\theta'}(F)\big)x_j &= \int \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})D_{\boldsymbol{\eta}_j}\big(\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)d\boldsymbol{\eta}\,d\mathbf{y} \\ &\quad - \frac{1}{2}\int \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\Big(\sum_k \theta_{jk}\boldsymbol{\eta}_k\Big)\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y} \end{aligned}$$

$$
\begin{aligned}
&= -\int \alpha^2_{\boldsymbol{\eta}}(D_{\xi_j}a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\big(\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&\quad -\int \alpha^2_{\boldsymbol{\eta}}(a)(D_{\boldsymbol{\eta}_j}\widehat{F})(\boldsymbol{\eta},\mathbf{y})\big(\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&\quad -\frac{1}{2}\int \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\Big(\sum_k \theta_{jk}\boldsymbol{\eta}_k\Big)\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\\
&= -\operatorname{Op}(D_{\xi_j}a)\lambda_{\theta,\theta'}(F) - \operatorname{Op}(a)\lambda_{\theta,\theta'}(\boldsymbol{\xi}_j F)\\
&\quad -\frac{1}{2}\sum_k \theta'_{jk}\operatorname{Op}(a)\lambda_{\theta,\theta'}(D_{\boldsymbol{\xi}_k}F)
\end{aligned}
$$

which is again in the set Ω. By induction, Ω is stable under right multiplication of polynomials $x^\beta\xi^\gamma$. By Proposition 3.4, we know that $\Omega \subset \mathcal{S}_{\theta,\theta'}$ because for all $\beta_1,\beta_2,\gamma_1,\gamma_2$

$$
\big\| D_x^{\beta_1}D_\xi^{\gamma_1}\big(\operatorname{Op}(a)\lambda_{\theta,\theta'}(F)\big)x^{\beta_2}\xi^{\gamma_2}\big\|_\infty < \infty.
$$

Moreover, one can track that these norms are controlled by the seminorms of $a \in \Sigma^0$ and $\lambda_{\theta,\theta'}(F) \in S_{\theta,\theta'}$. Thus we proved that $\operatorname{Op}(a) : \mathcal{S}_{\theta,\theta'} \to \mathcal{S}_{\theta,\theta'}$ is continuous for 0-order ΨDO. Now consider $b \in \Sigma^m$ with m being an even integer, we know that $b = b\langle\xi\rangle^{-m}\langle\xi\rangle^m$, $b\langle\xi\rangle^{-m}$ is a 0-order symbol, and $\langle\xi\rangle^m$ is a polynomial. Note that for $a \in \Sigma^0$,

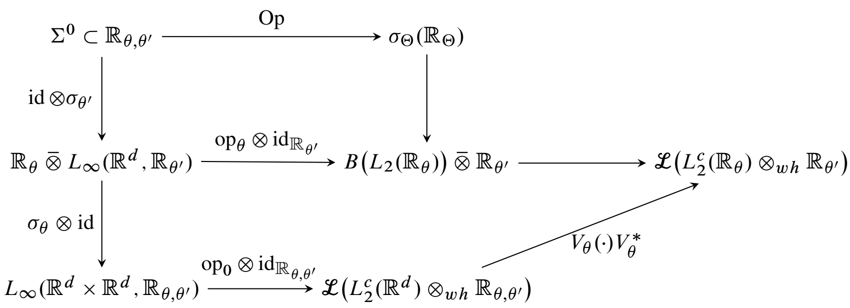
$$
\begin{aligned}
&\operatorname{Op}(a\xi_j)\lambda_{\theta,\theta'}(F)\\
&= \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a\xi_j)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^{2d}} (\xi_j+\boldsymbol{\eta}_j)\alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^{2d}} \xi_j\alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y} + \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\widehat{F}(\boldsymbol{\eta},\mathbf{y})\boldsymbol{\eta}_j\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \xi_j\operatorname{Op}(a)\lambda_{\theta,\theta'}(F) + \operatorname{Op}(a)\lambda_{\theta,\theta'}(D_{\mathbf{x}_j}F)
\end{aligned}
$$

which is again in Ω. Moreover, the continuity of $\operatorname{Op}(a\xi_j)$ follows from the continuity of $\operatorname{Op}(a)$. By induction, we obtain that $\operatorname{Op}(a) : \mathcal{S}_{\theta,\theta'} \to \mathcal{S}_{\theta,\theta'}$ is continuous for $\operatorname{Op}(a) \in \Sigma^m$ for all m. Finally, we verify the property that $\operatorname{Op}(a_1 \otimes a_2) = \sigma(a_1a_2)$. It suffices to consider test functions $\lambda_{\theta,\theta'}(F) = \lambda_\theta(f_1)\otimes\lambda_{\theta'}(f_2)$ with $F(\mathbf{x},\boldsymbol{\xi}) = f_1(\mathbf{x})f_2(\boldsymbol{\xi})$. Then

$$
\begin{aligned}
\operatorname{Op}(a_1\otimes a_2)\lambda_{\theta,\theta'}(F) &= \int \big(a_1\otimes\alpha_{\boldsymbol{\eta}}(a_2)\big)\hat{f}_1(\boldsymbol{\eta})\hat{f}_2(\mathbf{y})\big(\lambda_\theta(\boldsymbol{\eta})\otimes\lambda_{\theta'}(\mathbf{y})\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \int \hat{f}_1(\boldsymbol{\eta})a_1\lambda_\theta(\boldsymbol{\eta})\otimes\big(\alpha_{\boldsymbol{\eta}}(a_2)\lambda_{\theta'}(f_2)\big)d\boldsymbol{\eta}\\
&= W^*\Big(\int \hat{f}_1(\boldsymbol{\eta})a_1\lambda_\theta(\boldsymbol{\eta})\alpha_{\boldsymbol{\eta}}(a_2)\lambda_{\theta'}(f_2)d\boldsymbol{\eta}\Big)\\
&= W^*\Big(a_1a_2\int \hat{f}_1(\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(f_2)d\boldsymbol{\eta}\Big)\\
&= W^*\big(a_1a_2\lambda_\theta(f_1)\lambda_{\theta'}(f_2)\big) = W^*(a_1a_2)W\big(\lambda_\theta(f_1)\otimes\lambda_{\theta'}(f_2)\big).
\end{aligned}
$$

Here we use the fact that, for $a_2 \in \mathcal{M}_{\theta'}$, $a_2 \lambda_\theta(\boldsymbol{\eta}) = \lambda_\theta(\boldsymbol{\eta}) \alpha_{\boldsymbol{\eta}}(a_2)$. This property is easily verified for $a_2 \in \mathcal{S}_{\theta'}$ and then extends to $\mathcal{M}_{\theta'}$. ■

The connection between our setting and ΨDOs on $\mathbb{R}^d$ and $\mathbb{R}_\theta$ can be made explicit via the commuting diagram



Here $\sigma_\theta, \sigma_{\theta'}, \sigma_\Theta$ are the co-representation maps discussed in Section 4.2. The composition $\sigma_\Theta \circ \mathrm{Op}$ gives Definition 4.9. On the second row, the co-representation

$$\mathrm{id} \otimes \sigma_{\theta'}(a)(\boldsymbol{\eta}) = \alpha^2_{\boldsymbol{\eta}}(a)$$

gives $\mathbb{R}_{\theta'}$-valued symbol, and Definition 4.9 is then coincides with the $\mathbb{R}_{\theta'}$-valued operator map $\mathrm{op}_\theta \otimes \mathrm{id}$ on $\mathbb{R}_\theta$ in (4.7). Via the identification $B(L_2(\mathbb{R}_\theta)) \bar{\otimes} \mathbb{R}_{\theta'} \cong \mathcal{L}(L_2(\mathbb{R}_\theta)^c \otimes_{wh} \mathbb{R}_{\theta'})$ [27], this also gives operators on Hilbert $\mathbb{R}_{\theta'}$-module $L_2(\mathbb{R}_\theta)^c \otimes_{wh} \mathbb{R}_{\theta'}$. On the bottom row, we have an $\mathbb{R}_{\theta,\theta'}$-valued classical symbol $\sigma_\theta \otimes \sigma'_\theta(a)(\mathbf{x}, \boldsymbol{\xi}) = \alpha^1_{\mathbf{x}} \alpha^2_{\boldsymbol{\xi}}(a)$, and $\mathrm{op}_0 \otimes \mathrm{id}_{\theta,\theta'}$ is the $\mathbb{R}_{\theta,\theta'}$-valued operator map on $\mathbb{R}^d$ in (4.6). The ΨDOs are $\mathbb{R}_{\theta,\theta'}$-linear operators on the Hilbert module $L_2(\mathbb{R}^d)^c \otimes_{wh} \mathbb{R}_{\theta,\theta'}$. By Proposition 4.8, we have the Hilbert space isometry

$$V_\theta \otimes \mathrm{id}_{\mathbb{R}_{\theta'}} : L_2(\mathbb{R}_\theta)^c \otimes_{wh} \mathbb{R}_{\theta'} \to L_2(\mathbb{R}^d)^c \otimes_{wh} \mathbb{R}_{\theta,\theta'}.$$

Moreover, for a symbol $a \in \Sigma^0$, the operator $\mathrm{Op}(a)$ can be viewed as a restriction of the $\mathbb{R}_{\theta,\theta'}$-valued ΨDO $\mathrm{op}_0 \otimes \mathrm{id}(\sigma_{\theta,\theta'}(a))$ as follows:

$$\begin{aligned}
&\mathrm{op}_0 \otimes \mathrm{id}\,\big(\sigma_\theta \otimes \sigma'_\theta(a)\big)\big(V_\theta \otimes \mathrm{id}\,\big(\lambda_{\theta,\theta'}(F)\big)\big) \\
&\quad = (2\pi)^{-d} \int e^{i\mathbf{x}\boldsymbol{\xi}} \alpha^1_{\mathbf{x}} \alpha^2_{\boldsymbol{\xi}}(a) \widehat{F}(\boldsymbol{\xi}, \mathbf{y}) \lambda_{\theta,\theta'}(\boldsymbol{\xi}, \mathbf{y}) d\boldsymbol{\xi}\, d\mathbf{y} \\
&\quad = \alpha_{\mathbf{x}} \bigg((2\pi)^{-d} \int \alpha^2_{\boldsymbol{\xi}}(a) \widehat{F}(\boldsymbol{\xi}, \mathbf{y}) \lambda_{\theta,\theta'}(\boldsymbol{\xi}, \mathbf{y}) d\boldsymbol{\xi}\, d\mathbf{y} \bigg) = V_\theta \otimes \mathrm{id}\,\big(\mathrm{Op}(a) \lambda_{\theta,\theta'}(F)\big).
\end{aligned}$$

This enables us to reduce the L_2-boundedness to the operator-valued case. For that we recall the operator-valued Calderón–Vaillancourt theorem proved by Merklen in [33].

Theorem 4.11 ([33, Theorem 2.1]). *Let $\mathcal{A}$ be a C^*-algebra and let $CB^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{A})$ be the set of smooth $\mathcal{A}$-valued functions with bounded derivatives of all orders. Then, for* *any* $a \in CB^{\infty}(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{A})$,

$$\operatorname{op}(a) f(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\xi} a(\mathbf{x}, \boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad f \in \mathcal{S}(\mathbb{R}^d, \mathcal{A}),$$

extends to a bounded operator on the Hilbert $\mathcal{A}$*-module* $L_2(\mathbb{R}^d, \mathcal{A})$*. Moreover, there exists a constant* C *independent of* a*, such that*

$$\|\operatorname{op}(a)\| \leq C \sup\left\{\|D_{\mathbf{x}}^{\alpha} D_{\boldsymbol{\xi}}^{\beta}(a)\|_{\infty} \mid 0 \leq \alpha, \beta \leq (1, 1, \ldots, 1)\right\}.$$

Then L_2-boundedness theorem in our setting follows from the commuting diagram.

Theorem 4.12 (L_2-boundedness). *Let* $a \in \Sigma^0$ *be a symbol of order* 0*. Then* $\operatorname{Op}(a)$ *extends to a bounded operator in* $\sigma_{\Theta}(\mathbb{R}_{\Theta}) \subset B(L_2(\mathbb{R}_{\theta,\theta'}))$.

Proof. By definition of Σ^0, a and all its derivatives $D_x^{\alpha} D_{\xi}^{\beta}(a)$ are in $\mathbb{R}_{\theta,\theta'}$. Then $\sigma_{\theta} \otimes \sigma_{\theta'}(a) \in L_{\infty}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}_{\theta,\theta'})$ and for any α, β,

$$\|D_{\mathbf{x}}^{\alpha} D_{\boldsymbol{\xi}}^{\beta}(\sigma_{\theta,\theta'}(a))\| = \|\sigma_{\theta,\theta'}(D_x^{\alpha} D_{\xi}^{\beta}(a))\|$$

are bounded. Thus $\sigma_{\theta,\theta'}(a)$ is an $\mathbb{R}_{\theta,\theta'}$-valued symbol with all derivatives bounded. Then, by Theorem 4.11, we know that $\operatorname{op}_0 \otimes \operatorname{id}(\sigma_{\theta,\theta'}(a))$ is a bounded element in $B(L_2(\mathbb{R}^d)) \mathbin{\bar{\otimes}} \mathbb{R}_{\theta,\theta'}$. By diagram chasing,

$$\|\operatorname{Op}(a)\| = \|V_{\theta} \operatorname{Op}(a) V_{\theta}^*\|_{B(L_2(\mathbb{R}_{\theta}))\bar{\otimes}\mathbb{R}_{\theta'}} \leq \|\operatorname{op}(\sigma_{\theta} \otimes \sigma_{\theta}'(a))\|_{\mathcal{L}(L_2(\mathbb{R}^d, \mathbb{R}_{\theta,\theta'}))}$$

and the norm estimates follow from Theorem 4.11. ■

We now discuss the composition formula. Let us first identify the formula by a heuristic argument. Given two classical operator-valued symbols $a, b \in C^{\infty}(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{A})$, the composition symbol in the usual Euclidean case is

$$c(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} a(\mathbf{x}, \boldsymbol{\eta}) b(\mathbf{y}, \boldsymbol{\xi}) e^{i(\boldsymbol{\eta}-\boldsymbol{\xi})\cdot(\mathbf{x}-\mathbf{y})} \, d\boldsymbol{\eta} \, d\mathbf{y}.$$

Given symbols a, b affiliated to $\mathbb{R}_{\theta,\theta'}$, the co-representation $\sigma_{\theta,\theta'}$ gives us operator-valued symbols

$$\sigma_{\theta,\theta'}(a)(\mathbf{x}, \boldsymbol{\xi}) = \alpha_{\mathbf{x}}^1 \alpha_{\boldsymbol{\xi}}^2(a), \quad \sigma_{\theta,\theta'}(b)(\mathbf{x}, \boldsymbol{\xi}) = \alpha_{\mathbf{x}}^1 \alpha_{\boldsymbol{\xi}}^2(b).$$

The operator-valued composition symbol is

$$\begin{aligned} C(\mathbf{x}, \boldsymbol{\xi}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \alpha_{\mathbf{x}}^1 \alpha_{\boldsymbol{\eta}}^2(a) \alpha_{\mathbf{y}}^1 \alpha_{\boldsymbol{\xi}}^2(b) e^{i(\boldsymbol{\eta}-\boldsymbol{\xi})\cdot(\mathbf{x}-\mathbf{y})} \, d\boldsymbol{\eta} \, d\mathbf{y} \\ &= \alpha_{\mathbf{x}}^1 \alpha_{\boldsymbol{\xi}}^2 \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \alpha_{\boldsymbol{\eta}-\boldsymbol{\xi}}^2(a) \alpha_{\mathbf{y}-\mathbf{x}}^1(b) e^{i(\boldsymbol{\eta}-\boldsymbol{\xi})\cdot(\mathbf{x}-\mathbf{y})} \, d\boldsymbol{\eta} \, d\mathbf{y} \right) \\ &= \alpha_{\mathbf{x}}^1 \alpha_{\boldsymbol{\xi}}^2 \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \alpha_{\boldsymbol{\eta}}^2(a) \alpha_{\mathbf{y}}^1(b) e^{-i\boldsymbol{\eta}\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} \right) = \sigma_{\theta,\theta'}(c), \end{aligned}$$

where c is an $\mathcal{M}_{\theta,\theta'}$-valued singular integral,

$$c = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \alpha^2_{\boldsymbol{\eta}}(a)\alpha^1_{\mathbf{y}}(b)e^{-i\boldsymbol{\eta}\cdot\mathbf{y}}\, d\boldsymbol{\eta}\, d\mathbf{y}.$$

We first justify this singular integral and prove its formal series of the following definition.

Definition 4.13. Let m_j, $j \geq 0$, be a decreasing sequence of real numbers and $a_j \in \Sigma^{m_j}$. We write an m_0-order symbol $a \sim \sum_{j\geq 0} a_j$ if for any N, $a - \sum_{N\leq m_j} a_j \in \Sigma^N$.

The proof adapts the argument for the classical case by Stein [41] to the operator-valued setting.

Theorem 4.14 (Composition formula). *Let $a \in \Sigma^m$ and $b \in \Sigma^n$. Then there exists a symbol $c \in \Sigma^{m+n}$ such that* $\mathrm{Op}(c) = \mathrm{Op}(a)\,\mathrm{Op}(b)$ *and*

$$c \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi(a) D^\alpha_x(b).$$

Proof. Let ϕ be a positive function on $\mathbb{R}^d$ such that $\phi(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$ and $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| > 2$. For each $\varepsilon > 0$, denote by $b_\varepsilon(\mathbf{y}) = \phi(\varepsilon\mathbf{y})\alpha^2_{\mathbf{y}}(b)$ as Σ^n-valued function. Define the symbol

$$c_\varepsilon = \frac{1}{(2\pi)^d} \int \alpha^2_{\boldsymbol{\eta}}(a) b_\varepsilon(\mathbf{y}) e^{-i\boldsymbol{\eta}\cdot\mathbf{y}}\, d\boldsymbol{\eta}\, d\mathbf{y}.$$

This is a Bochner integral because the integrand function $(\boldsymbol{\eta}, \mathbf{y}) \mapsto \alpha^2_{\boldsymbol{\eta}}(a) b_\varepsilon(\mathbf{y}) e^{-i\boldsymbol{\eta}\cdot\mathbf{y}}$ is smooth in the Fréchet space Σ^{m+n} by Proposition 4.4. We split our proofs into three steps.

Step 1. For any $\varepsilon > 0$, c_ε converges in Σ^{m+n} and there exists remainder $R_\varepsilon \in \Sigma^{m+n-N-1}$ such that

$$c_\varepsilon = \sum_{|\beta|\leq N} \frac{i^{|\beta|}}{\beta!} D^\beta_\xi a D^\beta_x b + R_\varepsilon.$$

For the compactly supported $b_\varepsilon \in C(\mathbb{R}^d, \Sigma^n)$, the Fourier transform for functions valued in the Fréchet space Σ^n is well defined:

$$\hat{b}_\varepsilon(\boldsymbol{\eta}) = \int b_\varepsilon(\mathbf{y}) e^{-i\mathbf{y}\boldsymbol{\eta}}\, d\mathbf{y}, \quad \int \hat{b}_\varepsilon(\mathbf{y}) e^{-i\boldsymbol{\eta}\mathbf{y}}\, d\boldsymbol{\eta}\, d\mathbf{y} = (2\pi)^d b_\varepsilon(0) = (2\pi)^d b(0).$$

Then, for any β,

$$\begin{aligned}\int \boldsymbol{\eta}^\beta \hat{b}_\varepsilon(\boldsymbol{\eta}) d\boldsymbol{\eta} &= (-1)^{|\beta|} \int b_\varepsilon(\mathbf{y}) D^\beta_{\mathbf{y}}(e^{-i\mathbf{y}\boldsymbol{\eta}}) d\mathbf{y}\, d\boldsymbol{\eta} = \int D^\beta_{\mathbf{y}}\big(\phi(\varepsilon\mathbf{y})\alpha^1_{\mathbf{y}}(b)\big) e^{-i\mathbf{y}\boldsymbol{\eta}}\, d\mathbf{y}\, d\boldsymbol{\eta} \\ &= \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1, \beta_2} \int \varepsilon^{|\beta_1|}(D^{\beta_1}\phi)(\varepsilon\mathbf{y})\alpha^1_{\mathbf{y}}(D^{\beta_2}_x b) e^{-i\mathbf{y}\boldsymbol{\eta}}\, d\mathbf{y}\, d\boldsymbol{\eta} \\ &= (2\pi)^d \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1, \beta_2} \varepsilon^{|\beta_1|}(D^{\beta_1}\phi)(0) D^{\beta_2}_x b = (2\pi)^d D^\beta_x b. \quad (4.9)\end{aligned}$$

We also have

$$D_x^\beta D_\xi^\gamma(\hat{b}_\varepsilon(\boldsymbol{\eta})) = D_x^\beta D_\xi^\gamma\Big(\int \phi_\varepsilon(\mathbf{y})\alpha_{\mathbf{y}}(b)e^{-i\mathbf{y}\boldsymbol{\eta}}\,d\mathbf{y}\Big) = \int \phi_\varepsilon(\mathbf{y})\alpha_{\mathbf{y}}(D_x^\beta D_\xi^\gamma b)e^{-i\mathbf{y}\boldsymbol{\eta}}d\mathbf{y} = \widehat{D_x^\beta D_\xi^\gamma b_\varepsilon}(\boldsymbol{\eta}).$$

By Proposition 4.4, we use the Taylor expansion for functions valued in the Fréchet space Σ^m,

$$\alpha_{\boldsymbol{\eta}}(a) = \sum_{|\beta|\le N}\frac{i^{|\beta|}(D_\xi^\beta a)\boldsymbol{\eta}^\beta}{\beta!} + (N+1)\sum_{|\beta|=N+1}\frac{i^{|\beta|}}{\beta!}\boldsymbol{\eta}^\beta\int_0^1 \alpha_{t\boldsymbol{\eta}}(D_\xi^\beta a)(1-t)^N\,dt. \tag{4.10}$$

Using the calculation (4.9), the first part leads to

$$\frac{1}{(2\pi)^d}\int\sum_{|\beta|\le N}\frac{D_\xi^\beta a}{\beta!}\boldsymbol{\eta}^\beta\hat{b}_\varepsilon(\boldsymbol{\eta})d\boldsymbol{\eta} = \sum_{|\beta|\le N}\frac{i^{|\beta|}}{\beta!}D_\xi^\beta a D_x^\beta b$$

which gives the leading terms. For the second term in (4.10), we have $|\beta| = N+1$ and

$$\begin{aligned}
&\Big\|\int_0^1 \alpha_{t\boldsymbol{\eta}}^2(D_\xi^\alpha a)(1-t)^N dt\,\langle\xi\rangle^{-m+N+1}\Big\| \\
&\le \int_0^1 (1-t)^N\big\|\alpha_{t\boldsymbol{\eta}}^2\big(D_\xi^\beta(a)\langle\xi\rangle^{-m+N+1}\big)\big\|\cdot\big\|\langle\xi+t\boldsymbol{\eta}\rangle^{m-N-1}\langle\xi\rangle^{-m+N+1}\big\|dt \\
&\le \int_0^1 (1-t)^N\big\|D_\xi^\beta(a)\langle\xi\rangle^{-m+N+1}\big\|\cdot\big\|\langle\xi+t\boldsymbol{\eta}\rangle^{m-N-1}\langle\xi\rangle^{-m+N+1}\big\|dt \\
&\lesssim \int_0^1 (1-t)^N\big(t\langle\boldsymbol{\eta}\rangle\big)^{\lceil -m+N+1\rceil}\,dt \le A_{N,m}\langle\boldsymbol{\eta}\rangle^{\lceil -m+N+1\rceil}.
\end{aligned}$$

Here we used Lemma 3.5 with $A_{N,m}$ being some positive constant that only depends on N, m and $\lceil r\rceil$ as the smallest even integer greater than $|r|$. On the other hand, for any β,

$$\hat{b}_\varepsilon(\boldsymbol{\eta})\boldsymbol{\eta}^\beta = \sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}\int D_{\mathbf{y}}^{\beta_1}\phi_\varepsilon(\mathbf{y})\alpha_{\mathbf{y}}^2\big(D_x^{\beta_2}(b)\big)e^{-i\mathbf{y}\boldsymbol{\eta}}\,d\mathbf{y}.$$

For each term,

$$\begin{aligned}
&\big\|\langle\xi\rangle^{m-N-1}D_{\mathbf{y}}^{\beta_1}\phi_\varepsilon(\mathbf{y})\alpha_{\mathbf{y}}^1\big(D_x^{\beta_2}(b)\big)\langle\xi\rangle^{-n-m+N+1}\big\| \\
&\quad\le \big|D_{\mathbf{y}}^{\beta_1}\phi_\varepsilon(\mathbf{y})\big|\cdot\big\|\alpha_{\mathbf{y}}^1\big(\langle\xi\rangle^{m-N-1}D_x^{\beta_2}(b)\langle\xi\rangle^{-n-m+N+1}\big)\big\|.
\end{aligned}$$

Here we used the assumption that $b, D_x^{\beta_2}(b)\in\Sigma^n$. Because $D_{\mathbf{y}}^{\beta_1}(\phi_\varepsilon(\mathbf{y}))$ is a compactly supported function of $\mathbf{y}$, we have for any positive integer l,

$$\big\|\langle\xi\rangle^{m-N-1}\hat{b}_\varepsilon(\boldsymbol{\eta})\langle\xi\rangle^{-n-m+N+1}\big\| \le B_{n,m,N}\big(1+|\boldsymbol{\eta}|\big)^{-l},$$

where $B_{l,n,m,N}$ is a constant depending on (l, n, m, N) and ε. Thus, by choosing large enough l,

$$\left\| \int_{\mathbb{R}^d} \left(\int_0^1 \alpha_{t\boldsymbol{\eta}}(D_\xi^\beta a)(1-t)^N dt \right) \boldsymbol{\eta}^\beta \hat{b}_\varepsilon(\boldsymbol{\eta}) d\eta \langle \xi \rangle^{-m-n+N+1} \right\| \lesssim \int \langle \boldsymbol{\eta} \rangle^{\lceil m-N-1 \rceil} \big(1 + |\boldsymbol{\eta}|\big)^{-l} d\boldsymbol{\eta} < \infty.$$

A similar argument applies for derivatives:

$$D_x^{\gamma_1} D_\xi^{\gamma_2} \bigg(\int_{\mathbb{R}^d} \left(\int_0^1 \alpha_{t\eta}^2 (D_\xi^\beta a)(\boldsymbol{\eta})(1-t)^N \, dt \right) \boldsymbol{\eta}^\beta \hat{b}_\varepsilon(\boldsymbol{\eta}) d\boldsymbol{\eta} \bigg).$$

Therefore we obtain that

$$c_\varepsilon = \sum_{|\beta| \le N} \frac{i^{|\beta|}}{\beta!} D_\xi^\beta a D_x^\beta b + R_\varepsilon,$$

where R_ε is a remainder term in $\Sigma^{n+m-N-1}$.

Step 2. The limit $\lim_{\varepsilon \to 0} c_\varepsilon = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^d} \int \alpha_{\boldsymbol{\eta}}^2(a) b_\varepsilon(\mathbf{y}) e^{-i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y}$ converges in Σ^{n+m}.

Now take $\varepsilon' < \varepsilon$ and

$$b_2(\mathbf{y}) := b_{\varepsilon'}(\mathbf{y}) - b_\varepsilon(\mathbf{y}) = \big(\phi(\varepsilon'\mathbf{y}) - \phi(\varepsilon\mathbf{y})\big)\alpha_{\mathbf{y}}(b),$$

which is supported on $1/\varepsilon < |\mathbf{y}| < 2/\varepsilon'$. Note that in the above argument, we actually show that the singular integral $\int \alpha_{\boldsymbol{\eta}}(a) b(\mathbf{y}) e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y}$ converges absolutely if b is compactly supported. Then for each j, we can use integration by parts:

$$\begin{aligned} \int \alpha_{\boldsymbol{\eta}}(a) \mathbf{y}_j |\mathbf{y}|^{-2} b_2(\mathbf{y}) e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} &= \int \alpha_{\boldsymbol{\eta}}(a) |\mathbf{y}|^{-2} b_2(\mathbf{y}) D_{\boldsymbol{\eta}_j} e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} \\ &= \int D_{\boldsymbol{\eta}_j}(\alpha_{\boldsymbol{\eta}})(a) |\mathbf{y}|^{-2} b_2(\mathbf{y}) e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} \\ &= \int \alpha_{\boldsymbol{\eta}}(D_{\xi_j} a) |\mathbf{y}|^{-2} b_2(\mathbf{y}) e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y}. \end{aligned}$$

Here we used the property $D_{\boldsymbol{\eta}_j}(\alpha_{\boldsymbol{\eta}})(a) = \alpha_{\boldsymbol{\eta}}(D_{\xi_j} a)$. Denote $\Delta_{\boldsymbol{\eta}} = \sum_j D_{\boldsymbol{\eta}_j}^2$, $\Delta_\xi = \sum_j D_{\xi_j}^2$, and $\Delta_{\mathbf{y}} = \sum_j D_{\mathbf{y}_j}^2$. Because $\Delta_{\boldsymbol{\eta}}(\alpha_{\boldsymbol{\eta}}^1(a)) = \alpha_{\boldsymbol{\eta}}^1(\Delta_\xi a)$, using integration by parts we have

$$\begin{aligned} &\int \alpha_{\boldsymbol{\eta}}(a) b_2(\mathbf{y}) e^{i\boldsymbol{\eta}\cdot\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} \\ &\quad = \int \alpha_{\boldsymbol{\eta}}(\Delta_\xi^{m_1} a) |\mathbf{y}|^{-2m_1} b_2(\mathbf{y}) e^{-i\boldsymbol{\eta}\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y} \\ &\quad = \int \alpha_{\boldsymbol{\eta}}(\Delta_\xi^{m_1} a)(1 + \Delta_{\mathbf{y}})^{m_2} \big(|\mathbf{y}|^{-2m_1} b_2(\mathbf{y})\big) \langle \boldsymbol{\eta} \rangle^{-2m_2} e^{-i\boldsymbol{\eta}\mathbf{y}} \, d\boldsymbol{\eta} \, d\mathbf{y}. \end{aligned}$$

Here $|\mathbf{y}|^{-2m_1}b_2(\mathbf{y})$ has no singularity because b_2 is supported away from $\mathbf{y}=0$. Because $a\in\Sigma^m, b\in\Sigma^n$,

$$\Delta_\xi^{m_1}(a)\in\Sigma^{m-2m_1},\quad (1+\Delta_{\mathbf{y}})^{m_2}\big(|\mathbf{y}|^{-2m_1}b_2(\mathbf{y})\big)\in\Sigma^n.$$

For the symbol $a\in\Sigma^m$, we have

$$\begin{aligned}\big\|\alpha_{\boldsymbol\eta}(\Delta_\xi^{m_1}a)\langle\xi\rangle^{-m+2m_1}\big\| &\le \big\|\alpha_{\boldsymbol\eta}\big(\Delta_\xi^{m_1}a\langle\xi\rangle^{-m+2m_1}\big)\big\|\,\big\|\langle\xi-\boldsymbol\eta\rangle^{m-2m_1}\langle\xi\rangle^{-m+2m_1}\big\|\\ &\le \widetilde{A}_{m,m_1}\langle\boldsymbol\eta\rangle^{\lceil -m+2m_1\rceil},\end{aligned}$$

for some constants $\widetilde{A}_{m,m_1}$. For the symbol $b\in\Sigma^n$, because $|\mathbf{y}|^{-2m_1}b_2(\mathbf{y})$ is supported on $\frac{1}{\varepsilon}<|\mathbf{y}|<\frac{2}{\varepsilon'}$,

$$\begin{aligned}&\big\|\langle\xi\rangle^{m-2m_1}(1+\Delta_{\mathbf{y}})^{m_2}\big(|\mathbf{y}|^{-2m_1}b_2(\mathbf{y})\big)\langle\xi\rangle^{-m+2m_1-n}\big\|\\ &\quad\le \widetilde{B}_{m,m_1,n}\langle\mathbf{y}\rangle^{\lceil m-2m_1\rceil+\lceil 2m_1-m-n\rceil-2m_1}\chi_{\{\frac{1}{\varepsilon}<|\mathbf{y}|<\frac{2}{\varepsilon'}\}}\end{aligned}$$

for some constants $\widetilde{B}_{m,m_1,n}$ and here χ is the characteristic function. We first choose m_1 large enough with $2m_1-\lceil m-2m_1\rceil-\lceil 2m_1-m-n\rceil>0$ and then m_2 large enough such that the integral

$$\begin{aligned}&\left\|\int\alpha_{\boldsymbol\eta}(a)b_2(\mathbf{y})e^{-i\boldsymbol\eta\mathbf{y}}\,d\boldsymbol\eta\,d\mathbf{y}\cdot\langle\xi\rangle^{-m-n+N+1}\right\|\\ &\le\int_{\frac{1}{\varepsilon}<|\mathbf{y}|<\frac{2}{\varepsilon'}}\langle\boldsymbol\eta\rangle^{\lceil -m+2m_1\rceil}\langle\boldsymbol\eta\rangle^{-2m_2}\langle\mathbf{y}\rangle^{\lceil m-2m_1\rceil+\lceil 2m_1-m-n\rceil-2m_1}\,d\boldsymbol\eta\,d\mathbf{y}<\infty\end{aligned}\tag{4.11}$$

converges absolutely. The argument for the derivatives is similar. Hence

$$\int\alpha_{\boldsymbol\eta}(a)b_2(\mathbf{y})e^{-i\boldsymbol\eta\mathbf{y}}\,d\boldsymbol\eta\,d\mathbf{y}\in\Sigma^{n+m-N-1},$$

which is of lower order of the leading terms. Note that when $\varepsilon\to 0$, the norm estimates (4.11) go to 0 uniformly for $\varepsilon'<\varepsilon$. This implies that the remainder R_ε converges to 0 in $\Sigma^{n+m-N-1}$.

Step 3. For any $\lambda_{\theta,\theta'}(F)\in\mathcal{S}_{\theta,\theta'}$,

$$\mathrm{Op}(a)\,\mathrm{Op}(b)\lambda_{\theta,\theta'}(F)=\lim_{\varepsilon\to 0}\mathrm{Op}(c_\varepsilon)\lambda_{\theta,\theta'}(F)=\mathrm{Op}(c)\lambda_{\theta,\theta'}(F).$$

Hence $\mathrm{Op}(a)\,\mathrm{Op}(b)=\mathrm{Op}(c)$.

Indeed, since the integral in c_ε converges absolutely, then

$$\begin{aligned}&\mathrm{Op}(c_\varepsilon)\lambda_{\theta,\theta'}(F)\\ &=\int\alpha^2_{\boldsymbol\eta_1}\Big(\int\phi(\varepsilon\mathbf{y})\alpha^2_{\boldsymbol\eta}(a)\alpha^1_{\mathbf{y}}(b)e^{-i\boldsymbol\eta\mathbf{y}}\,d\boldsymbol\eta\,d\mathbf{y}\Big)\widehat{F}(\boldsymbol\eta_1,\mathbf{y}_1)\lambda_{\theta,\theta'}(\boldsymbol\eta_1,\mathbf{y}_1)d\boldsymbol\eta_1\,d\mathbf{y}_1\\ &=\int\phi(\varepsilon\mathbf{y})e^{-i\boldsymbol\eta\mathbf{y}}\alpha^2_{\boldsymbol\eta+\boldsymbol\eta_1}(a)\alpha^1_{\mathbf{y}}\alpha^2_{\boldsymbol\eta}(b)\widehat{F}(\boldsymbol\eta_1,\mathbf{y}_1)\lambda_{\theta,\theta'}(\boldsymbol\eta_1,\mathbf{y}_1)d\boldsymbol\eta_1\,d\mathbf{y}_1\,d\boldsymbol\eta\,d\mathbf{y}\end{aligned}$$

$$= \int \phi(\varepsilon \mathbf{y}) e^{-i(\boldsymbol{\xi}-\boldsymbol{\eta}_1)\mathbf{y}} \alpha^2_{\boldsymbol{\xi}}(a) \alpha^1_{\mathbf{y}} \alpha^2_{\boldsymbol{\eta}_1}(b) \widehat{F}(\boldsymbol{\eta}_1, \mathbf{y}_1) \lambda_{\theta,\theta'}(\boldsymbol{\eta}_1, \mathbf{y}_1) d\boldsymbol{\eta}_1\, d\mathbf{y}_1\, d\boldsymbol{\xi}\, d\mathbf{y}$$

$$= \int \phi(\varepsilon \mathbf{y}) e^{-i\boldsymbol{\xi}\mathbf{y}} \alpha^2_{\boldsymbol{\xi}}(a) \alpha^1_{\mathbf{y}} \Big(\int \alpha^2_{\boldsymbol{\eta}_1}(b) \widehat{F}(\boldsymbol{\eta}_1, \mathbf{y}_1) \lambda_{\theta,\theta'}(\boldsymbol{\eta}_1, \mathbf{y}_1) d\boldsymbol{\eta}_1\, d\mathbf{y}_1 \Big) d\boldsymbol{\xi}\, d\mathbf{y}$$

$$= \int \phi(\varepsilon \mathbf{y}) e^{-i\boldsymbol{\xi}\mathbf{y}} \alpha^2_{\boldsymbol{\xi}}(a) \alpha^1_{\mathbf{y}} \big(\mathrm{Op}(b) \lambda_{\theta,\theta'}(F) \big) d\boldsymbol{\xi}\, d\mathbf{y}.$$

Thus it suffices to show that for $\lambda_{\theta,\theta'}(G) = \mathrm{Op}(b)\lambda_{\theta,\theta'}(F)$,

$$\lim_{\varepsilon \to 0} \int \phi(\varepsilon \mathbf{y}) e^{-i\boldsymbol{\xi}\mathbf{y}} \alpha^2_{\boldsymbol{\xi}}(a) \alpha^1_{\mathbf{y}} \big(\lambda_{\theta,\theta'}(G) \big) d\boldsymbol{\eta}\, d\mathbf{y} = \mathrm{Op}(a) \lambda_{\theta,\theta'}(G).$$

Let $\widehat{\phi}$ be the Fourier transform of ϕ, then

$$\begin{aligned} \int \phi(\varepsilon \mathbf{y}) e^{-i\boldsymbol{\xi}\mathbf{y}} \alpha^1_{\mathbf{y}} \big(\lambda_{\theta,\theta'}(G) \big) d\mathbf{y} &= \int \phi(\varepsilon \mathbf{y}) e^{-i(\boldsymbol{\xi}-\boldsymbol{\eta}_1)\mathbf{y}} \widehat{G}(\boldsymbol{\eta}_1, \mathbf{y}_1) \lambda_{\theta,\theta'}(\boldsymbol{\eta}_1, \mathbf{y}_1) d\mathbf{y}\, d\mathbf{y}_1\, d\boldsymbol{\eta}_1 \\ &= \int \frac{1}{\varepsilon^d} \widehat{\phi}\Big(\frac{\boldsymbol{\xi} - \boldsymbol{\eta}_1}{\varepsilon} \Big) \widehat{G}(\boldsymbol{\eta}_1, \mathbf{y}_1) \lambda_{\theta,\theta'}(\boldsymbol{\eta}_1, \mathbf{y}_1) d\mathbf{y}_1\, d\boldsymbol{\eta}_1. \end{aligned}$$

Here $\frac{1}{\varepsilon^d} \widehat{\phi}(\frac{\cdot}{\varepsilon})$ approximates the delta function,

$$\begin{aligned} \int \phi(\varepsilon \mathbf{y}) e^{-i\boldsymbol{\xi}\mathbf{y}} \alpha^2_{\boldsymbol{\xi}}(a) \alpha^1_{\mathbf{y}} \big(\lambda_{\theta,\theta'}(G) \big) d\boldsymbol{\eta}\, d\mathbf{y} &= \int \frac{1}{\varepsilon^d} \widehat{\phi}\Big(\frac{\boldsymbol{\xi}}{\varepsilon} \Big) \mathrm{Op}(\alpha^2_{\boldsymbol{\xi}} a) \lambda_{\theta,\theta'}(G) d\boldsymbol{\xi} \\ &= \int \frac{1}{\varepsilon^d} \widehat{\phi}\Big(\frac{\boldsymbol{\xi}}{\varepsilon} \Big) \alpha^2_{\boldsymbol{\xi}} \big(\mathrm{Op}(a) \alpha^2_{-\boldsymbol{\xi}} \lambda_{\theta,\theta'}(G) \big) d\boldsymbol{\xi}. \end{aligned}$$

Since $\boldsymbol{\xi} \to \alpha^2_{\boldsymbol{\xi}}(\mathrm{Op}(a)\alpha^2_{-\boldsymbol{\xi}} \lambda_{\theta,\theta'}(G))$ is continuous in $\mathcal{S}_{\theta,\theta'}$, the above integral converges to $\mathrm{Op}(a)\lambda_{\theta,\theta'}(G)$ in $\mathcal{S}_{\theta,\theta'}$ as $\varepsilon \to 0$. ■

4.4. Integrability and trace formula

In the rest of this section, we discuss the integrability of ΨDOs whose symbol is integrable in the first component $\mathbb{R}_\theta$.

Definition 4.15 (Tame symbols). An element $a \in \mathcal{M}_{\theta,\theta'}$ is a *tame symbol* of order m if there exists an $r > d$ such that for any α, β, and γ,

$$\langle x \rangle^r D^\alpha_x D^\beta_\xi (a) \langle \xi \rangle^{|\beta| - m}$$

extends to a bounded element in $\mathbb{R}_{\theta,\theta'}$. We write Σ^m_{tame} as the set of all tame symbols of order m and $\Sigma^{-\infty}_{\mathrm{tame}} := \bigcap_r \Sigma^r_{\mathrm{tame}}$.

Proposition 4.16. *A symbol $a \in \Sigma^m_{\mathrm{tame}}$ if and only if there exists $r > d$ such that for all α, β, $D^\alpha_x D^\beta_\xi(a) \in O^{-r, m-|\beta|}$. Moreover, if $b \in \Sigma^n$, then $ab, ba \in \Sigma^{n+m}_{\mathrm{tame}}$.*

Proof. This is a direct consequence of Theorem 4.3. ■

Lemma 4.17. *Let $a \in L_2(\mathbb{R}_\theta)$ and $b \in L_2(\mathbb{R}_{\theta'})$. Then $ab \in L_2(\mathbb{R}_\Theta)$ and*

$$\|ab\|_{L_2(\mathbb{R}_\Theta)} = \|a\|_{L_2(\mathbb{R}_\theta)}\|b\|_{L_2(\mathbb{R}_{\theta'})}.$$

Proof. Note that, for $\lambda_\theta(\boldsymbol{\eta}), \lambda_{\theta'}(\mathbf{y}) \in \mathbb{R}_\Theta$,

$$\lambda_\theta(\boldsymbol{\eta})\lambda_{\theta'}(\mathbf{y}) = e^{-\frac{i}{2}\boldsymbol{\eta}\cdot\mathbf{y}}\lambda_\Theta\big((\boldsymbol{\eta},\mathbf{0}) + (\mathbf{0},\mathbf{y})\big) = e^{-\frac{i}{2}\boldsymbol{\eta}\cdot\mathbf{y}}\lambda_\Theta(\boldsymbol{\eta},\mathbf{y}),$$

where $\lambda_\Theta(\boldsymbol{\eta},\mathbf{y})$ is the quantization of $\mathbb{R}_\Theta$ as in (2.2). For $f \in \mathcal{S}_\theta, g \in \mathcal{S}_{\theta'}$, we have

$$\begin{aligned}\lambda_\theta(f)\lambda_{\theta'}(g) &= \frac{1}{(2\pi)^{2d}}\bigg(\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\eta})\lambda_\theta(\boldsymbol{\eta})d\boldsymbol{\eta}\bigg)\bigg(\int_{\mathbb{R}^d} \hat{g}(\mathbf{y})\lambda_{\theta'}(\mathbf{y})d\mathbf{y}\bigg)\\ &= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\eta})\hat{g}(\mathbf{y})e^{-\frac{i}{2}\boldsymbol{\eta}\cdot\mathbf{y}}\lambda_\Theta(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}.\end{aligned}$$

Thus we have

$$\tau_\Theta\big(\lambda_\theta(f)\lambda_{\theta'}(g)\big) = \hat{f}(0)\hat{g}(0) = \tau_\theta\big(\lambda_\theta(f)\big)\tau_{\theta'}\big(\lambda_\theta(g)\big).$$

Therefore,

$$\begin{aligned}\big\|\lambda_\theta(f)\lambda_{\theta'}(g)\big\|^2_{L_2(\mathbb{R}_\Theta)} &= \tau_\Theta\big(\lambda_{\theta'}(g)^*\lambda_\theta(f)^*\lambda_\theta(f)\lambda_{\theta'}(g)\big)\\ &= \tau_\Theta\big(\lambda_\theta(f)^*\lambda_\theta(f)\lambda_{\theta'}(g)\lambda_{\theta'}(g)^*\big)\\ &= \tau_\theta\big(\lambda_\theta(f)^*\lambda_\theta(f)\big)\tau_{\theta'}\big(\lambda_{\theta'}(g)\lambda_{\theta'}(g)^*\big)\\ &= \big\|\lambda_\theta(f)\big\|^2_{L_2(\mathbb{R}_\theta)}\big\|\lambda_{\theta'}(g)\big\|^2_{L_2(\mathbb{R}_{\theta'})}.\end{aligned}$$

The assertion for general $a \in L_2(\mathbb{R}_\theta), b \in L_2(\mathbb{R}_{\theta'})$ follows from density. ∎

In the following and also Section 5, we identify $\sigma_\Theta(\mathbb{R}_\Theta)$ with $\mathbb{R}_\Theta$ and viewing our ΨDOs $\mathrm{Op}(a)$ as an operator affiliated to $\mathbb{R}_\Theta \subset B(L_2(\mathbb{R}_\Theta))$.

Corollary 4.18. *Let $a \in \Sigma^m_{\mathrm{tame}}$. Then*

(i) $\mathrm{Op}(a) \in L_2(\mathbb{R}_\Theta)$ *if* $m < -\frac{d}{2}$;

(ii) $\mathrm{Op}(a) \in L_1(\mathbb{R}_\Theta)$ *if* $m < -d$.

Proof. We know from the algebraic property that $\mathrm{Op}(\lambda_\theta(f_1) \otimes \lambda_{\theta'}(f_2)) = \lambda_\theta(f_1)\lambda_{\theta'}(f_2)$ for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$. The Op is an L_2-isometry and trace preserving on $\mathcal{S}_{\theta,\theta'}$. Let $a \in \Sigma^m_{\mathrm{tame}}$. Then for some $r > d$,

$$\begin{aligned}\mathrm{Op}(a) &= \langle x\rangle^{-r}\langle\xi\rangle^m\langle\xi\rangle^{-m}\langle x\rangle^r\,\mathrm{Op}(a) = \langle x\rangle^{-r}\langle\xi\rangle^m\langle\xi\rangle^{-m}\,\mathrm{Op}\big(\langle x\rangle^r a\big)\\ &= \big(\langle x\rangle^{-r}\langle\xi\rangle^m\big)\big(\langle\xi\rangle^{-m}\,\mathrm{Op}\big(\langle x\rangle^r a\big)\big).\end{aligned}$$

By symbol calculus, $\langle\xi\rangle^{-m}\,\mathrm{Op}(\langle x\rangle^r a)$ is a ΨDO of order 0 hence in $\mathbb{R}_\Theta$. For $m < -d/2$, $\|\langle\xi\rangle^m\|_{L_2(\mathbb{R}_{\theta'})} < \infty$ and $\|\langle x\rangle^{-r}\|_{L_2(\mathbb{R}_\theta)} < \infty$. Then $\langle x\rangle^{-r}\langle\xi\rangle^m \in L_2(\mathbb{R}_\Theta)$ and

$$\big\|\,\mathrm{Op}(a)\big\|_2 \leq \big\|\langle x\rangle^{-r}\langle\xi\rangle^m\big\|_2\big\|\langle\xi\rangle^{-m}\,\mathrm{Op}\big(\langle x\rangle^r a\big)\big\|_\infty.$$

For $m < -d$, choose $n = \frac{m}{2}$,

$$\mathrm{Op}(a) = \big(\langle x\rangle^n \langle\xi\rangle^n\big)\big(\langle\xi\rangle^{-n}\,\mathrm{Op}\,\big(\langle x\rangle^{-n} a\big)\big).$$

$\langle\xi\rangle^{-n}\,\mathrm{Op}(\langle x\rangle^{-n} a)$ is a tame ΨDO of order less than $d/2$ hence in $L_2(\mathbb{R}_\Theta)$ and $\langle x\rangle^{-n}\langle\xi\rangle^{-n}$ is also in $L_2(\mathbb{R}_\Theta)$ by the discussion in (i). ■

We end this section with the trace formula.

Proposition 4.19. *Suppose a symbol $a \in L_1(\mathbb{R}_{\theta,\theta'})$ and its operator* $\mathrm{Op}(a) \in L_1(\mathbb{R}_\Theta)$. *Then*

$$\tau_\Theta\big(\mathrm{Op}(a)\big) = \tau_{\theta,\theta'}(a).$$

Proof. Using the definition of $\mathrm{Op}(a)$,

$$\begin{aligned}
\tau_\Theta\big(\mathrm{Op}(a)\lambda_\Theta(F)\big) &= \tau_{\theta,\theta'}\Big(\int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})\alpha^2_{\boldsymbol{\eta}}(a)\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})d\boldsymbol{\eta}\,d\mathbf{y}\Big)\\
&= \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})\big(\tau_{\theta,\theta'}\big(\alpha^2_{\boldsymbol{\eta}}(a)\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})\tau_{\theta,\theta'}\big(a\alpha^2_{-\boldsymbol{\eta}}\big(\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \int_{\mathbb{R}^{2d}} \widehat{F}(\boldsymbol{\eta},\mathbf{y})e^{-i\boldsymbol{\eta}\mathbf{y}}\big(\tau_{\theta,\theta'}\big(a\lambda_{\theta,\theta'}(\boldsymbol{\eta},\mathbf{y})\big)\big)d\boldsymbol{\eta}\,d\mathbf{y}\\
&= \tau_{\theta,\theta'}\big(a\lambda_{\theta,\theta'}(F')\big),
\end{aligned}$$

where F' has the Fourier transform $\widehat{F}'(\boldsymbol{\eta},\mathbf{y}) = \widehat{F}(\boldsymbol{\eta},\mathbf{y})e^{-i\boldsymbol{\eta}\mathbf{y}}$. Here we use the Fubini theorem because $a \in L_1(\mathbb{R}_{\theta,\theta'})$. Let $F_n \in \mathcal{S}(\mathbb{R}^{2d})$ be a sequence of Schwartz function in Proposition 3.7. Then $\lambda_\Theta(F_n)$ (resp. $\lambda_{\theta,\theta'}(F_n)$) is an approximation of identity in $L_1(\mathbb{R}_\Theta)$ (resp. $L_1(\mathbb{R}_{\theta,\theta'})$). Take $F_n' \in \mathcal{S}(\mathbb{R}^{2d})$ such that $\widehat{F}_{n'}(\boldsymbol{\eta},\mathbf{y}) = \widehat{F}_n(\boldsymbol{\eta},\mathbf{y})e^{-i\boldsymbol{\eta}\mathbf{y}}$. Note that $\|\widehat{F}_n\|_1 = 1$ and $\widehat{F}_n$ is supported in $|(\boldsymbol{\eta},\mathbf{y})| \le \frac{1}{n}$. When $n \to 1$,

$$\big\|\lambda_{\theta,\theta'}(F_n) - \lambda_{\theta,\theta'}(F_n')\big\|_\infty \le \big\|\widehat{F}_n' - \widehat{F}_n\big\|_1 = \int_{\mathbb{R}^{2d}} \widehat{F}_n(\boldsymbol{\eta},\mathbf{y})|1 - e^{-i\boldsymbol{\eta}\mathbf{y}}|d\boldsymbol{\eta}\,d\mathbf{y} \to 0.$$

Therefore,

$$\begin{aligned}
\tau_\Theta\big(\mathrm{Op}(a)\big) &= \lim_{n\to\infty} \tau_\Theta\big(\mathrm{Op}(a)\lambda_\Theta(F_n)\big) = \lim_{n\to\infty} \tau_{\theta,\theta'}\big(a\lambda_{\theta,\theta'}(F_n')\big)\\
&= \lim_{n\to\infty} \tau_{\theta,\theta'}\big(a\lambda_{\theta,\theta'}(F_n)\big) = \tau_{\theta,\theta'}(a).
\end{aligned}$$ ■

5. Local index formula

In this section, we discuss the spectral triple structure on $\mathbb{R}_\theta$ equipped with noncommuting partial derivatives. We first recall the definitions of semifinite spectral triple from [9].

We shall show that the noncommuting derivatives in Section 4 give a natural example of a semifinite spectral triple. The main result of this chapter is a simplified index formula and we calculate it for the Bott projector as an example.

5.1. Semifinite spectral triple

Let $\mathcal{N}$ be a von Neumann algebra equipped with a normal faithful semifinite trace τ. The τ-compact operator $\mathcal{K}(\mathcal{N},\tau)$ is defined to be the $\mathcal{N}$-norm completion of $L_1(\mathcal{N},\tau)\cap\mathcal{N}$ in $\mathcal{N}$. In our case, $\mathcal{K}(\mathbb{R}_\theta,\tau_\theta)=\mathbb{E}_\theta$. The following definition of semifinite spectral triple is from [9, Definition 2.1].

Definition 5.1. A semifinite spectral triple $(\mathcal{A},H,D)$, relative to a semifinite tracial von Neumann algebra $(\mathcal{N},\tau)$, consists of a Hilbert space H on which $\mathcal{N}$ is faithfully represented, a $*$-subalgebra $\mathcal{A}$ of $\mathcal{N}$ acting on H, and a densely defined unbounded self-adjoint operator D affiliated to $\mathcal{N}$ such that

(i) $a\cdot\operatorname{dom}D\subset\operatorname{dom}D$ for all $a\in\mathcal{A}$, so that $da:=[D,a]$ is densely defined. Moreover, da extends to a bounded operator in $\mathcal{N}$ for all $a\in\mathcal{A}$;

(ii) $a(1+D^2)^{-1/2}\in\mathcal{K}(\mathcal{N},\tau)$.

$(\mathcal{A},H,D)$ is *even* if there is an operator $\gamma\in\mathcal{N}$ such that for all $a\in\mathcal{A}$,

$$\gamma=\gamma^*,\quad \gamma^2=1,\quad \gamma a=a\gamma,\quad \text{and}\quad D\gamma+\gamma D=0.$$

$(\mathcal{A},H,D)$ is *finitely summable* if there exists $s>0$ such that $a(1+D^2)^{-\frac{s}{2}}\in L_1(\mathcal{N},\tau)$ for all $a\in\mathcal{A}$. Then

$$p=\inf\big\{s>0\mid \text{ for all } a\in\mathcal{A},\ a(1+D^2)^{-\frac{s}{2}}\in L_1(\mathcal{N},\tau)\big\}$$

is called the spectral dimension of $(\mathcal{A},H,D)$.

The subalgebra $\mathcal{A}$ plays the role of smooth functions. A spectral triple is often called compact if $\mathcal{A}$ contains the identity operator on H. In this case, the condition (ii) simplifies to the assumption that $(1+D^2)^{-1/2}$ is compact. We recall the following sufficient condition for the *smooth summability* of a semifinite spectral triple and refer to [9] for a detailed discussion of the definition.

Proposition 5.2 ([9, Proposition 2.21]). *Let $(\mathcal{A},H,D)$ be a spectral triple of spectral dimension p relative to $(\mathcal{N},\tau)$. If for all $a\in\mathcal{A}\cup[D,\mathcal{A}]$, $k\in\mathbb{N}^+$, and $s>p$,*

$$(1+D^2)^{-\frac{s}{4}}L^k(a)(1+D^2)^{-\frac{s}{4}}\in L_1(\mathcal{N},\tau),$$

then (A,H,D) is smoothly summable. *Here $L(T):=(1+D^2)^{-\frac{1}{2}}[D^2,T]$ and $L^k(T)=L(L^{k-1}(T))$.*

A quantum Euclidean space $\mathbb{R}_\theta$ equipped with its natural partial derivatives D_j was studied as the prototypical example of noncommutative semifinite spectral triple in [9,18]. The rest of this subsection is to show that a further deformation of $\mathbb{R}^d$ associated with noncommuting spatial coordinates and noncommuting derivatives also gives a semifinite spectral triple structure for $\mathbb{R}_\theta$. First, we choose the smooth subalgebra $\mathcal{A}$ to be the noncommutative Sobolev space

$$W^{1,\infty}(\mathbb{R}_\theta)=\big\{a\mid D^\alpha(a)\in L_1(\mathbb{R}_\theta)\text{ for all }\alpha\big\}.$$

In the classical case, $W^{1,\infty}(\mathbb{R}^d) \subset C_0^\infty(\mathbb{R}^d)$ by Sobolev embedding theorem (cf. [8, Chapter 4, Corollary 21]). The next lemma is a weaker analog on $\mathbb{R}_\theta$.

Lemma 5.3. *If $D^\alpha(a) \in L_1(\mathbb{R}_\theta)$ for all α, then $D^\alpha(a) \in L_p(\mathbb{R}_\theta)$ for all $1 \le p \le \infty$ and α. In particular, the unitalization $W^{1,\infty}(\mathbb{R}_\theta)^\sim := (W^{1,\infty}(\mathbb{R}_\theta) + \mathbb{C})$ is a dense $*$-subalgebra of $\mathbb{E}_\theta^\sim$ closed under holomorphic functional calculus.*

Proof. Denote $\Delta = \sum_j D_{x_j}^2$. For $\lambda_\theta(f) \in \mathcal{S}_\theta$,

$$(1+\Delta)\lambda_\theta(f) = \lambda_\theta\big((1+\Delta)f\big) = \int \langle \eta \rangle^2 \hat{f}(\eta)\lambda_\theta(\eta)d\eta.$$

Choose an integer $2n > d$, then we have that $(1+\Delta)^{-n} : L_2(\mathbb{R}_\theta) \to L_\infty(\mathbb{R}_\theta)$ is bounded because

$$\begin{aligned}\big\|(1+\Delta)^{-n}\lambda_\theta(f)\big\| &= \Big\| \int \langle \eta\rangle^{-n} \hat{f}(\eta)\lambda_\theta(\eta)d\eta \Big\| \le \big\|\langle\eta\rangle^{-n}\hat{f}\big\|_1 \\ &\le \|\langle\eta\rangle^{-n}\|_2 \big\|\hat{f}\big\|_2 = \big\|\langle\eta\rangle^{-n}\big\|_2 \big\|\lambda_\theta(f)\big\|_2.\end{aligned}$$

By duality, we also have that $(1+\Delta)^{-n} : L_1(\mathbb{R}_\theta) \to L_2(\mathbb{R}_\theta)$ is bounded. Indeed, for any $\lambda_\theta(f), \lambda_\theta(g) \in \mathcal{S}_\theta$,

$$\begin{aligned}\big\langle \lambda_\theta(g), (1+\Delta)^{-n}\lambda_\theta(f)\big\rangle_{\tau_\theta} &= \big\langle (1+\Delta)^{-n}\lambda_\theta(g), \lambda_\theta(f)\big\rangle_{\tau_\theta} \\ &\le \big\|(1+\Delta)^{-n}\lambda_\theta(g)\big\|_\infty \big\|\lambda_\theta(f)\big\|_1 \le C\big\|\lambda_\theta(g)\big\|_2 \big\|\lambda_\theta(f)\big\|_1.\end{aligned}$$

Here we have used the fact $(1+\Delta)^{-n}$ admits self-adjoint extension when initially defined on $\mathcal{S}_\theta$. Thus we have that $(1+\Delta)^{-n} : L_1(\mathbb{R}_\theta) \to L_\infty(\mathbb{R}_\theta)$ is continuous. If $D^\alpha(a) \in L_1(\mathbb{R}_\theta)$ for all $|\alpha| \le 2n$, then $(1+\Delta)^n(a) \in L_1(\mathbb{R}^d)$ and hence $a \in L_\infty(\mathbb{R}_\theta)$. Therefore $W^{1,\infty}(\mathbb{R}_\theta)$ is closed under product hence a subalgebra of $\mathbb{E}_\theta$. It is dense because $\mathcal{S}_\theta \subset W^{1,\infty}(\mathbb{R}_\theta)$. To show that $W^{1,\infty}(\mathbb{R}_\theta)$ is closed under holomorphic calculus, it suffices to consider the resolvent $(\lambda - a)^{-1}$ for $\lambda \notin \mathrm{Spec}(a)$. Indeed, $(\lambda - a)^{-1}$ is bounded and

$$\lambda^{-1} - (\lambda - a)^{-1} = \lambda^{-1}\big((\lambda - a) - \lambda\big)(\lambda - a)^{-1} = -\lambda^{-1}a(\lambda - a)^{-1} \in L_1(\mathbb{R}_\theta).$$

For the derivatives,

$$\big[D_j, (\lambda - a)^{-1}\big] = (\lambda - a)^{-1}[D_j, a](\lambda - a)^{-1} \in L_1.$$

For higher-order derivatives D^α, we use induction and the Leibniz rule

$$\begin{aligned}&D^\alpha\big((\lambda - a)^{-1}\big) \\ &\quad = D^\alpha\big((\lambda - a)^{-1}(\lambda - a)(\lambda - a)^{-1}\big) \\ &\quad = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} D^{\alpha_1}\big((\lambda - a)^{-1}\big) D^{\alpha_2}(\lambda - a) D^{\alpha_3}\big((\lambda - a)^{-1}\big).\end{aligned}$$ ∎

The above lemma implies that the inclusion $W^{1,\infty}(\mathbb{R}_\theta) \subset \mathbb{E}_\theta$ induces an isomorphism on K-groups (cf. [10, p. 292]). In particular, every projection (resp. unitary) in $\mathbb{E}_\theta^\sim$ or

$M_n(\mathbb{E}_\theta^\sim)$ can be approximated using projections (resp. unitary) in $W^{1,\infty}(\mathbb{R}_\theta)^\sim$. To verify the finite and smooth summability conditions, we need the following lemma.

Lemma 5.4. *Let $a \in W^{1,\infty}(\mathbb{R}_\theta)$. Then $\langle\xi\rangle^{-\frac{r}{2}} a \langle\xi\rangle^{-\frac{r}{2}}, a\langle\xi\rangle^{-r} \in L_1(\mathbb{R}_\Theta)$ if $r > d$.*

Proof. We write a as $a = a_1 a_2$ with $a_1, a_2 \in L_2(\mathbb{R}_\theta)$. Then

$$\langle\xi\rangle^{-\frac{r}{2}} a \langle\xi\rangle^{-\frac{r}{2}} = \big(\langle\xi\rangle^{-\frac{r}{2}} a_1\big)\big(a_2 \langle\xi\rangle^{-\frac{r}{2}}\big) \in L_1(\mathbb{R}_\Theta)$$

because

$$\begin{aligned} \big\|\langle\xi\rangle^{-\frac{r}{2}} a_1\big\|_{L_2(\mathbb{R}_\theta)} &= \big\|\langle\xi\rangle^{-\frac{r}{2}}\big\|_{L_2(\mathbb{R}_{\theta'})} \|a_1\|_{L_2(\mathbb{R}_\theta)}, \\ \big\|a_2 \langle\xi\rangle^{-\frac{r}{2}}\big\|_{L_2(\mathbb{R}_\theta)} &= \big\|\langle\xi\rangle^{-\frac{r}{2}}\big\|_{L_2(\mathbb{R}_{\theta'})} \|a_2\|_{L_2(\mathbb{R}_\theta)}. \end{aligned}$$

Note that $\langle\xi\rangle^{-\frac{r}{2}}[a, \langle\xi\rangle^{-\frac{r}{2}}] = \langle\xi\rangle^{-\frac{r}{2}} a \langle\xi\rangle^{-\frac{r}{2}} - a\langle\xi\rangle^{-r}$. To show that $\langle\xi\rangle^{-\frac{r}{2}}[a, \langle\xi\rangle^{-\frac{r}{2}}] \in L_1(\mathbb{R}_\Theta)$, choose n such that $2n > \frac{r}{2}$ and write $s = \frac{r}{4n}$. Using the fractional power formula,

$$\begin{aligned} &\langle\xi\rangle^{-\frac{r}{2}}\big[a, \langle\xi\rangle^{-\frac{r}{2}}\big] \\ &= C_s \langle\xi\rangle^{-\frac{r}{2}} \int_0^\infty t^{-s}\big[a, \big(t + \langle\xi\rangle^{2n}\big)^{-1}\big] dt \\ &= C_s \langle\xi\rangle^{-\frac{r}{2}} \int_0^\infty t^{-s}\big(t + \langle\xi\rangle^{2n}\big)^{-1}\big[a, t + \langle\xi\rangle^{2n}\big]\big(t + \langle\xi\rangle^{2n}\big)^{-1}\, dt \\ &= C_s \int_0^\infty t^{-s}\big(t + \langle\xi\rangle^{2n}\big)^{-1}\big(\langle\xi\rangle^{-\frac{r}{2}}\big[a, \langle\xi\rangle^{2n}\big]\langle\xi\rangle^{-2n}\big)\langle\xi\rangle^{2n}\big(t + \langle\xi\rangle^{2n}\big)^{-1}\, dt. \end{aligned}$$

Here C_s is some positive constant depending on s. Since $[a, \langle\xi\rangle^{2n}]$ is a linear combination of a's derivatives, we know that

$$\langle\xi\rangle^{-\frac{r}{2}}\big[a, \langle\xi\rangle^{2n}\big]\langle\xi\rangle^{-2n} \in L_1(\mathbb{R}_\Theta).$$

Then the integral converges in L_1-norm,

$$\begin{aligned} &\big\|\langle\xi\rangle^{-\frac{r}{2}}\big[a, \langle\xi\rangle^{-\frac{r}{2}}\big]\big\|_1 \\ &\lesssim \int_0^\infty t^{-s}\big\|\big(t+\langle\xi\rangle^{2n}\big)^{-1}\big\|_\infty \big\|\langle\xi\rangle^{-\frac{r}{2}}\big[a, \langle\xi\rangle^{2n}\big]\langle\xi\rangle^{-2n}\big\|_1 \big\|\langle\xi\rangle^{2n}\big(t+\langle\xi\rangle^{2n}\big)^{-1}\big\|_\infty\, dt \\ &\lesssim \int_0^\infty t^{-s}(t+1)^{-1} dt < \infty. \end{aligned}$$

■

Recall that the Clifford algebra Cl^d is generated by d self-adjoint operators $c_1, \ldots, c_d$ satisfying the anti-commutation relation $c_j c_k + c_k c_j = 2\delta_{j,k}$. For $d = 2n$ even, Cl^d is isomorphic to the $N \times N$ matrix algebra M_N with $N = 2^n$. For $d = 2n+1$ odd, Cl^d is isomorphic to $M_{2^n} \oplus M_{2^n} \subset M_N$ with $N = 2^{n+1}$. When d is even, Cl^d is $\mathbb{Z}_2$ graded with the parity element $\gamma = (-i)^{\frac{d}{2}} c_1 \cdots c_d$.

Theorem 5.5. *$(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta)\otimes\mathbb{C}^N, \sum_j \xi_j\otimes c_j)$ relative to $(\mathbb{R}_\Theta\otimes M_N, \tau_\Theta\otimes \mathrm{tr})$ is a smooth summable semifinite spectral triple with spectral dimension d. Moreover, it is even if $d = 2n$ is even, and $\gamma = (-i)^{\frac{d}{2}} c_1\cdots c_d$.*

Proof. Note that

$$D^2 = \sum_{j,k}\xi_j\xi_k\otimes c_jc_k = \sum_j \xi_j^2 - \frac{i}{2}\sum_{j,k}\theta'_{j,k}c_jc_k.$$

Denote $\omega = \frac{i}{2}\sum_{j,k}\theta'_{j,k}c_jc_k$. Then $1 + D^2 = \langle\xi\rangle^2 - \omega$. Since $\omega\in M_N$ commutes with $\mathbb{R}_\Theta$, to verify summability it is equivalent to replace $1 + D^2$ by $\langle\xi\rangle^2$. Lemma 5.4 shows that $a\langle\xi\rangle^{-r}\in L_1(\mathbb{R}_\Theta)$ if $r > d$. For the converse direction, suppose that, for some $r > 0$, $a\langle\xi\rangle^{-r}\in L_1(\mathbb{R}_\Theta)$ for any $a\in W^{\infty,1}(\mathbb{R}_\theta)$. We know that $\langle\xi\rangle^{-r}\le 1$, $\langle\xi\rangle^{-r}\in L_s(\mathbb{R}_{\theta'})$ for $s > d/r$. Take e_n as the spectral projection of $\langle\xi\rangle^{-r}$ on the interval $[1/n, 1]$ and $b_n := e_n\langle\xi\rangle^{-r}$. We have $\tau(e_n) < \infty$, $\tau(b_n) < \infty$, and $\tau(b_n)$ is monotonically increasing. For any nonzero $a\in\mathcal{S}_\theta$ and $n\ge 1$, by Lemma 4.17

$$\begin{aligned}\|a\|^2_{L_2(\mathbb{R}_\theta)}\|b_n^{1/2}\|^2_{L_2(\mathbb{R}_{\theta'})} &= \|ab_n^{1/2}\|^2_{L_2(\mathbb{R}_\Theta)}\\ &= \tau_\Theta(ab_na^*) \le \tau_\Theta\big(a\langle\xi\rangle^{-r}a^*\big) \le \big\|a\langle\xi\rangle^{-r}\big\|_1\|a^*\|_\infty < \infty.\end{aligned}$$

By Fatou's lemma, This implies that

$$\big\|\langle\xi\rangle^{-r}\big\|_{L_1(\mathbb{R}'_\theta)} = \lim_{n\to\infty}\tau_{\theta'}\big(e_n\langle\xi\rangle^{-r}\big) = \lim_{n\to\infty}\|b_n^{1/2}\|^2_{L_2(\mathbb{R}_{\theta'})} < \infty.$$

By Proposition 2.6, this implies that $r > d$. Thus we prove that the spectral dimension is d. For smooth summability, we know that $[\langle\xi\rangle^2, a]\in L_1(\mathbb{R}_\theta)$ and by Lemma 5.4 again,

$$(1 + D^2)^{-\frac{s}{2}}L(a)(1 + D^2)^{-\frac{s}{2}}\in L_1(\mathbb{R}_\Theta)$$

if $s > d$. The arguments for $L^k(a)$ are similar. ∎

5.2. Local index formula

We briefly recall the local index formula for the even case and refer to [9, 13] for detailed information. Let $(\mathcal{A}, H, D)$ be an even spectral triple relative to $(\mathcal{N}, \tau)$ and γ is the parity element. Denote $H_+ = \frac{\gamma+1}{2}H$ and $H_- = \frac{1-\gamma}{2}H$. For $\mu > 0$, define $D_\mu = \left[\begin{smallmatrix} D & \mu\\ \mu & D\end{smallmatrix}\right]$ on $H\oplus H$. Write $F_\mu = D_\mu|D_\mu|^{-1}$ and

$$(F_\mu)_+ = \left(\frac{1+\gamma}{2}\otimes I_2\right)F_\mu : H_+\oplus H_+\to H_-\oplus H_-. \tag{5.1}$$

Here and in the following, I_n represents the n-dimensional identity matrix. For a projection $e\in M_n(\mathcal{A}^\sim)$, denote $\hat{e} = \left[\begin{smallmatrix} e & 0\\ 0 & 1_e\end{smallmatrix}\right]\in M_{2n}(\mathcal{A}^\sim)$, where $1_e\in M_n(\mathbb{C})$ is the rank element of e. Following [9, Definition 2.12 and Proposition 2.13], the numerical index pairing between the $K_0(\mathcal{A})$ element $[e] - [1_e]$ and the even spectral triple $(\mathcal{A}, H, D)$ is given by

$$\big\langle [e] - [1_e], (\mathcal{A}, H, D)\big\rangle = \mathrm{index}_{\tau\otimes\mathrm{tr}_{2n}}\big(\hat{e}(F_{\mu,+}\otimes I_n)\hat{e}\big).$$

Here the numerical index $\text{index}_\tau(F) = \tau(\ker F) - \tau(\text{coker } F)$ is defined as the trace of kernel subtracting the trace of cokernel. Both quantities are invariants under homotopy. The local index formula express the index pairings by the following residue cocycle formulas.

Definition 5.6. $(\mathcal{A}, H, D)$ has *isolated spectral dimension* if for all $a_0, \dots, a_m \in \mathcal{A}$, the zeta function

$$\zeta(z) = \tau(\gamma a_0 da_1^{(k_1)} \cdots da_m^{(k_m)} (1 + D^2)^{-|k|-m/2-z})$$

has an analytic continuation to a deleted neighborhood of $z = 0$.

Here we introduce the notation

$$da := [D, a], \quad da^{(k)} := \underbrace{[D^2, [D^2, \dots, [D^2}_{k\text{-times}}, da]]].$$

Let $(\mathcal{A}, H, D)$ be a smoothly summable semifinite spectral triple with spectral dimension d and let M be the largest integer in $[0, d + 1]$. Suppose that $\mathcal{A}$ has an isolated spectral dimension. The residue cocycle $\phi_m : \mathcal{A}^{\otimes m+1} \to \mathbb{C}$ is the $(m + 1)$-linear form given by

$$\phi_0(a_0) = \text{Res}_{z=0}\, z^{-1} \tau\big(\gamma a_0 (1 + D^2)^{-z}\big), \tag{5.2}$$

$$\begin{aligned} \phi_m(a_0, \dots, a_m) = \sum_{|k|=0}^{M-m} (-1)^{|k|} \alpha(k) \sum_{j=0}^{|k|+m/2} \sigma_{|k|+m/2, j}\, \text{Res}_{z=0}\, z^{j-1} \\ \times \tau\big(\gamma a_0 da_1^{(k_1)} \cdots da_m^{(k_m)} (1 + D^2)^{-|k|-m/2-z}\big), \end{aligned} \tag{5.3}$$

where $\alpha(k)$, $\sigma_{|k|+m/2,j}$ are the constant defined as follows. For a multi-index $k = (k_1, \dots, k_m)$,

$$\alpha(k) = k_1! k_2! \cdots k_m! / (k_1 + 1)(k_1 + k_2 + 2) \cdots \big(|k| + m\big) \tag{5.4}$$

$\sigma_{n,j}$ are the nonnegative constant given by the equation

$$\prod_{j=0}^{n-1} (z + j) = \sum_{j=1} \sigma_{n,j} z^j. \tag{5.5}$$

In particular, $\alpha(0) = m!$ and $\sigma_{n,1} = (n - 1)!$. The terms in ϕ_m is a linear combination of residue and higher-order residue of the zeta function

$$\zeta(z) = \tau\big(\gamma a_0 da_1^{(k_1)} \cdots da_m^{(k_m)} (1 + D^2)^{-|k|-m/2-z}\big).$$

The isolated spectral dimension condition assumes that these residues are well defined.

Theorem 5.7 ([9, Theorem 3.33] (even case)). *Let $(\mathcal{A}, H, D)$ relative to $(\mathcal{N}, \tau)$ be an even smoothly summable semifinite spectral triple. Suppose that $(\mathcal{A}, H, D)$ has an isolated spectral dimension. Then the numerical index pairing can be computed by*

$$\big\langle [e] - [1_e], \big[(\mathcal{A}, H, D)\big] \big\rangle = \sum_{m=0,\ even}^{M} \phi_m\big(Ch^m(e) - Ch^m(1_e)\big),$$

where for a projection $e \in M_n(\mathcal{A}^\sim)$, $Ch_0(e) = (e)$ *and*

$$Ch^{2k}(e) = (-1)^k \frac{2k!}{k!}\left(e - \frac{1}{2}\right) \otimes e \otimes \cdots \otimes e \in \mathcal{A}^{\otimes 2k+1}.$$

We shall now calculate the local index formula for the spectral triple $(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, \sum_j \xi_j \otimes c_j)$. Recall that $\omega = \frac{i}{2}\sum \theta'_{jk} c_j c_k$ is the analog of curvature form. Let us denote the super trace on Cl^d as $\mathrm{str}(a) = \mathrm{tr}(\gamma a)$ and the corresponding super trace on $\mathbb{R}_\Theta \otimes \mathrm{Cl}^d$ (resp. $\mathbb{R}_\theta \otimes \mathrm{Cl}^d$) as $\mathrm{Str}_\Theta = \tau_\Theta \otimes \mathrm{str}$ (resp. $\mathrm{Str}_\theta = \tau_\theta \otimes \mathrm{str}$).

Theorem 5.8. *Let d be even. The spectral triple* $(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, \sum_j \xi_j \otimes c_j)$ *has an isolated spectral dimension. Moreover, for* $a_0, \ldots, a_m \in W^{\infty,1}(\mathbb{R}_\theta)^\sim$,

$$\phi_m(a_0, \ldots, a_m) = \begin{cases} \frac{\pi^{\frac{d}{2}}}{m!}\mathrm{Str}_\theta\left(a_0 da_1 \cdots da_m \frac{\omega^{\frac{d-m}{2}}}{\frac{(d-m)}{2}!}\right), & \text{if } m \text{ even},\\ 0, & \text{if } m \text{ odd}.\end{cases}$$

Denote $\Psi_k = a_0 da_1^{(k_1)} \cdots da_m^{(k_m)}$. The cocycle ϕ_m is a linear combination of residue of the following zeta functions at $z = 0$:

$$\zeta_k(z) = \mathrm{Str}_\Theta\left(\Psi_k (1 + D^2)^{-|k| - \frac{m}{2} - z}\right).$$

We first show that the residue for nonzero k vanishes.

Lemma 5.9. *For* $k \neq 0$, $\mathrm{Res}_{z=0}\, \zeta_k(z) = 0$.

Proof. Denote $\Delta(a) = \sum_j D_j^2(a)$. For $a \in W^{1,\infty}(\mathbb{R}_\theta)$,

$$\left[|\xi|^2, D_j(a)\right] = \Delta(D_j a) + 2\sum_k D_k D_j(a)\xi_k \in \Sigma^1,$$

$$[D^2, da] = \left[|\xi|^2 - \omega, \sum_j D_j(a) \otimes c_j\right]$$

$$= \sum_j \left[|\xi|^2, D_j(a)\right] \otimes c_j + \sum_j D_j(a) \otimes [\omega, c_j] \in \Sigma^1 \otimes \mathrm{Cl}^d$$

is a ΨDO of order 1. Note that for any j_1, j_2, and j_3, $[c_{j_1} c_{j_2}, c_{j_3}] = 0$ or of Clifford order 1. Then $da^{(1)}$ is of Clifford order 1 and similarly for $da^{(k_0)}$. Thus $\Psi_k = a_0 da_1^{(k_1)} \cdots da_m^{(k_m)} \in \Sigma^k$ with Clifford term of at most order m. Moreover, by using the commutator relation $[\xi_j, a] = D_j(a)$, $\Psi_k = \sum_{|\beta| \le k} b_\beta \xi_j^\beta \otimes \nu_\beta$ for some $b_\beta \in W^{1,\infty}(\mathbb{R}_\theta)$ and $\nu_\beta \in \mathrm{Cl}^d$ of Clifford order at most m. Thus it suffices to show that the zeta function

$$\rho(z) := \mathrm{Str}_\Theta\left((b\xi^\beta \otimes \nu)(1 + D^2)^{-|k| - \frac{m}{2} - z}\right) \tag{5.6}$$

has zero residue at $z = 0$, for any $b \in W^{1,\infty}(\mathbb{R}_\theta)$, $|\beta| \le k$, and ν of Clifford order at most m. Recall that $1 + D^2 = \langle\xi\rangle^2 - \omega$ commutes with $\langle\xi\rangle$. Then, for any $r > 0$,

$$\begin{aligned}\left\|\langle\xi\rangle^r (1 + D^2)^{-r/2}\right\|_\infty &= \left\|\langle\xi\rangle^r (1 + D^2)^{-r} \langle\xi\rangle^r\right\|_\infty\\ &\le \left(1 + \|\omega\|_\infty\right)^r \left\|\langle\xi\rangle^r \left(1 + \|\omega\|_\infty + D^2\right)^{-r} \langle\xi\rangle^r\right\|_\infty\\ &\le \left(1 + \|\omega\|_\infty\right)^r \left\|\langle\xi\rangle^r \langle\xi\rangle^{-2r} \langle\xi\rangle^r\right\|_\infty = \left(1 + \|\omega\|_\infty\right)^r.\end{aligned}$$

By Theorem 3.2, $\langle\xi\rangle^{r-|\beta|}\xi^\beta\langle\xi\rangle^{-r}$ is bounded. Combined with Lemma 5.4, we have

$$(b\xi^\beta\otimes\nu)(1+D^2)^{-r}=\big(b\xi^\beta\langle\xi\rangle^{-2r}\otimes\nu\big)\big(\langle\xi\rangle^{2r}(1+D^2)^{-r}\big)\in L_1(\mathbb{R}_\Theta\otimes M_N)$$

for $2r-|\beta|>d$. This implies that it suffices to consider the nonzero residue of (5.6) at $z=0$ for $m+2|k|\le d+|\beta|$. Applying Cahen–Mellin integral,

$$(1+D^2)^{-|k|-\frac{m}{2}-z}=\frac{1}{\Gamma(|k|+\frac{m}{2}+z)}\int_0^\infty e^{-s(1+D^2)}s^{|k|+\frac{m}{2}+z-1}\,ds. \tag{5.7}$$

For $2r-|\beta|>d$,

$$\big\|(b\xi^\beta\otimes\nu)e^{-s(1+D^2)}\big\|_{L_1(\mathbb{R}_\Theta\otimes M_N)}\le e^{-s}\big\|(b\xi^\beta\otimes\nu)(1+D^2)^{-r}\big\|_1\big\|(1+D^2)^re^{-sD^2}\big\|_\infty.$$

By functional calculus,

$$\big\|(1+D^2)^re^{-sD^2}\big\|_\infty\le\max\left\{1,\frac{r^r}{s^r}\right\}.$$

Then the integral

$$\int_0^\infty\big\|(b\xi^\beta\otimes\nu)e^{-s(1+D^2)}\big\|_{L_1(\mathbb{R}_\Theta\otimes M_N)}s^{|k|+\frac{m}{2}+z-1}\,ds$$

converges for $|k|+\frac{m}{2}+\mathrm{Re}(z)>(d+|\beta|)/2$. Hence by Fubini's theorem

$$\rho(z)=\int_0^\infty\mathrm{Str}_\Theta\left((b\xi^\beta\otimes\nu)e^{-s(1+D^2)}\right)s^{|k|+m/2+z-1}\,ds.$$

Using the trace formula from Proposition 4.19,

$$\begin{aligned}\mathrm{Str}_\Theta\left((b\xi^\beta\otimes\nu)e^{-s(1+D^2)}\right)&=\mathrm{Str}_\Theta\left((b\xi^\beta\otimes\nu)(e^{-s(1+|\xi|^2)}\otimes e^{-s\omega})\right)\\&=\tau_{\theta'}(\xi^\beta e^{-s(1+|\xi|^2)})\,\mathrm{Str}_\theta(b\otimes\nu e^{s\omega})\\&=\sum_n\mathrm{Str}_\theta\left(b\otimes\nu\frac{\omega^n}{n!}\right)s^ne^{-s}\tau_{\theta'}(\xi^\beta e^{-s|\xi|^2}).\end{aligned}$$

Applying Theorem 3.2 and Proposition 2.6,

$$\begin{aligned}\tau_{\theta'}(\xi^\beta e^{-s|\xi|^2})&\le\big\|\xi^\beta\langle\xi\rangle^{-|\beta|}\big\|_\infty\big\|\langle\xi\rangle^{-|\beta|}e^{-\frac{s}{2}|\xi|^2}\big\|_\infty\tau_\theta(e^{-\frac{s}{2}|\xi|^2})\\&\le\big\|\xi^\beta\langle\xi\rangle^{-|\beta|}\big\|_\infty\Big(\sup_{w\ge0}(1+w)^{-|\beta|/2}e^{-sw}\Big)(s/2)^{-\frac{d}{2}}\det\left(\frac{\pi i\frac{s}{2}\theta}{\sinh(i\frac{s}{2}\theta)}\right)^{1/2}\\&\le C_{\beta,\theta'}s^{-\frac{|\beta|}{2}}s^{-\frac{d}{2}}=C_{\beta,\theta'}s^{-\frac{d+|\beta|}{2}},\end{aligned}$$

where $C_{\beta,\theta'}$ is a constant only depending on β and θ'. Then for $n\ge0$, the integral

$$\int_0^\infty\big|e^{-s}s^n\tau_{\theta'}(\xi^\beta e^{-s|\xi|^2})s^{|k|+m/2+z-1}\big|\,ds\le\int_0^\infty e^{-s}s^{n-(d+|\beta|)/2+|k|+m/2+\mathrm{Re}(z)-1}\,ds$$

converges absolutely if $\mathrm{Re}(z) > (d + |\beta|)/2 - n - |k| - m/2$. Since $|\beta| \leq |k|$, we have the residue

$$\mathrm{Res}_{z=0} \int_0^\infty \mathrm{Str}_\theta(b \otimes \nu\omega^n)\tau_{\theta'}(\xi^\beta e^{-s|\xi|^2}) s^{n-\frac{d}{2}+|k|+m/2+z-1} e^{-s}\, ds = 0$$

if $n + |k|/2 + m/2 - d/2 > 0$. On the other hand, $\nu\omega^n$ contains Clifford elements of order at most $m + 2n$, then the super trace $\mathrm{Str}_\theta(b \otimes \nu\omega^n) = 0$ for $2n + m < d$. Hence for $|k| > 0$, the above residue vanishes for every n and hence

$$\begin{aligned} \mathrm{Rez}_{z=0}\, \rho(z) &= \mathrm{Rez}_{z=0} \int_0^\infty \mathrm{Str}_\Theta \left((b\xi^\beta \otimes \nu)e^{-s(1+D^2)}\right) s^{|k|+m/2+z-1}\, ds \\ &= \sum_n \frac{1}{n!} \mathrm{Rez}_{z=0} \int_0^\infty \sum_n \mathrm{Str}_\theta(b \otimes \nu\omega^n) s^n e^{-s} \tau_{\theta'}(\xi^\beta e^{-s|\xi|^2}) ds = 0. \quad \blacksquare \end{aligned}$$

We shall now calculate the residue for $k = 0$.

Proof of Theorem 5.8. We first consider the case $m > 0$. Denote $\Psi_0 = a_0 da_1 \cdots da_m \in W^{1,\infty}(\mathbb{R}^\theta) \otimes \mathrm{Cl}^d$. Using the trace formula from Proposition 4.19,

$$\begin{aligned} \mathrm{Str}_\Theta(\Psi_0 e^{-s(1+D^2)}) &= \mathrm{Str}_\Theta\left(\Psi_0(e^{-s(1+|\xi|^2)} \otimes e^{-s\omega})\right) = \mathrm{tr}_{\theta'}(e^{-s(1+|\xi|^2)})\, \mathrm{Str}_\theta(\Psi_0 e^{s\omega}) \\ &= \sum_n \mathrm{Str}_\theta\left(\Psi_0 \frac{\omega^n}{n!}\right) \pi^{\frac{d}{2}} e^{-s} s^{n-\frac{d}{2}} h(s). \end{aligned}$$

Here we used again the calculation in Proposition 2.6 that

$$\mathrm{tr}_{\theta'}(e^{-s|\xi|^2}) = s^{-\frac{d}{2}} \det\left(\frac{i\pi s\theta'}{\sinh is\theta'}\right)^{\frac{1}{2}} = s^{-\frac{d}{2}} \pi^{\frac{d}{2}} h(s),$$

where

$$h(s) = \det\left(\frac{is\theta'}{\sinh is\theta'}\right) = \prod_{j=1}^{l} \frac{\lambda_j s}{\sinh \lambda_j s},$$

where $i\lambda_1, -i\lambda_1, \ldots, i\lambda_l, -i\lambda_l$ are the nonzero eigenvalues of θ'. Using L'Hospital's rule, we know $\lim_{s\to 0} s^{-1}(h(s) - 1) = 0$. Then we split the residue into two parts:

$$\begin{aligned} \mathrm{Res}_{z=0}\, \zeta_0(z) &= \mathrm{Res}_{z=0}\, \mathrm{Str}_\Theta(\Psi_0(1 + D^2)^{-m/2-z}) \\ &= \mathrm{Res}_{z=0} \frac{1}{\Gamma(m/2 + z)} \int_0^\infty \mathrm{Str}_\Theta(\Psi_0 e^{-s(1+D^2)}) s^{m/2+z-1}\, ds \\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma(m/2)} \sum_n \frac{1}{n!} \mathrm{Str}_\theta(\Psi_0\omega^n) \bigg(\mathrm{Res}_{z=0} \int_0^\infty e^{-s} s^{n-\frac{d}{2}+m/2+z-1}\, ds \\ &\quad + \mathrm{Res}_{z=0} \int_0^\infty (h(s) - 1) e^{-s} s^{n-\frac{d}{2}+m/2+z-1}\, ds \bigg). \end{aligned}$$

Since $\Psi_0\omega^n$ contains Clifford elements of order at most $m + 2n$, the super trace

$$\mathrm{Str}_\theta(\Psi_0\omega^n) = 0 \quad \text{for } 2n + m < d.$$

On one hand, for $2n + m \geq d$,

$$\operatorname{Res}_{z=0} \int_0^\infty (h(s) - 1) e^{-s} s^{n-\frac{d}{2}} s^{m/2+z-1}\, ds$$
$$= \operatorname{Res}_{z=0} \int_0^\infty \frac{h(s)-1}{s} e^{-s} s^{n-\frac{d}{2}+m/2+z}\, ds = 0 \tag{5.8}$$

because the integral converges absolutely for $Re(z) > -1 \geq -n + \frac{d}{2} - m/2 - 1$. For the other residue

$$\operatorname{Res}_{z=0} \int_0^\infty e^{-s} s^{n-\frac{d}{2}+m/2+z-1}\, ds = \operatorname{Res}_{z=0} \Gamma\Big(n - \frac{d}{2} + m/2 + z\Big)$$

is zero if $n - \frac{d}{2} + m/2 > 0$. Therefore, the only nonzero residue is at $2n + m - d = 0$ and it is a simple pole. Then ϕ_m vanishes for odd m and for even $m \geq 2$,

$$\begin{aligned}
\phi_m(a_0, \ldots, a_m) &= \alpha(0)\sigma_{\frac{m}{2},1} \operatorname{Res}_{z=0} \zeta_0(z) \\
&= \frac{\Gamma(m/2)}{m!} \frac{\pi^{\frac{d}{2}}}{\Gamma(m/2)} \operatorname{Res}_{z=0} \Gamma(z) \operatorname{Str}_\theta \left(\Psi_0 \frac{\omega^{(d-m)/2}}{\frac{d-m}{2}!}\right) \\
&= \frac{\pi^{\frac{d}{2}}}{m!} \operatorname{Str}_\theta \left(a_0 da_1 \cdots da_m \frac{\omega^{(d-m)/2}}{\frac{d-m}{2}!}\right).
\end{aligned}$$

For $m = 0$, we follow the same argument:

$$\begin{aligned}
\phi_0(a_0) &= \operatorname{Res}_{z=0} z^{-1} \operatorname{Str}_\Theta(a_0(1 + D^2)^{-z}) \\
&= \operatorname{Res}_{z=0} z^{-1} \frac{1}{\Gamma(z)} \int_0^\infty \operatorname{Str}_\Theta(a_0 e^{-s(1+D^2)}) s^{z-1}\, ds \\
&= \operatorname{Res}_{z=0} \frac{1}{z\Gamma(z)} \int_0^\infty \tau_\theta(a_0)\tau_{\theta'}(e^{-s|\xi|^2}) \operatorname{str}(e^{s\omega}) e^{-s} s^{z-1}\, ds \\
&= \tau_\theta(a_0) \operatorname{Res}_{z=0} \frac{1}{\Gamma(z+1)} \int_0^\infty \sum_{n=0} \frac{\operatorname{str}(\omega^n)}{n!} h(s) e^{-s} \pi^{\frac{d}{2}} s^{n-\frac{d}{2}+z-1}\, ds \\
&= \pi^{\frac{d}{2}} \tau_\theta(a_0) \sum_{n=0} \frac{\operatorname{str}(\omega^n)}{n!} \Big(\operatorname{Res}_{z=0} \int_0^\infty e^{-s} s^{n-\frac{d}{2}+z}\, ds \\
&\quad + \operatorname{Res}_{z=0} \int_0^\infty \big(h(s) - 1\big) e^{-s} s^{n-\frac{d}{2}+z-1}\, ds\Big).
\end{aligned}$$

The only nonzero residue is the first term with $n = d/2$ and

$$\operatorname{Res}_{z=0} \int_0^\infty e^{-s} s^{n-\frac{d}{2}+z-1}\, ds = \operatorname{Res}_{z=0} \Gamma(z) = 1.$$

Therefore, $\phi_0(a_0) = \pi^{d/2} \operatorname{Str}_\theta(a_0 \frac{\omega^{d/2}}{(d/2)!})$. ∎

For compact spin manifolds, the isolated spectral dimension condition always holds and the only nonzero residues are when $j = 0$ and $k = 0$. This simplification recovers the Atiyah–Singer index theorem for a Dirac operator associated with a spin structure (see [13, 25, 37]). The above theorem gives a simplification of the cocycle formula for

$$\Big(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, \sum \xi_j \otimes c_j\Big)$$

to the terms only for $|k| = j = 0$. As a consequence, the local index formula for $\mathbb{R}_\theta$ simplifies too. We can see the term ω playing the role of the curvature form.

Corollary 5.10. *For any projection $e \in M_n(W^{\infty,1}(\mathbb{R}_\theta))$ and with $F_{\mu,+}$ defined as in* (5.1),

$$\mathrm{Index}\,\big(e(F_{\mu,+} \otimes \mathrm{id}_n)e\big) = \pi^{\frac{d}{2}}\,\mathrm{Str}_\theta\left((e - 1_e)\frac{\omega^{\frac{d}{2}}}{\frac{d}{2}!} + \sum_{m=2,even}^{d} \frac{1}{m!} e(de)^m \frac{\omega^{\frac{d-m}{2}}}{(\frac{d-m}{2})!}\right).$$

5.3. A concrete example for $d = 2$

We shall now calculate a concrete example in dimension $d = 2$. In the classical case, a canonical generator for $K_0(C_0(\mathbb{R}^2))$ is the Bott projector

$$e_B(\mathbf{x},\mathbf{y}) = \frac{1}{1 + \mathbf{x}^2 + \mathbf{y}^2}\begin{bmatrix} 1 & \mathbf{x} - i\mathbf{y} \\ \mathbf{x} + i\mathbf{y} & \mathbf{x}^2 + \mathbf{y}^2 \end{bmatrix} \in M_2\big(C_0(\mathbb{R}^2)^\sim\big),$$
$$1_{e_B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{C}).$$

Now let θ be a real number and $\mathbb{R}_\theta$ is the Moyal plane generated by two self-adjoint elements x, y with $[x, y] = -i\theta$. We consider an analog of Bott projection for $\mathbb{R}_\theta$. Write $z = x + iy$, $R = (1 + z^*z)^{-1}$, and $u = \left[\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right]$. Then $e := u\left[\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right]u^* = \left[\begin{smallmatrix} R & Rz^* \\ zR & zRz^* \end{smallmatrix}\right]$ is a projection because $u^*Ru = 1$. The only drawback of e is that it does not belong to $M_2(W^{\infty,1}(\mathbb{R}_\theta)^\sim)$. Indeed, by Proposition 2.6 and Theorem 3.2, we know that

$$R, zR, zRz^* \notin L_1(\mathbb{R}_\theta).$$

Nevertheless, $dede$ and $\mathrm{id} \otimes \mathrm{tr}_2(e - 1_e) = R + zRz^* - 1$ do belong to L_1 so that the cocycle formula in Corollary 5.10 is well defined. The next lemma shows that by approximation the cocycle formula remains valid for e.

Lemma 5.11. *There exists a sequence of projection $e_n \in M_2(W^{\infty,1}(\mathbb{R}_\theta)^\sim)$ such that $1_{e_n} = 1_e$ and $\lim_{n\to\infty} \|e_n - e\|_\infty = 0$, $\lim_{n\to\infty} \|\mathrm{id} \otimes \mathrm{tr}_2(e_n - e)\|_1 = 0$. As a consequence,*

$$\Big\langle [e] - [1_e], \Big(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, \sum \xi_j \otimes c_j\Big)\Big\rangle$$
$$= \pi\,\mathrm{Str}_\theta\big((e - 1_e)\omega\big) + \pi\,\mathrm{Str}_\theta(edede).$$

Proof. Let $\lambda_\theta(\phi_n)$ be the approximation identity in Proposition 3.7. Define

$$\tilde{e}_n := \big(\lambda_\theta(\phi_n) \otimes 1\big)(e - 1_e) + 1_e \in M_2\big(W^{\infty,1}(\mathbb{R}_\theta)\big).$$

Because $e - 1_e \in \mathbb{E}_\theta$ and $\mathrm{id} \otimes \mathrm{tr}_2(e - 1_e) \in L_1(\mathbb{R}_\theta)$, we have

$$\begin{aligned}\|\tilde{e}_n - e\|_\infty = \big\|\big(\lambda_\theta(\phi_n) \otimes 1\big)(e - 1_e) - (e - 1_e)\big\|_\infty \to 0,\\ \|\,\mathrm{id} \otimes \mathrm{tr}_2(\tilde{e}_n - 1_e) - \mathrm{id} \otimes \mathrm{tr}_2(e - 1_e)\|_1 \to 0.\end{aligned}$$

Using holomorphic functional calculus, we can make projections $e_n \in M_2(W^{\infty,1}(\mathbb{R}_\theta))$ from $\tilde{e}_n$ which satisfies the same limits above. It is known that if two projections e, f satisfy that $\|e - f\| < 1$, then e is homotopic to f and hence $[e] = [f]$ (see, e.g., [39]). Then by the homotopy invariance of index pairing, we know that for n large enough

$$\begin{aligned}\big\langle [e] - [1_e], (\mathcal{A}, H, D)\big\rangle &= \big\langle [e_n] - [1_{e_n}], (\mathcal{A}, H, D)\big\rangle\\ &= \phi_0(e_n - 1_{e_n}) + \phi_2\Big(e_n - \frac{1}{2}, e_n, e_n\Big)\\ &= \pi\,\mathrm{Str}_\theta(e_n - 1_{e_n}\omega) + \pi\,\mathrm{Str}_\theta\Big(\Big(e_n - \frac{1}{2}\Big)de_n de_n\Big).\end{aligned}$$

Taking the limit $n \to \infty$,

$$\lim_{n\to\infty} \mathrm{Str}_\theta\big((e_n - 1_{e_n})\omega\big) = \mathrm{Str}_\theta\big((e - 1_e)\omega\big).$$

For the second term, we first note that $\mathrm{Str}_\theta(de_n de_n) = \mathrm{Str}_\theta(-de_n de_n) = 0$ because $de_n\gamma = -\gamma de_n$. For the same reason, we have the cyclicity that

$$\begin{aligned}\mathrm{Str}_\theta(ede_n de) &= \mathrm{Str}_\theta\big(d(ee_n)de\big) - \mathrm{Str}_\theta\big(d(e)e_n de\big) = \mathrm{Str}_\theta(e_n dede),\\ \mathrm{Str}_\theta(e_n dede_n) &= \mathrm{Str}_\theta\big(d(e_n e)de_n\big) - \mathrm{Str}_\theta\big(d(e_n)ede_n\big) = \mathrm{Str}_\theta\big(ede_n d(e_n)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Str}_\theta(edede) - \mathrm{Str}_\theta(e_n de_n de_n)\\ &= \mathrm{Str}_\theta(edede - e_n dede) + \mathrm{Str}_\theta(e_n dede - e_n de_n de) + \mathrm{Str}_\theta(e_n de_n de - e_n de_n de_n)\\ &= \mathrm{Str}_\theta(edede - e_n dede) + \mathrm{Str}_\theta(edede_n - e_n dee_n) + \mathrm{Str}_\theta(ede_n de_n e_n de_n de_n)\\ &= \mathrm{Str}_\theta\big((e - e_n)dede\big) + \mathrm{Str}_\theta\big((e - e_n)dede_n\big) + \mathrm{Str}_\theta\big((e - e_n)de_n de_n\big).\end{aligned}$$

All the three terms above converge to 0, since $\|e - e_n\|_\infty \to 0$ and $dede, dede_n, de_n de_n$ are in $M_2(L_1(\mathbb{R}_\theta))$. ■

Theorem 5.12. *For any θ, θ',*

$$\Big\langle [e] - [1_e], \Big(W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes \mathbb{C}^N, \sum \xi_j \otimes c_j\Big)\Big\rangle = 4\pi^2(1 - \theta\theta').$$

In particular, $[e]$ is a generator of $K_0(\mathbb{E}_\theta) = \mathbb{Z}$.

Proof. The super trace $\mathrm{Str}_\theta(edede)$ is of eight terms:

$$\begin{aligned}\mathrm{Str}_\theta(edede) &= \mathrm{Str}_\theta \otimes \mathrm{tr}_2 \left(\begin{bmatrix} R & Rz^* \\ zR & zRz^* \end{bmatrix} \begin{bmatrix} dR & d(Rz^*) \\ d(zR) & d(zRz^*) \end{bmatrix} \begin{bmatrix} dR & d(Rz^*) \\ d(zR) & d(zRz^*) \end{bmatrix} \right) \\ &= \mathrm{Str}_\theta \big(Rd(R)d(R) + Rd(Rz^*)d(zR) + Rz^*d(zR)d(R) \\ &\quad + Rz^*d(zRz^*)d(zR) + zRd(R)d(Rz^*) + zRd(Rz^*)d(zRz^*) \\ &\quad + zRz^*d(zR)d(Rz^*) + zRz^*d(zRz^*)d(zRz^*)\big).\end{aligned}$$

We will repeatedly use the Leibniz rule and cyclicity of trace (in the strong sense [7, Theorem 17]) that

$$d(a_1a_2) = (da_1)a_2 + a_1da_2, \quad \mathrm{Str}_\theta \big(da_1(da_2)a_3\big) = \mathrm{Str}_\theta(a_3da_1da_2).$$

Denote $\tau = \mathrm{Str}_\theta$ in short. For the first and fifth terms,

$$\begin{aligned}&\tau\big(Rd(R)d(R) + zRd(R)d(Rz^*)\big) \\ &\quad = \tau\big(d(R)d(R)R + d(R)d(Rz^*)zR\big) \\ &\quad = \tau\big(d(R)d(R)R + d(R)d(R)z^*zR + d(R)Rd(z^*)zR\big) \\ &\quad = \tau\big(d(R)d(R)R + d(R)d(R)(1-R) + d(R)Rd(z^*)zR\big) \\ &\quad = \tau\big(d(R)d(R) + d(R)Rd(z^*)zR\big).\end{aligned}$$

Similarly, we have for the second and sixth terms, third and seventh terms, fourth and eighth terms,

$$\begin{aligned}&\tau\big(Rd(Rz^*)d(zR) + zRd(Rz^*)d(zRz^*)\big) = \tau\big(d(Rz^*)d(zR) + zRd(Rz^*)zRdz^*\big), \\ &\tau\big(Rz^*d(zR)d(R) + zRz^*d(zR)d(Rz^*)\big) = \tau\big(z^*d(zR)dR + zRz^*d(zR)Rdz^*\big), \\ &\tau\big(Rz^*d(zRz^*)d(zR) + zRz^*d(zRz^*)d(zRz^*)\big) \\ &\qquad = \tau\big(z^*d(zRz^*)d(zR) + zRz^*d(zRz^*)zRdz^*\big).\end{aligned}$$

Recoupling these terms,

$$\begin{aligned}&\tau\big(dRdR + z^*d(zR)dR\big) = \tau\big(R^{-1}dRdR + z^*(dz)RdR\big), \\ &\tau\big(zR(dR)Rdz^* + zRz^*d(zR)Rdz^*\big) = \tau\big(z(dR)Rdz^* + zRz^*dzR^2dz^*\big), \\ &\tau\big(d(Rz^*)d(zR) + z^*d(zRz^*)d(zR)\big) = \tau\big(R^{-1}d(Rz^*)d(zR) + z^*(dz)Rz^*d(zR)\big), \\ &\tau\big(zRd(Rz^*)zRdz^* + zRz^*d(zRz^*)zRdz^*\big) \\ &\qquad = \tau\big(zd(Rz^*)zRdz^* + zRz^*(dz)Rz^*zRdz^*\big).\end{aligned}$$

On the right-hand side, there are only three terms that still contain derivatives of products. We again use the Leibniz rule,

$$\begin{aligned}\tau\big(R^{-1}d(Rz^*)d(zR)\big) &= \tau\big(R^{-1}d(R)z^*d(zR) + dz^*d(zR)\big) \\ &= \tau\big(d(R)z^*d(z) + R^{-1}d(R)(R^{-1}-1)dR + dz^*d(z)R + dz^*zdR\big),\end{aligned}$$

$$\tau\big(z^*(dz)Rz^*d(zR)\big)=\tau\big(z^*(dz)(1-R)dR+z^*(dz)Rz^*d(z)R\big),$$
$$\tau\big(zd(Rz^*)zRdz^*\big)=\tau\big(z^*Rdz^*zRdz^*+zdR(1-R)dz^*\big).$$

Gathering all the terms we have that

$$\begin{aligned}&\big((dR)z^*dz+z^*dzdR\big)+(dz^*zdR+zdRdz^*)\\&\quad+\big(zR(dz^*)zRdz^*+R^{-1}dRR^{-1}dR+(dz)Rz^*(dz)Rz^*\big)\\&\quad+Rdz^*dz+zRz^*(dz)Rdz^*.\end{aligned}$$

Here only the last two terms has nonzero trace. This is because for any $a_1, a_2, a_3, b_1, b_2 b_3$

$$\begin{aligned}&\mathrm{Str}_\theta\,\big(a_1(da_2)a_3b_1(db_2)b_3\big)=-\,\mathrm{Str}_\theta\,\big(b_1(db_2)b_3a_1(da_2)a_3\big),\\&\mathrm{Str}_\theta\,\big(a_1(da_2)a_3a_1(da_2)a_3\big)=0.\end{aligned}$$

This follows from the fact that $a_1(da_2)a_3$ has Clifford term of order 1 and hence $a_1(da_2)a_3\gamma=-\gamma a_1(da_2)a_3$. It remains to calculate the trace of $Rdz^*dz+zRz^*dzRdz^*$. Note that $zz^*=z^*z-2\theta=R^{-1}-1-2\theta$, $dz=-ic_1+c_2$, and $dz^*=-ic_1-c_2$. Then

$$\mathrm{Str}_\theta\,\big(Rdz^*dz+zRz^*(dz)Rdz^*\big)=4\tau_\theta(R-zRz^*R).$$

Finally, we use the spectrum of quantum harmonic oscillator to compute the above trace. Assume that $d=2$ and $\det(\theta)>0$. Denote $x_1=x$ and $x_2=y$. By Proposition 2.4, there is a trace preserving $*$-isomorphism $\pi:\mathbb{R}_\theta\to B(L_2(\mathbb{R}))$ such that (up to a factor $2\pi\theta$)

$$x\mapsto\sqrt{\det(\theta)}D_{\mathbf{x}},\quad y\mapsto\sqrt{\det(\theta)}\mathbf{x}.$$

Recall that $H=D_{\mathbf{x}}^2+\mathbf{x}^2$ is the Hamiltonian of 1-dimensional quantum harmonic oscillator which has eigenbasis $|n\rangle, n\geq 0$ with $H|n\rangle=(2n+1)|n\rangle$. For the creation operator $a^*=D_{\mathbf{x}}+i\mathbf{x}$ and the annihilation $a=D_{\mathbf{x}}-i\mathbf{x}$,

$$a^*|n\rangle=\sqrt{2n+2}|n+1\rangle,\quad a|n\rangle=\sqrt{2n}|n-1\rangle.$$

Now take $z=\sqrt{\theta}a^*$, $z^*=\sqrt{\theta}a$, and $R^{-1}=1+2\theta+zz^*=\theta(H+1)+1$. We have that

$$\begin{aligned}4\tau_\theta(R-zRz^*R)&=2\theta\pi\cdot4\sum_{k=0}\frac{1}{1+2\theta+2k\theta}-\frac{1}{1+2k\theta}\frac{2k\theta}{1+2\theta+2k\theta}\\&=8\theta\pi\sum_{k=0}\frac{1}{1+2k\theta}\frac{1}{1+2\theta+2k\theta}=4\pi.\end{aligned}$$

For ϕ_0, we have that

$$\begin{aligned}\phi_0(e-1_e)&=\mathrm{Str}_\theta\,\big((e-1_e)\omega\big)\\&=\tau_\theta(R+zRz^*-1)\,\mathrm{tr}(\gamma\omega)=2\theta'\tau_\theta(R+zRz^*-1).\end{aligned}$$

Note that $R^{-1} = 1 + z^*z = 1 + \theta + x^2 + y^2$ and $[R^{-1}, z] = [x^2 + y^2, x + iy] = 2\theta z$. Then

$$\begin{aligned} R + zRz^* - 1 &= R(1 + z^*z) - 1 + [z, Rz^*] = [z, Rz^*] \\ &= [z, R]z^* + R[z, z^*] = R[R^{-1}, z]Rz^* - 2\theta R \\ &= 2\theta(RzRz^* - R). \end{aligned}$$

We have calculated that $\tau_\theta(R - RzRz^*) = 2\pi$. So $\mathrm{Str}_\theta((e - 1_e)\omega) = -\theta\theta' 4\pi$. To conclude, we have the index pairing

$$\begin{aligned} \big\langle [e] - [1_e], (W^{\infty,1}(\mathbb{R}_\theta), L_2(\mathbb{R}_\Theta) \otimes M_N, D)\big\rangle &= \pi\, \mathrm{Str}_\theta\big((e - 1_e)\omega\big) + \pi\, \mathrm{Str}_\theta(edede) \\ &= -4\pi^2\theta\theta' + 4\pi^2 = 4\pi^2(1 - \theta\theta'). \end{aligned}$$

Recall for $d = 2$ that

$$\Theta = \begin{bmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ 1 & 0 & 0 & \theta' \\ 0 & 1 & -\theta' & 0 \end{bmatrix}.$$

When $\det\Theta = (1 - \theta\theta')^2 \neq 0$, we have that $\mathbb{R}_\Theta$ is $*$-isomorphic to $B(L_2(\mathbb{R}^2))$ with the trace differing by a factor $\tau_\Theta = (2\pi)^2|1 - \theta\theta'|\,\mathrm{tr}$, which is exactly the normalization constant we obtained. In other words, if we replace τ_Θ with the standard operator trace tr, $\mathrm{Index}_{\mathrm{tr}}(eF_{\mu,+}e) = 1$ (or -1). Since for every θ, we can choose θ' such that $\theta\theta' \neq 1$, then the index pairing shows that $e \in M_2(\widetilde{\mathbb{E}_\theta})$ is a representative of generator of the $K_0(\mathbb{E}_\theta) = \mathbb{Z}$. ■

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Li Gao
Department of Mathematics, Texas A&M University, College Station, TX 77843, USA;
lgao12@uh.edu

Marius Junge
Department of Mathematics, University of Illinois, Urbana, IL 61801, USA;
mjunge@illinois.edu

Edward McDonald
School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia; edward.mcdonald@unsw.edu.au