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The relative Mishchenko–Fomenko higher index and almost flat bundles II: Almost flat index pairing

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Abstract. This is the second part of a series of papers which bridges the Chang–Weinberger–Yu relative higher index and geometry of almost flat Hermitian vector bundles on manifolds with boundary. In this paper, we apply the description of the relative higher index given in part I to establish the relative version of the Hanke–Schick theorem, which relates the relative higher index with the index pairing of K-homology cycles and almost flat relative vector bundles. We also deal with the quantitative version and the dual problem of this theorem.

Contents

1. Introduction

This paper is a sequel of [27]. In this part II, we apply the Mishchenko–Fomenko description of the Chang–Weinberger–Yu relative higher index developed in part I to the index pairing with almost flat bundles on manifolds with boundary. Here, we also make use of the foundations of almost flat (stably) relative bundles prepared in [26].

The notion of almost flat bundle is introduced as a geometric counterpart of the higher index theory by Connes–Gromov–Moscovici [8] for the purpose of proving the Novikov conjecture for a large class of groups. It also plays a fundamental role in the study of positive scalar curvature metrics in [15, 16]. Its central concept is the almost monodromy correspondence, that is, the rough one-to-one correspondence between almost flat bundles and quasi-representations of the fundamental group. In [26], the author introduces the notion of almost flatness for an element of the relative K^0-group of a pair of topological

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spaces. Here, the relative or stably relative vector bundles are employed as representatives of an element of the relative K^0-group. Moreover, the almost monodromy correspondence is generalized to this relative setting.

The relation between the role of almost flat index pairing and the C^*-algebraic higher index theory is clearly understood in the work of Hanke and Schick. In [19,20], it is proved that the higher index $\alpha_\Gamma([M])$ of the K-homology fundamental class $[M] \in \mathrm{K}_*(M)$ of an enlargeable closed spin manifold M with $\pi_1(M) = \Gamma$ does not vanish without any assumption on the fundamental group concerned with the Baum–Connes conjecture. As is reorganized in [17], this is essentially a consequence of the fact that $\alpha_\Gamma([M]) \neq 0$ if M admits an almost flat bundle with non-trivial index pairing. The idea of Hanke and Schick relies on the fact that the dual higher index is related to the monodromy correspondence of flat bundles of Hilbert C^*-modules.

Recall that the Mishchenko–Fomenko higher index map α_Γ is given by the Kasparov product with the KK-class $\ell_\Gamma \in \mathrm{KK}(\mathbb{C}, C(M) \otimes C^*\Gamma)$ represented by the Mishchenko line bundle $\widetilde{M} \times_\Gamma C^*\Gamma$. Let P be a finitely generated projective A-module and let $\pi\colon \Gamma \to \mathbb{B}(P)$ be a unitary representation. Then, the dual higher index map, defined by the Kasparov product with ℓ_Γ over $C^*\Gamma$, maps the element $[\pi] \in \mathrm{KK}(C^*\Gamma, A)$ to the associated bundle $[\mathcal{P} := \widetilde{M} \times_\Gamma P] \in \mathrm{KK}(\mathbb{C}, C(M) \otimes A)$. Hence, the associativity of the Kasparov product relates the index pairing $[\mathcal{P}] \otimes_{C(M)} [M]$ with the higher index as

$$\begin{aligned} \alpha_\Gamma\big([M]\big) \otimes_{C^*\Gamma} [\pi] &= \ell_\Gamma \otimes_{C(M)} [M] \otimes_{C^*\Gamma} [\pi] \\ &= [\mathcal{P}] \otimes_{C(M)} [M]. \end{aligned} \tag{1.1}$$

An essential ingredient of the works of Hanke–Schick is their construction of a nice flat Hilbert C^*-module bundle from a family of almost flat bundles.

The first purpose of this paper, studied in Section 3, is to establish a relative version of the result of Hanke–Schick. This is based on the following two works of the author: the foundation of almost flat bundles on manifolds with boundary (particularly the almost monodromy correspondence) developed in [26] and the relative version of index pairing (1.1) given in this paper. Here, the higher index is replaced with the Chang–Weinberger–Yu relative higher index map [6], which is a homomorphism

$$\alpha_{\Gamma,\Lambda}\colon \mathrm{K}_*(X,Y) \to \mathrm{K}_*\big(C^*(\Gamma,\Lambda)\big),$$

defined for a pair of connected CW-complexes (X, Y) with $\pi_1(X) = \Gamma$ and $\pi_1(Y) = \Lambda$ (for more details on the definition, see Section 2.1). It is proved in [27] that this map is given by the Kasparov product with an element $\ell_{\Gamma,\Lambda} \in \mathrm{KK}(\mathbb{C}, C_0(X^\circ) \otimes C^*(\Gamma,\Lambda))$. The key observation is the following theorem.

Theorem 3.3. *The Kasparov product*

$$\ell_{\Gamma,\Lambda} \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi} \in \mathrm{KK}\big(\mathbb{C}, C_0(X^\circ) \otimes A\big) \cong \mathrm{K}^0(X, Y; A)$$

is represented by the stably relative bundle $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_1 u)$ *on* (X, Y).

The precise statement, particularly the definition of $\mathbf{\Pi}$ and $(\mathscr{P}_1, \mathscr{P}_2, \mathscr{Q}, V_1 u)$, is given in Section 3.1. Roughly speaking, the theorem claims that the Kasparov product with $\ell_{\Gamma,\Lambda}$ maps the KK-element $\mathbf{\Pi}$ represented by a relative representation of (Γ, Λ) (i.e., a pair of representations of Γ which is identified on Λ) to the associated relative bundle. To realize the concept in full generality, we employ the equivalence relation generated by unitary equivalence, stabilization, and homotopy as the "identification on Λ."

It is an immediate consequence of Theorem 3.3 that, the same argument as (1.1) using the associativity of the Kasparov product can be applied to the relative higher index pairing. The main theorem of the paper, a relative version of the Hanke–Schick theorem, is now obtained in the same way as in [19, 20] with the help of the relative almost monodromy correspondence.

Theorem 3.5. *Let M be a compact connected spin manifold with boundary N. Let $\Gamma := \pi_1(M)$, $\Lambda := \pi_1(N)$, and let ϕ be the homomorphism induced from the inclusion $N \to M$.*

(1) *If M has an infinite stably relative C^*-*K*-area, then the relative higher index $\mu_*^{\Gamma,\Lambda}([M, N])$ does not vanish.*
(2) *If M has an infinite relative C^*-*K*-area, then the relative higher index $\mu_*^{\Gamma,\phi(\Lambda)}([M, N])$ does not vanish.*

In addition, there is another application of Theorem 3.3 to the index theoretic refinement of the Hanke–Pape–Schick codimension 2 index obstruction for the existence of a positive scalar curvature metric [18], which is discussed in Section 3.3. Here, the higher index of a codimension 2 submanifold N of M (with a condition on homotopy groups) is related to the relative higher index of the manifold $M \setminus U$, where U is a tubular neighborhood of N.

In the rest part of the paper, we discuss in-depth problems related to the relative index theory of almost flat bundles. In Section 4, we study the quantitative version of Theorem 3.5. Recall that a key idea of [19] is to treat an infinite family of almost flat bundles simultaneously and relate the asymptotics of the index pairings with the higher index. On the other hand, if we consider the $\ell^1(\Gamma)$-valued higher index instead of the usual $C^*(\Gamma)$-valued one, then it is mapped by a single quasi-representation to a projection up to a small correction. This map is studied in [8] and compared with the index pairing with the associated almost flat bundle. In [12], Dadarlat gives an alternative approach using Lafforgue's Banach KK-theory. Here, we reformulate the result of [12] in terms of the quantitative K-theory introduced by Oyono-Oyono–Yu [30] instead of Banach KK-theory. By using this formulation, we generalize the result of Connes–Gromov–Moscovici and Dadarlat to the relative setting.

In Section 5, we study the dual problem of Theorem 3.5, in other words, the relative version of the problem proposed by Gromov in [15, Section $4\frac{2}{3}$]. It is a consequence of the almost monodromy correspondence that any almost flat bundle is obtained as the pullback of a bundle on the classifying space $B\Gamma$ (cf. [26, Corollary 6.13]). Then, it is a natural question whether any element of $\mathrm{K}^0(B\Gamma)$ (or $\mathrm{K}^0(B\Gamma) \otimes \mathbb{Q}$) is represented by an almost flat bundle. This question is first considered in [15, Section $8\frac{14}{15}$] geometrically in the case that Γ is the fundamental group of a Riemannian manifold with a non-positive sectional curvature. Later, Dadarlat gives a KK-theoretic approach to this problem in [13]. Here, we follow the approach of Dadarlat to study the subgroup of almost flat K-theory classes for the pair $(B\Gamma, B\Lambda)$. The celebrated Tikuisis–White–Winter theorem [36] in the theory of C^*-algebras enables us to include a large class of residually amenable groups in the range of our discussion. We show that any element of the range of the dual higher index map

$$\beta_{\Gamma,\Lambda}\colon \mathrm{K}^0\big(C^*(\Gamma,\Lambda)\big) \to \mathrm{K}^0(X,Y),$$

i.e., the Kasparov product with $\ell_{\Gamma,\Lambda}$ over $C^*(\Gamma,\Lambda)$, is represented by an almost flat stably relative vector bundle. Moreover, we also show that such elements are represented by an almost flat relative vector bundle if $\phi\colon \Lambda \to \Gamma$ is injective.

Notation 1.2. Throughout this paper, we use the following notations.

- For a C^*-algebra A, let A^+ denote its unitization $A + \mathbb{C} \cdot 1$.
- For a C^*-algebra A, let $\mathcal{M}(A)$ denote its multiplier C^*-algebra and $\mathcal{Q}(A) := \mathcal{M}(A)/A$.
- For a C^*-algebra A and $a < b \in \mathbb{R} \cup \{\pm\infty\}$, let $A(a,b) := A \otimes C_0(a,b)$. Similarly, we define $A[a,b)$ and $A[a,b]$. For a Hilbert A-module E, let $E(a,b)$ denote the Hilbert $A(a,b)$-module $E \otimes C_0(a,b)$.
- For a $*$-homomorphism $\phi\colon A \to B$, let $C\phi$ denote the mapping cone C^*-algebra defined as
$$C\phi = \big\{(a, b_s) \in A \oplus B[0,1) \mid \phi(a) = b_0\big\}.$$
- For a Hilbert A-module E, let $\mathbb{B}(E)$ and $\mathbb{K}(E)$ denote the C^*-algebra of bounded adjointable and compact operators on E, respectively. Let $\mathrm{U}(E)$ denote the unitary group of $\mathbb{B}(E)$.
- For a compact space X and a Hilbert A-module P, let $\underline{P}_X$ denote the trivial bundle $X \times P$ on X.
- For a pair (X,Y) of locally compact Hausdorff spaces, we write
$$Y_r := \begin{cases} Y \times [0,r] & \text{for } r \in [0,\infty), \\ Y \times [0,\infty) & \text{for } r = \infty, \end{cases}$$
$$X_r := X \sqcup_Y Y_r.$$
For $r \in [1,\infty)$, let $Y_r' := Y \times [1,r] \subset X_\infty$. We write X_r°, Y_r°, and $(Y_r')^\circ$ for the interiors of X_r, Y_r, and Y_r' as subspaces of X_∞.
- For an open cover $\mathcal{U} := \{U_\mu\}_{\mu\in I}$ of a topological space X, we write $U_{\mu\nu} := U_\mu \cap U_\nu$ for $\mu,\nu \in I$.
- For a C^*-algebra A and $a_1,\dots,a_n \in A$, $\mathrm{diag}(a_1,\dots,a_n)$ denotes the diagonal matrix in $A \otimes \mathbb{M}_n$.

2. Preliminaries

In this section, we summarize the results of [26, 27] which will be used in this paper. Throughout this paper, we only treat the complex coefficient K-theory, C^*-algebra, vector bundle, and so on.

2.1. Relative Mishchenko–Fomenko higher index

Let (Γ, Λ) be a pair of discrete groups with a homomorphism $\phi\colon \Lambda \to \Gamma$. Throughout the paper, we use the same letter ϕ for the induced $*$-homomorphism between maximal group C^*-algebras; $\phi\colon C^*\Lambda \to C^*\Gamma$. Note that ϕ induces $B\phi\colon B\Lambda \to B\Gamma$ (we may assume that $B\phi$ is an inclusion by replacing $B\Gamma$ with the mapping cylinder $B\Gamma \sqcup_{B\phi} B\Lambda \times [0,1]$).

Let (X, Y) be a pair of finite CW-complexes with a reference map $f\colon (X, Y) \to (B\Gamma, B\Lambda)$, to which a Γ-covering $\tilde{X} \to X$ and a Λ-covering $\tilde{Y} \to Y$ are associated. The *Chang–Weinberger–Yu relative higher index* is a group homomorphism

$$\mu_*^{\Gamma,\Lambda}\colon \mathrm{K}_*(X, Y) \to \mathrm{K}_*\big(C^*(\Gamma, \Lambda)\big),$$

where $C^*(\Gamma, \Lambda)$ is the relative (maximal) group C^*-algebra defined as

$$C^*(\Gamma, \Lambda) := SC(\phi\colon C^*\Lambda \to C^*\Gamma).$$

In [27, Section 3], the author gives a definition of $\mu_*^{\Gamma,\Lambda}$ inspired from the Mishchenko–Fomenko index pairing. Let us write the Mishchenko line bundles on X and Y as $\mathcal{V} := \tilde{X} \times_\Gamma C^*\Gamma$ and $\mathcal{W} := \tilde{Y} \times_\Lambda C^*\Lambda$, respectively, and let $\mathcal{X} := \tilde{Y} \times_\Lambda C\phi$. For simplicity of notation, we use the same letter $\mathcal{X}$ for the pull-back of $\mathcal{X}$ to $Y_r = Y \times [0, r]$ by the projection to the first component. We define

$$\begin{aligned} \mathcal{E}_2 &:= SC(X, \mathcal{V}) \oplus_{C(Y,\mathcal{X})} C_0\big(Y \times [0,2), \mathcal{X}\big) \\ &= \big\{(\xi, \eta) \in C(X, S\mathcal{V}) \oplus C_0\big(Y \times [0,2), \mathcal{X}\big) \mid \psi_Y(\xi|_Y) = \eta|_{Y \times \{0\}}\big\}, \end{aligned} \tag{2.1}$$

where $\psi_Y\colon S\mathcal{V}|_Y \to \mathcal{X}$ is the bundle map induced from the standard inclusion $\psi\colon SC^*\Gamma \to C\phi$, and

$$\rho(r, s) = \rho_s(r) := \min\{1, 2s + 2r - 3\} \in C\big([1,2] \times [0,1]\big).$$

We regard this ρ as a continuous function on $X_2 \times [0,1]$ by $\rho(x, s) := 2s - 1$ for $x \in X_1$ and $\rho(y, r, s) := \rho(r, s)$ for $(r, y) \in Y_2'$. Then, ρ acts on $\mathcal{E}_2$ by multiplication such that $\rho \in \mathbb{B}(\mathcal{E}_2)$ is a self-adjoint operator with $\rho^2 - 1 \in \mathbb{K}(\mathcal{E}_2)$ (as is seen in Figure 1).

The *relative Mishchenko line bundle* $\ell_{\Gamma,\Lambda}$ is defined by an odd Kasparov bimodule (cf. [3, Section 17.5.2]) as

$$\ell_{\Gamma,\Lambda} := [\mathcal{E}_2, 1, \rho] \in \mathrm{KK}_{-1}\big(\mathbb{C}, C_0(X_2^\circ) \otimes C\phi\big).$$

Definition 2.2 ([27, Definition 3.3]). The *relative Mishchenko–Fomenko higher index* $\mu_*^{\Gamma,\Lambda}$ is defined by the Kasparov product

$$\ell_{\Gamma,\Lambda} \mathbin{\hat{\otimes}}_{C_0(X_2^\circ)} \cdot\colon \mathrm{KK}_*\big(C_0(X_2^\circ), \mathbb{C}\big) \to \mathrm{K}_*\big(C^*(\Gamma, \Lambda)\big).$$

We also use the symbol $\alpha_{\Gamma,\Lambda}$ for this homomorphism.

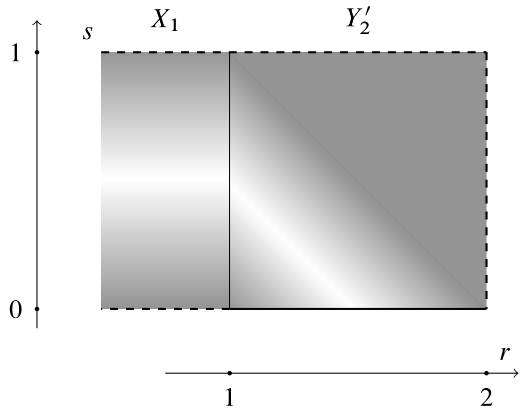

Figure 1. The shading shows the value of $|\rho(r,s)|$.

Note that the pair (X,Y) is homotopy equivalent to $(X_2, Y\times\{2\})$ and hence there is an isomorphism $\mathrm{K}^*(X,Y)\cong \mathrm{K}_{-*}(C_0(X_2^\circ))$.

Proposition 2.3 ([27, Proposition 3.6]). *The dual relative higher index map*

$$\beta_{\Gamma,\Lambda}\colon \mathrm{KK}(C^*(\Gamma,\Lambda),\mathbb{C})\to \mathrm{K}^*(X,Y)$$

is defined as the Kasparov product $\ell_{\Gamma,\Lambda}\mathbin{\hat{\otimes}}_{C^*(\Gamma,\Lambda)}\cdot$. *It satisfies*

$$\big\langle \alpha_{\Gamma,\Lambda}(x),\xi\big\rangle = \big\langle x,\beta_{\Gamma,\Lambda}(\xi)\big\rangle \in \mathrm{KK}(\mathbb{C},\mathbb{C})\cong\mathbb{Z},$$

where the bracket $\langle\cdot,\cdot\rangle$ *denotes the index pairing, i.e., the Kasparov product of the* K*-homology and* K*-cohomology groups of a* C^**-algebra.*

Here, we give a presentation of the relative Mishchenko line bundle $\ell_{\Gamma,\Lambda}$, which is an element of the K_1-group $\mathrm{K}_1(C_0(X_2^\circ)\otimes C\phi)$, by using a unitary of $C_0(X_2^\circ)\otimes C\phi$. Let $\mathcal{U}:=\{U_\mu\}_{\mu\in I}$ be a finite open cover of X such that the restriction of $\widetilde{X}$ to each U_μ is a trivial bundle. We choose a local trivialization $\theta_\mu\colon \widetilde{X}|_{U_\mu}\cong U_\mu\times\Gamma$ and let $\gamma_{\mu\nu}$ denote the transformation function $\theta_\nu(x)\theta_\mu^*(x)$ (which is independent of $x\in U_{\mu\nu}$).

Let $\{\eta_\mu\}_{\mu\in I}$ be a family of continuous functions such that $\mathrm{supp}(\eta_\mu)\subset U_\mu$, $0\le \eta_\mu(x)\le 1$ and $\sum\eta_\mu^2=1$. We write $\mathbb{M}_I$ for the matrix algebra on $\mathbb{C}^I$ and let $\{e_{\mu\nu}\}_{\mu,\nu\in I}$ denote the matrix unit. Then,

$$P_{\mathcal{V}} := \sum_{\mu,\nu\in I}\eta_\mu\eta_\nu\otimes u_{\gamma_{\mu\nu}}\otimes e_{\mu\nu}\in C(X)\otimes C^*(\Gamma)\otimes\mathbb{M}_I \tag{2.4}$$

is a projection whose support is isomorphic to $\mathcal{V}$ as Hilbert $C^*\Gamma$-module bundles on X. This means that $\ell_\Gamma=[P_{\mathcal{V}}]$.

Lemma 2.5. *The element* $\ell_{\Gamma,\Lambda} \in \mathrm{K}_{-1}(C_0(X_2^\circ) \otimes C\phi)$ *is represented by the unitary* $(U_{\mathcal{W}}, V_{\mathcal{V},s}) \in (C_0(X_2^\circ) \otimes C\phi)^+$, *where*

$$U_{\mathcal{W}} := -e^{-\pi i \rho_0} P_{\mathcal{W}} + 1 - P_{\mathcal{W}} \in \big(C_0(Y_2^\circ) \otimes C^*\Lambda \otimes \mathbb{M}_I\big)^+,$$
$$V_{\mathcal{V},s} := -e^{-\pi i \rho_s} P_{\mathcal{V}} + 1 - P_{\mathcal{V}} \in \big(C_0(X_2^\circ) \otimes C^*\Gamma \otimes \mathbb{M}_I\big)^+.$$

Proof. Let $i_*: C_0(Y_2^\circ) \to C_0(X_2^\circ)$ denote the $*$-homomorphism induced from the open embedding and let $\mathcal{C}$ denote the fiber sum C^*-algebra

$$\mathcal{C} := C_0(Y_2^\circ) \otimes C^*\Lambda \oplus_{C_0(X_2^\circ)\otimes C^*\Gamma} C\big(C_0(X_2^\circ) \otimes C^*\Gamma\big)$$

by the $*$-homomorphisms

$$i_* \otimes \phi: C_0(Y_2^\circ) \otimes C^*(\Lambda) \to C_0(X_2^\circ) \otimes C^*\Gamma,$$
$$\mathrm{ev}_0: C\big(C_0(X_2^\circ) \otimes C^*\Gamma\big) \to C_0(X) \otimes C^*\Gamma.$$

This is an ideal of

$$C_0(X_2^\circ) \otimes C\phi \cong C_0(X_2^\circ) \otimes C^*\Lambda \oplus_{C_0(X_2^\circ)\otimes C^*\Gamma} C\big(C_0(X_2^\circ) \otimes C(C\Gamma)\big).$$

Let $\iota: \mathcal{C} \to C_0(X_2^\circ) \otimes C\phi$ denote the inclusion. Then, $P_{\mathcal{E}} := (P_{\mathcal{W}}, P_{\mathcal{V}})$ determines an element of the multiplier C^*-algebra $\mathcal{M}(\mathbb{M}_I(\mathcal{C}))$ such that the exterior tensor product $(P_{\mathcal{E}}\mathcal{C}^{\oplus |I|}) \otimes_\iota (C_0(X) \otimes C\phi)$ is isomorphic to $\mathcal{E}_2$ as a Hilbert $C(X) \otimes C\phi$-module.

Recall that the identification of $\mathrm{K}_1(\mathcal{C}) \cong \mathrm{KK}_1(\mathbb{C}, \mathcal{C})$ is given (in [3, Proposition 17.5.6]) by the composition of isomorphisms

$$\mathrm{K}_1(\mathcal{C}) \xleftarrow{\partial} \mathrm{K}_0\big(\mathcal{Q}(\mathcal{C})\big) \xrightarrow{\cong} \mathrm{KK}_1(\mathbb{C}, \mathcal{C}).$$

Here, for $p \in \mathcal{Q}(\mathcal{C}) \otimes \mathbb{M}_n$ and its lift $\tilde{p} \in \mathcal{M}(\mathcal{C}) \otimes \mathbb{M}_n$, the right isomorphism is given by $[p] \mapsto [\mathcal{C}^n, 1, \tilde{p}]$. In particular, the projection $P_{\mathcal{E}} \cdot (\frac{\rho+1}{2}) + (1 - P_{\mathcal{E}}) \in \mathcal{Q}(\mathcal{C}) \otimes \mathbb{M}_I$ corresponds to both $[P_{\mathcal{E}}\mathcal{C}^{\oplus |I|}, 1, \rho] \in \mathrm{KK}_1(\mathbb{C}, \mathcal{C})$ and

$$\big[e^{-2\pi i (P_{\mathcal{E}} \cdot \frac{\rho+1}{2} + (1 - P_{\mathcal{E}}))}\big] = [-e^{-\pi i \rho} P_{\mathcal{E}} + 1 - P_{\mathcal{E}}] \in \mathrm{K}_1(\mathcal{C}).$$

This finishes the proof since $[P_{\mathcal{E}}\mathcal{C}^{\oplus |I|}, 1, \rho] \otimes [\iota] = \ell_{\Gamma,\Lambda}$ and $\iota(-e^{-\pi i \rho} P_{\mathcal{E}} + 1 - P_{\mathcal{E}}) = (U_{\mathcal{V}}, V_{\mathcal{V},s})$. ∎

2.2. Rational surjectivity of the dual relative higher index map

The rational injectivity of the relative higher index map and the rational surjectivity of its dual are studied in [27, Section 6]. In Section 5, we will apply the latter to prove the existence of an almost flat (stably) relative vector bundle representing an arbitrary element of relative K^0-group of the pair $(B\Gamma, B\Lambda)$.

We consider the following assumptions for (Γ, Λ).

(2.6) The group Γ has the γ-element γ_Γ.

(2.7) For any finite subgroup $K \subset \Gamma$, the subgroup $\phi^{-1}(K) \leq \Lambda$ satisfies $\gamma = 1$.

(2.8) The subgroup $\ker\phi$ is torsion-free.

For example, the condition (2.6) is satisfied if Γ is coarsely embeddable into a separable Hilbert space ([34] and [37, Theorem 3.3]) and the condition (2.7) is satisfied if $\ker\phi$ has the Haagerup property [22, Theorem 8.6].

We also consider a stronger variant of (2.7).

(2.7′) The subgroup $\ker\phi$ of Λ is amenable.

If (2.7′) is satisfied, the group homomorphism ϕ induces a $*$-homomorphism $\phi_r\colon C_r^*\Lambda \to C_r^*\Gamma$ between the reduced group C^*-algebras. Indeed, the unitary representation $\lambda_\Gamma \circ \phi$ (where λ_Γ denotes the left regular representation) is a direct sum of copies of the induced representation $\mathrm{Ind}_N^\Lambda 1_N$, where $N := \ker\phi$. By amenability of N, it is weakly contained in $\mathrm{Ind}_N^\Lambda \lambda_N = \lambda_\Lambda$. (We refer to [2, Appendix F.4] for group C^*-algebras and weak containment of representations.) Hence, the reduced relative group C^*-algebra

$$C_r^*(\Gamma,\Lambda) := SC\big(\phi_r\colon C_r^*\Lambda \to C_r^*\Gamma\big) \tag{2.9}$$

is defined. We write $\epsilon_{\Gamma,\Lambda}\colon C^*(\Gamma,\Lambda) \to C_r^*(\Gamma,\Lambda)$ for the quotient.

We write j_ϕ for the functor from the category of Γ-C^*-algebras to the category of C^*-algebras mapping A to the relative crossed product defined as

$$A \rtimes (\Gamma,\Lambda) := SC(\mathrm{id}_A \rtimes \phi\colon A \rtimes \Lambda \to A \rtimes \Gamma).$$

By the universality of the equivariant Kasparov category [29, Theorem 6.6], this j_ϕ gives rise to the functor $j_\phi\colon \mathfrak{KK}^\Gamma \to \mathfrak{KK}$, which maps the γ-element of Γ to $j_\phi(\gamma_\Gamma) \in \mathrm{KK}(C^*(\Gamma,\Lambda), C^*(\Gamma,\Lambda))$.

Theorem 2.10 ([27, Theorem 6.6, Proposition 6.10]). *Let $\phi\colon \Lambda \to \Gamma$ be a homomorphism of groups.*

(1) *If (2.6), (2.7), and (2.8) are satisfied, then the composition*

$$\beta_{\Gamma,\Lambda} \circ j_\phi(\gamma_\Gamma)\colon \mathrm{K}^*\left(C^*(\Gamma,\Lambda)\right) \to \mathrm{K}^*(B\Gamma, B\Lambda)$$

is rationally surjective.

(2) *If (2.7′) is satisfied, then* $\mathrm{Im}(\epsilon_{\Gamma,\Lambda}^*) \subset \mathrm{K}^*(C^*(\Gamma,\Lambda))$ *includes* $\mathrm{Im}\, j_\phi(\gamma_\Gamma)$.

Therefore, if (2.6), (2.7′), and (2.8) are satisfied, then $\beta_{\Gamma,\Lambda} \circ \epsilon_{\Gamma,\Lambda}^*$ is rationally surjective.

Remark 2.11. Theorem 2.10 (1) is a relative analogue of the following statement: let $\beta_\Gamma\colon \mathrm{K}^*(C^*\Gamma) \to \mathrm{K}^*(B\Gamma)$ denote the dual higher index map, i.e., the Kasparov product $\ell_\Gamma \otimes_{C^*\Gamma} \cdot$. If Γ has the γ-element, then $\beta_\Gamma \circ j_\Gamma(\gamma_\Gamma)\colon \mathrm{K}^*(C^*(\Gamma)) \to \mathrm{K}^*(B\Gamma)$ is rationally surjective. This is proved in the same way as [27, Theorem 6.6] by using the Dirac-dual Dirac method and the rational injectivity of the higher index map $\alpha_\Gamma\colon \mathrm{K}_*(B\Gamma) \to \mathrm{K}_*(C^*\Gamma)$ shown in [1, Section 15]. Also, it is shown, in the same way as [26, Proposition 6.10], that the image $\mathrm{Im}\, j_\Gamma(\gamma_\Gamma)$ is included in $\mathrm{Im}\,\epsilon_\Gamma^*$, where $\epsilon_\Gamma\colon C^*(\Gamma) \to C_r^*(\Gamma)$ is the quotient.

Remark 2.12. As is pointed out in [27, Remark 6.8], we do not need to restrict the situation to the case that $B\Gamma$ and $B\Lambda$ have the homotopy type of finite CW-complexes in the statement of Theorem 2.10.

2.3. Almost flat relative bundles

Here, we briefly review the foundation of almost flat (stably) relative vector bundle and the (stably) relative almost monodromy correspondence. Let X be a connected finite CW-complex with a good open cover $\mathcal{U} := \{U_\mu\}_{\mu \in I}$. As is noted in Notation 1.2, $U_{\mu\nu}$ denotes $U_\mu \cap U_\nu$. Then, the fundamental group $\Gamma := \pi_1(X)$ is generated by $\mathcal{G} := \{\gamma_{\mu\nu}\}_{\mu,\nu \in I}$ once we fix the collection of translation functions $\{\gamma_{\mu\nu}\}_{\mu,\nu\in I}$ of the Γ-Galois covering $\widetilde{X}$. Note that $\mathcal{G}$ is symmetric; i.e., $\gamma^{-1} \in \mathcal{G}$ for any $\gamma \in \mathcal{G}$.

Definition 2.13. Let X, $\mathcal{U}$, Γ, and $\mathcal{G}$ be as above. Let A be a unital C^*-algebra, let P be a finitely generated projective Hilbert A-module, and let T be a maximal subtree of the 1-skeleton $N_{\mathcal{U}}^{(1)}$ of the nerve of $\mathcal{U}$.

- A $\mathrm{U}(P)$-valued Čech 1-cocycle $\mathbf{v} = \{v_{\mu\nu}\}_{\mu,\nu\in I}$ on $\mathcal{U}$ is an $(\varepsilon, \mathcal{U})$-*flat bundle* on X with the typical fiber P if $\|v_{\mu\nu}(x) - v_{\mu\nu}(y)\| < \varepsilon$ for any $x, y \in U_{\mu\nu}$. It is said to be normalized on T if $\|v_{\mu\nu} - 1\| < \varepsilon$ for any $\langle \mu, \nu\rangle \in T$.
- A map $\pi\colon \Gamma \to \mathrm{U}(P)$ is a $(\varepsilon, \mathcal{G})$-*representation* of Γ on P if $\pi(e) = 1$ and
$$\big\|\pi(g)\pi(h) - \pi(gh)\big\| < \varepsilon$$
for any $g, h \in \mathcal{G}$.

We write $\mathrm{Bdl}_P^{\varepsilon,\mathcal{U}}(X)$ for the set of $(\varepsilon, \mathcal{U})$-flat bundles with the typical fiber P and $\mathrm{qRep}_P^{\varepsilon,\mathcal{G}}(\Gamma)$ for the set of $(\varepsilon, \mathcal{G})$-representations of Γ on P. We define the metrics on $\mathrm{Bdl}_P^{\varepsilon,\mathcal{U}}(X)$ and $\mathrm{qRep}_P^{\varepsilon,\mathcal{G}}(\Gamma)$ as

$$d(\mathbf{v}, \mathbf{v}') := \max_{\mu,\nu} \|v_{\mu\nu} - v'_{\mu\nu}\| \quad \text{and} \quad d(\pi, \pi') = \sup_{\gamma\in\mathcal{G}} \big\|\pi(\gamma) - \pi'(\gamma)\big\|,$$

respectively.

Remark 2.14. The bundle $E_{\mathbf{v}}$ associated to a $\mathrm{U}(P)$-valued Čech 1-cocycle is constructed as follows: as in (2.4), let $\{\eta_\mu\}_{\mu\in I}$ be a family of positive continuous functions on X such that $\sum_{\mu\in I} \eta_\mu^2 = 1$ and let $e_{\mu\nu} \in \mathbb{M}_I$ denote the matrix element; i.e., $e_{\mu\nu}e_\sigma = \delta_{\nu,\sigma}e_\mu$, where $\{e_\mu\}_{\mu\in I}$ is the standard basis of $\mathbb{C}^I$. Let

$$p_{\mathbf{v}}(x) := \sum_{\mu,\nu} \eta_\mu(x)\eta_\nu(x)v_{\mu\nu}(x) \otimes e_{\mu\nu} \in C(X) \otimes \mathbb{B}(P) \otimes \mathbb{M}_I,$$
$$\psi_\mu^{\mathbf{v}}(x) := \sum_{\nu} \eta_\nu(x)v_{\nu\mu}(x) \otimes e_\nu \in C_b(U_\mu) \otimes \mathbb{B}(P) \otimes \mathbb{C}^I.$$

Then, we have $p_{\mathbf{v}}(x)\psi_\mu^{\mathbf{v}}(x) = \psi_\mu(x)$ for $x \in U_\mu$ and $\psi_\mu^{\mathbf{v}}(x)^*\psi_\nu^{\mathbf{v}}(x) = v_{\mu\nu}(x)$ for $x \in U_{\mu\nu}$. That is, $p_{\mathbf{v}}$ is a projection with the support $E_{\mathbf{v}}$ and $\psi_\mu^{\mathbf{v}}$ is a local trivialization of $E_{\mathbf{v}}$.

It is essentially proved in [5, Theorems 3.1 and 3.2] (see also [26, Lemma 6.9]) that there is a constant $C > 0$ depending only on $\mathcal{U}$ and maps

$$\begin{aligned} \alpha &: \mathrm{Bdl}_P^{\varepsilon,\mathcal{U}}(X)_T \to \mathrm{qRep}_P^{C\varepsilon,\mathcal{G}}(\Gamma), \\ \beta &: \mathrm{qRep}_P^{\varepsilon,\mathcal{G}}(\Gamma) \to \mathrm{Bdl}_P^{C\varepsilon,\mathcal{U}}(X)_T, \end{aligned} \tag{2.15}$$

satisfying

- $d(\alpha(\mathbf{v}), \alpha(\mathbf{v}')) \le d(\mathbf{v}, \mathbf{v}') + C\varepsilon$, $d(\beta \circ \alpha(\mathbf{v}), \mathbf{v}) \le C\varepsilon$ for any $\mathbf{v}, \mathbf{v}' \in \mathrm{Bdl}_P^{\varepsilon,\mathcal{U}}(X)$,
- $d(\beta(\pi), \beta(\pi')) \le d(\pi, \pi') + C\varepsilon$, $d(\alpha \circ \beta(\pi), \pi) \le C\varepsilon$ for any $\pi, \pi' \in \mathrm{qRep}_P^{\varepsilon,\mathcal{G}}(\Gamma)$.

Remark 2.16. The construction of the map β is essentially given in [26, Lemma 4.4] (see also [26, Definition 6.7]). Here, it is mentioned that, for a $(\varepsilon, \mathcal{G})$-representation π of Γ, the associated bundle $\mathbf{v} := \beta(\pi)$ satisfies $\|v_{\mu\nu}(x) - \pi(\gamma_{\mu\nu})\| < 4\varepsilon$. Indeed, this inequality characterizes $\beta(\pi)$ up to a small correction.

Definition 2.17. Let (X, Y) be a pair of compact spaces. A *stably relative bundle* on (X, Y) with the typical fiber (P, Q) is a quadruple (E_1, E_2, E_0, u), where E_1 and E_2 are P-bundles on X, E_0 is a Q-bundle on Y, and u is a unitary bundle isomorphism $E_1|_Y \oplus E_0 \to E_2|_Y \oplus E_0$.

A stably relative bundle of Hilbert $\mathbb{C}$-modules with the typical fiber $(\mathbb{C}^n, \mathbb{C}^m)$ is simply called a stably relative vector bundle of rank (n, m). We simply call a stably relative bundle of the form $(E_1, E_2, 0, u)$ a *relative bundle*.

Remark 2.18. We associate to a stably relative bundle an element of the relative K^0-group $\mathrm{K}^0(X, Y; A) := \mathrm{K}_0(C_0(X_2^\circ) \otimes A)$ in the following way. Let $f_1(r) := \min\{1, \max\{0, 1-3r\}\}$ and $f_2(r) := \min\{1, \max\{0, 3r-2\}\}$. The inverse of κ is given by mapping (E_1, E_2, E_0, u) to

$$[E_1, E_2, E_0, u] := \left[\mathcal{E}_1 \oplus \mathcal{E}_2^{\mathrm{op}}, 1, \begin{pmatrix} 0 & \tilde{u}^* \\ \tilde{u} & 0 \end{pmatrix} \right] \in \mathrm{KK}\big(\mathbb{C}, C_0(X_1^\circ) \otimes A\big),$$

where

$$\begin{aligned} \mathcal{E}_1 &:= C_0(X_2^\circ, E_1) \oplus C_0\big((Y_2')^\circ, E_0\big), \\ \mathcal{E}_2 &:= C_0(X_2^\circ, E_2) \oplus C_0\big((Y_2')^\circ, E_0\big), \\ \tilde{u} &:= f_1(r) 1_{E_0} + f_2(r) u \in \mathbb{B}(\mathcal{E}_1, \mathcal{E}_2). \end{aligned}$$

In particular, $[E_1, E_2, E_0, u] = 0$ if $E_1 = E_2$ and $u = 1_{E_1|_Y \oplus E_0}$.

Let (X, Y) be a pair of connected finite CW-complexes. We say that a good open cover of (X, Y) is a good open cover $\mathcal{U} = \{U_\mu\}_{\mu \in I}$ of X such that $\mathcal{U}|_Y := \{Y \cap U_\mu\}$ is also a good open cover of Y.

Definition 2.19. Let (X, Y) and $\mathcal{U}$ be as above and let P be a finitely generated Hilbert A-module. Let T be a maximal subtree of the 1-skeleton of $N(\mathcal{U})$ such that $T|_{N(\mathcal{U}|_Y)}$ is also a maximal subtree.

- For two $(\varepsilon, \mathcal{U})$-flat bundles $\mathbf{v}_1 = \{v^1_{\mu\nu}\}$ and $\mathbf{v}_2 = \{v^2_{\mu\nu}\}$, a *morphism of* $(\varepsilon, \mathcal{U})$*-flat bundles* $\mathbf{u} \in \mathrm{Hom}_\varepsilon(\mathbf{v}_1, \mathbf{v}_2)$ is a family of unitaries $\mathbf{u} = \{u_\mu\}_{\mu \in I} \in \mathrm{U}(P)^I$ such that

$$\sup_{\mu,\nu \in I} \sup_{x \in U_{\mu\nu}} \left\| u_\mu v^1_{\mu\nu}(x) u_\nu^* - v^2_{\mu\nu}(x) \right\| < \varepsilon.$$

- A $(\varepsilon, \mathcal{U})$*-flat stably relative bundle* on (X, Y) with the typical fiber (P, Q) is a quadruple $\mathfrak{v} := (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0, \mathbf{u})$, where
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are $(\varepsilon, \mathcal{U})$-flat P-bundles on X,
  - $\mathbf{v}_0$ is a $(\varepsilon, \mathcal{U}_Y)$-flat Q-bundle on Y, and
  - $\mathbf{u} \in \mathrm{Hom}_\varepsilon(\mathbf{v}_1|_Y \oplus \mathbf{v}_0, \mathbf{v}_2|_Y \oplus \mathbf{v}_0)$.

  It is said to be normalized on T if $\mathbf{v}_1, \mathbf{v}_2$ are normalized on T and $\mathbf{v}_0$ is normalized on $T \cap Y$.

We write the set of $(\varepsilon, \mathcal{U})$-flat stably relative bundles on (X, Y) normalized on T with the typical fiber (P, Q) as $\mathrm{Bdl}^{\varepsilon,\mathcal{U}}_{P,Q}(X, Y)_T$. We define the metric on $\mathrm{Bdl}^{\varepsilon,\mathcal{U}}_{P,Q}(X, Y)_T$ as

$$d(\mathfrak{v}, \mathfrak{v}') := \max \left\{ d(\mathbf{v}_1, \mathbf{v}_1'), d(\mathbf{v}_2, \mathbf{v}_2'), d(\mathbf{v}_0, \mathbf{v}_0'), d(\mathbf{u}, \mathbf{u}') \right\},$$

where $d(\mathbf{u}, \mathbf{u}') := \max_\mu \|u_\mu - u'_\mu\|$.

Remark 2.20. For sufficiently small $\varepsilon > 0$, a $(\varepsilon, \mathcal{U})$-flat stably relative bundle $\mathfrak{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0, \mathbf{u})$ associates an element $[\mathfrak{v}]$ of the relative K^0-group $\mathrm{K}^0(X, Y; A)$.

(1) The definition of $[\mathfrak{v}]$ is as follows (see [26, Definition 3.9] for the precise definition): firstly, let $E_{\mathbf{v}_i} \to X$, for $i = 1, 2$, and $E_{\mathbf{v}_0} \to Y$ be the bundles associated to $\mathbf{v}_1$, $\mathbf{v}_2, \mathbf{v}_0$ as in Remark 2.14. It is proved in [26, Lemma 3.4] that there is a collection of continuous maps $\{\bar{u}_\mu : U_\mu \to \mathrm{U}(P \oplus Q)\}_{\mu \in I}$ such that $\bar{u}_\mu (v^1_{\mu\nu} \oplus v^0_{\mu\nu}) \bar{u}_\nu^* = v^2_{\mu\nu} \oplus v^0_{\mu\nu}$ and $\|\bar{u}_\mu - u_\mu\| < C\varepsilon$, where $C > 0$ is a constant depending only on $\mathcal{U}$. This family $\{\bar{u}_\mu\}_{\mu \in I}$ induces a bundle map $\bar{u} : E_{\mathbf{v}_1|_Y} \oplus E_{\mathbf{v}_0} \to E_{\mathbf{v}_2|_Y} \oplus E_{\mathbf{v}_0}$. Now, the quadruple $(E_{\mathbf{v}_1}, E_{\mathbf{v}_2}, E_{\mathbf{v}_0}, \bar{u})$ is a stably relative bundle on (X, Y) with the typical fiber (P, Q) and hence associates an element of $\mathrm{K}^0(X, Y; A)$ as in Remark 2.18.

(2) If $\varepsilon > 0$ is sufficiently small and $\mathfrak{v}, \mathfrak{v}' \in \mathrm{Bdl}^{\varepsilon,\mathcal{U}}_{P,Q}(X, Y)$ satisfies $d(\mathfrak{v}, \mathfrak{v}') < \varepsilon$, then we have $[\mathfrak{v}] = [\mathfrak{v}']$ [26, Lemma 6.11].

(3) Let $\{\eta_\mu\}$ and $\{e_{\mu\nu}\}$ be as in Remark 2.14. The element

$$\bar{w} := \sum \eta_\mu \eta_\nu \cdot \left(v^2_{\mu\nu} \oplus v^0_{\mu\nu}\right) \bar{u}_\nu \otimes e_{\mu\nu} \in C(Y) \otimes \mathbb{B}(P) \otimes \mathbb{M}_I$$

is a partial isometry such that $\bar{w}^* \bar{w} = p_{\mathbf{v}_1|_Y} \oplus p_{\mathbf{v}_0}$, $\bar{w} \bar{w}^* = p_{\mathbf{v}_2|_Y} \oplus p_{\mathbf{v}_0}$, and

$$(\psi_{\mathbf{v}_2|_Y \oplus \mathbf{v}_0})^* \bar{w} (\psi_{\mathbf{v}_1|_Y \oplus \mathbf{v}_0}) = \bar{u}_\mu.$$

That is, $\bar{w}$ is identified with $\bar{u}$ in (1) under the canonical isomorphism $E_{\mathbf{v}_i|_Y} \cong p_{\mathbf{v}_1|_Y} \underline{P}^I_Y$ for $i = 1, 2$ and $E_{\mathbf{v}_0} \cong p_{\mathbf{v}_0} \underline{Q}^I_Y$. We remark that this $\bar{w}$ satisfies

$$\left\| \bar{w} - p_{\mathbf{v}_2} \cdot \mathrm{diag}(u_\mu)_{\mu \in I} \right\| \leq |I|^2 \cdot \sup_{\mu \in I} \|\bar{u}_\mu - u_\mu\| < |I|^2 \varepsilon,$$

where $\mathrm{diag}(u_\mu)_{\mu \in I}$ is a unitary in $\mathbb{B}(P) \otimes \mathbb{M}_I$.

We say that an element $\xi \in \mathrm{K}^0(X, Y; A)$ is *(resp. stably) almost flat* with respect to a good open cover $\mathcal{U}$ if for any $\varepsilon > 0$ there is a $(\varepsilon, \mathcal{U})$-flat (resp. stably) relative bundle $\mathfrak{v}$ of finitely generated projective Hilbert A-modules such that $x = [\mathfrak{v}]$. It is shown in [26, Corollary 3.16] that (stably) almost flatness does not depend on the choice of good open covers $\mathcal{U}$. Similarly, we say that an element of $\mathrm{K}^0(X, Y; A)_{\mathbb{Q}} := \mathrm{K}^0(X, Y; A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is (resp. stably) almost flat if it is a $\mathbb{Q}$-linear combination of (resp. stably) almost flat elements.

Definition 2.21. Let (X, Y) be a pair of connected finite CW-complexes.

(1) We write $\mathrm{K}^0_{\mathrm{af}}(X, Y; A)$, $\mathrm{K}^0_{\mathrm{af}}(X, Y; A)_{\mathbb{Q}}$, $\mathrm{K}^0_{\text{s-af}}(X, Y; A)$, and $\mathrm{K}^0_{s\text{-af}}(X, Y; A)_{\mathbb{Q}}$ for the subgroup of (stably) almost flat elements.

(2) We say that a K-homology class $\xi \in \mathrm{K}_*(X, Y)$ has an *infinite (resp. stably) relative* K-*area* if there is a (resp. stably) almost flat K-theory class $x \in \mathrm{K}^0(M, N)$ such that the index pairing $\langle x, \xi \rangle$ is non-zero.

(3) We say that $\xi \in \mathrm{K}_*(X, Y)$ has an *infinite (resp. stably) relative C^*-*K*-area* if for any $\varepsilon > 0$ there is a C^*-algebra A_ε and a (resp. stably) relative $(\varepsilon, \mathcal{U})$-flat bundle $\mathfrak{v}$ of finitely generated projective Hilbert A_ε-modules such that the index pairing $\langle [\mathfrak{v}], \xi \rangle \in \mathrm{K}_0(A_\varepsilon)$ is non-zero.

In particular, we say that a spin manifold M with the boundary N has a (stably) relative infinite (C^*-)K-area if the K-homology fundamental class $[M, N] \in \mathrm{K}_*(M, N)$ has a (stably) relative infinite (C^*-)K-area.

Theorem 2.22 ([26, Theorem 5.1]). *Let (M, g) be a Riemannian spin manifold with a collared boundary N. If the infinite cylinder M_∞ is area-enlargeable, then (M, N) has an infinite stably relative C^*-K-area.*

Finally, we review the almost monodromy correspondence in the relative setting.

Definition 2.23. Let (Γ, Λ) be a pair of discrete groups and let $\phi \colon \Lambda \to \Gamma$ be a homomorphism. Let $\mathcal{G} = (\mathcal{G}_\Gamma, \mathcal{G}_\Lambda)$ be a symmetric generating set of (Γ, Λ) in the following sense: $\mathcal{G}_\Gamma \subset \Gamma$ and $\mathcal{G}_\Lambda \subset \Lambda$ are symmetric generating sets and $\phi(\mathcal{G}_\Lambda) \subset \mathcal{G}_\Gamma$.

- Let π_1 and π_2 be $(\varepsilon, \mathcal{G})$-representations of Γ. A *ε-intertwiner* $u \in \mathrm{Hom}_\varepsilon(\pi_1, \pi_2)$ is a unitary $u \in \mathrm{U}(P)$ such that $\|u\pi_1(\gamma)u^* - \pi_2(\gamma)\| < \varepsilon$ for any $\gamma \in \mathcal{G}$.
- A *stably relative $(\varepsilon, \mathcal{G})$-representation* of (Γ, Λ) is a quadruple $\boldsymbol{\pi} := (\pi_1, \pi_2, \pi_0, u)$, where
  - $\pi_1 \colon \Gamma \to \mathrm{U}(P)$ and $\pi_2 \colon \Gamma \to \mathrm{U}(P)$ are $(\varepsilon, \mathcal{G}_\Gamma)$-representations,
  - $\pi_0 \colon \Lambda \to \mathrm{U}(Q)$ is a $(\varepsilon, \mathcal{G}_\Lambda)$-representation, and
  - $u \in \mathrm{Hom}_\varepsilon(\pi_1 \circ \phi \oplus \pi_0, \pi_2 \circ \phi \oplus \pi_0)$.

We write $\mathrm{qRep}^{\varepsilon,\mathcal{G}}_{P,Q}(\Gamma, \Lambda)$ for the set of stably relative $(\varepsilon, \mathcal{G})$-representations of (Γ, Λ) on (P, Q). We define the metric on $\mathrm{qRep}^{\varepsilon,\mathcal{G}}_{P,Q}(\Gamma, \Lambda)$ as

$$d(\boldsymbol{\pi}, \boldsymbol{\pi}') := \max\left\{d(\pi_1, \pi_1'), d(\pi_2, \pi_2'), d(\pi_0, \pi_0'), \|u - u'\|\right\}.$$

Theorem 2.24 ([26, Definition 6.11, Theorem 6.12]). *There are a constant $C_{\mathrm{am}} > 0$ depending only on $\mathcal{U}$ and continuous maps*

$$\boldsymbol{\alpha}\colon \mathrm{Bdl}_{P,Q}^{\varepsilon,\mathcal{U}}(X,Y)_T \to \mathrm{qRep}_{P,Q}^{C_{\mathrm{am}}\varepsilon,\mathcal{G}}(\Gamma,\Lambda),$$
$$\boldsymbol{\beta}\colon \mathrm{qRep}_{P,Q}^{\varepsilon,\mathcal{G}}(\Gamma,\Lambda) \to \mathrm{Bdl}_{P,Q}^{C_{\mathrm{am}}\varepsilon,\mathcal{U}}(X,Y)_T,$$

which satisfy the following

(1) *for $\mathfrak{v},\mathfrak{v}' \in \mathrm{Bdl}_{P,Q}^{\varepsilon,\mathcal{U}}(X,Y)_T$, one has $d(\boldsymbol{\alpha}(\mathfrak{v}),\boldsymbol{\alpha}(\mathfrak{v}')) \le d(\mathfrak{v},\mathfrak{v}') + C_{\mathrm{am}}\varepsilon$ and $d(\boldsymbol{\beta}\circ\boldsymbol{\alpha}(\mathfrak{v}),\mathfrak{v}) \le C_{\mathrm{am}}\varepsilon$;*

(2) *for $\boldsymbol{\pi},\boldsymbol{\pi}' \in \mathrm{qRep}_{P,Q}^{\varepsilon,\mathcal{G}}(\Gamma,\Lambda)$, one has $d(\boldsymbol{\beta}(\boldsymbol{\pi}),\boldsymbol{\beta}(\boldsymbol{\pi}')) \le d(\boldsymbol{\pi},\boldsymbol{\pi}') + C_{\mathrm{am}}\varepsilon$ and $d(\boldsymbol{\alpha}\circ\boldsymbol{\beta}(\boldsymbol{\pi}),\boldsymbol{\pi}) \le C_{\mathrm{am}}\varepsilon$.*

For the latter, we only recall the definition of $\boldsymbol{\beta}$ given in [26, Definition 6.10]. For a $(\varepsilon,\mathcal{G})$-representation $\boldsymbol{\pi} = (\pi_1,\pi_2,\pi_0,u)$ of (Γ,Λ), set

$$\boldsymbol{\beta}(\boldsymbol{\pi}) := \big(\beta(\pi_1),\beta(\pi_2),\beta(\pi_0),\Delta_I(u)\big), \tag{2.25}$$

where β is the map in (2.15) and $\Delta_I\colon \mathrm{U}(P\oplus Q) \to \mathrm{U}(P\oplus Q)^I$ is the diagonal embedding.

3. Relative index pairing with coefficient in a C^*-algebra

In this section, we establish an obstruction for the relative higher index to vanish arising from an index pairing with coefficient in a C^*-algebra. It has two applications: a relative version of the Hanke–Schick theorem [19,20] and the non-vanishing of the relative higher index in the setting of Hanke–Pape–Schick [18].

3.1. Index pairing with stably h-relative representations

Let M be a closed connected spin manifold, let $\Gamma := \pi_1(M)$, and let π be a representation of Γ on a finitely generated projective Hilbert A-module P; i.e., a homomorphism $\pi\colon\Gamma\to\mathrm{U}(P)$. Then, π gives rise to a $*$-homomorphism $\pi\colon C^*\Gamma\to\mathbb{B}(P)$ and hence determines an element $[\pi]\in\mathrm{KK}(C^*\Gamma,\mathbb{B}(P))\cong\mathrm{KK}(C^*\Gamma,A)$. The Kasparov product $\alpha_\Gamma([M])\,\hat{\otimes}_{C^*\Gamma}[\pi]\in\mathrm{K}_0(A)$ coincides with the index pairing with the associated flat P-bundle $\widetilde{M}\times_\pi P$.

Here, we develop a relative version of this argument. The relative counterpart of π is a pair of representations of Γ whose restrictions to Λ are identified "up to stabilization and homotopy" in the following sense.

Definition 3.1. Let A be a unital C^*-algebra and let P_1, P_2, Q be finitely generated projective Hilbert A-modules. A *stably h-relative representation* of (Γ,Λ) on (P_1,P_2,Q) is a quintuple $\Pi := (\pi_1,\pi_2,\pi_0,u,\tilde{\pi})$, where

- $\pi_i\colon\Gamma\to\mathrm{U}(P_i)$, for $i=1,2$, and $\pi_0\colon\Lambda\to\mathrm{U}(Q)$ are representations,

- $u\colon P_1 \oplus Q \to P_2 \oplus Q$ is a unitary, and
- $\widetilde{\pi} = \{\widetilde{\pi}_\kappa\}_{\kappa \in [1,2]}$ is a continuous family of representations of Λ to $P_2 \oplus Q$ (that is, $\widetilde{\pi}$ is a homomorphism from Λ to $\mathrm{U}(\mathbb{B}(P_2 \oplus Q)[1,2])$) such that $\widetilde{\pi}_1 = \mathrm{Ad}(u) \circ (\pi_1 \circ \phi \oplus \pi_0)$ and $\widetilde{\pi}_2 = \pi_2 \circ \phi \oplus \pi_0$.

We associate the following two objects to a stably h-relative representation. First, let $\mathcal{P}_i := \widetilde{X} \times_{\Gamma,\pi_i} P$ for $i = 1,2$, let $\mathcal{Q} := \widetilde{Y} \times_{\Lambda,\pi_0} Q$, and let V_κ be a continuous family of bundle isomorphisms

$$V_\kappa\colon \widetilde{Y} \times_{\Lambda,\widetilde{\pi}_\kappa} (P_1 \oplus Q) \to \widetilde{Y} \times_{\Lambda,\widetilde{\pi}_2} (P_2 \oplus Q)$$

for $\kappa \in [1,2]$ such that V_2 is the identity. Note that such V_κ exists by a standard argument showing that two bundles are isomorphic if and only if they are homotopic (see, for example, [24, Theorem 4.3]). Moreover, such V_κ is unique up to homotopy. Indeed, another choice V'_κ corresponds one-to-one to a continuous path $\{V'_\kappa V_\kappa^*\}_{\kappa \in [0,1]}$ of endomorphisms on $\widetilde{Y} \times_{\Lambda,\widetilde{\pi}_2} (P_2 \oplus Q)$ with $V'_2 V_2^* = \mathrm{id}$, which is homotopic to the constant path. Now, $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_1 u)$ is a stably relative Hilbert A-module bundle with the typical fiber (P, Q).

Second, let $\widetilde{P}_i$ denote the Hilbert $A(-1,1)$-module $\widetilde{P}_i := P_i(-1,1) \oplus Q(-1,0)$. We define a KK-class

$$\mathbf{\Pi} = \left[\widetilde{P}_1 \oplus \widetilde{P}_2, \Pi_1 \oplus \Pi_2, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right] \in \mathrm{KK}\big(C\phi, A(-1,1)\big), \tag{3.2}$$

where

$$\Pi_1(a, b_s)(s) := \begin{cases} (\pi_1 \oplus \pi_0 \circ \phi)(a) & s \in (-1,0), \\ \pi_1(b_s) & s \in [0,1), \end{cases}$$

$$\Pi_2(a, b_s)(s) := \begin{cases} \widetilde{\pi}_{2+s}(a) & s \in (-1,0), \\ \pi_2(b_s) & s \in [0,1), \end{cases}$$

and U is defined by using functions f_1 and f_2 used in Remark 2.18 as

$$U := f_1(-s)1_Q + f_2(-s)\bar{u} \in \mathbb{B}(\widetilde{P}).$$

By a reparametrization of $\widetilde{\pi}_\kappa$, we may assume that $\widetilde{\pi}_\kappa = \widetilde{\pi}_1$ for $\kappa \in [1, \frac{4}{3}]$ and $\widetilde{\pi}_\kappa = \widetilde{\pi}_2$ for $\kappa \in [\frac{5}{3}, 2]$. Then, U intertwines Π_1 with Π_2; that is, $U\Pi_1(x) = \Pi_2(x)U$ for any $x \in C\phi$.

Theorem 3.3. *The Kasparov product*

$$\ell_{\Gamma,\Lambda} \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi} \in \mathrm{KK}\big(\mathbb{C}, C_0(X^\circ) \otimes A\big) \cong \mathrm{K}^0(X, Y; A)$$

is represented by the stably relative bundle $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_1 u)$ on (X, Y).

For the proof, we use the following lemma.

Lemma 3.4 ([27, Lemma A.2]). *Let A, B, and D be σ-unital C^*-algebras such that A is separable, let (E_1, π_1, T_1) be an odd Kasparov A-B bimodule, and let (E_2, φ_2, F_2) be a Kasparov B-D bimodule. Set $E := E_1 \otimes_B E_2$ with the trivial $\mathbb{Z}_2$-grading, $\pi := \pi_1 \otimes_B 1$, and $\widetilde{T}_1 := T_1 \otimes_B 1$. Let $G = \left(\begin{smallmatrix} 0 & G_0^* \\ G_0 & 0 \end{smallmatrix}\right) \in \mathbb{B}(E)$ be an odd F-connection and assume that $[\pi(A), T] \subset \mathbb{K}(E)$, where*

$$T = \begin{pmatrix} \widetilde{T}_1 & (1-\widetilde{T}_1^2)^{1/4} G_0^* (1-\widetilde{T}_1^2)^{1/4} \\ (1-\widetilde{T}_1^2)^{1/4} G_0 (1-\widetilde{T}_1^2)^{1/4} & -\widetilde{T}_1 \end{pmatrix} \in \mathbb{B}(E).$$

Then, the odd Kasparov A-D bimodule (E, π, T) represents the Kasparov product $[E_1, \pi_1, T_1] \hat{\otimes}_B [E_2, \pi_2, F_2]$.

Proof of Theorem 3.3. Let $\mathcal{E}_2$ be as in (2.1). The Hilbert $C_0(X_2^\circ) \otimes A$-module $\mathcal{E}_2 \otimes_{\Pi_2} \widetilde{P}_2$ is the section space of the continuous field

$$\widetilde{\mathcal{P}}_2 := \bigsqcup_{s \in (0,1)} \mathcal{P}_2 \cup \bigsqcup_{s \in (-1,0]} \widetilde{Y} \times_{\widetilde{\pi}_{2+s}} (P \oplus Q)$$

of Hilbert A-modules over $X_2^\circ \times (-1,1)$. Let Z denote its support; that is, $Z := X_2^\circ(0,1) \cup (Y_2')^\circ(-1,0]$.

For $i = 1, 2$, set

$$\overline{\mathcal{P}}_i := C_0(Z, \mathcal{P}_i) \oplus C_0\big((Y_2')^\circ, \mathcal{Q}\big).$$

Then, $\overline{\mathcal{P}}_1$ is canonically identified with $\mathcal{E}_2 \otimes_{\widetilde{\Pi}_1} \widetilde{P}$ and

$$\bar{V}(\varphi)(x,s) = \begin{cases} \varphi(x,s), & s \in (0,1),\ x \in X_2^\circ, \\ V_{2+s}\big(\varphi(x,s)\big), & s \in (-1,0],\ x \in (Y_2')^\circ, \end{cases}$$

gives a unitary isomorphism

$$\bar{V} \colon \mathcal{E}_2 \otimes_{\widetilde{\Pi}_2} \widetilde{P} \to \overline{\mathcal{P}}_2.$$

Moreover, since U intertwines Π_1 with Π_2, it induces an operator

$$\bar{U} \colon \mathcal{E}_2 \otimes_{\widetilde{\Pi}_1} \widetilde{P} \to \mathcal{E}_2 \otimes_{\widetilde{\Pi}_2} \widetilde{P}.$$

In particular, $\bar{U}$ is a U-connection. By Lemma 3.4, we obtain that

$$\begin{aligned} \ell_{\Gamma,\Lambda} \otimes_{C\phi} \mathbf{\Pi} &= \left[(\mathcal{E}_2 \otimes_{\widetilde{\Pi}_1} \widetilde{P}_1) \oplus (\mathcal{E}_2 \otimes_{\widetilde{\Pi}_2} \widetilde{P}_2)^{\mathrm{op}}, 1, \begin{pmatrix} \bar{\rho} & \bar{\sigma}^2 \bar{U}^* \\ \bar{\sigma}^2 \bar{U} & -\bar{\rho} \end{pmatrix} \right] \\ &= \left[\overline{\mathcal{P}}_1 \oplus \overline{\mathcal{P}}_2, 1, \begin{pmatrix} \bar{\rho} & \bar{\sigma}^2 \bar{U}^* \bar{V}^* \\ \bar{\sigma}^2 \bar{V} \bar{U} & -\bar{\rho} \end{pmatrix} \right], \end{aligned}$$

where

$$\bar{\rho}(x,s) := (\rho \otimes_{\Pi_i} 1)(x,s) = \begin{cases} \rho_s(x) & (x,s) \in X_2^\circ(0,1), \\ \rho_0(x) & (x,s) \in (Y_2')^\circ(-1,0], \end{cases}$$

and $\bar{\sigma} = (1-\bar{\rho}^2)^{1/4}$. Note that $\bar{V}\bar{U} = f_1(-s) 1_{\mathcal{Q}} + f_2(-s) V_1 u$.

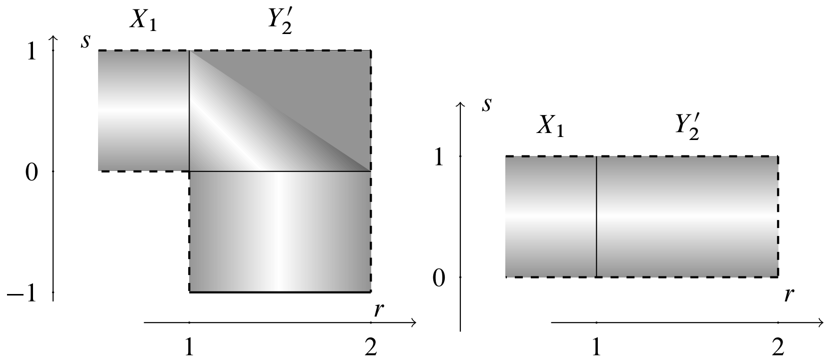

Figure 2. The shading shows the value of $|\rho(r,s)|$ on Z and $|2s-1|$ on $X(0,1)$, respectively.

On the other hand, let $\widetilde{\mathcal{P}}_i := C_0(X_2^\circ, \mathcal{P}_i) \oplus C_0((Y_2')^\circ, \mathcal{Q})$ for $i = 1, 2$ and $\widetilde{U} := f_1(r-1)1_{\mathcal{Q}} + f_2(r-1)V_1 u$. As is mentioned in Remark 2.18, we have

$$[\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_1 u] = \left[\widetilde{\mathcal{P}}_1 \oplus \widetilde{\mathcal{P}}_2^{\mathrm{op}}, 1, \begin{pmatrix} 0 & \widetilde{U} \\ \widetilde{U}^* & 0 \end{pmatrix}\right].$$

Hence, Lemma 3.4 implies that

$$\beta \otimes [\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_2] = \left[\widetilde{\mathcal{P}}_1(0,1) \oplus \widetilde{\mathcal{P}}_2^{\mathrm{op}}(0,1), 1, \begin{pmatrix} 2s-1 & \tau^2 \widetilde{U}^* \\ \tau^2 \widetilde{U} & 1-2s \end{pmatrix}\right],$$

where $\tau = (1-(2s-1)^2)^{1/4}$.

Let $\iota\colon Z \to X(-1,1)$ denote the open embedding. We define a continuous map $f\colon Z \to X(0,1)$ by $f(x,s) = (x,s)$ for $(x,s) \in X_1^\circ(0,1)$ and

$$f(y,r,s) = \begin{cases} \left(y, 1-s, \frac{\bar{\rho}(r,s)+1}{2}\right) & (y,r,s) \in (Y_2')^\circ(-1,0), \\ \left(y, 1, \frac{\bar{\rho}(r,s)+1}{2}\right) & (y,r,s) \in (Y_2')^\circ(0,1). \end{cases}$$

Then, the $*$-homomorphism $f^*\colon C_0(X_2^\circ(0,1)) \to C_0(Z)$ satisfies $f^*(2s-1) = \bar{\rho} \in C_0(Z)$ (see Figure 2). Moreover, by construction, there are unitaries $\Phi_i\colon \widetilde{\mathcal{P}}_i \otimes_{f^*} C_0(Z) \to \bar{\mathcal{P}}_i$ of Hilbert $C_0(Z) \otimes A$-modules for $i = 1, 2$ such that $\Phi_2(\widetilde{U} \otimes_{f^*} 1)\Phi_1^* = \bar{V}_1 \bar{U}$. Consequently, we obtain that

$$\ell_{\Gamma,\Lambda} \otimes \mathbf{\Pi} = \left(\beta \otimes [\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}, V_2]\right) \otimes [f^*] \otimes [\iota_*].$$

This concludes the proof since $\iota_* \circ f^*\colon C_0(X_1^\circ(0,1)) \to C_0(X_1^\circ(-1,1))$ is homotopic to the inclusion $X_2^\circ(0,1) \to X_2^\circ(-1,1)$. ■

3.2. Relative Hanke–Schick obstruction

We apply Theorem 3.3 to show a relative version of [17, Theorem 3.9]. Here, we identify the topological K-group $\mathrm{K}^0(M,N)$ with the C^*-algebra K-group $\mathrm{K}_0(C_0(M_2^\circ))$.

Theorem 3.5. *Let M be a connected compact connected spin manifold with boundary N. Let $\Gamma := \pi_1(M)$, let $\Lambda := \pi_1(N)$, and let $\phi\colon \Lambda \to \Gamma$ be the homomorphism induced from the inclusion $N \to M$.*

(1) *If M has an infinite stably relative C^*-*K*-area, then the relative higher index $\mu_*^{\Gamma,\Lambda}([M,N])$ does not vanish.*

(2) *If M has an infinite relative C^*-*K*-area, then the relative higher index $\mu_*^{\Gamma,\phi(\Lambda)}([M,N])$ does not vanish.*

Proof. First, we show (1). By assumption, for each $n \in \mathbb{N}$ there is a C^*-algebra A_n, a pair of finitely generated projective Hilbert A_n-modules (P_n, Q_n) and a $(\frac{1}{n}, \mathcal{U})$-flat stably relative bundle $\mathfrak{v}_n := (\mathbf{v}_n^1, \mathbf{v}_n^2, \mathbf{v}_n^0, \mathbf{u}_n)$ with the typical fiber (P_n, Q_n) such that $\langle [\mathfrak{v}_n], [M,N] \rangle \neq 0 \in \mathrm{K}_0(A_n)$. Set

$$B := \prod_{n\in\mathbb{N}} \mathbb{B}(P_n \oplus Q_n),$$
$$p := \prod 1_{P_n}, \quad P = pB,$$
$$q := \prod 1_{Q_n}, \quad Q = qB.$$

We define the stably relative bundle $\mathfrak{v} = (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^0, \mathbf{u})$ with the typical fiber (P, Q) as $\mathbf{v}^i = \{v^i_{\mu\nu}\}_{\mu,\nu\in I}$, $\mathbf{v}^0 = \{v^0_{\mu\nu}\}_{\mu,\nu\in I}$ and $\mathbf{u} = \{u_\mu\}_{\mu\in I}$, where

$$v^i_{\mu\nu}(x) := \prod_{n\in\mathbb{N}} (v^i_n)_{\mu\nu}(x) \in \mathbb{B}(P),$$
$$v^0_{\mu\nu}(y) := \prod_{n\in\mathbb{N}} (v^0_n)_{\mu\nu}(y) \in \mathbb{B}(Q),$$
$$u_\mu := \prod_{n\in\mathbb{N}} (u_n)_\mu \in \mathbb{B}(P \oplus Q),$$

for $i = 1, 2$, $x \in U_{\mu\nu}$ and $y \in U_{\mu\nu} \cap N$.

Let $J := \bigoplus_{n\in\mathbb{N}} \mathbb{B}(P_n \oplus Q_n)$, let $D = B/J$, and let $\tau\colon B \to D$ denote the quotient. Then, we have

- $v^i_{\mu\nu}(x) v^i_{\nu\sigma}(x) - v^i_{\mu\sigma}(x) \in J$ and
- $(v^1_{\mu\nu} \oplus v^0_{\mu\nu})(y) - u_\mu (v^2_{\mu\nu} \oplus v^0_{\mu\nu})(y) u_\nu^* \in J$;

that is,

$$\tau_* \mathfrak{v} := \big(\{\tau(v^1_{\mu\nu})\}, \{\tau(v^2_{\mu\nu})\}, \{\tau(v^0_{\mu\nu})\}, \{\tau(u_\mu)\}\big)$$

is a stably relative flat bundle. Let $\mathbf{\Pi} \in \mathrm{KK}(C^*(\Gamma, \Lambda), D)$ denote the Kasparov bimodule associated to the stably relative representation $\boldsymbol{\alpha}(\tau_* \mathfrak{v}) = (\pi_1, \pi_2, \pi_0, u)$ as in Theorem 2.24; that is, each π_i is the monodromy representation of $\tau(\mathbf{v}^i)$ on the fiber at a fixed basepoint $x \in N$ and u is the restriction of the bundle map induced from $\{u_\mu\}$ to the fibers at x. By Theorem 3.3, we obtain that

$$\alpha_{\Gamma,\Lambda}\big([M,N]\big) \mathbin{\hat\otimes} \mathbf{\Pi} = \ell_{\Gamma,\Lambda} \otimes_{C_0(M^\circ)} [M,N] \mathbin{\hat\otimes}_{C^*(\Gamma,\Lambda)} \mathbf{\Pi}$$

$$= [\tau_* \mathfrak{v}] \,\widehat{\otimes}_{C_0(M^\circ)} [M, N] = \tau_*\big([\mathfrak{v}] \,\widehat{\otimes}_{C_0(M^\circ)} [M, N]\big)$$
$$= \tau_*\Big(\prod_n \big\langle [\mathfrak{v}_n], [M, N] \big\rangle\Big).$$

It is non-zero because $\ker \tau_*$ is identified with $\bigoplus \mathrm{K}_0(A_n)$ through the injective homomorphism $\mathrm{K}_0(B) \subset \prod \mathrm{K}_0(A_n)$.

The claim (2) is proved in the same way. We only remark that in this case Π is a relative representation of (Γ, Λ), which is actually a relative representation of $(\Gamma, \phi(\Lambda))$ by [26, Remark 6.3]. ■

Together with Theorem 2.22, Theorem 3.5 implies the following relative version of the result of [19, 20].

Corollary 3.6. *Let (M, g) be a compact Riemannian spin manifold with a collared boundary N. If M_∞ is area-enlargeable, then $\mu_*^{\Gamma,\Lambda}([M, N])$ does not vanish.*

3.3. The Hanke–Pape–Schick codimension 2 obstruction

The second application of Theorem 3.3 is concerned with the codimension 2 obstruction of positive scalar curvature metric which is first introduced by Gromov–Lawson [16, Theorem 7.5] and generalized by Hanke–Pape–Schick [18, Theorem 4.3]. Here, we show the following theorem.

Theorem 3.7. *Let M be an n-dimensional closed connected spin manifold with an embedded connected codimension 2 submanifold N satisfying that*

- *the induced map $\pi_1(N) \to \pi_1(M)$ is injective,*
- *the induced map $\pi_2(N) \to \pi_2(M)$ is surjective, and*
- *the normal bundle of N is trivial.*

Let $W \cong N \times \mathbb{D}^2$ be a closed tubular neighborhood of N, let $M_0 := M \setminus W^\circ$, let $N_0 := \partial M_0$, let $\Gamma := \pi_1(M)$, and let $\Lambda := \pi_1(N)$. Then, $\mu_{n-2}^{\Lambda}([N]) \neq 0$ implies that $\mu_n^{\Gamma,\Lambda}([M_0, N_0]) \neq 0$.

Remark 3.8. It is proved in [27, Corollary 5.3] that if there are discrete groups $\Gamma_1, \Gamma_2, \Lambda$, injective homomorphisms $\Lambda \to \Gamma_i$, and a partitioned manifold $M = M_1 \sqcup_N M_2$ equipped with reference maps $f_i \colon (M_i, N) \to (B\Gamma_i, B\Lambda)$, then the non-vanishing of $\mu^{\Gamma,\Lambda}([M_i, N])$ implies that $\mu_*^{\Gamma_1 *_\Lambda \Gamma_2}([M]) \neq 0$. We apply this theorem to $M = N \times \mathbb{D}^2 \sqcup_{N \times S^1} M_0$, $\Gamma_1 = \pi_1(N \times \mathbb{D}_2)$, $\Lambda = \pi_1(N)$, and $\Gamma_2 = \pi_1(M_0)$ in the setting of Theorem 3.7. Then, the conclusion of Theorem 3.7 implies the non-vanishing of $\mu^{\Gamma}([M])$. In particular, we obtain that M does not admit any metric with positive scalar curvature, as is proved in [18, Theorem 4.3].

As is remarked in the introduction of [18], a combination of the stable Gromov–Lawson–Rosenberg conjecture proved by Rosenberg–Stolz [32] and [18, Theorem 4.3] also implies the non-vanishing of $\mu_n([M])$ if Γ satisfies the Baum–Connes injectivity. Here, we give a direct proof of this fact without the assumption of Baum–Connes injectivity.

For the proof, we prepare general lemmas on the boundary map of K-theory.

Remark 3.9. Recall that a pair of projections (q_1, q_2) of the multiplier algebra $\mathcal{M}(A)$ of a C^*-algebra A such that $q_1 - q_2 \in A$ represents the difference class $[q_1, q_2]$ of the $\mathrm{K}_0(A)$ (cf. [10, p. 64]) in the following way. There is an isomorphism $\mathrm{K}_0(\mathcal{M}(A) \oplus_A \mathcal{M}(A)) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_0(\mathcal{M}(A))$ induced from the split exact sequence $0 \to A \to \mathcal{M}(A) \oplus_A \mathcal{M}(A) \to \mathcal{M}(A) \to 0$. Let $[q_1, q_2]$ denote the $\mathrm{K}_0(A)$-component of $[(q_1, q_2)] \in \mathrm{K}_0(\mathcal{M}(A) \oplus_A \mathcal{M}(A))$.

Lemma 3.10. *Let $0 \to I \to D \to D/I \to 0$ be an exact sequence of C^*-algebras. For a pair of projections $(q_1, q_2) \in \mathcal{M}(D/I)^{\oplus 2}$ with $q_1 - q_2 \in D/I$, the image $\partial[q_1, q_2] \in \mathrm{K}_1(I)$ of the difference class by the boundary map is represented by a unitary*

$$\exp(-2\pi i \tilde{q}_1) \exp(2\pi i \tilde{q}_2) \in 1 + I,$$

where each $\tilde{q}_i \in \mathcal{M}(D)$ is a self-adjoint lift of q_i such that $\tilde{q}_1 - \tilde{q}_2 \in D$.

Proof. Let $\mathcal{I}$ denote the kernel of the homomorphism $\mathcal{M}(D) \to \mathcal{M}(D/I)$. It includes I as an ideal and $\mathcal{I} \cap D = I$ holds. Consider the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & D/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I} \oplus_{\mathcal{I}/I} \mathcal{I} & \longrightarrow & \mathcal{M}(D) \oplus_{\mathcal{Q}(D)} \mathcal{M}(D) & \longrightarrow & \mathcal{M}(D/I) \oplus_{\mathcal{Q}(D/I)} \mathcal{M}(D/I) & \longrightarrow & 0. \end{array}$$

The vertical morphisms are inclusions into the first component. Now, the projection $(q_1, q_2) \in \mathcal{M}(D/I) \oplus_{\mathcal{Q}(D/I)} \mathcal{M}(D/I)$ has a self-adjoint lift $(\tilde{q}_1, \tilde{q}_2) \in \mathcal{M}(D) \oplus_{\mathcal{Q}(D)} \mathcal{M}(D)$, and hence

$$\begin{aligned} \partial\big[(q_1, q_2)\big] &= \big[(e^{-2\pi i \tilde{q}_1}, e^{-2\pi i \tilde{q}_2})\big] \\ &= \iota_*[e^{-2\pi i \tilde{q}_1} e^{2\pi i \tilde{q}_2}] + [e^{-2\pi i \tilde{q}_2}, e^{-2\pi i \tilde{q}_2}] \in \mathrm{K}_1(\mathcal{I} \oplus_{\mathcal{I}/I} \mathcal{I}). \end{aligned}$$

This shows the lemma by commutativity of the boundary map and the isomorphism

$$\mathrm{K}_*(\mathcal{I} \oplus_I \mathcal{I}) \cong \mathrm{K}_*(\mathcal{I}) \oplus \mathrm{K}_*(I)$$

induced from the split exact sequence $0 \to \mathcal{I} \to \mathcal{I} \oplus_I \mathcal{I} \to \mathcal{I} \to 0$, where the splitting $\mathcal{I} \to \mathcal{I} \oplus_I \mathcal{I}$ is given by the diagonal map. ∎

Let A be a C^*-algebra, let $B := \mathbb{B}(\mathcal{H}_A)$, and let $J := \mathbb{K}(\mathcal{H}_A)$. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be bundles of infinitely generated projective Hilbert A-modules with the typical fiber Z_1 and Z_2, respectively. Then,

$$\bar{\mathcal{Z}}_i := \mathbb{B}(\mathcal{Z}_i, \mathcal{H}_A)/\mathbb{K}(\mathcal{Z}_i, \mathcal{H}_A)$$

(where $\mathbb{B}(\mathcal{Z}_i, \mathcal{H}_A)$ denotes the set of adjointable bounded operators from $\mathcal{H}_A$ to $\mathcal{Z}_i$) is a Hilbert B/J-module bundle with $\mathbb{B}(\bar{\mathcal{Z}}_i) \cong \mathcal{Q}(Z_i)$ (the B/J-action from the right, the $\mathcal{Q}(Z_i)$-action from the left, and the inner product are induced from the product of operators). Suppose that there is a bundle homomorphism $U \colon \mathcal{Z}_1|_{N_0} \to \mathcal{Z}_2|_{N_0}$ such that

$U^*U - 1 \in \mathbb{K}(C(N_0, \mathcal{Z}_1))$ and $UU^* - 1 \in \mathbb{K}(C(N_0, \mathcal{Z}_2))$. Then, it induces a unitary operator $\bar{U}: \bar{\mathcal{Z}}_1 \to \bar{\mathcal{Z}}_2$.

We write $[\partial_{B/J}] \in \mathrm{KK}_1(B/J, J)$ and $[\partial_{C(N_0)}] \in \mathrm{KK}_1(C(N_0), C_0(M_0^\circ))$ for the KK-classes corresponding to the extensions $0 \to J \to B \to B/J \to 0$ and $0 \to C_0(M_0^\circ) \to C(M_0) \to C(N_0) \to 0$, respectively.

Lemma 3.11. *Let $\mathcal{Z}_i$, U, $\bar{\mathcal{Z}}_i$, and $\bar{U}$ be as above. Then, one has*

$$[\bar{\mathcal{Z}}_1, \bar{\mathcal{Z}}_2, \bar{U}] \hat{\otimes}_{B/J} [\partial_{B/J}] = -[\mathcal{Z}_1|_{N_0}, \mathcal{Z}_2|_{N_0}, U] \hat{\otimes}_{C(N_0)} [\partial_{C(N_0)}] \tag{3.12}$$

under the isomorphism $\mathrm{KK}(\mathbb{C}, C_0(M_0^\circ) \otimes J) \cong \mathrm{KK}(\mathbb{C}, C_0(M_0^\circ) \otimes A)$ *given by the Kasparov product with the imprimitivity bimodule* $[\mathcal{H}_A] \in \mathrm{KK}(J, A)$.

Proof. First, notice that there are isometries $V_i: \mathcal{Z}_i \to \underline{\mathcal{H}_A}$ such that $V_2^* V_1 - U \in \mathbb{K}(\mathcal{Z}_1, \mathcal{Z}_2)$. Indeed, let S denote a unitary lift of $\begin{pmatrix} 0 & \bar{U}^* \\ \bar{U} & 0 \end{pmatrix}$ and let $W: \mathcal{Z}_1 \oplus \mathcal{Z}_2 \to \underline{\mathcal{H}_A}$ be an isometry (which exists by the Kasparov stabilization theorem [25, Theorem 2]). Then, $V_1 := WV_1'$ and $V_2 := WSV_2'$, where $V_i': \mathcal{Z}_i \to \mathcal{Z}_1 \oplus \mathcal{Z}_2$ denotes the embedding to the ith direct summand, are desired isometries. Moreover, by a pull-back with respect to a deformation retract of N_0, we may assume that $P_1 = P_2$ on a neighborhood O of N_0. Let ψ be a continuous function supported on O such that $0 \le \psi \le 1$ and $\psi|_{N_0} \equiv 1$ and let $P' := \psi P_1 + (1-\psi)P_2$.

Now, we apply Lemma 3.10 to determine the left- and right-hand sides of (3.12). Since (P_1, P') is a self-adjoint lift of $(q(P_1), q(P_2)) \in \mathcal{M}(C_0(M_0^\circ) \otimes B/J)^{\oplus 2}$ to $\mathcal{M}(C_0(M_0^\circ) \otimes B)^{\oplus 2}$ such that $P_1 - P' \in C_0(M_0^\circ) \otimes B$, we get

$$\begin{aligned}[\bar{\mathcal{Z}}_1, \bar{\mathcal{Z}}_2, \bar{U}] \otimes_{B/J} [\partial_{B/J}] &= \partial\big[q(P_1), q(P_2)\big] \\ &= \big[\exp(-2\pi i P_1)\exp(2\pi i P')\big] = \big[\exp(2\pi i P')\big].\end{aligned}$$

Similarly, since (P', P_2) is a self-adjoint lift of $(P_1|_{N_0}, P_2|_{N_0}) \in \mathcal{M}(C(N_0) \otimes J)^{\oplus 2}$ to $\mathcal{M}(C(M_0) \otimes J)^{\oplus 2}$ such that $P' - P_2 \in C(M_0) \otimes J$, we get

$$\begin{aligned}[\mathcal{Z}_1|_{N_0}, \mathcal{Z}_2|_{N_0}, U] \otimes_{C(N_0)} [\partial_{C(N_0)}] &= \partial[P_1|_{N_0}, P_2|_{N_0}] \\ &= \big[\exp(-2\pi i P')\exp(2\pi i P_2)\big] = \big[\exp(-2\pi i P')\big].\end{aligned}$$

This completes the proof of the lemma. ∎

We fix a base point $x_0 \in N_0$ in order to consider the Galois correspondence of covering spaces. Let $\tilde{M}$ denote the universal covering of M. Set $\bar{M} := \tilde{M}/\Lambda = \tilde{M} \times_\Gamma \Gamma/\Lambda$ and let $\bar{\pi}: \bar{M} \to M$, $\tilde{\pi}: \tilde{M} \to \bar{M}$ denote the projections. Then, $\bar{\pi}^{-1}(W)$ is the disjoint union of coverings of W indexed by $\Lambda g \Lambda \in \Lambda \setminus \Gamma/\Lambda$, each of which has the fundamental group $\Lambda \cap g\Lambda g^{-1}$. In particular, the connected component $\bar{W}$ including the base point x_0 is diffeomorphic to W by $\bar{\pi}$. Let $\bar{N}_0 := \partial \bar{W}$.

An essential ingredient of the codimension 2 obstruction theorem, which is given in the proof of [18, Theorem 4.3], is the existence of a nice $\Lambda \times \mathbb{Z}$-Galois covering on $\bar{M} \setminus \bar{W}^\circ$. Here, we restate it for our convenience.

Lemma 3.13. *There is a $\mathbb{Z}$-Galois covering $\breve{M}_0$ over $\widetilde{M}_0 := (\tilde{\pi} \circ \bar{\pi})^{-1}(M_0)$ with the following properties:*

- *its restriction to $\tilde{\pi}^{-1}(\bar{N}_0) \cong \tilde{N} \times S^1$ is the universal covering;*
- *its restriction to $\tilde{\pi}^{-1}(\bar{\pi}^{-1}(N_0) \setminus \bar{N}_0)$ is trivial.*

Proof. We write γ for the closed loop $\{x_0\} \times S^1 \subset N \times S^1 \cong N_0$. Then, γ generates the second component of $\pi_1(N_0) \cong \Lambda \times \mathbb{Z}[\gamma]$. Let $i \colon \bar{N} \to \bar{M}_0$ and $j \colon \bar{M}_0 \to \bar{M}$ denote the inclusions. It is proved in [18, Theorem 4.3] that there is a splitting

$$r \colon \pi_1(\bar{M} \setminus \bar{W}^\circ) \to \Lambda \times \mathbb{Z}$$

of i_*; that is, $r \circ i_* = \mathrm{id}_{\Lambda \times \mathbb{Z}}$.

Then, the homomorphism $\mathrm{pr}_\Lambda \circ r$ (where $\mathrm{pr}_\Lambda \colon \Lambda \times \mathbb{Z} \to \Lambda$ is the projection) is equal to j_*. Indeed, both $\mathrm{pr}_\Lambda \circ r$ and j_* map $[\gamma]$ to the trivial element and the induced homomorphisms from $\pi_1(\bar{M} \setminus \bar{W}^\circ)/\langle [\gamma] \rangle$ to Λ are the inverse of the composition

$$\Lambda \hookrightarrow \Lambda \times \mathbb{Z} \xrightarrow{i_*} \pi_1(\bar{M} \setminus \bar{W}^\circ) \to \pi_1(\bar{M} \setminus \bar{W}^\circ)\big/\big\langle [\gamma] \big\rangle.$$

Therefore, the covering $\breve{M}_0$ of $\bar{M}_0$ associated to r satisfies $\breve{M}_0/\mathbb{Z} = \breve{M}_0 \times_{\Lambda \times \mathbb{Z}} \Lambda \cong \widetilde{M}_0$. That is, $\breve{M}_0$ is a $\mathbb{Z}$-Galois covering on $\tilde{\pi}^{-1}(\bar{M}_0)$.

The equality $r \circ i_* = \mathrm{id}_{\Lambda \times \mathbb{Z}}$ means that the restriction of $\breve{M}_0$ to $\bar{N}_0$ is the universal covering $\tilde{N} \times \mathbb{R}$ of $N \times S^1$. That is, the restriction of the $\mathbb{Z}$-Galois covering $\breve{M}_0$ to $\tilde{\pi}^{-1}(\bar{N}) \cong \tilde{N} \times S^1$ is the universal covering. At the same time, the restriction of the $\mathbb{Z}$-Galois covering $\breve{M}_0$ to each connected component of $\tilde{\pi}^{-1}(\bar{\pi}^{-1}(N) \setminus \bar{N})$ is trivial because it is extended to a connected component of $(\tilde{\pi} \circ \bar{\pi})^{-1}(W)$, which is simply connected. ∎

Lemma 3.14. *Under the assumption of Theorem 3.7, $\bar{M}$ is an infinite covering; that is, Γ/Λ is an infinite set.*

Proof. Assume that $\bar{M}$ is a finite covering of M, and hence a closed manifold. The $\Lambda \times \mathbb{Z}$-Galois covering $\breve{M}_0 \to \bar{\pi}^{-1}(M_0)$ constructed in Lemma 3.13 extends to a $\Lambda \times \mathbb{Z}$-Galois covering on a spin manifold $\bar{M} \setminus \bar{W}^\circ$. Since its restriction to the boundary $\bar{N}_0 \cong N_0$ is isomorphic to the universal covering of N_0, we obtain that $[N_0, f] = 0 \in \Omega^{\mathrm{spin}}_{n-1}(B(\Lambda \times \mathbb{Z}))$ (where f is the reference map associated to the universal covering). This contradicts the assumption $\mu^{\Lambda}_{n-2}([N]) \neq 0$ (which implies that $\mu^{\Lambda \times \mathbb{Z}}_{n-1}([N_0]) \neq 0$). ∎

Proof of Theorem 3.7. Let $A := C^*(\Lambda \times \mathbb{Z})$. We consider two bundles

- $\mathcal{V}_1 := \breve{M}_0 \times_{\Lambda \times \mathbb{Z}} C^*(\Lambda \times \mathbb{Z})$ and
- $\mathcal{V}_2 := \widetilde{M}_0 \times_{\Lambda} C^*(\Lambda \times \mathbb{Z})$ (here, Λ acts on $C^*(\Lambda \times \mathbb{Z})$ from the left through the inclusion $\Lambda \to \Lambda \times \mathbb{Z}$)

of Hilbert A-modules over $\bar{M}_0$, where $\breve{M}_0$ is as in Lemma 3.13. We associate to them bundles

$$\mathcal{Z}_i := \bar{\pi}_! \mathcal{V}_i = \bigsqcup_{x \in M_0} \bigoplus_{\bar{\pi}(\bar{x}) = x} (\mathcal{V}_i)_{\bar{x}}$$

of infinitely generated (by Lemma 3.14) Hilbert A-modules on M_0, which are equipped with the canonical flat structures. Let $Z_i := \bigoplus_{\bar{\pi}(\bar{x})=x_0} (\mathcal{V}_i)_{\bar{x}}$ be the fiber of $\mathcal{Z}_i$ on x_0 and let $\sigma_i \colon \bar{\Gamma} \to \mathrm{U}(Z_i)$ denote the associated monodromy representation. Note that σ_2 factors through Γ.

By the construction of $\check{M}_0$ in Lemma 3.13, we have an isomorphism of flat A-module bundles between the restrictions of $\mathcal{V}_1$ and $\mathcal{V}_2$ on $\bar{\pi}^{-1}(N_0) \setminus \bar{N}_0$. It induces a partial isometry $U \colon \mathcal{Z}_1|_{N_0} \to \mathcal{Z}_2|_{N_0}$ such that $\ker U = \mathcal{V}_1|_{\bar{N}_0} \subset \mathcal{Z}_1$, $\ker U^* = \mathcal{V}_2|_{\bar{N}_0} \subset \mathcal{Z}_2$, and

$$\sigma_2(g) U_{x_0} = U_{x_0} \sigma_1(g)$$

for any $g \in \Lambda \times \mathbb{Z}$, where U_{x_0} is a restriction of U to $\bar{\pi}^{-1}(x_0)$.

As in Lemma 3.11, let $\bar{\mathcal{Z}}_i$ denote the bundle $\mathbb{B}(\mathcal{Z}_i, \mathcal{H}_A)/\mathbb{K}(\mathcal{Z}_i, \mathcal{H}_A)$ of Hilbert B/J-modules and let $\bar{Z}_i := (\bar{\mathcal{Z}}_i)_{x_0} = \mathbb{B}(Z_i, \mathcal{H}_A)/\mathbb{K}(Z_i, \mathcal{H}_A)$ for $i = 1, 2$. Then, σ_i and U_{x_0} above induce $\bar{\sigma}_i \colon \bar{\Gamma} \to \mathrm{U}(\mathcal{Q}(Z_i)) \cong \mathrm{U}(\bar{Z}_i)$ and $\bar{U}_{x_0} \colon \bar{Z}_1 \to \bar{Z}_2$, respectively. Then, $\bar{U}_{x_0}$ is a unitary and $\bar{U}_{x_0} \bar{\sigma}_1(g) \bar{U}_{x_0}^* = \bar{\sigma}_2(g)$ holds for any $g \in \Lambda \times \mathbb{Z}$. This particularly implies that $\bar{\sigma}_1(\gamma) = 1$ (where γ is the generator of $\mathbb{Z} \subset \Lambda \times \mathbb{Z}$); that is, $\bar{\sigma}_1 \colon \bar{\Gamma} \to \mathrm{U}(\bar{Z}_1)$ factors through Γ.

Consequently, we obtain that the triplet $\Pi := (\bar{\sigma}_1, \bar{\sigma}_2, \bar{U}_{x_0})$ is a relative representation of (Γ, Λ) and its associated relative B/J-module bundle (in the sense of Theorem 3.3) is $(\bar{\mathcal{Z}}_1, \bar{\mathcal{Z}}_2, \bar{U})$. Let $\mathbf{\Pi}$ denote the KK-element of $\mathrm{KK}(C\phi, (B/J)(-1,1))$ associated to Π as in (3.2). Now, we apply Theorem 3.3 and Lemma 3.11 to get

$$\begin{aligned}
&\big((\ell_{\Gamma,\Lambda} \otimes_{C_0(M_0^\circ)} [M_0, N_0]) \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi} \big) \otimes_{B/J} [\partial_{B/J}] \\
&\quad = \big(\ell_{\Gamma,\Lambda} \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi} \big) \otimes_{B/J} [\partial_{B/J}] \otimes_{C_0(M_0^\circ)} [M_0, N_0] \\
&\quad = [\bar{\mathcal{Z}}_1, \bar{\mathcal{Z}}_2, \bar{U}] \otimes_{B/J} [\partial_{B/J}] \otimes_{C_0(M_0^\circ)} [M_0, N_0] \\
&\quad = -\big([\mathcal{Z}_1|_{N_0}, \mathcal{Z}_2|_{N_0}, U] \otimes_{C(N_0)} [\partial_{C(N_0)}] \big) \otimes_{C_0(M_0^\circ)} [M_0, N_0] \\
&\quad = \big(-[\mathcal{V}_1|_{\bar{N}_0}] + [\mathcal{V}_2|_{\bar{N}_0}] \big) \otimes_{C(N_0)} [N_0] \\
&\quad = -\mu_{n-1}^{\Lambda \times \mathbb{Z}}([N \times S^1]) + \mu_{n-1}^{\Lambda}([N \times S^1]) \\
&\quad = -\mu_{n-2}^{\Lambda}([N]) + 0 \neq 0.
\end{aligned}$$

The last equality is considered under the identification of $\mathrm{K}_{n-2}(C^*(\Lambda))$ with the second direct summand of

$$\mathrm{K}_{n-1}\big(C^*(\Lambda \times \mathbb{Z})\big) = \mathrm{K}_{n-1}\big(C^*\Lambda \otimes C^*(\mathbb{Z})\big) \cong \mathrm{K}_{n-1}\big(C^*(\Lambda)\big) \oplus \mathrm{K}_{n-1}\big(C^*(\Lambda) \otimes S^{0,1}\big).$$

For the fourth equality, we use "the boundary of Dirac is Dirac principle" $[\partial_{C(N_0)}] \otimes_{C_0(M_0^\circ)} [M_0, N_0] = [N_0]$ (for the proof, see, for example, [23, Proposition 11.2.15]). ∎

4. Relative quantitative index pairing

In this section, we reformulate the index theorems of Connes–Gromov–Moscovici [8] and Dadarlat [12] and generalize them to the relative setting. Instead of Lafforgue's Banach

KK-theory, on which the formulation of [12] is based, we use the quantitative K-theory introduced by Oyono-Oyono and Yu [30].

4.1. Quantitative K-theory and almost $*$-homomorphism

We start with a quick review of the quantitative K-theory. The standard reference is [30]. We say that a filtered C^*-algebra is a C^*-algebra A equipped with an increasing family $\{A_r\}_{r\in[0,\infty)}$ of closed subspaces of A such that $A_r^* = A_r$, $A_r \cdot A_{r'} \subset A_{r+r'}$ and $\bigcup_r A_r \subset A$ is dense.

For a unital filtered C^*-algebra A, $0 \le \varepsilon \le \frac{1}{4}$, and $r > 0$, let

$$\mathrm{P}_n^{\varepsilon,r}(A) := \big\{p \in \mathbb{M}_n(A_r) \mid p = p^*, \|p^2 - p\| < \varepsilon\big\},$$
$$\mathrm{U}_n^{\varepsilon,r}(A) := \big\{u \in \mathbb{M}_n(A_r) \mid \|u^*u - 1\| < \varepsilon, \|uu^* - 1\| < \varepsilon\big\}$$

and let $\mathrm{P}_\infty^{\varepsilon,r}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{P}_n^{\varepsilon,r}(A)$, $\mathrm{U}_\infty^{\varepsilon,r}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{U}_n^{\varepsilon,r}(A)$. For $k \in \mathbb{N}$, let 1_k denote the unit of $\mathbb{M}_k \subset A^+ \otimes \mathbb{M}_k$. We introduce the equivalence relation to $\mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,r}(A)$ as

- $(p,k) \sim (q,l)$ if $\mathrm{diag}(p, 1_l)$ and $\mathrm{diag}(q, 1_k)$ are connected by a continuous path in $\mathrm{P}_\infty^{\varepsilon,r}(A)$,
- $u \sim v$ if u and v are connected by a continuous path in $\mathrm{U}_\infty^{3\varepsilon,2r}(A)$.

The quantitative K-groups are defined by

$$\mathrm{K}_0^{\varepsilon,r}(A) := \mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}/\sim,$$
$$\mathrm{K}_1^{\varepsilon,r}(A) := \mathrm{U}_\infty^{\varepsilon,r}(A)/\sim.$$

We write the elements of quantitative K_*-groups represented by $(p,l) \in \mathrm{P}_\infty^{\varepsilon,r}(A)$ and $u \in \mathrm{U}_\infty^{\varepsilon,r}(A)$ as $[p,l]_{\varepsilon,r}$ and $[u]_{\varepsilon,r}$, respectively. The summations $[p,k]_{\varepsilon,r} + [q,l]_{\varepsilon,r} := [\mathrm{diag}(p,q), k+l]_{\varepsilon,r}$ and $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$ make $\mathrm{K}_0^{\varepsilon,r}(A)$ and $\mathrm{K}_1^{\varepsilon,r}(A)$ into abelian groups (for the proof, see [30, Lemmas 1.14, 1.15, and 1.16]).

For a non-unital filtered C^*-algebra A, the unitization A^+ is also equipped with the structure of filtered C^*-algebra by $A_r^+ := A_r + \mathbb{C}1$. Let $\rho\colon A^+ \to \mathbb{C}$ denote the quotient. The quantitative K-group is defined by

$$\mathrm{K}_0^{\varepsilon,r}(A) := \ker\big(\rho_*\colon \mathrm{K}_0^{\varepsilon,r}(A^+) \to \mathrm{K}_0^{\varepsilon,r}(\mathbb{C}) \cong \mathbb{Z}\big)$$

and $\mathrm{K}_1^{\varepsilon,r}(A) := \mathrm{K}_1^{\varepsilon,r}(A^+)$. For any (ε, r), we write ι_A for the canonical homomorphism from $\mathrm{K}_*^{\varepsilon,r}(A)$ to $\mathrm{K}_*(A)$.

Remark 4.1. Hereafter, we often use the norm estimates $\|p\| \le 1+\varepsilon$ for $p \in \mathrm{P}_\infty^{\varepsilon,r}(A)$ and $\|u\| \le 1 + \varepsilon/2$ for $\mathrm{U}_\infty^{\varepsilon,r}(A)$ (cf. [30, Remark 1.4]).

Next, we introduce the notion of a complete almost $*$-homomorphism between filtered C^*-algebras.

Definition 4.2. Let A and D be filtered C^*-algebras. A bounded linear map $\pi\colon A_r \to D_{\kappa r}$ is a complete (ε, r, κ)-$*$-homomorphism if $\pi(a^*) = \pi(a)^*$ for any $a \in A_r$ and

$$\left\|\pi_n(ab) - \pi_n(a)\pi_n(b)\right\| \leq \varepsilon\|a\|\|b\|$$

holds for any $n \in \mathbb{N}$ and $a, b \in A_r \otimes \mathbb{M}_n$, where $\pi_n := \pi \otimes \mathrm{id}_{\mathbb{M}_n}$.

Remark 4.3. Let $\pi\colon A_r \to D_{\kappa r}$ be a complete (ε, r, κ)-$*$-homomorphism. Then, $\pi_n(a^*) = \pi_n(a)^*$ also holds for any $a \in A \otimes \mathbb{M}_n$. Moreover, for $a \in A_r \otimes \mathbb{M}_n$ with $\|a\| = 1$ and $\|\pi_n(a)\| > \|\pi_n\| - \varepsilon'$, we have

$$\begin{aligned}(\|\pi_n\| - \varepsilon')^2 &< \left\|\pi_n(a)^*\pi_n(a)\right\| \\ &\leq \left\|\pi_n(a^*a) - \pi_n(a)^*\pi_n(a)\right\| + \left\|\pi_n(a^*a)\right\| \\ &\leq \varepsilon + \|\pi_n\|.\end{aligned}$$

This means that $\|\pi_n\|^2 < \|\pi_n\| + \varepsilon$ and hence $\|\pi_n\| < 1 + \varepsilon/2$. That is, π is a completely bounded map between operator spaces (a reference on completely bounded maps and operator spaces is [4, Appendix B]). In particular, $\pi \otimes \mathrm{id}_B\colon A_r \otimes B \to D_{\kappa r} \otimes B$ is a well-defined completely bounded map for any nuclear C^*-algebra B [4, Corollary B.8].

A C^*-algebra is said to be *quasi-diagonal* if it admits a faithful representation $\varphi\colon A \to \mathbb{B}(\mathcal{H})$ with an increasing sequence p_n of finite rank projections in $\mathbb{B}(\mathcal{H})$ such that $[\varphi(a), p_n] \to 0$ as $n \to \infty$ for any $a \in A$ (for more details, see, for example, [4, Section 7]). Note that

$$A \ni a \mapsto \psi(a) := \big(p_n\varphi(a)p_n\big) \in \frac{\prod \mathbb{B}(p_n\mathcal{H})}{\oplus \mathbb{B}(p_n\mathcal{H})} \cong \frac{\prod \mathbb{M}_{k_n}}{\bigoplus \mathbb{M}_{k_n}},$$

where $k_n := \operatorname{rank} p_n$ is a faithful $*$-homomorphism.

Lemma 4.4. *Let $\pi\colon A \to D$ be a complete (ε, r, κ)-homomorphism and let B be a quasi-diagonal C^*-algebra. Then, $\pi \otimes \mathrm{id}_B\colon A_r \otimes B \to D_{\kappa r} \otimes B$ is a complete (ε, r, κ)-$*$-homomorphism.*

Proof. First, for a sequence of positive numbers $\{k_n\}_{n\in\mathbb{N}}$, $\pi \otimes \mathrm{id}_{\prod_{n\in\mathbb{N}} \mathbb{M}_{k_n}}$ is a complete (ε, r, κ)-$*$-homomorphism since $A \otimes (\prod_n \mathbb{M}_{k_n})$ is canonically isomorphic to $\prod_n (A \otimes \mathbb{M}_{k_n})$. Since there is an isomorphism

$$\Big(\prod_{n\in\mathbb{N}} \mathbb{M}_{k_n}\Big)\Big/\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{M}_{k_n}\Big) \cong \varinjlim_{N\to\infty} \Big(\prod_{n\geq N} \mathbb{M}_{k_n}\Big),$$

we obtain that $\pi \otimes \mathrm{id}_{\prod \mathbb{M}_{k_n}/\bigoplus \mathbb{M}_{k_n}}$ is also a complete (ε, r, κ)-$*$-homomorphism.

Since B is quasi-diagonal, there is a faithful $*$-homomorphism

$$\psi\colon B \to \prod_{n\in\mathbb{N}} \mathbb{M}_{k_n}\Big/\bigoplus_{n\in\mathbb{N}} \mathbb{M}_{k_n} \quad \text{for some } \{k_n\}_{n\in\mathbb{N}}.$$

Since the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\pi \otimes \mathrm{id}_B} & D \otimes B \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes \psi} & & \downarrow{\scriptstyle \mathrm{id}_D \otimes \psi} \\ A \otimes \frac{\prod \mathbb{M}_{k_n}}{\bigoplus \mathbb{M}_{k_n}} & \xrightarrow{\pi \otimes \mathrm{id}_{\prod \mathbb{M}_{k_n} / \bigoplus \mathbb{M}_{k_n}}} & D \otimes \frac{\prod \mathbb{M}_{k_n}}{\bigoplus \mathbb{M}_{k_n}} \end{array}$$

commutes, $\pi \otimes \mathrm{id}_B$ is also a complete (ε, r, κ)-$*$-homomorphism. ∎

Proposition 4.5. *Let A, B be two unital filtered C^*-algebras and let $\pi \colon A_r \to B_{\kappa r}$ be a unital complete (ε, r, κ)-$*$-homomorphism. Then, for any $\delta \geq 0$ such that $\varepsilon + (1+3\varepsilon)\delta < \frac{1}{4}$, π gives rise to continuous maps*

$$\pi \colon \mathrm{P}_n^{\delta,r}(A) \to \mathrm{P}_n^{\varepsilon+(1+2\varepsilon)\delta,\kappa r}(B),$$
$$\pi \colon \mathrm{U}_n^{\delta,r}(A) \to \mathrm{U}_n^{\varepsilon+(1+3\varepsilon)\delta,\kappa r}(B)$$

and hence induces homomorphisms

$$\pi_\sharp \colon \mathrm{K}_*^{\delta,r}(A) \to \mathrm{K}_*^{\varepsilon+(1+3\varepsilon)\delta,\kappa r}(B).$$

Proof. Let $p \in \mathrm{P}_n^{\delta,r}(A)$ and $u \in \mathrm{U}_n^{\delta,r}(A)$. Then, we have

$$\begin{aligned} \left\|\pi_n(p)^2 - \pi_n(p)\right\| &\leq \left\|\pi_n(p)^2 - \pi_n(p^2)\right\| + \left\|\pi_n(p^2 - p)\right\| \\ &\leq \varepsilon\|p\|^2 + \|\pi\|_{\mathrm{cb}}\|p^2 - p\| \\ &\leq \varepsilon(1+\delta) + (1+\varepsilon/2)\delta \\ &\leq \varepsilon + (1+2\varepsilon)\delta, \\ \left\|\pi_n(u)^*\pi_n(u) - 1\right\| &\leq \left\|\pi_n(u)^*\pi_n(u) - \pi_n(u^*u)\right\| + \left\|\pi_n(u^*u - 1)\right\| \\ &\leq \varepsilon\|u^*\|\|u\| + \|\pi\|_{\mathrm{cb}}\delta \\ &\leq \varepsilon(1+\delta)^2 + (1+\varepsilon/2)\delta \\ &\leq \varepsilon + (1+3\varepsilon)\delta. \end{aligned}$$

Similarly, we also have $\|\pi_n(u)\pi_n(u)^* - 1\| \leq \varepsilon + (1+3\varepsilon)\delta$. ∎

Remark 4.6. For possibly non-unital filtered C^*-algebras A, B and a (ε, r, κ)-$*$-homomorphism π, it is straightforward to see that the unitization $*$-homomorphism $\pi^+ \colon A^+ \to B^+$ defined by $\pi^+|_A = \pi$ and $\pi^+(1_A) = 1_B$ is also a complete (ε, r, κ)-$*$-homomorphism. Therefore, π induces a homomorphism of quantitative K-groups by Proposition 4.5.

4.2. Quantitative index pairing

Let Γ be a finitely generated discrete group and let $e \in \mathcal{G}_\Gamma \subset \Gamma$ be a finite set generating Γ. We assume that $\mathcal{G}_\Gamma$ is symmetric; i.e., $\gamma^{-1} \in \mathcal{G}_\Gamma$ for any $\gamma \in \mathcal{G}_\Gamma$. Let l_Γ denote the word length function on Γ with respect to $\mathcal{G}_\Gamma$. Since l_Γ satisfies $l_\Gamma(\gamma \cdot \gamma') \le l_\Gamma(\gamma) + l_\Gamma(\gamma')$, it gives the structure of a filtered C^*-algebra on the group C^*-algebra $C^*\Gamma$; that is,

$$C^*(\Gamma)_r := \Big\{ \sum_{\ell_\Gamma(\gamma)\le r} c_\gamma u_\gamma \in \mathbb{C}[\Gamma] \Big\} \subset C^*(\Gamma)$$

forms an increasing family of closed subspaces of $C^*(\Gamma)$ such that $C^*(\Gamma)_r \cdot C^*(\Gamma)_{r'} \subset C^*(\Gamma)_{r+r'}$ and $\bigcup C^*(\Gamma)_r = \mathbb{C}[\Gamma]$ is dense in $C^*(\Gamma)$. For $r \in \mathbb{Z}_{>0}$, we write $\mathcal{G}_\Gamma^r$ for the set $\{\gamma_1 \cdots \gamma_r \mid \gamma_i \in \mathcal{G}_\Gamma\}$.

For an $(\varepsilon, \mathcal{G}_\Gamma^r)$-representation π of Γ on P, we use the same letter π for the linear map $C^*(\Gamma)_r \to B := \mathbb{B}(P)$ given by $\pi(\sum c_\gamma u_\gamma) := \sum c_\gamma \pi(\gamma)$. We say that π is *self-adjoint* if $\pi(\gamma^{-1}) = \pi(\gamma)^*$ holds for any $\gamma \in \mathcal{G}_\Gamma^r$. Note that for any $(\varepsilon, \mathcal{G}_\Gamma^r)$-representation π, there is a self-adjoint $(70\varepsilon, \mathcal{G}_\Gamma^r)$-representation $\breve{\pi}$ with $d(\pi, \breve{\pi}) < 20\varepsilon$ [5, Proposition 5.6].

Proposition 4.7. *Let π be a self-adjoint $(\varepsilon, \mathcal{G}_\Gamma^r)$-representation of Γ on P. Then, π is a unital complete $(|\mathcal{G}_\Gamma^r|^2\varepsilon, r, 1)$-$*$-homomorphism.*

Proof. Let $x = \sum_{\gamma\in\mathcal{G}_\Gamma^r} a_\gamma u_\gamma$ and $y = \sum_{\gamma\in\mathcal{G}_\Gamma^r} b_\gamma u_\gamma$ be elements in $C^*(\Gamma)_r \otimes \mathbb{M}_n$, where a_γ and b_γ are elements of $\mathbb{M}_n$. We remark that $\|a_\gamma\| \le \|x\|$ and $\|b_\gamma\| \le \|y\|$ for any $\gamma \in \Gamma$. Indeed, let $\tau\colon C^*\Gamma \to \mathbb{C}$ denote the tracial state given by $\tau(\sum c_\gamma u_\gamma) := c_e$. Then, we have

$$\|a_\gamma\| = \big\|(\tau \otimes \mathrm{id}_{\mathbb{M}_n})(x u_{\gamma^{-1}})\big\| \le \|x u_{\gamma^{-1}}\| = \|x\|.$$

Now, we obtain that

$$\begin{aligned} \big\|\pi_n(x)\pi_n(y) - \pi_n(xy)\big\| &= \Big\| \sum_{\gamma,\gamma'\in\mathcal{G}_\Gamma^r} a_\gamma b_{\gamma'}\big(\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\big)\Big\| \\ &\le \sum_{\gamma,\gamma'\in\mathcal{G}_\Gamma^r} \|a_\gamma\| \cdot \|b_{\gamma'}\| \cdot \big\|\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\big\| \\ &\le \Big(\sum_{\gamma\in\mathcal{G}_\Gamma^r} \|a_\gamma\|\Big)\Big(\sum_{\gamma'\in\mathcal{G}_\Gamma^r} \|b_{\gamma'}\|\Big)\varepsilon \\ &\le |\mathcal{G}_\Gamma^r|^2 \|x\|\|y\|\varepsilon. \end{aligned}$$ ∎

Let X be a connected finite CW-complex and let $\Gamma := \pi_1(X)$ (and hence Γ is a finitely presented discrete group). Let $\mathcal{U} := \{U_\mu\}_{\mu\in I}$ be a good cover of X and let $\{\gamma_{\mu\nu}\}_{\mu,\nu\in I}$ be a collection of flat transition functions of the universal covering $\widetilde{X} \to X$. Let $\mathcal{G}_\Gamma := \{\gamma_{\mu\nu}\}_{\mu,\nu\in I}$. Let $\mathbf{v} = \{v_{\mu\nu}\}$ be a $\mathrm{U}(P)$-valued Čech 1-cocycle. As are mentioned in (2.4) and Remark 2.14, the projections

$$\begin{aligned} P_{\mathcal{V}} &:= \sum_{\mu,\nu\in I} \eta_\mu\eta_\nu \otimes u_{\gamma_{\mu\nu}} \otimes e_{\mu\nu} \in C(X) \otimes (C^*\Gamma)_1 \otimes \mathbb{M}_I, \\ p_{\mathbf{v}} &:= \sum_{\mu,\nu\in I} \eta_\mu\eta_\nu v_{\mu\nu} \otimes e_{\mu\nu} \in C(X) \otimes B \otimes \mathbb{M}_I \end{aligned}$$

have the support isomorphic to $\mathcal{V}$ and $E_{\mathbf{v}}$, respectively.

Remark 4.8. For the latter, we give two remarks on Cuntz's quasi-homomorphism picture of the KK-theory [9]. Here, a KK-element $\xi \in \mathrm{KK}(A, \mathbb{C})$ is represented by a quasi-homomorphism $[\varphi_1, \varphi_2]\colon A \to \mathbb{B}(\mathcal{H}) \rhd \mathbb{K}(\mathcal{H})$, i.e., a pair of $*$-homomorphisms $A \to \mathbb{B}(\mathcal{H})$, such that $\varphi_1(f) - \varphi_2(f) \in \mathbb{K}(\mathcal{H})$ (strictly speaking, this pair should be called a pre-quasihomomorphism).

(1) Let $\mathcal{B} := \mathbb{B}(\mathcal{H}) \oplus_{\mathcal{Q}(\mathcal{H})} \mathbb{B}(\mathcal{H})$. Then, a quasi-homomorphism $[\varphi_1, \varphi_2]$ corresponds to a $*$-homomorphism $\Phi := (\varphi_1, \varphi_2)\colon A \to \mathcal{B}$. The projection to the first and second components determines a quasi-homomorphism $[\mathrm{pr}_1, \mathrm{pr}_2] \in \mathrm{KK}(\mathcal{B}, \mathbb{C})$ such that $[\varphi_1, \varphi_2] = [\Phi] \otimes_{\mathcal{B}} [\mathrm{pr}_1, \mathrm{pr}_2]$.

(2) Let D be another C^*-algebra and let $p \in A \otimes D \otimes \mathbb{M}_n$ be a projection. Then, $(\varphi_1(p), \varphi_2(p))$ is a pair of projections such that $\varphi_1(p) - \varphi_2(p) \in \mathbb{K} \otimes D$. The Kasparov product $[p] \otimes_A ([\varphi_1, \varphi_2] \otimes \mathrm{id}_D) \in \mathrm{K}_0(D)$ is equal to the difference class $[\varphi_1(p), \varphi_2(p)]$ as in Remark 3.9 since

$$\begin{aligned}[p] \otimes_A [\varphi_1, \varphi_2] &= [p] \otimes_A [\Phi] \otimes_{\mathcal{B}} [\mathrm{pr}_1, \mathrm{pr}_2] \\ &= \big[\big(\varphi_1(p), \varphi_2(p)\big)\big] \otimes_{\mathcal{B}} [\mathrm{pr}_1, \mathrm{pr}_2].\end{aligned}$$

(3) A 1-parameter family of quasi-homomorphisms $[\varphi_1^t, \varphi_2^t]$ is said to be continuous if $t \mapsto \varphi_i^t(a)$ is strongly continuous and $t \mapsto \varphi_1(a) - \varphi_2(a)$ is norm continuous for any $a \in A$. If $[\varphi_1, \varphi_2] = [\varphi_1', \varphi_2']$, then $\mathbb{K}(\mathcal{H} \oplus \mathcal{H})$-valued quasi-homomorphisms $[\varphi_1 \oplus 0, \varphi_2 \oplus 0]$ and $[\varphi_1' \oplus 0, \varphi_2' \oplus 0]$ are homotopic. This is a consequence of [9, Proposition 2.4] and Kasparov's stabilization theorem [25, Theorem 2].

Proposition 4.9. *There is a group homomorphism*

$$\alpha_\Gamma^{\mathrm{alg}}\colon \mathrm{K}_0(X) \to \mathrm{K}_0^{0,3}\left(\mathbb{K}(\mathcal{H}) \otimes C^*(\Gamma)\right)$$

such that $\iota_{C^*(\Gamma)}(\alpha_\Gamma^{\mathrm{alg}}(\xi)) = \alpha_\Gamma(\xi) \in \mathrm{K}_0(C^*(\Gamma))$ *for any* $\xi \in \mathrm{K}_0(X)$.

Proof. Let $[\varphi_1, \varphi_2]$ be a quasi-homomorphism representing $\xi \in \mathrm{KK}(C(X), \mathbb{C})$. Let $P_1 := \varphi_1(P_{\mathcal{V}})$ and $P_2 := \varphi_2(P_{\mathcal{V}})$. Then, the Kasparov product $[P_{\mathcal{V}}] \otimes_{C(X)} \xi \in \mathrm{K}_0(\mathbb{K} \otimes C^*\Gamma)$ is represented by a pair of projections $[P_1, P_2]$ by Remark 4.8 (2).

Set

$$V := \begin{pmatrix} P_2 & 1_I - P_2 \\ 1_I - P_2 & P_2 \end{pmatrix} \in \mathbb{M}_2\big(\mathbb{B}(\mathcal{H}) \otimes C^*(\Gamma)_1 \otimes \mathbb{M}_I\big).$$

Then, V is a self-adjoint unitary and $V \operatorname{diag}(P_2, 1_I - P_2) V = \operatorname{diag}(1_I, 0)$ holds. This implies that

$$V \begin{pmatrix} P_1 & 0 \\ 0 & 1_I - P_2 \end{pmatrix} V - \begin{pmatrix} 1_I & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2\big(\mathbb{K}(\mathcal{H}) \otimes C^*(\Gamma)_3 \otimes \mathbb{M}_I\big);$$

that is, the pair $(V \operatorname{diag}(P_1, 1_I - P_2)V, \operatorname{diag}(1_I, 0))$ determines a difference class

$$\big[V \operatorname{diag}(P_1, 1_I - P_2)v, \operatorname{diag}(1_I, 0)\big] \in \mathrm{K}_0\left(\mathbb{K}(\mathcal{H}) \otimes C^*(\Gamma)\right).$$

Moreover, we have the equality of difference classes as

$$[P_1, P_2] = \left[\begin{pmatrix} P_1 & 0 \\ 0 & 1_I - P_2 \end{pmatrix}, \begin{pmatrix} P_2 & 0 \\ 0 & 1_I - P_2 \end{pmatrix} \right] = \left[V \begin{pmatrix} P_1 & 0 \\ 0 & 1_I - P_2 \end{pmatrix} V, \begin{pmatrix} 1_I & 0 \\ 0 & 0 \end{pmatrix} \right] \in \mathrm{K}_0(\mathbb{K} \otimes C^*\Gamma).$$

Now, we define the map $\alpha_\Gamma^{\mathrm{alg}}$ as

$$\alpha_\Gamma^{\mathrm{alg}}(\xi) := \big[V \operatorname{diag}(P_1, 1_I - P_2)V, |I|\big]_{0,3}.$$

Then, the above discussion means that this $\alpha_\Gamma^{\mathrm{alg}}$ satisfies $\iota_{C^*(\Gamma)} \circ \alpha_\Gamma^{\mathrm{alg}} = \alpha_\Gamma$. Note that the definition of $\alpha_\Gamma^{\mathrm{alg}}$ is well defined since it is independent of the choice of a representative $[\varphi_1, \varphi_2]$. To see this, let $[\varphi_1', \varphi_2']$ be another representative of ξ, to which the $(0,3)$-projection $V' \operatorname{diag}(P_1', 1_I - P_2')V'$ is associated. By Remark 4.8 (3), there is a continuous path $[\varphi_1^t, \varphi_2^t]$ connecting $[\varphi_1 \oplus 0, \varphi_2 \oplus 0]$ and $[\varphi_1' \oplus 0, \varphi_2' \oplus 0]$. Now, the same construction for $[\varphi_1^t, \varphi_2^t]$ provides a desired norm continuous path of $(0,3)$-projections connecting $V \operatorname{diag}(P_1, 1_I - P_2)V \oplus 1_I$ with $V' \operatorname{diag}(P_1', 1_I - P_2')V' \oplus 1_I$. ∎

Definition 4.10. We call the map $\alpha_\Gamma^{\mathrm{alg}}$ as in Proposition 4.9 the *algebraic Mishchenko–Fomenko higher index*. For $r > 3$, we call the composition $\alpha_\Gamma^{\delta,r} := \iota_{0,3}^{\varepsilon,r} \circ \alpha_\Gamma^{\mathrm{alg}} \colon \mathrm{K}_0(X) \to \mathrm{K}_0^{\delta,r}(C^*\Gamma)$ the *quantitative higher index*.

Now, we reformulate [12, Theorem 3.2] in the framework of quantitative K-theory.

Theorem 4.11. *There is a constant $C_1 = C_1(\mathcal{U})$ depending only on $\mathcal{U}$ that the following holds: for $0 < \varepsilon < (4C_1)^{-1}$, a self-adjoint quasi-representation $\pi \in \mathrm{qRep}_P^{\varepsilon, \mathcal{G}_\Gamma^3}(\Gamma)$, and $\xi \in \mathrm{K}_0(X)$, one has*

$$\iota_B \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \pi)_\sharp\big(\alpha_\Gamma^{\mathrm{alg}}(\xi)\big) = \big\langle [\beta(\pi)], \xi \big\rangle \in \mathrm{K}_0(B).$$

Remark 4.12. Here, we discuss the usage of Theorem 4.11 for the study of the K-theory of group C^*-algebras. For any (δ, r) with $|\mathcal{G}_\Gamma^r|^2\varepsilon + (1 + 3|\mathcal{G}_\Gamma^r|^2\varepsilon)\delta < 1/4$, $(\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \pi)_\sharp$ is defined on $\mathrm{K}_0^{\delta,r}(\mathbb{K}(\mathcal{H}) \otimes C^*\Gamma)$. Hence, the left-hand side of Theorem 4.11 is written as $\iota_B \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \pi)_\sharp \circ \alpha_\Gamma^{\delta,r}(\xi)$. Let $\xi \in \mathrm{K}_0(X)$ be a K-homology class satisfying $\alpha_\Gamma(\xi) = 0$. Then, there is (δ, r) with $\delta < 1/4$ such that $\alpha_\Gamma^{\delta,r}(\xi) = 0$, and hence

$$\iota_B \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \pi)_\sharp \circ \alpha_\Gamma^{\delta,r}(\xi) = 0$$

for any $\pi \in \mathrm{qRep}_P^{\varepsilon,\mathcal{G}_\Gamma^r}(\Gamma)$ with $\varepsilon < \min\{\frac{1}{4C_1}, \frac{1/4-\delta}{|\mathcal{G}_\Gamma^r|^2(1+3\varepsilon)}\}$. By Theorem 4.11, we obtain $\langle [\beta(\pi)], \xi \rangle = 0$. This is a quantitative version of the Hanke–Schick theorem [17, Theorem 3.9].

For the proof of Theorem 4.11, first of all, let $[\varphi_1, \varphi_2] \colon C(X) \to \mathbb{B}(\mathcal{H}) \rhd \mathbb{K}(\mathcal{H})$ be a quasi-homomorphism representing $\xi \in \mathrm{K}_0(X)$ such that φ_1 is ample; i.e., $\varphi_1^{-1}(\mathbb{K}(\mathcal{H})) = 0$. Set

$$\mathcal{D} := \mathbb{K}(\mathcal{H}) + \varphi_1\big(C(X)\big).$$

Remark 4.13. Note that $\mathcal{D}$ is quasi-diagonal. Indeed, let p_n be an increasing sequence of finite rank projections on $\mathcal{H}$ such that $\|[\varphi_1(f), p_n]\| \to 0$ as $n \to \infty$. There exists such p_n by Voiculescu's theorem (see, for example, [4, Theorem 7.2.5]) and the fact that any commutative C^*-algebra is quasi-diagonal [4, Proposition 7.1.5]. This sequence of projections also satisfies $\|[p_n, x]\| \to 0$ for any $x \in \mathbb{K}(\mathcal{H})$.

Let π denote a self-adjoint $(\varepsilon, \mathcal{G}_\Gamma)$-representation of Γ and let $\mathbf{v} := \beta(\pi)$. Let P_1, P_2, and V be as in the proof of Proposition 4.9. Set

$$p_\pi := \big(\mathrm{id}_{C(X)\otimes\mathbb{M}_I} \otimes \pi\big)(P_{\mathbf{v}}) \in C(X) \otimes B \otimes \mathbb{M}_I.$$

Moreover, let $p_{\pi,i} := (\varphi_i \otimes \mathrm{id})(p_\pi)$, $p_{\mathbf{v},i} := (\varphi_i \otimes \mathrm{id})(p_{\mathbf{v}}) \in \mathcal{D} \otimes B \otimes \mathbb{M}_I$ (for $i = 1, 2$), and

$$v_\pi := \begin{pmatrix} p_{\pi,2} & 1_n - p_{\pi,2} \\ 1_n - p_{\pi,2} & p_{\pi,2} \end{pmatrix}, \quad v_{\mathbf{v}} := \begin{pmatrix} p_{\mathbf{v},2} & 1_n - p_{\mathbf{v},2} \\ 1_n - p_{\mathbf{v},2} & p_{\mathbf{v},2} \end{pmatrix}.$$

Lemma 4.14. *For $0 < \varepsilon < (60|\mathcal{G}_\Gamma^3|^2)^{-1}$, both $(\mathrm{id}_{\mathcal{D}} \otimes \pi)(V \operatorname{diag}(P_1, 1 - P_2)V)$ and $v_\pi \operatorname{diag}(p_{\pi,1}, 1 - p_{\pi,2})v_\pi$ are $(15|\mathcal{G}_\Gamma^3|\varepsilon, 3)$-projections and*

$$\begin{aligned}&\big[(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1 - P_2)V\big), |I|\big]_{15|\mathcal{G}_\Gamma^3|\varepsilon,3} \\ &\quad= \big[v_\pi \operatorname{diag}(p_{\pi,1}, 1 - p_{\pi,2})v_\pi, |I|\big]_{15|\mathcal{G}_\Gamma^3|^3\varepsilon,3}\end{aligned}$$

holds.

Proof. By Lemma 4.4, $\mathrm{id}_{\mathcal{D}} \otimes \pi$ is a $(|\mathcal{G}_\Gamma^3|^2\varepsilon, 3, 1)$-$*$-homomorphism. Hence, by Proposition 4.5, we have

$$(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1 - P_2)V\big) \in \mathrm{P}_{|I|}^{|\mathcal{G}_\Gamma^3|^2\varepsilon,3}\big(\mathbb{K}(\mathcal{H}) \otimes B\big).$$

Moreover, since

$$(\mathrm{id}_{\mathcal{D}} \otimes \pi)(\varphi_i \otimes \mathrm{id}_{C^*(\Gamma)}) = (\varphi_i \otimes \mathrm{id}_{C^*(\Gamma)})(\mathrm{id}_{C(X)} \otimes \pi)$$

as completely bounded maps, we have $(\mathrm{id}_{\mathcal{D}} \otimes \pi)(V) = v_\pi$ and $(\mathrm{id}_{\mathcal{D}} \otimes \pi)(P_i) = p_{\pi,i}$ for $i = 1, 2$. Therefore, Proposition 4.7 implies that

$$\begin{aligned}&\big\|(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1 - P_2)V\big) - v_\pi \operatorname{diag}(p_{\pi,1}, 1 - p_{\pi,2})v_\pi\big\| \\ &\le \big\|(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1 - P_2)V\big) - (\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1 - P_2)\big)(\mathrm{id}_{\mathcal{D}} \otimes \pi)(V)\big\| \\ &\quad + \big\|(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(V \operatorname{diag}(P_1, 1-P_2)\big)v_\pi - (\mathrm{id}_{\mathcal{D}} \otimes \pi)(V)(\mathrm{id}_{\mathcal{D}} \otimes \pi)\big(\operatorname{diag}(P_1, 1-P_2)\big)v_\pi\big\| \\ &\le \big\|V \operatorname{diag}(P_1, 1 - P_2)\big\| \cdot \|V\| \cdot |\mathcal{G}_\Gamma^3|^2\varepsilon + \|v_\pi\| \cdot \big\|\operatorname{diag}(P_1, 1 - P_2)\big\| \cdot \|V\| \cdot |\mathcal{G}_\Gamma^3|^2\varepsilon \\ &\le 3|\mathcal{G}_\Gamma^3|^2\varepsilon.\end{aligned}$$

This shows the lemma by [30, Lemma 1.7], which claims that if p is a (ε, r)-projection and $\|p - q\| < \varepsilon$, then q is a $(5\varepsilon, r)$-projection and $[p]_{5\varepsilon,r} = [q]_{5\varepsilon,r}$. ∎

Lemma 4.15. *For $0 < \varepsilon < (800|I|^2)^{-1}$, both the elements $v_\pi \operatorname{diag}(p_{\pi,1}, 1_I - p_{\pi,2})v_\pi$ and $v_{\mathbf{v}} \operatorname{diag}(p_{\mathbf{v},1}, 1_I - p_{\mathbf{v},2})v_{\mathbf{v}}$ are $(200|I|^2\varepsilon, 3)$-projections and*

$$\big[v_\pi \operatorname{diag}(p_{\pi,1}, p_{\pi,2})v_\pi, |I|\big]_{200|I|^2\varepsilon} = \big[v_{\mathbf{v}} \operatorname{diag}(p_{\mathbf{v},1}, p_{\mathbf{v},2})v_{\mathbf{v}}, |I|\big]_{200|I|^2\varepsilon,r}$$

holds.

Proof. As recalled in Remark 2.16, the Čech 1-cocycle $\mathbf{v} = \beta(\pi)$ satisfies

$$\big\| v_{\mu\nu}(x) - \pi(\gamma_{\mu\nu})\big\| < 4\varepsilon.$$

Then, we have

$$\big\| p_\pi(x) - p_{\mathbf{v}}(x)\big\| = \Big\| \sum_{\mu,\nu} \eta_\mu(x)\eta_\nu(x)\big(\pi(\gamma_{\mu\nu}) - v_{\mu\nu}(x)\big) \otimes e_{\mu\nu}\Big\| \le 4|I|^2\varepsilon. \tag{4.16}$$

This implies that $\| p_{\pi,i} - p_{\mathbf{v},i}\| = \|\varphi_i(p_\pi - p_{\mathbf{v}})\| \le 4|I|^2\varepsilon$ and hence

$$\|v_\pi - v_{\mathbf{v}}\| = \left\| \begin{pmatrix} p_{\pi,2} - p_{\mathbf{v},2} & p_{\mathbf{v},2} - p_{\pi,2} \\ p_{\mathbf{v},2} - p_{\pi,2} & p_{\pi,2} - p_{\mathbf{v},2}\end{pmatrix} \right\| \le 2 \cdot 4|I|^2\varepsilon = 8|I|^2\varepsilon.$$

Therefore, we get

$$\begin{aligned}
&\big\|v_\pi \operatorname{diag}(p_{\pi,1}, 1_I - p_{\pi,2})v_\pi - v_{\mathbf{v}} \operatorname{diag}(p_{\mathbf{v},1}, 1_I - p_{\mathbf{v},2})v_{\mathbf{v}}\big\| \\
&\quad \le \big\|(v_\pi - v_{\mathbf{v}}) \operatorname{diag}(p_{\mathbf{v},1}, 1_I - p_{\mathbf{v},2})v_{\mathbf{v}}\big\| + \big\|v_\pi \operatorname{diag}(p_{\mathbf{v},1}, 1_I - p_{\mathbf{v},2})(v_{\mathbf{v}} - v_\pi)\big\| \\
&\qquad + \big\|v_\pi \operatorname{diag}(p_{\pi,1} - p_{\mathbf{v},1}, p_{\mathbf{v},2} - p_{\pi,2})v_\pi\big\| \\
&\quad \le \|v_\pi - v_{\mathbf{v}}\| + \|v_\pi\| \cdot \|v_\pi - v_{\mathbf{v}}\| + \|v_\pi\|^2 \max\big\{\|p_{\pi,1} - p_{\mathbf{v},1}\|, \|p_{\pi,2} - p_{\mathbf{v},2}\|\big\} \\
&\quad \le 8|I|^2\varepsilon + 2 \cdot 8|I|^2\varepsilon + 2^2 \cdot 4|I|^2\varepsilon = 40|I|^2\varepsilon.
\end{aligned}$$

Here, we use the fact that $\|p_{\mathbf{v},i}\| = 1$, $\|v_{\mathbf{v}}\| = 1$, and $\|v_\pi\| \le 2$, which follows from $\|v_\pi^2 - 1_{2I}\| \le 4|I|^2\varepsilon \le 1$. Now, [30, Lemma 1.7] concludes the proof since the element $v_{\mathbf{v}} \operatorname{diag}(p_{\mathbf{v},1}, 1_I - p_{\mathbf{v},2})v_{\mathbf{v}}$ is a projection. ∎

Proof of Theorem 4.11. Let $C_1 := \max\{15|\mathcal{G}_\Gamma^3|^2, 200|I|^2\}$. Then, Lemmas 4.14 and 4.15 conclude the proof as

$$\begin{aligned}
\iota_B(\mathrm{id}_{\mathfrak{D}} \otimes \pi)_\sharp\big(\alpha_\Gamma^{\mathrm{alg}}(\xi)\big) &= \iota_B\big[\pi\big(V \operatorname{diag}(P_1, 1 - P_2)V\big), |I|\big]_{C_1\varepsilon,3} \\
&= \iota_B\big[v_\pi \operatorname{diag}(p_{\pi,1}, 1 - p_{\pi,2})v_\pi, |I|\big]_{C_1\varepsilon,3} \\
&= \iota_B\big[v_{\mathbf{v}} \operatorname{diag}(p_{\mathbf{v},1}, p_{\mathbf{v},2})v_{\mathbf{v}}, |I|\big]_{C_1\varepsilon,3} \\
&= [p_{\mathbf{v},1}, p_{\mathbf{v},2}] = \big\langle [p_{\mathbf{v}}], \xi\big\rangle = \big\langle [\beta(\pi)], \xi\big\rangle,
\end{aligned}$$

where $[p_{\mathbf{v},1}, p_{\mathbf{v},2}] \in \mathrm{K}_0(B)$ denotes the difference class. ∎

Theorem 4.11 is related to the Connes–Gromov–Moscovici index formula [8, Théorème 10], which is generalized in [12]. Let τ be a tracial state on a C^*-algebra A. For a bundle E of finitely generated Hilbert A-modules, let $\mathrm{ch}_\tau(E) \in \Omega^{\mathrm{even}}(M)$ denote the

Chern character defined in [33, Definition 5.1]. In particular, if $A = \mathbb{C}$ and τ is the identity map, then $\mathrm{ch}_\tau(E)$ is the usual Chern character.

Corollary 4.17 (cf. [12, Theorem 3.6]). *Let $\pi \in \mathrm{qRep}_P^{\varepsilon,\mathcal{G}_\Gamma^3}(\Gamma)$ be a self-adjoint $(\varepsilon, \mathcal{G}_\Gamma^3)$-representation for $\varepsilon < (4C_1)^{-1}$ and let τ be a trace on A. Then, for any elliptic operator D on M with the principal symbol $\sigma(D)$, one has*

$$(\tau \circ \pi_\sharp)\big(\alpha_{\Gamma,\Lambda}^{\delta,r}([M])\big) = \int_{T^*M} \mathrm{ch}_\tau(E_{\beta(\pi)})\,\mathrm{ch}\big(\sigma(D)\big)\,\mathrm{Td}(T_{\mathbb{C}}M).$$

Proof. Apply Schick's L^2-index theorem [33, Theorem 6.10] for the index pairing $\tau(\langle \mathbf{v}, [M]\rangle) = \tau(\mathrm{ind}\, D_{E_{\beta(\pi)}})$. ∎

4.3. Relative quantitative index pairing

Now, we establish a relative version of the quantitative index pairing in Section 4.2. Let $\mathcal{G} = (\mathcal{G}_\Gamma, \mathcal{G}_\Lambda)$ be a finite symmetric generating set of (Γ, Λ) in the sense of Definition 2.23. We write $\mathcal{G}^r := (\mathcal{G}_\Gamma^r, \mathcal{G}_\Lambda^r)$ and $|\mathcal{G}^r| := \max\{|\mathcal{G}_\Gamma^r|, |\mathcal{G}_\Lambda^r|\}$. Let l_Γ and l_Λ denote the word length function on Γ and Λ with respect to $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Lambda$, respectively. Then, the assumption $\phi(\mathcal{G}_\Lambda) \subset \mathcal{G}_\Lambda$ implies that $\phi(C^*(\Lambda)_r) \subset C^*(\Gamma)_r$. We put the structure of a filtered C^*-algebra on $C\phi$ as

$$(C\phi)_r := \big\{(a, b_s) \in C\phi \mid a \in C^*(\Lambda)_r, b_s \in C^*(\Gamma)_r\big\}.$$

Let (X, Y) be a pair of connected finite CW-complexes. As in Lemma 2.5, let

$$U_{\mathcal{W}} := -e^{-\pi i \rho_0} P_{\mathcal{W}} + 1 - P_{\mathcal{W}} \in \big(C_0\big((Y_2')^\circ\big) \otimes C^*(\Lambda)_1 \otimes \mathbb{M}_I\big)^+,$$
$$V_{\mathcal{V},s} := -e^{-\pi i \rho_s} P_{\mathcal{V}} + 1 - P_{\mathcal{V}} \in \big(C_0(X_2^\circ) \otimes (C^*\Gamma)_1 \otimes \mathbb{M}_I\big)^+.$$

Then, $(U_{\mathcal{W}}, V_{\mathcal{V},s})$ is a $(0,1)$-unitary of $(C_0(X_2^\circ) \otimes C\phi)^+$ such that $[(U_{\mathcal{W}}, V_{\mathcal{V},s})] = \ell_{\Gamma,\Lambda}$.

Proposition 4.18. *There is a group homomorphism*

$$\alpha_{\Gamma,\Lambda}^{\mathrm{alg}} \colon \mathrm{K}_0(X,Y) \to \mathrm{K}_1^{0,2}\big(\mathbb{K}(\mathcal{H}) \otimes C\phi\big)$$

such that $\iota_{C\phi}(\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}(\xi)) = \alpha_{\Gamma,\Lambda}(\xi)$ for any $\xi \in \mathrm{K}_0(X,Y)$.

Proof. Let $[\varphi_1, \varphi_2] \colon C_0(X_2^\circ) \to \mathbb{B}(\mathcal{H}) \rhd \mathbb{K}(\mathcal{H})$ be a quasi-homomorphism representing $\xi \in \mathrm{K}_0(X,Y)$. Let $U_i := \varphi_i(U_{\mathcal{W}})$ and $V_{i,s} := \varphi_i(V_{\mathcal{V},s})$ for $i = 1, 2$. Then,

$$(U_1U_2^*, V_{1,s}V_{2,s}^*) \in \big(\mathbb{K}(\mathcal{H}) \otimes C\phi\big)^+$$

is a $(0,2)$-unitary. Now, we define the map $\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}$ as

$$\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}(\xi) := \big[(U_1U_2^*, V_{1,s}V_{2,s}^*)\big] \in \mathrm{K}_1^{0,2}\big(\mathbb{K}(\mathcal{H}) \otimes C\phi\big).$$

Then, it is straightforward to check that $\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}$ satisfies $\iota_{C\phi} \circ \alpha_{\Gamma,\Lambda}^{\mathrm{alg}} = \alpha_{\Gamma,\Lambda}$ in a similar fashion to Proposition 4.9. It is also checked in the same way as in Proposition 4.9 that the map $\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}$ is well defined since it is independent of the choice of a representative $[\varphi_1, \varphi_2]$. ∎

Definition 4.19. We call $\alpha^{\mathrm{alg}}_{\Gamma,\Lambda}$ as in Proposition 4.18 the *algebraic relative Mishchenko–Fomenko higher index*. For $r > 2$, we call the composition $\alpha^{\delta,r}_{\Gamma,\Lambda} := \iota^{\varepsilon,r}_{0,2} \circ \alpha^{\mathrm{alg}}_{\Gamma,\Lambda} \colon \mathrm{K}_0(X,Y) \to \mathrm{K}_1^{\delta,r}(\mathbb{K}(\mathcal{H}) \otimes C\phi)$ the *quantitative relative higher index*.

Next, we construct a complete (ε, r, κ)-$*$-homomorphism from $C\phi$ to a certain C^*-algebra associated to a stably relative quasi-representation. Hereafter, let (X, Y) be a pair of connected finite CW-complexes with a good open cover $\mathcal{U}$. Let $\Gamma := \pi_1(X)$ and $\Lambda := \pi_1(Y)$. Moreover, we choose a collection of translation functions $\{\gamma_{\mu\nu}\}_{\mu,\nu\in I}$ of $\widetilde{X}$ and $\{\lambda_{\mu\nu}\}_{\mu,\nu\in I}$ of $\widetilde{Y}$ such that $\phi(\lambda_{\mu\nu}) = \gamma_{\mu\nu}$ for $\mu, \nu \in I$ such that $U_{\mu\nu} \cap Y \neq \emptyset$. Let $\mathcal{G}_\Gamma = \{\gamma_{\mu\nu}\}$ and $\mathcal{G}_\Lambda = \{\lambda_{\mu\nu}\}$. We write $\mathcal{G}^r := (\mathcal{G}^r_\Gamma, \mathcal{G}^r_\Lambda)$ and $|\mathcal{G}^r| := \max\{|\mathcal{G}^r_\Gamma|, |\mathcal{G}^r_\Lambda|\}$.

Let

$$\mathcal{S} := \begin{pmatrix} C_0[-1,1) & C_0[-1,0) \\ C_0[-1,0) & C[-1,0] \end{pmatrix}, \quad \mathcal{S}_0 := \begin{pmatrix} C_0(-1,1) & C_0(-1,0) \\ C_0(-1,0) & C_0(-1,0) \end{pmatrix}$$

and let $\widehat{\mathcal{S}} := \{(f,g) \in \mathcal{S} \oplus \mathcal{S} \mid f - g \in \mathcal{S}_0\}$. Then, the embedding $C_0(-1,1) \to \mathcal{S}_0$ to the left upper component induces a KK-equivalence and hence $\mathrm{K}_*(\mathcal{S}_0 \otimes D) \cong \mathrm{K}_{*-1}(D)$ for any C^*-algebra D. We write θ for the quasi-homomorphism $[\mathrm{pr}_1, \mathrm{pr}_2] \colon \widehat{\mathcal{S}} \to \mathcal{S} \rhd \mathcal{S}_0$, where pr_i (for $i = 1, 2$) denotes the projection to the ith component.

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_0, u) \in \mathrm{qRep}^{\varepsilon,\mathcal{G}^r}_{P,Q}(\Gamma, \Lambda)$ be a self-adjoint stably relative $(\varepsilon, \mathcal{G}^r)$-representation (we say that $\boldsymbol{\pi}$ is self-adjoint if each π_i is a self-adjoint representation). Pick a continuous path $\{\bar{u}_s\}_{s\in[1,2]}$ of unitaries in $\mathrm{U}(\mathbb{B}((P \oplus Q)^{\oplus 2}))$ such that $\bar{u}_1 = \mathrm{diag}(u, u^*)$ and $\bar{u}_2 = 1$. We associate to $\boldsymbol{\pi}$ continuous families of maps $\widetilde{\pi}_{1,s}, \widetilde{\pi}_{2,s} \colon \mathcal{G}^r_\Lambda \to \mathbb{B}((P \oplus Q)^{\oplus 2})$ parametrized by $s \in [1, 2]$ defined as

$$\begin{aligned} \widetilde{\pi}'_{1,s}(\gamma) &:= (s-1)\big(\mathrm{diag}\,\big(\pi_1\big(\phi(\gamma)\big), \pi_0(\gamma), 1_{P\oplus Q}\big)\big) \\ &\qquad + (2-s)\bar{u}_1^*\big(\mathrm{diag}\,\big(\pi_2\big(\phi(\gamma)\big), \pi_0(\gamma), 1_{P\oplus Q}\big)\big)\bar{u}_1, \\ \widetilde{\pi}_{2,s}(\gamma) &:= \bar{u}_s^*\big(\mathrm{diag}\,\big(\pi_2\big(\phi(\gamma)\big), \pi_0(\gamma), 1_{P\oplus Q}\big)\big)\bar{u}_s, \end{aligned}$$

and $\widetilde{\pi}_{1,s}(\gamma) := \widetilde{\pi}'_{1,s}(\gamma)(\widetilde{\pi}'_{1,s}(\gamma)^*\widetilde{\pi}'_{1,s}(\gamma))^{-1/2}$. Then,

$$\bar{\pi}(a,b)(s) := \begin{cases} \big(\pi_1(b_s), \pi_2(b_s)\big) & s \in (0,1), \\ \big(\widetilde{\pi}_{1,2+s}(a), \widetilde{\pi}_{2,2+s}(a)\big) & s \in (-1,0] \end{cases}$$

determines a linear map $\bar{\pi} \colon (C\phi)_r \to B \otimes \widehat{\mathcal{S}}$.

Lemma 4.20. *For any $\boldsymbol{\pi} \in \mathrm{qRep}^{\varepsilon,\mathcal{G}^r}_{P,Q}(\Gamma, \Lambda)$ that is self-adjoint, the above $\bar{\pi}$ is a complete $(10|\mathcal{G}^r|^2\varepsilon, r, 1)$-$*$-homomorphism.*

Proof. Since $\|\widetilde{\pi}_{1,2}(\gamma) - \widetilde{\pi}'_{1,s}(\gamma)\| < \varepsilon$, we have $\|1 - \widetilde{\pi}'_{1,s}(\gamma)^*\widetilde{\pi}'_{1,s}(\gamma)\| < 2\varepsilon$ and hence

$$\begin{aligned} \big\|\widetilde{\pi}_{1,s}(\gamma) - \widetilde{\pi}_{1,2}(\gamma)\big\| &\leq \big\|\widetilde{\pi}'_{1,s}(\gamma) - \widetilde{\pi}_{1,2}(\gamma)\big\| + \big\|1 - \big(\widetilde{\pi}'_{1,s}(\gamma)^*\widetilde{\pi}'_{1,s}(\gamma)\big)^{-1/2}\big\| \\ &< 3\varepsilon. \end{aligned}$$

Then, we obtain that

$$\begin{aligned}\big\|\widetilde{\pi}_{1,2+s}(\gamma)\widetilde{\pi}_{1,2+s}(\gamma') - \widetilde{\pi}_{1,2+s}(\gamma\gamma')\big\| &\le \big\|\widetilde{\pi}_{1,2+s}(\gamma)\widetilde{\pi}_{1,2+s}(\gamma') - \widetilde{\pi}_{1,2}(\gamma)\widetilde{\pi}_{1,2}(\gamma')\big\| \\ &\quad + \big\|\widetilde{\pi}_{1,2}(\gamma)\widetilde{\pi}_{1,2}(\gamma') - \widetilde{\pi}_{1,2}(\gamma\gamma')\big\| \\ &\quad + \big\|\widetilde{\pi}_{1,2+s}(\gamma\gamma') - \widetilde{\pi}_{1,2}(\gamma\gamma')\big\| \\ &\le 2\cdot 3\varepsilon + \varepsilon + 3\varepsilon = 10\varepsilon; \end{aligned} \tag{4.21}$$

that is, each $\widetilde{\pi}_{i,2+s}$ is a $(10\varepsilon, \mathcal{G}^r)$-representation.

Now, Lemma 4.7 implies that each evaluation $\mathrm{ev}_s \circ \overline{\pi} \colon (C\phi)_r \to \mathbb{M}_2 \oplus \mathbb{M}_2$ is a complete $(10|\mathcal{G}^r|^2\varepsilon, r, 1)$-$*$-homomorphism, which finishes the proof. ∎

Therefore, by Proposition 4.5 we get a homomorphism

$$\theta \circ \iota_B \circ (\mathrm{id} \otimes \overline{\pi})_\sharp \colon \mathrm{K}_1^{\delta,r}(C\phi) \to \mathrm{K}_1(\mathcal{S}_0 \otimes B) \cong \mathrm{K}_0(B)$$

for $\delta > 0$ such that $\varepsilon + (1+3\varepsilon)\delta < 1/4$ and $r > 0$.

Theorem 4.22. *There is a constant $C_2 = C_2(\mathcal{U})$ depending only on $\mathcal{U}$ so that the following holds: for $0 < \varepsilon < (4C_2)^{-1}$, $\boldsymbol{\pi} \in \mathrm{qRep}_{P,Q}^{\varepsilon,\mathcal{G}^2}(\Gamma,\Lambda)$, and $\xi \in \mathrm{K}_0(X,Y)$, one has*

$$\big(\theta \circ \iota_B \circ (\mathrm{id}_{\mathbb{K}} \otimes \overline{\pi})_\sharp\big)\big(\alpha_{\Gamma,\Lambda}^{\mathrm{alg}}(\xi)\big) = \big\langle [\boldsymbol{\beta}(\boldsymbol{\pi})], \xi \big\rangle \in \mathrm{K}_0(B).$$

Remark 4.23. Here is a remark parallel to Remark 4.12. For any (δ, r) with $10|\mathcal{G}^r|^2\varepsilon + (1+40|\mathcal{G}^r|^2\varepsilon)\delta < 1/4$, the left-hand side of Theorem 4.22 is written as $\iota_B \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \pi)_\sharp \circ \alpha_{\Gamma,\Lambda}^{\delta,r}(\xi)$. Hence, if a K-homology class $\xi \in \mathrm{K}_0(X)$ satisfies $\alpha_{\Gamma,\Lambda}(\xi) = 0$, then there is (δ, r) with $\delta < 1/4$ such that $\alpha_\Gamma^{\delta,r}(\xi) = 0$. By Theorem 4.22, we have $\langle [\boldsymbol{\beta}(\boldsymbol{\pi})], \xi\rangle = 0$ for any $\boldsymbol{\pi} \in \mathrm{qRep}_{P,Q}^{\varepsilon,\mathcal{G}^r}(\Gamma,\Lambda)$ with $\varepsilon < \min\{\frac{1}{4C_2}, \frac{1/4-\delta}{10|\mathcal{G}_\Gamma^r|^2(1+3\varepsilon)}\}$. This is a quantitative version of Theorem 3.5.

Let $[\varphi_1, \varphi_2] \colon C_0(X_2^\circ) \to \mathbb{B}(\mathcal{H}) \rhd \mathbb{K}(\mathcal{H})$ be a quasi-homomorphism representing $\xi \in \mathrm{K}_0(X,Y)$ such that φ_1 is ample and let $\mathcal{D} := \mathbb{K}(\mathcal{H}) + \varphi_1(C_0(X_2^\circ))$. Then, $\mathcal{D}$ is nuclear and quasi-diagonal as is mentioned in Remark 4.13. Let U^i, V_s^i be as in the proof of Proposition 4.18. We consider the element

$$u_{\boldsymbol{\pi}} = (u_{\boldsymbol{\pi},s})_{s\in(-1,1)} := (\mathrm{id}_{C_0(X_2^\circ)\otimes\mathbb{M}_I} \otimes \overline{\pi})(U, V_s) \in C_0(X_2^\circ) \otimes \widehat{\mathcal{S}} \otimes B \otimes \mathbb{M}_I$$

and set $u_{\boldsymbol{\pi}}^i := (\varphi_i \otimes \mathrm{id})(u_{\boldsymbol{\pi}})$ for $i = 1, 2$.

Lemma 4.24. *For $0 < \varepsilon < (160|\mathcal{G}^2|^2)^{-1}$, both $(\mathrm{id}_{\mathbb{K}} \otimes \overline{\pi})((U^1, V_s^1)(U^2, V_s^2)^*)$ and $u_{\boldsymbol{\pi}}^1(u_{\boldsymbol{\pi}}^2)^*$ are $(40|\mathcal{G}^2|^2\varepsilon, 2)$-unitaries and*

$$\big[(\mathrm{id}_{\mathbb{K}} \otimes \overline{\pi})\big((U_1, V_{1,s})(U_2, V_{2,s})^*\big)\big]_{40|\mathcal{G}^2|^2\varepsilon,2} = \big[u_{\boldsymbol{\pi},s}^1(u_{\boldsymbol{\pi},s}^2)^*\big]_{40|\mathcal{G}^2|^2\varepsilon,2}$$

holds.

Proof. By Proposition 4.5 and Lemma 4.20, the element

$$(\mathrm{id}_{\mathbb{K}} \otimes \overline{\pi})\big((U_1, V_{1,s})(U_2, V_{2,s})^*\big)$$

is a $10|\mathcal{G}^2|^2\varepsilon$-unitary. By Lemma 4.4, the tensor product $(\mathrm{id}_{\mathfrak{D}} \otimes \overline{\pi})$ is well defined as a completely bounded map and $(\mathrm{id}_{\mathfrak{D}} \otimes \overline{\pi})(U^i, V^i_s)(s) = u^i_{\boldsymbol{\pi},s}$ holds for $s \in (-1, 1)$. Hence, we have

$$\big\|(\mathrm{id}_{\mathbb{K}} \otimes \overline{\pi})\big((U_1, V_{1,s})(U_2, V_{2,s})^*\big) - u^1_{\boldsymbol{\pi},s}(u^2_{\boldsymbol{\pi},s})^*\big\| \leq 10|\mathcal{G}^2|^2\varepsilon.$$

This shows the lemma by [30, Lemma 1.7], which claims that if u is a (ε, r)-unitary and $\|u - v\| < \varepsilon$ holds, then v is a $(4\varepsilon, r)$-unitary and $[u]_{4\varepsilon,r} = [v]_{4\varepsilon,r}$. ■

For the proof of Theorem 4.22, it is convenient to rephrase the proof of Theorem 3.3 in terms of unitaries $(U_{\mathcal{W}}, V_{\mathcal{V},s})$. Let

$$\mathcal{C}(X,Y) := \begin{pmatrix} C(X_2) & C_0\big(Y(1,2]\big) \\ C_0\big(Y(1,2]\big) & C_0\big(Y[1,2]\big)\end{pmatrix}, \quad \mathcal{C}_0(X,Y) := \begin{pmatrix} C_0(X_2^\circ) & C_0((Y_2')^\circ) \\ C_0\big((Y_2')^\circ\big) & C_0\big((Y_2')^\circ\big)\end{pmatrix}$$

and let $\widehat{\mathcal{C}}(X,Y) := \{(f,g) \in \mathcal{C}(X,Y) \oplus \mathcal{C}(X,Y) \mid f - g \in \mathcal{C}_0(X,Y)\}$. Then, the embedding $C_0(X_2^\circ) \to \mathcal{C}_0(X,Y)$ to the left upper component induces a KK-equivalence. Let $\theta_{X,Y}$ denote the quasi-homomorphism $[\mathrm{pr}_1, \mathrm{pr}_2]\colon \widehat{\mathcal{C}}(X,Y) \to \mathcal{C}(X,Y) \triangleright \mathcal{C}_0(X,Y)$. Then, the continuous map f and ι as in Theorem 3.3 induce

$$\iota_* \circ f^*\colon \mathcal{C}_0(X,Y)(0,1) \to C_0(X_2^\circ) \otimes \mathcal{S}_0,$$

which extends to a $*$-homomorphism from $\mathcal{C}(X,Y)(0,1)$ to $C_0(X_2^\circ) \otimes \mathcal{S}$ denoted by the same letter $\iota_* \circ f^*$.

Let $\mathfrak{v} := \boldsymbol{\beta}(\boldsymbol{\pi})$, $\mathbf{v}_j := \beta(\pi_j)$, and $\mathbf{v}_{j,s} := \beta(\tilde{\pi}_{j,s})$ for $j = 1,2$ and $s \in [1,2]$. Let $\tilde{p}_{\mathfrak{v},j} \in \mathcal{C}(X_1, Y_1) \otimes B \otimes \mathbb{M}_I$ for $j = 1,2$ denote the projections

$$\tilde{p}_{\mathfrak{v},j}(x) := \begin{cases} p_{\mathbf{v}_j}(x) & x \in X_1^\circ, \\ p_{\mathbf{v}_{j,r}}(y) & x = (y,r) \in Y_2'. \end{cases}$$

Then, $\tilde{p}_{\mathfrak{v},1} - \tilde{p}_{\mathfrak{v},2} \in \mathcal{C}_0(X,Y) \otimes B \otimes \mathbb{M}_I$; that is, $\tilde{p}_{\mathfrak{v}} := (\tilde{p}_{\mathfrak{v},1}, \tilde{p}_{\mathfrak{v},2})$ is a projection in $\widehat{\mathcal{C}}(X,Y) \otimes B \otimes \mathbb{M}_I$, such that $\theta_{X,Y}[(\tilde{p}_{\mathfrak{v},1}, \tilde{p}_{\mathfrak{v},2})] = [\mathfrak{v}]$. Now, the element

$$u_{\mathfrak{v},s} := (\iota_* \circ f^*)(\tilde{p}_{\mathfrak{v}} e^{2\pi i \rho_s} + 1 - \tilde{p}_{\mathfrak{v}}) \in C(X_2^\circ) \otimes \widehat{\mathcal{S}} \otimes B \otimes \mathbb{M}_I$$

is a unitary satisfying

$$\theta[u_{\mathfrak{v}}] = (\iota_* \circ f^*)[\mathfrak{v}] \otimes \beta \in \mathrm{K}_1\left(C_0(X_2^\circ) \otimes \mathcal{S}_0 \otimes B\right).$$

Lemma 4.25. *For $0 < \varepsilon < (1280|I|^2)^{-1}$, both the elements $u^1_{\boldsymbol{\pi},s}(u^2_{\boldsymbol{\pi},s})^*$ and $u^1_{\mathfrak{v},s}(u^2_{\mathfrak{v},s})^*$ are $(320|I|^2\varepsilon, 2)$-unitaries and*

$$\big[u^1_{\boldsymbol{\pi},s}(u^2_{\boldsymbol{\pi},s})^*\big]_{320|I|^2\varepsilon,2} = \big[u^1_{\mathfrak{v},s}(u^2_{\mathfrak{v},s})^*\big]_{320|I|^2\varepsilon,2}$$

holds.

Proof. By the definitions of f, $u_{\boldsymbol{\pi},s}$, and $u_{\mathfrak{v},s}$, we have

$$u_{\boldsymbol{\pi},s}(x) = -e^{-\pi i\rho(s,r)}\big(\pi_1(P_{\mathcal{V}}),\pi_2(P_{\mathcal{V}})\big)(x) + 1 - \big(\pi_1(P_{\mathcal{V}}),\pi_2(P_{\mathcal{V}})\big)(x),$$
$$u_{\mathfrak{v},s}(x) = -e^{-\pi i\rho(s,r)}(p_{\mathbf{v}_1},p_{\mathbf{v}_2})(x) + 1 - (p_{\mathbf{v}_1},p_{\mathbf{v}_2})(x),$$

for $(x,s)\in X_2(0,1)$ and

$$\begin{aligned} u_{\boldsymbol{\pi},s}(y,r) &= e^{2\pi i(r-1)}\big(\tilde{\pi}_{1,2+s}(P_{\mathcal{W}}),\tilde{\pi}_{2,2+s}(P_{\mathcal{W}})\big)(y) \\ &\quad + 1 - \big(\tilde{\pi}_{1,2+s}(P_{\mathcal{W}}),\tilde{\pi}_{2,2+s}(P_{\mathcal{W}})\big)(y), \\ u_{\mathfrak{v},s}(y,r) &= e^{2\pi i(r-1)}(p_{\mathbf{v}_{1,2+s}},p_{\mathbf{v}_{2,2+s}})(y) + 1 - (p_{\mathbf{v}_{1,2+s}},p_{\mathbf{v}_{2,2+s}})(y), \end{aligned}$$

for $(y,r,s)\in Y_2'(-1,0]$. Hence, (4.16) and (4.21) imply that

$$\begin{aligned} \big\|u_{\boldsymbol{\pi},s} - \iota_* f^*(u_{\mathfrak{v},s})\big\| &\le \big\|e^{2\pi i(r-1)}\big\|\,\big\|\big(\tilde{\pi}_{1,2+s}(P_{\mathcal{W}}),\tilde{\pi}_{2,2+s}(P_{\mathcal{W}})\big) - (p_{\mathbf{v}_{1,2+s}},p_{\mathbf{v}_{2,2+s}})\big\| \\ &\quad + \big\|\big(1-\big(\tilde{\pi}_{1,2+s}(P_{\mathcal{W}}),\tilde{\pi}_{2,2+s}(P_{\mathcal{W}})\big)\big) - \big(1-(p_{\mathbf{v}_{1,2+s}},p_{\mathbf{v}_{2,2+s}})\big)\big\| \\ &\le 2\cdot 4|I|^2\cdot 10\varepsilon = 80|I|^2\varepsilon \end{aligned}$$

for $s\in(-1,0]$. By the same argument, we also see that $\|u_{\boldsymbol{\pi},s} - \iota_* f^*(u_{\mathfrak{v},s})\| < 80|I|^2\varepsilon$ for $s\in[0,1)$. Again by [30, Lemma 1.7], this concludes the proof. ∎

Proof of Theorem 4.22. Let $C_2 := \max\{320|I|^2, 40|\mathcal{G}^2|^2\}$. Then, Lemmas 4.24 and 4.25 prove the theorem as

$$\begin{aligned} \theta\circ\iota_B(\mathrm{id}\otimes\bar{\pi})_\sharp\big(\alpha^{\mathrm{alg}}_{\Gamma,\Lambda}(\xi)\big) &= \theta\circ\iota_B\big[(\mathrm{id}_{\mathbb{K}}\otimes\bar{\pi})\big((U_1,V_{1,s})(U_2,V_{2,s})^*\big)\big]_{C_2\varepsilon,2} \\ &= \theta\circ\iota_B\big[u^1_{\boldsymbol{\pi},s}(u^2_{\boldsymbol{\pi},s})^*\big]_{C_2\varepsilon,2} = \theta\circ\iota_B\big[u^1_{\mathfrak{v},s}(u^2_{\mathfrak{v},s})^*\big]_{C_2\varepsilon,2} \\ &= \theta\big[u^1_{\mathfrak{v},s}(u^2_{\mathfrak{v},s})^*\big] = \theta\big\langle[u_{\mathfrak{v}}],\xi\big\rangle = \big\langle(\iota_*\circ f^*)[\mathfrak{v}]\otimes\beta,\xi\big\rangle \\ &= \big\langle[\mathfrak{v}],\xi\big\rangle\otimes\beta\in \mathrm{K}_1(B\otimes S). \end{aligned}$$ ∎

Corollary 4.26. *Let D be an elliptic differential operator on M, let $\varepsilon < (4C_2)^{-1}$, let $\pi\in\mathrm{qRep}^{\varepsilon,\mathcal{G}^2}_{P,Q}(\Gamma,\Lambda)$ be a self-adjoint stably relative $(\varepsilon,\mathcal{G}^2)$-representation, and let τ be a trace on A. Then, one has*

$$\big(\tau\circ\theta\circ\iota_{C_\phi}\circ(\mathrm{id}_{\mathbb{K}}\otimes\bar{\pi})_\sharp\big)\big(\mu_0^{\Gamma,\Lambda}([D])\big) = \int_{T^*M}\mathrm{ch}_\tau\big(\boldsymbol{\beta}(\boldsymbol{\pi})\big)\,\mathrm{ch}\big(\sigma(D)\big)\,\mathrm{Td}(T_{\mathbb{C}}M).$$

Proof. Let $\hat{D}$ be an elliptic operator on the invertible double $\hat{M} = M\sqcup_N(-M)$ with the principal symbol $\sigma(\hat{D})|_{\pm M} = \sigma(D)$. Let $i\colon M^\circ\to M$ denote the open embedding and let E_1, E_2 be vector bundles on $\hat{M}$ such that $i_*\boldsymbol{\beta}(\boldsymbol{\pi}) = [E_1]-[E_2]$. Then, Theorem 4.22 and the L^2-index theorem [33, Theorem 6.10] for the index pairing

$$\tau\big(\langle\boldsymbol{\beta}(\boldsymbol{\pi}),[D]\rangle\big) = \tau\big(\langle i_*\boldsymbol{\beta}(\boldsymbol{\pi}),[\hat{D}]\rangle\big) = \tau(\mathrm{ind}\,\hat{D}_{E_1} - \mathrm{ind}\,\hat{D}_{E_2})$$

show the corollary since the Chern character form $\mathrm{ch}_\tau(i_*\boldsymbol{\beta}(\boldsymbol{\pi})) = \mathrm{ch}_\tau(E_1)-\mathrm{ch}_\tau(E_2)$ is a compactly supported differential form on M° cohomologous to $\mathrm{ch}_\tau(\boldsymbol{\beta}(\boldsymbol{\pi}))$ in $H^*_c(M^\circ)$. ∎

5. Dual assembly map and almost flat bundles

In this section, we relate the dual higher index map $\beta_{\Gamma,\Lambda}$ defined in Proposition 2.3 with the almost monodromy correspondence, i.e., Theorem 2.24. The goal of this section is to show that the index pairing with elements of the subgroup $\mathrm{K}^0_{\text{s-af}}(X, Y)$ of almost flat K-theory class (in the sense of Definition 2.21) has rich information enough to detect the non-vanishing of the relative higher index under certain assumptions on the fundamental groups.

5.1. K-homology group of mapping cone C^*-algebras

Let A and B be separable C^*-algebras and let $\phi\colon A \to B$ be a $*$-homomorphism. Let us choose unital $*$-representations of unitization C^*-algebras $\sigma\colon A^+ \to \mathbb{B}(\mathcal{H})$ and $\tau\colon B^+ \to \mathbb{B}(\mathcal{K})$ such that τ and $\bar{\sigma} := \sigma \oplus \tau \circ \phi$ are ample representations; that is, $\tau^{-1}(\mathbb{K}(\mathcal{K})) = 0$ and $\bar{\sigma}^{-1}(\mathbb{K}(\bar{\mathcal{H}})) = 0$ (where $\bar{\mathcal{H}} := \mathcal{H} \oplus \mathcal{K}$). Note that we can choose σ as the zero representation if ϕ is injective.

For a C^*-algebra D, let $C_u(\mathfrak{T}, D)$ denote the C^*-algebra of bounded D-valued uniformly continuous functions on $\mathfrak{T} := [0, \infty)$. Hereafter, we identify $\mathfrak{T}$ with $[0, 1)$ by a reparametrization $t \mapsto s = t(1 + t^2)^{-1/2}$. Following [14], we define the C^*-algebras

$$\begin{aligned}
\mathfrak{D}(A) &:= \{T \in \mathbb{B}(\bar{\mathcal{H}}) \mid [T, \bar{\sigma}(a)] \in \mathbb{K}(\bar{\mathcal{H}})\ \forall a \in A\},\\
\mathfrak{D}(B) &:= \{T \in \mathbb{B}(\mathcal{K}) \mid [T, \tau(b)] \in \mathbb{B}(\mathcal{K})\ \forall b \in B\},\\
\mathfrak{C}(A) &:= \{T \in \mathfrak{D}(A) \mid T\bar{\sigma}(a) \in \mathbb{K}(\bar{\mathcal{H}})\ \forall a \in A\},\\
\mathfrak{D}_L(A) &:= \{T_s \in C_u(\mathfrak{T}, \mathfrak{D}(A)) \mid [T_s, \bar{\sigma}(a)] \in C_0([0,1), \mathbb{K}(\bar{\mathcal{H}}))\ \forall a \in A\},\\
\mathfrak{C}_L(A) &:= C_u(\mathfrak{T}, \mathfrak{C}(A)) \cap \mathfrak{D}_L(A),\\
\mathfrak{D}^0_L(A) &:= \{T_s \in \mathfrak{D}_L(A) \mid T_0 = 0\},\\
\mathfrak{C}^0_L(A) &:= \{T_s \in \mathfrak{C}_L(A) \mid T_0 = 0\}.
\end{aligned} \tag{5.1}$$

Note that $\mathfrak{D}(B) \subset \mathfrak{D}(A)$ as C^*-subalgebras of $\mathbb{B}(\bar{\mathcal{H}})$. We write $\phi_{\mathfrak{D}}$ for this inclusion.

Lemma 5.2. *The inclusions*

- $\iota_1\colon \mathfrak{C}^0_L(A) \to \mathfrak{C}_L(A)$,
- $\iota_2\colon \mathfrak{C}^0_L(A) \to \mathfrak{D}^0_L(A)$*, and*
- $\iota_3\colon \mathfrak{D}(A)(0,1) \to \mathfrak{D}^0_L(A)$

induce isomorphisms of K*-groups.*

Proof. Note that ι_3 is homotopic to the inclusion of $\mathfrak{D}(A)(0,1) \cong \mathfrak{D}(A)(0, \frac{1}{2})$ into $\mathfrak{D}^0_L(A)$. They follow from the vanishing of K-groups of $\mathfrak{C}_L(A)/\mathfrak{C}^0_L(A) \cong \mathfrak{C}(A)$, $\mathfrak{D}^0_L(A)/\mathfrak{C}^0_L(A)$, and $\mathfrak{D}^0_L(A)/\mathfrak{D}(A)(0, \frac{1}{2}) \cong \mathfrak{D}_L(A)$, which are proved in [23, Proposition 5.3.7], [14, Proposition 4.3 (b)], and [14, Proposition 4.3 (a)], respectively. ∎

We consider two homomorphisms

$$\Theta_{A,*}\colon \mathrm{K}_{1-*}\left(\mathfrak{D}^0_L(A)\right) \to \mathrm{KK}_*\left(A, C_0(0,1)\right)$$

for $* = 0, 1$ given by

$$\Theta_{A,0}\big([u_s]\big) := \left[\bar{\mathcal{H}}(0,1) \oplus \bar{\mathcal{H}}(0,1)^{\mathrm{op}}, \sigma \oplus \sigma, \begin{pmatrix} 0 & u_s^* \\ u_s & 0 \end{pmatrix}\right],$$
$$\Theta_{A,1}\big([p_s]\big) := \big[\bar{\mathcal{H}}(0,1), \bar{\sigma}, 2p_s - 1\big],$$

for $u_s \in \mathrm{U}(\mathbb{M}_N(\mathfrak{D}_L(A)^0)^+)$ and $p_s \in \mathrm{P}(\mathbb{M}_N(\mathfrak{D}_L^0(A)^+))$.

Lemma 5.3. *The above $\Theta_{A,0}$ and $\Theta_{A,1}$ are isomorphisms.*

Proof. By Lemma 5.2, it suffices to show that the composition

$$\Psi_{A,*}\colon \mathrm{K}_{1-*}\left(\mathfrak{D}(A)(0,1)\right) \xrightarrow{(\iota_3)_*} \mathrm{K}_{1-*}\left(\mathfrak{D}_L^0(A)\right) \xrightarrow{\Theta_{A,*}} \mathrm{KK}_*\left(A, C_0(0,1)\right)$$

is an isomorphism.

For a locally compact space X, let

$$\mathfrak{D}(A,X) := \big\{T \in C_b^{\mathrm{st}}\big(X, \mathbb{B}(\bar{\mathcal{H}})\big) \mid \big[T, \sigma(a)\big] \in C_0\big(X, \mathbb{K}(\bar{\mathcal{H}})\big)\big\},$$
$$\mathfrak{D}_0(A,X) := \overline{C_0(X) \cdot \mathfrak{D}(A,X)},$$

where $C_b^{\mathrm{st}}(X, \mathbb{B}(\mathcal{H}))$ denotes the C^*-algebra of bounded strictly continuous $\mathbb{B}(\mathcal{H})$-valued functions on X, which is isomorphic to the bounded operator algebra on the Hilbert $C_0(X)$-module $\mathcal{H} \otimes C_0(X)$. By Kasparov's generalized Voiculescu theorem [25, Theorem 5], the representation $\bar{\sigma} \otimes 1\colon A \to \mathbb{B}(\bar{\mathcal{H}} \otimes C_0(X)) \cong C_b^{\mathrm{st}}(X, \mathbb{B}(\bar{\mathcal{H}}))$ is absorbing. Hence, the duality of KK-theory [35, Theorem 3.2] implies that the homomorphisms $\tilde{\Theta}_{A,X,*}\colon \mathrm{K}_{1-*}(\mathfrak{D}(A,X)) \to \mathrm{KK}_*(A, C_0(X))$ given by

$$\tilde{\Theta}_{A,X,0}\big([u_x]\big) := \left[C_0(X, \bar{\mathcal{H}} \oplus \bar{\mathcal{H}}^{\mathrm{op}}), \sigma \oplus \sigma, \begin{pmatrix} 0 & u_x^* \\ u_x & 0 \end{pmatrix}\right],$$
$$\tilde{\Theta}_{A,X,1}\big([p_x]\big) := \big[C_0(X, \bar{\mathcal{H}}), \sigma, 2p_x - 1\big],$$

are isomorphic.

The remaining task is to show that the inclusions

(1) $\mathfrak{D}(A)(0,1) \to \mathfrak{D}_0(A,(0,1))$ and

(2) $\mathfrak{D}_0(A,(0,1)) \to \mathfrak{D}(A,(-1,2))$

induce isomorphisms of K-groups. Indeed, the composition of these two inclusions is homotopic to the inclusion $\mathfrak{D}(A)(0,1) \to \mathfrak{D}(A,(0,1))$.

For (1), apply the five lemma for the map between long exact sequences of K-groups associated to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{D}(A)(0,1) & \longrightarrow & \mathfrak{D}(A)[0,1) & \longrightarrow & \mathfrak{D}(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathfrak{D}_0\big(A,(0,1)\big) & \longrightarrow & \mathfrak{D}_0\big(A,[0,1)\big) & \longrightarrow & \mathfrak{D}(A) & \longrightarrow & 0. \end{array}$$

Note that $\mathfrak{D}(A)[0,1)$ and $\mathfrak{D}_0(A,[0,1))$ have trivial K-groups since they are contractible (indeed, the continuous path of $*$-endomorphisms $\psi_t(T)(s) := T(\max\{s+t,1\})$ connects the identity and the zero map). For (2), observe that

$$\mathfrak{D}\big(A,(-1,2)\big)\big/\mathfrak{D}_0\big(A,(0,1)\big) \cong \mathfrak{D}\big(A,(-1,0]\big) \oplus \mathfrak{D}\big(A,[1,2)\big)$$

and

$$\mathrm{K}_*\left(\mathfrak{D}\big(A,[0,1)\big)\right) \cong \mathrm{KK}_{1-*}\left(A, C_0[0,1)\right) = 0. \qquad \blacksquare$$

It is proved in [14, Proposition 4.2] that $\mathfrak{D}_L(A)/\mathfrak{C}_L(A)$ is canonically isomorphic to $C_u(\mathfrak{T},\mathfrak{D}(A))/C_u(\mathfrak{T},\mathfrak{C}(A))$. Hence, the $*$-homomorphism $\mathfrak{D}(A) \to C_u([0,1),\mathfrak{D}(A))$ mapping $T \in \mathfrak{D}(A)$ to the constant function with the value T induces a $*$-homomorphism

$$c\colon \mathfrak{D}(A) \to C_u\big(\mathfrak{T},\mathfrak{D}(A)\big)\big/C_u^0\big(\mathfrak{T},\mathfrak{C}(A)\big) \cong \mathfrak{D}_L(A)/\mathfrak{C}_L^0(A),$$

where $C_u^0(\mathfrak{T},\mathfrak{C}(A)) := \{T_s \in C_u(\mathfrak{T},\mathfrak{C}(A)) \mid T_0 = 0\}$. Set

$$\mathfrak{D}_L(\phi) := \big\{T_s \in \mathfrak{D}_L(A) \mid T_0 \in \mathfrak{D}(B), T_s - T_0 \in \mathfrak{C}(A)\big\}.$$

Then, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{C}_L^0(A) & \longrightarrow & \mathfrak{D}_L(\phi) & \longrightarrow & \mathfrak{D}(B) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle c\circ\phi_{\mathfrak{D}}} & & \\
0 & \longrightarrow & \mathfrak{C}_L^0(A) & \longrightarrow & \mathfrak{D}_L(A) & \longrightarrow & \mathfrak{D}_L(A)/\mathfrak{C}_L^0(A) & \longrightarrow & 0.
\end{array}$$

Let ι_4 denote the inclusion $\mathfrak{C}_L^0(A) \to \mathfrak{D}_L(\phi)$ and let q denote the quotient $\mathfrak{D}_L(\phi) \to \mathfrak{D}(B)$.

Lemma 5.4. *The diagram*

$$\begin{array}{ccc}
\mathrm{K}_*\left(\mathfrak{D}(B)(0,1)\right) & \xrightarrow{\ \partial\ } & \mathrm{K}_*\left(\mathfrak{C}_L^0(A)\right) \\
\downarrow{\scriptstyle \Psi_{B,*}} & & \downarrow{\scriptstyle \Theta_{A,*}\circ(\iota_2)_*} \\
\mathrm{KK}_{1-*}\left(B, C_0(0,1)\right) & \xrightarrow{\ \phi^*\ } & \mathrm{KK}_{1-*}\left(A, C_0(0,1)\right)
\end{array}$$

commutes.

Proof. Let $k\colon \mathfrak{C}_L^0(A) \to \mathfrak{D}_L(\phi)$ denote the inclusion. We regard an element $f \in Ck$ as a $\mathfrak{D}(A)$-valued continuous function on $[0,1]_t \times [0,1)_s$ such that $f(0,\cdot) \in \mathfrak{C}_L^0(\phi)$, $f(t,\cdot) \in \mathfrak{D}_L(\phi)$ for $t \in (0,1)$ and $f(1,\cdot) = 0$. Let

$$\varphi\colon Ck \to \big(\mathfrak{D}_L(\phi)/\mathfrak{C}_L^0(A)\big)(0,1) \cong \mathfrak{D}(B)(0,1)$$

denote the quotient (in other words, the evaluating homomorphism at $s = 0$) and let $l\colon Ck \to \mathfrak{C}^0_L(A)$ denote the evaluating $*$-homomorphism at $t = 0$. Since φ_* is an isomorphism and $l_* \circ (\varphi_*)^{-1} = \partial$, it suffices to show that the diagram

$$\begin{array}{ccc} \mathrm{K}_*(Ck) & \xrightarrow{l_*} & \mathrm{K}_*\big(\mathfrak{C}^0_L(A)\big) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle (\iota_2)_*} \\ \mathrm{K}_*\big(\mathfrak{D}(B)(0,1)\big) & \xrightarrow{\iota_3\circ\phi_{\mathfrak{D}}} & \mathrm{K}_*\big(\mathfrak{D}^0_L(A)\big) \\ \downarrow{\scriptstyle \Psi_{A,*}} & & \downarrow{\scriptstyle \Theta_{A,*}} \\ \mathrm{K}_*\big(\mathfrak{D}(B,(0,1))\big) & \xrightarrow{\phi^*} & \mathrm{K}_*\big(\mathfrak{D}(A,(0,1))\big) \end{array}$$

commutes. The lower square commutes by definition. Since the continuous path

$$\theta_\kappa(f)(s) = \begin{cases} f(s, 2\kappa s) & \kappa \in [0,1/2], \\ f(2\kappa s, s) & \kappa \in [1/2,1] \end{cases}$$

of $*$-homomorphisms from Ck to $\mathfrak{D}^0_L(A)$ for $\kappa \in [0,1]$ satisfies $\theta_0 = \iota_2 \circ l$ and $\theta_1 = \iota_3 \circ \phi_{\mathfrak{D}} \circ \varphi$, we obtain that the upper square also commutes. ∎

Let $\widetilde{\mathcal{H}}$ denote the Hilbert $C_0(-1,1)$-module $\mathcal{H}(-1,0) \oplus \mathcal{K}(-1,1)$. We define the $*$-homomorphism $\widetilde{\sigma}\colon C\phi \to \mathbb{B}(\widetilde{\mathcal{H}})$ by

$$\pi(a, b_s)(s) = \begin{cases} \overline{\sigma}(a) & s \in (-1,0), \\ \sigma(b_s) & s \in [0,1), \end{cases}$$

and the group homomorphism

$$\Theta_\phi\colon \mathrm{K}_1\big(\mathfrak{D}_L(\phi)\big) \to \mathrm{KK}\big(C\phi, C_0(\mathbb{R})\big)$$

by

$$\Theta_\phi\big([u_s]\big) := \left[\widetilde{\mathcal{H}} \oplus \widetilde{\mathcal{H}}^{\mathrm{op}}, \pi \oplus \pi, \begin{pmatrix} 0 & u_{-s}^* \\ u_{-s} & 0 \end{pmatrix}\right]. \tag{5.5}$$

Here, we extend u_s to $(-1,1)$ as $u_s = u_0$ for $s < 0$.

Lemma 5.6. *The diagram*

$$\begin{array}{ccccc} \mathrm{K}_1\big(\mathfrak{C}^0_L(A)\big) & \xrightarrow{(\iota_4)_*} & \mathrm{K}_1\big(\mathfrak{D}_L(\phi)\big) & \xrightarrow{q_*(\cdot)\,\widehat{\otimes}_{\mathbb{C}}\,\beta} & \mathrm{K}_0\big(\mathfrak{D}(B)(0,1)\big) \\ \downarrow{\scriptstyle \Theta_{A,0}\circ(\iota_2)_*} & & \downarrow{\scriptstyle \Theta_\phi} & & \downarrow{\scriptstyle \Psi_{B,1}} \\ \mathrm{KK}\big(A, C_0(-1,1)\big) & \xrightarrow{\theta^*} & \mathrm{KK}\big(C\phi, C_0(-1,1)\big) & \xrightarrow{\beta\,\widehat{\otimes}_{C_0(0,1)}\,\psi^*(\cdot)} & \mathrm{KK}_1\big(B, C_0(-1,1)\big) \end{array}$$

commutes.

Proof. Let $u_s \in \mathbb{M}_N(\mathfrak{C}_L^0(A))^+$ be a unitary. Then, we have

$$\begin{aligned}(\theta^* \circ (\iota_2)_* \circ \Theta_{A,0})([u_s]) &= \left[\bar{\mathcal{H}}(0,1) \oplus \bar{\mathcal{H}}^{\mathrm{op}}(0,1), \bar{\sigma} \oplus \bar{\sigma}, \begin{pmatrix} 0 & u_s^* \\ u_s & 0 \end{pmatrix}\right] \\ &= -\left[\bar{\mathcal{H}}(-1,0) \oplus \bar{\mathcal{H}}^{\mathrm{op}}(-1,0), \bar{\sigma} \oplus \bar{\sigma}, \begin{pmatrix} 0 & u_{-s}^* \\ u_{-s} & 0 \end{pmatrix}\right] \\ &= -\left[\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^{\mathrm{op}}, \tilde{\sigma} \oplus \tilde{\sigma}, \begin{pmatrix} 0 & u_{-s}^* \\ u_{-s} & 0 \end{pmatrix}\right] \\ &= -(\Theta_\phi \circ (\iota_4)_*)([u_s]).\end{aligned}$$

This means that the left square commutes.

Next, let $v_s \in \mathbb{M}_N(\mathfrak{D}_L(\phi))$ be a unitary. Let $\tilde{\tau}$ denote the $*$-homomorphism from $B(0,1)$ to $\mathbb{B}(\mathcal{K}(-1,1))$ given by $\tilde{\tau}(b)(s) = \sigma(b_s)$ for $b = (b_s)_{s \in (0,1)} \in B(0,1)$. Then, we have

$$\begin{aligned}(\psi^* \circ \Theta_\phi)([v_s]) &= \left[\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^{\mathrm{op}}, \tilde{\sigma}|_{B(0,1)} \oplus \tilde{\sigma}|_{B(0,1)}, \begin{pmatrix} 0 & v_{-s}^* \\ v_{-s} & 0 \end{pmatrix}\right] \\ &= \left[\mathcal{K}(0,1) \oplus \mathcal{K}(0,1)^{\mathrm{op}}, \tilde{\tau} \oplus \tilde{\tau}, \begin{pmatrix} 0 & v_0^* \\ v_0 & 0 \end{pmatrix}\right] \\ &= \tilde{\Theta}_{B,\mathrm{pt},0}([v_0]) \otimes j_* \in \mathrm{KK}\left(B(0,1), C_0(-1,1)\right),\end{aligned}$$

where $j \colon C_0(0,1) \to C_0(-1,1)$ is the inclusion (note that j induces a KK-equivalence). Now, we recall that

$$\Psi_{B,1}([v_0] \otimes \beta) = \tilde{\Theta}_{B,\mathrm{pt},0}([v_0]) \otimes \beta \in \mathrm{KK}_{-1}\left(B, C_0(-1,1)\right)$$

by the definition of $\tilde{\Theta}_{B,\mathrm{pt},0}$ and $\Psi_{B,1}$. Therefore, we get

$$\beta \mathbin{\hat{\otimes}_{C_0(0,1)}} (\psi^* \circ \Theta_\phi)([v_s]) = \tilde{\Theta}_{B,\pi,0}([v_0]) \otimes \beta = \Psi_{B,1}(q_*([v_s]) \otimes \beta).$$

This means that the right square commutes. ∎

Theorem 5.7. *The homomorphism Θ_ϕ is an isomorphism.*

Proof. Here, we write $S := C_0(-1,1)$ and $SD := D(-1,1)$ for any C^*-algebra D. Apply the five lemma to the diagram of exact sequences

$$\begin{array}{ccccccccc} \mathrm{K}_1\left(S\mathfrak{D}(B)\right) & \to & \mathrm{K}_1\left(\mathfrak{C}_L^0(A)\right) & \to & \mathrm{K}_1\left(\mathfrak{D}_L(\phi)\right) & \to & \mathrm{K}_0\left(S\mathfrak{D}(B)\right) & \to & \mathrm{K}_0\left(\mathfrak{C}_L^0(A)\right) \\ \downarrow{\scriptstyle \Psi_{B,0}} & & \downarrow{\scriptstyle \Theta_{A,0}\circ(\iota_2)_*} & & \downarrow{\scriptstyle \Theta_\phi} & & \downarrow{\scriptstyle \Psi_{B,1}} & & \downarrow{\scriptstyle \Theta_{A,1}\circ(\iota_2)_*} \\ \mathrm{KK}(B,S) & \longrightarrow & \mathrm{KK}(A,S) & \longrightarrow & \mathrm{KK}(C\phi,S) & \longrightarrow & \mathrm{KK}_1(B,S) & \longrightarrow & \mathrm{KK}_1(A,S), \end{array}$$

which commutes by Lemmas 5.4 and 5.6. ∎

Lastly, we consider the case that A and B are unital and $\phi\colon A \to B$ preserves the unit. Let $(\sigma, \mathcal{H})$ and $(\tau, \mathcal{K})$ be unital ample $*$-representations of A and B, respectively, and $(\bar{\sigma}, \bar{\mathcal{H}}) := (\sigma \oplus \tau, \mathcal{H} \oplus \mathcal{K})$. Then, the $*$-representations $\sigma^+ := \sigma \oplus 0_{\mathcal{H}}$ onto $\mathcal{H}^+ := \mathcal{H}^{\oplus 2}$ and $\tau^+ := \tau \oplus 0_{\mathcal{K}}$ onto $\mathcal{K}^+ := \mathcal{K}^{\oplus 2}$ (where $0_{\mathcal{H}}$ is the zero representation to $\mathcal{H}$) extend to unital ample representations of A^+ and B^+, respectively. Here, we use σ^+ and τ^+ for the definition of C^*-algebras as in (5.1). We also define the C^*-algebras $\mathfrak{D}_L^{\mathrm{u}}(\phi)$ as

$$\mathfrak{D}_L^{\mathrm{u}}(\phi) := \mathfrak{D}_L(\phi) \cap C_u\big(\mathfrak{T}, \mathbb{B}(\bar{\mathcal{H}})\big) = p\mathfrak{D}_L(\phi)p,$$

where p denotes the projection onto the first direct summand $\bar{\mathcal{H}} \subset \bar{\mathcal{H}}^+$; i.e., $p = \bar{\sigma}(1)$.

Lemma 5.8. *The corner embedding $\mathfrak{D}_L^{\mathrm{u}}(\phi) \to \mathfrak{D}_L(\phi)$ induces an isomorphism of* K-*theory.*

Proof. Since the commutators $[\sigma(1_B), T_0]$ and $[\bar{\sigma}(1_A), T_s] \in \mathbb{K}(\bar{\mathcal{H}})$ are compact operators, the off-diagonal part $p\mathfrak{D}_L(\phi)(1-p)$ is of the form

$$\mathfrak{C} := \big\{T_s \in C_0[0,1) \otimes \mathbb{K}(\bar{\mathcal{H}}) \mid T_0 \in \mathbb{K}(\mathcal{H})\big\},$$

which has trivial K-groups. Similarly, the corner subalgebra $(1-p)\mathfrak{D}_L(\phi)(1-p)$ is of the form

$$\mathfrak{B} := \big\{T_s \in C_u\big(\mathfrak{T}, \mathbb{B}(\bar{\mathcal{H}})\big) \mid T_s - T_0 \in \mathbb{K}(\mathcal{H})\big\}.$$

By the six-term exact sequence associated to the extension

$$0 \to \big\{T_s \in C_u\big(\mathfrak{T}, \mathbb{K}(\mathcal{H})\big) \mid T_0 = 0\big\} \to \mathfrak{B} \to \mathbb{B}(\mathcal{H}) \to 0,$$

the K-group of $\mathfrak{B}$ turns out to be zero. Hence, the composition

$$\mathfrak{D}_L^{\mathrm{u}}(\phi) \to \mathfrak{D}_L(\phi) \to \mathfrak{D}_L(\phi)/\mathbb{M}_2\mathfrak{C} \cong \begin{pmatrix} \mathfrak{D}_L^{\mathrm{u}}(\phi)/\mathfrak{C} & 0 \\ 0 & \mathfrak{B}/\mathfrak{C} \end{pmatrix}$$

induces an isomorphism of K-theory. This finishes the proof since the quotient $\mathfrak{D}_L(\phi) \to \mathfrak{D}_L(\phi)/\mathbb{M}_2\mathfrak{C}$ also induces the isomorphism of K-theory. ■

5.2. Range of the dual assembly map

Let (X, Y) be a pair of connected finite CW-complexes. Now, we determine the rational relative and (stably) almost flat K^0-groups $\mathrm{K}^0_{\mathrm{af}}(X, Y)_{\mathbb{Q}}$ and $\mathrm{K}_{\text{s-af}}(X, Y)_{\mathbb{Q}}$ under the assumption that $\Gamma := \pi_1(X)$ and $\Lambda := \pi_1(Y)$ satisfy (2.6), (2.7′), and (2.8) and

(5.9) Both Γ and Λ are residually amenable.

A discrete group Γ is said to be residually amenable (cf. [7, Definition 1.3]) if for any non-trivial element $\gamma \in \Gamma$ there is a homomorphism from Γ to an amenable group Γ' which maps γ to a non-trivial element. For example, all residually finite groups are residually amenable. In particular, all finitely generated linear groups [28] and 3-manifold groups [21] (thanks to Perelman's proof of the geometrization theorem) are examples of residually amenable groups (note that they also satisfy the condition (2.6)).

Lemma 5.10. *Let Γ be a residually amenable group and let $\mathcal{A}$ denote the family of unitary representations of Γ factoring through amenable quotients of Γ. Then, the completion $C^*_{\mathcal{A}}(\Gamma)$ of $\mathbb{C}[\Gamma]$ by the norm $\|x\|_{\mathcal{A}} := \sup_{\pi \in \mathcal{A}} \|\pi(x)\|$ is an intermediate completion; that is, there are quotient maps*

$$C^*_{\max}(\Gamma) \xrightarrow{\epsilon^\Gamma_{\max,\mathcal{A}}} C^*_{\mathcal{A}}(\Gamma) \xrightarrow{\epsilon^\Gamma_{\mathcal{A},r}} C^*_r(\Gamma)$$

such that $\epsilon^\Gamma_{\mathcal{A},r} \circ \epsilon^\Gamma_{\max,\mathcal{A}} = \epsilon^\Gamma$.

Proof. Since Γ is residually amenable, there is a decreasing sequence N_n of normal subgroups of Γ such that $\Gamma_n := \Gamma/N_n$ is amenable and $\bigcap_n N_n = \{e\}$ (for the proof, see [7, Section 1]). Let λ_n denote the left regular representation $\Gamma \to \mathrm{U}(\ell^2(\Gamma_n))$ and let λ denote the left regular representation $\Gamma \to \mathrm{U}(\ell^2(\Gamma))$. Now, it suffices to show that λ is weakly contained in $\bigoplus_n \lambda_n$. (Again we refer to [2, Appendix F.4] for group C^*-algebras and weak containment of representations.)

Let $\varepsilon > 0$, let $F \subset \Gamma$ be a finite subset, and let $\xi \in L^2(\Gamma)$. Pick a compactly supported function $\eta \in c_c(\Gamma) \subset \ell^2(\Gamma)$ such that $\|\eta\| \le \|\xi\|$ and $\|\xi - \eta\| < (2\|\xi\|)^{-1}\varepsilon$. For a sufficiently large n, the restriction of the quotient $q_n\colon \Gamma \to \Gamma_n$ to $(\operatorname{supp}\eta)^{-1} \cdot F \cdot (\operatorname{supp}\eta)$ is injective. Let us choose a section $s\colon q_n(\operatorname{supp}\eta) \to \operatorname{supp}\eta$ of q_n. Then, we have

$$\begin{aligned}\big|\big(\lambda(\gamma)\xi,\xi\big) - \big(\lambda_n(\gamma)s^*\eta, s^*\eta\big)\big| &= \big|\big(\lambda(\gamma)\xi,\xi\big) - \big(\lambda(\gamma)\eta,\eta\big)\big| \\ &\le 2\|\xi\| \cdot \big(2\|\xi\|\big)^{-1}\varepsilon = \varepsilon\end{aligned}$$

for any $\gamma \in F$. This concludes the proof. ∎

Lemma 5.11. *For a residually amenable group Γ, the intermediate completion $C^*_{\mathcal{A}}(\Gamma)$ is quasi-diagonal. Moreover, a homomorphism $\phi\colon \Lambda \to \Gamma$ between residually amenable groups induces the $*$-homomorphism $\phi_{\mathcal{A}}\colon C^*_{\mathcal{A}}(\Lambda) \to C^*_{\mathcal{A}}(\Gamma)$.*

Proof. Let Γ_n and λ_n be as in Lemma 5.10. By the Tikuisis–White–Winter theorem [36], the group C^*-algebra $C^*(\Gamma_n)$ is quasi-diagonal. Pick a dense sequence $\{a_n\}_{n\in\mathbb{N}}$ of $C^*_{\mathcal{A}}(\Gamma)$. Then, for each $n \in \mathbb{N}$, there is an increasing sequence $\{p_{n,m} \in \mathbb{B}(\ell^2(\Gamma_n))\}_{n \le m}$ of finite rank projections such that $\|[\lambda_n(a_l), p_{n,m}]\| < 2^{-m}$ for all $l \le m$. Now, $p_m := \bigoplus p_{n,m}$ is an increasing sequence of finite rank projections in $\bigoplus \ell^2(\Gamma_n)$ such that $\|[\bigoplus_n \lambda_n(a_l), p_m]\| \to 0$ for all $l \in \mathbb{N}$. Since $\bigoplus_n \lambda_n$ is a faithful representation of $C^*_{\mathcal{A}}(\Gamma)$, the proof of the first part of the lemma is completed.

The second part follows from the fact that $\phi^*(\mathcal{A}_\Gamma) \subset \mathcal{A}_\Lambda$ since amenability is passed to subgroups [2, Corollary G.3.4]. ∎

Theorem 5.12 ([13, Corollary 4.4]). *Let Γ be a residually amenable group. Then, for any finite CW-complex X with a reference map $f\colon X \to B\Gamma$, any element in $\mathrm{Im}(\beta_\Gamma \circ \epsilon^\Gamma) \subset \mathrm{K}^0(X)$ is almost flat. Moreover, if Γ has the γ-element (e.g., Γ is coarsely embeddable into a Hilbert space), any element of $\mathrm{Im}(f^*_{\mathbb{Q}}) \subset \mathrm{K}^0(X)_{\mathbb{Q}}$ is almost flat.*

Proof. By Lemma 5.11, any element in the image of

$$(\epsilon_{\mathcal{A}}^{\Gamma})^*\colon \mathrm{KK}\big(C^*_{\mathcal{A}}(\Gamma), \mathbb{C}\big) \to \mathrm{KK}(C^*\Gamma, \mathbb{C})$$

is quasi-diagonal in the sense of [13, Definition 2.2] and hence is mapped to an almost flat element in $\mathrm{K}^*(X)$ by [13, Corollary 4.4]. Now, Remark 2.11 concludes the proof. ∎

Now, we develop the relative version of Theorem 5.12. Let us define the intermediate relative group C^*-algebra

$$C^*_{\mathcal{A}}(\Gamma, \Lambda) := SC(\phi_{\mathcal{A}}\colon C^*_{\mathcal{A}}\Lambda \to C^*_{\mathcal{A}}\Gamma),$$

where $\phi_{\mathcal{A}}$ is a $*$-homomorphism given in Lemma 5.11. We discuss finite rank approximation of a representative of each element $x \in \mathrm{KK}(C^*_{\mathcal{A}}(\Gamma, \Lambda), \mathbb{C})$. Let $(\sigma, \mathcal{H})$ and $(\tau, \mathcal{K})$ be unital $*$-representations of $C^*_{\mathcal{A}}(\Lambda)$ and $C^*_{\mathcal{A}}(\Gamma)$, respectively, such that τ and $\bar{\sigma} := \sigma \oplus \tau \circ \phi_{\mathcal{A}}$ are ample. By Theorem 5.7 and Lemma 5.8, the KK-group $\mathrm{KK}(C^*_{\mathcal{A}}(\Gamma, \Lambda), \mathbb{C})$ is isomorphic to the K-group of $\mathfrak{D}^{\mathrm{u}}_{L}(\phi_{\mathcal{A}})$ by the map Θ_ϕ.

As in Remark 4.8, let $\mathcal{B} := \mathbb{B}(\bar{\mathcal{H}}) \oplus_{\mathcal{Q}(\bar{\mathcal{H}})} \mathbb{B}(\bar{\mathcal{H}})$. Note that the inclusion $\iota\colon \mathbb{K}(\bar{\mathcal{H}}) \to \mathcal{B}$ to the first component induces the isomorphism of K-groups. Let $p, q \in \mathbb{B}(\bar{\mathcal{H}})$ denote the projection onto $\mathcal{H}$ and $\mathcal{K}$ and set $P := (p, p)\mathcal{B}$, $Q := (q, q)\mathcal{B}$ (note that $Q = 0$ if τ is the zero representation). Let $\Pi_u := (\pi_1, \pi_2, \pi_0, \tilde{\pi}, 1)$ denote the stably h-relative representation of (Γ, Λ) on (P, Q) defined by $\pi_1 := (\mathrm{Ad}(u_0) \circ \sigma, \sigma)$, $\pi_2 := (\sigma, \sigma)$, $\pi_0 := (\tau, \tau)$ and $\tilde{\pi}_\kappa$ is a continuous family of representations of Λ onto $P \oplus Q$ defined as

$$\tilde{\pi}_\kappa(\gamma) := \begin{cases} \big(u_0 \bar{\sigma}(\gamma) u_0^*, \bar{\sigma}(\gamma)\big) & \kappa = 1, \\ \big(u_0 u_{2-\kappa}^* \bar{\sigma}(\gamma) u_{2-\kappa} u_0^*, \bar{\sigma}(\gamma)\big) & \kappa \in (1, 2]. \end{cases}$$

We write $\mathbf{\Pi}_u$ for the element of $\mathrm{KK}(C\phi, \mathcal{B}(-1, 1))$ associated to Π_u as in (3.2). Since σ and τ factor through $C^*_{\mathcal{A}}(\Gamma)$ and $C^*_{\mathcal{A}}(\Lambda)$, respectively, the Kasparov bimodule representing $\mathbf{\Pi}_u$ actually determines an element of $\mathrm{KK}(C\phi_{\mathcal{A}}, \mathcal{B}(-1, 1))$.

Lemma 5.13. *One has* $\Theta_\phi[u_s] \otimes \iota = \mathbf{\Pi}_u$.

Proof. Firstly, the unitary adjoint $\mathrm{Ad}\,\mathrm{diag}(u_0, u_0 u_{-s}^*)$ identifies $\Theta_\phi([u_s])$ with the KK-element represented by the quasi-homomorphism

$$\big[\mathrm{Ad}(u_0) \circ \tilde{\sigma}, \mathrm{Ad}(u_0 u_{-s}^*)\tilde{\sigma}\big]\colon C\phi \to \mathbb{B}(\tilde{\mathcal{H}}) \rhd \mathbb{K}(\tilde{\mathcal{H}}).$$

At the same time, $\mathbf{\Pi}_u$ is also represented by a quasi-homomorphism $[\Pi_1, \Pi_2]$ associated to Π_u defined as in (3.2); that is, $\Pi_1 = (\mathrm{Ad}(u_0) \circ \tilde{\sigma}, \tilde{\sigma})$ and $\Pi_2 = (\mathrm{Ad}(u_0 u_{-s}^*)\tilde{\sigma}, \tilde{\sigma})$. This is observed as

$$\mathbf{\Pi}_u = \left[\tilde{P}_1 \oplus \tilde{P}_2, \Pi_1 \oplus \Pi_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = [\Pi_1, \Pi_2] \in \mathrm{KK}\big(C\phi_{\mathcal{A}}, \mathcal{B}(-1, 1)\big),$$

since the operator U as in (3.2) satisfies $U - 1 \in \mathbb{K}(\tilde{P})$ in our setting, namely, in the case that $\bar{u} = 1$.

Now, we have $\iota(\mathrm{Ad}(u_0)\widetilde{\sigma}(a) - \mathrm{Ad}(u_0 u_{-s}^*)\widetilde{\sigma}(a)) = \Pi_1(a) - \Pi_2(a)$ for any $a \in C\phi_{\mathcal{A}}$. That is, $[\mathrm{Ad}(u_0) \circ \widetilde{\sigma}, \mathrm{Ad}(u_0 u_{-s}^*)\widetilde{\sigma}]$ and $[\Pi_1, \Pi_2]$ coincide as $*$-homomorphisms from $q(C^*\phi_{\mathcal{A}})$ (following [11], qA denotes the kernel of $A * A \to A$, where $A * A$ denotes the free product C^*-algebra). Consequently, $\Theta_\phi[u_s] \otimes \iota$ and $\mathbf{\Pi}_u$ determine the same element by the description of the KK-group given in [11, Definition 1.5]. ■

On the other hand, for a sufficiently large $s \in [0, 1)$, we have

$$\left\| u_s u_0^* (u_0 \bar{\sigma}(\gamma) u_0^*) u_0 u_s^* - \bar{\sigma}(\gamma) \right\| < \varepsilon$$

for all $\gamma \in \mathcal{G}_\Lambda$. That is, $\boldsymbol{\pi}_{u,\varepsilon} := (\pi_1, \pi_2, \pi_0, u_s u_0^*)$ is a stably relative $(\varepsilon, \mathcal{G})$-representation of (Γ, Λ) onto (P, Q). Note that π_1, π_2, π_0 are genuine representations and only the intertwiner $u_s u_0^*$ breaks the condition of genuine stable relative representation.

Lemma 5.14. *For a unitary $u \in \mathrm{U}(\mathbb{M}_N(\mathfrak{D}_L^{\mathrm{u}}(\phi_{\mathcal{A}})))$ and any $\varepsilon < (4 + 4|I|^2)^{-1}$, the* KK-*cycle $\mathbf{\Pi}_u$ satisfies*

$$\ell_{\Gamma,\Lambda} \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi}_u = \left[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})\right] \in \mathrm{K}_0(X, Y; \mathcal{B}).$$

Proof. We write $\mathbf{v}_i$ for the Čech 1-cocycle $\beta(\pi_i)$ for $i = 1, 2, 0$, where β is as in (2.15), and let $p_{\mathbf{v}_i}$ be the corresponding projection as in Remark 2.14. Since $\widetilde{\pi}_\kappa = \mathrm{Ad}(u_0 u_{2-\kappa}^*) \circ (\pi_2 \oplus \pi_0)$, $\mathrm{Ad}(u_0 u_{2-\kappa}^* \otimes 1_{\mathbb{M}_I})(p_{\mathbf{v}_2} \oplus p_{\mathbf{v}_0})$ gives a continuous family of projections connecting $p_{\mathbf{v}_1} \oplus p_{\mathbf{v}_0}$ and $p_{\mathbf{v}_2} \oplus p_{\mathbf{v}_0}$. Therefore, by a standard argument in C^*-algebra K-theory (see, for example, [31, Proposition 2.2.6]), we obtain a continuous path of partial isometries $(v_s)_{s \in [1,2]}$ such that

- $v_s v_s^* = p_{\mathbf{v}_2} \oplus p_{\mathbf{v}_0}$,
- $v_s^* v_s = \mathrm{Ad}(u_0 u_{2-s}^* \otimes 1_{\mathbb{M}_I})(p_{\mathbf{v}_2} \oplus p_{\mathbf{v}_0})$ for $s \in (1, 2]$,
- $v_1^* v_1 = p_{\mathbf{v}_1} \oplus p_{\mathbf{v}_0}$, and
- $v_2 = p_{\mathbf{v}_2} \oplus p_{\mathbf{v}_0}$.

By the continuity of v_s, there is $s_0 \in (1, 2]$ such that $\|v_{s_1} - v_{s_2}\| < \varepsilon$ for any $s_1, s_2 \in [1, s_0]$. Set

$$w_s := \begin{cases} (p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0})(u_{2-s} u_0^* \otimes 1_{\mathbb{M}_I}) & s \in [s_0, 2], \\ (p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0})(u_{2-s_0} u_0^* \otimes 1_{\mathbb{M}_I}) v_{s_0}^* v_s & s \in [1, s_0]. \end{cases}$$

Then, w_s also satisfies $w_s w_s^* = p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0}$, $w_s^* w_s = \mathrm{Ad}(u_0 u_{2-s}^* \otimes 1_{\mathbb{M}_I})(p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0})$ for $s \in (1, 2]$, $w_1^* w_1 = p_{\mathbf{v}_1}|_Y \oplus p_{\mathbf{v}_0}$, and $w_2 = p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0}$.

Let $E_{\mathbf{v}_i}$ denote the P-bundle $p_{\mathbf{v}_i} \underline{P_X^I} = \widetilde{X} \times_{\pi_i} P$. Then,

$$\ell_{\Gamma,\Lambda} \otimes_{C^*(\Gamma,\Lambda)} \mathbf{\Pi}_u = [E_{\mathbf{v}_1}, E_{\mathbf{v}_2}, E_{\mathbf{v}_0}, w_1] \in \mathrm{K}^0(X, Y; \mathcal{B})$$

by Theorem 3.3. At the same time, we also have

$$\left[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})\right] = [E_{\mathbf{v}_1}, E_{\mathbf{v}_2}, E_{\mathbf{v}_0}, w_1].$$

Indeed, as is mentioned in (2.25) we have $\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon}) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0, \Delta_I(u_s u_0^*))$. Hence, $[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})] = [(E_{\mathbf{v}_1}, E_{\mathbf{v}_2}, E_{\mathbf{v}_0}, \bar{w})]$, where $\bar{w} \in C(Y) \otimes \mathbb{B}(P \oplus Q) \otimes \mathbb{M}_I$ is the partial isometry constructed in Remark 2.20 (3). In particular, $\bar{w}$ satisfies the inequality

$$\left\| \bar{w} - (p_{\mathbf{v}_2|_Y \oplus \mathbf{v}_0})(u_{2-s_0} u_0^* \otimes 1_{\mathbb{M}_I}) \right\| < |I|^2 \varepsilon.$$

On the other hand, we have

$$\begin{aligned}
&\left\| w_1 - (p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0})(u_{2-s_0} u_0^* \otimes 1_{\mathbb{M}_I}) \right\| \\
&\quad = \left\| (p_{\mathbf{v}_2}|_Y \oplus p_{\mathbf{v}_0})(u_{2-s_0} u_0^* \otimes 1_{\mathbb{M}_I})(v_{s_0}^* v_1 - 1) \right\| \\
&\quad \le \| v_{s_0}^* v_1 - 1 \| < \varepsilon,
\end{aligned}$$

and hence $\| w_1 - \bar{w} \| < (1 + |I|^2)\varepsilon < 1/4$. This shows that w_1 and $\bar{w}$ are homotopic as unitary isomorphisms of $\mathcal{B}$-module bundles. ∎

Theorem 5.15. *Let $\phi\colon \Lambda \to \Gamma$ be a homomorphism between countable discrete groups. Assume that (Γ, Λ) satisfies (2.6), (2.7′), and (5.9). Let (X, Y) be a pair of finite CW-complexes with a reference map $f\colon (X, Y) \to (B\Gamma, B\Lambda)$. Then, any element $x \in \mathrm{Im}(\beta_{\Gamma,\Lambda} \circ j_\phi(\gamma_\Gamma)) \subset \mathrm{K}^0(X, Y)$ is stably almost flat. Moreover, it is almost flat if ϕ is injective.*

Proof. By the assumption (2.7′), the reduced relative group C^*-algebra $C^*_r(\Gamma, \Lambda)$ is defined as in (2.9). The C^*-algebra $C^*_{\mathcal{A}}(\Gamma, \Lambda)$ is an intermediate completion of relative group C^*-algebras in the sense that there are quotient maps

$$C^*_{\max}(\Gamma, \Lambda) \xrightarrow{\epsilon^{\Gamma,\Lambda}_{\max,\mathcal{A}}} C^*_{\mathcal{A}}(\Gamma, \Lambda) \xrightarrow{\epsilon^{\Gamma,\Lambda}_{\mathcal{A},r}} C^*_r(\Gamma, \Lambda).$$

By Theorem 2.10 (2), it suffices to show that any element of $\mathrm{Im}(\beta_{\Gamma,\Lambda} \circ \epsilon^{\Gamma,\Lambda}_{\max,\mathcal{A}}) \subset \mathrm{K}^0(X, Y)$ is stably almost flat. By Theorem 5.7 and Lemmas 5.8, 5.13, and 5.14, any element of $\mathrm{Im}(\beta_{\Gamma,\Lambda} \circ \epsilon^{\Gamma,\Lambda}_{\max,\mathcal{A}})$ is of the form $[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})]$ by some unitary $u \in \mathrm{U}(\mathbb{M}_N(\mathfrak{D}^{\mathrm{u}}_L(\phi_{\mathcal{A}})))$ and small $\varepsilon > 0$, under the identification $\mathrm{K}^0(X, Y; \mathcal{B}) \cong \mathrm{K}^0(X, Y)$. Here, we show that $[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})]$ is represented by a $(5C_{\mathrm{am}}\varepsilon, \mathcal{U})$-flat stably relative vector bundle $\mathfrak{v}$ on (X, Y) for any small $\varepsilon > 0$.

By Lemma 5.11 and the fact that $u := u_{2-s_0} u_0^*$ satisfies $u - 1 \in \mathbb{K}(\bar{\mathcal{H}})$, there are finite rank projections $e \in \mathbb{K}(\mathcal{H})$ and $f \in \mathbb{K}(\mathcal{K})$ such that

- $\| [\pi_1(\gamma), e] \| < \varepsilon$ for $\gamma \in \mathcal{G}_\Gamma$,
- $\| (\pi_1(\gamma) - \pi_2(\gamma)) e^\perp \| < \varepsilon$ and $\| e^\perp (\pi_1(\gamma) - \pi_2(\gamma)) \| < \varepsilon$ for any $\gamma \in \mathcal{G}_\Gamma$,
- $\| [\pi_0(\gamma), f] \| < \varepsilon$ for $\gamma \in \mathcal{G}_\Lambda$,
- $\| [u, e \oplus f] \| < \varepsilon$ and $\| (e^\perp \oplus f^\perp)(u - 1)(e^\perp \oplus f^\perp) \| < \varepsilon$.

We define the map $\pi_i^e\colon \mathcal{G}_\Gamma \to e\mathcal{B}e = \mathbb{B}(e\mathcal{B})$ as $\pi_i^e(\gamma) := e\pi_i(\gamma)e \in e\mathcal{B}e$. Similarly, we also define $\pi_i^{e^\perp}$, π_0^f, and $\pi_0^{f^\perp}$. Let $u^{e \oplus f}$ denote the unitary component of the polar decomposition of $(e \oplus f)u(e \oplus f)$; namely,

$$u^{e \oplus f} := (e \oplus f)u(e \oplus f)\big((e \oplus f)u^*(e \oplus f)u(e \oplus f)\big)^{-1/2} \in (e \oplus f)\mathcal{B}(e \oplus f).$$

Similarly, we also define $u^{e^\perp \oplus f^\perp} := (e^\perp \oplus f^\perp) u (e^\perp \oplus f^\perp)$. Then, we have

(i) π_i^e and $\pi_i^{e^\perp}$ are $(2\varepsilon, \mathcal{G}_\Gamma)$-representation of Γ for $i = 1, 2$,

(ii) π_0^f and $\pi_0^{f^\perp}$ are $(2\varepsilon, \mathcal{G}_\Lambda)$-representation of Λ,

(iii) $u^{e\oplus f} \in \mathrm{Hom}_{5\varepsilon}(\pi_1^e\phi \oplus \pi_0^f, \pi_2^e\phi \oplus \pi_0^f)$ and $u^{e^\perp \oplus f^\perp} \in \mathrm{Hom}_{5\varepsilon}(\pi_1^{e^\perp}\phi \oplus \pi_0^{f^\perp}, \pi_2^{e^\perp}\phi \oplus \pi_0^{f^\perp})$,

(iv) $\|\pi_1^{e^\perp}(\gamma) - \pi_2^{e^\perp}(\gamma)\| < \varepsilon$ for any $\gamma \in \Gamma$ and $\|u^{e^\perp \oplus f^\perp} - 1\| < \varepsilon$.

(i), (ii), and (iv) are straightforward. Here, we check (iii). For simplicity of notations, let $\bar{e} := e \oplus f$. Since $\|\bar{e}u^*\bar{e}u\bar{e} - \bar{e}\| < \varepsilon$, we have

$$\big\|u^{e\oplus f} - \bar{e}u\bar{e}\big\| = \big\|\bar{e}u\bar{e}\big(1 - (\bar{e}u^*\bar{e}u\bar{e})^{-1/2}\big)\big\| \le \varepsilon.$$

This inequality and

$$\begin{aligned}&\big\|\bar{e}u\bar{e}\big((\pi_1\phi \oplus \pi_0)(\gamma)\big)\bar{e}u^*\bar{e} - \bar{e}\big((\pi_2\phi \oplus \pi_0)(\gamma)\big)\bar{e}\big\| \\ &\quad \le 2\big\|[u, \bar{e}]\big\| + \big\|\bar{e}\big(u(\pi_1\phi \oplus \pi_0)(\gamma)u^* - (\pi_2\phi \oplus \pi_0)(\gamma)\big)\bar{e}\big\| < 3\varepsilon\end{aligned}$$

conclude that

$$\begin{aligned}&\big\|u^{e\oplus f}\big((\pi_1^e\phi \oplus \pi_0^f)(\gamma)\big)(u^{e\oplus f})^* - \bar{e}\big((\pi_2^e\phi \oplus \pi_0^f)(\gamma)\big)\bar{e}\big\| \\ &\le 2\|u^{e\oplus f} - \bar{e}u\bar{e}\| + \big\|\bar{e}u\bar{e}\big((\pi_1\phi \oplus \pi_0)(\gamma)\big)\bar{e}u^*\bar{e} - \bar{e}\big((\pi_2\phi \oplus \pi_0)(\gamma)\big)\bar{e}\big\| < 5\varepsilon.\end{aligned}$$

Now, (i), (ii), and (iii) say that

$$\begin{aligned}\boldsymbol{\pi}^{e,f} &:= (\pi_1^e, \pi_2^e, \pi_0^f, u^{e\oplus f}),\\ \boldsymbol{\pi}^{e^\perp, f^\perp} &:= (\pi_1^{e^\perp}, \pi_2^{e^\perp}, \pi_0^{f^\perp}, u^{e^\perp \oplus f^\perp})\end{aligned}$$

are stably relative $(5\varepsilon, \mathcal{G})$-representations of (Γ, Λ) and

$$d(\boldsymbol{\pi}_{u,\varepsilon}, \boldsymbol{\pi}^{e,f} \oplus \boldsymbol{\pi}^{e^\perp, f^\perp}) < \varepsilon.$$

Moreover, (iv) implies that

$$d\big(\boldsymbol{\pi}^{e^\perp, f^\perp}, (\pi_1^{e^\perp}, \pi_1^{e^\perp}, \pi_0^{f^\perp}, 1)\big) < \varepsilon.$$

By Theorem 2.24, we obtain that $\boldsymbol{\beta}(\boldsymbol{\pi}^{e,f})$ is a $(5C_{\mathrm{am}}\varepsilon, \mathcal{U})$-flat stably relative bundle on (X, Y) and

$$\begin{aligned}d\big(\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon}), \boldsymbol{\beta}(\boldsymbol{\pi}^{e,f}) \oplus \boldsymbol{\beta}(\boldsymbol{\pi}^{e^\perp, f^\perp})\big) &< 5C_{\mathrm{am}}\varepsilon,\\ d\big(\boldsymbol{\beta}(\boldsymbol{\pi}^{e^\perp, f^\perp}), (\beta(\pi_1^{e^\perp}), \beta(\pi_1^{e^\perp}), \beta(\pi_0^{f^\perp}), 1)\big) &< 5C_{\mathrm{am}}\varepsilon.\end{aligned}$$

The second inequality together with Remark 2.18 and Remark 2.20 (2) implies that

$$\big[\boldsymbol{\beta}(\boldsymbol{\pi}^{e^\perp, f^\perp})\big] = \big[\big(\beta(\pi_1^{e^\perp}), \beta(\pi_1^{e^\perp}), \beta(\pi_0^{f^\perp}), 1\big)\big] = 0$$

if $\varepsilon > 0$ is sufficiently small. Consequently, we obtain that

$$[\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})] = [\boldsymbol{\beta}(\boldsymbol{\pi}^{e,f})] + [\boldsymbol{\beta}(\boldsymbol{\pi}^{e^\perp,f^\perp})] = [\boldsymbol{\beta}(\boldsymbol{\pi}^{e,f})]$$

for sufficiently small $\varepsilon > 0$.

Since e and f are finite rank projections in $\mathbb{K}(\bar{\mathcal{H}}) \subset \mathcal{B}$, the quadruple $(\pi_1^e, \pi_2^e, \pi_0^f, u^{e\oplus f})$ also determines a $(5\varepsilon, \mathcal{G})$-representation of (Γ, Λ) on a pair of finite rank vector spaces $(e\mathcal{H}, f\mathcal{K})$, which is denoted by $\boldsymbol{\pi}'$. Now,

$$\iota_*[\boldsymbol{\beta}(\boldsymbol{\pi}')] = [\boldsymbol{\beta}(\boldsymbol{\pi}^{e,f})] = [\boldsymbol{\beta}(\boldsymbol{\pi}_{u,\varepsilon})] \in \mathrm{K}^0(X, Y; \mathcal{B})$$

finishes the proof.

As is remarked at the beginning of Section 5, we can choose τ as the zero representation if ϕ is injective. Then, the projection f in the above argument is the zero projection, and hence the obtained $\boldsymbol{\beta}(\boldsymbol{\pi}')$ is a $(\varepsilon, \mathcal{U})$-flat relative vector bundle on (X, Y). Therefore, a given element $x \in \mathrm{Im}(\beta_{\Gamma,\Lambda} \circ j_\phi(\gamma_\Gamma))$ is almost flat. ∎

For a pair of connected (not necessarily finite) CW-complexes (X, Y), we say that an element x of $\mathrm{K}^0(X, Y)$ or $\mathrm{K}^0(X, Y)_{\mathbb{Q}}$ is (resp. stably) almost flat if f^*x is (resp. stably) almost flat for any continuous map f from a pair of connected finite CW-complexes (Z, W) to (X, Y).

Then, Theorem 5.15, together with Theorem 2.10 (2), implies the following.

Corollary 5.16. *Let $\phi\colon \Lambda \to \Gamma$ be a homomorphism between countable discrete groups. Assume that (Γ, Λ) satisfy (2.6), (2.7′), (2.8), and (5.9).*

(1) *Any element $x \in \mathrm{K}^0(B\Gamma, B\Lambda)_{\mathbb{Q}}$ is stably almost flat.*

(2) *If ϕ is injective, any element $x \in \mathrm{K}^0(B\Gamma, B\Lambda)_{\mathbb{Q}}$ is almost flat.*

Equivalently, we characterize infiniteness of K-area by the characteristic class.

Corollary 5.17. *Let M be a compact spin manifold with a boundary N such that $\Gamma := \pi_1(M)$ and $\Lambda := \pi_1(N)$ satisfies (2.6), (2.7′), (2.8), and (5.9). Let f denote the reference map from (M, N) to $(B\Gamma, B\Lambda)$.*

(1) *Then, (M, N) has an infinite stably relative* K*-area if and only if* $\mathrm{ch}(f_*[M, N]) = 0 \in H_{\mathrm{ev}}(B\Gamma, B\Lambda; \mathbb{Q})$.

(2) *If $\phi\colon \Lambda \to \Gamma$ is injective, then (M, N) has an infinite relative* K*-area if and only if* $\mathrm{ch}(f_*[M, N]) = 0 \in H_{\mathrm{ev}}(B\Gamma, B\Lambda; \mathbb{Q})$.

Proof. It immediately follows from Corollary 5.16. We only remark that the Chern character gives an isomorphism between $\mathrm{K}^0(B\Gamma, B\Lambda)_{\mathbb{Q}}$ and

$$H^{\mathrm{ev}}(B\Gamma, B\Lambda; \mathbb{Q}) := \prod_{n\in\mathbb{N}} H^{2n}(B\Gamma, B\Lambda; \mathbb{Q}) \cong \Big(\bigoplus_{n\in\mathbb{N}} H_{2n}(B\Gamma, B\Lambda; \mathbb{Q})\Big)^*.$$ ∎

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References

[1] P. Baum and A. Connes, Chern character for discrete groups. In *A Fête of Topology*, pp. 163–232, Academic Press, Boston, MA, 1988 Zbl 0656.55005 MR 928402

[2] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's Property (T)*. New Math. Monogr. 11, Cambridge University Press, Cambridge, 2008 Zbl 1146.22009 MR 2415834

[3] B. Blackadar, *K-Theory For Operator Algebras*. 2nd edn., Math. Sci. Res. Inst. Publ. 5, Cambridge University Press, Cambridge, 1998 Zbl 0913.46054 MR 1656031

[4] N. P. Brown and N. Ozawa, *C^*-Algebras and Finite-Dimensional Approximations*. Grad. Stud. Math. 88, Amer. Math. Soc., Providence, RI, 2008 Zbl 1160.46001 MR 2391387

[5] J. Carrión and M. Dadarlat, Almost flat K-theory of classifying spaces. *J. Noncommut. Geom.* **12** (2018), no. 2, 407–438 Zbl 1402.46048 MR 3825191

[6] S. Chang, S. Weinberger, and G. Yu, Positive scalar curvature and a new index theory for noncompact manifolds. *J. Geom. Phys.* **149** (2020), 103575 Zbl 1437.58017 MR 4045309

[7] B. Clair, Residual amenability and the approximation of L^2-invariants. *Michigan Math. J.* **46** (1999), no. 2, 331–346 Zbl 0967.58017 MR 1704205

[8] A. Connes, M. Gromov, and H. Moscovici, Conjecture de Novikov et fibrés presque plats. *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), no. 5, 273–277 Zbl 0693.53007 MR 1042862

[9] J. Cuntz, Generalized homomorphisms between C^*-algebras and KK-theory. In *Dynamics and Processes (Bielefeld, 1981)*, pp. 31–45, Lecture Notes in Math. 1031, Springer, Berlin, 1983 Zbl 0561.46034 MR 733641

[10] J. Cuntz, K-theory and C^*-algebras. In *Algebraic K-Theory, Number Theory, Geometry and Analysis (Bielefeld, 1982)*, pp. 55–79, Lecture Notes in Math. 1046, Springer, Berlin, 1984 Zbl 0548.46056 MR 750677

[11] J. Cuntz, A new look at KK-theory. *K-Theory* **1** (1987), no. 1, 31–51 Zbl 0636.55001 MR 899916

[12] M. Dadarlat, Group quasi-representations and index theory. *J. Topol. Anal.* **4** (2012), no. 3, 297–319 Zbl 1258.46029 MR 2982445

[13] M. Dadarlat, Group quasi-representations and almost flat bundles. *J. Noncommut. Geom.* **8** (2014), no. 1, 163–178 Zbl 1296.46058 MR 3275029

[14] M. Dadarlat, R. Willett, and J. Wu, Localization C^*-algebras and K-theoretic duality. *Ann. K-Theory* **3** (2018), no. 4, 615–630 Zbl 1409.46044 MR 3892961

[15] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional Analysis on the Eve of the 21st Century, Vol. II (New Brunswick, NJ, 1993)*, pp. 1–213, Progr. Math. 132, Birkhäuser, Boston, MA, 1996 Zbl 0945.53022 MR 1389019

[16] M. Gromov and H. B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.* (1983), no. 58, 83–196 Zbl 0538.53047 MR 720933

[17] B. Hanke, Positive scalar curvature, K-area and essentialness. In *Global Differential Geometry*, pp. 275–302, Springer Proc. Math. 17, Springer, Heidelberg, 2012 Zbl 1252.53039 MR 3289846

[18] B. Hanke, D. Pape, and T. Schick, Codimension two index obstructions to positive scalar curvature. *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 6, 2681–2710 Zbl 1344.58012 MR 3449594

[19] B. Hanke and T. Schick, Enlargeability and index theory. *J. Differential Geom.* **74** (2006), no. 2, 293–320 Zbl 1122.58011 MR 2259056

[20] B. Hanke and T. Schick, Enlargeability and index theory: infinite covers. *K-Theory* **38** (2007), no. 1, 23–33 Zbl 1128.58012 MR 2353861

[21] J. Hempel, Residual finiteness for 3-manifolds. In *Combinatorial Group Theory and Topology (Alta, Utah, 1984)*, pp. 379–396, Ann. of Math. Stud. 111, Princeton University Press, Princeton, NJ, 1987 Zbl 0772.57002 MR 895623

[22] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.* **144** (2001), no. 1, 23–74 Zbl 0988.19003 MR 1821144

[23] N. Higson and J. Roe, *Analytic K-Homology*. Oxford Math. Monogr., Oxford University Press, Oxford, 2000 Zbl 0968.46058 MR 1817560

[24] D. Husemoller, *Fibre Bundles*. 3rd edn., Grad. Texts in Math. 20, Springer, New York, 1994 Zbl 0794.55001 MR 1249482

[25] G. G. Kasparov, Hilbert C^*-modules: theorems of Stinespring and Voiculescu. *J. Operator Theory* **4** (1980), no. 1, 133–150 Zbl 0456.46059 MR 587371

[26] Y. Kubota, Almost flat relative vector bundles and their monodromy. 2019, arXiv:1908.10575

[27] Y. Kubota, The relative Mishchenko–Fomenko higher index and almost flat bundles. I. The relative Mishchenko–Fomenko index. *J. Noncommut. Geom.* **14** (2020), no. 3, 1209–1244 Zbl 07358371 MR 4170653

[28] A. Malcev, On isomorphic matrix representations of infinite groups. *Rec. Math. [Mat. Sbornik] N.S.* **8 (50)** (1940), 405–422 Zbl 0025.00804 MR 0003420

[29] R. Meyer, Equivariant Kasparov theory and generalized homomorphisms. *K-Theory* **21** (2000), no. 3, 201–228 Zbl 0982.19004 MR 1803228

[30] H. Oyono-Oyono and G. Yu, On quantitative operator K-theory. *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 2, 605–674 Zbl 1329.19009 MR 3449163

[31] M. Rørdam, F. Larsen, and N. Laustsen, *An Introduction to K-Theory for C^*-Algebras*. London Math. Soc. Stud. Texts 49, Cambridge University Press, Cambridge, 2000 Zbl 0967.19001 MR 1783408

[32] J. Rosenberg and S. Stolz, A "stable" version of the Gromov-Lawson conjecture. In *The Čech Centennial (Boston, MA, 1993)*, pp. 405–418, Contemp. Math. 181, Amer. Math. Soc., Providence, RI, 1995 Zbl 0818.53057 MR 1321004

[33] T. Schick, L^2-index theorems, KK-theory, and connections. *New York J. Math.* **11** (2005), 387–443 Zbl 1093.58009 MR 2188248

[34] G. Skandalis, J. L. Tu, and G. Yu, The coarse Baum-Connes conjecture and groupoids. *Topology* **41** (2002), no. 4, 807–834 Zbl 1033.19003 MR 1905840

[35] K. Thomsen, On absorbing extensions. *Proc. Amer. Math. Soc.* **129** (2001), no. 5, 1409–1417 Zbl 0968.46059 MR 1814167

[36] A. Tikuisis, S. White, and W. Winter, Quasidiagonality of nuclear C^*-algebras. *Ann. of Math. (2)* **185** (2017), no. 1, 229–284 Zbl 1367.46044 MR 3583354

[37] J.-L. Tu, The gamma element for groups which admit a uniform embedding into Hilbert space. In *Recent Advances in Operator Theory, Operator Algebras, and Their Applications*, pp. 271–286, Oper. Theory Adv. Appl. 153, Birkhäuser, Basel, 2005 Zbl 1074.46049 MR 2105483

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