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Lifting theorems for completely positive maps

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Abstract. We prove lifting theorems for completely positive maps going out of exact C^*-algebras, where we remain in control of which ideals are mapped into which. A consequence is, that if X is a second countable topological space, $\mathfrak{A}$ and $\mathfrak{B}$ are separable, nuclear C^*-algebras over X, and the action of X on $\mathfrak{A}$ is continuous, then $E(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cong KK(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ naturally. As an application, we show that a separable, nuclear, strongly purely infinite C^*-algebra $\mathfrak{A}$ absorbs a strongly self-absorbing C^*-algebra $\mathcal{D}$ if and only if $\mathfrak{I}$ and $\mathfrak{I} \otimes \mathcal{D}$ are KK-equivalent for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$. In particular, if $\mathfrak{A}$ is separable, nuclear, and strongly purely infinite, then $\mathfrak{A} \otimes \mathcal{O}_2 \cong \mathfrak{A}$ if and only if every two-sided, closed ideal in $\mathfrak{A}$ is KK-equivalent to zero.

1. Introduction

Arveson was perhaps the first to recognise the importance of lifting theorems for completely positive maps. In [1], he uses a lifting theorem to give a simple and operator theoretic proof of the fact that the Brown–Douglas–Fillmore semigroup $\mathrm{Ext}(\mathsf{X})$ is actually a group. This was already proved by Brown, Douglas, and Fillmore in [5], but the proof was somewhat complicated and very topological in nature. All the known lifting theorems at that time were generalised by Choi and Effros [8], when they proved that any nuclear map going out of a separable C^*-algebra is liftable. This result, together with the dilation theorem of Stinespring [31] and the Weyl–von Neumann type theorem of Voiculescu [33], was used by Arveson [2] to prove that the (generalised) Brown–Douglas–Fillmore semigroup $\mathrm{Ext}(\mathfrak{A})$ defined in [6] is a group for any unital, separable, nuclear C^*-algebra $\mathfrak{A}$. When doing this, Arveson included a simplified proof of the lifting theorem of Choi and Effros, a proof which in many ways illustrates, that the Choi–Effros lifting theorem is a non-commutative analogue of the selection theorems of Michael [25].

Kasparov [18] used the same idea as Arveson to prove that for any separable, nuclear C^*-algebra $\mathfrak{A}$ and any σ-unital C^*-algebra $\mathfrak{B}$, the semigroup $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ is in fact a group. It was also an application of the Choi–Effros lifting theorem, which allowed Kasparov to prove that the functor $KK(\mathfrak{A}, -)$ is half-exact for any separable, nuclear C^*-algebra $\mathfrak{A}$, and thus induces a six-term exact sequence for any short exact sequence of σ-unital C^*-algebras. This fails if one does not assume $\mathfrak{A}$ to be nuclear, which is basically due to the

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fact, that we can not lift completely positive maps in general. So KK-theory lacks certain desirable properties such as excision, i.e. that short exact sequences of C^*-algebras induce six-term exact sequences of KK-groups. In an attempt to fix this “defect” of KK-theory, Higson [16] constructed E-theory, which resembles KK-theory quite a bit, but which is always half-exact. As a consequence of the half-exactness of $KK(\mathfrak{A}, -)$ for separable, nuclear C^*-algebras $\mathfrak{A}$, it follows that $E(\mathfrak{A}, \mathfrak{B}) \cong KK(\mathfrak{A}, \mathfrak{B})$ naturally, for such $\mathfrak{A}$.

We say that a topological space X acts on a C^*-algebra $\mathfrak{A}$, if there is an order preserving map from the lattice $\mathbb{O}(\mathsf{X})$ of open subsets of X, to the lattice $\mathbb{I}(\mathfrak{A})$ of two-sided, closed ideals in $\mathfrak{A}$. A map between such C^*-algebras is X-equivariant, if it respects the action. Kirchberg introduced a modified version of KK-theory for C^*-algebras with an action of X, and proved the very deep result [20] (see also [13]), that all separable, nuclear, strongly purely infinite C^*-algebras with a tight action of X, are classified by $KK(\mathsf{X})$-theory. Here tight refers to the action $\mathbb{O}(\mathsf{X}) \to \mathbb{I}(\mathfrak{A})$ being a lattice isomorphism. As it turns out, the functor $KK(\mathsf{X}; \mathfrak{A}, -)$ is not half-exact in general, not even when $\mathfrak{A}$ is nuclear. This is mainly due to the lack of lifting theorems for completely positive maps, for which we preserve the action of X.

In [10], Dadarlat and Meyer construct a version of E-theory for C^*-algebras with an action of X, which is half-exact, and which also possesses other nice properties which $KK(\mathsf{X})$-theory does not enjoy. Thus, it would be desirable to find sufficient criteria for when $E(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cong KK(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ naturally, as it is known that nuclearity of $\mathfrak{A}$ does not suffice. This is the main motivation of this paper. We show that if $\mathfrak{A}$ and $\mathfrak{B}$ are nuclear, and if the actions of X on $\mathfrak{A}$ and $\mathfrak{B}$ satisfy certain continuity properties, then $E(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cong KK(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ naturally. This is done by proving that we may lift X-equivariant completely positive maps, while preserving the X-equivariant structure.

Combining this result with the deep classification result of Kirchberg [20], it follows that all separable, nuclear, strongly purely infinite C^*-algebras with a tight action of X, are classified by $E(\mathsf{X})$-theory. We apply this to show that if $\mathfrak{A}$ is a separable, nuclear, strongly purely infinite C^*-algebra, and $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, then $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{D}$ if and only if $\mathfrak{I}$ and $\mathfrak{I} \otimes \mathcal{D}$ are KK-equivalent for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.

In particular, let $\mathfrak{A}$ be a separable, nuclear, strongly purely infinite C^*-algebra, let M_{n^∞} denote the UHF algebra of type n^∞, $\mathcal{Q}$ denote the universal UHF algebra, and $\mathcal{O}_2$ denote the Cuntz algebra. We show that:

- If all two-sided, closed ideals in $\mathfrak{A}$ satisfy the UCT, then $\mathfrak{A} \cong \mathfrak{A} \otimes M_{n^\infty}$ if and only if $K_*(\mathfrak{I})$ is uniquely n-divisible for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.
- If all two-sided, closed ideals in $\mathfrak{A}$ satisfy the UCT, then $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Q}$ if and only if $K_*(\mathfrak{I})$ is uniquely divisible for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.
- $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_2$ if and only if every two-sided, closed ideal in $\mathfrak{A}$ is KK-equivalent to zero.

The author has been made aware that Eberhard Kirchberg announced results partially overlapping with results presented here, at the 2009 Oberwolfach meeting "C*-Algebren", cf. [21], and thanks Ralf Meyer for pointing this out.

2. A Hahn–Banach separation theorem for closed operator convex cones

Definition 2.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $CP(\mathfrak{A},\mathfrak{B})$ denote the convex cone of all completely positive (c.p.) maps from $\mathfrak{A}$ to $\mathfrak{B}$. A subset $\mathfrak{C}$ of $CP(\mathfrak{A},\mathfrak{B})$ is called an *operator convex cone* if it satisfies the following:

(1) $\mathfrak{C}$ is a convex cone,

(2) if $\phi \in \mathfrak{C}$ and b in $\mathfrak{B}$ then $b^*\phi(-)b \in \mathfrak{C}$,

(3) if $\phi \in \mathfrak{C}$, $a_1,\dots,a_n \in \mathfrak{A}$, and $b_1,\dots,b_n \in \mathfrak{B}$ then the map

$$\sum_{i,j=1}^{n} b_i^*\phi(a_i^*(-)a_j)b_j \tag{2.1}$$

is in $\mathfrak{C}$.

We equip $\mathfrak{C}$ with the point-norm topology, and say that it is a *closed* operator convex cone, if it is closed as a subspace of $CP(\mathfrak{A},\mathfrak{B})$.

We will almost only be considering operator convex cones which are closed.

Example 2.2. A c.p. map is called *factorable* if it factors through a matrix algebra by c.p. maps. The set $CP_{\text{fact}}(\mathfrak{A},\mathfrak{B}) \subseteq CP(\mathfrak{A},\mathfrak{B})$ of all factorable maps is an operator convex cone.

Checking (1) in the definition amounts to the observation, that there exists a conditional expectation $M_{k+l} \to M_k \oplus M_l$, so if two c.p. maps factor through M_k and M_l respectively, then their sum factors through M_{k+l}. Condition (2) is obvious, so only (3) remains to be checked. Let $a_1,\dots,a_n \in \mathfrak{A}$ and $b_1,\dots,b_n \in \mathfrak{B}$ be given, and $\phi^{(n)}\colon M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ be the amplification of ϕ. Let $r \in M_{1,n}(\mathfrak{A})$ be the row vector $r = (a_1 \cdots a_n)$, and $c \in M_{n,1}(\mathfrak{B})$ be the column vector $(b_1 \cdots b_n)^t$. The map in equation (2.1) is exactly $c^*\phi^{(n)}(r^*(-)r)c$, which is factorable since $\phi^{(n)}$ is (clearly) factorable. Hence $CP_{\text{fact}}(\mathfrak{A},\mathfrak{B})$ is an operator convex cone.

A c.p. map is called *nuclear* if it can be approximated point-norm by factorable maps, i.e. if it is in the point-norm closure of $CP_{\text{fact}}(\mathfrak{A},\mathfrak{B})$. Thus, the set $CP_{\text{nuc}}(\mathfrak{A},\mathfrak{B})$ of nuclear c.p. maps is a closed operator convex cone.

The above definition of nuclearity agrees with the one often used in the literature (for contractive maps), e.g. the definition used in the book by Brown and Ozawa [7, Definition 2.1.1], in which the maps going in and out of the matrix algebras are assumed to be contractive. This has been well known for a long time, and a proof of this is presented in [14, Lemma 2.3] (alternatively, see [13, Lemma 3.7]).

Observation 2.3. Note that by our definition of a nuclear map, it follows immediately that the composition of any c.p. map with a nuclear map, is again nuclear. We will use this fact several times without mentioning.

The following is a well-known, very basic result on c.p. maps using the Hahn–Banach separation theorem.

Lemma 2.4. *Let $\mathfrak{C} \subseteq CP(\mathfrak{A}, \mathfrak{B})$ be a point-norm closed convex subset. If $\phi \in CP(\mathfrak{A}, \mathfrak{B})$ is in the point-weak closure $\overline{\mathfrak{C}}^{\text{pt-weak}} \subseteq CP(\mathfrak{A}, \mathfrak{B}^{**})$, then $\phi \in \mathfrak{C}$, i.e. if for every $a_1, \ldots, a_n \in \mathfrak{A}$, every $f_1, \ldots, f_n \in \mathfrak{B}^*$ (or in the state space $S(\mathfrak{B})$) and every $\varepsilon > 0$ there is a $\psi \in \mathfrak{C}$, such that*

$$|f_i(\phi(a_i)) - f_i(\psi(a_i))| < \varepsilon, \quad \textit{for } i = 1, \ldots, n,$$

then $\phi \in \mathfrak{C}$.

Proof. This is an easy Hahn–Banach separation argument. In fact, let $a_1, \ldots, a_n \in \mathfrak{A}$. The set

$$\{(\psi(a_1), \ldots, \psi(a_n)) : \psi \in \mathfrak{C}\}$$

is a norm-closed convex subset of $\mathfrak{B}^n$. Hence, by the Hahn–Banach separation theorem (since we can not separate $(\phi(a_1), \ldots, \phi(a_n))$ from the above set by linear functionals) we must have $(\phi(a_1), \ldots, \phi(a_n))$ is in the above set. Now the result follows trivially since $\mathfrak{C}$ is point-norm closed. ∎

Kirchberg and Rørdam show in [23, Proposition 4.2], that if $\mathfrak{C} \subseteq CP(\mathfrak{A}, \mathfrak{B})$ is a closed operator convex cone, where $\mathfrak{A}$ is separable and nuclear, and $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is any c.p. map, then $\phi \in \mathfrak{C}$ if and only if $\phi(a) \in \overline{\mathfrak{B}\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{B}}$ for every $a \in \mathfrak{A}$. We refer to this as a Hahn–Banach separation theorem for closed operator convex cones, as one obtains a separation of ϕ from a closed operator convex cone, and since the result relies heavily on the Hahn–Banach separation theorem.

We generalise the result of Kirchberg and Rørdam to exact C^*-algebras and nuclear c.p. maps, and where we only take positive elements in $\mathfrak{A}$. The proof is virtually identical to the proof of [23, Proposition 4.2], but we fill in the proof for completion.

Theorem 2.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ exact, and let $\mathfrak{C} \subseteq CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Suppose that $\mathfrak{C} \subseteq CP_{\mathrm{nuc}}(\mathfrak{A}, \mathfrak{B})$ and let $\phi \in CP_{\mathrm{nuc}}(\mathfrak{A}, \mathfrak{B})$. Then $\phi \in \mathfrak{C}$ if and only if $\phi(a) \in \overline{\mathfrak{B}\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{B}}$ for every positive $a \in \mathfrak{A}$.*

Proof. "Only if" is obvious. For "if", suppose $\phi(a) \in \overline{\mathfrak{B}\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{B}}$ for every positive $a \in \mathfrak{A}$. By Lemma 2.4 it suffices to show, that given $a_1, \ldots, a_n \in \mathfrak{A}$, $\varepsilon > 0$ and $f_1, \ldots, f_n \in \mathfrak{B}^*$, there is a $\psi \in \mathfrak{C}$ such that

$$|f_i(\phi(a_i)) - f_i(\psi(a_i))| < \varepsilon, \quad \text{for } i = 1, \ldots, n.$$

By [22, Lemma 7.17 (i)] we may find a cyclic representation $\pi\colon \mathfrak{B} \to \mathbb{B}(\mathscr{H})$ with cyclic vector $\xi \in \mathscr{H}$, and elements $c_1, \ldots, c_n \in \pi(\mathfrak{B})' \cap \mathbb{B}(\mathscr{H})$, such that $f_i(b) = \langle \pi(b)c_i\xi, \xi\rangle$

for $i = 1, \dots, n$. Let $\mathfrak{C} = C^*(c_1, \dots, c_n)$ and $\iota\colon \mathfrak{C} \hookrightarrow \mathbb{B}(\mathcal{H})$ be the inclusion. For any c.p. map $\rho\colon \mathfrak{A} \to \mathfrak{B}$ there is an induced positive linear functional on $\mathfrak{A} \otimes_{\max} \mathfrak{C}$ given by the composition

$$\mathfrak{A} \otimes_{\max} \mathfrak{C} \xrightarrow{\rho\otimes \mathrm{id}_{\mathfrak{C}}} \mathfrak{B} \otimes_{\max} \mathfrak{C} \xrightarrow{\pi\times\iota} \mathbb{B}(\mathcal{H}) \xrightarrow{\omega_\xi} \mathbb{C},$$

where ω_ξ is the vector functional induced by ξ, i.e. $\omega_\xi(T) = \langle T\xi, \xi\rangle$. If ρ is nuclear, then $\rho \otimes \mathrm{id}_{\mathfrak{C}}$ above factors through the spatial tensor product $\mathfrak{A} \otimes \mathfrak{C}$ (see e.g. [7, Lemma 3.6.10]), so if ρ is nuclear it induces a positive linear functional η_ρ on $\mathfrak{A} \otimes \mathfrak{C}$.

Let $\mathcal{K}$ be the weak-$*$ closure of $\{\eta_\psi : \psi \in \mathcal{C}\} \subseteq (\mathfrak{A} \otimes \mathfrak{C})^*$. It suffices to show that $\eta_\phi \in \mathcal{K}$ since, if $|\eta_\phi(a_i \otimes c_i) - \eta_\psi(a_i \otimes c_i)| < \varepsilon$ for some $\psi \in \mathcal{C}$, then

$$\begin{aligned} f_i(\phi(a_i)) &= \langle \pi(\phi(a_i))c_i\xi, \xi\rangle = \eta_\phi(a_i \otimes c_i) \\ &\approx_\varepsilon \eta_\psi(a_i \otimes c_i) = \langle \pi(\psi(a_i))c_i\xi, \xi\rangle = f_i(\psi(a_i)), \end{aligned}$$

for $i = 1, \dots, n$, which is what we want to prove. It is easily verified (e.g. by checking on elementary tensors $a \otimes c$) that $\eta_{\psi_1} + \eta_{\psi_2} = \eta_{\psi_1+\psi_2}$, and that $t\eta_\psi = \eta_{t\psi}$ for $t \in \mathbb{R}_+$. Hence $\mathcal{K}$ is a weak-$*$ closed convex cone of positive linear functionals.

We want to show that if $\eta \in \mathcal{K}$ and $d \in \mathfrak{A} \otimes \mathfrak{C}$, then $d^*\eta d := \eta(d^*(-)d) \in \mathcal{K}$. Since $\mathcal{K}$ is weak-$*$ closed, it suffices to show this for $\eta = \eta_\psi$ where $\psi \in \mathcal{C}$, and $d = \sum_{j=1}^k x_j \otimes y_j$ where $x_1, \dots, x_k \in \mathfrak{A}$ and $y_1, \dots, y_k \in \mathfrak{C}$. For $a \in \mathfrak{A}$ and $c \in \mathfrak{C}$ we have

$$\eta_\psi(d^*(a \otimes c)d) = \sum_{j,l=1}^k \eta_\psi((x_j^* a x_l) \otimes (y_j^* c y_l)) = \sum_{j,l=1}^k \langle \pi(\psi(x_j^* a x_l)) c y_l \xi, y_j \xi\rangle.$$

Since ξ is cyclic for π we may, for any $\delta > 0$, find $b_1, \dots, b_k \in \mathfrak{B}$ such that $\|\pi(b_j)\xi - y_j\xi\| < \delta$. Thus, by choosing δ sufficiently small we may approximate $d^*\eta_\psi d$ in the weak-$*$ topology by

$$\sum_{j,l=1}^k \langle \pi(\psi(x_j^* a x_l)) c \pi(b_l)\xi, \pi(b_j)\xi\rangle = \Big\langle \pi\Big(\sum_{j,l=1}^k b_j^* \psi(x_j^* a x_l) b_l\Big) c\xi, \xi\Big\rangle = \eta_{\psi_0}(a \otimes c),$$

where $\psi_0 = \sum_{j,l=1}^k b_j^* \psi(x_j^*(-)x_l) b_l \in \mathcal{C}$. Thus, $d^*\eta d \in \mathcal{K}$ for any $\eta \in \mathcal{K}$ and $d \in \mathfrak{A} \otimes \mathfrak{C}$. Let $\mathfrak{J}$ be the subset of $\mathfrak{A} \otimes \mathfrak{C}$ consisting of elements d such that $\eta(d^*d) = 0$ for all $\eta \in \mathcal{K}$. By [22, Lemma 7.17 (ii)] it follows that $\mathfrak{J}$ is a closed two-sided ideal in $\mathfrak{A} \otimes \mathfrak{C}$, and that $\eta_\phi \in \mathcal{K}$ if $\eta_\phi(d^*d) = 0$ for all $d \in \mathfrak{J}$.

Since $\mathfrak{A}$ is exact, $\mathfrak{J}$ is the closed linear span of all elementary tensors $x \otimes y$ for which $x \otimes y \in \mathfrak{J}$ (see e.g. [7, Corollary 9.4.6]). Recall that the left kernel of η_ϕ, i.e. the set of all d such that $\eta_\phi(d^*d) = 0$, is a closed linear subspace of $\mathfrak{A} \otimes \mathfrak{C}$. Hence it suffices to show, that when $x \in \mathfrak{A}$ and $y \in \mathfrak{C}$ are such that $x \otimes y \in \mathfrak{J}$, then $\eta_\phi(x^*x \otimes y^*y) = 0$. Fix such x and y.

By assumption $\phi(x^*x) \in \overline{\mathfrak{B}\{\psi(x^*x) : \psi \in \mathcal{C}\}\mathfrak{B}}$. Thus, for any $\delta > 0$ we may choose $\psi_1, \dots, \psi_m \in \mathcal{C}$ and $b_1, \dots, b_m \in \mathfrak{B}$ such that

$$\left\| \phi(x^*x) - \Big(\sum_{j=1}^{m} b_j^* \psi_j(x^*x) b_j \Big) \right\| < \delta.$$

Let $\psi = \sum_{j=1}^m b_j^* \psi_j(-) b_j$ which is in $\mathcal{C}$, such that $\|\phi(x^*x) - \psi(x^*x)\| < \delta$. Since $\eta_\psi(x^*x \otimes y^*y) = 0$ we get that

$$\begin{aligned} |\eta_\phi(x^*x \otimes y^*y)| &= |\eta_\phi(x^*x \otimes y^*y) - \eta_\psi(x^*x \otimes y^*y)| \\ &= |\langle \pi(\phi(x^*x) - \psi(x^*x)) y\xi, y\xi \rangle| \\ &< \delta \|y\xi\|^2. \end{aligned}$$

Since δ was arbitrary we get that $\eta_\phi(x^*x \otimes y^*y) = 0$ which finishes the proof. ∎

2.1. An abstract lifting result

The main goal of this paper, is to obtain lifting results for c.p. maps, where we remain in control of the lift, in the sense that we may choose a lift in a given closed operator convex cone. This can be obtained as an application of the Hahn–Banach type theorem. First we need a lemma, which is essentially due to Arveson.

Lemma 2.6. *Let $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ be C^*-algebras with $\mathfrak{A}$ separable, and let $\pi : \mathfrak{B} \to \mathfrak{C}$ be a surjective $*$-homomorphism. Let $\mathcal{C} \subseteq CP(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. Then*

$$\pi(\mathcal{C}) := \{\pi \circ \psi : \psi \in \mathcal{C}\}$$

is a closed operator convex cone.

Proof. Clearly $\pi(\mathcal{C})$ is an operator convex cone. That $\pi(\mathcal{C})$ is point-norm closed is essentially the same proof as [2, Theorem 6] (that the set of c.p. maps with contractive c.p. lifts is point-norm closed). However, to run Arveson's argument we must show that if $\phi : \mathfrak{A} \to \mathfrak{C}$ is a contractive c.p. map which is a point-norm limit of a net of (not necessarily contractive) maps $\pi \circ \psi_\lambda$ with $\psi_\lambda \in \mathcal{C}$, then there is a sequence of contractive maps $\widetilde{\psi}_n \in \mathcal{C}$ such that $\pi \circ \widetilde{\psi}_n \to \phi$ point-norm. Let $(a_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ be a dense sequence and an approximate identity respectively in $\mathfrak{A}$. For each $n \in \mathbb{N}$ we fix λ_n such that $\|\pi(\psi_{\lambda_n}(e_n x e_n)) - \phi(e_n x e_n)\| < 1/n$ for $x \in \{1, a_1, \dots, a_n\}$. We may pick a positive contraction $f_n \in \ker \pi$ such that $\|(1 - f_n)\psi_{\lambda_n}(e_n^2)(1 - f_n)\| < \|\phi(e_n^2)\| + 1/n \leq \frac{n+1}{n}$. Let $\widetilde{\psi}_n := \frac{n}{n+1}(1 - f_n)\psi_{\lambda_n}(e_n(-)e_n)(1 - f_n) \in \mathcal{C}$ which is contractive. It is easy to check that $\pi \circ \widetilde{\psi}_n \to \phi$ point-norm. Now the exact same proof as [2, Theorem 6] (alternatively, see [7, Lemma C.2]) provides $\psi \in \mathcal{C}$ such that $\pi \circ \psi = \phi$. ∎

Proposition 2.7. *Let $\mathfrak{A}$ be a separable, exact C^*-algebra, let $\mathfrak{B}$ be a C^*-algebra with a two-sided, closed ideal $\mathfrak{J}$, and let $\pi : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the quotient map. Let $\mathcal{C} \subseteq CP_{\mathrm{nuc}}(\mathfrak{A}, \mathfrak{B})$ be a closed operator convex cone. A c.p. map $\phi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ lifts to a c.p. map in $\mathcal{C}$ if and only if ϕ is nuclear and $\phi(a) \in \pi(\overline{\mathfrak{B}\{\psi(a) : \psi \in \mathcal{C}\}\mathfrak{B}})$, for every positive $a \in \mathfrak{A}$.*

Proof. If ϕ lifts to a map $\psi \in \mathfrak{C}$, then $\phi = \pi \circ \psi$ is nuclear (as ψ is nuclear), and $\phi(a) = \pi(\psi(a)) \in \pi(\overline{\mathfrak{B}\{\psi'(a) : \psi' \in \mathfrak{C}\}\mathfrak{B}})$ for every positive $a \in \mathfrak{A}$.

Suppose that ϕ is nuclear, and that $\phi(a) \in \pi(\overline{\mathfrak{B}\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{B}})$ for every positive $a \in \mathfrak{A}$. By Lemma 2.6, the set $\pi(\mathfrak{C})$ of c.p. maps that lift to $\mathfrak{C}$, is a closed operator convex cone consisting only of nuclear maps. Thus, by our Hahn–Banach type separation theorem (Theorem 2.5), $\phi \in \pi(\mathfrak{C})$ if and only if

$$\begin{aligned}\phi(a) &\in \overline{(\mathfrak{B}/\mathfrak{J})\{\eta(a) : \eta \in \pi(\mathfrak{C})\}(\mathfrak{B}/\mathfrak{J})} \\ &= \overline{\pi(\mathfrak{B})\{\pi \circ \psi(a) : \psi \in \mathfrak{C}\}\pi(\mathfrak{B})} \\ &= \pi\big(\overline{\mathfrak{B}\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{B}}\big).\end{aligned}$$

■

3. Exact C^*-algebras and nuclear maps

In this short section, we prove a few well-known results about exact C^*-algebras. For a C^*-algebra $\mathfrak{B}$, we let $\mathcal{M}(\mathfrak{B})$ denote its multiplier algebra, and $\mathcal{Q}(\mathfrak{B}) := \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ its corona algebra.

Definition 3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, and $\phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ be a c.p. map. We say that ϕ is *weakly nuclear* if the c.p. maps $b^*\phi(-)b\colon \mathfrak{A} \to \mathfrak{B}$ are nuclear for all $b \in \mathfrak{B}$.

Recall, that a C^*-algebra is exact if and only if the C^*-algebra has a faithful representation on a Hilbert space which is nuclear. By Arveson's extension theorem, this is equivalent to *any* representation on a Hilbert space being nuclear. We need the following other characterisation of exactness.

Proposition 3.2. *A C^*-algebra $\mathfrak{A}$ is exact if and only if it holds that for any σ-unital C^*-algebra $\mathfrak{B}$ and any weakly nuclear map $\phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$, ϕ is nuclear.*

Proof. Suppose that any weakly nuclear map from $\mathfrak{A}$ into a multiplier algebra of a σ-unital C^*-algebra is nuclear. To show that $\mathfrak{A}$ is exact, it suffices to show that every separable C^*-subalgebra is exact. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}$ be a separable C^*-subalgebra, and let $\pi\colon \mathfrak{A}_0 \to \mathcal{M}(\mathbb{K})$ be a faithful representation. By Arveson's extension theorem, we may extend this map to a c.p. map $\tilde{\pi}\colon \mathfrak{A} \to \mathcal{M}(\mathbb{K})$, which is nuclear by assumption. Thus, π is nuclear and hence $\mathfrak{A}_0$ is exact. It follows that $\mathfrak{A}$ is exact.

Now suppose that $\mathfrak{A}$ is exact, that $\mathfrak{B}$ is any σ-unital C^*-algebra and $\phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ is weakly nuclear. By standard arguments we may assume that $\mathfrak{A}$ and ϕ are unital. It suffices to show that for any unital, separable C^*-subalgebra $\mathfrak{A}_0$ the restriction $\phi|_{\mathfrak{A}_0}$ is nuclear. Let $\iota\colon \mathfrak{A}_0 \hookrightarrow \mathcal{M}(\mathbb{K})$ be a unital inclusion. Since $\mathfrak{A}_0$ is a C^*-subalgebra of an exact C^*-algebra, it is itself exact, and thus ι is nuclear. Let Φ be the composition

$$\mathfrak{A}_0 \xrightarrow{\iota} \mathcal{M}(\mathbb{K}) \xrightarrow{1 \otimes \mathrm{id}} \mathcal{M}(\mathfrak{B}) \otimes \mathcal{M}(\mathbb{K}) \hookrightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K}),$$

which is nuclear. It basically follows from a result of Kasparov in [17] (see [9] for details on generalising Kasparov's result to the case which we are considering) that Φ absorbs any unital weakly nuclear c.p. map. In particular, it absorbs the map ϕ_0, defined as the composition

$$\mathfrak{A}_0 \xrightarrow{\phi|_{\mathfrak{A}_0}} \mathcal{M}(\mathfrak{B}) \xrightarrow{\mathrm{id}\otimes e_{11}} \mathcal{M}(\mathfrak{B}) \otimes \mathcal{M}(\mathbb{K}) \hookrightarrow \mathcal{M}(\mathfrak{B} \otimes \mathbb{K}).$$

Thus, there is a sequence of isometries (v_n) in $\mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ such that $v_n^* \Phi(-) v_n$ converges point-norm to ϕ_0. Since Φ is nuclear it follows that ϕ_0 is nuclear. There is a conditional expectation Ψ given by the composition

$$\mathcal{M}(\mathfrak{B} \otimes \mathbb{K}) \xrightarrow{(1\otimes e_{11})(-)(1\otimes e_{11})} \mathcal{M}(\mathfrak{B}) \otimes e_{11} \cong \mathcal{M}(\mathfrak{B})$$

such that $\Psi \circ \phi_0 = \phi|_{\mathfrak{A}_0}$ and thus $\phi|_{\mathfrak{A}_0}$ is nuclear. ■

Recall, that when $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ is an extension of C^*-algebras, there is an induced $*$-homomorphism $\tau\colon \mathfrak{A} \to \mathcal{Q}(\mathfrak{B}) := \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ called the *Busby map*. Also, there is a canonical isomorphism from $\mathfrak{E}$ onto the pull-back

$$\mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}) := \{(a, m) \in \mathfrak{A} \oplus \mathcal{M}(\mathfrak{B}) : \tau(a) = m + \mathfrak{B}\}.$$

An interesting observation can be made on extensions of exact C^*-algebras by nuclear C^*-algebras. This will be used in Theorem 5.6 to prove an Effros–Haagerup type lifting result, cf. [12].

Corollary 3.3. *Let $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be an extension of C^*-algebras with Busby map τ. Suppose that $\mathfrak{A}$ is exact and $\mathfrak{B}$ is σ-unital and nuclear. Then $\mathfrak{E}$ is exact if and only if τ is nuclear.*

Proof. If $\mathfrak{E}$ is non-unital we may consider the unitised extension $0 \to \mathfrak{B} \to \mathfrak{E}^\dagger \to \mathfrak{A}^\dagger \to 0$. Since τ is nuclear if and only if the unitisation $\tau^\dagger$ is nuclear, and $\mathfrak{E}$ is exact if and only if $\mathfrak{E}^\dagger$ is exact, we may assume that $\mathfrak{E}$ is unital. It is well known (see e.g. [7, Exercise 3.9.8]) that the extension algebra of an extension of exact C^*-algebras is exact if the extension is locally split. The converse is also true, and follows from [12]. If τ is nuclear then for any finite dimensional operator system $E \subseteq \mathfrak{A}$, there is a c.p. lift $\tilde{\tau}\colon E \to \mathcal{M}(\mathfrak{B})$ of $\tau|_E$ by the Choi–Effros lifting theorem [8]. If $\iota\colon E \to \mathfrak{A}$ is the inclusion, then $(\iota, \tilde{\tau})\colon E \to \mathfrak{A} \oplus_{\mathcal{Q}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}) \cong \mathfrak{E}$ is a c.p. lift of ι. Hence the extension is locally split and thus $\mathfrak{E}$ is exact.

If $\mathfrak{E}$ is exact then it is locally split as noted above. Since $\mathfrak{B}$ is nuclear it follows from [12] that for any separable C^*-subalgebra $\mathfrak{A}_0 \subseteq \mathfrak{A}$ there is a c.p. lift $\tilde{\tau}\colon \mathfrak{A}_0 \to \mathcal{M}(\mathfrak{B})$ of $\tau|_{\mathfrak{A}_0}$. Since $\mathfrak{B}$ is nuclear it follows that $\tilde{\tau}$ is weakly nuclear, and since $\mathfrak{A}_0$ is exact it follows from Proposition 3.2 that $\tilde{\tau}$ is nuclear. Hence $\tau|_{\mathfrak{A}_0} = \pi \circ \tilde{\tau}$ is nuclear. Since $\mathfrak{A}_0$ was arbitrarily chosen, it follows that τ is nuclear. ■

4. Ideal related completely positive selections

In this section, we prove ideal related selection results for completely positive maps, where we by ideal related mean X-equivariant as defined below. The purpose of these selection results, is to construct "many" X-equivariant c.p. maps between two X-C^*-algebras, which is important when one wishes to lift X-equivariant c.p. maps to X-equivariant c.p. maps.

4.1. Actions of topological spaces on C^*-algebras

When X is a topological space, we let $\mathbb{O}(\mathsf{X})$ denote the complete lattice of open subsets of X. Also, for a C^*-algebra $\mathfrak{A}$, we let $\mathbb{I}(\mathfrak{A})$ denote the complete lattice of two-sided, closed ideals in $\mathfrak{A}$.

Definition 4.1. Let X be a topological space. An *action of* X *on a* C^**-algebra* $\mathfrak{A}$ is an order preserving map $\psi\colon \mathbb{O}(\mathsf{X}) \to \mathbb{I}(\mathfrak{A})$, i.e. a map such that if $\mathsf{U} \subseteq \mathsf{V}$ in $\mathbb{O}(\mathsf{X})$ then $\psi(\mathsf{U}) \subseteq \psi(\mathsf{V})$.

A C^*-algebra $\mathfrak{A}$ together with an action ψ of X on $\mathfrak{A}$, is called an X-C^**-algebra*. It is customary to suppress the action ψ in the notation, by simply saying that $\mathfrak{A}$ is an X-C^*-algebra, and defining $\mathfrak{A}(\mathsf{U}) := \psi(\mathsf{U})$ for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

A map $\phi\colon \mathfrak{A} \to \mathfrak{B}$ of C^*-algebras with actions of X is called X-*equivariant* if $\phi(\mathfrak{A}(\mathsf{U})) \subseteq \mathfrak{B}(\mathsf{U})$ for every $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

Remark 4.2. If X is a space acting on the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, then the set $CP(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ of X-equivariant c.p. maps $\mathfrak{A} \to \mathfrak{B}$ is a closed operator convex cone.

In particular, the set of all nuclear, X-equivariant c.p. maps $\mathfrak{A} \to \mathfrak{B}$ is a closed operator convex cone, as this is the set $CP(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cap CP_{\mathrm{nuc}}(\mathfrak{A}, \mathfrak{B})$, and since being a closed operator convex cone is preserved under intersections.

It is often necessary to impose stronger conditions on our actions.

Definition 4.3. Let $\mathfrak{A}$ be an X-C^*-algebra. We say that $\mathfrak{A}$ is

- *finitely lower semicontinuous* if $\mathfrak{A}(\mathsf{X}) = \mathfrak{A}$, and if it respects finite infima, i.e. for open subsets U and V of X we have
$$\mathfrak{A}(\mathsf{U}) \cap \mathfrak{A}(\mathsf{V}) = \mathfrak{A}(\mathsf{U} \cap \mathsf{V}),$$
- *lower semicontinuous* if $\mathfrak{A}(\mathsf{X}) = \mathfrak{A}$, and if it respects arbitrary infima, i.e. for any family (U_λ) of open subsets of X we have
$$\bigcap_\lambda \mathfrak{A}(\mathsf{U}_\lambda) = \mathfrak{A}(\mathsf{U}),$$
where U is the interior of $\bigcap_\lambda \mathsf{U}_\lambda$,
- *finitely upper semicontinuous* if $\mathfrak{A}(\emptyset) = 0$, and if it respects finite suprema, i.e. for open subsets U and V of X we have
$$\mathfrak{A}(\mathsf{U}) + \mathfrak{A}(\mathsf{V}) = \mathfrak{A}(\mathsf{U} \cup \mathsf{V}),$$

- *monotone upper semicontinuous* if it respects monotone suprema, i.e. for any increasing net (U_λ) of open subsets of X we have

$$\overline{\bigcup_\lambda \mathfrak{A}(\mathsf{U}_\lambda)} = \mathfrak{A}\Big(\bigcup_\lambda \mathsf{U}_\lambda\Big),$$

- *upper semicontinuous* if it is finitely and monotone upper semicontinuous.

Note that an upper semicontinuous X-C^*-algebra $\mathfrak{C}$ satisfies $\mathfrak{C}(\emptyset) = 0$. This condition ensures that the map $\Psi\colon \mathbb{I}(\mathfrak{C}) \to \mathbb{O}(\mathsf{X})$ given by

$$\Psi(\mathfrak{I}) = \bigcup \{\mathsf{U} \in \mathbb{O}(\mathsf{X}) : \mathfrak{C}(\mathsf{U}) \subseteq \mathfrak{I}\}$$

is well defined. This will be used in the proof of Proposition 4.11. That a lower semicontinuous X-C^*-algebra $\mathfrak{C}$ satisfies $\mathfrak{C}(\mathsf{X}) = \mathfrak{C}$, is for a similar reason.

Definition 4.4. Let $\mathfrak{A}$ be an X-C^*-algebra, $a \in \mathfrak{A}$ and $\mathsf{U} \in \mathbb{O}(\mathsf{X})$. We say that a is U-*full*, if U is minimal amongst open sets $\mathsf{V} \in \mathbb{O}(\mathsf{X})$ for which $a \in \mathfrak{A}(\mathsf{V})$, i.e. $a \in \mathfrak{A}(\mathsf{U})$ and whenever $\mathsf{V} \in \mathbb{O}(\mathsf{X})$ such that $a \in \mathfrak{A}(\mathsf{V})$ then $\mathsf{U} \subseteq \mathsf{V}$.

If a is U-full, then the set U is unique.

Notation 4.5. Whenever $a \in \mathfrak{A}$ is U-full for some $\mathsf{U} \in \mathbb{O}(\mathsf{X})$, then we denote by $\mathsf{U}_a := \mathsf{U}$.

Any element $a \in \mathfrak{A}$ in a C^*-algebra generates a two-sided closed ideal $\overline{\mathfrak{A}a\mathfrak{A}}$ which corresponds uniquely to an open subset U of $\operatorname{Prim}\mathfrak{A}$. If $\mathfrak{A}$ is equipped with the canonical action $\mathbb{O}(\operatorname{Prim}\mathfrak{A}) \xrightarrow{\cong} \mathbb{I}(\mathfrak{A})$, then a is U-full for this set $\mathsf{U} \in \mathbb{O}(\operatorname{Prim}\mathfrak{A})$, so $\mathfrak{A}(\mathsf{U}_a) = \overline{\mathfrak{A}a\mathfrak{A}}$. If $\mathfrak{A}$ is a general X-C^*-algebra, and $a \in \mathfrak{A}$ is U_a-full, then one should think of U_a as being the open subset of X generated by a.

We will use the following result from [14]. For the sake of completion, we give a proof.

Proposition 4.6. *Let $\mathfrak{A}$ be an X-C^*-algebra. Then $\mathfrak{A}$ is lower semicontinuous if and only if every element $a \in \mathfrak{A}$ is U_a-full for some (unique) $\mathsf{U}_a \in \mathbb{O}(\mathsf{X})$.*

Proof. If $\mathfrak{A}$ is lower semicontinuous and $a \in \mathfrak{A}$, let U_a be the interior of the intersection of all open sets $\mathsf{U} \subseteq \mathsf{X}$ for which $a \in \mathfrak{A}(\mathsf{U})$. As $\mathfrak{A}(\mathsf{X}) = \mathfrak{A}$, this construction is well defined. By lower semicontinuity, $a \in \mathfrak{A}(\mathsf{U}_a)$, so a is U_a-full, as U_a is minimal amongst $\mathsf{U} \in \mathbb{O}(\mathsf{X})$ for which $a \in \mathfrak{A}(\mathsf{U})$.

Suppose every $a \in \mathfrak{A}$ is U_a-full, let (U_λ) be a family of sets in $\mathbb{O}(\mathsf{X})$, and U be the interior of the intersection of (U_λ). Clearly $\mathfrak{A}(\mathsf{U}) \subseteq \bigcap \mathfrak{A}(\mathsf{U}_\lambda)$. Let $a \in \bigcap \mathfrak{A}(\mathsf{U}_\lambda)$. As $a \in \mathfrak{A}(\mathsf{U}_\lambda)$ for all λ, it follows that $\mathsf{U}_a \subseteq \mathsf{U}_\lambda$ for all λ, and thus $\mathsf{U}_a \subseteq \mathsf{U}$. So $a \in \mathfrak{A}(\mathsf{U})$ and thus $\mathfrak{A}(\mathsf{U}) = \bigcap \mathfrak{A}(\mathsf{U}_\lambda)$. Finally, suppose $a \in \mathfrak{A} \setminus \mathfrak{A}(\mathsf{X})$. Then $a \in \mathfrak{A}(\mathsf{U}_a) \subseteq \mathfrak{A}(\mathsf{X})$, a contradiction, so $\mathfrak{A} = \mathfrak{A}(\mathsf{X})$. ∎

Example 4.7. Let X be a locally compact Hausdorff space. A $C_0(\mathsf{X})$-*algebra* is a C^*-algebra $\mathfrak{A}$ together with an essential $*$-homomorphism Φ from $C_0(\mathsf{X})$ into the centre of $\mathcal{M}(\mathfrak{A})$.

Essential means that $\overline{\Phi(C_0(\mathsf{X}))\mathfrak{A}} = \mathfrak{A}$. As described in [24, Section 2.1] there is a one-to-one correspondence between such $*$-homomorphisms, and actions of X on $\mathfrak{A}$ which are finitely lower semicontinuous and upper semicontinuous. The induced action is given by $\mathfrak{A}(\mathsf{U}) = \overline{\mathfrak{A}\Phi(C_0(\mathsf{U}))}$ for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$. A $C_0(\mathsf{X})$-algebra is called *continuous* if for every $a \in \mathfrak{A}$, the set $\mathsf{U}_a := \{x \in \mathsf{X} : \|a + \mathfrak{A}(\mathsf{X} \setminus \{x\})\| > 0\}$ is open. If this is the case it is easily seen that a is U_a-full, and conversely, if a is U_a-full, then $\mathsf{U}_a = \{x \in \mathsf{X} : \|a + \mathfrak{A}(\mathsf{X} \setminus \{x\})\| > 0\}$. Thus, $\mathfrak{A}$ as an X-C^*-algebra is continuous if and only if $\mathfrak{A}$ as a $C_0(\mathsf{X})$-algebra is continuous.

Observation 4.8. Let $\mathfrak{A}$ and $\mathfrak{B}$ be X-C^*-algebras with $\mathfrak{A}$ lower semicontinuous. Then a map $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is X-equivariant if and only if for all (positive) $a \in \mathfrak{A}$, $\phi(a) \in \mathfrak{B}(\mathsf{U}_a)$.

When $\mathfrak{B}$ is an X-C^*-algebra and $\mathfrak{D}$ is any C^*-algebra, then the spatial tensor product $\mathfrak{B} \otimes \mathfrak{D}$ is canonically an X-C^*-algebra by the action $\mathsf{U} \mapsto \mathfrak{B}(\mathsf{U}) \otimes \mathfrak{D}$.

Lemma 4.9. *Let $\mathfrak{B}$ be an X-C^*-algebra and $\mathfrak{D}$ be a C^*-algebra. Whenever $\mathfrak{B}$ is monotone (resp. finitely) upper semicontinuous, then so is $\mathfrak{B} \otimes \mathfrak{D}$.*

Suppose, moreover, that $\mathfrak{B}$ or $\mathfrak{D}$ is exact. If $\mathfrak{B}$ is (finitely) lower semicontinuous, then so is $\mathfrak{B} \otimes \mathfrak{D}$.

Proof. Monotone upper semicontinuity: this is clearly preserved when tensoring with $\mathfrak{D}$.

Finite upper semicontinuity: Clearly $(\mathfrak{B} \otimes \mathfrak{D})(\emptyset) = 0$. If $\mathfrak{I}$ and $\mathfrak{J}$ are two-sided, closed ideals in $\mathfrak{B}$, then $(\mathfrak{I} + \mathfrak{J}) \otimes \mathfrak{D}$ is the closed linear span of elementary tensors. Since any element in $\mathfrak{I} + \mathfrak{J}$ can be written as $x + y$ with $x \in \mathfrak{I}$ and $y \in \mathfrak{J}$ it easily follows that $(\mathfrak{I} + \mathfrak{J}) \otimes \mathfrak{D} = \mathfrak{I} \otimes \mathfrak{D} + \mathfrak{J} \otimes \mathfrak{D}$. Thus, finite upper semicontinuity is preserved when tensoring with $\mathfrak{D}$.

(Finite) lower semicontinuity: Clearly $(\mathfrak{B} \otimes \mathfrak{D})(\mathsf{X}) = \mathfrak{B} \otimes \mathfrak{D}$. Let $(\mathfrak{J}_\lambda)$ be a family of two-sided, closed ideals in $\mathfrak{B}$, let $\mathfrak{J} = \bigcap \mathfrak{J}_\lambda$, and let $\mathfrak{I} = \bigcap (\mathfrak{J}_\lambda \otimes \mathfrak{D})$. Clearly $\mathfrak{J} \otimes \mathfrak{D} \subseteq \mathfrak{I}$. By [7, Corollary 9.4.6], $\mathfrak{I}$ is the closed linear span of all elementary tensors $b \otimes d$ with $b \in \mathfrak{B}$, $d \in \mathfrak{D}$ and $b \otimes d \in \mathfrak{I}$. For such b, d it easily follows that $b \in \mathfrak{J}$, so $\mathfrak{I} = \mathfrak{J} \otimes \mathfrak{D}$. It clearly follows that (finite) lower semicontinuity is preserved when tensoring with $\mathfrak{D}$. ∎

4.2. Selection results

In this subsection, we will be applying a variation of one of the remarkable selection theorems of Michael [25]. To do this we need some notation. Let Y and Z be topological spaces. A *carrier* from Y to Z is a map $\Gamma\colon \mathsf{Y} \to 2^{\mathsf{Z}}$, where 2^{Z} is the set of non-empty subsets of Z. We say that Γ is *lower semicontinuous* if for every open subset U of Z, the set

$$\{y \in \mathsf{Y} : \Gamma(y) \cap \mathsf{U} \neq \emptyset\}$$

is open in Y. One of Michael's selection theorems [26, Theorem 1.2] implies that if Y is a paracompact T_1-space (e.g. a second countable, locally compact Hausdorff space), if $(\mathsf{Z}^*)_1$ is the unit ball of the dual space of a separable Banach space Z, and if Γ is a lower semicontinuous carrier from Y to $(\mathsf{Z}^*)_1$ such that $\Gamma(y)$ is a weak*-closed convex set in

$(\mathsf{Z}^*)_1$ for all $y \in \mathsf{Y}$, then there exists a continuous map $\gamma\colon \mathsf{Y} \to (\mathsf{Z}^*)_1$ such that $\gamma(y) \in \Gamma(y)$ for all y.

When Y is a locally compact Hausdorff space, $y \in \mathsf{Y}$ we let $\mathrm{ev}_y\colon C_0(\mathsf{Y}) \to \mathbb{C}$ denote the $*$-homomorphism which is evaluation in y.

We will use the following ideal related selection result. A very similar result can be found in the preprint [15, Lemma A.15].[1]

Lemma 4.10. *Let Y be a second countable, locally compact Hausdorff space, and let $\mathfrak{A}$ be a lower semicontinuous Y-C^*-algebra. For any distinct points $y_1, \dots, y_n \in \mathsf{Y}$, and any quasi-states η_k on $\mathfrak{A}/\mathfrak{A}(\mathsf{Y} \setminus \{y_k\})$, there is a contractive Y-equivariant c.p. map $\phi\colon \mathfrak{A} \to C_0(\mathsf{Y})$, such that*

$$\mathrm{ev}_{y_k}(\phi(a)) = \eta_k(a + \mathfrak{A}(\mathsf{Y} \setminus \{y_k\}))$$

for all $a \in \mathfrak{A}$.

Proof. First note that the forced unitisation $\mathfrak{A}^\dagger$ has a canonical lower semicontinuous action of Y given by $\mathfrak{A}^\dagger(\mathsf{V}) = \mathfrak{A}(\mathsf{V})$ when $\mathsf{V} \neq \mathsf{Y}$ and $\mathfrak{A}^\dagger(\mathsf{Y}) = \mathfrak{A}^\dagger$. Let $P(\mathfrak{A}^\dagger)$ be the space of pure states on $\mathfrak{A}^\dagger$. Let $\Gamma\colon \mathsf{Y} \to 2^{P(\mathfrak{A}^\dagger)}$ be the carrier given by

$$\Gamma(y) = \left\{\eta \in P(\mathfrak{A}^\dagger) : \eta(\mathfrak{A}^\dagger(\mathsf{Y} \setminus \{y\})) = 0\right\} = \left\{\eta \in P(\mathfrak{A}^\dagger) : \eta(\mathfrak{A}(\mathsf{Y} \setminus \{y\})) = 0\right\}.$$

We claim that Γ is lower semicontinuous. To see this, first recall (e.g. [28, Theorem 4.3.3]) that the continuous map $P(\mathfrak{A}^\dagger) \to \operatorname{Prim} \mathfrak{A}^\dagger$ given by $\eta \mapsto \ker \pi_\eta$, where π_η is the GNS representation, is an open map. Thus, this induces a map

$$\mathbb{O}(P(\mathfrak{A}^\dagger)) \to \mathbb{O}(\operatorname{Prim} \mathfrak{A}^\dagger) \cong \mathbb{I}(\mathfrak{A}^\dagger).$$

We may construct a map $\mathbb{I}(\mathfrak{A}^\dagger) \to \mathbb{O}(\mathsf{Y})$, by

$$\mathfrak{J} \mapsto \Bigg(\bigcap_{\substack{\mathsf{V} \in \mathbb{O}(\mathsf{Y}),\\ \mathfrak{J} \subseteq \mathfrak{A}^\dagger(\mathsf{V})}} \mathsf{V} \Bigg)^{\circ}.$$

Since $\mathfrak{A}^\dagger$ is lower semicontinuous, $\mathfrak{J}$ is mapped to the unique smallest open subset V of Y for which $\mathfrak{J} \subseteq \mathfrak{A}^\dagger(\mathsf{V})$. Let Φ denote the composition

$$\mathbb{O}(P(\mathfrak{A}^\dagger)) \to \mathbb{O}(\operatorname{Prim} \mathfrak{A}^\dagger) \cong \mathbb{I}(\mathfrak{A}^\dagger) \to \mathbb{O}(\mathsf{Y}).$$

The map Φ can be described as follows: Let $\mathsf{U} \in \mathbb{O}(P(\mathfrak{A}^\dagger))$. Then there is a unique two-sided, closed ideal $\mathfrak{J}_{\mathsf{U}}$ in $\mathfrak{A}^\dagger$, such that

$$\left\{\ker \pi_\eta : \eta \in \mathsf{U}\right\} = \left\{\mathfrak{p} \in \operatorname{Prim} \mathfrak{A}^\dagger : \mathfrak{J}_{\mathsf{U}} \not\subseteq \mathfrak{p}\right\},$$

and $\Phi(\mathsf{U})$ is the unique smallest open subset of Y such that $\mathfrak{J}_{\mathsf{U}} \subseteq \mathfrak{A}^\dagger(\Phi(\mathsf{U}))$.

[1] In [15, Lemma A.15] they assume that $\mathfrak{A}$ is separable and Y is any locally compact Hausdorff space. In their proof they use an unspecified selection theorem of Michael from [25]. The selection theorem with weakest preliminary conditions in [25] requires Y to be normal. However, there are examples of locally compact Hausdorff spaces (not second countable) which are not normal. Thus, the proof of [15, Lemma A.15] requires more arguments than are given, if one should apply the selection theorems of Michael.

We claim that for any $\mathsf{U} \in \mathbb{O}(P(\mathfrak{A}^\dagger))$ we have

$$\{y \in \mathsf{Y} : \Gamma(y) \cap \mathsf{U} \neq \emptyset\} = \Phi(\mathsf{U})$$

and thus Γ is a lower semicontinuous carrier. That this is true follows from

$$\begin{aligned}\Gamma(y) \cap \mathsf{U} = \emptyset &\Leftrightarrow \text{for every } \eta \in \mathsf{U} \text{ we have } \mathfrak{A}^\dagger(\mathsf{Y} \setminus \{y\}) \nsubseteq \ker \pi_\eta \\ &\Leftrightarrow \{\mathfrak{p} \in \operatorname{Prim} \mathfrak{A}^\dagger : \mathfrak{J}_\mathsf{U} \nsubseteq \mathfrak{p}\} \subseteq \{\mathfrak{p} \in \operatorname{Prim} \mathfrak{A}^\dagger : \mathfrak{A}^\dagger(\mathsf{Y} \setminus \{y\}) \nsubseteq \mathfrak{p}\} \\ &\Leftrightarrow \mathfrak{J}_\mathsf{U} \subseteq \mathfrak{A}^\dagger(\mathsf{Y} \setminus \{y\}) \\ &\Leftrightarrow \Phi(\mathsf{U}) \subseteq \mathsf{Y} \setminus \{y\} \\ &\Leftrightarrow y \notin \Phi(\mathsf{U}).\end{aligned}$$

Let $\mathfrak{Q}(\mathfrak{A}) \subseteq \mathfrak{A}^*$ denote the quasi-state space of $\mathfrak{A}$ and

$$K_y := \{\eta \in \mathfrak{Q}(\mathfrak{A}) : \eta(\mathfrak{A}(\mathsf{Y} \setminus \{y\})) = 0\}$$

for every $y \in \mathsf{Y}$. Recall, that the restriction map $(\mathfrak{A}^\dagger)^* \to \mathfrak{A}^*$ induces a homeomorphism $P(\mathfrak{A}^\dagger) \to \{0\} \cup P(\mathfrak{A})$. Moreover, under this identification, the closed convex hull of $\Gamma(y)$ is exactly K_y. Thus, it follows from [25, Propositions 2.3 and 2.6] that the carrier $\Gamma_1 \colon \mathsf{Y} \to 2^{(\mathfrak{A}^*)_1}$ given by $\Gamma_1(y) = K_y$, is lower semicontinuous.

Let $\mathsf{A} = \{y_1, \ldots, y_n\} \subseteq \mathsf{Y}$ which is (obviously) a closed subspace, and let $\pi_y \colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(\mathsf{Y} \setminus \{y\})$ be the quotient map for each $y \in \mathsf{Y}$. The map $g_0 \colon \mathsf{A} \to \mathfrak{Q}(\mathfrak{A})$ given by $g_0(y_k) = \eta_k \circ \pi_{y_k}$ is clearly continuous and $g_0(y_k) \in K_{y_k} = \Gamma_1(y_k)$. Thus, it follows from [25, Example 1.3*] that the carrier $\Gamma_2 \colon \mathsf{Y} \to 2^{(\mathfrak{A}^*)_1}$ given by

$$\Gamma_2(y) = \begin{cases} \{g_0(y)\}, & \text{if } y \in \mathsf{A}, \\ K_y, & \text{otherwise} \end{cases}$$

is lower semicontinuous. Since Y is a paracompact T_1-space, and $\Gamma_2(y)$ is a closed convex space for every $y \in \mathsf{Y}$, it follows from [26, Theorem 1.2] that there exists a continuous map $g \colon \mathsf{Y} \to (\mathfrak{A}^*)_1$, such that $g(y) \in \Gamma_2(y)$ for all $y \in \mathsf{Y}$.

Now, let $\widehat{\phi} \colon \mathfrak{A} \to C_b(\mathsf{Y})$ be given by $\mathrm{ev}_y \circ \widehat{\phi}(a) = g(y)(a)$. Since $\mathrm{ev}_y \circ \widehat{\phi}$ is a contractive c.p. map (a quasi-state) for every $y \in \mathsf{Y}$, it follows that $\widehat{\phi}$ is a contractive c.p. map. Pick a positive contraction f in $C_0(\mathsf{Y})$ such that $f(y_k) = 1$ for $k = 1, \ldots, n$. Then $\phi \colon \mathfrak{A} \to C_0(\mathsf{Y})$ given by $\phi(a) = f \cdot \widehat{\phi}(a)$ is again a contractive c.p. map. Moreover, we clearly have

$$\mathrm{ev}_{y_k}(\phi(a)) = f(y_k) \cdot g(y_k)(a) = \eta_k(a + \mathfrak{A}(\mathsf{Y} \setminus \{y_k\})).$$

Thus, it remains to show that ϕ is Y-equivariant.

Let $\mathsf{V} \in \mathbb{O}(\mathsf{Y})$ and $a \in \mathfrak{A}(\mathsf{V})$. For every $y \notin \mathsf{V}$ we have $\mathsf{V} \subseteq \mathsf{Y} \setminus \{y\}$ and thus $a \in \mathfrak{A}(\mathsf{Y} \setminus \{y\})$. Since $\mathrm{ev}_y \circ \phi(a) \in K_y$ it follows that $\mathrm{ev}_y \circ \phi(a) = 0$, and thus $\phi(a) \in C_0(\mathsf{Y} \setminus \{y\})$. Hence we have

$$\phi(a) \in \bigcap_{y \notin \mathsf{V}} C_0(\mathsf{Y} \setminus \{y\}) = C_0(\mathsf{V}),$$

which implies that ϕ is Y-equivariant. ■

The above lemma lets us prove the following selection result for X-equivariant maps. Recall, that when $\mathfrak{A}$ is a lower semicontinuous X-C^*-algebra and $a \in \mathfrak{A}$, then U_a denotes the unique smallest open subset of X for which $a \in \mathfrak{A}(\mathsf{U}_a)$.

Proposition 4.11. *Let $\mathfrak{A}$ be a lower semicontinuous X-C^*-algebra and let $\mathfrak{C}$ be a separable, commutative, upper semicontinuous X-C^*-algebra. For every positive $a \in \mathfrak{A}$ there exists an X-equivariant c.p. map $\phi\colon \mathfrak{A} \to \mathfrak{C}$ such that $\phi(a)$ is strictly positive in $\mathfrak{C}(\mathsf{U}_a)$.*

Proof. Let $\mathsf{Y} = \operatorname{Prim}\mathfrak{C}$ such that $\mathfrak{C} = C_0(\mathsf{Y})$. To avoid confusion, we will write $\mathfrak{C}$ when we are using the X-C^*-algebra structure, and write $C_0(\mathsf{Y})$ when we consider $\mathfrak{C} = C_0(\mathsf{Y})$ with the tight Y-C^*-algebra structure. The idea of the proof, is to construct a lower semicontinuous action $\widetilde{\Psi}$ of Y on $\mathfrak{A}$, such that a c.p. map $\mathfrak{A} \to \mathfrak{C}$ is X-equivariant if and only if the same map $(\mathfrak{A}, \widetilde{\Psi}) \to C_0(\mathsf{Y})$ is Y equivariant. When this is done we can apply Lemma 4.10 to construct X-equivariant c.p. maps $\mathfrak{A} \to \mathfrak{C}$.

Construct a map $\Psi\colon \mathbb{O}(\mathsf{Y}) \to \mathbb{O}(\mathsf{X})$ given by

$$\Psi(\mathsf{V}) = \bigcup \{\mathsf{U} \in \mathbb{O}(\mathsf{X}) : \mathfrak{C}(\mathsf{U}) \subseteq C_0(\mathsf{V})\}.$$

Since the action of X on $\mathfrak{C}$ is upper semicontinuous, $\Psi(\mathsf{V})$ is the *unique largest* open subset of X such that $\mathfrak{C}(\Psi(\mathsf{V})) \subseteq C_0(\mathsf{V})$, in the sense that $\mathfrak{C}(\Psi(\mathsf{V})) \subseteq C_0(\mathsf{V})$ and if $\mathsf{U} \in \mathbb{O}(\mathsf{X})$ satisfies $\mathfrak{C}(\mathsf{U}) \subseteq C_0(\mathsf{V})$ then $\mathsf{U} \subseteq \Psi(\mathsf{V})$. We clearly have that Ψ is order preserving and that $\Psi(\mathsf{Y}) = \mathsf{X}$. We want to show that whenever (V_α) is a family of open subsets of Y, and V is the interior of $\bigcap \mathsf{V}_\alpha$, then $\Psi(\mathsf{V})$ is the interior of $\bigcap \Psi(\mathsf{V}_\alpha)$. For now, we let W denote the interior of $\bigcap \Psi(\mathsf{V}_\alpha)$.

Since Ψ is order preserving we clearly have that $\Psi(\mathsf{V}) \subseteq \mathsf{W}$. For the converse inclusion we have that $\mathfrak{C}(\mathsf{W}) \subseteq \mathfrak{C}(\Psi(\mathsf{V}_\alpha)) \subseteq C_0(\mathsf{V}_\alpha)$ for each α. Hence $\mathfrak{C}(\mathsf{W}) \subseteq \bigcap C_0(\mathsf{V}_\alpha) = C_0(\mathsf{V})$. It follows from the definition of Ψ that $\mathsf{W} \subseteq \Psi(\mathsf{V})$, and thus we have equality.

Let $\widetilde{\Psi}\colon \mathbb{O}(\mathsf{Y}) \to \mathbb{I}(\mathfrak{A})$ be the action of Y on $\mathfrak{A}$ given by $\widetilde{\Psi}(\mathsf{V}) = \mathfrak{A}(\Psi(\mathsf{V}))$. It follows that $\widetilde{\Psi}(\mathsf{Y}) = \mathfrak{A}$, and since the action of X on $\mathfrak{A}$ is lower semicontinuous, so is the action $\widetilde{\Psi}$, by what we have proven above. Thus, $(\mathfrak{A}, \widetilde{\Psi})$ is a lower semicontinuous Y-C^*-algebra.

We will prove that $CP(\mathsf{X}; \mathfrak{A}, \mathfrak{C}) = CP(\mathsf{Y}; (\mathfrak{A}, \widetilde{\Psi}), C_0(\mathsf{Y}))$. To see this, first note that $\mathfrak{C}(\Psi(\mathsf{V})) \subseteq C_0(\mathsf{V})$ for all $\mathsf{V} \in \mathbb{O}(\mathsf{Y})$. Thus, if ϕ is X-equivariant then

$$\phi(\widetilde{\Psi}(\mathsf{V})) = \phi(\mathfrak{A}(\Psi(\mathsf{V}))) \subseteq \mathfrak{C}(\Psi(\mathsf{V})) \subseteq C_0(\mathsf{V})$$

and thus ϕ is Y-equivariant. For $\mathsf{U} \in \mathbb{O}(\mathsf{X})$ let $\mathsf{V}^{\mathsf{U}} \in \mathbb{O}(\mathsf{Y})$ be such that $\mathfrak{C}(\mathsf{U}) = C_0(\mathsf{V}^{\mathsf{U}})$. Since $\Psi(\mathsf{V}^{\mathsf{U}})$ is the unique largest open subset of X such that $\mathfrak{C}(\Psi(\mathsf{V}^{\mathsf{U}})) \subseteq C_0(\mathsf{V}^{\mathsf{U}}) = \mathfrak{C}(\mathsf{U})$ it follows that $\mathsf{U} \subseteq \Psi(\mathsf{V}^{\mathsf{U}})$. Thus, if ψ is Y-equivariant then

$$\psi(\mathfrak{A}(\mathsf{U})) \subseteq \psi(\mathfrak{A}(\Psi(\mathsf{V}^{\mathsf{U}}))) = \psi(\widetilde{\Psi}(\mathsf{V}^{\mathsf{U}})) \subseteq C_0(\mathsf{V}^{\mathsf{U}}) = \mathfrak{C}(\mathsf{U}).$$

Hence it follows that $CP(\mathsf{X}; \mathfrak{A}, \mathfrak{C}) = CP(\mathsf{Y}; (\mathfrak{A}, \widetilde{\Psi}), C_0(\mathsf{Y}))$.

Fix a positive $a \in \mathfrak{A}$. Recall that U_a is the open subset of X such that a is U_a-full, when considering $\mathfrak{A}$ with the X-C^*-algebra structure. Since $(\mathfrak{A}, \widetilde{\Psi})$ is a lower semicontinuous

Y-C^*-algebra, we may find a unique open subset V_a of Y such that a is V_a-full when considered with the Y-C^*-algebra structure. We will show that $\mathfrak{C}(\mathsf{U}_a) = C_0(\mathsf{V}_a)$.

Since $a \in \widetilde{\Psi}(\mathsf{V}_a) = \mathfrak{A}(\Psi(\mathsf{V}_a))$ it follows from U_a-fullness that $\mathsf{U}_a \subseteq \Psi(\mathsf{V}_a)$ and thus $\mathfrak{C}(\mathsf{U}_a) \subseteq \mathfrak{C}(\Psi(\mathsf{V}_a)) \subseteq C_0(\mathsf{V}_a)$. Let $\mathsf{W} \in \mathbb{O}(\mathsf{Y})$ be such that $C_0(\mathsf{W}) = \mathfrak{C}(\mathsf{U}_a)$. Then $\mathsf{U}_a \subseteq \Psi(\mathsf{W})$ by the definition of Ψ. This implies that $a \in \mathfrak{A}(\mathsf{U}_a) \subseteq \mathfrak{A}(\Psi(\mathsf{W})) = \widetilde{\Psi}(\mathsf{W})$. By V_a-fullness it follows that $\mathsf{V}_a \subseteq \mathsf{W}$ and thus $C_0(\mathsf{V}_a) \subseteq C_0(\mathsf{W}) = \mathfrak{C}(\mathsf{U}_a)$. This shows that $\mathfrak{C}(\mathsf{U}_a) = C_0(\mathsf{V}_a)$.

Our goal is to construct an X-equivariant c.p. map $\psi\colon \mathfrak{A} \to \mathfrak{C}$ such that $\psi(a)$ is strictly positive in $\mathfrak{C}(\mathsf{U}_a)$. Equivalently, by what we have shown above, we should construct a Y-equivariant c.p. map $\psi\colon (\mathfrak{A}, \widetilde{\Psi}) \to C_0(\mathsf{Y})$ such that $\psi(a)$ is strictly positive in $C_0(\mathsf{V}_a)$.

Suppose that $\mathsf{V}_a = \emptyset$. Then $C_0(\mathsf{V}_a) = 0$, and thus letting ψ be the zero map will suffice. Thus, suppose that $\mathsf{V}_a \neq \emptyset$. For each $y \in \mathsf{V}_a$ we have that $\|a + \mathfrak{A}(\mathsf{Y} \setminus \{y\})\| > 0$. In fact, if $a \in \mathfrak{A}(\mathsf{Y} \setminus \{y\})$ then we would have

$$a \in \mathfrak{A}(\mathsf{V}_a) \cap \mathfrak{A}(\mathsf{Y} \setminus \{y\}) = \mathfrak{A}(\mathsf{V}_a \setminus \{y\})$$

which contradicts that a is V_a-full. Let η_y be a state on $\mathfrak{A}/\mathfrak{A}(\mathsf{Y} \setminus \{y\})$ such that $\eta_y(a + \mathfrak{A}(\mathsf{Y} \setminus \{y\})) = \|a + \mathfrak{A}(\mathsf{Y} \setminus \{y\})\|$. By Lemma 4.10 there is a contractive Y-equivariant c.p. map $\psi_y\colon (\mathfrak{A}, \widetilde{\Psi}) \to C_0(\mathsf{Y})$ such that $\mathrm{ev}_y \circ \psi_y(a) = \eta_y(a + \mathfrak{A}(\mathsf{Y} \setminus \{y\})) > 0$. Let $\mathsf{W}_y \subseteq \mathsf{V}_a$ be an open neighbourhood of y such that $\mathrm{ev}_z \circ \psi_y(a) > 0$ for all $z \in \mathsf{W}_y$. Then $(\mathsf{W}_y)_{y \in \mathsf{V}_a}$ is an open cover of V_a. Since Y is second countable (as $\mathfrak{C}$ is separable) V_a is σ-compact, so we may find a sequence (y_n) of points in V_a such that $(\mathsf{W}_{y_n})_{n \in \mathbb{N}}$ covers V_a. Let $\psi = \sum_{n=1}^{\infty} 2^{-n} \psi_{y_n}$ which is clearly a contractive Y-equivariant c.p. map. Clearly $0 < \mathrm{ev}_y(\psi(a))$ for every $y \in \mathsf{V}_a$. Since $\psi(a) \in C_0(\mathsf{V}_a)$ by Y-equivariance, it follows that $\psi(a)$ is strictly positive in $C_0(\mathsf{V}_a)$. ■

4.3. Property (UBS)

Definition 4.12. Let $\mathfrak{B}$ be an X-C^*-algebra. If $\mathfrak{C}$ is a separable, commutative, upper semicontinuous X-C^*-algebra, we will say that $\mathfrak{B}$ has *Property (UBS) with respect to* $\mathfrak{C}$ if there exists a c.p. map $\Phi\colon \mathfrak{C} \to \mathcal{M}(\mathfrak{B})$ such that $\mathfrak{B}(\mathsf{U}) = \overline{\mathfrak{B}\Phi(\mathfrak{C}(\mathsf{U}))\mathfrak{B}}$ for all $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

We will say that $\mathfrak{B}$ has *Property (UBS)* if it has Property (UBS) with respect to $\mathfrak{C}$ for some separable, commutative, upper semicontinuous X-C^*-algebra $\mathfrak{C}$.

Remark 4.13. If $\mathfrak{B}$ in the above definition is σ-unital, then we may assume that the c.p. map Φ factors through $\mathfrak{B}$. In fact, one may simply replace Φ in the above definition with $b\Phi(-)b$ for some strictly positive element $b \in \mathfrak{B}$.

The name (UBS) has been chosen, since these X-C^*-algebras resemble the upper semicontinuous C^*-bundles over a second countable, locally compact Hausdorff space, as seen in the following example.

Example 4.14. Let X be a second countable, locally compact Hausdorff space. It was shown in [27] that any upper semicontinuous C^*-bundle over X, may be considered, in a natural way, as a $C_0(\mathsf{X})$-algebra, i.e. as a C^*-algebra $\mathfrak{B}$ together with an essential

$*$-homomorphism $\Phi\colon C_0(\mathsf{X}) \to \mathcal{ZM}(\mathfrak{B})$, where $\mathcal{ZM}(\mathfrak{B})$ is the centre of the multiplier algebra. This induces an action of X on $\mathfrak{B}$ given by

$$\mathfrak{B}(\mathsf{U}) = \overline{\mathfrak{B}\Phi(C_0(\mathsf{U}))} = \overline{\mathfrak{B}\Phi(C_0(\mathsf{U}))\mathfrak{B}}.$$

Thus, $\mathfrak{B}$ with this action has Property (UBS) with respect to $C_0(\mathsf{X})$.

Example 4.15. Let X be a finite space, and $\mathfrak{B}$ be an upper semicontinuous X-C^*-algebra such that $\mathfrak{B}(\mathsf{U})$ is σ-unital for each $\mathsf{U} \in \mathbb{O}(\mathsf{X})$. Then $\mathfrak{B}$ has Property (UBS). Such X-C^*-algebras are considered in [14].

To see that $\mathfrak{B}$ has Property (UBS), let $\mathfrak{C} = \bigoplus_{x\in\mathsf{X}} \mathbb{C}$ with the action of X given by $\mathfrak{C}(\mathsf{U}) = \bigoplus_{x\in\mathsf{U}} \mathbb{C}$ for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$. This is easily seen to be an upper semicontinuous X-C^*-algebra. For $x \in \mathsf{X}$ let U^x be the smallest open subset of X containing x and let h_x be a strictly positive element in $\mathfrak{B}(\mathsf{U}^x)$. The c.p. map $\Phi\colon \mathfrak{C} \to \mathfrak{B}$, which maps 1 in the coordinate corresponding to x to h_x, satisfies the condition in Definition 4.12.

The following is the reason that we are interested in Property (UBS).

Proposition 4.16. *Let $\mathfrak{A}$ be a lower semicontinuous X-C^*-algebra and $\mathfrak{B}$ be a σ-unital X-C^*-algebra with Property (UBS). For any positive $a \in \mathfrak{A}$, there exists a nuclear, X-equivariant c.p. map $\phi\colon \mathfrak{A} \to \mathfrak{B}$ such that $\overline{\mathfrak{B}\phi(a)\mathfrak{B}} = \mathfrak{B}(\mathsf{U}_a)$.*

Proof. As $\mathfrak{B}$ has Property (UBS), we may find a separable, commutative, upper semicontinuous X-C^*-algebra $\mathfrak{C}$, and a c.p. map $\Phi\colon \mathfrak{C} \to \mathfrak{B}$ such that $\mathfrak{B}(\mathsf{U}) = \overline{\mathfrak{B}\Phi(\mathfrak{C}(\mathsf{U}))\mathfrak{B}}$ for all $\mathsf{U} \in \mathbb{O}(\mathsf{X})$. Clearly Φ is X-equivariant.

Fix $a \in \mathfrak{A}$ positive. By Proposition 4.11, there is an X-equivariant c.p. map $\psi\colon \mathfrak{A} \to \mathfrak{C}$ such that $\psi(a)$ is strictly positive in $\mathfrak{C}(\mathsf{U}_a)$. Let $\phi = \Phi \circ \psi$, which is X-equivariant as both ψ and Φ are, and nuclear since it factors through a commutative C^*-algebra. Also,

$$\overline{\mathfrak{B}\phi(a)\mathfrak{B}} = \overline{\mathfrak{B}\Phi(\mathfrak{C}(\mathsf{U}_a))\mathfrak{B}} = \mathfrak{B}(\mathsf{U}_a). \qquad \blacksquare$$

To give (many) more examples of X-C^*-algebras with Property (UBS), we will use the following lemma. Recall, that we let $\otimes$ denote the spatial tensor product.

Lemma 4.17. *Let $\mathfrak{D}$ be a separable, exact C^*-algebra. Then there exists a state η on $\mathfrak{D}$ with the following property: for any C^*-algebra $\mathfrak{B}$ and any two-sided, closed ideal $\mathfrak{J}$ in $\mathfrak{B}$, it holds for any $x \in \mathfrak{B} \otimes \mathfrak{D}$ that $x \in \mathfrak{J} \otimes \mathfrak{D}$ if and only if $(\mathrm{id} \otimes \eta)(x^*x) \in \mathfrak{J}$.*

Proof. Let (η_n) be a weak-$*$ dense sequence in the state space of $\mathfrak{D}$ and $\eta = \sum_{n=1}^\infty 2^{-n}\eta_n$. Let $\mathfrak{J}, \mathfrak{B}$ and x be given. By [3, Corollary IV.3.4.2], we have $x \in \mathfrak{J} \otimes \mathfrak{D}$ if and only if $(\mathrm{id} \otimes \eta')(x^*x) \in \mathfrak{J}$ for every state η' on $\mathfrak{D}$. Clearly it suffices to only consider the case where η' runs through all η_n since these sit densely in the state space. But since $(\mathrm{id} \otimes \eta_n)(x^*x)$ is positive for each n, and $\mathfrak{J}$ is a hereditary C^*-subalgebra of $\mathfrak{B}$, it follows that $(\mathrm{id} \otimes \eta_n)(x^*x) \in \mathfrak{J}$ for all n if and only if

$$\sum_{n=1}^\infty 2^{-n}(\mathrm{id} \otimes \eta_n)(x^*x) = (\mathrm{id} \otimes \eta)(x^*x) \in \mathfrak{J}. \qquad \blacksquare$$

Proposition 4.18. *Let $\mathfrak{B}$ be a σ-unital X-C^*-algebra, let $\mathfrak{D}$ be a separable, exact C^*-algebra, and let $\mathfrak{C}$ be a separable, commutative, upper semicontinuous X-C^*-algebra. Then $\mathfrak{B}$ has Property (UBS) with respect to $\mathfrak{C}$ if and only if $\mathfrak{B} \otimes \mathfrak{D}$ has Property (UBS) with respect to $\mathfrak{C}$.*

Proof. If $\mathfrak{B}$ has Property (UBS) with respect to $\mathfrak{C}$, and $\tilde{\Phi}: \mathfrak{C} \to \mathcal{M}(\mathfrak{B})$ is a c.p. map as in Definition 4.12, then

$$\Phi = \tilde{\Phi} \otimes 1_{\mathcal{M}(\mathfrak{D})}: \mathfrak{C} \to \mathcal{M}(\mathfrak{B}) \otimes \mathcal{M}(\mathfrak{D}) \hookrightarrow \mathcal{M}(\mathfrak{B} \otimes \mathfrak{D})$$

is a c.p. map satisfying $\overline{(\mathfrak{B} \otimes \mathfrak{D})\Phi(\mathfrak{C}(\mathsf{U}))(\mathfrak{B} \otimes \mathfrak{D})} = \mathfrak{B}(\mathsf{U}) \otimes \mathfrak{D}$.

Conversely, suppose that $\mathfrak{B} \otimes \mathfrak{D}$ has Property (UBS) with respect to $\mathfrak{C}$. Clearly $\mathfrak{B} \otimes \mathfrak{D}$ is σ-unital since it has a countable approximate identity, so we may find $\tilde{\Phi}: \mathfrak{C} \to \mathfrak{B} \otimes \mathfrak{D}$ as in Remark 4.13. Let η be a state on $\mathfrak{D}$ as given by Lemma 4.17. Define $\Phi: \mathfrak{C} \to \mathfrak{B}$ to be the composition

$$\mathfrak{C} \xrightarrow{\tilde{\Phi}} \mathfrak{B} \otimes \mathfrak{D} \xrightarrow{\mathrm{id}_{\mathfrak{B}} \otimes \eta} \mathfrak{B}.$$

Let $\mathfrak{J}_{\mathsf{U}} := \overline{\mathfrak{B}\Phi(\mathfrak{C}(\mathsf{U}))\mathfrak{B}}$. Since

$$\Phi(\mathfrak{C}(\mathsf{U})) = (\mathrm{id}_{\mathfrak{B}} \otimes \eta)(\tilde{\Phi}(\mathfrak{C}(\mathsf{U}))) \subseteq (\mathrm{id}_{\mathfrak{B}} \otimes \eta)(\mathfrak{B}(\mathsf{U}) \otimes \mathfrak{D}) = \mathfrak{B}(\mathsf{U}),$$

it follows that $\mathfrak{J}_{\mathsf{U}} \subseteq \mathfrak{B}(\mathsf{U})$. By Lemma 4.17 any element in $\tilde{\Phi}(\mathfrak{C}(\mathsf{U}))$ will be in $\mathfrak{J}_{\mathsf{U}} \otimes \mathfrak{D}$. This implies that $\mathfrak{B}(\mathsf{U}) \otimes \mathfrak{D} \subseteq \mathfrak{J}_{\mathsf{U}} \otimes \mathfrak{D}$. It follows that $\mathfrak{B}(\mathsf{U}) = \mathfrak{J}_{\mathsf{U}}$ which finishes the proof. ∎

The following proposition, which uses somewhat heavy machinery of Kirchberg and Rørdam, shows that almost all X-C^*-algebras of interest have Property (UBS).

Proposition 4.19. *Any separable, nuclear, upper semicontinuous X-C^*-algebra has Property (UBS). Moreover, we may choose that it has Property (UBS) with respect to a $\mathfrak{C}$, where the covering dimension of* $\operatorname{Prim}\mathfrak{C}$ *is at most* 1.

Although we do not need the covering dimension of $\mathfrak{C}$ to be at most 1 in this paper, the author believes that this could be important in future applications.

Proof. Let $\mathfrak{B}$ be a separable, nuclear, upper semicontinuous X-C^*-algebra. A C^*-subalgebra $\mathfrak{C} \subseteq \mathfrak{B}$ is called *regular* if $(\mathfrak{C} \cap \mathfrak{I}) + (\mathfrak{C} \cap \mathfrak{J}) = \mathfrak{C} \cap (\mathfrak{I} + \mathfrak{J})$, and if $\mathfrak{C} \cap \mathfrak{I} = \mathfrak{C} \cap \mathfrak{J}$ implies $\mathfrak{I} = \mathfrak{J}$ for all $\mathfrak{I}, \mathfrak{J} \in \mathbb{I}(\mathfrak{B})$. By [23, Theorem 6.11][2], $\mathfrak{B} \otimes \mathcal{O}_2$ contains a regular, commutative C^*-subalgebra $\mathfrak{C}$ such that $\operatorname{Prim}\mathfrak{C}$ has covering dimension at most 1. Clearly $\mathfrak{C}$ is separable since $\mathfrak{B}$ is. Equip $\mathfrak{C}$ with the action of X given by $\mathfrak{C}(\mathsf{U}) = \mathfrak{C} \cap \mathfrak{B}(\mathsf{U}) \otimes \mathcal{O}_2$ for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

Since $\mathfrak{C}$ is a regular C^*-subalgebra of $\mathfrak{B} \otimes \mathcal{O}_2$, which is upper semicontinuous by Lemma 4.9, $\mathfrak{C}$ is clearly upper semicontinuous.

[2]Note that the proof of [23, Theorem 6.11] does *not* require any of the classification results of [20] although other results in the paper do. Thus, if one's goal is to reprove the results in [20], one may still use this result.

Let $\mathfrak{I} \in \mathbb{I}(\mathfrak{B} \otimes \mathcal{O}_2)$ and let $\mathfrak{J}$ be the two-sided, closed ideal in $\mathfrak{B} \otimes \mathcal{O}_2$ generated by $\mathfrak{C} \cap \mathfrak{I}$. Then $\mathfrak{C} \cap \mathfrak{I} = \mathfrak{C} \cap \mathfrak{J}$ which implies that $\mathfrak{I} = \mathfrak{J}$. Thus,

$$\mathfrak{B}(\mathsf{U}) \otimes \mathcal{O}_2 = \overline{(\mathfrak{B} \otimes \mathcal{O}_2)\mathfrak{C}(\mathsf{U})(\mathfrak{B} \otimes \mathcal{O}_2)}$$

for all $\mathsf{U} \in \mathbb{O}(\mathsf{X})$, so $\mathfrak{B} \otimes \mathcal{O}_2$ has Property (UBS) with respect to $\mathfrak{C}$. By Proposition 4.18, $\mathfrak{B}$ has Property (UBS) with respect to $\mathfrak{C}$. ■

5. The ideal related lifting theorems

In this section, we prove X-equivariant versions of the Choi–Effros lifting theorem and of the Effros–Haagerup lifting theorem. As a consequence, we show that extensions of nuclear X-C^*-algebras have X-equivariant c.p. splittings, as long as the actions on the ideal and the quotient are sufficiently nice. Such results are closely related to ideal related KK-theory.

If $\mathfrak{B}$ is a C^*-algebra, and $\mathfrak{J}$ is a closed, two-sided ideal in $\mathfrak{B}$, then there are induced ideals in the multiplier algebra and the corona algebra, given by

$$\mathcal{M}(\mathfrak{B}, \mathfrak{J}) = \{x \in \mathcal{M}(\mathfrak{B}) : x\mathfrak{B} \subseteq \mathfrak{J}\},$$
$$\mathcal{Q}(\mathfrak{B}, \mathfrak{J}) = \pi(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$$

where $\pi \colon \mathcal{M}(\mathfrak{B}) \to \mathcal{Q}(\mathfrak{B})$ is the quotient map.

If $\mathfrak{B}$ is a stable C^*-algebra, then there exist isometries $s_1, s_2, \ldots \in \mathcal{M}(\mathfrak{B})$ such that $\sum_{k=1}^\infty s_k s_k^*$ converges strictly to $1_{\mathcal{M}(\mathfrak{B})}$. By an *infinite repeat* x_∞ of an element $x \in \mathcal{M}(\mathfrak{B})$, we mean $x_\infty = \sum_{k=1}^\infty s_k x s_k^*$, for some $s_1, s_2, \ldots$ as above. Infinite repeats are unique up to unitary equivalence. In fact, if $t_1, t_2, \ldots \in \mathcal{M}(\mathfrak{B})$ are also isometries as above, then $u = \sum_{k=1}^\infty s_k t_k^*$ is a unitary in $\mathcal{M}(\mathfrak{B})$ satisfying $u^*(\sum_{k=1}^\infty s_k x s_k^*)u = \sum_{k=1}^\infty t_k x t_k^*$.

Lemma 5.1. *Let $\mathfrak{J}$ be a σ-unital ideal in a stable C^*-algebra $\mathfrak{B}$. Then $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ contains a (norm-)full projection P.*

Proof. As $\mathfrak{J}$ is an essential ideal in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ there is an induced embedding $\iota \colon \mathcal{M}(\mathfrak{B}, \mathfrak{J}) \hookrightarrow \mathcal{M}(\mathfrak{J})$. The image of ι is easily seen to be a hereditary C^*-subalgebra of $\mathcal{M}(\mathfrak{B})$. In fact, let $x_1, x_2 \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ and $y \in \mathcal{M}(\mathfrak{J})$. We define a multiplier $z \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$ by

$$zb := x_1(y(x_2 b)), \quad bz := ((bx_1)y)x_2, \qquad b \in \mathfrak{B}.$$

Then $\iota(z) = \iota(x_1) y \iota(x_2)$, so $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ is hereditary in $\mathcal{M}(\mathfrak{J})$.

As $\mathfrak{B}$ is stable, $\mathfrak{B} \cong \mathfrak{B} \otimes \ell^2(\mathbb{N})$, as Hilbert $\mathfrak{B}$-modules. By Kasparov's stabilisation theorem [17, Theorem 2], the Hilbert $\mathfrak{B}$-module $\mathfrak{J} \oplus \mathfrak{B}$ is isomorphic to $\mathfrak{B}$. Thus, there is a projection $Q \in \mathbb{B}(\mathfrak{B}) = \mathcal{M}(\mathfrak{B})$ (corresponding to $1 \oplus 0 \in \mathbb{B}(\mathfrak{J} \oplus \mathfrak{B})$), such that $Q\mathfrak{B} \cong \mathfrak{J}$ as Hilbert $\mathfrak{B}$-modules. As $\langle Q\mathfrak{B}, Q\mathfrak{B}\rangle = \mathfrak{J}$, it follows that $Q\mathfrak{B}Q = \mathbb{K}(Q\mathfrak{B})$ is full in $\mathfrak{J}$.

Let P be an infinite repeat of Q in $\mathcal{M}(\mathfrak{B})$. Clearly $P \in \mathcal{M}(\mathfrak{B}, \mathfrak{J})$, and it is easy to see[3] that $\iota(P)$ is an infinite repeat of $\iota(Q)$ in $\mathcal{M}(\mathfrak{J})$. Thus, it follows from a result of Brown [4, Lemma 2.5], that $\iota(P)$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathfrak{J})}$. As $\iota(\mathcal{M}(\mathfrak{B}, \mathfrak{J}))$ is a hereditary C^*-subalgebra of $\mathcal{M}(\mathfrak{J})$, it follows that P is full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$. ∎

For a positive element x in a C^*-algebra, we let $(x - \varepsilon)_+ := g_\varepsilon(x)$ defined by functional calculus, where $g_\varepsilon\colon [0, \infty) \to [0, \infty)$ is given by $g_\varepsilon(t) = \max\{0, t - \varepsilon\}$.

Lemma 5.2. *Let $\mathfrak{B}$ be a separable, stable C^*-algebra, $x \in \mathcal{M}(\mathfrak{B})$ be a positive element, and let x_∞ denote an infinite repeat of x. For any $\varepsilon > 0$, let $\mathfrak{J}_\varepsilon = \overline{\mathfrak{B}(x - \varepsilon)_+\mathfrak{B}}$. Then the ideal $\overline{\mathcal{M}(\mathfrak{B})x_\infty\mathcal{M}(\mathfrak{B})}$ contains the ideal $\mathcal{M}(\mathfrak{B}, \mathfrak{J}_\varepsilon)$.*

Proof. Fix an $\varepsilon > 0$, and let $f_\varepsilon\colon [0, \infty) \to [0, \infty)$ be the continuous function

$$f_\varepsilon(t) = \begin{cases} 0, & t = 0, \\ 1, & t \geq \varepsilon, \\ \text{affine}, & 0 \leq t \leq \varepsilon. \end{cases}$$

Let $y = (x - \varepsilon)_+$ and $\mathfrak{J}_\varepsilon = \overline{\mathfrak{B}y\mathfrak{B}}$. As $\mathfrak{B}$ is separable, $\mathfrak{J}_\varepsilon$ is σ-unital, so we may fix a full projection $P \in \mathcal{M}(\mathfrak{B}, \mathfrak{J}_\varepsilon)$ by Lemma 5.1. Fix a strictly positive element $h \in \mathfrak{B}$ of norm 1. We wish to recursively construct integers $n_1 < n_2 < \cdots < n_k$ and $a_1, \ldots, a_{n_k} \in \mathfrak{B}$ such that if $z_k = \sum_{i=1}^{n_k} a_i^* y a_i$ then $\|z_k\| \leq 1$ and $\|(1 - z_k)Ph\| < 1/k$.

Suppose we have constructed such up to the stage k. As $(1 - z_k)^{1/2}Ph \in \mathfrak{J}_\varepsilon$, we may choose a positive element $e \in \mathfrak{J}_\varepsilon$ with $\|e\| < 1$, such that $\|(1 - e)(1 - z_k)^{1/2}Ph\| < 1/(k + 1)$. Let $\delta > 0$ be small enough so that $\|e\| + \delta \leq 1$ and $\|(1 - e)(1 - z_k)^{1/2}Ph\| + \delta < 1/(k + 1)$. We may find $m \in \mathbb{N}$, and $c_1, \ldots, c_m \in \mathfrak{B}$ such that $z' := \sum_{i=1}^m c_i^* y c_i \approx_\delta e$. In particular, $\|z'\| \leq 1$ and

$$\|(1 - z_k)^{1/2}(1 - z')(1 - z_k)^{1/2}Ph\| < 1/(k + 1).$$

Letting $n_{k+1} = n_k + m$ and $a_{n_k+i} = c_i(1 - z_k)^{1/2}$ does the trick.

Then $\|hP(1 - \sum_{i=1}^{n_k} a_i^* y a_i)Ph\| \to 0$ for $k \to \infty$. As $hP(1 - \sum_{i=1}^{n} a_i^* y a_i)Ph$ is monotonely decreasing and has a subsequence tending to 0, the sequence itself will tend to zero for $n \to \infty$. Thus, $\|(P - \sum_{i=1}^n b_i^* y b_i)h\| \to 0$ for $n \to \infty$ where $b_i := a_i P$. As h is strictly positive it follows that $\sum_{i=1}^\infty b_i^* y b_i$ converges strictly to P.

Let $s_1, s_2, \ldots$ be isometries in $\mathcal{M}(\mathfrak{B})$ such that $\sum_{i=1}^\infty s_i s_i^*$ converges strictly to $1_{\mathcal{M}(\mathfrak{B})}$. Then (up to unitary equivalence) $x_\infty = \sum_{i=1}^\infty s_i x s_i^*$, and $f_\varepsilon(x_\infty) = \sum_{i=1}^\infty s_i f_\varepsilon(x) s_i^*$.

Define the element $d = \sum_{i=1}^\infty s_i y^{1/2} b_i$ (strict convergence). We check that this is well defined, i.e. that $\sum_{i=1}^\infty s_i y^{1/2} b_i$ converges strictly. To see that dh converges, note that

$$\left\| \sum_{i=n}^m s_i y^{1/2} b_i h \right\| = \left\| h \sum_{i=n}^m b_i^* y b_i h \right\|^{1/2} \to 0, \quad \text{for } n, m \to \infty.$$

[3]The embedding ι extends to a strictly continuous, unital $*$-homomorphism $\iota\colon \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{J})$, and if $s_1, s_2, \ldots$ are isometries in $\mathcal{M}(\mathfrak{B})$ defining an infinite repeat, then $\iota(s_1), \iota(s_2), \ldots$ are isometries in $\mathcal{M}(\mathfrak{J})$ which induce an infinite repeat by strict continuity.

Thus, $\sum_{i=1}^{n} s_i y^{1/2} b_i h$ is a Cauchy sequence, and thus converges. As h is strictly positive, $\sum_{i=1}^{\infty} s_i y^{1/2} b_i b$ converges for every $b \in \overline{h\mathfrak{B}} = \mathfrak{B}$.

Similarly, let $h_0 := \sum_{k=1}^{\infty} 2^{-k} s_k h s_k^*$. We get that

$$\left\| h_0 \sum_{k=n}^{m} s_k y^{1/2} b_k \right\| = \left\| \sum_{k=n}^{m} 2^{-k} s_k h y^{1/2} b_i \right\| \leq \sum_{k=n}^{m} 2^{-k} \|h\| \|b_i^* y b_i\|^{1/2}$$
$$\leq \sum_{k=n}^{m} 2^{-k} \|h\| \to 0$$

for $n, m \to \infty$. Here we used that $b_i^* y b_i \leq \sum_{k=1}^{\infty} b_k^* y b_k \leq 1$. As above, $b \sum_{i=1}^{\infty} s_i y^{1/2} b_i$ converges for $b \in \overline{Bh_0}$. Thus, if h_0 is strictly positive, it will follow that d is well defined.

To see that h_0 is strictly positive, let $\delta > 0$, pick $N \in \mathbb{N}$ such that

$$\sum_{j,k=1}^{\infty} s_j s_j^* h s_k s_k^* \approx_\delta \sum_{j,k=1}^{N} s_j s_j^* h s_k s_k^*,$$

and pick $c_{j,k} \in \mathfrak{B}$ such that $c_{j,k} h \approx_{\delta/N^2} s_j^* h s_k$, which is doable as h is strictly positive. Then

$$h = \left(\sum_{j=1}^{\infty} s_j s_j^* \right) h \left(\sum_{k=1}^{\infty} s_k s_k^* \right) = \sum_{j,k=1}^{\infty} s_j s_j^* h s_k s_k^*$$
$$\approx_\delta \sum_{j,k=1}^{N} s_j s_j^* h s_k s_k^* \approx_\delta \sum_{j,k=1}^{N} s_j c_{j,k} h s_k^*$$
$$= \left(\sum_{j,k=1}^{N} 2^j s_j c_{j,k} s_k^* \right) h_0.$$

As $\delta > 0$ was arbitrary, it follows that $h \in \overline{\mathfrak{B}h_0}$ which implies that h_0 is strictly positive, since h is strictly positive. Hence d is well defined.

Then, as $f_\varepsilon(x) y^{1/2} = y^{1/2}$, we get

$$d^* f_\varepsilon(x_\infty) d = \left(\sum_{j=1}^{\infty} b_i^* y^{1/2} s_j^* \right) \left(\sum_{k=1}^{\infty} s_k f_\varepsilon(x) s_k^* \right) \left(\sum_{l=1}^{\infty} s_l y^{1/2} b_l \right)$$
$$= \sum_{i=1}^{\infty} b_i^* y^{1/2} f_\varepsilon(x) y^{1/2} b_i$$
$$= \sum_{i=1}^{\infty} b_i^* y b_i = P.$$

As P was full in $\mathcal{M}(\mathfrak{B}, \mathfrak{J}_\varepsilon)$, and as x_∞ and $f(x_\infty)$ generate the same ideal in $\mathcal{M}(\mathfrak{B})$, it follows that $\mathcal{M}(\mathfrak{B}, \mathfrak{J}_\varepsilon) \subseteq \overline{\mathcal{M}(\mathfrak{B}) x_\infty \mathcal{M}(\mathfrak{B})}$. ∎

Given any c.p. map $\phi\colon \mathfrak{A} \to \mathfrak{B}$ with $\mathfrak{A}$ separable, and $\mathfrak{B}$ σ-unital and stable, Kasparov showed in [17, Theorem 3] that there is a Stinespring-type dilation, in the sense that there is a $*$-homomorphism $\Phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ and an element $V \in \mathcal{M}(\mathfrak{B})$ such that $V^*\Phi(-)V = \phi$. The pair (Φ, V) is called the *Kasparov–Stinespring dilation* of ϕ, and the construction could be done as follows:

Construct the (right) Hilbert $\mathfrak{B}$-module $E := \mathfrak{A} \otimes_\phi \mathfrak{B}$, by defining a pre-inner product on the algebraic tensor product $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$ given on elementary tensors by $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle = b_1^*\phi(a_1^*a_2)b_2$, quotienting out length zero vectors, and completing to obtain E. Let $\widetilde{\Phi}\colon \mathfrak{A} \to \mathbb{B}(E) \subseteq \mathbb{B}(E \oplus \mathfrak{B})$ be the $*$-homomorphism given by left multiplication on the left tensors. As $\mathfrak{B}$ is stable, $\mathfrak{B} \cong \mathfrak{B} \otimes \ell^2(\mathbb{N})$ as Hilbert $\mathfrak{B}$-modules. Thus, by Kasparov's stabilisation theorem [17, Theorem 2], there is a unitary $u \in \mathbb{B}(\mathfrak{B}, E \oplus \mathfrak{B})$. We let

$$\Phi := u^*\widetilde{\Phi}(-)u\colon \mathfrak{A} \to \mathbb{B}(\mathfrak{B}) = \mathcal{M}(\mathfrak{B}).$$

If $W \in \mathbb{B}(\mathfrak{B}, E \oplus \mathfrak{B})$ is the adjointable operator $W(b) = (1 \otimes b, 0)$, and $V := u^*W$, then $V^*\Phi(-)V = \phi$.

Whenever X acts on $\mathfrak{B}$, there is an induced action on $\mathcal{M}(\mathfrak{B})$ and $\mathcal{Q}(\mathfrak{B})$ given by

$$\mathcal{M}(\mathfrak{B})(\mathsf{U}) := \mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathsf{U})), \qquad \mathcal{Q}(\mathfrak{B})(\mathsf{U}) := \mathcal{Q}(\mathfrak{B}, \mathfrak{B}(\mathsf{U})).$$

Lemma 5.3. *Let $\mathfrak{A}$ be a separable, exact X-C^*-algebra, and $\mathfrak{B}$ be a σ-unital, stable X-C^*-algebra. Suppose that $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is a nuclear, X-equivariant c.p. map, and let (Φ, V) be the Kasparov–Stinespring dilation constructed above. Then $\Phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ is nuclear and X-equivariant.*

Proof. An element $x \in \mathcal{M}(\mathfrak{B})$ is in $\mathcal{M}(\mathfrak{B}, \mathfrak{J})$ if and only if $b^*xb \in \mathfrak{J}$ for every $b \in \mathfrak{B}$. Thus, Φ is X-equivariant if and only if $b^*\Phi(-)b\colon \mathfrak{A} \to \mathfrak{B}$ is X-equivariant for every $b \in \mathfrak{B}$. Moreover, as $\mathfrak{A}$ is exact and $\mathfrak{B}$ is σ-unital, Φ is nuclear if and only if Φ is weakly nuclear, i.e. the maps $b^*\Phi(-)b$ are nuclear for every $b \in \mathfrak{B}$, by Proposition 3.2. Thus, it suffices to prove that $b^*\Phi(-)b$ is nuclear and X-equivariant for every $b \in \mathfrak{B}$. Clearly it suffices to check this latter condition only on for b in dense subset of $\mathfrak{B}$.

Note that $b^*\Phi(-)b = b^*u^*(1 \oplus 0)\widetilde{\Phi}(-)(1 \oplus 0)ub$ for any $b \in \mathfrak{B}$. As any element $(1 \oplus 0)ub \in \mathbb{K}(\mathfrak{B}, E \oplus \mathfrak{B})$ can be approximated[4] by an element of the form

$$T = \sum_{i=1}^n \theta_{(x_i,0),c_i},$$

where $x_i = \sum_{k=1}^{m_i} a_k^{(i)} \otimes b_k^{(i)} \in E$ (as such elements are dense in E) and $c_i \in \mathfrak{B}$. Thus, it suffices to check that $T^*\widetilde{\Phi}(-)T$ is nuclear and X-equivariant for such $T \in \mathbb{K}(\mathfrak{B}, E \oplus \mathfrak{B})$.

[4]We let $\theta_{x,y} \in \mathbb{K}(F, F')$ denote the "rank 1" operator $\theta_{x,y}(z) = x\langle y, z\rangle$, for $x \in F'$ and $y, z \in F$ (F and F' Hilbert modules), and recall that any element in $\mathbb{K}(F, F')$ can be approximated by sums of such "rank 1" operators.

Observe, that

$$\begin{aligned}\langle (x_i,0), \widetilde{\Phi}(-)(x_j,0)\rangle &= \sum_{k=1}^{m_i}\sum_{l=1}^{m_j} \langle a_k^{(i)} \otimes b_k^{(i)}, (-)a_l^{(j)} \otimes b_l^{(j)}\rangle_E \\ &= \sum_{k=1}^{m_i}\sum_{l=1}^{m_j} b_k^{(i)*}\phi(a_k^{(i)*}(-)a_l^{(j)})b_l^{(j)}. \end{aligned} \tag{5.1}$$

An easy computation shows that $\theta_{x,y}^* S\theta_{z,w} = \theta_{y\langle x,Sz\rangle,w}$, so

$$\begin{aligned} T^*\widetilde{\Phi}(-)T &= \sum_{i,j=1}^{n} \theta_{(x_i,0),c_i}^* \widetilde{\Phi}(-)\theta_{(x_j,0),c_j} \\ &= \sum_{i,j=1}^{n} \theta_{c_i\langle (x_i,0),\widetilde{\Phi}(-)(x_j,0)\rangle, c_j} \\ &\overset{(5.1)}{=} \sum_{i,j=1}^{n}\sum_{k=1}^{m_i}\sum_{l=1}^{m_j} \theta_{c_i b_k^{(i)*}\phi(a_k^{(i)*}(-)a_l^{(j)})b_l^{(j)}, c_j}. \end{aligned}$$

Under the canonical identification of $\mathbb{K}(\mathfrak{B})$ and $\mathfrak{B}$, the "rank 1" operator θ_{d_1,d_2} corresponds to $d_1d_2^*$. Thus, the map above, under this identification, is exactly the map

$$T^*\widetilde{\Phi}(-)T = \sum_{i,j=1}^{n}\sum_{k=1}^{m_i}\sum_{l=1}^{m_j} c_i b_k^{(i)*}\phi(a_k^{(i)*}(-)a_l^{(j)})b_l^{(j)}c_j^* \colon \mathfrak{A} \to \mathfrak{B}.$$

As ϕ is nuclear and X-equivariant, and as the set of nuclear, X-equivariant c.p. maps is a closed operator convex cone, it follows that the above map is nuclear and X-equivariant from Definition 2.1 (3). Because T was chosen arbitrarily in a dense subset of $(1 \oplus 0)\mathbb{K}(\mathfrak{B}, E \oplus \mathfrak{B})$, and as $\widetilde{\Phi} = (1 \oplus 0)\widetilde{\Phi}(-)(1 \oplus 0)$, it follows that $b^*\Phi(-)b$ is nuclear and X-equivariant for all $b \in \mathfrak{B}$. As seen above, this implies that Φ is nuclear and X-equivariant. ■

Proposition 5.4. *Let* X *be a topological space,* $\mathfrak{A}$ *be a separable, exact, lower semicontinuous* X-C^**-algebra, and let* $\mathfrak{B}$ *be a separable* X-C^**-algebra with property (UBS). Then any nuclear,* X*-equivariant c.p. map* $\eta\colon \mathfrak{A} \to \mathfrak{Q}(\mathfrak{B})$ *lifts to a nuclear,* X*-equivariant c.p. map* $\tilde{\eta}\colon \mathfrak{A} \to \mathfrak{M}(\mathfrak{B})$.

Proof. We start by proving the result under the additional assumption that $\mathfrak{B}$ is stable. Let $\mathfrak{C}$ denote the set of all nuclear, X-equivariant c.p. maps $\mathfrak{A} \to \mathfrak{M}(\mathfrak{B})$, which is a closed operator convex cone. By Proposition 2.7, it suffices to show that

$$\eta(a) \in \pi\left(\overline{\mathfrak{M}(\mathfrak{B})\{\psi(a) : \psi \in \mathfrak{C}\}\mathfrak{M}(\mathfrak{B})}\right) \tag{5.2}$$

for every positive $a \in \mathfrak{A}$. Fix $a \in \mathfrak{A}_+$. By Proposition 4.16, there are nuclear, X-equivariant c.p. maps $\phi_n\colon \mathfrak{A} \to \mathfrak{B}$ for $n \in \mathbb{N}$, such that $\mathfrak{B}(\mathsf{U}_{(a-1/n)_+}) = \overline{\mathfrak{B}\phi_n((a-1/n)_+)\mathfrak{B}}$ for every

$n \in \mathbb{N}$. We may assume that each map ϕ_n is contractive. Let $\phi := \sum_{n=1}^{\infty} 2^{-n}\phi_n$ which is nuclear and X-equivariant, and for which $\mathfrak{B}(\mathsf{U}_{(a-1/n)_+}) = \overline{\mathfrak{B}\phi((a-1/n)_+)\mathfrak{B}}$ for $n \in \mathbb{N}$.

Let (Φ, V) be the Kasparov–Stinespring dilation of ϕ. By Lemma 5.3, $\Phi\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ is a nuclear, X-equivariant $*$-homomorphism. We get $\overline{\mathfrak{B}\Phi((a-1/n)_+)\mathfrak{B}} \subseteq \mathfrak{B}(\mathsf{U}_{(a-1/n)_+})$ since Φ is X-equivariant. Also,

$$\begin{aligned}\overline{\mathfrak{B}\Phi((a-1/n)_+)\mathfrak{B}} &\subseteq \overline{\mathfrak{B}V^*\Phi((a-1/n)_+)V\mathfrak{B}}\\ &= \overline{\mathfrak{B}\phi((a-1/n)_+)\mathfrak{B}} = \mathfrak{B}(\mathsf{U}_{(a-1/n)_+}),\end{aligned}$$

so it follows that $\mathfrak{B}(\mathsf{U}_{(a-1/n)_+}) = \overline{\mathfrak{B}\Phi((a-1/n)_+)\mathfrak{B}}$ for each $n \in \mathbb{N}$.

Let $s_1, s_2, \ldots \in \mathcal{M}(\mathfrak{B})$ be isometries such that $\sum_{k=1}^{\infty} s_k s_k^*$ converges strictly to $1_{\mathcal{M}(\mathfrak{B})}$, and let $\Phi_\infty = \sum_{k=1}^{\infty} s_k \Phi(-) s_k^*$ (convergence strictly). As $\mathfrak{A}$ is exact, and

$$b^*\Phi_\infty(-)b = \sum_{k=1}^{\infty} b^* s_k \Phi(-) s_k^* b \colon \mathfrak{A} \to \mathfrak{B}$$

is nuclear and X-equivariant for all $b \in \mathfrak{B}$, it follows from Proposition 3.2 that Φ_∞ is nuclear and (clearly) X-equivariant.

Recall, that our goal is to show (5.2). It suffices to show that

$$\eta((a-1/n)_+) \in \pi\big(\overline{\mathcal{M}(\mathfrak{B})\Phi_\infty(a)\mathcal{M}(\mathfrak{B})}\big),$$

for every $n \in \mathbb{N}$. As η is X-equivariant,

$$\eta((a-1/n)_+) \in \mathcal{Q}(\mathfrak{B})(\mathsf{U}_{(a-1/n)_+}) = \pi(\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathsf{U}_{(a-1/n)_+}))).$$

So, it suffices to show that $\mathcal{M}(\mathfrak{B}, \mathfrak{B}(\mathsf{U}_{(a-1/n)_+})) \subseteq \overline{\mathcal{M}(\mathfrak{B})\Phi_\infty(a)\mathcal{M}(\mathfrak{B})}$ for every $n \in \mathbb{N}$. However, as

$$\overline{\mathfrak{B}(\Phi(a)-1/n)_+\mathfrak{B}} = \overline{\mathfrak{B}\Phi((a-1/n)_+)\mathfrak{B}} = \mathfrak{B}(\mathsf{U}_{(a-1/n)_+}),$$

this follows from Lemma 5.2, and finishes the proof, under the assumption that $\mathfrak{B}$ is stable.

If $\mathfrak{B}$ is not stable, consider the nuclear, X-equivariant map $\eta \otimes e_{11} \colon \mathfrak{A} \to \mathcal{Q}(\mathfrak{B}) \otimes \mathbb{K} \hookrightarrow \mathcal{Q}(\mathfrak{B} \otimes \mathbb{K})$. By what we proved above, this lifts to a nuclear, X-equivariant map $\tilde{\eta}' \colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$. Let $P = 1_{\mathcal{M}(\mathfrak{B})} \otimes e_{11}$. The map $\tilde{\eta} := P\tilde{\eta}'(-)P \colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B}) \otimes e_{11} = \mathcal{M}(\mathfrak{B})$ is a nuclear and X-equivariant lift of η. ∎

An extension of X-C^*-algebras is a short exact sequence $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ in the category of X-C^*-algebras such that for every open subset U of X, the sequence $0 \to \mathfrak{B}(\mathsf{U}) \to \mathfrak{E}(\mathsf{U}) \to \mathfrak{D}(\mathsf{U}) \to 0$ is a short exact sequence.

When $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ is an extension of C^*-algebras, and $\tau\colon \mathfrak{D} \to \mathcal{Q}(\mathfrak{B})$ is the Busby map, we can construct the pull-back

$$\mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D} := \big\{(x, d) \in \mathcal{M}(\mathfrak{B}) \oplus \mathfrak{D} : \pi(x) = \tau(d)\big\}.$$

It is well known that $(\sigma, p)\colon \mathfrak{E} \xrightarrow{\cong} \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}$ is an isomorphism, where $p\colon \mathfrak{E} \to \mathfrak{D}$ is the quotient map, and $\sigma\colon \mathfrak{E} \to \mathcal{M}(\mathfrak{B})$ is the canonical $*$-homomorphism.

It was shown in [14, Proposition 5.20], that $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ is an extension of X-C^*-algebras if and only if τ is X-equivariant and $(\sigma, p)\colon \mathfrak{E} \xrightarrow{\cong} \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}$ is an isomorphism of X-C^*-algebras (i.e. the map *and its inverse* are X-equivariant). Here we equipped the pull-back with the action

$$(\mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D})(\mathsf{U}) := (\mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}) \cap (\mathcal{M}(\mathfrak{B})(\mathsf{U}) \oplus \mathfrak{D}(\mathsf{U})), \quad \mathsf{U} \in \mathbb{O}(\mathsf{X}),$$

which is well defined whenever τ is X-equivariant. We fill in the proof for completion.

Proposition 5.5. *Let $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ be a short exact sequence in the category of X-C^*-algebras. The sequence is an extension of X-C^*-algebras if and only if the Busby map τ is X-equivariant and the canonical isomorphism $\mathfrak{E} \to \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}$ is an isomorphism of X-C^*-algebras.*

Proof. If the sequence is an extension of X-C^*-algebras, then $\mathfrak{B}(\mathsf{U}) = \mathfrak{B} \cdot \mathfrak{E}(\mathsf{U})$, and $\mathfrak{D}(\mathsf{U}) = p(\mathfrak{E}(\mathsf{U}))$, where $p\colon \mathfrak{E} \to \mathfrak{D}$ is the quotient map. It follows that $\sigma\colon \mathfrak{E} \to \mathcal{M}(\mathfrak{B})$ is X-equivariant and thus τ is also X-equivariant. We have

$$\begin{aligned}(\mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D})(\mathsf{U}) &= \mathcal{M}(\mathfrak{B})(\mathsf{U}) \oplus_{\mathcal{Q}(\mathfrak{B})(\mathsf{U})} \mathfrak{D}(\mathsf{U}) \\ &\overset{(*)}{=} \mathcal{M}(\mathfrak{B}(\mathsf{U})) \oplus_{\mathcal{Q}(\mathfrak{B}(\mathsf{U}))} \mathfrak{D}(\mathsf{U}),\end{aligned} \tag{5.3}$$

for $\mathsf{U} \in \mathbb{O}(\mathsf{X})$, where $(*)$ is easily verified, e.g. by uniqueness of pull-backs, and is left for the reader. It follows that the isomorphism $\mathfrak{E} \xrightarrow{\cong} \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}$ restricts to an isomorphism $\mathfrak{E}(\mathsf{U}) \xrightarrow{\cong} (\mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D})(\mathsf{U})$ for every U. Thus, it is an isomorphism of X-C^*-algebras.

Conversely, suppose τ is X-equivariant, and $\mathfrak{E} \to \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{D}$ is an isomorphism of X-C^*-algebras. As (5.3) holds, it follows that $0 \to \mathfrak{B}(\mathsf{U}) \to \mathfrak{E}(\mathsf{U}) \to \mathfrak{D}(\mathsf{U}) \to 0$ is exact, so $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ is an extension of X-C^*-algebras. ∎

We can now prove an ideal related lifting theorem. Part (i) in the theorem is an X-equivariant version of the Choi–Effros lifting theorem [8], and part (ii) is an X-equivariant version of the Effros–Haagerup lifting theorem [12].

Theorem 5.6. *Let X be a topological space, let $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{D} \to 0$ be an extension of X-C^*-algebras, for which $\mathfrak{B}$ is separable and has Property (UBS) (in particular, $\mathfrak{B}$ could be upper semicontinuous and nuclear), and let $\mathfrak{A}$ be a separable, exact, lower semicontinuous X-C^*-algebra. Let $\phi\colon \mathfrak{A} \to \mathfrak{D}$ be an X-equivariant c.p. map. There exists an X-equivariant c.p. lift $\tilde{\phi}\colon \mathfrak{A} \to \mathfrak{E}$, if one of the following hold:*

(i) *ϕ is nuclear,*

(ii) *$\mathfrak{E}$ is exact and $\mathfrak{B}$ is nuclear.*

Proof. Let τ denote the Busby map of our given extension. By Proposition 5.5, τ is X-equivariant and $\mathfrak{E} \xrightarrow{\cong} \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{A}$ is an isomorphism of X-C^*-algebras.

Suppose that $\tau \circ \phi$ is nuclear. As $\tau \circ \phi$ is X-equivariant, we may lift $\tau \circ \phi$ to an X-equivariant c.p. map $\psi \colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$, by Proposition 5.4. The c.p. map

$$\tilde{\phi} = (\psi, \phi) \colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B}) \oplus_{\mathcal{Q}(\mathfrak{B})} \mathfrak{A} \cong \mathfrak{E}$$

is an X-equivariant lift of ϕ. So it suffices to show that $\tau \circ \phi$ is nuclear if either (i) or (ii) holds. If ϕ is nuclear (i.e. (i) holds), then $\tau \circ \phi$ is nuclear, as compositions of a nuclear c.p. map with any c.p. map is nuclear. If $\mathfrak{E}$ is exact and $\mathfrak{B}$ is nuclear (i.e. (ii) holds), then $\mathfrak{D}$ is exact, as quotients of exact C^*-algebras are exact [19]. Thus, by Corollary 3.3, τ is nuclear and thus $\tau \circ \phi$ is nuclear.

The "in particular" part follows from Proposition 4.19. ∎

A consequence of Theorem 5.6 is the following result, which says that in most cases of interest, an extension of X-C^*-algebra will be *semisplit*, i.e. it will have an X-equivariant c.p. splitting, as long as the quotient is lower semicontinuous.

Theorem 5.7. *Let X be a topological space, and $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be an extension of separable, nuclear X-C^*-algebras. Suppose that $\mathfrak{B}$ is upper semicontinuous and $\mathfrak{A}$ is lower semicontinuous. Then there is an X-equivariant c.p. splitting $\mathfrak{A} \to \mathfrak{E}$.*

Proof. Apply Theorem 5.6, with $\mathfrak{A} = \mathfrak{D}$ and $\phi = \mathrm{id}_{\mathfrak{A}}$, to find the X-equivariant c.p. splitting. ∎

Remark 5.8. It is well known that the above theorem fails if we remove the lower semicontinuity assumption of $\mathfrak{A}$. E.g., the extension $0 \to C_0((0,1]) \to C([0,1]) \to \mathbb{C} \to 0$ of $[0,1]$-C^*-algebras (with the obvious actions) can never have an $[0,1]$-equivariant c.p. splitting (or even $[0,1]$-equivariant non-c.p. splitting), as the only $[0,1]$-equivariant map $\mathbb{C} \to C([0,1])$ is the zero map.

6. Comparing ideal related KK-theory and E-theory

Recall, that a *C^*-algebra over* X, is an X-C^*-algebra for which the action is finitely lower semicontinuous and upper semicontinuous. In [10], Dadarlat and Meyer construct E-theory for separable C^*-algebras over X when X is second countable. We will sketch the construction.

An *asymptotic morphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a map $\phi \colon \mathfrak{A} \to C_b([0,\infty), \mathfrak{B})$, such that the composition of this map with the quotient map onto $\mathfrak{B}_\infty := C_b([0,\infty), \mathfrak{B}) / C_0([0,\infty), \mathfrak{B})$, call this composition $\dot{\phi}$, is a $*$-homomorphism. If ϕ and ϕ' are asymptotic morphisms, we say that they are *equivalent* if $\dot{\phi} = \dot{\phi}'$. If $\mathfrak{A}$ and $\mathfrak{B}$ are X-C^*-algebras, then $\mathfrak{B}_\infty$ is an X-C^*-algebra by

$$\mathfrak{B}_\infty(\mathsf{U}) = \frac{C_b([0,\infty), \mathfrak{B}(\mathsf{U})) + C_0([0,\infty), \mathfrak{B})}{C_b([0,\infty), \mathfrak{B})}.$$

We say that an asymptotic morphism ϕ is *approximately* X*-equivariant* if the induced $*$-homomorphism $\dot{\phi}$ is X-equivariant. Note that this does not imply that the asymptotic morphism is X-equivariant. We say that two approximately X-equivariant asymptotic morphisms ϕ_0, ϕ_1 from $\mathfrak{A}$ to $\mathfrak{B}$ are *homotopic* if there is an approximately X-equivariant asymptotic morphism Φ from $\mathfrak{A}$ to $C([0,1], \mathfrak{B})$ such that $\mathrm{ev}_i \circ \Phi = \phi_i$ for $i = 0, 1$. We let $[\![\mathfrak{A}, \mathfrak{B}]\!]_{\mathsf{X}}$ denote the set of homotopy classes of approximately X-equivariant asymptotic morphisms.

For separable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over X, where X is second countable, we define

$$E(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) := [\![C_0(\mathbb{R}) \otimes \mathfrak{A} \otimes \mathbb{K}, C_0(\mathbb{R}) \otimes \mathfrak{B} \otimes \mathbb{K}]\!]_{\mathsf{X}}.$$

This comes equipped with an abelian group structure, as well as a bilinear composition product. Thus, $E(\mathsf{X}; -, -)$ is a bivariant functor from the category of separable C^*-algebras over X to the category of abelian groups.

Similarly, consider asymptotic morphisms $\phi\colon \mathfrak{A} \to C_b([0,\infty), \mathfrak{B})$, such that ϕ is an X-equivariant contractive c.p. map. Note that these are actually X-equivariant and not just approximately X-equivariant. By again taking homotopies only of this form we may construct the set $[\![\mathfrak{A}, \mathfrak{B}]\!]_{\mathsf{X}}^{\mathrm{cp}}$ of homotopy classes of such asymptotic morphisms.

By [10, Theorem 5.2], when X is second countable, and $\mathfrak{A}$ and $\mathfrak{B}$ are separable C^*-algebras over X, there is a natural isomorphism

$$KK(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cong [\![C_0(\mathbb{R}) \otimes \mathfrak{A} \otimes \mathbb{K}, C_0(\mathbb{R}) \otimes \mathfrak{B} \otimes \mathbb{K}]\!]_{\mathsf{X}}^{\mathrm{cp}}.$$

Remark 6.1. Although C^*-algebras over X are commonly thought of as the "correct" generalisation of C^*-algebras when one wants to incorporate ideal structure, there are given examples in [14] of why it is not always convenient only to consider C^*-algebras over X instead of more general X-C^*-algebras. $E(\mathsf{X})$-theory can easily be generalised to monotone upper semicontinuous X-C^*-algebras (which will be important in future work by the author), however, for simplicity we will mainly work with C^*-algebras over X in this section.

Recall, that an X-C^*-algebra (or a C^*-algebra over X) is called *continuous* if it is lower and upper semicontinuous.

Theorem 6.2. *Let X be second countable, and let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, nuclear C^*-algebras over X. If $\mathfrak{A}$ is continuous, then $E(\mathsf{X}; \mathfrak{A}, \mathfrak{B}) \cong KK(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ naturally.*

Proof. Let ϕ be an approximately X-equivariant asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$, and

$$\dot{\phi}\colon \mathfrak{A} \to \mathfrak{B}_\infty := \frac{C_b([0,\infty), \mathfrak{B})}{C_0([0,\infty), \mathfrak{B})}$$

be the induced X-equivariant $*$-homomorphism. Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0([0,\infty), \mathfrak{B}) & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{A} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \dot{\phi}} & & \\ 0 & \longrightarrow & C_0([0,\infty), \mathfrak{B}) & \longrightarrow & C_b([0,\infty), \mathfrak{B}) & \longrightarrow & \mathfrak{B}_\infty & \longrightarrow & 0, \end{array}$$

and observe that the top row is an extension of X-C^*-algebras. By Lemma 4.9, we have that $C_0([0,\infty),\mathfrak{B})$ is a C^*-algebra over X. Thus, by Theorem 5.7, there is an X-equivariant contractive c.p. split $\psi\colon \mathfrak{A} \to \mathfrak{E}$. It follows that $\sigma \circ \psi$ is an X-equivariant, contractive c.p. asymptotic morphism which is equivalent to ϕ. By replacing $\mathfrak{B}$ with $C([0,1],\mathfrak{B})$, it follows that any approximately X-equivariant asymptotic homotopy may be replaced by an X-equivariant, contractive c.p. asymptotic homotopy. Thus,

$$[\![\mathfrak{A},\mathfrak{B}]\!]_{\mathsf{X}} = [\![\mathfrak{A},\mathfrak{B}]\!]_{\mathsf{X}}^{\mathrm{cp}}.$$

Let $\mathfrak{A}_0 = C_0(\mathbb{R},\mathfrak{A}) \otimes \mathbb{K}$ and $\mathfrak{B}_0 = C_0(\mathbb{R},\mathfrak{B}) \otimes \mathbb{K}$, which are nuclear C^*-algebras over X. By what we proved above and by [10, Theorem 5.2] it follows that

$$KK(\mathsf{X};\mathfrak{A},\mathfrak{B}) \cong [\![\mathfrak{A}_0,\mathfrak{B}_0]\!]_{\mathsf{X}}^{\mathrm{cp}} = [\![\mathfrak{A}_0,\mathfrak{B}_0]\!]_{\mathsf{X}} = E(\mathsf{X};\mathfrak{A},\mathfrak{B}),$$

where the isomorphism is natural. ■

Remark 6.3. The proof above can easily be modified so that we only require that $\mathfrak{B}$ has Property (UBS) instead of being nuclear. Thus, if X is finite or if it is locally compact and Hausdorff, we do not need nuclearity of $\mathfrak{B}$ in Theorem 6.2.

7. Absorption of strongly self-absorbing C^*-algebras

In this section, we give a few easy applications of Theorem 6.2. Using this result we can weaken the deep classification result of Kirchberg [20]. A proof of this theorem can be found in [13] by the author.

A C^*-algebra is called *strongly purely infinite*, if it has a certain comparability property defined by Kirchberg and Rørdam in [22, Definition 5.1]. As we do not need the exact definition in this paper, we simply mention that a separable, nuclear C^*-algebra $\mathfrak{A}$ is strongly purely infinite if and only if $\mathfrak{A} \otimes \mathcal{O}_\infty \cong \mathfrak{A}$, by [22, Theorem 8.6] and [32, Corollary 3.2].

Recall that an action $\mathbb{O}(\mathsf{X}) \to \mathbb{I}(\mathfrak{A})$ is *tight* if it is a lattice isomorphism.

Theorem 7.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, nuclear, stable, strongly purely infinite, tight X-C^*-algebras. Then any invertible element in $E(\mathsf{X};\mathfrak{A},\mathfrak{B})$ lifts to an X-equivariant $*$-isomorphism $\mathfrak{A} \to \mathfrak{B}$.*

Proof. By Theorem 6.2, $E(\mathsf{X};\mathfrak{A},\mathfrak{B}) \cong KK(\mathsf{X};\mathfrak{A},\mathfrak{B})$ and $E(\mathsf{X};\mathfrak{B},\mathfrak{A}) \cong KK(\mathsf{X};\mathfrak{B},\mathfrak{A})$ naturally. Thus, any invertible element in $E(\mathsf{X};\mathfrak{A},\mathfrak{B})$ lifts to an invertible element in $KK(\mathsf{X};\mathfrak{A},\mathfrak{B})$, which in turn lifts to an X-equivariant $*$-isomorphism $\mathfrak{A} \to \mathfrak{B}$ by a very deep theorem of Kirchberg [20] (alternatively, see [13, Theorem G]). ■

It turns out (cf. [10, Theorem 4.6]) that ideal related E-theory is (a priori) much more well-behaved with respect to K-theory than ideal related KK-theory, and thus it should be easier to apply the above theorem for K-theoretic classification than the original result of Kirchberg.

As is customary, we say that a separable C^*-algebra *satisfies the UCT*, if it satisfies the universal coefficient theorem of Rosenberg and Schochet [29]. This is equivalent to the C^*-algebra being KK-equivalent to a commutative C^*-algebra.

For any $\alpha \in E(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ there is an induced element $\alpha_{\mathsf{U}} \in E(\mathfrak{A}(\mathsf{U}), \mathfrak{B}(\mathsf{U}))$. In particular, this also induces a homomorphism in K-theory $K_*(\alpha_{\mathsf{U}})\colon K_*(\mathfrak{A}(\mathsf{U})) \to K_*(\mathfrak{B}(\mathsf{U}))$. The following result of Dadarlat and Meyer gives a very effective way of determining when an $E(\mathsf{X})$-element is invertible as a "point-wise" condition.

Theorem 7.2 ([10, Theorems 3.10 and 4.6]). *Let X be a second countable space, and let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras over X. An element $\alpha \in E(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ is invertible if and only if the induced element $\alpha_{\mathsf{U}} \in E(\mathfrak{A}(\mathsf{U}), \mathfrak{B}(\mathsf{U}))$ is invertible for each $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.*

In particular, if $\mathfrak{A}(\mathsf{U})$ and $\mathfrak{B}(\mathsf{U})$ satisfy the UCT of Rosenberg and Schochet for each $\mathsf{U} \in \mathbb{O}(\mathsf{X})$, then $\alpha \in E(\mathsf{X}; \mathfrak{A}, \mathfrak{B})$ is invertible if and only if $K_*(\alpha_{\mathsf{U}})\colon K_*(\mathfrak{A}(\mathsf{U})) \to K_*(\mathfrak{B}(\mathsf{U}))$ is an isomorphism for each $\mathsf{U} \in \mathbb{O}(\mathsf{X})$.

Definition 7.3 (Toms–Winter [32]). A separable, unital C^*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if $\mathcal{D} \not\cong \mathbb{C}$ and if there exists an isomorphism $\phi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the $*$-homomorphism $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$.

The following are all known examples of strongly self-absorbing C^*-algebras: the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, all UHF algebras of infinite type, the Jiang–Su algebra $\mathcal{Z}$, and any UHF algebra of infinite type tensor $\mathcal{O}_\infty$. Any strongly self-absorbing C^*-algebra that satisfies the UCT of Rosenberg and Schochet is one of the above. For more information, see [35] for a good overview.

Proposition 7.4. *Let $\mathfrak{A}$ be a separable, nuclear, strongly purely infinite C^*-algebra, and let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. Then $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{D}$ if and only if $\mathfrak{I}$ and $\mathfrak{I} \otimes \mathcal{D}$ are KK-equivalent for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.*

Proof. If $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{D}$, then $\mathfrak{I} \cong \mathfrak{I} \otimes \mathcal{D}$ for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$. In particular, $\mathfrak{I}$ and $\mathfrak{I} \otimes \mathcal{D}$ are KK-equivalent.

Suppose that $\mathfrak{I}$ and $\mathfrak{I} \otimes \mathcal{D}$ are KK-equivalent, and let $\alpha \in KK(\mathfrak{I}, \mathfrak{I} \otimes \mathcal{D})$ be invertible. The Kasparov product (composition)

$$\mathfrak{I} \xrightarrow{\alpha} \mathfrak{I} \otimes \mathcal{D} \xrightarrow{\mathrm{id}_{\mathfrak{I}} \otimes \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}} \mathfrak{I} \otimes \mathcal{D} \otimes \mathcal{D} \xrightarrow{\alpha^{-1} \otimes \mathrm{id}_{\mathcal{D}}} \mathfrak{I} \otimes \mathcal{D}$$

is exactly $\mathrm{id}_{\mathfrak{I}} \otimes 1_{\mathcal{D}}\colon \mathfrak{I} \to \mathfrak{I} \otimes \mathcal{D}$. Clearly α and $\alpha^{-1} \otimes \mathrm{id}_{\mathcal{D}}$ are invertible. By [11, Theorem 2.2], $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is *asymptotically* unitarily equivalent to an isomorphism ϕ (as any strongly self-absorbing C^*-algebra is K_1-injective by [34, Remark 3.3]). Thus, $\mathrm{id}_{\mathfrak{A}} \otimes \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$ is invertible in KK-theory, and hence $\mathrm{id}_{\mathfrak{I}} \otimes 1_{\mathcal{D}}\colon \mathfrak{I} \to \mathfrak{I} \otimes \mathcal{D}$ induces a KK-equivalence, as it is a composition of KK-equivalences.

Let $\mathsf{X} = \operatorname{Prim} \mathfrak{A}$. Equip $\mathfrak{A} \otimes \mathcal{D}$ with the action $\mathbb{O}(\mathsf{X}) \to \mathbb{I}(\mathfrak{A} \otimes \mathcal{D})$ given by $(\mathfrak{A} \otimes \mathcal{D})(\mathsf{U}) = \mathfrak{A}(\mathsf{U}) \otimes \mathcal{D}$. By [32, Theorem 1.6], $\mathcal{D}$ is simple and nuclear, and thus $\mathfrak{A} \otimes \mathcal{D}$ is a separable, nuclear, strongly purely infinite, tight X-C^*-algebra. Note that $\mathrm{id}_{\mathfrak{A}} \otimes 1_{\mathcal{D}} \colon \mathfrak{A} \to \mathfrak{A} \otimes \mathcal{D}$ is X-equivariant, and thus induces an $E(\mathsf{X})$-element $\alpha \in E(\mathsf{X}; \mathfrak{A}, \mathfrak{A} \otimes \mathcal{D})$. As

$$\mathrm{id}_{\mathfrak{A}(\mathsf{U})} \otimes 1_{\mathcal{D}} \colon \mathfrak{A}(\mathsf{U}) \to (\mathfrak{A} \otimes \mathbb{K})(\mathsf{U}) \otimes \mathcal{D}$$

induces an invertible KK-element, and thus also an invertible E-element, which is α_{U}, for every $\mathsf{U} \in \mathbb{O}(\mathsf{X})$, it follows from Theorem 7.2 that α is invertible. Using Theorem 7.1 and the fact that ideal-related E-theory is stable, we obtain an isomorphism $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{A} \otimes \mathcal{D} \otimes \mathbb{K}$. By [32, Corollary 3.2], $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{D}$. ∎

Definition 7.5. We say that an abelian group G is *uniquely n-divisible* for an integer $n \geq 2$, if for every $g \in G$ there is a unique element $h \in G$ such that $n \cdot h = g$.

We say that G is *uniquely divisible* if it is uniquely n-divisible for every $n \geq 2$.

Note that an abelian group G is uniquely n-divisible if and only if $G \cong G \otimes \mathbb{Z}[\frac{1}{n}]$. For any $n \geq 2$ we let $M_{n^\infty} = M_n \otimes M_n \otimes \cdots$ denote the UHF algebra of type n^∞. We let $\mathcal{Q} = \bigotimes_{k \in \mathbb{N}} M_k$ denote the universal UHF algebra.

Theorem 7.6. *Let $\mathfrak{A}$ be a separable, nuclear, strongly purely infinite C^*-algebra, for which every two-sided, closed ideal satisfies the UCT. For $n \geq 2$, it holds that $\mathfrak{A} \cong \mathfrak{A} \otimes M_{n^\infty}$ if and only if $K_*(\mathfrak{I})$ is uniquely n-divisible for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.*

In particular, $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Q}$ if and only if $K_*(\mathfrak{I})$ is uniquely divisible for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$.

Proof. If $\mathfrak{A} \cong \mathfrak{A} \otimes M_{n^\infty}$ then $\mathfrak{I} \cong \mathfrak{I} \otimes M_{n^\infty}$ for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$, and thus

$$K_i(\mathfrak{I}) \cong K_i(\mathfrak{I} \otimes M_{n^\infty}) \cong K_i(\mathfrak{I}) \otimes \mathbb{Z}[\tfrac{1}{n}]$$

by the Künneth theorem [30] for $i = 0, 1$. Hence $K_*(\mathfrak{I})$ is uniquely n-divisible.

Conversely, suppose that $K_*(\mathfrak{I})$ is uniquely n-divisible for every two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$. Then, as above, $K_i(\mathfrak{I}) \cong K_i(\mathfrak{I}) \otimes \mathbb{Z}[\frac{1}{n}] \cong K_i(\mathfrak{I} \otimes M_{n^\infty})$ for $i = 0, 1$. As $\mathfrak{I}$ and $\mathfrak{I} \otimes M_{n^\infty}$ satisfy the UCT, it follows that $\mathfrak{I}$ and $\mathfrak{I} \otimes M_{n^\infty}$ are KK-equivalent. Thus, $\mathfrak{A} \cong \mathfrak{A} \otimes M_{n^\infty}$ by Proposition 7.4.

The "in particular" part follows since $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Q}$ if and only if $\mathfrak{A} \cong \mathfrak{A} \otimes M_{n^\infty}$ for every $n \geq 2$. ∎

Recall, that a separable C^*-algebra is *KK-contractible* if it is KK-equivalent to 0. Note that $\mathfrak{A}$ is KK-contractible if and only if it satisfies the UCT and $K_*(\mathfrak{A}) = 0$.

Theorem 7.7. *Let $\mathfrak{A}$ be a separable, nuclear, strongly purely infinite C^*-algebra. Then $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_2$ if and only if every two-sided, closed ideal in $\mathfrak{A}$ is KK-contractible.*

Proof. If $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_2$ then $\mathfrak{I} \cong \mathfrak{I} \otimes \mathcal{O}_2$ for every closed, two-sided ideal $\mathfrak{I}$ in $\mathfrak{A}$. It follows that $\mathfrak{I}$ is KK-contractible.

Conversely, if $\mathfrak{I}$ is KK-contractible for each two-sided, closed ideal $\mathfrak{I}$ in $\mathfrak{A}$, then $\mathrm{id}_{\mathfrak{I}} \otimes 1_{\mathcal{O}_2} \colon \mathfrak{I} \to \mathfrak{I} \otimes \mathcal{O}_2$ induces a KK-equivalence. Hence $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_2$ by Proposition 7.4. ∎

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