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Universal deformation formula, formality and actions

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Abstract. In this paper we provide a quantization via formality of Poisson actions of a triangular Lie algebra $(\mathfrak{g}, r)$ on a smooth manifold M. Using the formality of E-polydifferential operators for Lie algebroids E over M, we obtain a deformation quantization of M together with a quantum group $\mathcal{U}_{\rho_\hbar}(\mathfrak{g})$ and a map of associated DGLA's. This motivates a definition of quantum action in terms of L_∞-morphisms which generalizes the well-known definition given by Drinfeld.

1. Introduction

The concept of deformation quantization has been introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in their seminal paper [4] based on the theory of associative deformations of algebras [22]. A formal star product on a Poisson manifold M is defined as a formal associative deformation of the algebra of smooth functions $\mathcal{C}^\infty(M)$ on M (the name comes from the notation $\star$ for the deformed product) and its existence has been proved as a corollary of the so-called *formality theorem* in [26] (for more details on deformation quantization we refer to the textbooks [17, 36]). On the other hand, Drinfeld introduced the notion of quantum groups in the setting of formal deformations, see e.g. the textbooks [8, 20] for a detailed discussion. Drinfeld also introduced the idea of using symmetries to get formal deformations. More explicitly, given an action by derivations of a Lie algebra $\mathfrak{g}$ on an associative algebra $(\mathcal{A}, m_{\mathcal{A}})$, the definition of the so-called *formal Drinfeld twist* $J \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[\![\hbar]\!]$ (see [12, 14]) allows us to obtain an associative formal deformation of $\mathcal{A}$ by means of a *universal deformation formula*

$$a \star_J b = m_{\mathcal{A}}(J \triangleright (a \otimes b)) \tag{1.1}$$

for $a, b \in \mathcal{A}[\![\hbar]\!]$. Here $\triangleright$ denotes the obvious extension of the action of $\mathfrak{g}$ on $\mathcal{A}$ to an action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{A}$. The deformed algebra $(\mathcal{A}[\![\hbar]\!], \star_J)$ is then a module-algebra for the quantum group

$$\mathcal{U}_J(\mathfrak{g}) := \big(\mathcal{U}(\mathfrak{g})[\![\hbar]\!], \Delta_J := J \Delta J^{-1}\big). \tag{1.2}$$

In other words, Drinfeld obtains a quantized action. Let us mention here that the relevance of deformations induced via symmetries has been deeply investigated in [23] and in a non-formal setting in [5].

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The aim of this paper is to obtain a more general notion of deformation through symmetry by using formality theory. We focus on the quantization of Lie algebra actions in the particular case of triangular Lie algebras. Such actions can be regarded as the infinitesimal version of Poisson Lie group actions, see e.g. [27, 33], and they are very important in the context of integrable systems. Triangular Lie algebras and their quantizations have been studied by many authors, see e.g. [7, 16, 37]. The idea of applying formality to actions has also been used in [2, 34], where the authors use Kontsevich's formality of polydifferential operators on a Poisson manifold to construct a derivation of the star product for each Poisson vector field. We recover this result.

The formality theorem states the existence of an L_∞-quasi-isomorphism from polyvectorfields to polydifferential operators on a manifold M. In [10, 11] Dolgushev proves the theorem for general M using the proof for $M = \mathbb{R}^n$. In order to construct such L_∞-quasi-isomorphisms, Dolgushev uses Fedosov's methods [21] concerning formal geometry, Kontsevich's quasi-isomorphism [26] and the twisting procedure inspired by Quillen [32]. Following the construction provided by Dolgushev, Calaque proved a formality theorem for Lie algebroids [6]. We consider an infinitesimal action of $\mathfrak{g}$ on M, i.e. a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \Gamma^\infty(TM)$. This can immediately be extended to a DGLA morphism

$$ {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}} \longrightarrow \mathrm{T}_{\mathrm{poly}}(M), \tag{1.3}$$

where ${}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}} = \wedge^\bullet \mathfrak{g}$ and $\mathrm{T}_{\mathrm{poly}}(M) = \Gamma^\infty(\wedge^\bullet TM)$ with the brackets extended via a Leibniz rule. Similarly, we can consider the DGLA ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$ of polydifferential operators on M and of the $\mathfrak{g}$-polydifferential operators on the point, ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$. From the formality theorem we know that we have the following L_∞-quasi-isomorphisms:

$$ {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}} \longrightarrow {}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}} \quad \text{and} \quad \mathrm{T}_{\mathrm{poly}}(M) \longrightarrow \mathrm{D}_{\mathrm{poly}}(M). \tag{1.4}$$

Using the quasi-invertibility of L_∞-quasi-isomorphisms we obtain the existence of an L_∞-morphism

$$ {}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}} \longrightarrow \mathrm{D}_{\mathrm{poly}}(M). \tag{1.5}$$

If the Lie algebra $\mathfrak{g}$ is endowed with an r-matrix, i.e. an element $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfying the Maurer–Cartan equation $[\![r, r]\!] = 0$, the action always induces a Poisson structure on M and it is automatically a Poisson action.

Lemma 1.1. (i) *Given the formal Maurer–Cartan element $\hbar r \in {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}}[\![\hbar]\!]$, we obtain via formality a Maurer–Cartan element $\rho_\hbar$. This yields a quantum group $\mathcal{U}_{\rho_\hbar}(\mathfrak{g})$ with deformed coproduct $\Delta_{1\otimes 1+\rho_\hbar}$.*

(ii) *Given the Maurer–Cartan element $\hbar\pi = \varphi \wedge \varphi(\hbar r) \in \mathrm{T}_{\mathrm{poly}}(M)[\![\hbar]\!]$, we obtain via formality a Maurer–Cartan element $B_\hbar$. This induces a formal deformation $(\mathcal{C}^\infty(M)[\![\hbar]\!], \star_{B_\hbar})$ of the Poisson algebra $(\mathcal{C}^\infty(M), \pi)$.*

The DGLA obtained from twisting the DGLA of $\mathfrak{g}$-polydifferential operators on the point, ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$ as given in [6], turns out to be a special case of a DGLA canonically associated to any Hopf algebra H, which we call H_{poly}. Given any Maurer–Cartan element

$F \in H_{\text{poly}}$ there is the associated Drinfeld twist $J = 1 \otimes 1 + F$ (in the formal sense). It turns out that the twisted DGLA H^F_{poly} is canonically isomorphic to $(H_J)_{\text{poly}}$ where H_J denotes the Hopf algebra twisted by J. Thus, using the twisting procedure on the L_∞-morphism (1.5) we prove the following theorem.

Theorem 1.2. *Let $\mathfrak{g}$ be a Lie algebra endowed with a classical r-matrix and a Lie algebra action $\varphi: \mathfrak{g} \to \Gamma^\infty(TM)$ inducing a Poisson structure on M by $\pi := \varphi \wedge \varphi(r)$. Then, there exists an L_∞-morphism $(\mathcal{U}_{\rho_\hbar}(\mathfrak{g})[[\hbar]])_{\text{poly}} \to C(\mathcal{A}_\hbar; \mathcal{A}_\hbar)$ between the DGLA associated to the quantum group $\mathcal{U}_{\rho_\hbar}(\mathfrak{g})[[\hbar]]$ and the Hochschild complex of the deformation quantization $\mathcal{A}_\hbar$ of $\mathcal{C}^\infty(M)$.*

This theorem motivates a definition, which generalizes Drinfeld's quantized action.

Definition 1.3 (Deformation symmetry). A deformation symmetry of a Hopf algebra H in a unital associative algebra $\mathcal{A}$ is a map of L_∞-algebras

$$\Phi: H_{\text{poly}} \longrightarrow C(\mathcal{A}). \tag{1.6}$$

Comparing the quantized structures obtained with our approach, it is easy to see that we recover Drinfeld's universal deformation formulas.

The paper is organized as follows. In Section 2 we recall the language of L_∞-algebras and the theorem, due to Kontsevich, stating the existence of an L_∞-quasi-isomorphism between polyvector fields and polydifferential operators on the formal completion at $0 \in \mathbb{R}^d$. In Section 3 we briefly discuss the proof of formality for Lie algebroids, following [7, 10, 11]. In particular, we recall the twisting procedure in the curved context. Section 4 contains the main results of the paper, i.e. the construction of an L_∞-morphism out of a Poisson action and the discussion on twisted structures and deformation symmetry. Finally, we compare our approach with Drinfeld's deformation formulas.

2. Preliminaries

Given a graded vector space $V^\bullet$ over $\mathbb{K}$ we denote the k-*shifted* vector space by $V^\bullet[k]$, it is given by

$$V^\bullet[k]^l = V^{l+k}. \tag{2.1}$$

2.1. L_∞-setting

We shall recall the definitions of L_∞-algebra and L_∞-morphism for the convenience of the reader (and to fix certain conventions). For the rest of this section we consider a field $\mathbb{K}$ of characteristic 0. Although many constructions will also allow for replacement of $\mathbb{K}$ by a PID containing the rationals.

Definition 2.1 (L_∞-algebra). A degree $+1$ coderivation Q on the co-unital conilpotent cocommutative co-algebra $S^c(\mathfrak{L})$ cofreely cogenerated by the graded vector space $\mathfrak{L}[1]^\bullet$ over $\mathbb{K}$ is called an L_∞-structure on the graded vector space $\mathfrak{L}$ if $Q^2 = 0$.

In more explicit terms we have

$$S^c(\mathfrak{L}) = \bigoplus_{k=0}^{\infty} \bigvee^k (\mathfrak{L}[1]), \tag{2.2}$$

where $\bigvee^k(\mathfrak{L}[1])$ is the space of coinvariants of $(\mathfrak{L}[1])^{\otimes k}$ for the action of the symmetric group S_k in k letters generated by

$$(i, i+1)(\gamma_1 \otimes \cdots \otimes \gamma_k) = (-1)^{|\gamma_i||\gamma_{i+1}|}(\gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_{i+1} \otimes \gamma_i \otimes \gamma_{i+2} \otimes \cdots \otimes \gamma_k),$$

where $|\gamma|$ denotes the degree of $\gamma \in \mathfrak{L}[1]$. We will usually denote the equivalence class of $\gamma_1 \otimes \cdots \otimes \gamma_k$ in $\bigvee^k \mathfrak{L}[1]$ by $\gamma_1 \vee \cdots \vee \gamma_k$. The space $S^c(\mathfrak{L})$ is equipped with the coproduct Δ given by

$$\Delta(1) = 1 \otimes 1, \tag{2.3}$$

$$\Delta(\gamma_1 \vee \cdots \vee \gamma_k) = 1 \otimes \gamma_1 \vee \cdots \vee \gamma_k + \gamma_1 \vee \cdots \vee \gamma_k \otimes 1 + \overline{\Delta}(\gamma_1 \vee \cdots \vee \gamma_k) \tag{2.4}$$

for $k \geq 1$ and any $\gamma_i \in \mathfrak{L}[1]$. Here we have

$$\overline{\Delta}(\gamma_1 \vee \cdots \vee \gamma_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in \mathrm{Sh}(i,k-i)} \varepsilon(\sigma) \gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(i)} \bigotimes \gamma_{\sigma(i+1)} \vee \cdots \vee \gamma_{\sigma(k)}, \tag{2.5}$$

where $\mathrm{Sh}(i, k-i)$ denotes the $(i, k-i)$ shuffles in the symmetric group S_k in k letters and the *Koszul sign* $\varepsilon(\sigma) = \varepsilon(\sigma, \gamma_1, \ldots, \gamma_k)$ is determined by the rule

$$\gamma_1 \vee \cdots \vee \gamma_k = \varepsilon(\sigma) \gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)}. \tag{2.6}$$

The co-unit on $S^c(\mathfrak{L})$ is given by the projection $\mathrm{pr}_{\mathbb{K}}$ onto the ground field $\mathbb{K}$.

Remark 2.2. A direct computation shows that, denoting the flip $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ by τ, we have

$$\Delta \circ (\cdot \vee \cdot) = (\cdot \vee \cdot) \otimes (\cdot \vee \cdot) \circ (\mathsf{id} \otimes \tau \otimes \mathsf{id}) \circ \Delta \otimes \Delta. \tag{2.7}$$

So we obtain the unital and co-unital bialgebra $(S^c(\mathfrak{L}), \cdot \vee \cdot, 1 \in \mathbb{K}, \Delta, \mathrm{pr}_{\mathbb{K}})$, i.e. $1 \vee X = X = X \vee 1$ for all $X \in S^c(\mathfrak{L})$. We sometimes abuse notation by omitting $\vee$ in favor of simple concatenation or superscripts, e.g. $ab := a \vee b$ and $x^3 := x \vee x \vee x$.

Lemma 2.3 (Characterization of coderivations). *Every degree $+1$ coderivation Q on $S^c(\mathfrak{L})$ is uniquely determined by the components*

$$Q_n \colon \bigvee^n (\mathfrak{L}[1]) \longrightarrow \mathfrak{L}[2] \tag{2.8}$$

by the formula

$$Q(\gamma_1 \vee \cdots \vee \gamma_n) = \sum_{k=0}^{n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \varepsilon(\sigma) Q_k(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)}) \vee \gamma_{\sigma(k+1)} \vee \cdots \vee \gamma_{\sigma(n)}, \tag{2.9}$$

where we use the conventions that $\mathrm{Sh}(n,0) = \mathrm{Sh}(0,n) = \{\mathsf{id}\}$ *and that the empty product equals the unit.*

Proof. It follows essentially by writing out both sides of the defining equation

$$\Delta \circ Q = (Q \otimes \mathsf{id} + \mathsf{id} \otimes Q) \circ \Delta. \qquad \blacksquare$$

Note that $Q_0(1)$ is of degree 1 in $\mathfrak{L}[1]$ (thus of degree 2 in $\mathfrak{L}$). The condition $Q^2 = 0$ can now be expressed in terms of a quadratic equation in the components Q_n.

Example 2.4 (Curved Lie algebras). Our main example of an L_∞-algebra is given by (curved) Lie algebras $(\mathfrak{L}, R, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ by setting

$$Q_0(1) = -R, \quad Q_1(\gamma) = -\mathrm{d}\gamma, \quad Q_2(\gamma \vee \mu) = -(-1)^{|\gamma|}[\gamma, \mu], \quad Q_i = 0 \quad \text{for } i \geq 3.$$

The condition $Q^2 = 0$ amounts to

- $\mathrm{d}R = 0$,
- $\mathrm{d}^2(\,\cdot\,) = [R, \,\cdot\,]$,
- d is a derivation of $[\,\cdot\,,\,\cdot\,]$,
- the graded Jacobi identity for $[\,\cdot\,,\,\cdot\,]$.

The signs in the expressions of Q_0, Q_1 and Q_2 are due to the fact that we pass through the decalage isomorphisms

$$\mathrm{dec}_n \colon \bigvee^n (\mathfrak{L}[1]) \longrightarrow (\Lambda^n \mathfrak{L})[n]$$

and an additional sign due to a convention related to the Maurer–Cartan equation introduced below.

Remark 2.5. We should note that our definition of L_∞-algebra is usually called *curved* L_∞-algebra, see e.g. [29]. Although this definition is also not set in stone, see for instance [24] for yet another notion of curved L_∞-algebra. For the purpose of this paper it is, however, more convenient to call the curved version simply L_∞-algebra. The only L_∞-algebras playing a role in this paper are, however, the flat L_∞-algebras, i.e. those having $Q_0 = 0$. The usual definition for an L_∞-algebra thus coincides with our definition of flat L_∞-algebra.

Remark 2.6. In the following we have to deal with various infinite sums. In order for this to make sense, we always consider only L_∞-algebras $\mathfrak{L}$ that are equipped with a decreasing filtration

$$\mathfrak{L} = \mathcal{F}^0\mathfrak{L} \supset \mathcal{F}^1\mathfrak{L} \supset \cdots \supset \mathcal{F}^k\mathfrak{L} \supset \cdots, \tag{2.10}$$

respecting the L_∞-structure and which is moreover *complete*, i.e.

$$\bigcap_k \mathcal{F}^k\mathfrak{L} = \{0\}. \tag{2.11}$$

This yields a corresponding complete metric topology and we consider convergence of infinite sums in terms of this topology.

Definition 2.7 (L_∞-morphisms). Let $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$ be L_∞-algebras. A degree 0 filtration preserving co-unital co-algebra morphism

$$F\colon S^c(\mathfrak{L}) \longrightarrow S^c(\widetilde{\mathfrak{L}}) \tag{2.12}$$

such that $FQ = \widetilde{Q}F$ is called an L_∞-morphism.

Lemma 2.8 (Characterization of co-algebra morphisms). *A co-unital co-algebra morphism F from $S^c(\mathfrak{L})$ to $S^c(\widetilde{\mathfrak{L}})$ is uniquely determined by its components, also called Taylor coefficients,*

$$F_n\colon \bigvee^n (\mathfrak{L}[1]) \longrightarrow \widetilde{\mathfrak{L}}[1], \tag{2.13}$$

where $n \geq 1$. Namely, we set $F(1) = 1$ and use the formula

$$\begin{aligned} &F(\gamma_1 \vee \cdots \vee \gamma_n) \\ &\quad = \sum_{p \geq 1} \sum_{\substack{k_1,\ldots,k_p \geq 1 \\ k_1+\cdots+k_p = n}} \sum_{\sigma \in \mathrm{Sh}(k_1,\ldots,k_p)} \frac{\varepsilon(\sigma)}{p!} F_{k_1}(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k_1)}) \vee \cdots \\ &\qquad \vee F_{k_p}(\gamma_{\sigma(n-k_p+1)} \vee \cdots \vee \gamma_{\sigma(n)}), \end{aligned} \tag{2.14}$$

where $\mathrm{Sh}(k_1,\ldots,k_p)$ denotes the set of $(k_1,\ldots,k_p)$-shuffles in S_n and $\mathrm{Sh}(n) = \{\mathrm{id}\}$.

Proof. Again, it essentially follows by writing out the defining equation

$$\Delta \circ F = F \otimes F \circ \Delta.$$ ∎

Note in particular that this definition does not allow for maps from flat L_∞-algebras to non-flat ones. It does however allow (in principle) for maps from non-flat L_∞-algebras to flat ones as long as $F(Q_0(1)) = 0$.

Example 2.9. Let $(\mathfrak{L}, R, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ and $(\mathfrak{L}', R', \mathrm{d}', [\,\cdot\,,\,\cdot\,]')$ be two curved Lie algebras and consider the morphism $C\colon \mathfrak{L} \to \mathfrak{L}'$ of curved Lie algebras, i.e. $C(R) = R'$, $C\,\mathrm{d} = \mathrm{d}'C$ and C is a homomorphism of the underlying Lie algebras. Then the map F given by applying the formula (2.14) to the components $F_1 = C$ and $F_i = 0$ for $i \geq 2$ is an L_∞-morphism. In general, if $F\colon \mathfrak{L} \to \mathfrak{L}'$ is an L_∞-morphism, then $F_1(R) = R'$, but we only have $\mathrm{d}'F_1(\gamma) = F_1\,\mathrm{d}(\gamma) + F_2(R \vee \gamma)$.

Note that, given an L_∞-morphism of *flat* L_∞-algebras $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$, we obtain the map of complexes

$$F_1\colon (\mathfrak{L}, Q_1) \longrightarrow (\widetilde{\mathfrak{L}}, \widetilde{Q}_1). \tag{2.15}$$

Definition 2.10 (L_∞-quasi-isomorphism). An L_∞-morphism F between flat L_∞-algebras is called L_∞-quasi-isomorphism if F_1 is a quasi-isomorphism of complexes.

The L_∞-quasi-isomorphisms we deal with in this paper happen to be the ones witnessing *formality*, let us therefore introduce the notion of formal L_∞-algebras here.

Definition 2.11 (Formal L_∞-algebra). An L_∞-algebra $\mathfrak{L}$ is called formal if it is flat and admits an L_∞-quasi-isomorphism

$$F\colon \mathrm{H}(\mathfrak{L}) \longrightarrow \mathfrak{L} \tag{2.16}$$

for the L_∞-structure canonically induced on the cohomology $\mathrm{H}(\mathfrak{L})$ of $\mathfrak{L}$.

Finally, a crucial concept for this paper (and deformation theory in general) is the concept of Maurer–Cartan elements, that we define below. It is important to remark that when L_∞-algebra $(\mathfrak{L}, Q)$ is a DGLA the following definition reduces to the standard Maurer–Cartan equation.

Definition 2.12 (Maurer–Cartan element). Given an L_∞-algebra $(\mathfrak{L}, Q)$, an element $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$ is called a Maurer–Cartan or MC element if it satisfies the equation

$$\sum_{n=0}^{\infty} \frac{Q_n(\pi^n)}{n!} = 0. \tag{2.17}$$

2.2. Local formality

Let us denote the formal completion at $0 \in \mathbb{R}^d$ by $\mathbb{R}^d_{\mathrm{formal}}$. Further, the smooth functions $\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}})$ on $\mathbb{R}^d_{\mathrm{formal}}$ are given by the algebra

$$\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}}) := \varprojlim_{k\to\infty} \mathcal{C}^\infty(\mathbb{R}^d)/\mathcal{I}_0^k, \tag{2.18}$$

where $\mathcal{I}_0$ denotes the ideal of functions vanishing at $0 \in \mathbb{R}^d$. Note that $\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}})$ comes equipped with the complete decreasing filtration

$$\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}}) \supset \mathcal{I}_0 \supset \mathcal{I}_0^2 \supset \cdots \tag{2.19}$$

and its corresponding (metric) topology. The Lie algebra of continuous derivations of $\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}})$ is denoted by $\mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$. By setting $\mathrm{T}^{-1}_{\mathrm{poly}} := \mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}})$ we obtain the Lie–Rinehart pair $(\mathrm{T}^{-1}_{\mathrm{poly}}, \mathrm{T}^0_{\mathrm{poly}})$ and the graded vector space

$$\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) := \bigoplus_{k\geq -1} \mathrm{T}^k_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}), \tag{2.20}$$

where $\mathrm{T}^k_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) := \Lambda^{k+1}\,\mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ for $k \geq 0$. Here, the tensor product is understood to be over $\mathrm{T}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and completed. Notice that there is no confusion about grading here although it may seem unnatural at first glance. It is actually obtained by shifting the natural grading. The natural structure is that of *Gerstenhaber algebra*, but we are only considering the underlying graded Lie algebra. The Lie bracket $[\![\cdot\,,\,\cdot]\!]$ on $\mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ extends to a graded Lie algebra structure on $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ by the rules

$$\begin{aligned}
[\![f, g]\!] &= 0,\\
[\![X_0, f]\!] &= X_0(f),\\
[\![X_0 \wedge \cdots \wedge X_k, Y]\!] &= \sum_{j=0}^{k} (-1)^{kl+j} [\![X_j, Y]\!] \wedge X_0 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge X_k
\end{aligned} \tag{2.21}$$

for all $f, g \in \mathrm{T}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$, $X_0, \ldots, X_k \in \mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $Y \in \mathrm{T}^l_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$.

We denote by $\mathrm{D}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ the universal enveloping algebra of the Lie–Rinehart pair $(\mathrm{T}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}), \mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}))$. Recall that $\mathrm{D}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ is naturally equipped with the structures of a bialgebra (see e.g. [30]). More precisely, $\mathrm{D}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ allows an $\mathbb{R}$-algebra structure $\cdot$ and an $\mathbb{R}$-co-algebra structure Δ. We extend the algebra structure in the obvious (componentwise) way to

$$\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) := \bigoplus_{k \geq -1} \mathrm{D}^k_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}), \tag{2.22}$$

where $\mathrm{D}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) := \mathrm{T}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $\mathrm{D}^k_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) := (\mathrm{D}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}))^{\otimes k+1}$. Again, the tensor product is understood to be over $\mathrm{D}^{-1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and completed. This allows us to define two $\mathbb{R}$-bilinear operations $\bullet$ and $[\cdot\,,\,\cdot]_G$ given by

$$P_1 \bullet P_2 := \sum_{i=0}^{k_1} (-1)^{ik_2} (\mathsf{id}^{\otimes i} \otimes \Delta^{(k_2)} \otimes \mathsf{id}^{\otimes k_1 - i})(P_1) \cdot (1^{\otimes i} \otimes P_2 \otimes 1^{\otimes k_1 - i}), \tag{2.23}$$

$$[P_1, P_2]_G := P_1 \bullet P_2 - (-1)^{k_1 k_2} P_2 \bullet P_1, \tag{2.24}$$

where $P_1 \in \mathrm{D}^{k_1}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$, $P_2 \in \mathrm{D}^{k_2}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $\Delta^{(k)}$ denotes the k-th iteration of Δ given by $(\Delta \otimes \mathsf{id}^{\otimes k-1})(\Delta \otimes \mathsf{id}^{\otimes k-2}) \cdots (\Delta \otimes \mathsf{id})\Delta$. Note that the bracket $[\cdot\,,\,\cdot]_G$ defines a graded Lie algebra structure on $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$.

Theorem 2.13 (Kontsevich [26]). *There exists an L_∞-quasi-isomorphism*

$$\mathcal{K}\colon \big(\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}), 0, [\![\cdot\,,\,\cdot]\!]\big) \longrightarrow \big(\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}), \partial, [\cdot\,,\,\cdot]_G\big) \tag{2.25}$$

between DGLA's, where $\partial = [\mu, \cdot]_G$ for $\mu = 1 \otimes 1 \in \mathrm{D}^1_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$. Moreover,

(i) *$\mathcal{K}$ is $\mathrm{GL}(d, \mathbb{R})$ equivariant;*

(ii) *$\mathcal{K}_n(X_1, \ldots, X_n) = 0$ for all $X_i \in \mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $n > 1$;*

(iii) *$\mathcal{K}_n(X, Y_2, \ldots, Y_n) = 0$ for all $Y_i \in \mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $n \geq 2$ whenever $X \in \mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ is induced by the action of $\mathfrak{gl}(d, \mathbb{R})$.*

3. Formality for Lie algebroids

In this section we recall the formality theorem for Lie algebroids, which is due to Calaque, see [6]. The proof of this theorem follows the lines of Dolgushev's construction [10,11] of the L_∞-quasi-isomorphism from polyvectorfields to polydifferential operators. The main ingredients are Fedosov's methods [21] concerning formal geometry, Kontsevich's quasi-isomorphism [26] and the twisting procedure inspired by Quillen [32] (although we use Dolgushev's version [11]). Since we only need the result and not in fact the details of the construction we are rather brief here and refer to [6] for details.

3.1. Fedosov resolutions

As a first step, Calaque constructs Fedosov resolutions of polyvector fields and polydifferential operators of Lie algebroids.

Let us recall that a Lie algebroid is a vector bundle $E \to M$ over a manifold M, equipped with a Lie bracket on sections $\Gamma^\infty(E)$ and an anchor map $\rho: E \to TM$, preserving the Lie bracket, such that

$$[v, fw]_E = f[v,w]_E + (\rho(v)f)w, \tag{3.1}$$

for any $v, w \in \Gamma^\infty(E)$ and $f \in \mathcal{C}^\infty(M)$. Equivalently, we can consider the algebra of E-differential forms $\Gamma^\infty(\wedge^\bullet E^*)$ endowed with the differential d_E given by $(\mathrm{d}_E f)(v) = \rho(v)(f)$ for $f \in \mathcal{C}^\infty(M)$ and $X \in \Gamma^\infty(E)$, by

$$(\mathrm{d}_E\alpha)(v,w) = \rho(v)(\alpha(w)) - \rho(w)(\alpha(v)) - \alpha([v,w]_E)$$

for $X, Y \in \Gamma^\infty(E)$ and $\alpha \in \Gamma^\infty(E^*)$ and extended as a derivation for the wedge product.

The definitions of the DGLA's $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ given in Section 2.2 go through mutatis mutandis to define the DGLA's ${}^E\mathrm{T}_{\mathrm{poly}}(M)$ and ${}^E\mathrm{D}_{\mathrm{poly}}(M)$ starting from the Lie–Rinehart pair $(\mathcal{C}^\infty(M), \Gamma^\infty(E))$. Notice that the resulting spaces ${}^E\mathrm{D}^k_{\mathrm{poly}}(M)$ can be identified with the spaces of E-polydifferential operators of order $k+1$. In order to extend the result of Theorem 2.13 to any Lie algebroid, we need to consider the so-called *Fedosov resolutions*. The idea (coming from formal geometry) consists in replacing the DGLA's $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ by quasi-isomorphic DGLA's (using DGLA morphisms in this case). For the rest of this section we consider a Lie algebroid E of rank d. We denote by ${}^E\mathcal{T}_{\mathrm{poly}}$ the bundle of formal fiberwise E-polyvector fields over M, this is the bundle associated to the principal bundle of general linear frames in E with fiber $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$. Similarly, the bundle ${}^E\mathcal{D}_{\mathrm{poly}}$ of formal fiberwise E-polydifferential operators is the bundle over M associated to the principal bundle of general linear frames in E with fiber $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$. The Fedosov resolutions are given on the level of vector spaces by the E-differential forms with values in ${}^E\mathcal{T}_{\mathrm{poly}}$ and ${}^E\mathcal{D}_{\mathrm{poly}}$ respectively. We denote these spaces by ${}^E\Omega(M; \mathcal{T}_{\mathrm{poly}})$ and ${}^E\Omega(M; \mathcal{D}_{\mathrm{poly}})$ respectively. Note that these spaces carry a natural DGLA structure, namely the one induced by the structure on fibers (which is $\mathrm{GL}(d, \mathbb{R})$-equivariant).

Lemma 3.1. *There exist* $\mathrm{GL}(d,\mathbb{R})$*-equivariant isomorphisms of algebras*

$$\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}}) \simeq \prod_{k=0}^{\infty} \mathrm{S}^k T_0^*\mathbb{R}^d \simeq \mathbb{R}[\![\hat{x}_1,\dots,\hat{x}_d]\!], \tag{3.2}$$

$$\mathrm{T}^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) \simeq \mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}}) \otimes T_0\mathbb{R}^d. \tag{3.3}$$

Here the Lie algebra structure on $\mathcal{C}^\infty(\mathbb{R}^d_{\mathrm{formal}}) \otimes T_0\mathbb{R}^d$ *is induced from the action of* $T_0\mathbb{R}^d$ *as derivations at* 0 *on* $\mathcal{C}^\infty(\mathbb{R}^d)$*.*

The proof of the above lemma can be found in [9, Prop. 2.1.10] and [31, Thm. 1.1.3]. It implies that

$$^E\Omega(M;\mathcal{T}_{\mathrm{poly}}) \simeq \Gamma^\infty\Big(\prod_{k=0}^{\infty} \Lambda^\bullet E \otimes \mathrm{S}^k E^* \otimes \Lambda^\bullet E^*\Big) \tag{3.4}$$

and similarly

$$^E\Omega(M;\mathcal{D}_{\mathrm{poly}}) \simeq \Gamma^\infty\Big(\prod_{k=0}^{\infty} \Big(\bigoplus_{l=0}^{\infty} \mathrm{S}^l E\Big)^{\otimes\bullet} \otimes \mathrm{S}^k E^* \otimes \Lambda^\bullet E^*\Big), \tag{3.5}$$

where Λ and S denote the anti-symmetric and symmetric algebra, respectively.

The next step consists in finding a differential on the Fedosov resolutions which is compatible with the graded Lie algebra structure and which makes them into DGLA's quasi-isomorphic to $^E\mathrm{T}_{\mathrm{poly}}(M)$ and $^E\mathrm{D}_{\mathrm{poly}}(M)$, respectively. Given some trivializing coordinate neighborhood $V \subset M$ of E, a local frame $e_1,\dots,e_d$ as well as its dual frame $(x^1,\dots,x^d)$, we can define the operators

$$\delta\colon {}^E\Omega(V;\mathcal{T}_{\mathrm{poly}}) \longrightarrow {}^E\Omega(V;\mathcal{T}_{\mathrm{poly}}) \tag{3.6}$$

by the formula

$$\delta(Y) = \sum_{i=1}^{d} x^i \wedge [\![e_i, Y]\!]. \tag{3.7}$$

In other words, $\delta = [\![A_{-1},\,\cdot\,]\!]$, where $A_{-1} \in {}^E\Omega^1(V;\mathcal{T}^0_{\mathrm{poly}})$ is the one-form $A_{-1} = \sum_{i=1}^d x^i \otimes e_i$. One easily checks that δ does not depend on the choice of coordinates, since A_{-1} is independent of coordinates, and therefore extends to all of M. By replacing $[\![\,\cdot\,,\,\cdot\,]\!]$ by $[\,\cdot\,,\,\cdot\,]_G$ in (3.7) we obtain the operators

$$\delta\colon {}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}) \longrightarrow {}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}). \tag{3.8}$$

Note that, since $[\![A_{-1},A_{-1}]\!] = [A_{-1},A_{-1}]_G = 0$, we have $\delta^2 = 0$. Furthermore, since it is given by an inner derivation and $\delta\mu = 0$, δ is compatible with the fiberwise Lie structures and thus yields DGLA structures.

The cohomology of the complexes $({}^E\Omega^l(M;\mathcal{T}_{\mathrm{poly}}),\delta)$ and $({}^E\Omega^l(M;\mathcal{D}_{\mathrm{poly}}),\delta)$ is given by the following proposition (proved e.g. in [6, Prop. 2.1]).

Proposition 3.2. *We have that*

$$\begin{aligned} \mathrm{H}^0({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}),\delta) &\simeq \Gamma^\infty(\Lambda^\bullet E),\\ \mathrm{H}^0({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}),\delta) &\simeq \Gamma^\infty\Big(\Big(\bigoplus_{l=0}^{\infty} S^l E\Big)^{\otimes\bullet}\Big) \end{aligned} \tag{3.9}$$

while

$$\begin{aligned} \mathrm{H}^{>0}({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}),\delta) &= 0,\\ \mathrm{H}^{>0}({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}),\delta) &= 0. \end{aligned} \tag{3.10}$$

Notice that

$$\begin{aligned} \mathrm{H}^0({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}),\delta) &\simeq {}^E\mathrm{T}_{\mathrm{poly}}(M),\\ \mathrm{H}^0({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}),\delta) &\simeq {}^E\mathrm{D}_{\mathrm{poly}}(M) \end{aligned} \tag{3.11}$$

as vector spaces. This does not provide us with the quasi-isomorphisms we are looking for, since $\mathrm{H}^0({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}),\delta)$ carries the trivial Lie algebra structure. To correct it, the idea is to construct a perturbation of the differential δ that does not affect the size of the cohomology, but only the Lie algebra structure on cohomology. Notice that the operator δ is of degree -1 in terms of the filtration and so we may start perturbing at order 0, i.e. adding a connection in the bundle E. The fact that the resulting perturbation should square to zero forces us to choose a torsion-free connection ∇. This gives us the operators

$$\begin{aligned} \nabla\colon {}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}) &\longrightarrow {}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}),\\ \nabla\colon {}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}) &\longrightarrow {}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}). \end{aligned} \tag{3.12}$$

Thus, we consider the corresponding operators $-\delta+\nabla$ and $-\delta+\nabla+\partial$. This leads to the problem that there is no reason to assume that we can find ∇ such that $\nabla^2=0$ (since not every Lie algebroid is flat). Following the idea of Fedosov, we correct $-\delta+\nabla$ by an inner derivation and make the ansatz

$$D := -\delta + \nabla + [A,\,\cdot\,] \tag{3.13}$$

with $A \in {}^E\Omega^1(M;\mathcal{T}^0_{\mathrm{poly}}) \hookrightarrow {}^E\Omega^1(M;\mathcal{D}^0_{\mathrm{poly}})$ and where $[A,\,\cdot\,]$ means $[\![A,\,\cdot\,]\!]$ or $[A,\,\cdot\,]_G$ depending on the situation. The trick is to find A such that $D^2=0$. The following lemma is proved in [6, Prop. 2.2].

Lemma 3.3. *There exists a unique A such that*

(i) $\delta A = R + \nabla A + \frac{1}{2}[A,A]$

(ii) $\delta^{-1}A = 0.$

Here R denotes the curvature of ∇ expressed in terms of the bundle ${}^E\mathcal{T}_{\mathrm{poly}}(M)$ (or ${}^E\mathcal{D}_{\mathrm{poly}}(M)$), i.e. it is given by the equation

$$\nabla^2 Y = [R, Y] \tag{3.14}$$

and δ^{-1} is a particular δ-homotopy from the projection onto degree 0, denoted σ, to the identity, i.e.

$$\delta^{-1}\delta + \delta\delta^{-1} + \sigma = \mathsf{id}\,. \tag{3.15}$$

The condition $\delta^{-1}A = 0$ is simply a normalization condition ensuring uniqueness of the solution.

For the proof of the following proposition we refer to [6, Thm. 2.3].

Proposition 3.4. *We have*

$$\mathrm{H}^{>0}({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}), D) = 0 \quad \textit{and} \quad \mathrm{H}^{>0}({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}), D) = 0. \tag{3.16}$$

Furthermore, we have

$$\begin{aligned} \mathrm{H}^{0}({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}), D) &\simeq \mathrm{H}^{0}({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}), \delta), \\ \mathrm{H}^{0}({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}), D) &\simeq \mathrm{H}^{0}({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}), \delta). \end{aligned} \tag{3.17}$$

Let us denote the isomorphisms from the above Proposition by τ. Then, using a Poincaré–Birkhoff–Witt-type isomorphism, Calaque constructs an isomorphism (see [6, Sec. 2.3])

$$\nu\colon \mathrm{Ker}\,\delta \cap {}^E\Omega^0(M;\mathcal{D}_{\mathrm{poly}}) \longrightarrow {}^E\mathrm{D}_{\mathrm{poly}}(M) \tag{3.18}$$

of filtered vector spaces.

Similarly, but in an easier way, we obtain an isomorphism

$$\nu\colon \mathrm{Ker}\,\delta \cap {}^E\Omega^0(M;\mathcal{T}_{\mathrm{poly}}) \longrightarrow {}^E\mathrm{T}_{\mathrm{poly}}(M) \tag{3.19}$$

of graded vector spaces. Finally, as proved in [6, Prop. 2.4–2.5], we get:

Theorem 3.5 (Fedosov resolutions). *The maps*

$$\lambda_D\colon \big({}^E\mathrm{D}_{\mathrm{poly}}(M), \partial\big) \longrightarrow \big({}^E\Omega(M;\mathcal{D}_{\mathrm{poly}}), \partial + D\big) \tag{3.20}$$

and

$$\lambda_T\colon \big({}^E\mathrm{T}_{\mathrm{poly}}(M), 0\big) \longrightarrow \big({}^E\Omega(M;\mathcal{T}_{\mathrm{poly}}), D\big) \tag{3.21}$$

both given by $\tau \circ \nu^{-1}$ are DGLA quasi-isomorphisms.

Let us sketch the remaining steps necessary to obtain the L_∞-quasi-isomorphisms from ${}^E\mathrm{T}_{\mathrm{poly}}(M)$ to ${}^E\mathrm{D}_{\mathrm{poly}}(M)$. As a second step, one notices that in a trivializing neighborhood $U \subset M$ of E the connection ∇ on both ${}^E\Omega(M;\mathcal{T}_{\mathrm{poly}})$ and ${}^E\Omega(M;\mathcal{D}_{\mathrm{poly}})$ is given by $\mathrm{d}_E + [B_U, \cdot\,]$ for some element $B_U \in {}^E\Omega^1(M;\mathcal{T}^0_{\mathrm{poly}}) \hookrightarrow {}^E\Omega^1(M;\mathcal{D}^0_{\mathrm{poly}})$. Thus, in this neighborhood, we have $D = \mathrm{d}_E + [\Gamma, \cdot\,]$, where Γ is a Maurer–Cartan element. We now observe that the map

$$\mathcal{U}\colon {}^E\Omega(U;\mathcal{T}_{\mathrm{poly}}) \longrightarrow {}^E\Omega(U;\mathcal{D}_{\mathrm{poly}}) \tag{3.22}$$

given by applying the map $\mathcal{K}$ from Theorem 2.13 fiberwise commutes with d_E. The next step consists in twisting this map by Γ to obtain L_∞-quasi-isomorphisms

$$\mathcal{U}^\Gamma \circ \lambda_T \colon {}^E\mathrm{T}_{\mathrm{poly}}(U) \longrightarrow {}^E\Omega(U; \mathcal{D}_{\mathrm{poly}}). \tag{3.23}$$

The twisting procedure is essential in our paper and will be discussed in full detail in Section 3.2. By using the properties of Kontsevich's quasi-isomorphism (2.25) and the fact that ∇ is a $\mathfrak{gl}(d, \mathbb{R})$ connection we find that these quasi-isomorphisms coincide on intersections and thus we obtain

$$\mathcal{U}^\Gamma \circ \lambda_T \colon {}^E\mathrm{T}_{\mathrm{poly}}(M) \longrightarrow {}^E\Omega(M; \mathcal{D}_{\mathrm{poly}}). \tag{3.24}$$

Remark 3.6. Although it may seem that we are being sloppy with notation by writing $\mathcal{U}^\Gamma$, since it is not a twist a priori, it is still possible to consider it as a twist in the context of curved L_∞-algebras. This construction is discussed in the paper [18].

Finally, we would like to define the L_∞-quasi-isomorphism $\lambda_D^{-1} \circ \mathcal{U}^\Gamma \circ \lambda_T \colon {}^E\mathrm{T}_{\mathrm{poly}} \to {}^E\mathrm{D}_{\mathrm{poly}}$. One problem remains and it is that, although λ_D is obviously injective, we cannot be assured that $\mathcal{U}^\Gamma \circ \lambda_T$ maps ${}^E\mathrm{T}_{\mathrm{poly}}$ into the image of λ_D. However, Dolgushev [10, Prop. 5] shows that we can always modify $\mathcal{U}^\Gamma \circ \lambda_T$ using a so-called *partial homotopy* to obtain a new quasi-isomorphism $\overline{\mathcal{U}}$ which maps into the image of λ_D. Thus we obtain the L_∞-quasi-isomorphism

$$F_E := \lambda_D^{-1} \circ \overline{\mathcal{U}} \colon {}^E\mathrm{T}_{\mathrm{poly}} \longrightarrow {}^E\mathrm{D}_{\mathrm{poly}}. \tag{3.25}$$

As a consequence, we obtain the formality theorem for a generic manifold M by considering the case $E = TM$ and formality for Lie algebras by considering the case $E = \mathfrak{g}$ over a point.

Remark 3.7. Note that the constructions of D, τ and so on are not unique, but they depend only on the choice of the torsion-free E-connection ∇.

3.2. Twisting procedure

In the following we recall the notions of twisting DGLA's and L_∞-morphisms by Maurer–Cartan elements. The idea of such twisting procedures comes from Quillen's seminal work [32]. Here we follow Dolgushev's approach as laid out in [11]. As an example we show how one obtains the local L_∞-quasi-isomorphisms $\mathcal{U}^\Gamma$ mentioned above.

Lemma 3.8. *Suppose $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$. Then the element*

$$\exp(\pi) := \sum_{n=0}^{\infty} \frac{\pi^k}{k!} \tag{3.26}$$

is well defined, invertible and group-like.

Proof. The element $\exp(\pi)$ is well defined, since the partial sums converge by virtue of π being in the first filtration (the filtration is respected by $\vee$). Invertibility follows from the usual direct computations showing that $\exp(-\pi)\exp(\pi) = 1 = \exp(\pi)\exp(-\pi)$. The fact that $\exp(\pi)$ is group-like can similarly be deduced from a direct computation using the definition of Δ given in Section 2.1. ∎

Given $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$ we define the π-twist of the L_∞-algebra $(\mathfrak{L}, Q)$ as the L_∞-algebra $\mathfrak{L}^\pi$ given by the pair $(\mathfrak{L}, Q^\pi)$ with

$$Q^\pi(a) := \exp(-\pi) \vee Q(\exp(\pi) \vee a). \tag{3.27}$$

Corollary 3.9. *Suppose* $(\mathfrak{L}, Q)$ *is an* L_∞*-algebra and* $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$. *Then the* π*-twist* $(\mathfrak{L}, Q^\pi)$ *is an* L_∞*-algebra.*

Example 3.10. Given a curved Lie algebra $(\mathfrak{L}, R, \mathrm{d}, [\cdot\,,\,\cdot])$ we find the twisted curved Lie algebra $(\mathfrak{L}, R^\pi, \mathrm{d} + [\pi, \cdot\,], [\cdot\,,\,\cdot])$, where

$$R^\pi := R + \mathrm{d}\pi + \frac{1}{2}[\pi, \pi]. \tag{3.28}$$

Note, in particular, that the π-twist is flat exactly when π satisfies the Maurer–Cartan equation.

Proposition 3.11. *Suppose* $\mathfrak{L}$ *is an* L_∞*-algebra and* $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$. *Then the* π*-twist of* $\mathfrak{L}$ *is flat if and only if* π *is a Maurer–Cartan element.*

Proof. We have

$$Q^\pi(1) = \exp(-\pi) \vee Q(\exp(\pi)) = \sum_{n=0}^{\infty} \frac{Q_n(\pi^n)}{n!}, \tag{3.29}$$

since all terms in $\bigoplus_{k=2}^{\infty} \bigvee^k(\mathfrak{L}[1])$ cancel out by virtue of the fact that

$$Q^\pi(1) = Q_0^\pi(1) \in \mathfrak{L}[1].$$ ∎

Example 3.12. For a DGLA $(\mathfrak{L}, \mathrm{d}, [\cdot\,,\cdot])$ definition (3.28) boils down to the usual Maurer–Cartan equation

$$\mathrm{d}\pi + \frac{1}{2}[\pi, \pi] = 0. \tag{3.30}$$

If we have similarly a curved Lie algebra with curvature $-R$ it comes down to the non-homogeneous equation

$$\mathrm{d}\pi + \frac{1}{2}[\pi, \pi] = R. \tag{3.31}$$

Lemma 3.13. *Suppose* $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$. *Then* π *is an MC element if and only if*

$$Q(\exp(\pi)) = 0.$$

Proof. The proof follows from the equation

$$\begin{aligned} Q(\exp(\pi)) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\sigma\in \mathrm{Sh}(k,n-k)} \varepsilon(\sigma)\frac{1}{n!}Q_k(\pi^k)\vee \pi^{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}Q_k(\pi^k)\vee \pi^{n-k} \\ &= \left(\sum_{n=0}^{\infty}\frac{Q_n(\pi^n)}{n!}\right)\vee \exp(\pi). \end{aligned}$$

■

Lemma 3.14. *Given an L_∞-morphism F from $\mathfrak{L}$ to $\mathfrak{L}'$ and an element $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$, we define the F-associated element $\pi_F \in \mathcal{F}^1\mathfrak{L}'[1]^0$ by the formula*

$$\pi_F := \sum_{n=1}^{\infty}\frac{F_n(\pi^n)}{n!}. \tag{3.32}$$

We have

$$F(\exp(\pi)) = \exp(\pi_F). \tag{3.33}$$

Proof. It follows from explicit computation using formula (2.14). ■

Lemmas 3.14 and 3.13 imply the following corollary.

Corollary 3.15. *If π is an MC element, then π_F is also an MC element.*

Let $F\colon (\mathfrak{L}, Q) \to (\widetilde{\mathfrak{L}}, \widetilde{Q})$ be an L_∞-morphism and $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$.

Definition 3.16 (π-twist morphism). The π-twist of F is the map

$$F^\pi\colon (\mathfrak{L}, Q^\pi) \longrightarrow (\widetilde{\mathfrak{L}}, \widetilde{Q}^\pi) \tag{3.34}$$

defined by

$$F^\pi(a) := \exp(\pi_F)\vee F(\exp(\pi)\vee a). \tag{3.35}$$

Corollary 3.17. *The π-twist of an L_∞-morphism F is an L_∞-morphism.*

Proof. Note that, by Lemma 3.8 and Remark 2.2, the operators of multiplication by $\exp(\pi)$ and $\exp(\pi_F)$ are co-algebra morphisms. Thus F^π is a co-algebra morphism. The relation $F^\pi Q^\pi = \widetilde{Q}^\pi F^\pi$ follows from the definitions. ■

Remark 3.18. Given two L_∞-morphisms F and G from $\mathfrak{L}$ to $\mathfrak{L}'$ and $\mathfrak{L}'$ to $\mathfrak{L}''$, respectively, and the elements $\pi, B \in \mathcal{F}^1\mathfrak{L}[1]^0$, we have that

$$(Q^\pi)^B = Q^{\pi+B} = (Q^B)^\pi, \tag{3.36}$$

$$(F^\pi)^B = F^{\pi+B} = (F^B)^\pi, \tag{3.37}$$

$$\pi_F + B_{F^\pi} = (\pi + B)_F = B_F + \pi_{F^B}, \tag{3.38}$$

$$(\pi_F)_G = \pi_{G\circ F}. \tag{3.39}$$

For the proof of the following proposition we refer to [11, Prop. 1].

Proposition 3.19. *Let $F\colon \mathfrak{L} \to \widetilde{\mathfrak{L}}$ be an L_∞-quasi-isomorphism such that the induced morphisms*

$$F|_{\mathcal{F}^k\mathfrak{L}}\colon \mathcal{F}^k\mathfrak{L} \longrightarrow \mathcal{F}^k\widetilde{\mathfrak{L}} \tag{3.40}$$

are also L_∞-quasi-isomorphisms for all k. Suppose further that $\pi \in \mathcal{F}^1\mathfrak{L}[1]^0$ is an MC element. Then the π-twist

$$F^\pi\colon \mathfrak{L}^\pi \longrightarrow \widetilde{\mathfrak{L}}^{\pi_F} \tag{3.41}$$

of F is also a quasi-isomorphism.

Remark 3.20. The proposition above says that the class of L_∞-quasi-isomorphisms is closed under the operation of twisting by a Maurer–Cartan element. This provides the method of showing that an L_∞-morphism is an L_∞-quasi-isomorphism by showing that it is the twist of a known L_∞-quasi-isomorphism.

Example 3.21 (Formality for $\mathbb{R}^d$). Here we generalize the result of Theorem 2.13 from $\mathbb{R}^d_{\text{formal}}$ to $\mathbb{R}^d$ by providing an example of the claim in Remark 3.20. From now on we set $E = TM$ and drop the E for notational convenience. Proposition 3.19 allows us to obtain an L_∞-quasi-isomorphism witnessing the formality of $\mathrm{D}_{\text{poly}}(M)$ for any manifold by twisting the formal quasi-isomorphism of Theorem 2.13. We set $M = \mathbb{R}^d$ and recall that we are looking for an L_∞-quasi-isomorphism

$$\mathcal{U}^\delta\colon (\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}}), D) \longrightarrow (\Omega(\mathbb{R}^d,\mathcal{D}_{\text{poly}}), \partial + D), \tag{3.42}$$

since this would complete the diagram

$$(\mathrm{T}_{\text{poly}}(\mathbb{R}^d), 0) \xrightarrow{\lambda_T} (\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}}), D) \xrightarrow{\mathcal{U}^\delta} (\Omega(\mathbb{R}^d,\mathcal{D}_{\text{poly}}), \partial + D) \xleftarrow{\lambda_D} (\mathrm{D}_{\text{poly}}(\mathbb{R}^d), \partial) \tag{3.43}$$

of L_∞-quasi-isomorphisms. Also, recall that $D := -\delta + \mathrm{d}$. We obtain this map $\mathcal{U}^\delta$ as follows. First we note that, by applying the map $\mathcal{K}$ from Theorem 2.13 fiberwise, we obtain the L_∞-morphism

$$\mathcal{U}\colon (\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}}), \mathrm{d}) \longrightarrow (\Omega(\mathbb{R}^d;\mathcal{D}_{\text{poly}}), \partial + \mathrm{d}). \tag{3.44}$$

By considering the filtrations by exterior degree on both of these algebras we construct spectral sequences which show that $\mathcal{U}$ is a quasi-isomorphism. Using this same filtration we may consider the Maurer–Cartan element $-A_{-1} \in \mathcal{F}^1\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}})$. Now note that $\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}})^{-A_{-1}}$ is exactly $(\Omega(\mathbb{R}^d;\mathcal{T}_{\text{poly}}), D)$ and $\Omega(\mathbb{R}^d;\mathcal{D}_{\text{poly}})^{(-A_{-1})_\mathcal{U}}$ is exactly $(\Omega(\mathbb{R}^d;\mathcal{D}_{\text{poly}}), D)$, since $(-A_{-1})_\mathcal{U} = -A_{-1}$ by point (ii) of Theorem 2.13. So we obtain the diagram (3.43) by setting $\mathcal{U}^\delta := \mathcal{U}^{-A_{-1}}$. This concludes the example of the claim in Remark 3.20. In order to obtain the quasi-isomorphism

$$\mathrm{T}_{\text{poly}}(\mathbb{R}^d) \longrightarrow \mathrm{D}_{\text{poly}}(\mathbb{R}^d) \tag{3.45}$$

we need to invert the final arrow of diagram (3.43). This arrow is actually an identification (by DGLA-morphism) with the kernel of D in exterior degree 0. Thus it can be inverted without problems if we can guarantee that the map $\mathcal{U}^{\delta} \circ \lambda_T$ maps $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d)$ into this kernel. We refer to [10, Sect. 4.2] for an explanation of a way to correct $\mathcal{U}^{\delta}$ to have this property.

Example 3.22 (Formality for Lie algebras). Let us conclude this section by providing the equivalent of the proof of formality for the case where $M = \{\mathrm{pt}\}$ is the connected 0-dimensional manifold and E is a d-dimensional Lie algebra $\mathfrak{g}$. The DGLA of polyvector fields is given by $\mathrm{CE}_{\bullet}(\mathfrak{g})$, the Chevalley–Eilenberg complex with the trivial differential. The complex of E-differential forms with values in the fiberwise polyvector fields is thus given by

$$\mathrm{CE}^{\bullet}(\mathfrak{g}; \mathrm{CE}_{\bullet}(\mathfrak{g}; \widehat{S}(\mathfrak{g}^*))), \tag{3.46}$$

where we have denoted $\widehat{S}(\mathfrak{g}^*) = \prod_{k \geq 0} S^k \mathfrak{g}^*$ and the differential $\delta - d_E$ coincides with the usual Chevalley–Eilenberg differential. A linear E-connection ∇ is simply given by a linear map

$$\nabla \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}. \tag{3.47}$$

The corresponding map $\nabla \colon \Lambda^{\bullet}\mathfrak{g}^* \to \Lambda^{\bullet+1}\mathfrak{g}^*$ is given by extending the formula

$$\nabla\alpha(X, Y) = -\alpha\Big(\nabla\big(\frac{1}{2}(X \otimes Y - Y \otimes X)\big)\Big) \tag{3.48}$$

from one-forms as a $\wedge$-derivation. Note that the $\mathfrak{g}$-differential $\mathrm{d}_{\mathfrak{g}}$ is simply given by $X \otimes Y \mapsto [X, Y]$. Suppose $\{e_i\}_{i=1}^d$ is a basis for $\mathfrak{g}$ with dual basis $\{e^i\}_{i=1}^d$. Then the torsion-freeness of the connection ∇ can be expressed as $\widetilde{\Gamma}_{ij}^k = \widetilde{\Gamma}_{ji}^k$ in terms of the Christoffel symbols $\widetilde{\Gamma}_{ij}^k \in \mathbb{R}$ defined by

$$\nabla(e_i \otimes e_j) = \widetilde{\Gamma}_{ij}^k e_k, \tag{3.49}$$

where we have used the Einstein summation convention. Let us consider also the *relative* Christoffel symbols Γ_{ij}^k defined by

$$\nabla(e_i \otimes e_j) - [e_i, e_j] = \Gamma_{ij}^k e_k, \tag{3.50}$$

i.e. $\Gamma_{ij}^k = \widetilde{\Gamma}_{ij}^k - \frac{1}{2}c_{ij}^k$, where c_{ij}^k are the structure constants. In terms of these, torsion-freeness is equivalent to the equation

$$\Gamma_{ij}^k - \Gamma_{ji}^k - c_{ij}^k = 0. \tag{3.51}$$

Note in particular, that the connection $\mathrm{d}_{\mathfrak{g}}$ is *not* torsion-free. The most obvious choice of torsion-free connection is given by $\Gamma_{ij}^k = \frac{1}{2}c_{ij}^k$, but we leave the choice of symmetric part open. Given any connection ∇ it is given on $\mathrm{CE}^{\bullet}(\mathfrak{g}; \mathrm{CE}_{\bullet}(\mathfrak{g}; \widehat{S}(\mathfrak{g}^*)))$ by the formula

$$\nabla = \mathrm{d}_{\mathfrak{g}} + [\Gamma_{ij}^k e^i \hat{e}^j e_k, \cdot], \tag{3.52}$$

where we have used the hat to signify that we consider $\hat{e}^j \in \widehat{S}(\mathfrak{g}^*)$. Similar statements hold for ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$. Now the example proceeds identically to the previous one.

4. Formality and deformation symmetries

In this section we prove the main result of this paper, which leads to a new perspective on Drinfeld's approach to deformation quantization. First we construct certain L_∞-algebras related to a Hopf algebra or more generally a unital bialgebra and show how one obtains deformations from Drinfeld twists and maps into a Hochschild cochain complex. Then we briefly recall the basic notions of Poisson action and triangular Lie algebra. We consider the particular case of a Poisson action of a triangular Lie algebra $(\mathfrak{g}, r)$ on a manifold M and we show that we can construct a corresponding L_∞-morphism between polydifferential operators ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$ and $\mathrm{D}_{\mathrm{poly}}(M)$. This morphism induces a DGLA morphism between a quantum group associated to our Lie algebra and a deformed algebra of smooth functions on M.

4.1. Deformation symmetries

In the following we define the concept of a deformation symmetry. This notion is inspired by Drinfeld's work on deformation through quantum actions and Drinfeld twists. Let us start by recalling the definition of Drinfeld twist. In this section we shall fix the Hopf algebra $(H, \Delta, \varepsilon, S)$ over the PID R containing $\mathbb{Q}$.

Definition 4.1 (Drinfeld twist, [12, 14]). An element $J \in H \otimes H$ is said to be a twist on H if the following three conditions are satisfied.

(i) J is invertible;

(ii) $(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J)$ and

(iii) $(\varepsilon \otimes 1)J = (1 \otimes \varepsilon)J = 1$.

In the following, we consider formal deformations. If we consider twists in $H[[\hbar]]$, the condition of invertibility and "co-invertibility" (condition (iii) in the above definition) may be replaced by a stronger condition which is easier to check. In fact, this condition may be formulated for any Hopf algebra equipped with a complete filtration $H = \mathcal{F}^0 H \supset \mathcal{F}^1 H \supset \cdots$.

Definition 4.2 (Formal Drinfeld twist). Let H be equipped with the complete filtration $H = \mathcal{F}^0 H \supset \mathcal{F}^1 H \supset \cdots$. Then an element $J \in H \otimes H$ is said to be a formal twist on H if J satisfies (ii) of Definition 4.1 and $J - 1 \otimes 1 \in \mathcal{F}^1(H \otimes H)$.

Corollary 4.3. *A formal twist on H is a twist on H.*

Proof. This follows immediately from the compatibility of the Hopf algebra structure with the complete filtration. ■

It turns out that the definition of formal twist coincides exactly with the definition of Maurer–Cartan element on a certain DGLA that we shall now define. The main observation is that the formulas (2.23) and (2.24) for the Gerstenhaber bracket on ${}^{E}\mathrm{D}_{\mathrm{poly}}$ only involve the structure of a unital bialgebra. From now on we denote

$$TH = \bigoplus_{k=0}^{\infty} T^k H \quad \text{with} \quad T^k H := H^{\otimes k}. \tag{4.1}$$

For $P_1 \in T^{k_1+1}H$ and $P_2 \in T^{k_2+1}H$ set

$$P_1 \bullet P_2 := \sum_{i=0}^{k_1} (-1)^{ik_2} (\mathsf{id}^{\otimes i} \otimes \Delta^{(k_2)} \otimes \mathsf{id}^{\otimes k_1 - i})(P_1) \cdot (1^{\otimes i} \otimes P_2 \otimes 1^{\otimes k_1 - i}) \tag{4.2}$$

and

$$[P_1, P_2]_H := P_1 \bullet P_2 - (-1)^{k_1 k_2} P_2 \bullet P_1. \tag{4.3}$$

Proposition 4.4. *The graded vector space $TH[1]$ equipped with the bracket $[\cdot\,,\,\cdot]_H$ is a graded Lie algebra.*

Proof. We can immediately extend $[\cdot\,,\,\cdot]_H$ to non-homogeneous elements, since $\bullet$ can be extended by bilinearity. Thus, the bilinearity and anti-symmetry of $[\cdot\,,\,\cdot]_H$ follow immediately from the bilinearity of $\bullet$, which follows in turn from the linearity of the coproduct and the bilinearity of the product. Finally, denote the associator of $\bullet$ by α, i.e.

$$\alpha(A, B, C) = A \bullet (B \bullet C) - (A \bullet B) \bullet C. \tag{4.4}$$

Then the average of α over the symmetric group S_3 is 0, i.e.

$$\sum_{\sigma \in S_3} \sigma^* \alpha = 0. \tag{4.5}$$

Here S_3 acts on $(TH[1])^{\otimes 3}$ through the usual signed permutation of tensor legs. The last equation is obviously equivalent to the Jacobi identity for $[\cdot\,,\,\cdot]_H$. ∎

Remark 4.5. Recall that the construction of multilinear operations, called braces, on the Hochschild complex of an associative algebra leads to the definition of a brace algebra on a graded vector space [25]. The structure $\bullet$ on $TH[1]$ is actually the pre-Lie structure coming from a brace algebra structure. As such, the identity (4.5) can actually be proved by showing the pre-Lie identity

$$\alpha(A, B, C) = (-1)^{|A||B|} \alpha(B, A, C). \tag{4.6}$$

The braces underlying the brace algebra structure are given by

$$\begin{aligned} P\langle Q_1, \dots, Q_r \rangle = \sum_{\substack{0 \le i_1 < i_2 \\ < \dots < i_r \le k}} & (-1)^{i_1 k_1 + i_2 k_2 + \dots + i_r k_r} \\ & \cdot \left(\mathsf{id}^{\otimes i_1} \otimes \Delta^{(k_1)} \otimes \mathsf{id}^{\otimes (i_2 - i_1)} \otimes \dots \otimes \Delta^{(k_r)} \otimes \mathsf{id}^{\otimes (k - i_r)} \right)(P) \\ & \cdot \left(1^{\otimes i_1} \otimes Q_1 \otimes 1^{\otimes (i_2 - i_1)} \otimes \dots \otimes 1^{\otimes (i_r - i_{r-1})} \otimes Q_r \otimes 1^{\otimes (k - i_r)} \right), \end{aligned}$$

where $P \in T^{k+1}H$ and $Q_j \in T^{k_j+1}H$ for all j. Note that restricting this brace algebra structure to $H = T^1 H = (TH[1])^0$ we find the brace algebra given in [1, Sect. 6].

Let us denote the twist of $TH[1]$ by the Maurer–Cartan element $1 \otimes 1$ as

$$(H^{\bullet}_{\mathrm{poly}}, [\cdot,\cdot], \partial) := (T^{\bullet+1}H, [\cdot,\cdot]_H, [1 \otimes 1, \cdot]_H). \tag{4.7}$$

Lemma 4.6. *An element $J \in T^2H$ is a formal twist on H if and only if $J - 1 \otimes 1$ is a Maurer–Cartan element in H_{poly}.*

Proof. Suppose first that $J \in T^2H$ is a formal twist on H. Then, by definition, $F := J - 1 \otimes 1$ is an element of $\mathcal{F}^1 H^1_{\mathrm{poly}}$ and

$$\begin{aligned} \partial F + \frac{1}{2}[F,F]_H &= \frac{1}{2}[J,J]_H \\ &= (\Delta \otimes \mathrm{id})(J)(J \otimes 1) - (\mathrm{id} \otimes \Delta)(J)(1 \otimes J) = 0 \end{aligned} \tag{4.8}$$

by condition (ii) of Definition 4.1. Conversely, suppose F is a Maurer–Cartan element in H_{poly}, then, for $J = 1 \otimes 1 + F$,

$$\begin{aligned} (\Delta \otimes \mathrm{id})(J)(J \otimes 1) - (\mathrm{id} \otimes \Delta)(J)(1 \otimes J) &= \frac{1}{2}[J,J]_H \\ &= \partial F + \frac{1}{2}[F,F]_H = 0 \end{aligned} \tag{4.9}$$

by the Maurer–Cartan equation. So J satisfies (ii) of 4.1, while clearly $J - 1 \otimes 1 = F \in \mathcal{F}^1(H \otimes H)$. ∎

Drinfeld discovered [13,14] that one can twist the Hopf algebra structure on H by any (formal) twist J. More explicitly, one obtains the twisted Hopf algebra H_J by changing only the coproduct Δ to Δ_J given by

$$\Delta_J(X) = J^{-1}\Delta(X)J. \tag{4.10}$$

Let us fix the (formal) twist J on H. It is convenient to introduce the following notation

$$\begin{aligned} J_k &:= \prod_{i=1}^{k} (\Delta^{(k-i)} \otimes \mathrm{id}^{\otimes i})(J \otimes 1^{\otimes i-1}) \\ &= (\Delta^{(k-1)} \otimes \mathrm{id})(J) \cdot (\Delta^{(k-2)} \otimes \mathrm{id} \otimes \mathrm{id})(J \otimes 1) \cdots \\ &\quad \cdot (\Delta \otimes \mathrm{id}^{\otimes k-1})(J \otimes 1^{\otimes k-2}) \cdot (J \otimes 1^{\otimes k-1}) \end{aligned}$$

and we set $J_0 = 1$. Note that $J_k \in H^{\otimes k+1}$ is invertible with J_k^{-1} given by reversing the order of terms above and replacing J by J^{-1}.

Lemma 4.7. *The iterates of Δ_J are given by the formula*

$$\Delta_J^{(k)}(X) = J_k^{-1}\Delta^{(k)}(X)J_k. \tag{4.11}$$

Proof. The proof is given by straightforward computation. ∎

We also need the following (slightly technical) lemma.

Lemma 4.8. *The J_k's satisfy the relation*

$$(\mathsf{id}^{\otimes i} \otimes \Delta^{(l)} \otimes \mathsf{id}^{\otimes k-i})(J_k) \cdot (1^{\otimes i} \otimes J_l \otimes 1^{\otimes k-i}) = J_{k+l} \tag{4.12}$$

for all $0 \le i \le k \in \mathbb{Z}$ *and* $l \in \mathbb{Z}_{\ge 0}$.

Proof. For $k = i = 0$, formula (4.12) reads

$$\Delta^{(l)}(1) \cdot J_l = J_l \tag{4.13}$$

which is obviously satisfied. For $k = 1$ and $i = 0$ we get

$$(\Delta^{(l)} \otimes \mathsf{id})(J) \cdot (J_l \otimes 1) = J_{l+1}, \tag{4.14}$$

which follows immediately from the definition of J_{l+1}. For $k = 1$, $i = 1$ and $l = 0$ we find

$$J \cdot (1 \otimes 1) = J. \tag{4.15}$$

We will fully establish the $k = 1$ case by induction now. So suppose the formula 4.12 holds for $k = 1$, $i = 1$, $l - 1 \in \mathbb{Z}_{\ge 0}$. Then

$$\begin{aligned}
(\mathsf{id} \otimes \Delta^{(l)})(J) \cdot (1 \otimes J_l) &= (\mathsf{id} \otimes \Delta^{(l-1)} \otimes \mathsf{id})((\mathsf{id} \otimes \Delta)(J) \cdot (1 \otimes J)) \cdot (1 \otimes J_{l-1} \otimes 1) \\
&= (\mathsf{id} \otimes \Delta^{(l-1)} \otimes \mathsf{id})((\Delta \otimes \mathsf{id})(J) \cdot (J \otimes 1)) \cdot (1 \otimes J_{l-1} \otimes 1) \\
&= (\Delta^{(l)} \otimes \mathsf{id})(J) \cdot (\mathsf{id} \otimes \Delta^{(l-1)} \otimes \mathsf{id})(J \otimes 1) \cdot (1 \otimes J_{l-1} \otimes 1) \\
&= (\Delta^{(l)} \otimes \mathsf{id})(J) \cdot \big(((\mathsf{id} \otimes \Delta^{(l-1)})(J) \cdot (1 \otimes J_{l-1})) \otimes 1\big) \\
&= (\Delta^{(l)} \otimes \mathsf{id})(J) \cdot (J_l \otimes 1) = J_{l+1}.
\end{aligned}$$

Thus we have established the $k = 0$ and $k = 1$ cases completely. To establish the $k \ge 2$ cases, we proceed by induction. Suppose that the formula (4.12) is satisfied for all triples $(\kappa, i, l) \in (\mathbb{Z}_{\ge 0})^3$, where $i \le \kappa$ and $\kappa < k$. Then we have

$$\begin{aligned}
&(\mathsf{id}^{\otimes i} \otimes \Delta^{(l)} \otimes \mathsf{id}^{\otimes k-i})(J_k) \cdot (1^{\otimes i} \otimes J_l \otimes 1^{\otimes k-i}) \\
&\quad = (\mathsf{id}^{\otimes i} \otimes \Delta^{(l)} \otimes \mathsf{id}^{k-i})\big((\mathsf{id} \otimes \Delta^{(k-1)})(J) \cdot (1 \otimes J_{k-1})\big) \cdot (1^{\otimes i} \otimes J_l \otimes 1^{\otimes k-i}) \\
&\quad = (\mathsf{id}^{\otimes i} \otimes \Delta^{(l)} \otimes \mathsf{id}^{\otimes k-i})((\mathsf{id} \otimes \Delta^{(k-1)})(J)) \\
&\qquad \cdot (1 \otimes \big((\mathsf{id}^{\otimes(i-1)} \otimes \Delta^{(l)} \otimes \mathsf{id}^{\otimes k-i})(J_{k-1})\big) \cdot (1^{\otimes i-1} \otimes J_l \otimes 1^{k-i})) \\
&\quad = (\mathsf{id}^{\otimes i} \otimes \Delta^{(l)} \otimes \mathsf{id}^{\otimes k-i})((\mathsf{id} \otimes \Delta^{(k-1)})(J)) \cdot (1 \otimes J_{k+l-1}) \\
&\quad = (\mathsf{id} \otimes \Delta^{(k+l-1)})(J) \cdot (1 \otimes J_{k+l-1}) = J_{k+l}.
\end{aligned}$$

Thus, equation (4.12) is satisfied for all triples (k, i, l). ∎

Proposition 4.9. *The map* $\mathfrak{J} \colon (H_J)_{\mathrm{poly}} \to (H_{\mathrm{poly}})^{J-1\otimes 1}$ *given by* $\mathfrak{J}(P) = J_k \cdot P$ *for* $P \in (H_J)^k_{\mathrm{poly}}$ *is a DGLA isomorphism.*

Proof. First, we note that $\mathcal{J}^{-1}(P) = J_k^{-1} P$ for $P \in H^k_{\mathrm{poly}}$ is obviously an inverse of $\mathcal{J}$. Furthermore, we observe that $\mathcal{J}(1 \otimes 1) = J$, which shows that we only need to check that $\mathcal{J}$ preserves the brackets. This follows by direct computation from Lemmas 4.7 and 4.8. ∎

Definition 4.10 (Deformation symmetry). Let $\mathcal{A}$ be a unital associative algebra over R equipped with a complete filtration $\mathcal{A} = \mathcal{F}^0\mathcal{A} \supset \mathcal{F}^1\mathcal{A} \supset \cdots$ and consider the Hochschild DGLA structure on the complex $C^\bullet(\mathcal{A}) = C^\bullet(\mathcal{A};\mathcal{A})[1]$. A deformation symmetry of H in $\mathcal{A}$ is a map

$$\Phi \colon H_{\mathrm{poly}} \longrightarrow C(\mathcal{A}) \tag{4.16}$$

of L_∞-algebras.

The previous proposition implies the following claim.

Corollary 4.11. *Any formal twist on H produces deformations of all algebras $\mathcal{A}$ which are equipped with a deformation symmetry of H.*

The notion of deformation symmetry is a generalization of the standard notion of universal deformation via Drinfeld twist, see e.g. [5, 19, 23]. Universal deformation formulas rely on the notion of Hopf algebra action that we recall in the following definition.

Definition 4.12 (Hopf algebra action). Let $\mathcal{A}$ be as in Definition 4.10. Then an action of the Hopf algebra H on $\mathcal{A}$ is defined as a map

$$\phi \colon H \otimes \mathcal{A} \longrightarrow \mathcal{A} \tag{4.17}$$

such that

$$\phi \circ (\mathsf{id}_H \otimes \mu) = \mu \circ (\phi \otimes \phi) \circ (\mathsf{id}_H \otimes \tau_{H\otimes\mathcal{A}} \otimes \mathsf{id}_{\mathcal{A}}) \circ (\Delta \otimes \mathsf{id}_{\mathcal{A}\otimes\mathcal{A}}), \tag{4.18}$$

$$\phi \circ (\mu_H \otimes \mathsf{id}_{\mathcal{A}}) = \phi \circ (\mathsf{id}_H \otimes \phi), \tag{4.19}$$

$$\phi \circ (\eta_H \otimes \mathsf{id}_{\mathcal{A}}) = \mu_{\mathcal{A}} \circ (\eta_{\mathcal{A}} \otimes \mathsf{id}_{\mathcal{A}}), \tag{4.20}$$

$$\phi \circ (\mathsf{id}_H \otimes \eta_{\mathcal{A}}) = \eta_{\mathcal{A}} \circ \mu_{\mathsf{R}} \circ (\varepsilon \otimes \mathsf{id}_{\mathsf{R}}). \tag{4.21}$$

Here μ denotes the multiplication of $\mathcal{A}$, μ_{R} denotes the multiplication of R, μ_H denotes the multiplication of H, $\eta_{\mathcal{A}} \colon \mathsf{R} \to \mathcal{A}$ denotes the unit of $\mathcal{A}$, $\eta_H \colon \mathsf{R} \to H$ denotes the unit of H and $\tau_{H\otimes\mathcal{A}}$ denotes the flip $h \otimes a \mapsto a \otimes h$.

Note that ϕ can be regarded as a map $H \to \mathsf{End}_{\mathsf{R}}(\mathcal{A}) = C^1(\mathcal{A};\mathcal{A})$ satisfying certain conditions.

Proposition 4.13. *Given a Hopf algebra action ϕ of H on $\mathcal{A}$, the map Φ defined by*

$$h_0 \otimes \cdots \otimes h_k \mapsto \mu_{\mathcal{A}}^{(k)} \circ (\phi(h_0) \otimes \phi(h_1) \otimes \cdots \otimes \phi(h_k)) \tag{4.22}$$

is a deformation symmetry. Here $\mu_{\mathcal{A}}^{(k)}$ denotes the k-th iteration $\mu_{\mathcal{A}} \circ (\mathsf{id} \otimes \mu_{\mathcal{A}}) \circ \cdots \circ (\mathsf{id}^{\otimes k-1} \otimes \mu_{\mathcal{A}})$ of $\mu_{\mathcal{A}}$.

Proof. We prove this proposition by showing that Φ is a map of DGLA's. Note that for $P = P_0 \otimes \cdots \otimes P_k$, $Q = Q_0 \otimes \cdots \otimes Q_l$ and $a = a_0 \otimes \cdots \otimes a_{k+l}$ we have

$$\begin{aligned}
\Phi(P \bullet Q)(a) &= \Phi\Big(\sum_{i=0}^{k}(-1)^{il} P_0 \otimes \cdots \otimes P_{i-1} \otimes \Delta^{(l)}(P_i)Q \otimes P_{i+1} \otimes \cdots \otimes P_k\Big)(a) \\
&= \sum_{i=0}^{k}(-1)^{il}\phi(P_0)(a_0)\cdots\phi(P_i^{(0)}Q_0)(a_i)\cdots\phi(P_i^{(l)}Q_l)(a_{i+l}) \\
&\quad \cdot \phi(P_{i+1})(a_{i+1+l})\cdots\phi(P_k)(a_{k+l}) \\
&= \sum_{i=0}^{k}(-1)^{il}\phi(P_0)(a_0)\cdots\phi(P_i)(\phi(Q_0)(a_i)\cdots\phi(Q_l)(a_{l+i})) \\
&\quad \cdot \phi(P_{i+1})(a_{l+i+1})\cdots\phi(P_k)(a_{l+k}) \\
&= \sum_{i=0}^{k}(-1)^{il}\phi(P_0)(a_0)\cdots\phi(P_{i-1})(a_{i-1}) \\
&\quad \cdot \phi(P_i)(\Phi(Q)(a_i \otimes \cdots \otimes a_{i+l}) \cdot \phi(P_{i+1})(a_{l+i+1})\cdots\phi(P_k)(a_{k+l}) \\
&= \sum_{i=0}^{k}(-1)^{il}\Phi(P)\big(a_0 \otimes \cdots \otimes a_{i-1} \otimes \Phi(Q)(a_i \otimes \cdots \otimes a_{i+l}) \\
&\quad \otimes a_{i+l+1} \otimes \cdots \otimes a_{k+l}\big) \\
&= \Phi(P)\{\Phi(Q)\}(a),
\end{aligned}$$

where we have used the *brace* notation, see e.g. [35]. Note that $[A, B]_G = A\{B\} - (-1)^{|A||B|}B\{A\}$. In short, the above computation boils down to

$$\Phi(P \bullet Q) = \Phi(P)\{\Phi(Q)\}. \tag{4.23}$$

Thus, Φ respects the brackets and with $\Phi(1 \otimes 1) = \mu_{\mathcal{A}}$ the proposition is proved. ∎

Remark 4.14. Recall that $TH[1]$ is endowed with a brace algebra structure, as described in Remark 4.5. The above proposition actually follows from the fact that a Hopf algebra action ϕ of H on $\mathcal{A}$ actually induces a brace algebra morphism

$$\Phi\colon TH[1] \longrightarrow C^{\bullet}(\mathcal{A}), \tag{4.24}$$

where the braces on $C^{\bullet}(\mathcal{A})$ are defined as usual by

$$\begin{aligned}
&P\{Q_1, \ldots, Q_r\}(a_0 \otimes \cdots \otimes a_{k+k_1+\cdots+k_r}) \\
&\quad = \sum_{0 \le i_1 < i_2 < \cdots < i_r \le k} (-1)^{i_1k_1+i_2k_2+\cdots+i_rk_r} P\big(a_0 \otimes \cdots \otimes Q_1(a_{i_1} \otimes \cdots \otimes a_{i_1+k_1}) \\
&\quad\quad \otimes a_{i_1+k_1+1} \otimes \cdots \otimes a_{i_2+k_1-1} \otimes Q_2(a_{i_2+k_1} \otimes \cdots \otimes a_{i_2+k_1+k_2}) \otimes \cdots \\
&\quad\quad \otimes Q_r(a_{i_r+k_1+\cdots+k_{r-1}} \otimes a_{i_r+k_1+\cdots+k_r}) \otimes \cdots \otimes a_{k+k_1+\cdots+k_r}\big),
\end{aligned}$$

for $P \in C^{k+1}(\mathcal{A};\mathcal{A})$, $Q_j \in C^{k_j+1}(\mathcal{A};\mathcal{A})$ for all j and $a_i \in \mathcal{A}$ for all i. So we have

$$\Phi(P\langle Q_1,\dots,Q_r\rangle) = \Phi(P)\{\Phi(Q_1),\dots,\Phi(Q_r)\} \tag{4.25}$$

for all $P, Q_1,\dots,Q_r \in TH[1]$.

4.2. Twisting Poisson actions

A Lie bialgebra is a pair $(\mathfrak{g},\gamma)$ where $\mathfrak{g}$ is a Lie algebra and γ is a 1-cocycle, $\gamma\colon \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$. In this paper we consider a particular class of Lie bialgebras. Recall that an element $r \in \mathfrak{g}\wedge\mathfrak{g}$ is called r-matrix if it satisfies the Maurer–Cartan equation $[\![r,r]\!] = 0$. It can be proved that r-matrices always induce a Lie bialgebra structure on $\mathfrak{g}$, by setting $\gamma = [\![r,\cdot]\!]$. We refer to the pair $(\mathfrak{g}, r)$ as *triangular Lie algebra*. For further details on Lie bialgebras we refer to [27].

Let us consider a Lie algebra action $\varphi : \mathfrak{g} \to \Gamma^\infty(TM)$.

Definition 4.15 (Poisson action). The action φ is Poisson if it satisfies

$$\mathrm{d}_\pi\varphi(X) = \varphi\wedge\varphi\circ\gamma(X), \tag{4.26}$$

where $\mathrm{d}_\pi = [\![\pi,\cdot]\!]$.

Proposition 4.16. *Let $\mathfrak{g}$ be a finite dimensional triangular Lie algebra and $\varphi : \mathfrak{g} \to \Gamma^\infty(TM)$ a Lie algebra action.*

(i) *The bitensor π defined as the image of r via φ is a Poisson tensor.*

(ii) *φ is a Poisson action with respect to $\pi = \varphi\wedge\varphi(r)$.*

Proof. Let us consider the r-matrix $r = \frac{1}{2}\sum_{i,j} r^{ij} e_i \wedge e_j$ and define

$$\pi := \frac{1}{2}\sum_{i,j} r^{ij}\varphi(e_i)\wedge\varphi(e_j). \tag{4.27}$$

From $[\![r,r]\!] = 0$, using the fact that φ is a Lie algebra morphism, it follows that $[\![\pi,\pi]\!] = 0$. The second claim is a straightforward computation. ∎

It is easy to see that the notion of Poisson action can be extended to a morphism of DGLA's

$$\wedge^\bullet\varphi : (\wedge^\bullet\mathfrak{g},\gamma,[\cdot,\cdot]) \longrightarrow (\Gamma^\infty(\wedge^\bullet TM),\mathrm{d}_\pi,[\![\cdot,\cdot]\!]). \tag{4.28}$$

We now show how one may use the formality of ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$ and $\mathrm{D}_{\mathrm{poly}}(M)$ to obtain a deformation symmetry of $\mathcal{U}(\mathfrak{g})$ in $\mathcal{C}^\infty(M)$ given an infinitesimal action φ of $\mathfrak{g}$ on M. Thus, given an r-matrix of $\mathfrak{g}$, we obtain deformations of all relevant structures and we comment on these. In Section 3 we have obtained the *formal* DGLA's ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$ and $\mathrm{D}_{\mathrm{poly}}(M)$. The classical r-matrix $r \in {}^{\mathfrak{g}}\mathrm{T}^1_{\mathrm{poly}}$ yields a Maurer–Cartan element. Although it satisfies the Maurer–Cartan equation $[\![r,r]\!] = 0$ it is in fact not an MC element according to our Definition 2.12 as we have neglected to consider any filtration. The filtration is needed since we go through formality, which means we encounter infinite sums. More precisely, we obtain a filtration by considering the formal power series ring $\mathbb{R}[\![\hbar]\!]$. Given a DGLA $\mathfrak{L}$ we denote the DGLA obtained by extending scalars to the formal power series ring by $\mathfrak{L}[\![\hbar]\!]$. We consider these DGLA's as filtered by the degree in $\hbar$. Note that, given L_∞-morphisms of DGLA's $\mathfrak{L} \to \mathfrak{L}'$, we obtain also L_∞-morphisms of the extended DGLA's $\mathfrak{L}[\![\hbar]\!] \to \mathfrak{L}'[\![\hbar]\!]$. In this way, L_∞-quasi-isomorphism go to L_∞-quasi-isomorphisms.

From Section 3 it follows that we have a *horse-shoe* diagram in the category of L_∞-algebras

$$\begin{array}{ccc} {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}}[\![\hbar]\!] & \xrightarrow{\ \varphi\ } & \mathrm{T}_{\mathrm{poly}}(M)[\![\hbar]\!] \\ {\scriptstyle F_{\mathfrak{g}}}\big\downarrow & & \big\downarrow{\scriptstyle F_M} \\ {}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}[\![\hbar]\!] & & \mathrm{D}_{\mathrm{poly}}(M)[\![\hbar]\!]. \end{array} \tag{4.29}$$

Here the vertical maps $F_{\mathfrak{g}}$ and F_M are L_∞-quasi-isomorphisms constructed as discussed in Section 3 and the horizontal arrow is induced by φ as in (4.28). Thus we obtain, given an r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$, the MC elements

$$\begin{aligned} \hbar r &\in {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}}[\![\hbar]\!], \\ \hbar\pi &= (\hbar r)_\varphi \in \mathrm{T}_{\mathrm{poly}}(M)[\![\hbar]\!], \\ \rho_\hbar &:= (\hbar r)_{F_{\mathfrak{g}}} \in {}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}[\![\hbar]\!], \\ B_\hbar &:= (\hbar\pi)_{F_M} \in \mathrm{D}_{\mathrm{poly}}(M)[\![\hbar]\!]. \end{aligned} \tag{4.30}$$

Remark 4.17. Notice that we obtain the deformed versions of $\mathcal{C}^\infty(M)$ and $\mathcal{U}(\mathfrak{g})$ by twisting by the Maurer–Cartan elements listed above. We would like then to obtain a deformation symmetry completing the square in the horse-shoe diagram above, such that it commutes. Commutativity ensures that the two formal deformations of $\mathcal{C}^\infty(M)$ induced by $\hbar r$ by transporting it along the bottom or the top to $\mathrm{D}_{\mathrm{poly}}(M)[\![\hbar]\!]$ coincide. An obvious candidate for such a map would be

$$X_1 \otimes \cdots \otimes X_k \mapsto \varphi(X_1) \otimes \cdots \otimes \varphi(X_k), \tag{4.31}$$

i.e. the map induced by the map of Lie–Rinehart pairs $(\mathfrak{g}, \mathbb{R}) \to (\Gamma^\infty(TM), \mathcal{C}^\infty(M))$. In other terms, the deformation symmetry is induced by the obvious Hopf algebra action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{C}^\infty(M)$ through Proposition 4.13. However, this map may not make always the diagram commute. In fact, it is the opinion of the authors that such commutation would involve some condition of compatibility of the connections used in defining these maps, see Remark 3.7.

We complete the horse-shoe diagram (4.29) to a commuting square by observing that L_∞-quasi-isomorphisms are *invertible*. The following lemma is essentially contained in [28, Chapt. 10.4]. We remind the reader that the field underlying L_∞-algebras is of characteristic 0.

Lemma 4.18. *Suppose $\mathfrak{L}$ and $\mathfrak{L}'$ are formal L_∞-algebras and $f\colon H(\mathfrak{L}) \to H(\mathfrak{L}')$ is an L_∞-morphism. Then there exists a lift $\tilde{f}\colon \mathfrak{L} \to \mathfrak{L}'$ of f. In other words, there exists a commuting diagram*

$$\begin{array}{ccc} H(\mathfrak{L}) & \xrightarrow{f} & H(\mathfrak{L}') \\ {\scriptstyle i_l}\downarrow & & \downarrow{\scriptstyle i_r} \\ \mathfrak{L} & \xrightarrow[\tilde{f}]{} & \mathfrak{L}'. \end{array} \tag{4.32}$$

in the category of L_∞-algebras such that the vertical arrows are quasi-isomorphisms.

Proof. Note that we start by hypothesis with the horse-shoe

$$\begin{array}{ccc} H(\mathfrak{L}) & \xrightarrow{f} & H(\mathfrak{L}') \\ {\scriptstyle i_l}\downarrow & & \downarrow{\scriptstyle i_r} \\ \mathfrak{L} & & \mathfrak{L}'. \end{array} \tag{4.33}$$

in the category of L_∞-algebras such that the vertical arrows are quasi-isomorphisms. As shown in [28, Sect. 10.4.4] we can always find a quasi-inverse $l_i\colon \mathfrak{L} \to H(\mathfrak{L})$ of i_l such that the induced maps in cohomology are inverse to each other. We define $\tilde{f} := i_r \circ f \circ l_i$. Note that this already proves that $\tilde{f} \circ i_l = i_r \circ f$ in cohomology (and thus "up to homotopy"). To get the stronger statement in the lemma we will need to consider the construction of the map l_i. This construction involves the notion of a so-called ∞-isomorphism. An ∞-isomorphism is a morphism of L_∞-algebras F such that F_1 is an *isomorphism*. The main observations for the construction of l_i are two-fold. First, the homotopy transfer theorem [28, Sect. 10.3] yields an L_∞-structure on $H(\mathfrak{L})$ which is unique up to ∞-isomorphism. It is obtained by picking a retraction of $\mathfrak{L}$ onto $H(\mathfrak{L})$, we may pick the retraction given by i_l. Secondly, in [28, Sect. 10.4.2] it is shown that any L_∞-algebra $\mathfrak{L}$ is ∞-isomorphic to the sum $H(\mathfrak{L}) \oplus K$, where K is an acyclic chain complex (with trivial Q_0, Q_2, Q_3 and so on). Now the construction of l_i follows by the fact that ∞-isomorphisms are invertible (shown in [28, Sect. 10.4.1]). Our lemma follows from the fact that, for a formal L_∞-algebra the L_∞-structure induced on $H(\mathfrak{L})$ by homotopy transfer equals the canonically induced structure up to ∞-isomorphism. Thus we may simply consider the splittings $\mathfrak{L} \simeq H(\mathfrak{L}) \oplus K_{\mathfrak{L}}$ and $\mathfrak{L}' \simeq H(\mathfrak{L}') \oplus K_{\mathfrak{L}'}$, where $\simeq$ means ∞-isomorphism, given by the retractions induced by i_l and i_r. Then the map $\tilde{f}$ simply maps $H(\mathfrak{L})$ to $H(\mathfrak{L}')$ by f. ∎

Corollary 4.19. *The diagrams*

$$\begin{array}{ccc} {}^{\mathfrak{g}}\mathrm{T}_{\mathrm{poly}}[\![\hbar]\!] & \xrightarrow{\varphi} & \mathrm{T}_{\mathrm{poly}}(M)[\![\hbar]\!] \\ {\scriptstyle F_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle F_M} \\ {}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}[\![\hbar]\!] & \xrightarrow{\tilde{\varphi}} & \mathrm{D}_{\mathrm{poly}}(M)[\![\hbar]\!] \end{array} \tag{4.34}$$

and

$$\begin{array}{ccc} {}^{\mathfrak{g}}\mathrm{T}^{\hbar r}_{\mathrm{poly}}[\![\hbar]\!] & \xrightarrow{\varphi^{\hbar r}} & \mathrm{T}^{\hbar\pi}_{\mathrm{poly}}(M)[\![\hbar]\!] \\ F_{\mathfrak{g}}^{\hbar r}\Big\downarrow & & \Big\downarrow F_M^{\hbar\pi} \\ {}^{\mathfrak{g}}\mathrm{D}^{\rho_\hbar}_{\mathrm{poly}}[\![\hbar]\!] & \xrightarrow{\tilde\varphi^{\rho_\hbar}} & \mathrm{D}^{B_\hbar}_{\mathrm{poly}}(M)[\![\hbar]\!] \end{array} \tag{4.35}$$

commute.

Proof. Applying the above lemma to our situation we immediately find that the diagram (4.34) commutes. Also, by applying the results of Section 3.2 we obtain diagram (4.35), which commutes thanks to Remark 3.18. ■

4.3. Twisted structures

In the following, we show that the twisted complexes obtained above are coming from a formal deformation quantization of $\mathcal{C}^\infty(M)$ (in the case of $\mathrm{D}^{B_\hbar}_{\mathrm{poly}}(M)$) and a deformation of $\mathcal{U}(\mathfrak{g})$ into a quantum group (in the case of ${}^{\mathfrak{g}}\mathrm{D}^{\rho_\hbar}_{\mathrm{poly}}$).

Proposition 4.20. *There is a formal deformation quantization $\mathcal{A}_\hbar$ of $(\mathcal{C}^\infty(M), \pi)$ such that*

$$\mathrm{D}^{B_\hbar}_{\mathrm{poly}}(M) \hookrightarrow C^\bullet(\mathcal{A}_\hbar, \mathcal{A}_\hbar), \tag{4.36}$$

i.e. $\mathrm{D}^{B_\hbar}_{\mathrm{poly}}(M)$ is a subcomplex of the Hochschild cochain complex of $\mathcal{A}_\hbar$.

Proof. Note that the Maurer–Cartan equation (2.17) matches exactly the associativity condition of $m + B_\hbar$, where m denotes the pointwise multiplication in $\mathcal{C}^\infty(M)[\![\hbar]\!]$. Since

$$B_\hbar = \sum_{n\geq 1} \frac{1}{n!}(F_M)_n(\hbar\pi, \dots, \hbar\pi) = \sum_{n\geq 1} \frac{\hbar^n}{n!}(F_M)_n(\pi, \dots, \pi), \tag{4.37}$$

we see that $m + B_\hbar$ defines a deformation quantization $\mathcal{A}_\hbar$. The differential on the Hochschild complex $C^\bullet(\mathcal{A}, \mathcal{A})$ is given by taking the Gerstenhaber bracket with the multiplication for any associative algebra $\mathcal{A}$. Thus, the twisted differential on $\mathrm{D}^{B_\hbar}_{\mathrm{poly}}(M)$ coincides with the differential of $C^\bullet(\mathcal{A}_\hbar, \mathcal{A}_\hbar)$. Finally, we recall that

$$B_\hbar = \sum_{n\geq 1} \frac{\hbar^n}{n!}(F_M)_n(\pi, \pi, \dots, \pi) = \hbar(F_M)_1(\pi) \mod \hbar^2. \tag{4.38}$$

Since $(F_M)_1$ is a quasi-isomorphism we find that the alternating part of $B_\hbar$ is $\hbar\pi$ modulo $\hbar^2$ and

$$\frac{[f, g]_\star}{\hbar} = \pi(f, g) + O(\hbar) \tag{4.39}$$

for all $f, g \in \mathcal{C}^\infty(M)$. Here $[\cdot, \cdot]_\star$ denotes the commutator bracket of $\mathcal{A}_\hbar$. ■

Thus, we obtain in particular the deformation quantization $\mathcal{A}_\hbar$. On the other hand, we also obtain the MC element $\rho_\hbar$ in ${}^{\mathfrak{g}}\mathrm{D}_{\mathrm{poly}}$. This yields the coproduct $\Delta_{J_\hbar}$ (where $J_\hbar = 1 \otimes 1 + \rho_\hbar$) thus establishing a quantum group, obtained by quantization of the Lie bialgebra $(\mathfrak{g}, [\![r, \cdot]\!])$. Finally, we obtain the following theorem as a corollary of Proposition 4.9 and Corollary 4.19.

Theorem 4.21. *Suppose $(\mathfrak{g}, r)$ is a triangular Lie algebra and $\varphi\colon \mathfrak{g} \to \Gamma^\infty(TM)$ is an action on the manifold M. Then there exist a formal deformation quantization $\mathcal{A}_\hbar$ of (M, π) (where $\varphi \wedge \varphi(r) = \pi$) and a quantization $\mathcal{U}_{\rho_\hbar}(\mathfrak{g})$ of $(\mathfrak{g}, r)$ which allow a deformation symmetry*

$$\tilde{\varphi}^{\rho_\hbar} \circ \mathcal{J}\colon \mathcal{U}_{\rho_\hbar}(\mathfrak{g})_{\mathrm{poly}} \longrightarrow C(\mathcal{A}_\hbar; \mathcal{A}_\hbar). \tag{4.40}$$

4.4. Comparison with Drinfeld's construction

First, let us briefly recall the original construction of Drinfeld (see [3, 15, 23]). Consider a formal twist J on $\mathcal{U}(\mathfrak{g})[\![\hbar]\!]$ and a generic $\mathcal{U}(\mathfrak{g})$-module algebra $\mathcal{A}$. Drinfeld proved that we can then always define an associative star product on $\mathcal{A}$. In particular, consider $\mathcal{A} = \mathcal{C}^\infty(M)$ with pointwise multiplication m. Given a Lie algebra action $\varphi : \mathfrak{g} \to \Gamma^\infty(TM)$ we obtain a Hopf algebra action

$$\triangleright\colon \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M), \tag{4.41}$$

which makes $\mathcal{C}^\infty(M)$ into a left $\mathcal{U}(\mathfrak{g})$-module algebra. More precisely, $X \triangleright f = \mathcal{L}_{\varphi(X)} f$, where $\mathcal{L}$ denotes the Lie derivative. The action $\triangleright$ extends to formal power series

$$\triangleright\colon \mathcal{U}(\mathfrak{g})[\![\hbar]\!] \times \mathcal{C}^\infty(M)[\![\hbar]\!] \longrightarrow \mathcal{C}^\infty(M)[\![\hbar]\!]. \tag{4.42}$$

Thus, the product defined by

$$f \star_J g = m(J \triangleright (f \otimes g)) = m(\mathcal{L}_{\varphi\otimes\varphi(J)}(f \otimes g)) \tag{4.43}$$

for $f, g \in \mathcal{C}^\infty(M)[\![\hbar]\!]$ is a star product. The classical limit of (4.43) is given by

$$\{f, g\} = m(r \triangleright (f \otimes g)), \tag{4.44}$$

where $r := \frac{J-\tau(J)}{\hbar}|_{\hbar=0}$ is the r-matrix associated to the twist J, here τ denotes the flip $X \otimes Y \mapsto Y \otimes X$. It is important to underline that the deformed algebra $(\mathcal{C}^\infty(M)[\![\hbar]\!], \star_J)$ is then a module-algebra for the quantum group

$$\mathcal{U}_J(\mathfrak{g}) := (\mathcal{U}(\mathfrak{g})[\![\hbar]\!], \Delta_J). \tag{4.45}$$

In other words, the action $\triangleright$ is a Hopf algebra action of the twisted Hopf algebra $\mathcal{U}_J(\mathfrak{g})$ on $(\mathcal{C}^\infty(M)[\![\hbar]\!], \star_J)$.

Thus, the construction takes a formal twist $J \in \mathcal{U}(\mathfrak{g})[\![\hbar]\!]$ and an infinitesimal action of $\mathfrak{g}$ on M as input and produces a deformation quantization $\mathcal{A}_\hbar$ together with an action of the quantum group $\mathcal{U}_J(\mathfrak{g})$ on it. In our approach one starts with an r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$

and an infinitesimal action of $\mathfrak{g}$ on M and obtains a formal twist J and a deformation symmetry $\widetilde{\varphi}$. These then also yield a deformation quantization given by

$$f \star_r g = \widetilde{\varphi}(J)(f \otimes g), \tag{4.46}$$

and another deformation symmetry $\widetilde{\varphi}^{\rho_\hbar} \circ \mathfrak{J}$ of the quantum group $\mathcal{U}_J(\mathfrak{g})$ in the deformed algebra $\mathcal{A}_\hbar$. The main difference of the two approaches is that the formal twist is taken as given in Drinfeld's approach while we obtain it through quantization of an r-matrix. The trade-off is, however, that we do not obviously obtain an action of a quantum group anymore, instead we obtain the deformation symmetry. A direct comparison of the two approaches will be nontrivial and it implies a study of the compatibility condition between connections mentioned in Remark 4.17. In particular, it is of interest whether the process of quantization so obtained can be made functorial for equivariant maps between the manifolds. We will come back to this in a future project.

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