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Immersions and the unbounded Kasparov product: embedding spheres into Euclidean space

Walter D. van Suijlekom and Luuk S. Verhoeven

Abstract. We construct an unbounded representative for the shriek class associated to the embeddings of spheres into Euclidean space. We equip this unbounded KK-cycle with a connection and compute the unbounded Kasparov product with the Dirac operator on $\mathbb{R}^{n+1}$. We find that the resulting spectral triple for the algebra $C(\mathbb{S}^n)$ differs from the Dirac operator on the round sphere by a so-called index cycle, whose class in $KK_0(\mathbb{C},\mathbb{C})$ represents the multiplicative unit. At all points we check that our construction involving the unbounded Kasparov product is compatible with the bounded Kasparov product using Kucerovsky's criterion and we thus capture the composition law for the shriek map for these immersions at the unbounded KK-theoretical level, while retaining the geometric information.

1. Introduction

In their 1984 paper on the longitudinal index theorem for foliations [6], Connes and Skandalis prove the wrong-way functoriality of the shriek map. The shriek, or wrong-way, map is a class $f_! \in KK(C(X),C(Y))$ associated to a K-oriented map $f : X \to Y$ [4]. Indeed, if $f : X \to Y$ and $g : Y \to Z$, then we have that

$$(g \circ f)_! = f_! \otimes_{C(Y)} g_!,$$

where $\otimes_{C(Y)}$ denotes the internal Kasparov product over $C(Y)$.

An interesting special case of the shriek map is the fundamental class $[X] \in KK(C(X),\mathbb{C})$ of a manifold, which is the shriek of the point map $\mathrm{pt}_X : X \to \{*\}$. Hence, whenever we have a K-oriented map $f : X \to Y$ we get a KK-theoretic factorization of fundamental classes

$$[X] = f_! \otimes_{C(Y)} [Y].$$

This is relevant to noncommutative geometry, since the canonical spectral triple of a manifold [5] is an unbounded representative for the fundamental class. The construction of $f_!$ given in [6] already has a strong unbounded character, so it seems natural to investigate how this factorization of spectral triples can be realized concretely in terms of unbounded KK-cycles in the sense of [1]. In fact, the advantage of working at the unbounded level is

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that geometric information remains in tact, while the relation with (bounded) KK-theory ensures that one is indeed refining the topological description of the pertinent spaces and maps between them.

When $\pi : M \to B$ is a submersion of compact manifolds, this factorization has already been investigated in [9]. There a vertical family of Dirac operators D_π was constructed, such that the Dirac operator D_M on M decomposes as the tensor sum

$$D_M = D_\pi \otimes 1 + 1 \otimes_\nabla D_B + \kappa, \tag{1.1}$$

in terms of the Dirac operator D_B on the base B lifted to M using a connection ∇, and a bounded operator κ which is related to the curvature of π. We stress that the geometrically valuable information encoded by the operator κ will disappear when passing to bounded KK-theory, in line with the remark in the previous paragraph.

When the map in question is an immersion $\iota : M \to N$, a similar factorization of Dirac operators should be available. Namely, it should be possible to write the Dirac operator D_M as an unbounded Kasparov product (or unbounded KK-product) of a shriek element corresponding to ι and the Dirac operator D_N. However, for this to work it is crucial to somehow be able to remove the vertical, or normal, part of the Dirac operator from D_N. Inspired by the bounded construction here the key ingredient is a Dirac-dual Dirac approach as in [10]; see also [7].

In this article, we will investigate whether, and how, this factorization works for a simple and concrete set of immersions given by the embeddings $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ for $n \geq 1$. We start by introducing and constructing the primary ingredients: the unbounded representatives of $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$, and the unbounded shriek cycle of ι which we also relate to the bounded shriek class $\iota_!$ constructed in [6]. Next, we investigate the interpretation of the shriek cycle as a dual Dirac, which yields a fourth unbounded KK-cycle which we will call the index cycle. Its bounded transform—the so-called index class—turns out to represent the multiplicative unit in KK-theory.

Once we have all ingredients, we use a connection on the unbounded shriek cycle to construct a candidate unbounded KK-cycle for the product, very much in the spirit of [8] and [13]. We then use the criterion in [11] to prove that this candidate indeed represents the Kasparov product of $\iota_!$ and $[\mathbb{R}^{n+1}]$ in KK-theory, and that it also represents the product of $[\mathbb{S}^n]$ and the index class, and hence $[\mathbb{S}^n]$ itself. This gives the desired factorization of the given immersion $\iota : \mathbb{S}^n \to R^{n+1}$ in terms of the unbounded KK-product.

2. The geometry of the spheres in Euclidean space

From the construction of the shriek class in [6] it is clear that the canonical spectral triple of a manifold M represents the fundamental class $[M]$ of that manifold in $KK(C_0(M),\mathbb{C})$. Our first goal is writing the Dirac operator for the embedded spin^c submanifold $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, $n \geq 1$, which of course coincides with the Dirac operator on the round sphere $\mathbb{S}^n$. Then we turn to the unbounded shriek cycle and show that its bounded transform is homotopic to the shriek class in $KK(C(\mathbb{S}^n), C_0(\mathbb{R}^{n+1}))$ that was considered in [6].

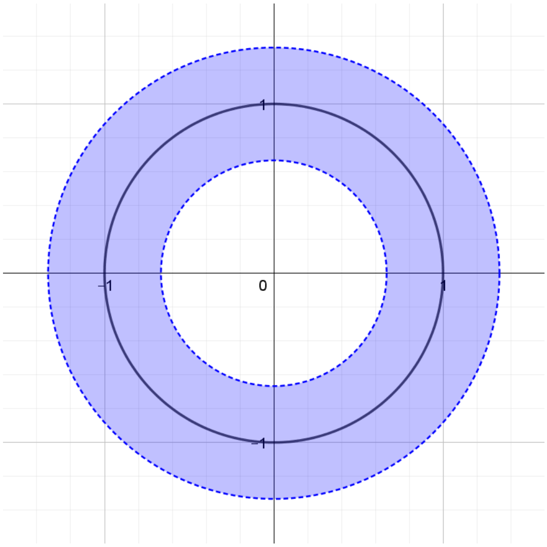

Figure 1. The tubular neighborhood around $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ for $n = 1$: the black line is the image of $\imath : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ and the blue band is the image of $\tilde{\imath} : \mathbb{S}^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$.

2.1. Spin geometry of $\mathbb{S}^n$ spheres and $\mathbb{R}^{n+1}$ Euclidean spaces

The first step in the construction of the Dirac operator on the embedded submanifold $\mathbb{S}^n \subseteq \mathbb{R}^n$ is to investigate the spin^c structure on $\mathbb{S}^n$ induced by restricting the standard spin^c structure on $\mathbb{R}^{n+1}$. This construction is well known (cf. [2, 3]) but we repeat it here in some detail since later on we will refer to some technical aspects of this construction.

Let $\imath : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ be the standard immersion of the n-dimensional sphere into $\mathbb{R}^{n+1}$. Choose some $0 < \varepsilon < 1$ and define a tubular neighborhood of this immersion by $\tilde{\imath} : \mathbb{S}^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ by using geodesic flow along the normal vector field $\partial_r = \frac{1}{r}(x^i \partial_{x^i})$; i.e., we have in spherical coordinates $\tilde{\imath}(\vec{\theta}, s) = (\vec{\theta}, s+1)$ (see Figure 1).

Let $\mathcal{S}$ denote the restriction of the spinor bundle on $\mathbb{R}^{n+1}$ to the image of $\tilde{\imath}$. We can define a Clifford action ρ of $T\mathbb{S}^n$ on $\mathcal{S}$ by setting $\rho(v)\psi = ic(v)c(\partial_r)\psi$, where $v \in T_x\mathbb{S}^n \subset T_x\mathbb{R}^{n+1}$, $\psi \in \mathcal{S}_x$, and c denotes the Clifford multiplication on $\mathbb{R}^{n+1}$. We will also write $\gamma_r = c(\partial_r)$.

In order to describe the induced spinor bundle on $\mathbb{S}^n$ explicitly, we need to distinguish between the odd- and the even-dimensional case.

Odd spheres

If n is odd, say $n = 2k-1$, the restriction of $\mathcal{S}$ to $\mathbb{S}^{2k-1}$ does not immediately yield a spinor bundle. But since in this case $n+1$ is even, $\mathcal{S}$ has a grading operator Γ which decomposes $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ into an even and an odd part (which are isomorphic). The decomposition along Γ is preserved by ρ, and the restriction ρ^+ of ρ to $\mathcal{S}^+$ turns $\mathcal{S}^+$ restricted to $\mathbb{S}^{2k-1}$ into a spinor bundle on $\mathbb{S}^{2k-1}$ [2, 3].

Using this spin^c structure on $\mathbb{S}^{2k-1}$, we get a Dirac operator $D^+_{\mathbb{S}^{2k-1}}$. In accordance to the discussion in Appendix A, we want to turn this into an even cycle, and we choose to use left doubling in this case to obtain

$$\big(L^2(\mathbb{S}^{2k-1}, \mathcal{S}^+|_{\mathbb{S}^{2k-1}}) \otimes \mathbb{C}^2, \widetilde{D_{\mathbb{S}^{2k-1}}} := D^+_{\mathbb{S}^{2k-1}} \otimes \gamma^2; 1 \otimes \gamma^3\big) \tag{2.1}$$

as an even unbounded $C(\mathbb{S}^{2k-1}) \otimes \mathbb{C}l_1$-$\mathbb{C}$ KK-cycle.

Remark 2.1. Note that equivalently we could have taken $\mathcal{S}^-$ as our defining spin^c structure, this would have yielded a different Dirac operator $D^-_{\mathbb{S}^{2k-1}}$. Under the isomorphism $\mathcal{S}^+ \cong \mathcal{S}^-$ given by γ_r we would have that $D^+_{\mathbb{S}^{2k-1}} = -D^-_{\mathbb{S}^{2k-1}}$.

The fact that the spin^c structure on $\mathbb{S}^{2k-1}$ is induced from $\mathbb{R}^{2k}$ allows us to relate the Dirac operators of $\mathbb{R}^{2k}$ and $\mathbb{S}^{2k-1}$. Choosing frames for $\mathcal{S}$ to identify $\mathcal{S} \cong \mathcal{S}^+ \otimes \mathbb{C}^2$, $\gamma_r \equiv 1 \otimes \gamma^1$, and $\Gamma \equiv 1 \otimes \gamma^3$, we get that

$$D_{\mathbb{R}^{2k}} = i\frac{1}{r} D^+_{\mathbb{S}^{2k-1}} \otimes \gamma^2 + i\frac{2k-1}{2r}(1 \otimes \gamma^1) + i\partial_r(1 \otimes \gamma^1). \tag{2.2}$$

(see also [3] and [2, Section 2]). This represents a class in $KK_0(C_0(\mathbb{R}^{2k+1}), \mathbb{C})$.

Even spheres

If n is even, say $n = 2k$, then $\mathcal{S}|_{\mathbb{S}^{2k}}$ immediately yields a spinor bundle on $\mathbb{S}^{2k}$ which is graded with the grading operator γ_r. So the representative for $[\mathbb{S}^{2k}]$ becomes simply $(L^2(\mathbb{S}^{2k}, \mathcal{S}|_{\mathbb{S}^{2k}}), D_{\mathbb{S}^{2k}}; \gamma_r)$. In this case, the relation between the Dirac operator on $\mathbb{R}^{2k+1}$ and $\mathbb{S}^{2k}$ is given by

$$D_{\mathbb{R}^{2k+1}} = i\frac{1}{r}\gamma_r D_{\mathbb{S}^{2k}} + i\frac{2k}{2r}\gamma_r + i\gamma_r\partial_r. \tag{2.3}$$

Finally, the spectral triple representing the Euclidean space will be the left-doubled version of the canonical spectral triple:

$$\big(L^2(\mathbb{R}^{2k+1}, \mathcal{S}) \otimes \mathbb{C}^2, \widetilde{D_{\mathbb{R}^{2k+1}}} := D_{\mathbb{R}^{2k+1}} \otimes \gamma^2; 1 \otimes \gamma^3\big),$$

representing a class in $KK_0(C_0(\mathbb{R}^{2k+1}) \otimes \mathbb{C}l_1, \mathbb{C})$.

2.2. The shriek class of the immersion

The class in $KK_1(C(\mathbb{S}^n), C_0(\mathbb{R}^{n+1}))$ that we want to associate to $\imath : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the shriek class, or "wrong-way" map. We will start by defining an odd unbounded KK-cycle between $C(\mathbb{S}^{n-1})$ and $C_0(\mathbb{R}^n)$, and then show in the next subsection that the corresponding bounded transform represents the shriek class as constructed by Connes and Skandalis.

Let $\mathcal{E}$ denote the vector space $C_0(\mathbb{S}^n \times (-\varepsilon, \varepsilon))$ and equip it with the $C_0(\mathbb{R}^{n+1})$-valued sesquilinear form

$$\langle \psi, \phi \rangle_{\mathcal{E}}(\vec{\theta}, r) := \begin{cases} \frac{1}{r^n}\overline{\psi}\big(\tilde{\imath}^{-1}(\vec{\theta}, r)\big)\phi\big(\tilde{\imath}^{-1}(\vec{\theta}, r)\big), & (\vec{\theta}, r) \in \tilde{\imath}\big(\mathbb{S}^n \times (-\varepsilon, \varepsilon)\big), \\ 0, & (\vec{\theta}, r) \notin \tilde{\imath}\big(\mathbb{S}^n \times (-\varepsilon, \varepsilon)\big). \end{cases} \tag{2.4}$$

Furthermore, equip $\mathcal{E}$ with a left- and right-action by $C(\mathbb{S}^n)$ and $C_0(\mathbb{R}^{n+1})$, respectively, by setting

$$(g \cdot \psi \cdot h)(\vec{\theta}, s) = g(\vec{\theta})\psi(\vec{\theta}, s)h\big(\tilde{\imath}(\vec{\theta}, s)\big), \tag{2.5}$$

for $g \in C(\mathbb{S}^n)$, $\psi \in \mathcal{E}$, and $h \in C_0(\mathbb{R}^{n+1})$. The reason for including the pre-factor $\frac{1}{r^n}$ in $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ is to "flatten" a neighborhood of the circle in $\mathbb{R}^{n+1}$ to a cylinder, as we will see in full detail in Section 3.1 below.

Lemma 2.2. *The sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ in (2.4) turns $\mathcal{E}$ into a Hilbert $C(\mathbb{S}^n)$-$C_0(\mathbb{R}^{n+1})$ bimodule, with left and right actions as in (2.5).*

Proof. The norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ is $\|\psi\|_{\mathcal{E}} = \|(1+s)^{-n/2}\psi\|_{\sup}$. Since $\frac{1}{\sqrt{1+s}}$ is bounded both from above and away from zero on $(-\varepsilon,\varepsilon)$, this immediately implies that the sesquilinear form is positive definite and that $\mathcal{E}$ is complete. The remaining properties are simple verifications. ∎

The self-adjoint and regular operator for our candidate unbounded KK-cycle representing the shriek class will be the multiplication operator by the function

$$f(s) = \alpha \tan(\alpha s),$$

where $\alpha = \frac{\pi}{2\varepsilon}$. More precisely, define

$$\operatorname{dom}(S) = \{\psi \in \mathcal{E} \,|\, f\psi \in \mathcal{E}\}, \quad (S\psi)(\theta, s) = f(s)\psi(\theta, s). \tag{2.6}$$

Lemma 2.3. *The operator S defined in (2.6) is self-adjoint, regular and has a compact resolvent.*

Proof. For self-adjointness and regularity it suffices to show that $S \pm i$ are surjective. Let $\psi \in \mathcal{E}$, then also $\phi := \frac{1}{f \pm i}\psi \in \mathcal{E}$ since $\frac{1}{f+i}$ is in $C_0(\mathbb{S}^{n-1} \times (-\varepsilon,\varepsilon))$. Clearly, $(S \pm i)\phi = \psi$, and hence $S \pm i$ is surjective.

To see that $(S \pm i)^{-1}$ are compact, recall that $\mathcal{K}(C_0(X)) = C_0(X)$ for any locally compact Hausdorff space X, where $C_0(X)$ is viewed as a Hilbert module over itself. Using the same equivalence of norms we saw in Lemma 2.2 we find that $\mathcal{K}(\mathcal{E}) = C_0(\mathbb{S}^{n-1} \times (-\varepsilon,\varepsilon))$, so that $(S \pm i)^{-1}$ is indeed compact. ∎

Proposition 2.4. *The data $(\mathcal{E}, S)$ defines an odd unbounded KK-cycle between $C(\mathbb{S}^n)$ and $C_0(\mathbb{R}^{n+1})$.*

Proof. We know from Lemma 2.2 that $\mathcal{E}$ is indeed a Hilbert bimodule between $C(\mathbb{S}^{n-1})$ and $C_0(\mathbb{R}^n)$ and from Lemma 2.3 that S has all properties to make $(\mathcal{E}, S)$ into an odd unbounded KK-cycle. ∎

Note that the unbounded KK-cycle $(\mathcal{E}, S)$ can be considered as the pushforward of an unbounded KK-representative for a class in $KK_1(C(\mathbb{S}^n), C_0(\mathbb{S}^n \times (-\varepsilon,\varepsilon)))$ by the inclusion $\imath : \mathbb{S}^n \times (-\varepsilon,\varepsilon) \to \mathbb{R}^{n+1}$.

We now have an unbounded KK-cycle, but we want to add one final piece of data. Namely, for the purpose of computing the product of this unbounded KK-cycle with $[\mathbb{R}^{n+1}]$ we also need a connection on $\mathcal{E}$ relative to $D_{\mathbb{R}^{2k}}$ if $n = 2k-1$ and to $\widetilde{D_{\mathbb{R}^{2k+1}}} = D_{\mathbb{R}^{2k+1}} \otimes \gamma^2$ if $n = 2k$. In the following we write, in an abuse of notation, $D_{\mathbb{R}^{n+1}}$ for both $D_{\mathbb{R}^{2k}}$ and $\widetilde{D_{\mathbb{R}^{2k+1}}}$.

Lemma 2.5. *Define* $\mathrm{dom}(\nabla^{\mathcal{E}}) := C_0^1(\mathbb{S}^n \times (-\varepsilon, \varepsilon))$ *as a dense subset in* $\mathcal{E}$. *Then the map*

$$\nabla^{\mathcal{E}} : \mathrm{dom}(\mathcal{E}) \to \mathcal{E} \otimes_{C_0(\mathbb{R}^{n+1})} \Omega^1_{D_{\mathbb{R}^{n+1}}}$$

given in local spherical coordinates $\vec{\theta} = (\theta^1, \ldots, \theta^n)$ *on* $\mathbb{S}^n$ *by*

$$\nabla^{\mathcal{E}}(\psi) = \left(\frac{\partial \psi}{\partial s} - \frac{n}{2(s+1)}\psi\right) \otimes [D_{\mathbb{R}^{n+1}}, r] + \sum_{i=1}^{n} \frac{\partial \psi}{\partial \theta^i} \otimes [D_{\mathbb{R}^{n+1}}, \theta^i]$$

is a metric connection on $\mathcal{E}$.

Proof. The connection property is a straightforward check. If we write ∇ for the "flat" connection on $\mathcal{E}$, that is, $\nabla^{\mathcal{E}}$ without the $-\frac{n}{2(s+1)}$ term, and $\langle \cdot, \cdot \rangle$ for the "flat" inner product, i.e., without the factor $\frac{1}{r^n}$, it follows from the fact that ∇ is a metric connection for $\langle \cdot, \cdot \rangle$ that

$$\begin{aligned}
&\big\langle \nabla^{\mathcal{E}}(\psi), \phi \big\rangle_{\mathcal{E}} + \big\langle \psi, \nabla^{\mathcal{E}}(\phi) \big\rangle_{\mathcal{E}} \\
&= \frac{1}{r^n}\big\langle \nabla(\psi), \phi \big\rangle + \frac{1}{r^n}\big\langle \nabla(\psi), \phi \big\rangle - \frac{n}{r^{n+1}}\langle \psi, \phi \rangle \otimes [D_{\mathbb{R}^{n+1}}, r] \\
&= \frac{1}{r^n}\big[D_{\mathbb{R}^{n+1}}, \langle \psi, \phi \rangle\big] + \left[D_{\mathbb{R}^{n+1}}, \frac{1}{r^n}\right]\langle \psi, \phi \rangle \\
&= \big[D_{\mathbb{R}^{n+1}}, \langle \psi, \phi \rangle_{\mathcal{E}}\big],
\end{aligned}$$

so that $\nabla^{\mathcal{E}}$ is a metric connection. ∎

As a final preparation for computing the products we need to use even KK-cycles, so we use doubled versions of the index class (cf. Appendix A). In the case where n is odd, we use left doubling, so our shriek cycle becomes

$$(\mathcal{E} \otimes \mathbb{C}^2, S \otimes \gamma^2; 1 \otimes \gamma^3; \nabla \otimes \mathrm{id})$$

representing a class in $KK_0(C(\mathbb{S}^n) \otimes \mathbb{C}l_1, C(\mathbb{R}^{n+1}))$. When n is even, we use right doubling, which makes our shriek cycle

$$(\mathcal{E} \otimes \mathbb{C}^2, S \otimes \gamma^1; 1 \otimes \gamma^3; \nabla \otimes \mathrm{id}),$$

this time representing a class in $KK_0(C(\mathbb{S}^n), C(\mathbb{R}^{n+1}) \otimes \mathbb{C}l_1)$. In both cases, we will denote the unbounded KK-cycle by $(\widetilde{\mathcal{E}}, \widetilde{S})$ and the shriek cycle $(\mathcal{E}, \mathfrak{b}(S))$ obtained in KK-theory by bounded transform by $\iota_!$.

2.3. Equivalence to bounded construction

We will now show that the bounded transform $(\mathcal{E}, \mathfrak{b}(S))$ is homotopic to the shriek cycle as constructed in [6]. This in fact already proves the factorization $[\mathbb{S}^n] = \iota_! \otimes [\mathbb{R}^{n+1}]$ as KK-classes, but we want to prove this factorization in full geometric detail in the unbounded KK-theoretic context.

In [6], one allows any map $\tilde{\iota}_{CS} : \mathbb{S}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ which is a diffeomorphism onto a tubular neighborhood of $\iota(\mathbb{S}^n) \subset \mathbb{R}^{n+1}$. For our purposes, we choose $\tilde{\iota}_{CS}$ such that $\tilde{\iota}_{CS}|_{\mathbb{S}^n \times (-\varepsilon,\varepsilon)} \equiv \tilde{\iota}$.

One defines a $C_0(\mathbb{R}^{n+1})$-valued sesquilinear form $\langle \cdot, \cdot \rangle_{CS}$ on $C_c(\mathbb{S}^n \times \mathbb{R})$ by setting

$$\langle \psi, \phi \rangle_{CS}(x) = \overline{\psi}\big(\tilde{\iota}_{CS}^{-1}(x)\big)\phi\big(\tilde{\iota}_{CS}^{-1}(x)\big)$$

for x in the tubular neighborhood, and $\langle \psi, \phi \rangle_{CS} = 0$ elsewhere. There is a left $C(\mathbb{S}^n)$ action and a right $C_0(\mathbb{R}^{n+1})$ action on $C_c(\mathbb{S}^n \times \mathbb{R})$ given by

$$(g \cdot \psi \cdot h)(\vec{\theta}, s) = g(\vec{\theta})\psi(\vec{\theta}, s)h\big(\tilde{\iota}_{CS}(\vec{\theta}, s)\big).$$

This turns $C_c(\mathbb{S}^n \times \mathbb{R})$ into a pre-Hilbert bimodule; denote by $\mathcal{E}_{CS}$ the corresponding Hilbert $C(\mathbb{S}^n)$-$C_0(\mathbb{R}^{n+1})$-bimodule. It is easy to see that $\mathcal{E}_{CS} = C_0(\mathbb{S}^n \times \mathbb{R})$.

Next, one chooses a function $M : [0, \infty) \to [0, 1]$ such that $M(0) = 1$ and M has a compact support. On $\mathcal{E}_{CS}$ define an operator $F : \mathcal{E}_{CS} \to \mathcal{E}_{CS}$ by

$$(F\psi)(\theta, s) = \sqrt{1 - M\big(|s|\big)}\,\frac{s}{|s|}\psi(\theta, s).$$

For instance, we may choose

$$M(s) = \begin{cases} \frac{1}{1+f(s)^2}, & s \in [0, \varepsilon), \\ 0, & s \geq \varepsilon, \end{cases}$$

so that

$$(F\psi)(\theta, s) = \begin{cases} -\psi(\theta, s), & s \leq -\varepsilon, \\ \frac{f(s)}{\sqrt{1+f(s)^2}}\psi(\theta, s), & s \in (-\varepsilon, \varepsilon), \\ \psi(\theta, s), & s \geq \varepsilon. \end{cases}$$

This already closely resembles $(\mathcal{E}, \mathfrak{b}(S))$; the major difference is that $\mathcal{E}$ uses $(-\varepsilon, \varepsilon)$ as fiber with an operator tending to 1 at the edge, while $\mathcal{E}_{CS}$ uses $\mathbb{R}$ as fiber with an operator that equals 1 outside $(-\varepsilon, \varepsilon)$. However, the two cycles represent the same class in KK-theory because of the following result.

Proposition 2.6. *The two bounded Kasparov cycles $(\mathcal{E}, \mathfrak{b}(S))$ and $(\mathcal{E}, F)$ are homotopic and, consequently, they define the same class in $KK(C(\mathbb{S}^n), C_0(\mathbb{R}^{n+1}))$.*

Proof. We will construct a bounded Kasparov cycle between $C(\mathbb{S}^n)$ and $C_0(\mathbb{R}^{n+1}) \otimes C([0, 1])$ such that evaluation at 0 yields a cycle unitarily equivalent to $(\mathcal{E}, \mathfrak{b}(S))$ and evaluation at 1 yields a cycle equivalent to $(\mathcal{E}_{CS}, F)$.

Let $R : [0, 1) \to \mathbb{R}$ be any increasing function such that $R(0) = \varepsilon$ and $R(x) \to \infty$ as $x \to 1$. Define $X \subset \mathbb{S}^n \times \mathbb{R} \times [0, 1]$ by $(\vec{\theta}, s, t) \in X$ if $t = 1$ or $|s| < R(t)$ for $t < 1$.

Set $\mathcal{F} = C_0(X)$, and define a $C_0(\mathbb{R}^{n+1}) \otimes C([0, 1]) = C_0(\mathbb{R}^{n+1} \times [0, 1])$-valued sesquilinear form on $\mathcal{F}$ by

$$\langle \psi, \phi \rangle_{CS}(\vec{\theta}, r, t) = \begin{cases} \bar{\psi}\big(\tilde{\imath}^{-1}(\vec{\theta}, r), t\big)\phi\big(\tilde{\imath}^{-1}(\vec{\theta}, r), t\big), & (\vec{\theta}, r) \in \tilde{\imath}\big(\mathbb{S}^n \times \big(-R(t), R(t)\big)\big), \\ 0, & (\vec{\theta}, r) \notin \tilde{\imath}\big(\mathbb{S}^n \times \big(-R(t), R(t)\big)\big). \end{cases}$$

We may equip $\mathcal{F}$ with a left-$C(\mathbb{S}^n)$ and a right-$C_0(\mathbb{R}^{n+1} \times [0, 1])$ module structure by setting

$$(f \cdot \psi \cdot g)(\vec{\theta}, s, t) = f(\vec{\theta})\psi(\vec{\theta}, s, t)g\big(\tilde{\imath}_{CS}(\vec{\theta}, s), t\big).$$

Note that the norm on $\mathcal{F}$ induced by this inner product is simply the sup-norm on $C_0(X)$, so that $\mathcal{F}$ is indeed a Hilbert bimodule. Then $\mathcal{L}(\mathcal{F}) = C_b(X)$ and $\mathcal{K}(\mathcal{F}) = C_0(X)$.

Now we define an operator G on $\mathcal{F}$ by

$$(G\psi)(\theta, s, t) = \begin{cases} -\psi(\theta, s, t), & s \leq -\varepsilon, \\ \frac{f(s)}{\sqrt{1+f(s)^2}}\psi(\theta, s, t), & s \in (-\varepsilon, \varepsilon), \\ \psi(\theta, s, t), & s \geq \varepsilon. \end{cases}$$

Note that $G^2 - 1$ is in $\mathcal{K}(\mathcal{F})$ since it is in $C_0(X)$.

We claim that $(\mathcal{F}, G)$ is a homotopy between $(\mathcal{E}, \mathfrak{b}(S))$ and $(\mathcal{E}_{CX}, F)$. Indeed, for $i = 0, 1$ we denote by B_i the Hilbert bimodule corresponding to the C^*-homomorphism $\phi_i : C_0(\mathbb{R}^{n+1} \times [0, 1]) \to C_0(\mathbb{R}^{n+1})$, $\phi(f)(\theta, r) = f(\theta, r, i)$. For the evaluation at $t = 0$ the map

$$\begin{aligned} &U : \mathcal{F} \otimes_{C_0(\mathbb{R}^{n+1}\times[0,1])} B_0 \to \mathcal{E}, \\ &U(\psi \otimes g)(\vec{\theta}, s) = (s+1)^{\frac{n}{2}}\psi(\vec{\theta}, s, 0)g\big(\tilde{\imath}(\vec{\theta}, s)\big) \end{aligned}$$

is a unitary equivalence between $(\mathcal{F} \otimes_{C_0(\mathbb{R}^{n+1}\times[0,1])} B_0, G \otimes 1)$ and $(\mathcal{E}, \mathfrak{b}(S))$.

At $t = 1$ the map

$$\begin{aligned} &V : \mathcal{F} \otimes_{C_0(\mathbb{R}^{n+1}\times[0,1])} B_1 \to \mathcal{E}_{CS} \\ &V(\psi \otimes g)(\vec{\theta}, s) = \psi(\vec{\theta}, s, 1)g\big(\tilde{\imath}_{CS}(\vec{\theta}, s)\big). \end{aligned}$$

is a unitary equivalence between $(\mathcal{F} \otimes_{C_0(\mathbb{R}^{n+1}\times[0,1])} B_1, G \otimes 1)$ and $(\mathcal{E}_{CS}, F)$. ∎

2.4. The index class

In our sought for KK-factorization of $D_{\mathbb{S}^n}$ in terms of $D_{\mathbb{R}^{n+1}}$, the cycle $(\widetilde{\mathcal{E}}, \widetilde{S})$ should in some way cancel out the normal, or radial, direction. This dimension reduction is, in bounded KK-theory, accomplished by a dual-Dirac element, as in [7]. In our case, $\widetilde{S}$ is expected to act as an unbounded dual-Dirac element, and this leads us to investigate the interaction between the radial derivative in $D_{\mathbb{R}^{n+1}}$ and the radial function defining $\widetilde{S}$.

So, let us define a symmetric operator T_0 on

$$\mathrm{dom}(T_0) = C_c^\infty\big((-\varepsilon,\varepsilon),\mathbb{C}^2\big) \subset L^2\big((-\varepsilon,\varepsilon),\mathbb{C}^2\big)$$

by

$$T_0\psi = i\gamma^1\partial_s\psi + \gamma^2 f(s)\psi = \begin{pmatrix} 0 & i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix}\psi,$$

where $f(s) = \alpha\tan(\alpha s)$ and $\alpha = \frac{\pi}{2\varepsilon}$ as before.

We want to show that the closure $T := \overline{T_0}$, together with Hilbert space $L^2((-\varepsilon,\varepsilon),\mathbb{C}^2)$ and grading γ^3, defines an even $\mathbb{C}$-$\mathbb{C}$ KK-cycle, and that this cycle represents the multiplicative unit in $KK_0(\mathbb{C},\mathbb{C})$. We will refer to $(L^2((-\varepsilon,\varepsilon),\mathbb{C}^2), T; \gamma^3)$ as the *index cycle*, and to the corresponding $KK_0(\mathbb{C},\mathbb{C})$ class as the *index class*, which we denote by $\mathbb{1}$.

In order to prove essential self-adjointness of T_0, we first find integrating factors I and J for the differential equation $(T_0 + \lambda i)u = g$. We then use these integrating factors to show that $\mathrm{ran}(T_0 + \lambda i)$ is the "orthogonal complement" of J and finally we show that this is dense. This argument is based on [12, Example 33.1].

Lemma 2.7. *Suppose that $u, g \in C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$ and $\lambda^2 = \alpha^2$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x} I_\lambda u = J_\lambda g$$

if and only if $g = (T_0 + \lambda i)u$, for

$$I_\lambda(s) = \begin{pmatrix} 1 + sf(s) & \lambda s \\ \frac{1}{\lambda} f(s) & 1 \end{pmatrix}, \quad J_\lambda(s) = -i\begin{pmatrix} \lambda s & 1 + sf(s) \\ 1 & \frac{1}{\lambda} f(s) \end{pmatrix}.$$

Proof. Using the differential equation

$$f(s)^2 - f'(s) + \alpha^2 = 0$$

which is satisfied by $f(s) = \alpha\tan(\alpha s)$, it is straightforward to show that

$$J_\lambda^{-1}\frac{\mathrm{d}}{\mathrm{d}x} I_\lambda = (T_0 + \lambda i). \qquad \blacksquare$$

The next step is to show that the range of $T_0 + \lambda i$ is the "orthogonal complement" of J_λ in $C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C})$.

Lemma 2.8. *For $\lambda = \pm\alpha$ and J_λ as in Lemma 2.7 one has that*

$$\mathrm{ran}(T_0 + \lambda i) = \left\{ g \in C_c^\infty\big((-\varepsilon,\varepsilon),\mathbb{C}^2\big) \;\middle|\; \int_{-\varepsilon}^{\varepsilon} J(x)g(x)\,\mathrm{d}x = 0 \right\}.$$

Proof. Suppose that $g = (T_0 + \lambda i)u$. Then by Lemma 2.7

$$\int_{-\varepsilon}^{\varepsilon} J(x)g(x)\,\mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} \left(\frac{\mathrm{d}}{\mathrm{d}x} I(x)u(x)\right)\mathrm{d}x = 0,$$

since $u \in C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$. Also, g is indeed in $C_c^\infty((-\varepsilon,\varepsilon),\mathbb{C}^2)$.

For the converse, suppose that $g \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ such that $\int Jg = 0$. Define

$$u(x) = I^{-1}(x) \int_{-\varepsilon}^{x} J(y)g(y)\,\mathrm{d}y.$$

Then certainly $u \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ and

$$\frac{\mathrm{d}}{\mathrm{d}x} I(x)u(x) = J(x)g(x).$$

Thus by Lemma 2.7 we have that $(T_0 + \lambda i)u = g$. ∎

Finally, we want to show that the range of $T_0 + \lambda i$ is dense for $\lambda = \pm\alpha$. The intuition here is that J_λ is "not L^2," so that the "orthogonal complement" of J_λ is dense. More precisely, we have the following result.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^d$ and $j \in C(\Omega, \mathbb{C}^n)$, $j \notin L^2(\Omega, \mathbb{C}^n)$. Then*

$$K_j = \left\{ g \in C_c^\infty(\Omega, \mathbb{C}^n) \;\middle|\; \int_\Omega \langle j(x), g(x) \rangle \mathrm{d}x = 0 \right\}$$

is dense in $L^2(\Omega, \mathbb{C}^n)$.

Proof. Define a linear functional $\langle j| : C_c^\infty(\Omega) \to \mathbb{C}$ by $\langle j|f = \int_\Omega \langle j(x), f(x)\rangle \mathrm{d}x$. Our first step is to prove that $\langle j|$ is unbounded.

Suppose that $\langle j|$ were bounded on $C_c^\infty(\Omega, \mathbb{C}^n)$ with respect to the $L^2(\Omega, \mathbb{C}^n)$-norm. Then $\langle j|$ extends to a bounded linear functional on $L^2(\Omega, \mathbb{C}^n)$, given by $\psi \mapsto \langle \tilde{j}, \psi\rangle$ for some $\tilde{j} \in L^2(\Omega, \mathbb{C}^n)$ by Riesz-representation. But then $\int_\Omega \langle j(x), g(x)\rangle \mathrm{d}x = \int_\Omega \langle \tilde{j}(x), g(x)\rangle$ for all $g \in C_c^\infty(\Omega)$, which implies that $j(x) = \tilde{j}(x)$. This is in contradiction with our assumption that $j \notin L^2(\Omega, \mathbb{C}^n)$.

So $\langle j|$ is an unbounded linear functional on $C_c^\infty(\Omega, \mathbb{C}^n)$. Therefore, there exists a sequence $(\delta_m)_{m\in\mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{C}^n)$ such that $\langle j|\delta_m = 1$ and $\|\delta_m\|_{L^2} < \frac{1}{m}$ for all $m \in \mathbb{N}$.

Let $\psi \in L^2(\Omega, \mathbb{C}^n)$ and $\varepsilon > 0$ be arbitrary. Then there is a $\psi_1 \in C_c^\infty(\Omega, \mathbb{C}^n)$ such that $\|\psi - \psi_1\|_{L^2} < \frac{1}{2}\varepsilon$. Define $\alpha = \langle j|\psi_1$ and find M such that $\frac{\alpha}{M} < \frac{1}{2}\varepsilon$. Set $\psi_2 = \psi_1 - \alpha\delta_M$, then $\|\psi - \psi_2\|_{L^2} < \varepsilon$ and $\langle j|\psi_2 = 0$, proving density of K_j. ∎

Proposition 2.10. *The range of $T_0 + \lambda i$ is dense for $\lambda = \pm\alpha$. Consequently, T_0 is essentially self-adjoint.*

Proof. Write j_1 and j_2 for the rows of J_λ, so

$$j_1(s) = i \begin{pmatrix} \lambda s \\ 1 + s f(s) \end{pmatrix}, \quad j_2(s) = -i \begin{pmatrix} 1 \\ \frac{1}{\lambda} f(s) \end{pmatrix}.$$

Then Lemma 2.8 tells us that the range of $T_0 + \lambda i$ is $K_{j_1} \cap K_{j_2}$, in the notation of Lemma 2.9.

To prove density of $K_{j_1} \cap K_{j_2}$ we use the same strategy as in Lemma 2.9 to obtain two sequences $(\delta_m^1)_{m\in\mathbb{N}}$ and $(\delta_m^2)_{m\in\mathbb{N}}$ such that $\langle j_i|\delta_m^i = 1$ and $\|\delta_m^i\|_{L^2} < \frac{1}{m}$.

Write $\delta^i_{m,1}$ and $\delta^i_{m,2}$ for the first and second components of δ^i_m, respectively, and $\tau : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$, $\tau(x) = -x$. Since the first component of j_1 is odd, while the second component is even we may replace δ^1_m by

$$\widetilde{\delta}^1_m = \frac{1}{2}\begin{pmatrix} \delta^1_{m,1} - \delta^1_{m,1} \circ \tau \\ \delta^1_{m,2} + \delta^1_{m,2} \circ \tau \end{pmatrix}.$$

On the other hand, the first component of j_2 is even, while the second component is odd, so we may replace δ^2_m by

$$\widetilde{\delta}^2_m = \frac{1}{2}\begin{pmatrix} \delta^2_{m,1} + \delta^2_{m,1} \circ \tau \\ \delta^2_{m,2} - \delta^2_{m,2} \circ \tau \end{pmatrix}.$$

Replacing δ^i_m by $\widetilde{\delta}^i_m$ does not change the values of $\langle j_i | \delta^i_m$, and it does not increase the norm of the δ^i_m. Furthermore, since the corresponding components of j_i and $\widetilde{\delta}^i_m$ now have opposite parity, then $\langle j_1 | \widetilde{\delta}^2_m = \langle j_2 | \widetilde{\delta}^1_m = 0$.

We can now complete the density proof similar to the final step in Lemma 2.9. Let $\psi \in L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ and $\varepsilon > 0$ be arbitrary. Then there is a $\psi_1 \in C^\infty_c((-\varepsilon, \varepsilon), \mathbb{C}^2)$ such that $\|\psi - \psi_1\|_{L^2} < \frac{1}{3}\varepsilon$. Let $\alpha_i = \langle j_i | \psi_1$ for $i = 1, 2$ and find M such that $\frac{\alpha_i}{M} < \frac{1}{3}\varepsilon$. Then $\psi_2 = \psi_1 - \alpha_1 \widetilde{\delta}^1_M - \alpha_2 \widetilde{\delta}^2_M$ satisfies both $\langle j_1 | \psi_2 = 0$, $\langle j_2 | \psi_2 = 0$, and $\|\psi - \psi_2\|_{L^2} < \varepsilon$. ■

The other property of T that we need is that of compact resolvent. Its proof is based on the following result.

Lemma 2.11. *The graph norm of $T \pm \lambda i$ is larger than the Sobolev norm of $\lambda^2 \geq \alpha^2$. Indeed, for $\psi \in C^\infty_c((-\varepsilon, \varepsilon), \mathbb{C}^2)$ one has that $\|\psi\|^2 + \|(T + i\lambda)\psi\|^2 > \|\psi\|^2 + \|\psi'\|^2$.*

Proof. We want to compute $\|(T + i\lambda)\psi\|^2$ for $\psi \in C^\infty_c((-\varepsilon, \varepsilon), \mathbb{C}^2)$, the domain of T_0. The claim then follows for T by continuity. Using the symmetry of T, this equals $\langle \psi, (T^2 + \lambda^2)\psi \rangle$, so let us compute T^2:

$$T^2 = \begin{pmatrix} -\partial_s^2 - f'(s) + f(s)^2 & 0 \\ 0 & -\partial_s^2 + f'(s) + f(s)^2 \end{pmatrix}.$$

Therefore,

$$\big\langle \psi, (T^2 + \lambda^2)\psi \big\rangle = \langle \psi, -\psi'' \rangle + \left\langle \psi, \begin{pmatrix} f(s)^2 - f'(s) + \lambda^2 & 0 \\ 0 & f(s)^2 + f'(s) + \lambda^2 \end{pmatrix} \psi \right\rangle.$$

For $\lambda^2 \geq \alpha^2$ both $f(s)^2 \pm f'(s) + \lambda^2 \geq 0$, so the second term on the right-hand side is positive. Hence

$$\big\langle \psi, (T^2 + \lambda^2)\psi \big\rangle \geq \langle \psi, -\psi'' \rangle.$$

By partial integration $\langle \psi, -\psi'' \rangle = \langle \psi', \psi' \rangle$, so we find that

$$\big\| (T + \lambda i)\psi \big\|^2 \geq \|\psi'\|^2.$$

■

Corollary 2.12. *The domain of the self-adjoint operator T is contained in the first-order Sobolev space $H^1((-\varepsilon, \varepsilon), \mathbb{C}^2)$.*

Proposition 2.13. *The resolvent* $(T + \lambda i)^{-1}$ *is compact for* $\lambda = \pm\alpha$ *and hence for all* $\lambda \in \rho(T)$.

Proof. Define $D = \{\psi \in L^2((-\varepsilon,\varepsilon),\mathbb{C}^2) \mid \|\psi\| \leq 1\}$ as the unit disc in L^2. We will prove that $M := (T + \lambda i)^{-1} D$ is pre-compact.

Let $\psi = (T + \lambda i)^{-1}\phi$, $\phi \in D$. Then $\|\psi\| \leq |\lambda|^{-1}$ since $\|(T + \lambda i)^{-1}\| \leq |\lambda|^{-1}$ and $\psi \in \mathrm{dom}(T) \subset H^1((-\varepsilon,\varepsilon),\mathbb{C}^2)$ by Corollary 2.12. Furthermore, Lemma 2.11 tells us that

$$\|\psi'\| \leq \big\|(T + \lambda i)\psi\big\| = \|\phi\| \leq 1.$$

Therefore,

$$M \subset \big\{\psi \in H^1\big((-\varepsilon,\varepsilon),\mathbb{C}^2\big) \mid \|\psi'\|, \|\psi\| \leq \max\big(1, |\lambda|^{-1}\big)\big\}.$$

By the Rellich embedding theorem the set on the right-hand side is compact in L^2, so that M is pre-compact in $L^2((-\varepsilon,\varepsilon),\mathbb{C}^2)$. Compactness of the resolvents for $\lambda \neq \pm\alpha$ follows from the first resolvent identity

$$(T + \lambda i)^{-1} = (T + \alpha i)^{-1} + (\lambda - \alpha)(T + \lambda i)^{-1}(T + \alpha i)^{-1}. \qquad \blacksquare$$

Remark 2.14. The operator $T^2 + \lambda^2$ can be interpreted as a Schrödinger-type operator on $L^2((-\varepsilon,\varepsilon),\mathbb{C}^2)$, which for λ large enough has positive potential. It is a classical result that Schrödinger operators with bounded potential on a bounded domain and Schrödinger operators on an unbounded domain with a confining potential have compact resolvents. The reason we did not use these classical results is that we are dealing with a combined case here: while $f(s)^2 - f'(s) + \lambda^2$ is a bounded potential, $f(s)^2 + f'(s) + \lambda^2$ is unbounded (it is, however, confining). Therefore, we have provided a direct proof along the lines of proofs for Schrödinger operators as found in [16].

Proposition 2.15. *The data* $(\mathbb{C}, L^2((-\varepsilon,\varepsilon),\mathbb{C}^2), T; \gamma^3)$ *is an even spectral triple that represents the multiplicative unit in* $KK_0(\mathbb{C},\mathbb{C})$.

Proof. We have already showed that T is self-adjoint and has compact resolvents. Also we have that $T\gamma^3 = -\gamma^3 T$ so that $(\mathbb{C}, L^2((-\varepsilon,\varepsilon),\mathbb{C}^2), T; \gamma^3)$ is an even spectral triple. So, since $\mathrm{Index} : KK_0(\mathbb{C},\mathbb{C}) \to \mathbb{Z}$ is an isomorphism of rings, it suffices to show that $\mathrm{Index}\, T = 1$ to conclude that $(L^2((-\varepsilon,\varepsilon),\mathbb{C}^2), T)$ is an unbounded representative for the multiplicative unit in $KK_0(\mathbb{C},\mathbb{C})$.

Write $T_+ = i\partial_s + if(s)$ and $T_- = i\partial_s - if(s) = T_+^*$ so that $T = \left(\begin{smallmatrix} 0 & T_+ \\ T_- & 0 \end{smallmatrix}\right)$. Let us then compute the index of T. First of all, $u \in \ker T_+$ if and only if u satisfies the differential equation

$$0 = iu'(s) + if(s)u(s).$$

This is a first-order, one-dimensional ODE, so all solutions are given by

$$u(s) = Ce^{-F(s)}$$

for $C \in \mathbb{C}$ and F a primitive function for f. But a primitive function for $f(s) = \alpha\tan(\alpha s)$ is $F(s) = -\ln(\cos(\alpha s))$, so the kernel of T_+ is given by constant multiples of $u_+(s) =$

$\cos(\alpha s)$, and then $\ker T_+ = \mathbb{C}u_+$. Similarly, we find that the kernel of T_- is given by constant multiples of $u_-(s) = \cos(\alpha s)^{-1}$. However, u_- is not an $L^2(-\varepsilon, \varepsilon)$ function so $\ker T_- = \{0\}$. Hence $\operatorname{Index}(T) = \dim\ker(T_+) - \dim\ker(T_-) = 1$. ■

3. Unbounded KK-product of the shriek cycle with the plane

In the spirit of [8, 13], we can use the connection on $(\widetilde{\mathcal{E}}, \widetilde{S})$ to construct a candidate unbounded KK-cycle for the product $\iota_! \otimes [\mathbb{R}^{n+1}]$.

We will begin by considering the product of the Hilbert bimodules

$$\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{n+1})} L^2(\mathbb{R}^{n+1}, \mathcal{S}),$$

followed by the computation of the product operator

$$D_\times = \widetilde{S} \otimes 1 + \gamma^3 \otimes_\nabla D_{\mathbb{R}^{n+1}} \tag{3.1}$$

with domain $\operatorname{dom}(\widetilde{S}) \otimes_{\mathrm{alg}} \operatorname{dom}(D_{\mathbb{R}^{n+1}})$, where we still use the notation $D_{\mathbb{R}^{n+1}}$ for $D_{\mathbb{R}^{2k}}$ and $\widetilde{D_{\mathbb{R}^{2k+1}}}$. We will then prove that the operator $D_\times$ on the balanced tensor product of $\widetilde{\mathcal{E}}$ and $L^2(\mathbb{R}^{n+1}, \mathcal{S})$ is an unbounded KK-cycle and that it represents not only $\iota_! \otimes [\mathbb{R}^{n+1}]$ but also the fundamental class $[\mathbb{S}^n]$.

3.1. Computation of the unbounded KK-product

The motivation for including the factor $\frac{1}{r^n}$ in $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ was to "flatten" a neighborhood of the circle in $\mathbb{R}^{n+1}$ to a cylinder. This is indeed accomplished, as we see in the computation of the balanced tensor product.

Proposition 3.1. *Let T be the index cycle defined in Section 2.4. Then for the odd- and even-dimensional spheres the following holds:*

- $n = 2k+1$*: There is a unitary isomorphism U from the Hilbert bimodule*

$$\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S}) \to L^2(\mathbb{S}^{2k-1}, \mathcal{S}^+ \otimes \mathbb{C}^2) \otimes L^2\big((-\varepsilon, \varepsilon), \mathbb{C}^2\big)$$

  such that

$$UD_\times U^* = \widetilde{D_{\mathbb{S}^{2k-1}}} \otimes \frac{1}{1+s} + \gamma_3 \otimes T. \tag{3.2}$$

- $n = 2k$*: There is a unitary isomorphism U from the Hilbert bimodule*

$$\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k+1})\otimes \mathbb{C}l_1} \big(L^2(\mathbb{R}^{2k+1}, \mathcal{S}) \otimes \mathbb{C}^2\big) \to L^2(\mathbb{S}^{2k}, \mathcal{S}) \otimes L^2\big((-\varepsilon, \varepsilon), \mathbb{C}^2\big)$$

  such that

$$UD_\times U^* = D_{\mathbb{S}^n} \otimes \frac{1}{1+s} + \gamma_r \otimes T.$$

Proof. The proof will be done for n odd; the same strategy works for n even. We will build the unitary equivalence in several steps, starting from the unitary map

$$V : \mathcal{E} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S}) \to L^2\big(\mathbb{S}^{2k-1} \times (-\varepsilon, \varepsilon), \mathcal{S}\big) \otimes \mathbb{C}^2,$$
$$V(g \otimes \psi)(\vec{\theta}, s) = g(\vec{\theta}, s)\psi(\vec{\theta}, s+1).$$

Let us first check that this is actually a unitary map:

$$\begin{aligned}
&\langle g\otimes\psi, g'\otimes\psi'\rangle_{\mathcal{E}\otimes L^2}\\
&\quad=\big\langle\psi,\langle g,g'\rangle_{\mathcal{E}}\cdot\psi'\big\rangle_{L^2(\mathbb{R}^{2k},\mathcal{S})}\\
&\quad=\int_{\mathbb{R}^{2k}}\big\langle\psi(\theta,r),\langle g,g'\rangle_{\mathcal{E}}(\theta,r)\psi'(\theta,r)\big\rangle_{\mathcal{S}}r^{2k-1}\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad=\int_{\mathbb{A}^{2k}}\Big\langle\psi(\theta,r),\frac{1}{r^{2k-1}}\overline{g(\theta,r-1)}g'(\theta,r-1)\psi'(\theta,r)\Big\rangle_{\mathcal{S}}r^{2k-1}\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad=\int_{\mathbb{S}^{2k-1}\times(-\varepsilon,\varepsilon)}\big\langle g(\theta,s)\psi(\theta,s+1),g'(\theta,s)\psi'(\theta,s+1)\big\rangle_{\mathcal{S}}\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad=\big\langle V(g\otimes\psi),V(g'\otimes\psi')\big\rangle_{L^2(\mathbb{S}^{2k-1}\times(-\varepsilon,\varepsilon),\mathcal{S})}.
\end{aligned}$$

Furthermore, V is surjective since $\mathcal{E}$ contains an approximate identity for $L^2(\mathbb{S}^{2k-1}\times(-\varepsilon,\varepsilon),\mathcal{S})$ consisting of bump functions with growing support.

We now apply the equivalence $\mathcal{S}\cong\mathcal{S}^+\otimes\mathbb{C}^2$ while moving from $\mathcal{E}$ to $\widetilde{\mathcal{E}}=\mathcal{E}\otimes\mathbb{C}^2$ to obtain a unitary equivalence from $\widetilde{\mathcal{E}}\otimes_{C_0(\mathbb{R}^{2k})}L^2(\mathbb{R}^{2k},\mathcal{S})$ to $L^2(\mathbb{S}^{2k-1}\times(-\varepsilon,\varepsilon),\mathcal{S}^+)\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. Note that the grading on this space is given by $1\otimes\gamma_3\otimes\gamma_3$.

Under this unitary equivalence, the operator $D_\times$ transforms as follows. The term $\widetilde{S}\otimes 1$ simply becomes $f(s)\otimes 1\otimes\gamma_2$, while $\gamma_3\otimes_{\nabla^{\widetilde{\mathcal{E}}}}D_{\mathbb{R}^{2k}}$ transforms to

$$\gamma_3\otimes_{\nabla^{\widetilde{\mathcal{E}}}}D_{\mathbb{R}^{2k}}\sim\frac{1}{r}D^+_{\mathbb{S}^{2k-1}}\otimes\gamma_2\otimes\gamma_3+i\partial_s\otimes\gamma_1\otimes\gamma_3,$$

where there is a crucial cancellation between the term $i\frac{2k-1}{2r}$ from $D_{\mathbb{R}^{2k}}$ against the $-\frac{2k-1}{2(s+1)}$ in the connection.

We now apply the following unitary transformation to the $\mathbb{C}^2\otimes\mathbb{C}^2$ component:

$$W=\frac{1}{\sqrt{2}}(1\otimes\gamma_3+\gamma_2\otimes\gamma_2).$$

The properties of the γ-matrices make it straightforward to check that W is a unitary such that

$$\begin{aligned}
W(\gamma_3\otimes\gamma_3)W^*&=\gamma_3\otimes\gamma_3, & W(1\otimes\gamma_2)W^*&=\gamma_2\otimes\gamma_3,\\
W(\gamma_1\otimes\gamma_3)W^*&=\gamma_1\otimes\gamma_3, & W(\gamma_2\otimes\gamma_3)W^*&=1\otimes\gamma_2.
\end{aligned}$$

This transforms the product operator into

$$D_\times\sim f(s)\otimes\gamma_2\otimes\gamma_3+\frac{1}{1+s}D^+_{\mathbb{S}^{2k-1}}\otimes 1\otimes\gamma_2+i\partial_s\otimes\gamma_1\otimes\gamma_3.$$

Finally, upon identifying that $L^2(\mathbb{S}^{2k-1}\times(-\varepsilon,\varepsilon),\mathcal{S}^+)\cong L^2(\mathbb{S}^{2k-1},\mathcal{S}^+)\otimes L^2((-\varepsilon,\varepsilon))$ we find that

$$\begin{aligned}
D_\times&\sim(D^+_{\mathbb{S}^{2k-1}}\otimes\gamma_2)\otimes\frac{1}{1+s}+(1\otimes\gamma_3)\otimes\big(\gamma_2 f(s)+i\gamma_1\partial_s\big)\\
&=\widetilde{D_{\mathbb{S}^{2k-1}}}\otimes\frac{1}{1+s}+\gamma_3\otimes T.
\end{aligned}$$

∎

This expression for $D_\times$ is essential for our further investigation and in fact already closely resembles the external product of the KK-cycle representing the sphere and the index cycle $(L^2((-\varepsilon,\varepsilon),\mathbb{C}^2),T)$. Secondly, this separated form allows us to investigate the analytical properties of $D_\times$ in terms of the already understood operators $\widetilde{D_{\mathbb{S}^{2k-1}}}$, $D_{\mathbb{S}^{2k}}$, and T.

3.2. Analysis of the product operator

We will now prove that $D_\times$ is essentially self-adjoint and that it has a compact resolvent.

For self-adjointness we use the concept of an adequate approximate identity introduced in [14]. The approach is similar to [17] where van den Dungen proves self-adjointness of a perturbed Dirac operator using an adequate approximate identity corresponding to the original Dirac operator. Let us recall the setup.

Definition 3.2. Let $D:\mathrm{dom}(D)\to H$ be a densely defined symmetric operator on some Hilbert space H. An *adequate approximate identity* for D is a sequential approximate identity $\{\phi_k\}_{k\in\mathbb{N}}$ on H such that $\phi_k\,\mathrm{dom}(D^*)\subset\mathrm{dom}(\bar{D})$, $[\bar{D},\phi_k]$ is bounded on $\mathrm{dom}(D)$, and $\sup_{k\in\mathbb{N}}\|\overline{[\bar{D},\phi_k]}\|<\infty$.

Remark 3.3. The definition of an adequate approximate identity is usually given in the context of Hilbert modules. We restrict our attention to the Hilbert space case, since it suffices for our purposes. All results, such as Proposition 3.4, still hold in the Hilbert module case.

The motivation for introducing these adequate approximate identities is the following proposition.

Proposition 3.4. *Let $D:\mathrm{dom}(D)\to H$ be a densely defined symmetric operator on a Hilbert space H and suppose that $\{\phi_k\}_{k\in\mathbb{N}}$ is an adequate approximate identity for D, then D is essentially self-adjoint.*

Proof. See [14]. ∎

We also have a converse.

Lemma 3.5. *Suppose that $D:\mathrm{dom}(D)\to H$ is a self-adjoint operator. Then $\phi_k:=(1+\frac{1}{k^2}D^2)^{-1}$ defines an adequate approximate identity $\{\phi_k\}_{k\in\mathbb{N}}$ for D. Furthermore, $\|(1+\frac{1}{k^2}D^2)^{-1}\|\le 1$ and $\|D(1+\frac{1}{k^2}D^2)^{-1}\|\le k$.*

Proof. The norm estimates, as well as the fact that $\{\phi_k\}$ defines an approximate unit, are in [15, Theorem 5.1.9]. Furthermore, this theorem tells us that $[D,\phi_k]=0$ on $\mathrm{dom}(D)$. The only remaining requirement is then that $\phi_k\,\mathrm{dom}(D)\subset\mathrm{dom}(D)$, we even have the stronger result that $\phi_k H\subset\mathrm{dom}(D)$ since $\phi_k=k^2(D+ki)^{-1}(D-ki)^{-1}$ and the resolvents map H into $\mathrm{dom}(D)$. ∎

We will show that, starting from adequate approximate identities for two self-adjoint operators D_1 and D_2, we can construct an adequate approximate identity for $D_1\otimes A+B\otimes D_2$ provided we have some control over the interaction between A and D_2, and B and D_1.

Proposition 3.6. *Let $D_1 : \mathrm{dom}(D_1) \to H_1$ and $D_2 : \mathrm{dom}(D_2) \to H_2$ be densely defined self-adjoint operators on Hilbert spaces H_1 and H_2. Let $A : H_2 \to H_2$ and $B : H_1 \to H_1$ be bounded, self-adjoint operators, such that*

$$\left\| \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} [B, D_1^2] \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \right\| \leq c_1 k,$$

$$\left\| \left(1 + \frac{1}{k^2} D_2^2\right)^{-1} [A, D_2^2] \left(1 + \frac{1}{k^2} D_2^2\right)^{-1} \right\| \leq c_2 k,$$

for some $c_1, c_2 \in \mathbb{R}$. Then $D_1 \otimes A + B \otimes D_2$ is essentially self-adjoint on $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$.

Proof. We will show that

$$\phi_k := \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \otimes \left(1 + \frac{1}{k^2} D_2^2\right)^{-1}$$

is an adequate approximate identity for $D_1 \otimes A + B \otimes D_2$, and then invoke Proposition 3.4. For ease of notation introduce $a = D_1 \otimes A + B \otimes D_2$.

First note that ϕ_k is an approximate identity for $H_1 \otimes H_2$, since it clearly is one on the dense subspace $H_1 \otimes_{\mathrm{alg}} H_2$.

Next, we show that $\phi_k \,\mathrm{dom}(a^*) \subset \mathrm{dom}(\bar{a})$; in fact we will show the stronger $\phi_k H_1 \otimes H_2 \subset \mathrm{dom}(\bar{a})$ similar to what we saw in Lemma 3.5. We will use that $a\phi_k$ is bounded on $H_1 \otimes_{\mathrm{alg}} H_2$; indeed

$$\begin{aligned} \|a\phi_k\| &\leq \left\| D_1 \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \otimes A \left(1 + \frac{1}{k^2} D_2^2\right)^{-1} \right\| \\ &\quad + \left\| B \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \otimes D_2 \left(1 + \frac{1}{k^2} D_2^2\right)^{-1} \right\| \\ &\leq k\|A\| + \|B\|k, \end{aligned}$$

and hence it extends to a bounded operator on $H_1 \otimes H_2$.

Let $z \in H_1 \otimes H_2$, $z = \lim_n z_n$, $z_n \in H_1 \otimes_{\mathrm{alg}} H_2$ and fix k. Clearly, $\phi_k z_n \in \mathrm{dom}(a) = \mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$ and $\phi_k z_n \to \phi_k z$ since ϕ_k is bounded. Moreover, as we just saw, $a\phi_k$ is bounded so that $a\phi_k z_n \to a\phi_k z$. Since we have a sequence in $\mathrm{dom}(a)$ converging to $\phi_k z$ for which the images under a also converge, we get $\phi_k z \in \mathrm{dom}(\bar{a})$.

Finally, we consider $[\bar{a}, \phi_k]$ on $\mathrm{dom}(a)$, and show that these commutators are bounded uniformly in k. Recall from the proof of Lemma 3.5 that $(1 + \frac{1}{k^2} D_i^2)^{-1}$ and D_i commute on $\mathrm{dom}(D_i)$. Then

$$\begin{aligned} [\bar{a}, \phi_k] &= D_1 \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \otimes \left[A, \left(1 + \frac{1}{k^2} D_2^2\right)^{-1} \right] \\ &\quad + \left[B, \left(1 + \frac{1}{k^2} D_1^2\right)^{-1} \right] \otimes D_2 \left(1 + \frac{1}{k^2} D_2^2\right)^{-1}. \end{aligned}$$

Since $\|D_i(1+\frac{1}{k^2}D_i^2)^{-1}\| \le k$ we want to find a bound of order $\frac{1}{k}$ for the commutators $[A,(1+\frac{1}{k^2}D_2^2)^{-1}]$ and $[B,(1+\frac{1}{k^2}D_1^2)^{-1}]$.

We start by rewriting these commutators in terms of the original operators

$$\left[B,\left(1+\frac{1}{k^2}D_1^2\right)^{-1}\right] = \left(1+\frac{1}{k^2}D_1^2\right)^{-1}\left[1+\frac{1}{k^2}D_1^2, B\right]\left(1+\frac{1}{k^2}D_1^2\right)^{-1}$$
$$= -\frac{1}{k^2}\left(1+\frac{1}{k^2}D_1^2\right)^{-1}[B,D_1^2]\left(1+\frac{1}{k^2}D_1^2\right)^{-1}.$$

By assumption there exists a c such that

$$\left\|\left(1+\frac{1}{k^2}D_1^2\right)^{-1}[B,D_1^2]\left(1+\frac{1}{k^2}D_1^2\right)^{-1}\right\| \le c_1 k$$

which implies that

$$\left\|\left[B,\left(1+\frac{1}{k^2}D_1^2\right)^{-1}\right]\right\| \le c_1\frac{1}{k}.$$

By the same reasoning we get

$$\left\|\left[A,\left(1+\frac{1}{k^2}D_2^2\right)^{-1}\right]\right\| \le c_2\frac{1}{k}.$$

Together this implies that $\|[\bar{a},\phi_k]\| \le c_1 + c_2$ which completes the proof. ∎

Corollary 3.7. *If D_1 and D_2 are essentially self-adjoint on* $\mathrm{dom}(D_1)$ *and* $\mathrm{dom}(D_2)$ *and satisfy the assumptions in Proposition 3.6, then $D_1 \otimes A + B \otimes D_2$ is essentially self-adjoint on* $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$.

Proof. Write $\mathrm{dom}(\overline{D_1})$ and $\mathrm{dom}(\overline{D_2})$ for the domains of self-adjointness of D_1 and D_2. Then we know that $D_1\otimes A + B\otimes D_2$ is essentially self-adjoint on $\mathrm{dom}(\overline{D_1})\otimes_{\mathrm{alg}}\mathrm{dom}(\overline{D_2})$. Write a_0 for the closure of $D_1 \otimes A + B \otimes D_2$ defined on $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$ and a for the closure on $\mathrm{dom}(\overline{D_1}) \otimes_{\mathrm{alg}} \mathrm{dom}(\overline{D_2})$.

Clearly, $a_0 \subset a$, so we want to show that $a \subset a_0$. This follows if we can show that $\mathrm{dom}(\overline{D_1}) \otimes_{\mathrm{alg}} \mathrm{dom}(\overline{D_2}) \subset \overline{\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)}$, with the closure taken in the graph norm of a. So suppose that $\psi\otimes\phi \in \mathrm{dom}(\overline{D_1})\otimes_{\mathrm{alg}}\mathrm{dom}(\overline{D_2})$. Then $\psi = \lim x_n$, $\phi = \lim y_n$ such that $D_1\psi = \lim D_1 x_n$ and $D_2\phi = \lim D_2 y_n$, with $x_n \in \mathrm{dom}(D_1)$ and $y_n \in \mathrm{dom}(D_2)$ since the D_i are essentially self-adjoint on the $\mathrm{dom}(D_i)$. But then

$$\begin{aligned}
&\big\|a(x_n\otimes y_n) - a(\psi\otimes\phi)\big\| \\
&\quad= \big\|(D_1\otimes A)(x_n\otimes y_n - \psi\otimes\phi) + (B\otimes D_2)(x_n\otimes y_n - \psi\otimes\phi)\big\| \\
&\quad\le \|D_1x_n\otimes Ay_n - D_1\psi\otimes Ay_n\| + \|D_1\psi\otimes Ay_n - D_1\psi\otimes A\phi\| \\
&\qquad+ \|Bx_n\otimes D_2y_n - Bx_n\otimes D_2\phi\| + \|Bx_n\otimes D_2\phi - B\psi\otimes D_2\phi\| \\
&\quad\le \big\|D_1(x_n-\psi)\big\|\cdot\|Ay_n\| + \|D_1\psi\|\cdot\big\|A(y_n-\phi)\big\| \\
&\qquad+ \|Bx_n\|\cdot\big\|D_2(y_n-\phi)\big\| + \big\|B(x_n-\psi)\big\|\cdot\|D_2\phi\|
\end{aligned}$$

tends to zero. Therefore, $\psi \otimes \phi \in \overline{\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)}$ (closure in the graph norm) so that $a \subset a_0$. ∎

Corollary 3.8. *The operator $D_\times$ is essentially self-adjoint on* $\mathrm{dom}(\widetilde{D^+_{\mathbb{S}^{2k-1}}}) \otimes_{\mathrm{alg}} \mathrm{dom}(T)$ *or* $\mathrm{dom}(D_{\mathbb{S}^{2k}}) \otimes_{\mathrm{alg}} \mathrm{dom}(T)$ *(depending on whether n is odd or even).*

Proof. Referring to the notation of Proposition 3.6 we have that $D_1 = \widetilde{D^+_{\mathbb{S}^{2k-1}}}$ or $D_1 = D_{\mathbb{S}^{2k}}$, $A = \frac{1}{1+s} 1_{\mathbb{C}^2}$, $B = \gamma_3$, and $D_2 = T$.

The commutator $[B, D_1^2] = 0$, and the commutator $[A, D_2]$ is bounded. Hence the required estimates hold by applying Lemma 3.5 and $[A, D_2^2] = [A, D_2]D_2 + D_2[A, D_2]$. ∎

Remark 3.9. In [8], self-adjointness of the product operator is proven by showing that $D_1 \otimes 1$ and $\gamma \otimes_\nabla D_2$ separately are (essentially) self-adjoint and that they anti-commute, which then proves that their sum is again (essentially) self-adjoint.

In our case, $\gamma^3 \otimes_{\nabla^\varepsilon} D_{\mathbb{R}^{n+1}}$ is not essentially self-adjoint on the domain $C_0^\infty(\mathbb{S}^{2k-1} \times (-\varepsilon, \varepsilon))$ (with the appropriate unitary transformations and spinor components), which should be the domain according to [8], so their results on using connections are not directly applicable.

We will use the following characterization to show that $D_\times^2$ has a compact resolvent, which implies that $D_\times$ has a compact resolvent as well.

Proposition 3.10. *Let $D : \mathrm{dom}(D) \to H$ be a self-adjoint operator that is bounded below. Then D has compact resolvents if and only if there exists a complete orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$, $\phi_n \in \mathrm{dom}(D)$ for H consisting of eigenvectors for D with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_n \to \infty$.*

Proof. See [16, Theorem XIII.64]. ∎

Proposition 3.11. *The operator $D_\times^2$ has a compact resolvent.*

Proof. In both the even- and the odd-dimensional case, $D_\times$ is of the form $D_1 \otimes A + \gamma \otimes T$, where D_1 is $\widetilde{D_{\mathbb{S}^{2k-1}}}$ or $D_{\mathbb{S}^{2k}}$, γ is the grading operator, and A is the operator corresponding to multiplication by $\frac{1}{1+s}$. For this proof we will write H_1 for $L^2(\mathbb{S}^{2k-1}, \mathcal{S}^+ \otimes \mathbb{C}^2)$ in the odd case and $L^2(\mathbb{S}^{2k}, \mathcal{S})$ in the even case.

Since D_1 is self-adjoint and has a compact resolvent, we may form an orthonormal basis $\{\psi_k\}_{k\in\mathbb{N}}$ for H_1 consisting of D_1^2 eigenvectors with eigenvalues λ_k^2. Since D_1^2 commutes with γD_1, we can additionally require that the ψ_k be eigenvectors for γD_1, for which they will have eigenvalues $\pm\lambda_k$.

Restricted to $\mathbb{C}\psi_k \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$, $D_\times = 1 \otimes (\lambda_k^2 A^2 \pm \lambda_k [A, T] + T^2) =: 1 \otimes R_k$. Since A and $[A, T]$ are bounded, this R_k is a bounded perturbation of T^2; hence it too is self-adjoint and has a compact resolvent. Moreover, it is bounded below by $\frac{1}{(1+\varepsilon)^2}\lambda_k^2 - \frac{1}{(1-\varepsilon)^2}\lambda_k$.

Let $\{\phi_{k,l}\}_{l\in\mathbb{N}}$ be an orthonormal basis for $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ consisting of eigenvectors for R_k with eigenvalues $\{\nu_{k,l}^2\}_{l\in\mathbb{N}}$. Then $\{\psi_k \otimes \phi_{k,l}\}_{(k,l)\in\mathbb{N}^2}$ is an orthonormal basis for

$H_1 \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ consisting of eigenvectors for $D_\times^2$. The corresponding eigenvalues $\nu_{k,l}^2$ tend to infinity, hence $D_\times^2$ has a compact resolvent. ■

3.3. Relation to the Kasparov product of $\iota_!$ and $[\mathbb{R}^{n+1}]$

Now that we have established the analytical properties of $D_\times$, it is time to turn to our primary goal and establish the unbounded factorization of $[\mathbb{S}^n]$ as the product of the unbounded shriek cycle and Euclidean space. This also provides, in a sense, a factorization of $D_{\mathbb{S}^n}$ as a product of S and $D_{\mathbb{R}^{n+1}}$, although we are left with the explicit remainder T, that becomes trivial in bounded KK-theory.

Theorem 1. *Let $n \geq 1$. Then the following holds:*

- *n odd: The data $(\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{n+1})} L^2(\mathbb{R}^{n+1}, \mathcal{S}), D_\times; \gamma^3 \otimes \gamma^3)$ defines an unbounded $C(\mathbb{S}^n) \otimes \mathbb{C}l_1$-$\mathbb{C}$ KK-cycle that represents both the Kasparov products $\iota_! \otimes [\mathbb{R}^{n+1}]$ and $[\mathbb{S}^n] \otimes \mathbb{1}$ in $KK_0(C(\mathbb{S}^n) \otimes \mathbb{C}l_1, \mathbb{C})$.*
- *n even: The data $(\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{n+1}) \otimes \mathbb{C}l_1} L^2(\mathbb{R}^{n+1}, \mathcal{S}), D_\times; \gamma^3 \otimes \gamma_r)$ defines an unbounded $C(\mathbb{S}^n)$-$\mathbb{C}$ KK-cycle that represents both the Kasparov products $\iota_! \otimes [\mathbb{R}^{n+1}]$ and $[\mathbb{S}^n] \otimes \mathbb{1}$ in $KK_0(C(\mathbb{S}^n), \mathbb{C})$.*

Proof. Again we will do the proof in the case n odd, however the same strategy works for the case where n is even.

We have proven that $D_\times$ is self-adjoint and has a compact resolvent in Section 3.2. Moreover, the commutators of $D_\times$ with $C^1(\mathbb{S}^{2k-1}) \otimes \mathbb{C}l_1$ are bounded so $(\widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S}), D_\times; \gamma_3 \otimes \gamma_3)$ is an unbounded KK-cycle.

The remainder of the proof deals with verifying Kucerovsky's criterion [11, Theorem 13] in both cases. In the case $\iota_! \otimes [\mathbb{R}^{2k}]$, we will use the expression in (3.1), and in the $[\mathbb{S}^{2k-1}] \otimes \mathbb{1}$ case we use the expression in (3.2).

Let us first consider Kucerovsky's connection condition for the product $\iota_! \otimes_{C_0(\mathbb{R}^{2k})} [\mathbb{R}^{2k}]$ where a general computation using the properties of a metric connection suffices. Indeed, let $\xi \in \widetilde{\mathcal{E}}$ be a homogeneous element of degree $\deg \xi$, and define $T_\xi : L^2(\mathbb{R}^{2k}, \mathcal{S}) \to \widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S})$ by $\psi \mapsto \xi \otimes \psi$. The adjoint is given by $\phi \otimes \psi \mapsto \langle \xi, \phi \rangle_{\widetilde{\mathcal{E}}} \cdot \psi$ for an elementary tensor $\phi \otimes \psi \in \widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S})$. The connection condition for the product $\iota_! \otimes [\mathbb{R}^{2k}]$ is, in this case, that the graded commutator

$$\left[\begin{pmatrix} D_\times & 0 \\ 0 & D_{\mathbb{R}^{2k}} \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix} \right]$$

is bounded for ξ in a dense subset of $\mathcal{E}$.

A simple calculation shows that this is equivalent to boundedness of

$$\left[\begin{pmatrix} \gamma^3 \otimes_{\nabla^{\widetilde{\mathcal{E}}}} D_{\mathbb{R}^{2k}} & 0 \\ 0 & D_{\mathbb{R}^{2k}} \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix} \right].$$

Evaluating the bottom-left component of the resulting matrix on the elementary tensor $\phi \otimes \psi \in \widetilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^{2k})} L^2(\mathbb{R}^{2k}, \mathcal{S})$ yields that

$$\begin{aligned} &D_{\mathbb{R}^{2k}} T_\xi^* - (-1)^{\deg \xi} T_\xi^* (\gamma^3 \otimes_{\nabla^{\widetilde{\mathcal{E}}}} D_{\mathbb{R}^{2k}}) \\ &\quad = D_{\mathbb{R}^{2k}} \big(\langle \xi, \phi \rangle_{\widetilde{\mathcal{E}}} \cdot \psi \big) - (-1)^{\deg \xi} T_\xi^* \big(\gamma^3 \phi \otimes D_{\mathbb{R}^{2k}} \psi + \nabla^{\widetilde{\mathcal{E}}} (\gamma^3 \phi) \cdot \psi \big) \\ &\quad = D_{\mathbb{R}^{2k}} \big(\langle \xi, \phi \rangle_{\widetilde{\mathcal{E}}} \cdot \psi \big) - \langle \xi, \phi \rangle_{\widetilde{\mathcal{E}}} \cdot D_{\mathbb{R}^{2k}} \psi - \big\langle \xi, \nabla^{\widetilde{\mathcal{E}}} (\phi) \big\rangle_{\widetilde{\mathcal{E}}} \cdot \psi \\ &\quad = \big[D_{\mathbb{R}^{2k}}, \langle \xi, \phi \rangle_{\widetilde{\mathcal{E}}} \big] \psi - \big\langle \xi, \nabla^{\widetilde{\mathcal{E}}} (\phi) \big\rangle_{\widetilde{\mathcal{E}}} \cdot \psi \\ &\quad = \big\langle \nabla^{\widetilde{\mathcal{E}}} (\xi), \phi \big\rangle_{\widetilde{\mathcal{E}}} \cdot \psi. \end{aligned}$$

This is bounded by $\|\nabla^{\widetilde{\mathcal{E}}}(\xi)\|$ which is indeed finite for a dense subset of $\widetilde{\mathcal{E}}$. Here $\langle \nabla^{\widetilde{\mathcal{E}}}(\xi), \phi \rangle_{\widetilde{\mathcal{E}}}$ acts on $L^2(\mathbb{R}^{2k}, \mathcal{S})$ in the way described in Lemma 2.5.

The top-right component is bounded by a similar computation and the diagonal components are 0. This computation is general for metric connections, in fact, whenever a product operator is constructed using a metric connection, the connection condition is automatically satisfied.

The compatibility condition is straightforward, simply by taking the domain of compatibility to be $\mathcal{W} = C_c^\infty(\mathbb{S}^{2k-1} \times (-\varepsilon, \varepsilon))$ (embedded appropriately in the respective spaces).

We then consider the positivity condition. Using symmetry of $\widetilde{S}$ and $D_\times$, we find that we need to prove that

$$\big\langle \psi, \big((\widetilde{S} \otimes 1) D_\times + D_\times (\widetilde{S} \otimes 1) \big) \psi \big\rangle \geq C \langle \psi, \psi \rangle$$

holds on $C_c^\infty(\mathbb{S}^{2k-1} \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ for some $C \in \mathbb{R}$. Using the (anti)-commutation properties of the γ-matrices, we find that

$$\begin{aligned} \big\langle \psi, \big((\widetilde{S} \otimes 1) D + D (\widetilde{S} \otimes 1) \big) \psi \big\rangle &= \big\langle \psi, \big(2 f(s)^2 + f'(s) \otimes \gamma_1 \otimes \gamma_1 \big) \psi \big\rangle \\ &= \big\langle \psi, f(s)^2 \psi \big\rangle + \big\langle \psi, \big(f(s)^2 + f'(s) \otimes \gamma_1 \otimes \gamma_1 \big) \psi \big\rangle \\ &\geq \big\langle \psi, \big(f(s)^2 - f'(s) \big) \psi \big\rangle \\ &= -\alpha^2 \langle \psi, \psi \rangle, \end{aligned}$$

so we may choose $C = -\alpha^2$.

Let us now turn to the product $\widetilde{[\mathbb{S}^{2k-1}]} \otimes \mathbb{1}$. The connection condition requires a more explicit computation. To avoid notational confusion between the maps T_ξ and the operator T we write D_2 for T in this computation, similar to the notation in [11]. In this case, the connection condition is that the commutator

$$\left[\begin{pmatrix} D_\times & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & D_\times T_\xi - (-1)^{\deg \xi} T_\xi D_2 \\ D_2 T_\xi^* - (-1)^{\deg \xi} T_\xi^* D_\times & 0 \end{pmatrix}$$

is bounded for $\xi \in C^1(\mathbb{S}^{2k-1}, \mathbb{C}^2)$.

As a first step, note that $T_\xi D_2 - (1 \otimes D_2)T_\xi = 0$ and that $D_2 T_\xi^* - T_\xi^*(1 \otimes D_2) = 0$. The grading factors introduced by the commutator cancel against the γ^3 appearing in $D_\times$ so that the connection condition reduces to

$$\begin{aligned}\Bigg[\begin{pmatrix} \widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes \frac{1}{1+s} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix}\Bigg] \\ = \begin{pmatrix} 0 & \big(\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes \frac{1}{1+s}\big)T_\xi \\ -(-1)^{\deg \xi} T_\xi^* \big(\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes \frac{1}{1+s}\big) & 0 \end{pmatrix}.\end{aligned}$$

Using the self-adjointness of $\widetilde{D^+_{\mathbb{S}^{2k-1}}}$ in the bottom-left, this equals

$$\frac{1}{1+s}\begin{pmatrix} 0 & T_{\widetilde{D^+_{\mathbb{S}^{2k-1}}}\xi} \\ -(-1)^{\deg \xi} T^*_{\widetilde{D^+_{\mathbb{S}^{2k-1}}}\xi} & 0 \end{pmatrix} \tag{3.3}$$

which is indeed bounded for $\xi \in C^1(\mathbb{S}^{2k-1}, \mathbb{C}^2)$.

The compatibility condition is again straightforward, while the positivity condition amounts to showing that

$$\big\langle \phi \otimes \psi, \big((\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1)D_\times + D_\times(\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1)\big)(\phi \otimes \psi)\big\rangle \geq C\langle \phi \otimes \psi, \phi \otimes \psi\rangle$$

for some $C \in \mathbb{R}$. Since $\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1$ anti-commutes with $\gamma_3 \otimes T$, this term drops out, while $\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1$ commutes with $\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes \frac{1}{1+s}$ to give

$$\begin{aligned}&\big\langle \phi \otimes \psi, \big((\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1)D_\times + D_\times(\widetilde{D^+_{\mathbb{S}^{2k-1}}} \otimes 1)\big)(\phi \otimes \psi)\big\rangle \\ &= 2\Big\langle \phi \otimes \psi, (\widetilde{D^+_{\mathbb{S}^{2k-1}}})^2 \phi \otimes \frac{1}{1+s}\psi\Big\rangle \\ &\geq \frac{2}{1-\varepsilon}\|\widetilde{D^+_{\mathbb{S}^{2k-1}}}\phi\|^2\|\psi\|^2 \geq 0.\end{aligned}$$

■

A. Unbounded *KK*-cycles: from odd to even

At several points in this paper we need to distinguish between the case n even and n odd. The fundamental class of a manifold M with $\dim(M)$ even will yield an even unbounded KK-cycle, while if $\dim(M)$ is odd we get an odd unbounded KK-cycle. However, we want to work with even cycles exclusively, since that is where Kucerovsky's criterion is applicable. We accomplish this by using the isomorphisms $KK_0(A \otimes \mathbb{C}l_1, B) \cong KK_1(A, B) \cong KK_0(A, B \otimes \mathbb{C}l_1)$, which at the level of concrete cycles are given by the following lemma.

Lemma A.1. *Let $(\mathcal{E}, D)$ be an odd unbounded A-B KK-cycle. Then*

(1) $(\mathcal{E} \otimes \mathbb{C}^2, D \otimes \gamma^2; 1 \otimes \gamma^3)$ *is an even unbounded $A \otimes \mathbb{C}l_1$-B KK-cycle, with $\mathbb{C}l_1$ acting by $1 \otimes \gamma^1$. We call this the left doubling of $(\mathcal{E}, D)$.*

(2) $(\mathcal{E} \otimes \mathbb{C}^2, D \otimes \gamma^1; 1 \otimes \gamma^3)$ *is an even unbounded A-$B \otimes \mathbb{C}l_1$ KK-cycle with $\mathbb{C}l_1$ acting by $1 \otimes \gamma^1$. We call this the right doubling of $(\mathcal{E}, D)$.*

Conversely, any even $A \otimes \mathbb{C}l_1$-B cycle is equivalent to the left doubling of an odd A-B cycle in $KK_0(A \otimes \mathbb{C}l_1, B)$ and any A-$B \otimes \mathbb{C}l_1$ cycle is the right doubling of the positive eigenspace of the non-trivial generator of $\mathbb{C}l_1$.

Proof. The only interesting claim in this lemma is that every even $A \otimes \mathbb{C}l_1$-B corresponds to an odd A-B cycle, since this requires the equivalence relations of KK-theory. The difficulty in this “halving” procedure is that the operator might not anti-commute with the action of $\mathbb{C}l_1$ as in the case of a doubled odd cycle. In [17, Theorem 5.1], van den Dungen shows that the operator can be modified such that it does anti-commute with the $\mathbb{C}l_1$ action, without changing the represented KK-class. ∎

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Walter D. van Suijlekom
Institute for Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands; waltervs@math.ru.nl

Luuk S. Verhoeven
Department of Mathematics, Middlesex College, University of Western Ontario, London, ON N6A 5B7, Canada; lverhoe@uwo.ca