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Lorentzian fermionic action by twisting Euclidean spectral triples

Pierre Martinetti and Devashish Singh

Abstract. We show how the twisting of spectral triples induces a transition from a Euclidean to a Lorentzian noncommutative geometry at the level of the fermionic action. More specifically, we compute the fermionic action for the twisting of a closed Euclidean manifold, then that of a two-sheet Euclidean manifold, and finally the twisting of the spectral triple of electrodynamics in Euclidean signature. We obtain the Weyl and the Dirac equations in Lorentzian signature (and in the temporal gauge). The twisted fermionic action is then shown to be invariant under an action of the Lorentz group. This permits us to interpret the field of 1-form that parametrises the twisted fluctuation of a manifold as the (dual) of the energy-momentum 4-vector.

1. Introduction

Noncommutative geometry [12] offers various ways to build models beyond the *standard model (SM) of elementary particles*, recently reviewed in [6, 21]. One of them [22, 23] consists in twisting the spectral triple of SM by an algebra automorphism, in the sense of Connes–Moscovici [17]. This provides a mathematical justification to the extra scalar field introduced in [7] to both fit the mass of the Higgs and stabilise the electroweak vacuum. A significant difference from the construction based on usual spectral triples without first-order condition [9, 10] is that the twist does not only yield an extra scalar field, but also a supplementary 1-form field (in [23], this field was improperly called *vector field*) whose meaning was rather unclear so far.

Connes' theory of spectral triples provides a spectral characterisation of compact Riemannian manifolds [15] along with the tools for their noncommutative generalisation [14]. Extending this program to the pseudo-Riemannian case is notoriously difficult. Although several interesting results in this context have been obtained recently, see e.g. [2, 24, 25, 36], there is no reconstruction theorem for pseudo-Riemannian manifolds in view, and it is still unclear how the spectral action should be handled in a pseudo-Riemannian signature.

Quite unexpectedly, the twist of the SM, which has been introduced in a purely Riemannian context, has something to do with the transition from the Euclidean signature to the

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Lorentzian one. In fact, the inner product induced by the twist on the Hilbert space of Euclidean spinors on a four-dimensional manifold $\mathcal{M}$ coincides with the Krein product of Lorentzian spinors [20]. This is not so surprising, for the twist ρ coincides with the automorphism that exchanges the two eigenspaces of the grading operator (in physicist's words: that exchanges the left and the right components of spinors). And this is nothing but the inner automorphism induced by the first Dirac matrix $\gamma^0 = c(dx^0)$. This explains why, by twisting, one is somehow able to single out the x_0 direction among the four Riemannian dimensions of $\mathcal{M}$. However, the promotion of this x_0 to a "time direction" is not fully accomplished, at least not in the sense of Wick rotation [19]. Indeed, regarding the Dirac matrices, the inner automorphism induced by γ^0 does not implement the Wick rotation (which maps the spatial Dirac matrices γ^j to $W(\gamma^j) := i\gamma^j$) but actually its square:

$$\rho(\gamma^j) = \gamma^0\gamma^j\gamma^0 = -\gamma^j = W^2(\gamma^j), \quad \text{for } j = 1, 2, 3. \tag{1.1}$$

Nevertheless, a transition from the Euclidean to the Lorentzian does occur, and the x_0 direction gets promoted to a time direction, but this happens at the level of the fermionic action. This is the main result of this paper, summarised in Propositions 4.5, 5.13, and their lorentz invariant version propositions 6.7 and 6.11.

More specifically, starting with the twisting of a *Euclidean* manifold, then that of a two-sheet Euclidean manifold, and finally the twisting of the spectral triple of electrodynamics in Euclidean signature [37], we show how the fermionic action for twisted spectral triples, proposed in [20], actually yields the Weyl and the Dirac equations in *Lorentzian signature*. In addition, the extra 1-form field acquires a clear interpretation as the dual of the energy-momentum 4-vector.

The following three aspects of the twisted fermionic action explain the change of signature.

- First, in order to guarantee that the fermionic action is symmetric when evaluated on Graßmann variables (which is an important requirement for the whole physical interpretation of the action formula, also in the non-twisted case [8]), one restricts the bilinear form that defines the action to the $+1$-eigenspace $\mathcal{H}_{\mathcal{R}}$ of the unitary operator $\mathcal{R}$ that implements the twist; whereas in the non-twisted case, the restriction is to the $+1$-eigenspace of the grading, in order to solve the fermion doubling problem. This different choice of eigenspace had been noticed in [20], but the physical consequences were not drawn. As already emphasised above, in the models relevant for physics, $\mathcal{R} = \gamma^0$, and once restricted to $\mathcal{H}_{\mathcal{R}}$, the bilinear form no longer involves a derivative in the x_0 direction. In other words, the restriction to $\mathcal{H}_{\mathcal{R}}$ projects the Euclidean fermionic action to what will constitute its spatial part in Lorentzian signature.
- Second, the twisted fluctuations of the Dirac operator of a four-dimensional Riemannian manifold are not necessarily zero [23, 28], in contrast with the non-twisted case where those fluctuations always vanish. These are parametrised by the above-mentioned 1-form field. By interpreting the zeroth component of this field as an energy, one recovers a derivative in the x_0 direction, but now in a Lorentzian signature.

- Third, we show that the twisted fermionic action is invariant under an action of the Lorentz group. From that follows the interpretation of the whole 1-form field (not only its zeroth component) as the dual of the energy-momentum 4-vector.

All this is detailed as follows. In Section 2, we review the known material regarding twisted spectral triples, their compatibility with the real structure (Section 2.1) and the new inner product they induce on the initial Hilbert space (Section 2.2). We discuss what a covariant Dirac operator is in the twisted context, and the corresponding gauge invariant fermionic action it defines (Section 2.3). We finally recall how to associate a twisted partner to graded spectral triples (Section 2.4).

In Section 3, we investigate the fermionic action for the minimal twist of a closed Euclidean manifold, that is, the twisted spectral triple having the same Hilbert space and Dirac operator as the canonical triple of the manifold, but whose algebra is doubled in order to make the twisting possible (Section 3.1). In Section 3.2, we show that twisted fluctuations of the Dirac operator are parametrised by a 1-form field of components X_μ, first discovered in [23]. In Section 3.3, we recall how to deal with gauge transformations in a twisted context, along the lines of [29]. We then compute the twisted fermionic action in Section 3.4 and show that it yields a Lagrangian density similar to that of the Weyl equations in Lorentzian signature, as soon as one interprets the zeroth component of X_μ as the time component of the energy-momentum 4-vector of fermions. However, there are not enough spinor degrees-of-freedom to deduce the Weyl equations for this Lagrangian density.

That is why in Section 4 we double the twisted manifold (Section 4.1), compute the twisted-covariant Dirac operator (Section 4.2), and obtain Weyl equations from the fermionic action (Section 4.3).

In Section 5, we apply the same construction to the spectral triple of electrodynamics proposed in [37]. Its minimal twist is written in Section 5.1, the twisted fluctuations are calculated in Section 5.2, for both the free part and the finite parts of the Dirac operator. The gauge transformations are studied in Section 5.3 and, finally, the Dirac equation in Lorentzian signature (and in the temporal gauge) is obtained in Section 5.4.

Section 6 deals with Lorentz invariance.

We conclude with some outlook and perspective. The appendices contain all the required notations for the Dirac matrices and for the Weyl and Dirac equations.

The Lorentz metric is $(+1, -1, -1, -1)$. We use Einstein convention for summing on alternate (up/down) indices: for instance, $\gamma^\mu \partial_\mu$ stands for $\sum_\mu \gamma^\mu \partial_\mu$.

2. Fermionic action for twisted spectral geometry

After an introduction to twisted spectral triples (Section 2.1), we recall how the inner product induced by the twist on the Hilbert space (Section 2.2) permits building a fermionic action (Section 2.3). The key difference with the usual (i.e., non-twisted) case is that one no longer restricts to the positive eigenspace of the grading Γ, but rather to that of the unitary $\mathcal{R}$ implementing the twist. Finally, we emphasise the twist-by-grading procedure that associates a twisted partner to any graded spectral triple whose representation is sufficiently faithful (Section 2.4).

2.1. Real twisted spectral triples

Twisted spectral triples have been introduced to build noncommutative geometries from type III algebras [17]. Later, they found applications in high energy physics describing extensions of SM, such as the Grand Symmetry model [22, 23].

Definition 2.1 (from [17]). A *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})_\rho$ is a unital $*$-algebra $\mathcal{A}$ that acts faithfully on a Hilbert space $\mathcal{H}$ as bounded operators,[1] along with a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ with compact resolvent, called the *Dirac operator*, and an automorphism ρ of $\mathcal{A}$ such that the *twisted commutator*, defined as

$$[\mathcal{D}, a]_\rho := \mathcal{D}a - \rho(a)\mathcal{D}, \tag{2.1}$$

is bounded for any $a \in \mathcal{A}$ (that is, $[\mathcal{D}, a]_\rho$ is well defined on the domain of $\mathcal{D}$ and extends to a bounded operator on $\mathcal{H}$).

A *graded* twisted spectral triple is one endowed with a self-adjoint operator Γ on $\mathcal{H}$ such that

$$\Gamma^2 = \mathbb{I}, \quad \Gamma\mathcal{D} + \mathcal{D}\Gamma = 0, \quad \Gamma a = a\Gamma, \quad \forall a \in \mathcal{A}. \tag{2.2}$$

The real structure [13] easily adapts to the twisted case [28]: as in the non-twisted case, one considers an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, such that

$$J^2 = \varepsilon\mathbb{I}, \quad J\mathcal{D} = \varepsilon'\mathcal{D}J, \quad J\Gamma = \varepsilon''\Gamma J, \tag{2.3}$$

where the signs $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$ determine the KO-dimension of the twisted spectral triple. In addition, J is required to implement an isomorphism between $\mathcal{A}$ and its opposite algebra $\mathcal{A}^\circ$,

$$b \mapsto b^\circ := Jb^*J^{-1}, \quad \forall b \in \mathcal{A}. \tag{2.4}$$

One requires this action of $\mathcal{A}^\circ$ on $\mathcal{H}$ to commute with that of $\mathcal{A}$ (the *zero-order condition*),

$$[a, b^\circ] = 0, \quad \forall a, b \in \mathcal{A}, \tag{2.5}$$

in order to define a right representation of $\mathcal{A}$ on $\mathcal{H}$:

$$\psi a := a^\circ\psi = Ja^*J^{-1}\psi, \quad \forall \psi \in \mathcal{H}. \tag{2.6}$$

[1] Wherever applicable, we use a to mean its representation $\pi(a)$. Thus a^* denotes $\pi(a^*) = \pi(a)^\dagger$, where $*$ is the involution of $\mathcal{A}$ and $\dagger$ is the Hermitian conjugation on $\mathcal{H}$.

The part of the real structure that is modified is the *first-order condition*. In the non-twisted case, it reads $[[D, a], b^\circ] = 0$, $\forall a, b \in \mathcal{A}$; while in the twisted case, it becomes [23, 28]

$$\big[[\mathcal{D}, a]_\rho, b^\circ\big]_{\rho^\circ} := [\mathcal{D}, a]_\rho b^\circ - \rho^\circ(b^\circ)[\mathcal{D}, a]_\rho = 0, \quad \forall a, b \in \mathcal{A}, \tag{2.7}$$

where ρ° is the automorphism induced by ρ on the opposite algebra:

$$\rho^\circ(b^\circ) = \rho^\circ(J b^* J^{-1}) := J\rho(b^*)J^{-1}. \tag{2.8}$$

Definition 2.2 (from [28]). A *real twisted spectral triple* is a graded twisted spectral triple, along with a real structure J satisfying (2.3), the zeroth and the first-order conditions (2.5), (2.7).

In case the automorphism ρ coincides with an inner automorphism of $\mathcal{B}(\mathcal{H})$, that is,

$$\pi\big(\rho(a)\big) = \mathcal{R}\pi(a)\mathcal{R}^\dagger, \quad \forall a \in \mathcal{A}, \tag{2.9}$$

where $\mathcal{R} \in \mathcal{B}(\mathcal{H})$ is unitary, then ρ is said to be *compatible with the real structure* J, as soon as

$$J\mathcal{R} = \varepsilon''' \mathcal{R} J, \quad \text{for } \varepsilon''' = \pm. \tag{2.10}$$

The inner automorphism, hence the unitary $\mathcal{R}$, is not necessarily unique. In that case, ρ is compatible with the real structure if there exists at least one $\mathcal{R}$ satisfying the above conditions.

Remark 2.3. In the original definition [17, eq. (3.4)], the automorphism is not required to be an $*$-automorphism, but rather to satisfy the regularity condition $\rho(a^*) = \rho^{-1}(a)^*$. If, however, one requires ρ to be an $*$-automorphism, then the regularity condition implies that

$$\rho^2 = \mathsf{Id}. \tag{2.11}$$

Other modifications of spectral triples by twisting the real structure have been proposed [4]. Interesting relations with the above real twisted spectral triples have been worked out in [5].

2.2. Twisted inner product

Given a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and an automorphism ρ of $\mathcal{B}(\mathcal{H})$, a *ρ-product* $\langle\cdot,\cdot\rangle_\rho$ is an inner product satisfying

$$\langle \phi, \mathcal{O}\xi\rangle_\rho = \big\langle \rho(\mathcal{O})^\dagger \phi, \xi\big\rangle_\rho, \quad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}) \text{ and } \phi, \xi \in \mathcal{H}, \tag{2.12}$$

where $\dagger$ is the Hermitian adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$. One calls

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger \tag{2.13}$$

the ρ-*adjoint* of the operator $\mathcal{O}$. If ρ is inner and implemented by a unitary operator $\mathcal{R}$ on $\mathcal{H}$ – that is, $\rho(\mathcal{O}) = \mathcal{R}\mathcal{O}\mathcal{R}^\dagger$ for any $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ – then, a canonical ρ-product is

$$\langle \phi, \xi \rangle_\rho = \langle \phi, \mathcal{R}\xi \rangle. \tag{2.14}$$

The ρ-adjointness is not necessarily an involution. If ρ is an $*$-automorphism (for instance, when ρ is inner), then $^+$ is an involution iff (2.11) holds, for

$$(\mathcal{O}^+)^+ = \rho(\mathcal{O}^+)^\dagger = \rho\big((\mathcal{O}^+)^\dagger\big) = \rho\big(\rho(\mathcal{O})\big). \tag{2.15}$$

Remark 2.4. The regularity condition in Remark 2.3 (written as $\rho(b)^* = \rho^{-1}(b^*)$ for any $b = a^* \in \mathcal{A}$) is equivalent to the ρ-adjointness $a^+ := \rho(a)^*$ being an involution, for

$$(a^+)^+ = \big(\rho(a)^*\big)^+ = \big(\rho\big(\rho(a)^*\big)\big)^* = \big(\rho\big(\rho^{-1}(a^*)\big)\big)^* = (a^*)^* = a. \tag{2.16}$$

Given a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\rho$ whose twisting automorphism ρ coincides with an automorphism of $\mathcal{B}(\mathcal{H})$, any choice of the unitary $\mathcal{R}$ implementing this automorphism induces a natural twisted inner product (2.14) on $\mathcal{H}$. These products are useful to define a gauge invariant fermionic action.

2.3. Twisted fermionic action

The *fermionic action* for a real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J, \Gamma)$ is [1, 8] $S(\mathcal{D}_\omega) := \mathfrak{A}_{\mathcal{D}_\omega}(\tilde{\xi}, \tilde{\xi})$, where

$$\mathfrak{A}_{\mathcal{D}_\omega}(\phi, \xi) := \langle J\phi, \mathcal{D}_\omega \xi \rangle, \quad \phi, \psi \in \mathcal{H}, \tag{2.17}$$

is a bilinear form defined by the covariant Dirac operator $\mathcal{D}_\omega := \mathcal{D} + \omega + \varepsilon' J\omega J^{-1}$ [14], where ω is a self-adjoint element of the set of generalised 1-forms

$$\Omega^1_{\mathcal{D}}(\mathcal{A}) := \Big\{ \sum_i a_i[\mathcal{D}, b_i],\ a_i, b_i \in \mathcal{A} \Big\}; \tag{2.18}$$

while $\tilde{\psi}$ is a Graßmann vector in the Fock space $\tilde{\mathcal{H}}_+$ of classical fermions, corresponding to the positive eigenspace $\mathcal{H}_+ \subset \mathcal{H}$ of the grading Γ; that is,

$$\tilde{\mathcal{H}}_+ := \{\tilde{\xi},\ \xi \in \mathcal{H}_+\}, \quad \text{where } \mathcal{H}_+ := \{\xi \in \mathcal{H},\ \Gamma\xi = \xi\}. \tag{2.19}$$

The fermionic action is invariant under a gauge transformation, that is, the simultaneous adjoint action of the group $\mathcal{U}(\mathcal{A})$ of unitaries of $\mathcal{A}$, both on $\mathcal{H} \ni \psi$,

$$(\mathrm{Ad}\, u)\psi := u\psi u^* = u(u^*)^\circ \psi = uJuJ^{-1} \quad u \in \mathcal{U}(\mathcal{A}) \tag{2.20}$$

and on the covariant Dirac operator: $\mathcal{D}_\omega \to (\mathrm{Ad}\, u)\mathcal{D}_\omega(\mathrm{Ad}\, u)^\dagger$.

Remark 2.5. The form (2.17) is antisymmetric in KO-dimensions $2, 4$ (Lemma 2.7 below), so $\mathfrak{A}_{\mathcal{D}_\omega}(\xi, \xi)$ vanishes when evaluated on vectors. However, it is non-zero when evaluated on Graßmann vectors [16, §I.16.2]. In particular, for the spectral triple of SM (of KO-dimension 2), the fermionic action contains the coupling of matter with fields (scalar, gauge, and gravitational).

In twisted spectral geometry, the fermionic action is constructed [20] substituting $\mathcal{D}_\omega$ with a twisted covariant Dirac operator

$$\mathcal{D}_{\omega_\rho} := \mathcal{D} + \omega_\rho + \varepsilon' J \omega_\rho J^{-1}, \tag{2.21}$$

where ω_ρ is an element of the set of twisted 1-forms [17],

$$\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho) := \Big\{ \sum_j a_j [\mathcal{D}, b_j]_\rho, \; a_j, b_j \in \mathcal{A} \Big\}, \tag{2.22}$$

such that $\mathcal{D}_{\omega_\rho}$ is self-adjoint,[2] and by replacing the inner product with the ρ-product (2.12). Instead of (2.17), one thus considers the bilinear form

$$\mathfrak{A}^\rho_{D_{\omega_\rho}}(\phi, \xi) := \langle J\phi, \mathcal{D}_{\omega_\rho} \xi \rangle_\rho. \tag{2.23}$$

A gauge transformation is given by the same action (2.20) of $\mathcal{U}(\mathcal{A})$ on $\mathcal{H}$, but the Dirac operator transforms in the following twisted manner [29]:

$$\mathcal{D}_{\omega_\rho} \to \big(\mathrm{Ad}\, \rho(u)\big) \mathcal{D}_{\omega_\rho} (\mathrm{Ad}\, u^*). \tag{2.24}$$

The r.h.s. of (2.24) is still a twisted covariant Dirac operator $D_{\omega^u_\rho}$, where [29, Prop. 4.2]

$$\omega^u_\rho := \rho(u)\big([D, u^*]_\rho + \omega_\rho u^*\big). \tag{2.25}$$

The transformation $\omega_\rho \to \omega^u_\rho$ is the twisted version of the law of transformation of the gauge potential in noncommutative geometry [14].

In case the twist ρ is compatible with the real structure in the sense of (2.10) for some unitary $\mathcal{R}$, the bilinear form (2.23) is invariant under the simultaneous transformation (2.20)–(2.24) [20, Prop. 4.1]. However, the antisymmetry of the form $\mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}$ is not guaranteed, unless one restricts to the positive eigenspace of $\mathcal{R}$, that is,

$$\mathcal{H}_\mathcal{R} := \{\xi \in \mathrm{Dom}\, \mathcal{D}, \; \mathcal{R}\xi = \xi\}. \tag{2.26}$$

This has been discussed in [20, Prop. 4.2] and led to the following definition.

Definition 2.6. For a real twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J)_\rho$, the fermionic action is

$$S_\rho(\mathcal{D}_{\omega_\rho}) := \mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}(\tilde{\xi}, \tilde{\xi}), \tag{2.27}$$

where $\tilde{\xi}$ is the Graßmann vector associated to $\xi \in \mathcal{H}_\mathcal{R}$.

[2] The domain of $\mathcal{D}_{\omega_\rho}$ coincides with the one of $\mathcal{D}$ (being $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ in $\mathcal{B}(\mathcal{H})$). By Kato–Relish theorem, $\mathcal{D}_{\omega_\rho}$ is self-adjoint iff $\omega_\rho + J\omega_\rho J^{-1}$ is self-adjoint. In [23], we required $\omega_\rho + J\omega_\rho J^{-1}$ to be self-adjoint without necessarily imposing the self-adjointness of ω_ρ. This is discussed in detail after Lemma 3.2 below.

In the spectral triple of SM, the restriction to $\mathcal{H}_+$ is there to solve the fermion doubling problem [30]. It also selects out the physically meaningful elements of $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_\mathcal{F}$, that is, those spinors whose chirality in $L^2(\mathcal{M}, \mathcal{S})$ coincides with their chirality as elements of the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$. In the twisted case, the restriction to $\mathcal{H}_\mathcal{R}$ is there to guarantee the antisymmetry of the bilinear form $\mathfrak{A}^\rho_{\mathcal{D}_{\omega_\rho}}$. However, the eigenvectors of $\mathcal{R}$ may not have a well-defined chirality. If fact, they cannot have it when the twist comes from the grading (see Section 2.4 below), since the unitary $\mathcal{R}$ implementing the twist (given in (2.35)) anticommutes with the chirality $\Gamma = \mathrm{diag}(\mathbb{I}_{\mathcal{H}_+}, -\mathbb{I}_{\mathcal{H}_-})$, so that

$$\mathcal{H}_+ \cap \mathcal{H}_\mathcal{R} = \{0\}. \tag{2.28}$$

From a physical standpoint, by restricting to $\mathcal{H}_\mathcal{R}$ rather than $\mathcal{H}_+$, one loses a clear interpretation of the elements of the Hilbert space: a priori, an element of $\mathcal{H}_\mathcal{R}$ is not physically meaningful since its chirality is ill-defined. However, we show in what follows that – at least in two examples: a manifold and the almost-commutative geometry of electrodynamics – the restriction to $\mathcal{H}_\mathcal{R}$ is actually meaningful, for it allows to obtain the Weyl and Dirac equations in the Lorentzian signature, even though one starts with a Riemannian manifold.

Before that, we conclude this section with two easy but useful lemmas. The first recalls how the symmetry properties of the bilinear form $\mathfrak{A}_D = \langle J\cdot, D\cdot\rangle$ do not depend on the explicit form of the Dirac operator, but solely on the signs $\varepsilon', \varepsilon''$ in (2.3). The second stresses that once restricted to $\mathcal{H}_\mathcal{R}$, the bilinear forms (2.17) and (2.23) differ only by a sign.

Lemma 2.7. *Let J be an antilinear isometry on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ such that $J^2 = \varepsilon\mathbb{I}$, and D a self-adjoint operator on $\mathcal{H}$ such that $JD = \varepsilon' DJ$. Then*

$$\langle J\phi, D\xi\rangle = \varepsilon\varepsilon'\langle J\xi, D\phi\rangle, \quad \forall \phi, \xi \in \mathcal{H}. \tag{2.29}$$

Proof. By definition, an antilinear isometry satisfies $\langle J\phi, J\xi\rangle = \overline{\langle\phi,\xi\rangle} = \langle\xi,\phi\rangle$. Thus

$$\langle J\phi, D\xi\rangle = \varepsilon\langle J\phi, J^2D\xi\rangle = \varepsilon\langle JD\xi, \phi\rangle = \varepsilon\varepsilon'\langle DJ\xi, \phi\rangle = \varepsilon\varepsilon'\langle J\xi, D\phi\rangle. \qquad \blacksquare$$

In particular, for KO-dimensions 2, 4 one has that $\varepsilon = -1$, $\varepsilon' = 1$, so $\mathfrak{A}_\mathcal{D}$ is antisymmetric. The same is true for $\mathfrak{A}_{\mathcal{D}_\omega}$ in (2.17), because the covariant operator $\mathcal{D}_\omega$ satisfies the same rules of sign (2.3) as $\mathcal{D}$.

Lemma 2.8. *Given D, and a unitary $\mathcal{R}$ compatible with $\mathcal{J}$ in the sense of (2.10), one has that*

$$\mathfrak{A}^\rho_D(\phi,\xi) = \varepsilon'''\,\mathfrak{A}_D(\phi,\xi), \quad \forall\phi,\xi \in \mathcal{H}_\mathcal{R}. \tag{2.30}$$

Proof. For any $\phi, \xi \in \mathcal{H}_\mathcal{R}$, we have that

$$\mathfrak{A}^\rho_D(\phi,\xi) = \langle J\phi, \mathcal{R}D\xi\rangle = \langle\mathcal{R}^\dagger J\phi, D\xi\rangle = \varepsilon'''\langle J\mathcal{R}^\dagger\phi, D\xi\rangle = \varepsilon'''\langle J\phi, D\xi\rangle, \tag{2.31}$$

where we used (2.10) as $\mathcal{R}^\dagger J = \varepsilon''' J\mathcal{R}^\dagger$ and (2.26) as $\mathcal{R}^\dagger\phi = \phi$. $\blacksquare$

2.4. Minimal twist by grading

The twisted spectral triples recently employed in physics are built by minimally twisting a usual spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The idea is to substitute the commutator $[\mathcal{D}, \cdot]$ with a twisted one $[\mathcal{D}, \cdot]_\rho$, while keeping the Hilbert space and the Dirac operator intact, because they encode the fermionic content of the theory and there is, so far, no experimental indications of extra fermions beyond those of the SM. However, for the spectral triples relevant for physics, $[\mathcal{D}, \cdot]$ and $[\mathcal{D}, \cdot]_\rho$ cannot be simultaneously bounded [28, §3.1]. So in order to be able to twist the commutator, one needs to play with the only object that remains available, namely the algebra.

Definition 2.9 (from [28]). A *minimal twist* of a spectral triple $(\mathcal{A}, H, D)$ by a unital $*$-algebra $\mathcal{B}$ is a twisted spectral triple $(\mathcal{A} \otimes B, H, D)_\rho$, where the initial representation π_0 of $\mathcal{A}$ on $\mathcal{H}$ is related to the representation π of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{H}$ by

$$\pi(a \otimes \mathbb{I}_{\mathcal{B}}) = \pi_0(a), \quad \forall a \in \mathcal{A}, \tag{2.32}$$

where $\mathbb{I}_{\mathcal{B}}$ is the identity of the algebra $\mathcal{B}$.

If the initial spectral triple is graded, a natural minimal twist may be obtained as follows. The grading Γ commutes with the representation of $\mathcal{A}$, so the latter is actually a direct sum of two representations on the positive and negative eigenspaces $\mathcal{H}_+$, $\mathcal{H}_-$ of Γ (see (2.19)). Therefore, one has enough space on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ to represent twice the algebra $\mathcal{A}$. It is tantamount to taking $\mathcal{B} = \mathbb{C}^2$ in Definition 2.9, with $\mathcal{A} \otimes \mathbb{C}^2 \simeq \mathcal{A} \oplus \mathcal{A} \ni (a, a')$ represented on $\mathcal{H}$ as

$$\pi(a, a') := \mathfrak{p}_+ \pi_0(a) + \mathfrak{p}_- \pi_0(a') = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a') \end{pmatrix}, \tag{2.33}$$

where $\mathfrak{p}_\pm := \frac{1}{2}(\mathbb{I}_{\mathcal{H}} \pm \Gamma)$ and $\pi_\pm(a) := \pi_0(a)_{|\mathcal{H}_\pm}$ are, respectively, the projections on $\mathcal{H}_\pm$ and the restrictions on $\mathcal{H}_\pm$ of π_0. If $\pi_\pm$ are faithful, then $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, \mathcal{D})_\rho$, with ρ the flip automorphism

$$\rho(a, a') := (a', a), \quad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2, \tag{2.34}$$

is indeed a twisted spectral triple, with grading Γ. Furthermore, if the initial spectral triple is real, then so is this minimal twist, with the same real structure [28].[3]

The flip ρ is an $*$-automorphism that satisfies (2.11), and coincides on $\pi(\mathcal{A} \otimes \mathbb{C}^2)$ with the inner automorphism of $\mathcal{B}(\mathcal{H})$ implemented by the unitary

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_{\mathcal{H}_+} \\ \mathbb{I}_{\mathcal{H}_-} & 0 \end{pmatrix} \quad \text{with } \mathbb{I}_{\mathcal{H}_\pm} \text{ the identity operator in } \mathcal{H}_\pm. \tag{2.35}$$

[3] The requirement that $\pi_\pm$ are faithful was not explicit in [28]. If it does not hold, then $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_\rho$ still satisfies all the properties of a twisted spectral triple, except that π in (2.33) might not be faithful.

As recalled in the next section, the canonical ρ-product (2.14) associated to the minimal twist of a closed *Riemannian* spin manifold of dimension 4 turns out to coincide with the *Lorentzian* Krein product on the space of Lorentzian spinors [20]. The aim of this paper is to show that a similar transition from the Euclidean to the Lorentzian also occurs for the fermionic action.

We first investigate how this idea comes about by studying in the next section the simplest example of the minimal twist of a manifold. Then, in the following sections, we show how to obtain the Weyl equations in the Lorentzian signature by doubling the twisted manifold and, finally, the Dirac equation by minimally twisting the spectral triple of electrodynamics in [37].

3. Preliminary: minimally twisted manifold

We compute the fermionic action for the minimal twist of a closed Euclidean spin manifold $\mathcal{M}$. Since we aim at finding back the Weyl and Dirac equations, we work in dimension 4, assuming that gravity is negligible (hence the flat metric). This is tantamount to choosing in (2.3)

$$\varepsilon = -1, \quad \varepsilon' = 1, \quad \varepsilon'' = 1. \tag{3.1}$$

3.1. Minimal twist of a Riemannian manifold

The minimal twist of $\mathcal{M}$ is the real, graded, twisted spectral triple

$$\left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2,\ L^2(\mathcal{M}, \mathcal{S}),\ \eth\right)_\rho, \tag{3.2}$$

where $C^\infty(\mathcal{M})$ is the algebra of smooth functions on $\mathcal{M}$, $L^2(\mathcal{M}, \mathcal{S})$ is the Hilbert space of square integrable spinors with inner product (dμ the volume form)

$$\langle \psi, \phi \rangle = \int_{\mathcal{M}} \mathrm{d}\mu\, \psi^\dagger \phi, \quad \text{for } \psi, \phi \in L^2(\mathcal{M}, \mathcal{S}), \tag{3.3}$$

and $\eth := -i\gamma^\mu \partial_\mu$ is the Euclidean Dirac operator with γ^μ the self-adjoint Euclidean Dirac matrices (see (A.2)). The real structure and grading are (cc denotes complex conjugation)

$$\mathcal{J} = i\gamma^0\gamma^2 cc = i\begin{pmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} cc, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \tag{3.4}$$

The representation (2.33) of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ on $L^2(\mathcal{M}, \mathcal{S}) = L^2(\mathcal{M}, \mathcal{S})_+ \oplus L^2(\mathcal{M}, \mathcal{S})_-$ is

$$\pi_{\mathcal{M}}(f, f') = \begin{pmatrix} f\,\mathbb{I}_2 & 0 \\ 0 & f'\,\mathbb{I}_2 \end{pmatrix}, \tag{3.5}$$

where each of the two copies of $C^\infty(\mathcal{M})$ acts independently and faithfully by pointwise multiplication on the eigenspaces $L^2(\mathcal{M}, \mathcal{S})_\pm$ of γ^5. The automorphism ρ of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ is the flip

$$\rho(f, f') = (f', f), \quad \forall f, f' \in C^\infty(\mathcal{M}). \tag{3.6}$$

It coincides with the inner automorphism of $\mathcal{B}(\mathcal{H})$ implemented by the unitary

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \tag{3.7}$$

which is nothing but the Dirac matrix γ^0 (this choice is not unique, as will be investigated in [3]). It is compatible with the real structure (2.10) with

$$\varepsilon''' = -1. \tag{3.8}$$

Lemma 3.1. *For any $a = (f, f') \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ and $\mu = 0, 1, 2, 3$, one has that*

$$\gamma^\mu a = \rho(a)\gamma^\mu, \quad \gamma^\mu \rho(a) = a\gamma^\mu, \quad \gamma^\mu \mathcal{J} = -\varepsilon' \mathcal{J}\gamma^\mu. \tag{3.9}$$

Proof. The first equation is checked by direct calculation, using the explicit form of γ^μ, along with (3.5) and (writing $\rho(a)$ for $\pi_\mathcal{M}(\rho(a))$)

$$\rho(a) = \begin{pmatrix} f'\,\mathbb{I}_2 & 0 \\ 0 & f\,\mathbb{I}_2 \end{pmatrix}. \tag{3.10}$$

The second follows from (2.11) and the third from (2.3), noticing that $\mathcal{J}$ commutes with ∂_μ, having constant components:

$$0 = \mathcal{J}\eth - \varepsilon'\eth\mathcal{J} = i\left(\mathcal{J}\gamma^\mu + \varepsilon'\gamma^\mu\mathcal{J}\right)\partial_\mu. \qquad \blacksquare$$

Corollary 3.1.1. *The boundedness of the twisted commutator follows immediately:*

$$[\eth, a]_\rho = -i\left(\gamma^\mu\partial_\mu a - \rho(a)\gamma^\mu\partial_\mu\right) = -i\gamma^\mu[\partial_\mu, a] = -i\gamma^\mu(\partial_\mu a) \quad \forall a \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2. \tag{3.11}$$

3.2. Twisted fluctuation for a manifold

Substituting, in a twisted spectral triple, $\mathcal{D}$ with the twisted covariant $\mathcal{D}_{\omega_\rho}$ (2.21) is called a *twisted fluctuation*. The minimally twisted manifold (3.2) has non-vanishing self-adjoint twisted fluctuations (2.21) of the form

$$\eth_\mathbf{X} := \eth + \mathbf{X}, \tag{3.12}$$

where

$$\mathbf{X} := -i\gamma^\mu X_\mu, \quad \text{with } X_\mu := f_\mu\gamma^5, \quad \text{for some } f_\mu \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{3.13}$$

This has been shown in [28, Prop. 5.3]; in contrast with the non-twisted case, where the self-adjoint fluctuation of $\eth$ always vanishes, irrespective of the dimension of the manifold $\mathcal{M}$ [14].

In [28], the self-adjointness of $\eth_\mathbf{X}$ was guaranteed by imposing the self-adjointness of $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$, but not necessarily the one of ω_ρ. One may worry that the non-vanishing of $\mathbf{X}$ is an artefact of this choice, and that $\mathbf{X}$ might actually vanish as soon as $\omega_\rho = \omega_\rho^\dagger$. The following lemma clarifies this point.

Lemma 3.2. *The twisted* 1*-forms* ω_ρ (2.18) *and the twisted fluctuations* $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ *of a minimally twisted four-dimensional closed Euclidean manifold are all of the kind*

$$\omega_\rho = -i\gamma^\mu W_\mu, \qquad \text{with } W_\mu = \mathsf{diag}(h_\mu \mathbb{I}_2,\ h'_\mu \mathbb{I}_2), \tag{3.14}$$

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = -i\gamma^\mu X_\mu, \quad \text{with } X_\mu = \mathsf{diag}(f_\mu \mathbb{I}_2,\ f'_\mu \mathbb{I}_2), \tag{3.15}$$

where $h_\mu, h'_\mu \in C^\infty(\mathcal{M})$*,* $f_\mu = 2\Re h_\mu$*, and* $f'_\mu = 2\Re h'_\mu$*. They are self-adjoint, respectively, iff*

$$h'_\mu = -\bar{h}_\mu \quad \text{and} \quad f'_\mu = -f_\mu. \tag{3.16}$$

Proof. By Lemma 3.1 and its corollary, one obtains for $a_i := (f_i, f'_i)$, $b_i := (g_i, g'_i) \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$,

$$\begin{aligned}\omega_\rho &= \sum_i b_i[\eth, a_i]_\rho = -i\gamma^\mu \sum_i \rho(b_i)(\partial_\mu a_i)\\ &= -i\gamma^\mu \sum_i \begin{pmatrix} g'_i \mathbb{I}_2 & 0 \\ 0 & g_i \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} (\partial_\mu f_i)\mathbb{I}_2 & 0 \\ 0 & (\partial_\mu f'_i)\mathbb{I}_2 \end{pmatrix},\end{aligned}$$

which is of the form (3.14) with $h_\mu := \sum_i g'_i(\partial_\mu f_i)$ and $h'_\mu := \sum_i g_i(\partial_\mu f'_i)$. The adjoint is

$$\omega_\rho^\dagger = iW_\mu^\dagger \gamma^\mu = i\gamma^\mu \rho(W_\mu^\dagger), \tag{3.17}$$

where the last equality follows from (3.9), applied to W_μ viewed as an element of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$. Thus ω_ρ is self-adjoint iff $\gamma^\mu\rho(W_\mu^\dagger) = -\gamma^\mu W_\mu$, that is, going back to the explicit form of γ^μ,

$$\sigma^\mu \bar{h}_\mu = -\sigma^\mu h'_\mu \quad \text{and} \quad \tilde{\sigma}^\mu \bar{h}'_\mu = -\tilde{\sigma}^\mu h_\mu. \tag{3.18}$$

Multiplying the first equation by σ^λ and using $\mathsf{Tr}(\sigma^\lambda\sigma^\mu) = 2\delta^{\mu\lambda}$ yield the first part of (3.16). Obviously, the latter implies both equations of (3.18). Hence, $\omega_\rho = \omega_\rho^\dagger$ is equivalent to the first equation of (3.16).

Further, we have that

$$\mathcal{J}\omega_\rho\mathcal{J}^{-1} = \mathcal{J}(-i\gamma^\mu W_\mu)\mathcal{J}^{-1} = i\mathcal{J}(\gamma^\mu W_\mu)\mathcal{J}^{-1} = -i\gamma^\mu \mathcal{J} W_\mu \mathcal{J}^{-1} = -i\gamma^\mu W_\mu^\dagger, \tag{3.19}$$

using $\mathcal{J}\gamma^\mu = -\gamma^\mu\mathcal{J}$ (from (3.1) and (3.9)), along with $\mathcal{J}W^\mu = W_\mu^\dagger \mathcal{J}$ (from (3.4) and the explicit form (3.14) of W_μ). Therefore,

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = -i\gamma^\mu(W_\mu + W_\mu^\dagger), \tag{3.20}$$

which is nothing but (3.15), identifying $X_\mu := W_\mu + W_\mu^\dagger = \mathsf{diag}((h_\mu + \bar{h}_\mu)\mathbb{I}_2, (h'_\mu + \bar{h}'_\mu)\mathbb{I}_2)$. One checks as above that $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ is self-adjoint iff the second equation of (3.16) holds. ■

Consequently, imposing that $\omega_\rho \neq 0$ is self-adjoint, that is, imposing (3.16) with $h_\mu \neq 0$, does not imply that X_μ vanishes (it does vanish only if h_μ is purely imaginary). In other words, as long as $h_\mu \notin i\mathbb{R}$, the self-adjointness of ω_ρ does not forbid a non-zero twisted fluctuation.

3.3. Gauge transformation

For a minimally twisted manifold, not only is the fermionic action (2.27) invariant under a gauge transformation (2.20), (2.24), but so is the operator $\mathcal{D}_{\omega_\rho}$ (in dimensions 0, 4) [29, Prop. 5.4]. We check it explicitly by studying how the field h_μ parametrising ω_ρ in (3.14) transforms.

A unitary of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ is $u := (e^{i\theta}, e^{i\theta'})$ with $\theta, \theta' \in C^\infty(\mathcal{M}, \mathbb{R})$. It (and its twist) acts on $\mathcal{H}$ according to (3.5) as (we omit the symbol of representation)

$$u = \begin{pmatrix} e^{i\theta}\mathbb{I}_2 & 0 \\ 0 & e^{i\theta'}\mathbb{I}_2 \end{pmatrix}, \quad \rho(u) = \begin{pmatrix} e^{i\theta'}\mathbb{I}_2 & 0 \\ 0 & e^{i\theta}\mathbb{I}_2 \end{pmatrix}. \tag{3.21}$$

Proposition 3.3. *Under a gauge transformation with unitary $u \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, the fields h_μ and h'_μ parametrising the twisted 1-form ω_ρ in (3.14) transform as*

$$h_\mu \to h_\mu - i\partial_\mu\theta, \quad h'_\mu \to h'_\mu - i\partial_\mu\theta'. \tag{3.22}$$

Proof. Under a gauge transformation, the twisted 1-form ω_ρ is mapped to (see (2.25))

$$\begin{aligned} \omega_\rho^u &= -i\rho(u)\big([\gamma^\mu\partial_\mu, u^*]_\rho + \gamma^\mu W_\mu u^*\big) = -i\rho(u)\gamma^\mu(\partial_\mu + W_\mu)u^* \\ &= -i\gamma^\mu(u\partial_\mu u^* + W_\mu), \end{aligned}$$

where we used (3.11) for $a = u^*$; namely

$$[\gamma^\mu\partial_\mu, u^*]_\rho = \gamma^\mu(\partial_\mu u^*), \tag{3.23}$$

as well as (3.9) for $a = u$, together with $uW_\mu u^* = W_\mu$ since u commutes with W_μ. Therefore, $W_\mu \to W_\mu + u\partial_\mu u^*$, which with the explicit representation of W_μ (3.14) and u (3.21) reads

$$\begin{pmatrix} h_\mu\mathbb{I}_2 & 0 \\ 0 & h'_\mu\mathbb{I}_2 \end{pmatrix} \to \begin{pmatrix} (h_\mu - i\partial_\mu\theta)\mathbb{I}_2 & 0 \\ 0 & (h'_\mu - i\partial_\mu\theta')\mathbb{I}_2 \end{pmatrix}. \qquad \blacksquare$$

Although h_μ, h'_μ transform in a nontrivial manner, their real parts $\frac{1}{2}f_\mu$, $\frac{1}{2}f'_\mu$ remain invariant. This explains why the fluctuation **X** in (3.14) is invariant under a gauge transformation (2.24). Furthermore, by simultaneously transforming spinors according to (2.20), the twisted fermionic action is invariant by construction. So one expects that any $\psi \in \mathcal{H}_R$ is unchanged under the adjoint action of $\mathrm{Ad}\, u$. This is true, as one checks from (3.4) that $u\mathcal{J}u\mathcal{J}^{-1} = \mathbb{I}$ for any unitary u.

3.4. Twisted fermionic action for a manifold

Let us first work out the positive eigenspace $\mathcal{H}_\mathcal{R}$ (2.26) for $\mathcal{R} = \gamma^0$ as in (3.7).

Lemma 3.4. *An eigenvector $\phi \in \mathcal{H}_\mathcal{R}$ is of the form $\phi := \binom{\varphi}{\varphi}$, where φ is a Weyl spinor.*

Proof. The $+1$-eigenspace of γ^0 is spanned by $\upsilon_1 = \binom{1}{0} \otimes \binom{1}{1}$, $\upsilon_2 = \binom{0}{1} \otimes \binom{1}{1}$. Therefore, a generic vector $\phi = \phi_1 \upsilon_1 + \phi_2 \upsilon_2$ in $\mathcal{H}_{\mathcal{R}}$ is as in the lemma, with $\varphi := \binom{\phi_1}{\phi_2}$. ■

We now compute the fermionic action (2.27) for a minimally twisted manifold.

Proposition 3.5. *Let $\eth_X$ be the twist-fluctuated Dirac operator (3.12). The bilinear form (2.23) restricted to $\mathcal{H}_{\mathcal{R}}$ (antisymmetric by Lemma 2.8) is*

$$\mathfrak{A}^{\rho}_{\eth_X}(\phi, \xi) = 2 \int_{\mathcal{M}} \mathrm{d}\mu \, \bar{\varphi}^{\dagger} \sigma_2 \left(i f_0 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \zeta, \tag{3.24}$$

where φ, ζ are, respectively, the Weyl components of the Dirac spinors $\phi, \xi \in \mathcal{H}_{\mathcal{R}}$, and f_0 is the zeroth component of the twisted fluctuation f_μ in (3.13).

Proof. One has that

$$\mathcal{J}\phi = i\gamma^0\gamma^2 \, cc \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} = i \begin{pmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} \bar{\varphi} \\ \bar{\varphi} \end{pmatrix} = i \begin{pmatrix} \tilde{\sigma}^2 \, \bar{\varphi} \\ \sigma^2 \, \bar{\varphi} \end{pmatrix}, \tag{3.25}$$

$$\eth\xi = -i\gamma^\mu \partial_\mu \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \partial_\mu \zeta \\ \partial_\mu \zeta \end{pmatrix} = -i \begin{pmatrix} \sigma^\mu \partial_\mu \zeta \\ \tilde{\sigma}^\mu \partial_\mu \zeta \end{pmatrix}, \tag{3.26}$$

$$\begin{aligned} \mathbf{X}\xi &= -i\gamma^\mu X_\mu \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} f_\mu \mathbb{I}_2 & 0 \\ 0 & -f_\mu \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \\ &= -i \begin{pmatrix} -f_\mu \sigma^\mu \zeta \\ f_\mu \tilde{\sigma}^\mu \zeta \end{pmatrix}. \end{aligned} \tag{3.27}$$

Hence, noticing that $(\tilde{\sigma}^2)^\dagger = -i\sigma_2$ and $\sigma^{2\dagger} = i\sigma_2$ (see Appendix B), and using

$$\sigma^\mu + \tilde{\sigma}^\mu = 2\mathbb{I}_2 \delta^{\mu 0}, \quad \sigma^\mu - \tilde{\sigma}^\mu = -2i\delta^{\mu j} \sigma_j, \tag{3.28}$$

one gets

$$\mathfrak{A}_{\eth}(\phi, \xi) = \langle \mathcal{J}\phi, \eth\xi \rangle = -\left(\bar{\varphi}^\dagger \tilde{\sigma}^{2\dagger}, \bar{\varphi}^\dagger \sigma^{2\dagger} \right) \begin{pmatrix} \sigma^\mu \partial_\mu \zeta \\ \tilde{\sigma}^\mu \partial_\mu \zeta \end{pmatrix} \tag{3.29}$$

$$= i \int_{\mathcal{M}} \mathrm{d}\mu \, \bar{\varphi}^\dagger \sigma_2 (\sigma^\mu - \tilde{\sigma}^\mu) \partial_\mu \zeta = 2 \int_{\mathcal{M}} \mathrm{d}\mu \, \bar{\varphi}^\dagger \sigma_2 \sum_{j=1}^{3} \sigma_j \partial_j \zeta; \tag{3.30}$$

$$\mathfrak{A}_{\mathbf{X}}(\phi, \xi) = \langle \mathcal{J}\phi, \mathbf{X}\xi \rangle = -\left(\bar{\varphi}^\dagger \tilde{\sigma}^{2\dagger}, \bar{\varphi}^\dagger \sigma^{2\dagger} \right) \begin{pmatrix} -f_\mu \sigma^\mu \zeta \\ f_\mu \tilde{\sigma}^\mu \, \zeta \end{pmatrix} \tag{3.31}$$

$$= -i \int_{\mathcal{M}} \mathrm{d}\mu \, \bar{\varphi}^\dagger \sigma_2 f_\mu (\sigma^\mu + \tilde{\sigma}^\mu) \partial_\mu \zeta = -2i \int_{\mathcal{M}} \mathrm{d}\mu \, f_0 \bar{\varphi}^\dagger \sigma_2 \, \zeta. \tag{3.32}$$

From Lemma 2.8 and (3.8), it follows that

$$\mathfrak{A}^{\rho}_{\eth_{\mathbf{X}}}(\phi, \xi) = -\mathfrak{A}_{\eth_{\mathbf{X}}}(\phi, \xi) = -\mathfrak{A}_{\eth}(\phi, \xi) - \mathfrak{A}_{\mathbf{X}}(\phi, \xi). \tag{3.33}$$

Hence the result. ■

The fermionic action is then obtained by substituting $\phi = \xi$ in (3.24) and replacing the components ζ of ξ by the associated Graßmann variable $\widetilde{\zeta}, \widetilde{\varphi}$:

$$S_\rho(\eth_{\mathbf{X}}) = \mathfrak{A}^\rho_{\eth_{\mathbf{X}}}(\widetilde{\xi}, \widetilde{\xi}) = 2 \int_{\mathcal{M}} \mathrm{d}\mu \left[\widetilde{\overline{\zeta}}^\dagger \sigma_2 \left(i f_0 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \widetilde{\zeta} \right]. \tag{3.34}$$

The striking fact about (3.34) is the disappearance of the derivative in the x_0 direction, and the appearance, instead, of the zeroth component of the real field f_μ parametrising the twisted fluctuation $\mathbf{X}$. This derivative, however, can be restored by interpreting $-i f_0 \zeta$ as $\partial_0 \zeta$, i.e., assuming that

$$\zeta(x_0, x_i) = \exp(-i f_0 x_0)\, \zeta(x_i) \tag{3.35}$$

with f_0 independent of x_0. Denoting by $\sigma^\mu_M = \{\mathbb{I}_2, \sigma_j\}$ the upper-right components of the Minkowskian Dirac matrices (see (A.4)), the integrand in the fermionic action then reads (with summation on the μ index)

$$-\widetilde{\overline{\zeta}}^\dagger \sigma^2_M (\sigma^\mu_M \partial_\mu) \widetilde{\zeta}, \tag{3.36}$$

which reminds of the Weyl Lagrangian densities (B.7)

$$\mathcal{L}^r_M = i \Psi^\dagger_r (\sigma^\mu_M \partial_\mu) \Psi_r, \tag{3.37}$$

but with the σ^2_M matrix that prevents to simultaneously identify $\widetilde{\zeta}$ with Ψ_r and $-\widetilde{\overline{\zeta}}^\dagger \sigma^2_M$ with $i \Psi^\dagger_r$.

To make such an identification possible, one needs more spinorial degrees of freedom. They are obtained in the next section, multiplying the manifold by a two-point space.

4. Doubled manifold and Weyl equations

In constructing a spectral triple for electrodynamics, the authors of [37, §3.2] first consider, as an intermediate step, the product of a manifold with the finite-dimensional spectral triple

$$\mathcal{A}_{\mathcal{F}} = \mathbb{C}^2, \quad \mathcal{H}_{\mathcal{F}} = \mathbb{C}^2, \quad D_{\mathcal{F}} = 0. \tag{4.1}$$

This model describes a $U(1)$ gauge theory, but fails to describe classical electrodynamics for two reasons, discussed at the end of [37, §3]: first, the finite Dirac operator is zero, so the electrons are massless; second, $\mathcal{H}_{\mathcal{F}}$ is not big enough to capture the required spinor degrees-of-freedom.

However, none of the above arises as an issue if one wishes to obtain the Weyl Lagrangian, since the Weyl fermions are massless anyway, and they only need half of the spinor degrees-of-freedom as compared to the Dirac fermions.

4.1. Minimal twist of a two-point almost-commutative geometry

The product – in the sense of spectral triple – of a four-dimensional closed Euclidean manifold $\mathcal{M}$ with the two-point space (4.1) is

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad D = \eth \otimes \mathbb{I}_2, \tag{4.2}$$

with real structure $J = \mathcal{J} \otimes J_{\mathcal{F}}$ and grading $\Gamma = \gamma^5 \otimes \gamma_{\mathcal{F}}$, where $\eth$, $\mathcal{J}$, γ^5 are as in (3.4), while

$$J_{\mathcal{F}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} cc, \quad \gamma_{\mathcal{F}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.3}$$

in the orthonormal basis $\{e, \bar{e}\}$ of $\mathcal{H}_{\mathcal{F}} = \mathbb{C}^2$. The algebra $\mathcal{A} \ni a := (f, g)$ acts on $\mathcal{H}$ as

$$\pi_0(a) := \begin{pmatrix} f\mathbb{I}_4 & 0 \\ 0 & g\mathbb{I}_4 \end{pmatrix}, \quad \forall f, g \in C^\infty(\mathcal{M}). \tag{4.4}$$

Following Section 2.4, the minimal twist of (4.2) is given by the algebra $\mathcal{A} \otimes \mathbb{C}^2$, acting on $\mathcal{H}$ as

$$\pi(a, a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g'\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_2 \end{pmatrix} =: \begin{pmatrix} F & 0 \\ 0 & G' \end{pmatrix}, \tag{4.5}$$

for $a := (f, g)$, $a' := (f', g') \in \mathcal{A}$; with twist

$$\pi\big(\rho(a, a')\big) = \pi(a', a) = \begin{pmatrix} f'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix} =: \begin{pmatrix} F' & 0 \\ 0 & G \end{pmatrix}. \tag{4.6}$$

In both of the equations above, we have denoted

$$\begin{aligned} F &:= \pi_{\mathcal{M}}(f, f'), \quad & F' &:= \pi_{\mathcal{M}}(f', f), \\ G &:= \pi_{\mathcal{M}}(g, g'), \quad & G' &:= \pi_{\mathcal{M}}(g', g), \end{aligned} \tag{4.7}$$

where $\pi_{\mathcal{M}}$ is the representation (3.5) of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ on $L^2(\mathcal{M}, \mathcal{S})$.

4.2. Twisted fluctuation of a doubled manifold

We begin with some notations and a technical lemma. Following (3.13) and (4.7), given any $Z_\mu = \pi_{\mathcal{M}}(f_\mu, f'_\mu)$ with $f_\mu, f'_\mu \in C^\infty(\mathcal{M})$, we denote $Z'_\mu = \pi_{\mathcal{M}}(f'_\mu, f_\mu)$ and

$$\mathbf{Z} := -i\gamma^\mu Z_\mu, \quad \mathbf{Z}' := -i\gamma^\mu Z'_\mu, \quad \bar{\mathbf{Z}} := -i\gamma^\mu \bar{Z}_\mu. \tag{4.8}$$

Notice that $\bar{\mathbf{Z}}$ *is not* the complex conjugate of $\mathbf{Z}$, since in (4.8), the complex conjugation acts neither on i nor on the Dirac matrices. This guarantees that $^{-}$ and $'$ commute not only for Z_μ, i.e., $\overline{Z'_\mu} = (\bar{Z}_\mu)' = \pi_{\mathcal{M}}(\overline{f'_\mu}, \overline{f_\mu})$, but also for $\mathbf{Z}$, i.e.,

$$(\bar{\mathbf{Z}})' = \bar{\mathbf{Z}}'. \tag{4.9}$$

The notation $\bar{\mathbf{Z}}'$ is thus unambiguous and denotes indistinctly the two members of (4.9).

Lemma 4.1. *For any F, G, Z_μ as in* (4.7), (4.8), *one has that*

$$F[\eth, G]_\rho = -i\gamma^\mu F'\partial_\mu G, \quad \mathcal{J}\mathbf{Z}\mathcal{J}^{-1} = \bar{\mathbf{Z}}, \quad \mathbf{Z}^\dagger = -\bar{\mathbf{Z}}'. \tag{4.10}$$

Proof. Equation (3.11) for $a = G$ yields $[\eth, G]_\rho = -i\gamma^\mu\partial_\mu G$, while (3.9) for $a = F'$ gives

$$F\gamma^\mu = \gamma^\mu F'. \tag{4.11}$$

Thus $F[\eth, G]_\rho = -iF\gamma^\mu\partial_\mu G = -i\gamma^\mu F'\partial_\mu G$. The second equation in (4.10) follows from

$$\mathcal{J}\mathbf{Z}\mathcal{J}^{-1} = i\mathcal{J}\gamma^\mu Z_\mu\mathcal{J}^{-1} = -i\gamma^\mu\mathcal{J}Z_\mu\mathcal{J}^{-1} = -i\gamma^\mu\bar{Z}_\mu = \bar{\mathbf{Z}}, \tag{4.12}$$

where we used (3.9) as well as (recalling that in KO-dimension 4, one has that $\mathcal{J}^{-1} = -\mathcal{J}$)

$$\mathcal{J}Z_\mu\mathcal{J}^{-1} = -i\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix} cc \begin{pmatrix} f_\mu\,\mathbb{I}_2 & 0\\ 0 & f'_\mu\,\mathbb{I}_2\end{pmatrix} i \begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix} cc, \tag{4.13}$$

$$= -\begin{pmatrix}\tilde{\sigma}^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix} \bar{f}_\mu\,\mathbb{I}_2 & 0\\ 0 & \bar{f}'_\mu\,\mathbb{I}_2\end{pmatrix}\begin{pmatrix}\bar{\tilde{\sigma}}^2 & 0\\ 0 & \bar{\sigma}^2\end{pmatrix} = \begin{pmatrix} \bar{f}_\mu\,\mathbb{I}_2 & 0\\ 0 & \bar{f}'_\mu\,\mathbb{I}_2\end{pmatrix} = \bar{Z}_\mu, \tag{4.14}$$

noticing that $\bar{\tilde{\sigma}}^2 = \tilde{\sigma}^2$ and $\bar{\sigma}^2 = \sigma^2$, so that $\tilde{\sigma}^2\bar{\tilde{\sigma}}^2 = \sigma^2\bar{\sigma}^2 = -\mathbb{I}_2$. The third equation in (4.10) follows from

$$\mathbf{Z}^\dagger = iZ_\mu^\dagger\gamma^\mu = i\bar{Z}_\mu\gamma^\mu = i\gamma^\mu(\bar{Z}_\mu)' = i\gamma^\mu\overline{Z'_\mu} = -\bar{\mathbf{Z}}', \tag{4.15}$$

where we notice that $Z_\mu^\dagger = \bar{Z}_\mu$, from the explicit form (3.5) of $\pi_{\mathcal{M}}$, then use (4.11). ∎

With this lemma, it is easy to compute the twisted fluctuation $\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1}$ for a generic twisted 1-form

$$\omega_\rho := \pi(a, a')\big[\eth\otimes\mathbb{I}_2,\ \pi(b, b')\big]_\rho \tag{4.16}$$

for $a = (f, g)$, $a' = (f', g')$, $b = (v, w)$, and $b' = (v', w')$ in $\mathcal{A}$.

Lemma 4.2. *One has that*

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \mathbf{X}\otimes\mathbb{I}_2 + i\mathbf{Y}\otimes\gamma_{\mathcal{F}}, \tag{4.17}$$

with $\mathbf{X} = -i\gamma^\mu X_\mu$, $\mathbf{Y} = -i\gamma^\mu Y_\mu$ for

$$X_\mu = \pi_{\mathcal{M}}(f_\mu, f'_\mu), \quad Y_\mu = \pi_{\mathcal{M}}(g_\mu, g'_\mu), \tag{4.18}$$

where f_μ, f'_μ and g_μ, g'_μ denote, respectively, the real and the imaginary parts of

$$z_\mu := f'\partial_\mu v + \bar{g}\partial_\mu\bar{w}' \quad \textit{and} \quad z'_\mu = f\partial_\mu v' + \bar{g}'\partial_\mu\bar{w}'. \tag{4.19}$$

Proof. Define

$$V := \pi_{\mathcal{M}}(v, v'), \quad V' := \pi_{\mathcal{M}}(v', v), \quad W := \pi_{\mathcal{M}}(w, w'), \quad W' = \pi_{\mathcal{M}}(w', w). \tag{4.20}$$

From (4.5)–(4.6), one gets

$$\left[\eth \otimes \mathbb{I}_2,\ \pi(b, b')\right]_\rho = \begin{pmatrix} [\eth, V]_\rho & 0 \\ 0 & [\eth, W']_\rho \end{pmatrix}, \tag{4.21}$$

so that, for (a, a') as in (4.5) and using (4.10), one finds that

$$\omega_\rho := \begin{pmatrix} F & 0 \\ 0 & G' \end{pmatrix} \begin{pmatrix} [\eth, V]_\rho & 0 \\ 0 & [\eth, W']_\rho \end{pmatrix} = \begin{pmatrix} -i\gamma^\mu P_\mu & 0 \\ 0 & -i\gamma^\mu Q'_\mu \end{pmatrix} = \begin{pmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q}' \end{pmatrix}, \tag{4.22}$$

with

$$P_\mu := F'\partial_\mu V, \quad Q'_\mu := G\partial_\mu W'. \tag{4.23}$$

The explicit form of the real structure and its inverse,

$$J = \mathcal{J} \otimes J_F = \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}, \quad J^{-1} = \begin{pmatrix} 0 & \mathcal{J}^{-1} \\ \mathcal{J}^{-1} & 0 \end{pmatrix}, \tag{4.24}$$

along with the second equation of (4.10), yield

$$J\omega_\rho J^{-1} = \begin{pmatrix} \mathcal{J}\mathbf{Q}'\mathcal{J}^{-1} & 0 \\ 0 & \mathcal{J}\mathbf{P}\mathcal{J}^{-1} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{Q}}' & 0 \\ 0 & \bar{\mathbf{P}} \end{pmatrix}. \tag{4.25}$$

Summing up (4.22) and (4.25), one obtains (4.29)

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \begin{pmatrix} \mathbf{Z} & 0 \\ 0 & \bar{\mathbf{Z}} \end{pmatrix}, \tag{4.26}$$

where $\mathbf{Z} := \mathbf{P} + \bar{\mathbf{Q}}' = -i\gamma^\mu Z_\mu$ with

$$\begin{aligned} Z_\mu &= P_\mu + \bar{Q}'_\mu = F'\partial_\mu V + \bar{G}\partial_\mu \bar{W}' \\ &= \begin{pmatrix} (f'\partial_\mu v + \bar{g}\partial_\mu \bar{w}')\mathbb{I}_2 & 0 \\ 0 & (f\partial_\mu v' + \bar{g}'\partial_\mu \bar{w})\mathbb{I}_2 \end{pmatrix} \end{aligned} \tag{4.27}$$

(the last equation follows from the explicit form (4.20) of V, W' and (4.7) of F', G). By (4.19), this reads as

$$Z_\mu = \pi_{\mathcal{M}}(z_\mu, z'_\mu) = \pi_{\mathcal{M}}(f_\mu, f'_\mu) + i\pi_{\mathcal{M}}(g_\mu, g'_\mu) = X_\mu + iY_\mu. \tag{4.28}$$

Similarly, $\bar{\mathbf{Z}} = -i\gamma^\mu \bar{Z}_\mu$ with $\bar{Z}_\mu = X_\mu - iY_\mu$. Hence, (4.26) yields

$$\omega_\rho + \mathcal{J}\omega_\rho\mathcal{J}^{-1} = \begin{pmatrix} -i\gamma^\mu(X_\mu + iY_\mu) & 0 \\ 0 & -i\gamma^\mu(X_\mu - iY_\mu) \end{pmatrix}, \tag{4.29}$$

which is nothing but (4.17). ∎

Proposition 4.3. *The self-adjoint twisted fluctuations of the Dirac operator of the doubled manifold are parametrised by two real fields f_μ and g_μ in $C^\infty(\mathcal{M}, \mathbb{R})$, and are of the form*

$$\eth_{\mathbf{X}} \otimes \mathbb{I}_2 + g_\mu \gamma^\mu \otimes \gamma_{\mathcal{F}}, \tag{4.30}$$

where $\eth_X$ is the twisted-covariant operator (3.12) of a manifold.

Proof. A generic twisted fluctuation (4.26) (adding a summation index i and redefining $\mathbf{Z} = \sum_i \mathbf{Z}_i$) is self-adjoint iff $\mathbf{Z} = \mathbf{Z}^\dagger$ and $\bar{\mathbf{Z}} = \bar{\mathbf{Z}}^\dagger$. By (4.9), and the third equation in (4.10), both conditions are equivalent to $\mathbf{Z} = -\bar{\mathbf{Z}}'$; that is, $-i\gamma^\mu(Z_\mu + \bar{Z}'_\mu) = 0$. As discussed below (3.18), this is equivalent to $Z_\mu = -\bar{Z}'_\mu$. From (4.27), this last condition is equivalent to $z_\mu = -\bar{z}'_\mu$; that is,

$$f_\mu = -f'_\mu \quad \text{and} \quad g_\mu = g'_\mu. \tag{4.31}$$

Substituting in (4.18), one obtains

$$X_\mu = \pi_{\mathcal{M}}(f_\mu, -f_\mu) = f_\mu \gamma^5, \quad Y_\mu = \pi_{\mathcal{M}}(g_\mu, g_\mu) = g_\mu \mathbb{I}_4, \tag{4.32}$$

so that (4.17) gives

$$\omega_\rho + \mathcal{J}\omega_\rho \mathcal{J}^{-1} = -i\gamma^\mu f_\mu \gamma^5 \otimes \mathbb{I}_2 + g_\mu \gamma^\mu \otimes \gamma_{\mathcal{F}}. \tag{4.33}$$

The result follows adding $\eth \otimes \mathbb{I}_2$. ∎

Self-adjointness is illustrated directly into the bold notation: by (4.31), $\mathbf{X} \otimes \mathbb{I}_2 + i\mathbf{Y} \otimes \gamma_{\mathcal{F}}$ is self-adjoint iff $\mathbf{X}' = -\mathbf{X}$ and $\mathbf{Y}' = \mathbf{Y}$. Since $\mathbf{X} = \bar{\mathbf{X}}$, $\mathbf{Y} = \bar{\mathbf{Y}}$ by construction, this is equivalent by the third equation of (4.10) to $\mathbf{X} = \mathbf{X}^\dagger$ and $\mathbf{Y} = -\mathbf{Y}^\dagger$.

4.3. Weyl equations from the twisted fermionic action

We show that the action defined by the component $\eth_{\mathbf{X}} \otimes \mathbb{I}_2$ of the twisted covariant Dirac operator (4.30) of the doubled manifold (i.e., we assume that $g_\mu = 0$) yields the Weyl equations. Non-vanishing g_μ will be taken into account in the spectral triple of electrodynamics.

Following the choice made in (3.7), we take as a unitary implementing the action of ρ on $\mathcal{H}$

$$\mathcal{R} = \gamma^0 \otimes \mathbb{I}_2. \tag{4.34}$$

It has eigenvalues ± 1 and is compatible with the real structure in the sense of (2.10) with $\varepsilon''' = -1$. A generic element η in the $+1$-eigenspace $\mathcal{H}_R$ is

$$\eta = \phi \otimes e + \xi \otimes \bar{e}, \quad \text{with } \phi := \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}, \ \xi := \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}, \tag{4.35}$$

where $\phi, \xi \in L^2(\mathcal{M}, \mathcal{S})$ are Dirac the eigenspinors of γ^0 (Lemma 3.4), with Weyl components φ, ζ.

Proposition 4.4. *The twisted fermionic action induced by* $\eth \otimes \mathbb{I}_2$ *on the doubled manifold is*

$$S_\rho(\eth_{\mathbf{X}} \otimes \mathbb{I}_2) = 2\,\mathfrak{A}^\rho_{\eth_{\mathbf{X}}}(\tilde{\phi}, \tilde{\xi}) = 4 \int_{\mathcal{M}} d\mu \left[\bar{\tilde{\varphi}}^\dagger \sigma_2 \Big(i f_0 - \sum_{j=1}^{3} \sigma_j \partial_j \Big) \tilde{\xi} \right]. \tag{4.36}$$

Proof. For $\eta, \eta' \in \mathcal{H}_{\mathcal{R}}$ given by (4.35), remembering that $J_{\mathcal{F}} e = \bar{e}$ and $J_{\mathcal{F}} \bar{e} = e$, one has that

$$J\eta = \mathcal{J}\phi \otimes \bar{e} + \mathcal{J}\xi \otimes e, \quad (\eth_X \otimes \mathbb{I}_2)\eta' = \eth_X \phi' \otimes e + \eth_X \xi' \otimes \bar{e}.$$

So, Lemma 2.8 with $\varepsilon''' = -1$ yields

$$\mathfrak{A}^\rho_{\eth_{\mathbf{X}} \otimes \mathbb{I}_2}(\eta, \eta') = -\big\langle J\eta, (\eth_{\mathbf{X}} \otimes \mathbb{I}_2)\eta' \big\rangle = -\langle \mathcal{J}\phi, \eth_X \xi' \rangle - \langle \mathcal{J}\xi, \eth_X \phi' \rangle, \tag{4.37}$$

$$= -\mathfrak{A}_{\eth_{\mathbf{X}}}(\phi, \xi') - \mathfrak{A}_{\eth_{\mathbf{X}}}(\xi, \phi') = \mathfrak{A}^\rho_{\eth_{\mathbf{X}}}(\phi, \xi') + \mathfrak{A}^\rho_{\eth_{\mathbf{X}}}(\xi, \phi'), \tag{4.38}$$

where the first inner product is in $\mathcal{H}$ and the second is in $L^2(\mathcal{M}, S)$. The action is then obtained substituting $\eta' = \eta$ and promoting ζ, φ to Graßmann variables. The antisymmetric bilinear form $\mathfrak{A}^\rho_{\eth_X}$ becomes symmetric when evaluated on Graßmann variables (as in the proof of [37, Prop. 4.3]), hence

$$\mathfrak{A}^\rho_{\eth_{\mathbf{X}} \otimes \mathbb{I}_2}(\tilde{\eta}, \tilde{\eta}) = 2\mathfrak{A}^\rho_{\eth_{\mathbf{X}}}(\tilde{\phi}, \tilde{\xi}). \tag{4.39}$$

The result then follows from Proposition 3.5. ∎

Identifying the physical Weyl spinors as

$$\psi := \tilde{\zeta}, \quad \psi^\dagger := \pm i \bar{\tilde{\varphi}}^\dagger \sigma_2 \tag{4.40}$$

(the sign is discussed below), the Lagrangian density in the action (4.36) becomes

$$\mathcal{L} = \mp 4 i \psi^\dagger \Big(i f_0 - \sum_j \sigma_j \partial_j \Big) \psi. \tag{4.41}$$

The Euler–Lagrange equation for $\psi^\dagger$ yields the equation of motion

$$\Big(i f_0 - \sum_j \sigma_j \partial_j \Big) \psi = 0. \tag{4.42}$$

Proposition 4.5. *For* f_0, *a non-zero constant, a plane wave solution of* (4.42) *coincides with the left-handed solutions of the Weyl equation with momentum* $p_0 = -f_0$, *or with the right-handed solution with momentum* $p = f_0$.

Proof. A plane wave solution (B.8) of (4.42) satisfies $(f_0 + \sum_j \sigma_j p_j)\psi_l = 0$. This is equivalent to the first equation of (B.9) with $p_0 = -f_0$, or to the second one of (B.9) with $p_0 = f_0$. ∎

One may also identify directly the Lagrangian (4.41) with the Weyl Lagrangians $\mathcal{L}^l_M$, $\mathcal{L}^r_M$ (B.6). Choosing the minus sign in (4.40) (that is the plus sign in (4.41)), then $\mathcal{L}$

coincides (up to a global factor 4) with $\mathcal{L}_M^l$ as soon as one imposes $\partial_0\psi = if_0\psi$ (meaning, for a plane wave solution, $p_0 = -f_0$). Choosing instead the plus sign, then $\mathcal{L}$ coincides with $\mathcal{L}_M^r$, as soon as one imposes $\partial_0\psi = -if_0\psi$ (meaning $p_0 = f_0$).

Proposition 4.5 gives weight to the observation made after Proposition 3.5: identifying x_0 with the time coordinate of Minkowski spacetime, then the fermionic action $S_\rho(\eth\otimes\mathbb{I}_2)$ of a twisted doubled manifold – without fluctuation – yields the spatial part of the Weyl equations (that is the Lagrangian (4.36) with $f_0 = 0$). For a non-zero but constant f_0, the twisted fluctuation does not only bring back a fourth component, but allows its interpretation as a time direction. It also provides a clear interpretation of f_0 as the zeroth component of the momentum, that is, an energy.

Even though the Lagrangian density is Lorentzian, one may argue the action is not the Weyl one, for the manifold over which one integrates is still Riemannian. We come back to this in the conclusion.

In these two examples – manifold and doubled manifold – the main difference between the twisted and the usual fermionic actions does not lay so much in the twist of the inner product than in the restriction to different subspaces. Indeed, by Lemma 2.8 the twist of the inner product just amounts to a global sign. As stressed in the following remark, this is the restriction to $\mathcal{H}_{\mathcal{R}}$ instead of $\mathcal{H}_+$ that explains the change of signature.

Remark 4.6. The disappearance of ∂_0 has no analogous counterpart in the non-twisted case. In that case, $\psi\in\mathcal{H}_+$ and there is no fluctuation $\mathbf{X}$, so that

- for a manifold, the usual fermionic action $\langle \mathcal{J}\widetilde{\psi}, \eth\widetilde{\psi}\rangle$ vanishes since $\eth\psi\in\mathcal{H}_-$ while $\mathcal{J}\psi\in\mathcal{H}_+$;
- for a doubled manifold, $\mathcal{H}_+$ is spanned by $\{\xi\otimes e, \phi\otimes\bar{e}\}$ with $\xi = \binom{c\zeta}{0}$, $\phi = \binom{c0}{\varphi}$. Then
$$S(\eth\otimes\mathbb{I}_2) = 2\langle\mathcal{J}\widetilde{\phi}, \eth\widetilde{\xi}\rangle = -2\int_{\mathcal{M}} \mathrm{d}\mu\, \widetilde{\bar{\varphi}}^\dagger\sigma^2\tilde{\sigma}^\mu\partial_\mu\tilde{\zeta}. \tag{4.43}$$
By (4.40), the integrand is the Euclidean version $\mathcal{L}_E^l := i\Psi_l^\dagger\tilde{\sigma}^\mu\partial_\mu\Psi_l$ of the Weyl Lagrangian $\mathcal{L}_M^l$.

Following the result of Section 3.3, one expects that the field f_μ remains invariant under a gauge transformation. In order not to make the paper too long, we do not check this here, but we will do it for the spectral triple of electrodynamics in Section 5.3. We will also give there the meaning of the other field g_μ that parametrises the twisted fluctuation in Proposition 4.3. As in the non-twisted case, this will identify with the $U(1)$ gauge field of electrodynamics.

5. Minimal twist of electrodynamics and Dirac equation

We first introduce the spectral triple of electrodynamics (as formalised in [37, 38]), then write down its minimal twist (Section 5.1) following the recipe prepared in Section 2.4.

We compute the twisted fluctuation in Section 5.2. Gauge transformations are investigated in Section 5.3: in addition to the X_μ field encountered already for the minimal twist of the (doubled) manifold, we obtain a $U(1)$ gauge field. Finally, we compute the fermionic action in Section 5.4 and derive the Lorentzian Dirac equation.

5.1. Minimal twist of electrodynamics

The spectral triple of electrodynamics is the product of a Riemannian manifold $\mathcal{M}$ (still assumed to be four-dimensional) by a two-point space like (4.1), except that $\mathcal{D}_{\mathcal{F}}$ is no longer zero (since fermions are massive). In order to satisfy the axioms of noncommutative geometry, this forces to enlarge $\mathcal{H}_{\mathcal{F}}$ from $\mathbb{C}^2$ to $\mathbb{C}^4$ (see [37, 38] for details). Hence

$$\begin{aligned} \mathcal{A}_{\text{ED}} &= C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4, \quad \mathcal{D} = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes \mathcal{D}_{\mathcal{F}}; \\ J &= \mathcal{J} \otimes J_{\mathcal{F}}, \quad \Gamma = \gamma^5 \otimes \gamma_{\mathcal{F}}, \end{aligned} \tag{5.1}$$

where $\eth$, $\mathcal{J}$, γ^5 are as in (3.4), $d \in \mathbb{C}$ is a constant parameter, and

$$D_{\mathcal{F}} = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix},$$

$$J_{\mathcal{F}} = \begin{pmatrix} 0 & 0 & cc & 0 \\ 0 & 0 & 0 & cc \\ cc & 0 & 0 & 0 \\ 0 & cc & 0 & 0 \end{pmatrix}, \tag{5.2}$$

$$\gamma_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

written in an orthonormal basis $\{e_L, e_R, \overline{e_L}, \overline{e_R}\}$ of $\mathcal{H}_{\mathcal{F}} = \mathbb{C}^4$. The algebra $\mathcal{A}_{\text{ED}} \ni a := (f, g)$ acts on $\mathcal{H}$ as

$$\pi_0(a) := \begin{pmatrix} f\mathbb{I}_4 & 0 & 0 & 0 \\ 0 & f\mathbb{I}_4 & 0 & 0 \\ 0 & 0 & g\mathbb{I}_4 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_4 \end{pmatrix}, \quad \forall f, g \in C^\infty(\mathcal{M}). \tag{5.3}$$

Inner fluctuations are parametrised by a single $U(1)$ gauge field $Y_\mu \in C^\infty(\mathcal{M}, \mathbb{R})$ [37, eq. (4.3)]:

$$D \to D_\omega = D + \gamma^\mu \otimes B_\mu, \quad B_\mu := \text{diag}(Y_\mu, Y_\mu, -Y_\mu, -Y_\mu), \tag{5.4}$$

carrying an adjoint action of a unitary $u := e^{i\theta} \in C^\infty(\mathcal{M}, U(1))$ on D_ω, implemented by

$$Y_\mu \to Y_\mu - iu\partial_\mu u^* = Y_\mu - \partial_\mu\theta, \quad \theta \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{5.5}$$

Computing the action (fermionic and bosonic, via the spectral action formula), one gets that this field is the $U(1)$ gauge potential of electrodynamics.

A minimal twist is obtained by replacing $\mathcal{A}_{\text{ED}}$ by $\mathcal{A} = \mathcal{A}_{\text{ED}} \otimes \mathbb{C}^2$ along with its flip automorphism ρ (2.34), with the representation π_0 of $\mathcal{A}$ defined by (2.33). Explicitly,

$$\Gamma = \gamma^5 \otimes \gamma_{\mathcal{F}} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{5.6}$$

so that the projections $\mathfrak{p}_\pm = \frac{1}{2}(\mathbb{I}_{16} \pm \Gamma)$ on the eigenspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ are

$$\begin{aligned} \mathfrak{p}_+ &= \mathsf{diag}(\mathbb{I}_2, 0_2, 0_2, \mathbb{I}_2, 0_2, \mathbb{I}_2, \mathbb{I}_2, 0_2), \\ \mathfrak{p}_- &= \mathsf{diag}(0_2, \mathbb{I}_2, \mathbb{I}_2, 0_2, \mathbb{I}_2, 0_2, 0_2, \mathbb{I}_2). \end{aligned} \tag{5.7}$$

Therefore, for $(a, a') \in \mathcal{A}$, where $a := (f, g)$, $a' := (f', g')$ with $f, g, f', g' \in C^\infty(\mathcal{M})$, one has that

$$\pi(a, a') = \mathfrak{p}_+\pi_0(a) + \mathfrak{p}_-\pi_0(a')$$
$$= \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f\mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix}$$
$$=: \begin{pmatrix} F & 0 & 0 & 0 \\ 0 & F' & 0 & 0 \\ 0 & 0 & G' & 0 \\ 0 & 0 & 0 & G \end{pmatrix}, \tag{5.8}$$

where F, F', G, and G' are as in (4.7). The image of $(a, a') \in \mathcal{A}$ under the flip ρ is represented by

$$\pi\big(\rho(a,a')\big) = \pi(a',a) = \begin{pmatrix} F' & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G' \end{pmatrix}. \tag{5.9}$$

In agreement with (3.7), we choose as unitary $\mathcal{R} \in \mathcal{B}(\mathcal{H})$ implementing the twist

$$\mathcal{R} = \gamma^0 \otimes \mathbb{I}_4 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \otimes \mathbb{I}_4. \tag{5.10}$$

It is compatible with the real structure in the sense of (2.10) with $\varepsilon''' = -1$, as before.

5.2. Twisted fluctuation of the Dirac operator

The twisted commutator $[D, a]_\rho$ being linear in D, we treat separately the free part $\eth \otimes \mathbb{I}_4$ and the finite part $\gamma^5 \otimes D_{\mathcal{F}}$ of the Dirac operator. The results are summarised in Proposition 5.6.

5.2.1. The free part. We show (Proposition 5.3 below) that self-adjoint twisted fluctuations of $\eth \otimes \mathbb{I}_4$ are parametrised by two real fields: X_μ arising from the minimal twist of a manifold (3.12) and the $U(1)$ gauge field Y_μ of electrodynamics. To arrive there, we need a couple of lemmas.

Lemma 5.1. *For $a = (f, g)$, $b = (v, w)$ in $\mathcal{A}_{ED}$, and similar definition for a', b', one has that*

$$\omega_{\rho_{\mathcal{M}}} := \pi(a,a')\big[\eth \otimes \mathbb{I}_4,\, \pi(b,b')\big]_\rho = \begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix}, \tag{5.11}$$

where we use the notation (4.8) for

$$P_\mu := F'\partial_\mu V, \quad P'_\mu := F\partial_\mu V', \quad Q_\mu := G'\partial_\mu W, \quad Q'_\mu := G\partial_\mu W', \tag{5.12}$$

with F, F', G, G' as in (4.7), and V, V', W, W' as in (4.20).

Proof. Using (5.8)–(5.9) written for (b, b'), one computes

$$\big[\eth \otimes \mathbb{I}_4,\, \pi(b,b')\big]_\rho =: \begin{pmatrix} [\eth, V]_\rho & 0 & 0 & 0 \\ 0 & [\eth, V']_\rho & 0 & 0 \\ 0 & 0 & [\eth, W']_\rho & 0 \\ 0 & 0 & 0 & [\eth, W]_\rho \end{pmatrix}. \tag{5.13}$$

The result follows multiplying by (5.8), then using (4.10). ■

Lemma 5.2. *With the same notations as in Lemma 5.1, one has that*

$$\mathcal{Z} := \omega_{\rho_\mathcal{M}} + J\omega_{\rho_\mathcal{M}} J^{-1} = \begin{pmatrix} \mathbf{Z} & 0 & 0 & 0 \\ 0 & \mathbf{Z}' & 0 & 0 \\ 0 & 0 & \bar{\mathbf{Z}} & 0 \\ 0 & 0 & 0 & \bar{\mathbf{Z}}' \end{pmatrix}, \tag{5.14}$$

with $\mathbf{Z} := \mathbf{P} + \bar{\mathbf{Q}}'$, $\mathbf{Z}' := \mathbf{P}' + \bar{\mathbf{Q}}$, $\bar{\mathbf{Z}} := \bar{\mathbf{P}} + \mathbf{Q}'$, *and* $\bar{\mathbf{Z}}' := \bar{\mathbf{P}}' + \mathbf{Q}$.

Proof. From (5.11), Lemma 4.1 and the explicit form of $J = \mathcal{J} \otimes J_\mathcal{F}$ with $J_\mathcal{F}$ as in (5.2), one gets

$$J\omega_{\rho_\mathcal{M}} J^{-1} = \begin{pmatrix} 0 & 0 & \mathcal{J} & 0 \\ 0 & 0 & 0 & \mathcal{J} \\ \mathcal{J} & 0 & 0 & 0 \\ 0 & \mathcal{J} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathcal{J}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{J}^{-1} \\ \mathcal{J}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{J}^{-1} & 0 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} \mathcal{J}\mathbf{Q}'\mathcal{J}^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{J}\mathbf{Q}\mathcal{J}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{J}\mathbf{P}\mathcal{J}^{-1} & 0 \\ 0 & 0 & 0 & \mathcal{J}\mathbf{P}'\mathcal{J}^{-1} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{Q}}' & 0 & 0 & 0 \\ 0 & \bar{\mathbf{Q}} & 0 & 0 \\ 0 & 0 & \bar{\mathbf{P}} & 0 \\ 0 & 0 & 0 & \bar{\mathbf{P}}' \end{pmatrix}. \tag{5.15}$$

Adding up with (5.11), the result follows. ∎

Proposition 5.3. *A self-adjoint twisted fluctuation* (5.14) *of the free Dirac operator* $\eth \otimes \mathbb{I}_4$ *is of the form*

$$\mathcal{Z} = \mathbf{X} \otimes \mathbb{I}' + i\mathbf{Y} \otimes \mathbb{I}'', \tag{5.16}$$

where $\mathbf{X} = -i\gamma^\mu X_\mu$, $\mathbf{Y} = -i\gamma^\mu Y_\mu$, $\mathbb{I}' := \mathsf{diag}(1, -1, 1, -1)$, $\mathbb{I}'' := \mathsf{diag}(1, 1, -1, -1)$ *with*

$$X_\mu := f_\mu \gamma^5, \quad Y_\mu := g_\mu \mathbb{I}_4, \quad f_\mu, g_\mu \in C^\infty(M, \mathbb{R}). \tag{5.17}$$

Proof. From (5.14), it follows that $\mathcal{Z}$ is self-adjoint iff $\mathbf{Z} = \mathbf{Z}^\dagger$, $\mathbf{Z}' = \mathbf{Z}'^\dagger$, $\bar{\mathbf{Z}} = \bar{\mathbf{Z}}^\dagger$, and $\bar{\mathbf{Z}}' = \bar{\mathbf{Z}}'^\dagger$. From (4.9) and the third equation of (4.10), these four conditions are equivalent to $\mathbf{Z} = -\bar{\mathbf{Z}}'$; i.e.,

$$Z_\mu = -\bar{Z}'_\mu. \tag{5.18}$$

By Lemma 5.2, one knows that

$$Z_\mu = P_\mu + \bar{Q}'_\mu = \begin{pmatrix} z^\mu \mathbb{I}_2 & 0 \\ 0 & z'_\mu \mathbb{I}_2 \end{pmatrix} \tag{5.19}$$

with $z_\mu = f'\partial_\mu v + \bar{g}\partial_\mu \bar{w}'$ and $z'_\mu = f\partial_\mu v' + \bar{g}'\partial_\mu \bar{w}$. Denoting f_μ, g_μ the real and imaginary parts of z_μ (and similarly for z'_μ), then (5.18) is equivalent to $f'_\mu = -f_\mu$ and $g'_\mu = g_\mu$; that is,

$$Z_\mu = \begin{pmatrix} (f_\mu + ig_\mu)\mathbb{I}_2 & 0 \\ 0 & (-f_\mu + ig_\mu)\mathbb{I}_2 \end{pmatrix}. \tag{5.20}$$

In other terms, $Z_\mu = X_\mu + iY_\mu$ with $X_\mu := f_\mu\gamma^5$, $Y_\mu := g_\mu \mathbb{I}_4$.

Going back to (5.14), one obtains

$$\begin{aligned}\mathcal{Z} &= \begin{pmatrix}\mathbf{Z} & 0 & 0 & 0\\ 0 & -\bar{\mathbf{Z}} & 0 & 0\\ 0 & 0 & \bar{\mathbf{Z}} & 0\\ 0 & 0 & 0 & -\mathbf{Z}\end{pmatrix} = \begin{pmatrix}-i\gamma^\mu Z_\mu & 0 & 0 & 0\\ 0 & i\gamma^\mu \bar{Z}_\mu & 0 & 0\\ 0 & 0 & -i\gamma^\mu \bar{Z}_\mu & 0\\ 0 & 0 & 0 & i\gamma^\mu Z_\mu\end{pmatrix}\\ &= \begin{pmatrix}-i\gamma^\mu (X_\mu + iY_\mu) & 0 & 0 & 0\\ 0 & i\gamma^\mu (X_\mu - iY_\mu) & 0 & 0\\ 0 & 0 & -i\gamma^\mu (X_\mu - iY_\mu) & 0\\ 0 & 0 & 0 & i\gamma^\mu (X_\mu + iY_\mu)\end{pmatrix} \\ &= -i\gamma^\mu X_\mu \otimes \mathbb{I}' + i(-i\gamma^\mu Y_\mu)\otimes \mathbb{I}''.\end{aligned} \tag{5.21}$$

■

Remark 5.4. Imposing the self-adjointness of the twisted 1-form $\omega_{\rho_\mathcal{M}}$ amounts to

$$\mathbf{P}^\dagger = \mathbf{P}, \quad \mathbf{Q}^\dagger = \mathbf{Q}. \tag{5.22}$$

This implies – but is not equivalent – to imposing the self-adjointness of $\omega_{\rho\mathcal{M}} + \mathcal{J}\omega_{\rho\mathcal{M}}\mathcal{J}^{-1}$,

$$\mathbf{Z}^\dagger = \mathbf{Z}. \tag{5.23}$$

As discussed below Lemma 3.2 for the minimal twist of a manifold, the relevant point is that the stronger condition (5.22) does not imply that the twisted fluctuation $\mathcal{Z}$ is zero. The final form of the twist-fluctuated operator is the same, whether one requires (5.22) or (5.23).

5.2.2. The finite part. In the spectral triple of electrodynamics, the finite part $\gamma^5 \otimes D_\mathcal{F}$ of the Dirac operator D (5.1) does not fluctuate [37], for it commutes with the representation π_0 (5.3) of $\mathcal{A}_{\text{ED}}$. The same is true for the minimal twist of electrodynamics.

Proposition 5.5. *The finite Dirac operator $\gamma^5 \otimes \mathcal{D}_\mathcal{F}$ has no twisted fluctuation.*

Proof. With the representations (5.8)–(5.9), one calculates that

$$\begin{aligned}\left[\gamma^5 \otimes \mathcal{D}_\mathcal{F}, \pi(a,a')\right]_\rho &= (\gamma^5 \otimes \mathcal{D}_\mathcal{F})\,\pi(a,a') - \pi(a',a)\,(\gamma^5\otimes\mathcal{D}_\mathcal{F})\\ &= \begin{pmatrix}0 & d\gamma^5 & 0 & 0\\ \bar{d}\gamma^5 & 0 & 0 & 0\\ 0 & 0 & 0 & \bar{d}\gamma^5\\ 0 & 0 & d\gamma^5 & 0\end{pmatrix}\begin{pmatrix}F & 0 & 0 & 0\\ 0 & F' & 0 & 0\\ 0 & 0 & G' & 0\\ 0 & 0 & 0 & G\end{pmatrix}\\ &\quad - \begin{pmatrix}F' & 0 & 0 & 0\\ 0 & F & 0 & 0\\ 0 & 0 & G & 0\\ 0 & 0 & 0 & G'\end{pmatrix}\begin{pmatrix}0 & d\gamma^5 & 0 & 0\\ \bar{d}\gamma^5 & 0 & 0 & 0\\ 0 & 0 & 0 & \bar{d}\gamma^5\\ 0 & 0 & d\gamma^5 & 0\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} 0 & d[\gamma^5, F'] & 0 & 0 \\ \bar{d}[\gamma^5, F] & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d}[\gamma^5, G] \\ 0 & 0 & d[\gamma^5, G'] & 0 \end{pmatrix} = 0,$$

where F, F', G, G' (denoted in (4.7)) being diagonal, commute with γ^5. ■

The results of this section summarise as follows.

Proposition 5.6. *The Dirac operator $\mathcal{D} = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes \mathcal{D}_{\mathcal{F}}$ of electrodynamics, under the minimal twist (5.8)–(5.10), twist fluctuates to*

$$\mathcal{D}_{\mathcal{Z}} := \mathcal{D} + \mathcal{Z}, \tag{5.24}$$

where $\mathcal{Z}$ is given by Proposition 5.3.

Remark 5.7. Expectedly, substituting $\rho = \mathsf{Id}$, one returns to the non-twisted case: the triviality of ρ is tantamount to equating (5.8) with (5.9), that is, to identify the "primed" functions with their "unprimed" partners. Hence, $\mathbf{Z}' = \mathbf{Z}$. Imposing self-adjointness, the third equation of (4.10) gives $\mathbf{Z} = -\bar{\mathbf{Z}}$. Going back to (5.20), this yields $f_\mu = 0$. Therefore, X_μ vanishes and remains only the $U(1)$ gauge field $\mathbf{Y}$. The latter is

$$i\mathbf{Y} \otimes \mathbb{I}'' = \gamma^\mu Y_\mu \otimes \mathbb{I}'' = \gamma^\mu \otimes g_\mu \mathbb{I}'' \tag{5.25}$$

and coincides with the gauge potential $\gamma^\mu \otimes B_\mu$ of the spectral triple of electrodynamics (5.4) in the non-twisted case.

5.3. Gauge transformation

We discuss the transformation of the fields $\mathbf{X}$ and $\mathbf{Y}$ parametrising the twisted fluctuation $\mathcal{Z}$, along the lines of Section 3.3. A unitary u of $\mathcal{A}_{\mathrm{ED}} \otimes \mathbb{C}^2$ is of the form $u = (v, v')$, where $v := (e^{i\alpha}, e^{i\beta})$, $v' := (e^{i\alpha'}, e^{i\beta'})$ are unitaries of $\mathcal{A}_{\mathrm{ED}}$, with $\alpha, \alpha', \beta, \beta' \in C^\infty(\mathcal{M}, \mathbb{R})$. It (and its twist) acts on $L^2(\mathcal{M}, S) \otimes \mathbb{C}^4$ as

$$\pi(u) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & 0 \\ 0 & 0 & B' & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \quad \pi\big(\rho(u)\big) = \pi(v', v) = \begin{pmatrix} A' & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B' \end{pmatrix}, \tag{5.26}$$

where we denote

$$\begin{aligned} A &:= \pi_{\mathcal{M}}(e^{i\alpha}, e^{i\alpha'}), \quad & A' &:= \rho(A) = \pi_{\mathcal{M}}(e^{i\alpha'}, e^{i\alpha}), \\ B &:= \pi_{\mathcal{M}}(e^{i\beta}, e^{i\beta'}), \quad & B' &:= \rho(B) = \pi_{\mathcal{M}}(e^{i\beta'}, e^{i\beta}). \end{aligned} \tag{5.27}$$

Proposition 5.8. *Under a gauge transformation (2.24), $\mathbf{X}$ remains invariant while $\mathbf{Y}$ is mapped to*

$$-i\gamma^\mu\left(Y^\mu - \begin{pmatrix} \partial_\mu \theta \mathbb{I}_2 & 0 \\ 0 & \partial_\mu \theta' \mathbb{I}_2 \end{pmatrix}\right) \tag{5.28}$$

for $\theta := \alpha - \beta'$, $\theta' = \alpha' - \beta$.

Proof. Since $\gamma_{\mathcal{F}} \otimes D_{\mathcal{F}}$ twist commutes with the algebra, in the transformation (2.25) of the gauge potential it is enough to consider $\eth \otimes \mathbb{I}_4$. So $\omega_{\rho_{\mathcal{M}}}$ in (5.14) transforms to

$$\omega^w_{\rho_{\mathcal{M}}} = \rho(u)\big([\eth \otimes \mathbb{I}_4, u^*]_\rho + \omega_{\rho_{\mathcal{M}}} u^*\big) = \rho(u)\big(\eth \otimes \mathbb{I}_4 + \omega_{\rho_{\mathcal{M}}}\big)u^*, \tag{5.29}$$

where we used $[\eth \otimes \mathbb{I}_4, u^*]_\rho = (\eth \otimes \mathbb{I}_4)u^*$ as in (3.23). By (5.26) and Lemma 5.1, this transformation writes

$$\begin{pmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P}' & 0 & 0 \\ 0 & 0 & \mathbf{Q}' & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{pmatrix} \to \begin{pmatrix} A'(\eth + \mathbf{P})\bar{A} & 0 & 0 & 0 \\ 0 & A(\eth + \mathbf{P}')\bar{A}' & 0 & 0 \\ 0 & 0 & B(\eth + \mathbf{Q}')\bar{B}' & 0 \\ 0 & 0 & 0 & B'(\eth + \mathbf{Q})\bar{B} \end{pmatrix}.$$

Since A', B' twist commute with γ^μ and A commutes with P_μ (and B with Q_μ), one has that P_μ is mapped to $P_\mu + A\partial_\mu \bar{A}$ and Q'_μ to $Q_\mu + B'\partial_\mu \bar{B}'$. Thus $Z_\mu = P_\mu + \bar{Q}'_\mu$ in (5.18) is mapped to $Z_\mu + (A\partial_\mu \bar{A} + \bar{B}'\partial_\mu B')$. With the representations (5.19) of Z_μ and (5.27) of A, B, this means

$$\begin{pmatrix} z_\mu \mathbb{I}_2 & 0 \\ 0 & z'_\mu \mathbb{I}_2 \end{pmatrix} \to \begin{pmatrix} (z_\mu - i\partial_\mu \theta)\mathbb{I}_2 & 0 \\ 0 & (z'_\mu - i\partial_\mu \theta')\mathbb{I}_2 \end{pmatrix}.$$

The result follows remembering that X^μ and Y^μ are the real and imaginary parts of Z^μ. ■

By imposing that both $\mathcal{Z}$ and its gauge transform are self-adjoint, that is, by Lemma 4.1, $z'_\mu = -\bar{z}_\mu$ and $z'_\mu - i\partial_\mu \theta' = -\overline{z_\mu - i\partial_\mu \theta}$, one is forced to identify $\theta' = \theta +$ constant. Then (5.28) means that $Y_\mu = g_\mu \mathbb{I}_4$ undergoes the transformation

$$g_\mu \to g_\mu - \partial_\mu \theta, \quad \theta \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{5.30}$$

This is a $U(1)$ gauge field, formally similar to the one in (5.5) of the (Euclidean) non-twisted case. By computing the twisted fermionic action, we show that this actually identifies with the $U(1)$ of electromagnetism, but now in Lorentzian signature.

Remark 5.9. For $\theta' - \theta$, a non-zero constant, the gauge transformation preserves the self-adjointness of the twisted fluctuation, even though u is not invariant by the twist. This is because such a u satisfies the weaker condition for preserving self-adjointness – pointed out in [29, §5.1] – namely $\rho(u)^* u$ twist commutes with $\mathcal{D}$.

5.4. Lorentzian Dirac equation from twisted fermionic action

To calculate the action, we first identify the eigenvectors of the unitary $\mathcal{R}$ implementing the twist.

Lemma 5.10. *Any η in the positive eigenspace $\mathcal{H}_{\mathcal{R}}$ (2.26) of the unitary $\mathcal{R}$ (5.10) is of the form*

$$\eta = \phi_1 \otimes e_L + \phi_2 \otimes e_R + \xi_1 \otimes \overline{e_L} + \xi_2 \otimes \overline{e_R}, \tag{5.31}$$

where $\phi_{k=1,2} := \binom{\varphi_k}{\varphi_k}$ and $\xi_{k=1,2} := \binom{\zeta_k}{\zeta_k}$ are Dirac spinors with Weyl components φ_k, ζ_k.

Proof. $\mathcal{R}$ has eigenvalues ± 1 and its eigenvectors corresponding to the eigenvalue $+1$ are

$$\varepsilon_1 = \upsilon_1 \otimes e_L, \quad \varepsilon_2 = \upsilon_2 \otimes e_L, \quad \varepsilon_3 = \upsilon_1 \otimes e_R, \quad \varepsilon_4 = \upsilon_2 \otimes e_R,$$
$$\varepsilon_5 = \upsilon_1 \otimes \overline{e_L}, \quad \varepsilon_6 = \upsilon_2 \otimes \overline{e_L}, \quad \varepsilon_7 = \upsilon_1 \otimes \overline{e_R}, \quad \varepsilon_8 = \upsilon_2 \otimes \overline{e_L},$$

where $\upsilon_1 := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $\upsilon_2 := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ denote the eigenvectors of γ^0. Thus

$$\begin{aligned}\eta = \sum_{j=1}^{8} \lambda_j \varepsilon_j &= (\lambda_1 \upsilon_1 + \lambda_2 \upsilon_2) \otimes e_L + (\lambda_3 \upsilon_1 + \lambda_4 \upsilon_2) \otimes e_R \\ &\quad + (\lambda_5 \upsilon_1 + \lambda_6 \upsilon_2) \otimes \overline{e_L} + (\lambda_7 \upsilon_1 + \lambda_8 \upsilon_2) \otimes \overline{e_R}, \\ &= \phi_1 \otimes e_L + \phi_2 \otimes e_R + \xi_1 \otimes \overline{e_L} + \xi_2 \otimes \overline{e_R},\end{aligned}$$

with $\varphi_1 := \left(\begin{smallmatrix}\lambda_1\\\lambda_2\end{smallmatrix}\right)$, $\varphi_2 := \left(\begin{smallmatrix}\lambda_3\\\lambda_4\end{smallmatrix}\right)$, $\xi_1 := \left(\begin{smallmatrix}\lambda_5\\\lambda_6\end{smallmatrix}\right)$, $\xi_2 := \left(\begin{smallmatrix}\lambda_7\\\lambda_8\end{smallmatrix}\right)$. ∎

The following lemma is useful to compute the contribution of $\gamma^5 \otimes \mathcal{D}_F$ and $\mathbf{Y}$ to the action.

Lemma 5.11. *For Dirac spinors $\phi := \left(\begin{smallmatrix}\varphi\\\varphi\end{smallmatrix}\right)$, $\xi := \left(\begin{smallmatrix}\zeta\\\zeta\end{smallmatrix}\right)$ in $L^2(\mathcal{M}, \mathcal{S})$, one has that*

$$\mathfrak{A}_{i\mathbf{Y}}(\phi, \xi) = 2i \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger \sigma_2 \Big(\sum_j \sigma_j g_j \Big) \zeta, \quad \mathfrak{A}_{\gamma^5}(\phi, \xi) = -2 \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger \sigma_2 \zeta. \qquad (5.32)$$

Proof. Using (5.17) for Y_μ and (A.2) for the Dirac matrices, one gets

$$i\mathbf{Y}\xi = \gamma^\mu Y_\mu \begin{pmatrix}\zeta\\\zeta\end{pmatrix} = \begin{pmatrix}0 & \sigma^\mu\\ \widetilde{\sigma}^\mu & 0\end{pmatrix} \begin{pmatrix} g_\mu \mathbb{I}_2 & 0\\ 0 & g_\mu \mathbb{I}_2\end{pmatrix} \begin{pmatrix}\zeta\\\zeta\end{pmatrix} = \begin{pmatrix} g_\mu \sigma^\mu \zeta\\ g_\mu \widetilde{\sigma}^\mu \zeta\end{pmatrix}.$$

Along with (3.25), recalling that $\sigma^{2\dagger} = i\sigma_2$ and $\widetilde{\sigma}^{2\dagger} = -i\sigma_2$ yields

$$\begin{aligned}\mathfrak{A}_{i\mathbf{Y}}(\phi, \xi) = (\mathcal{J}\phi)^\dagger (i\mathbf{Y}\xi) &= -i \begin{pmatrix}\widetilde{\sigma}^2 \overline{\varphi}\\ \sigma^2 \overline{\varphi}\end{pmatrix}^\dagger \begin{pmatrix} g_\mu \sigma^\mu \zeta\\ g_\mu \widetilde{\sigma}^\mu \zeta\end{pmatrix} \\ &= -i \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger (\widetilde{\sigma}^{2\dagger} \sigma^\mu + \sigma^{2\dagger} \widetilde{\sigma}^\mu) g_\mu \zeta \\ &= \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger \sigma_2 (-\sigma^\mu + \widetilde{\sigma}^\mu) g_\mu \zeta \\ &= 2i \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger \sigma_2 \Big(\sum_j \sigma_j g_j \Big) \zeta,\end{aligned}$$

where we used (3.28) and obtained the first equation of (5.32). The second one follows from

$$\begin{aligned}\mathfrak{A}_{\gamma^5}(\phi, \xi) = (\mathcal{J}\phi)^\dagger (\gamma^5 \xi) &= -i \begin{pmatrix}\widetilde{\sigma}^2 \overline{\varphi}\\ \sigma^2 \overline{\varphi}\end{pmatrix}^\dagger \begin{pmatrix}\zeta\\ -\zeta\end{pmatrix} \\ &= -i \int_{\mathcal{M}} \mathrm{d}\mu\, (\overline{\varphi}^\dagger \widetilde{\sigma}^{2\dagger} \zeta - \overline{\varphi}^\dagger \sigma^{2\dagger} \zeta) = -2 \int_{\mathcal{M}} \mathrm{d}\mu\, \overline{\varphi}^\dagger \sigma_2 \zeta.\end{aligned}$$ ∎

Proposition 5.12. *The fermionic action of the minimal twist of electrodynamics is the integral*

$$S_\rho(\mathcal{D}_\mathcal{Z}) = \mathfrak{A}^\rho_{D_\mathcal{Z}}(\widetilde{\eta},\widetilde{\eta}) = 4\int_\mathcal{M} \mathrm{d}\mu\, \mathcal{L}$$

of the Lagrangian density

$$\mathcal{L} := \bar{\widetilde{\varphi}}_1^\dagger \sigma_2\Big(if_0 - \sum_j \sigma_j \mathfrak{D}_j\Big)\widetilde{\xi}_1 - \bar{\widetilde{\varphi}}_2^\dagger \sigma_2\Big(if_0 + \sum_j \sigma_j \mathfrak{D}_j\Big)\widetilde{\xi}_2 \\ + \big(\bar{d}\bar{\widetilde{\varphi}}_1^\dagger \sigma_2 \widetilde{\xi}_2 + d\bar{\widetilde{\varphi}}_2^\dagger \sigma_2\widetilde{\xi}_1\big), \tag{5.33}$$

with $\mathfrak{D}_\mu := \partial_\mu - ig_\mu$ the covariant derivative associated to the electromagnetic four-potential (5.30).

Proof. Let $\mathfrak{A}^\rho_{\mathcal{D}_\mathcal{Z}}$ be the antisymmetric bilinear form (2.23) defined by the twisted-covariant Dirac operator (5.24). It breaks down into four terms:

$$\mathfrak{A}^\rho_{D_\mathcal{Z}} = \mathfrak{A}^\rho_{\eth\otimes\mathbb{I}_4} + \mathfrak{A}^\rho_{\mathbf{X}\otimes\mathbb{I}'} + \mathfrak{A}^\rho_{i\mathbf{Y}\otimes\mathbb{I}''} + \mathfrak{A}^\rho_{\gamma^5\otimes D_\mathcal{F}}. \tag{5.34}$$

For $\eta, \eta' \in \mathcal{H}_R$ as in (5.31) one gets

$$\begin{aligned} J\eta &= \mathcal{J}\phi_1 \otimes \overline{e_L} + \mathcal{J}\phi_2 \otimes \overline{e_R} + \mathcal{J}\xi_1 \otimes e_L + \mathcal{J}\xi_2 \otimes e_R,\\ (\eth\otimes\mathbb{I}_4)\eta' &= \eth\phi_1' \otimes e_L + \eth\phi_2' \otimes e_R + \eth\xi_1' \otimes \overline{e_L} + \eth\xi_2' \otimes \overline{e_R},\\ (\mathbf{X}\otimes\mathbb{I}')\eta' &= \mathbf{X}\phi_1' \otimes e_L - \mathbf{X}\phi_2' \otimes e_R + \mathbf{X}\xi_1' \otimes \overline{e_L} - \mathbf{X}\xi_2' \otimes \overline{e_R},\\ (i\mathbf{Y}\otimes\mathbb{I}'')\eta' &= i\mathbf{Y}\phi_1' \otimes e_L + i\mathbf{Y}\phi_2' \otimes e_R - i\mathbf{Y}\xi_1' \otimes \overline{e_L} - i\mathbf{Y}\xi_2' \otimes \overline{e_R},\\ (\gamma^5\otimes D_\mathcal{F})\eta' &= \gamma^5\phi_1' \otimes \bar{d}e_R + \gamma^5\phi_2' \otimes d e_L + \gamma^5\xi_1' \otimes d\,\overline{e_R} + \gamma^5\xi_2' \otimes \bar{d}\,\overline{e_L}, \end{aligned} \tag{5.35}$$

where the first and last equations come from the explicit forms (5.2) of $J_\mathcal{F}$ and $D_\mathcal{F}$, while the third and fourth follow from the explicit form (5.16) of $\mathbf{X}$ and $\mathbf{Y}$. These equations allow to reduce each of the four terms in (5.34) to a bilinear form on $L^2(\mathcal{M},\mathcal{S})$ rather than on the tensor product $L^2(\mathcal{M},\mathcal{S})\otimes\mathbb{C}^4$. More precisely, recalling Lemma 2.8 with $\varepsilon''' = -1$ (and noticing that $\eth\otimes\mathbb{I}_4$, $\mathbf{X}\otimes\mathbb{I}'$, $i\mathbf{Y}\otimes\mathbb{I}''$, $\gamma^5\otimes\mathcal{D}_\mathcal{F}$ are all self-adjoint), one computes

$$\begin{aligned} \mathfrak{A}^\rho_{\eth\otimes\mathbb{I}_4}(\eta,\eta') &= -\mathfrak{A}_{\eth\otimes\mathbb{I}_4}(\eta,\eta') = -\big\langle J\eta, (\eth\otimes\mathbb{I}_4)\eta'\big\rangle,\\ &= -\langle\mathcal{J}\phi_1, \eth\xi_1'\rangle - \langle\mathcal{J}\phi_2, \eth\xi_2'\rangle - \langle\mathcal{J}\xi_1, \eth\phi_1'\rangle - \langle\mathcal{J}\xi_2, \eth\phi_2'\rangle,\\ &= -\mathfrak{A}_\eth(\phi_1,\xi_1') - \mathfrak{A}_\eth(\phi_2,\xi_2') - \mathfrak{A}_\eth(\xi_1,\phi_1') - \mathfrak{A}_\eth(\xi_2,\phi_2'); \end{aligned} \tag{5.36}$$

$$\begin{aligned} \mathfrak{A}^\rho_{\mathbf{X}\otimes\mathbb{I}'}(\eta,\eta') &= -\mathfrak{A}_{\mathbf{X}\otimes\mathbb{I}'}(\eta,\eta') = -\big\langle J\eta, (\mathbf{X}\otimes\mathbb{I}')\eta'\big\rangle,\\ &= -\langle\mathcal{J}\phi_1, \mathbf{X}\xi_1'\rangle + \langle\mathcal{J}\phi_2, \mathbf{X}\xi_2'\rangle - \langle\mathcal{J}\xi_1, \mathbf{X}\phi_1'\rangle + \langle\mathcal{J}\xi_2, \mathbf{X}\phi_2'\rangle,\\ &= -\mathfrak{A}_\mathbf{X}(\phi_1,\xi_1') + \mathfrak{A}_\mathbf{X}(\phi_2,\xi_2') - \mathfrak{A}_\mathbf{X}(\xi_1,\phi_1') + \mathfrak{A}_\mathbf{X}(\xi_2,\phi_2'); \end{aligned} \tag{5.37}$$

$$\begin{aligned} \mathfrak{A}^\rho_{i\mathbf{Y}\otimes\mathbb{I}''}(\eta,\eta') &= -\mathfrak{A}_{i\mathbf{Y}\otimes\mathbb{I}''}(\eta,\eta') = -\big\langle J\eta, (i\mathbf{Y}\otimes\mathbb{I}'')\eta'\big\rangle,\\ &= \langle\mathcal{J}\phi_1, i\mathbf{Y}\xi_1'\rangle + \langle\mathcal{J}\phi_2, i\mathbf{Y}\xi_2'\rangle - \langle\mathcal{J}\xi_1, i\mathbf{Y}\phi_1'\rangle - \langle\mathcal{J}\xi_2, i\mathbf{Y}\phi_2'\rangle,\\ &= \mathfrak{A}_{i\mathbf{Y}}(\phi_1,\xi_1') + \mathfrak{A}_{i\mathbf{Y}}(\phi_2,\xi_2') - \mathfrak{A}_{i\mathbf{Y}}(\xi_1,\phi_1') - \mathfrak{A}_{i\mathbf{Y}}(\xi_2,\phi_2'); \end{aligned} \tag{5.38}$$

$$\begin{aligned}\mathfrak{A}^{\rho}_{\gamma^5\otimes D_{\mathcal{F}}}(\eta,\eta') &= -\mathfrak{A}_{\gamma^5\otimes D_{\mathcal{F}}}(\eta,\eta') = -\langle J\eta, (\gamma^5\otimes D_{\mathcal{F}})\eta'\rangle \\ &= -\bar{d}\,\langle \mathcal{J}\phi_1, \gamma^5\xi_2'\rangle - d\,\langle \mathcal{J}\phi_2, \gamma^5\xi_1'\rangle - d\,\langle \mathcal{J}\xi_1, \gamma^5\phi_2'\rangle - \bar{d}\,\langle \mathcal{J}\xi_2, \gamma^5\phi_1'\rangle, \\ &= -\bar{d}\,\mathfrak{A}_{\gamma^5}(\phi_1,\xi_2') - d\,\mathfrak{A}_{\gamma^5}(\phi_2,\xi_1') - d\,\mathfrak{A}_{\gamma^5}(\xi_1,\phi_2') - \bar{d}\,\mathfrak{A}_{\gamma^5}(\xi_2,\phi_1').\end{aligned} \tag{5.39}$$

Substituting $\eta = \eta'$, then going to Graßmann variables, the sum of (5.36), (5.37), and (5.39) is

$$\begin{aligned}&-2\,\mathfrak{A}_{\eth}(\widetilde{\phi}_1,\widetilde{\xi}_1) - 2\,\mathfrak{A}_{\eth}(\widetilde{\phi}_2,\widetilde{\xi}_2) - 2\,\mathfrak{A}_{\mathbf{X}}(\widetilde{\phi}_1,\widetilde{\xi}_1) + 2\,\mathfrak{A}_{\mathbf{X}}(\widetilde{\phi}_2,\widetilde{\xi}_2) \\ &-2\bar{d}\,\mathfrak{A}_{\gamma^5}(\widetilde{\phi}_1,\widetilde{\xi}_2) - 2d\,\mathfrak{A}_{\gamma^5}(\widetilde{\phi}_2,\widetilde{\xi}_1),\end{aligned} \tag{5.40}$$

where we used that $\mathfrak{A}_{\eth}$, $\mathfrak{A}_{\mathbf{X}}$, and $\mathfrak{A}_{\gamma^5}$ are antisymmetric on vectors (by Lemma 2.7, since $\eth$, $\mathbf{X}$, γ^5 all commute with $\mathcal{J}$: $\eth$ and γ^5 by (2.3) in KO-dimension 4; and $\mathbf{X}$ by (4.10)), and so symmetric when evaluated on Graßmann variables. On the other hand, (5.38) is symmetric on vectors (since $i\mathbf{Y}$ anticommutes with $\mathcal{J}$), while it is antisymmetric in Graßmann variables, so that (5.38) is equal to

$$2\mathfrak{A}_{i\mathbf{Y}}(\widetilde{\phi}_1,\widetilde{\xi}_1) + 2\,\mathfrak{A}_{i\mathbf{Y}}(\widetilde{\phi}_2,\widetilde{\xi}_2). \tag{5.41}$$

The Lagrangian (5.33) follows substituting all the bilinear forms in (5.40) and (5.41) with their explicit expressions given in (3.29), (3.31), and Lemma 5.11. ∎

In order to get Dirac equations, we have two possibilities for identifying the physical spinors:

$$\text{either}\quad \Psi = \begin{pmatrix}\psi_l\\ \psi_r\end{pmatrix} := \begin{pmatrix}\widetilde{\zeta}_1\\ \widetilde{\zeta}_2\end{pmatrix},\quad \Psi^\dagger = \begin{pmatrix}\psi_l^\dagger, & \psi_r^\dagger\end{pmatrix} := \begin{pmatrix}-i\bar{\widetilde{\varphi}}_1^\dagger\sigma_2, & i\bar{\widetilde{\varphi}}_2^\dagger\sigma_2\end{pmatrix} \tag{5.42}$$

$$\text{or}\quad \Psi' = \begin{pmatrix}\psi_l'\\ \psi_r'\end{pmatrix} := \begin{pmatrix}\widetilde{\zeta}_2\\ \widetilde{\zeta}_1\end{pmatrix},\quad \Psi'^\dagger = \begin{pmatrix}\psi_l'^\dagger, & \psi_r'^\dagger\end{pmatrix} := \begin{pmatrix}i\bar{\widetilde{\varphi}}_2^\dagger\sigma_2, & -i\bar{\widetilde{\varphi}}_1^\dagger\sigma_2\end{pmatrix}. \tag{5.43}$$

Imposing the complex parameter d to be purely imaginary as $d = im$, $m \in \mathbb{R}^*$ (in agreement with the non-twisted case [37, Rem. 4.4]), the Lagrangian (5.33) becomes

$$\begin{aligned}\text{either}\quad \mathcal{L} &= i\psi_l^\dagger\Big(if_0 - \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_l + i\psi_r^\dagger\Big(if_0 + \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_r \\ &\quad + m(\psi_l^\dagger\psi_r + \psi_r^\dagger\psi_l)\end{aligned} \tag{5.44}$$

$$\begin{aligned}\text{or}\quad \mathcal{L}' &= i\psi_r'^\dagger\Big(if_0 - \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_r' + i\psi_l'^\dagger\Big(if_0 + \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_l' \\ &\quad + m(\psi_r'^\dagger\psi_l' + \psi_l'^\dagger\psi_r').\end{aligned} \tag{5.45}$$

The Euler–Lagrange equations for $\psi_l^\dagger$, $\psi_r^\dagger$ and $\psi_l'^\dagger$, $\psi_r'^\dagger$ yield the equation of motion

$$i\Big(if_0 - \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_l + m\psi_r = 0,\quad i\Big(if_0 + \sum_j \sigma_j\mathfrak{D}_j\Big)\psi_r + m\psi_l = 0, \tag{5.46}$$

$$i\Big(if_0 + \sum_j \sigma_j \mathfrak{D}_j\Big)\psi'_l + m\psi'_r = 0, \quad i\Big(if_0 - \sum_j \sigma_j \mathfrak{D}_j\Big)\psi'_r + m\psi'_l = 0. \tag{5.47}$$

Which identification (5.42) or (5.43) is meaningful is fixed by the sign of m.

Proposition 5.13. *If $m < 0$ (resp. $m > 0$), then a plane wave solution of (5.46) (resp. (5.47)) coincides with a plane wave solution of the Dirac equation with electromagnetic potential g_μ, in Lorentzian signature and within Weyl temporal gauge (i.e., $\mathfrak{D}_0 = \partial_0$), with momentum p such that $p_0 = -f_0$ (resp. $p_0 = f_0$).*

Proof. A plane wave solution (B.3) of (5.46) satisfies

$$i\Big(if_0 + i\sum_j \sigma_j(p_j + g_j)\Big)\psi_l = -m\psi_r, \quad i\Big(if_0 - i\sum_j \sigma_j(p_j + g_j)\Big)\psi_r = -m\psi_l. \tag{5.48}$$

For $f_0 = -p_0$, this is equivalent to the system of equations (B.5) satisfied by a plane wave solution of the Dirac equation with mass $-m > 0$, having previously substituted in (B.2) the spatial derivative ∂_j with the covariant one $\mathfrak{D}_j$. Similarly, a plane wave solution of (5.47) satisfies

$$i\Big(if_0 - i\sum_j \sigma_j(p_j + g_j)\Big)\psi'_l = -m\psi'_r, \quad i\Big(if_0 + i\sum_j \sigma_j(p_j + g_j)\Big)\psi'_r = -m\psi'_l. \tag{5.49}$$

For $f_0 = p_0$, this is equivalent to the Dirac equations (B.5) for mass $m > 0$. ∎

Identifying x^0 with the time direction t of Minkowski space, then p_0 is the energy of the plane wave. As for the double manifold, the zeroth component of the twisted fluctuation of the spectral triple of electrodynamics gets interpreted as an energy.

As for the Weyl equations, one may directly identify the Lagrangian density (5.33) of the twisted fermionic action of *Euclidean* electrodynamic with the *Lorentzian* Dirac Lagrangian (B.1) (with covariant derivative $\mathfrak{D}_\mu$, in the temporal gauge $\mathfrak{D}_0 = \partial_0$):

- either considering (5.42) and imposing that $\partial_0\psi = if_0\psi$, so that (5.44) coincides with (B.1)
- or using (5.43) and imposing that $\partial_0\psi = -if_0\psi$, so that (5.45) coincides with (B.1).

Remark 5.14. The physical interpretation of f_0, g_μ is gauge invariant. From (5.26), one gets

$$U := \pi(u)J\pi(u)J^{-1} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & 0 \\ 0 & 0 & B' & 0 \\ 0 & 0 & 0 & B \end{pmatrix}\begin{pmatrix} \bar{B}' & 0 & 0 & 0 \\ 0 & \bar{B} & 0 & 0 \\ 0 & 0 & \bar{A} & 0 \\ 0 & 0 & 0 & \bar{A}' \end{pmatrix}$$
$$= \begin{pmatrix} \Theta & 0 & 0 & 0 \\ 0 & \Theta' & 0 & 0 \\ 0 & 0 & \bar{\Theta} & 0 \\ 0 & 0 & 0 & \bar{\Theta}' \end{pmatrix}, \tag{5.50}$$

where $\Theta := \text{diag}(e^{i\theta}\, e^{i\theta'})$, $\Theta' := \text{diag}(e^{i\theta'}\, e^{i\theta})$ with θ, θ' as in (5.28). Imposing the gauge transformation to preserve self-adjointness, that is, $\theta = \theta'$ (disregarding the constant), then U is simply the multiplication by a phase. This means that $U\eta$ is still in $\mathcal{H}_{\mathcal{R}}$, so that the computation of the fermionic action $\mathfrak{A}^{\rho}_{D_{\rho(U)Z\mathcal{U}}}(\widetilde{U\eta}, \widetilde{U\eta})$ is similar as above.

5.5. Identification of the physical degrees of freedom

The relation between the components $\xi := \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}$, $\phi := \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$ of the eigenvector η of $\mathcal{R}$ and the physical spinors $\Psi = \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix}$, $\Psi^\dagger = \begin{pmatrix} \psi_l^\dagger \\ \psi_r^\dagger \end{pmatrix}$ is encoded within the rule of identification (4.40) (with the sign discussed below Proposition 4.5) for the double manifold, that we write equivalently as

$$\Psi = \widetilde{\xi}, \quad \Psi^\dagger = i(\mathcal{J}\widetilde{\phi})^\dagger, \tag{5.51}$$

and within the rules (5.42), (5.43) for the spectral triple of electrodynamics, that we write equivalently

$$\begin{aligned} \Psi &= \widetilde{\Xi}, & \Psi^\dagger &= i(\mathcal{J}\widetilde{\phi})^\dagger, \\ \Psi' &= \gamma^0\widetilde{\Xi}, & \Psi'^\dagger &= i(\mathcal{J}\widetilde{\phi})^\dagger\gamma^0 = -i(\mathcal{J}\gamma^0\widetilde{\phi})^\dagger, \end{aligned} \quad \text{with } \Xi := \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},\ \phi := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{5.52}$$

In any case, the physical spinors are completely determined by the projection η_+ of η on the $+1$ eigenspace $\mathcal{H}_+$ of the grading operator; that is,

$$\eta_+ = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \otimes e + \begin{pmatrix} 0 \\ \zeta \end{pmatrix} \otimes \bar{e} \qquad \text{projecting (4.35)}, \tag{5.53}$$

$$\begin{aligned} \eta_+ = {} & \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \otimes e_L + \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \otimes e_R \\ & + \begin{pmatrix} 0 \\ \zeta_1 \end{pmatrix} \otimes \overline{e_L} + \begin{pmatrix} \zeta_2 \\ 0 \end{pmatrix} \otimes \overline{e_R} \qquad \text{projecting (5.31)}. \end{aligned} \tag{5.54}$$

This is similar to the non-twisted case, where the physical spinors are determined by an eigenvector in $\mathcal{H}_+$.

6. Lorentz invariance

So far, our results do not say anything on the components f_i of the twisted fluctuation for $i = 1, 2, 3$, because they do not appear in the Lagrangian (5.33). Since f_0 identifies with an energy, it is tempting to identify f_i with a momentum. This is actually achieved by acting with Lorentz transformations on the twisted fermionic action.

More precisely, we first define in (Section 6.1) an action of Lorentz boosts on the twisted spectral triple, which leaves the twisted fermionic action invariant. We then investigate the action from the point of view of boosted observers, both for the double manifold in Section 6.2 and for electrodynamics in Section 6.3. In both cases, we obtain equations of motion in which the components f_i of the twisted fluctuation get interpreted as momenta.

6.1. Lorentz invariance of the twisted fermionic action

As recalled in Appendix C, the Dirac equation on Minkowski spacetime is invariant under the action (C.3) of boosts simultaneously on spinors and on the Dirac operator. From a mathematical point of view, this action makes sense on a Euclidean spin manifold $\mathcal{M}$ as well: although this might seem physically non-relevant at first sight, we let boosts act on Euclidean spinors and on the Euclidean Dirac operator as

$$\phi \to \phi^\Lambda := S[\Lambda]\phi, \quad \forall \phi \in L^2(\mathcal{M}, \mathcal{S}), \tag{6.1}$$

$$\eth \to \eth^\Lambda := S[\Lambda]\,\eth\, S[\Lambda]^{-1}. \tag{6.2}$$

As an element of $\mathcal{B}(L^2(\mathcal{M}, S))$, the boost operator $S[\Lambda]$ is acted upon by the inner automorphism ρ induced by $\mathcal{R} = \gamma^0$ given in (3.7); namely

$$\rho\big(S[\Lambda]\big) = \gamma^0 \begin{pmatrix} \Lambda_- & 0_2 \\ 0_2 & \Lambda_+ \end{pmatrix} \gamma^0 = \begin{pmatrix} \Lambda_+ & 0_2 \\ 0_2 & \Lambda_- \end{pmatrix}. \tag{6.3}$$

Since Λ_+, Λ_- are inverse of one another and self-adjoint, one has that

$$\rho\big(S[\Lambda]\big) = S[\Lambda]^{-1}, \quad S[\Lambda]^+ = S[\Lambda]^{-1}. \tag{6.4}$$

Lemma 6.1. *The real structure $\mathcal{J}$ (introduced in (3.4)) twist commutes with boosts:*

$$\mathcal{J} S[\Lambda] = S[\Lambda]^{-1}\mathcal{J}. \tag{6.5}$$

Proof. Since σ_2 anticommutes with σ_1, σ_3 and commutes with itself, one has that

$$(\mathbf{n}\cdot\boldsymbol{\sigma})\sigma_2 = \sigma_2(-n_1\sigma_1 + n_2\sigma_2 - n_3\sigma_3) = -\sigma_2\overline{(\mathbf{n}\cdot\boldsymbol{\sigma})}, \tag{6.6}$$

where we use $\sigma_1, \sigma_3 = \bar\sigma_1, \bar\sigma_3, \bar\sigma_2 = -\sigma_2$. Hence, $\Lambda_\pm\sigma_2 = \sigma_2\bar\Lambda_\mp$. With $\mathcal{J} = \mathrm{diag}(-\sigma_2, \sigma_2)cc$, one gets

$$\mathcal{J} S[\Lambda] = \begin{pmatrix} -\sigma_2\bar\Lambda_- & 0 \\ 0 & \sigma_2\bar\Lambda_+ \end{pmatrix} cc = \begin{pmatrix} -\Lambda_+\sigma_2 & 0 \\ 0 & \Lambda_-\sigma_2 \end{pmatrix} cc = \quad S[\Lambda]^{-1}\mathcal{J}. \qquad \blacksquare$$

The inner product on $L^2(\mathcal{M}, S)$ is not invariant by (6.1); the twisted product is

$$\big\langle S[\Lambda]\phi, S[\Lambda]\xi\big\rangle_\rho = \big\langle \phi, S^+[\Lambda]\,S[\Lambda]\xi\big\rangle_\rho = \big\langle \phi, S[\Lambda]^{-1}S[\Lambda]\xi\big\rangle_\rho = \langle \phi, \xi\rangle_\rho \tag{6.7}$$

for any $\psi, \phi \in L^2(\mathcal{M}, S)$. This is not a surprise, being the twisted product of the Krein product of Lorentzian spinors (see Section 2). Yet, the bilinear form $\mathfrak{A}^\rho_\eth$ is not invariant:

$$\begin{aligned}\mathfrak{A}^\rho_{\eth^\Lambda}(\phi^\Lambda, \xi^\Lambda) &= \big\langle \mathcal{J}\, S[\Lambda]\phi, \eth_\Lambda S[\Lambda]\xi\big\rangle_\rho = \big\langle S[\Lambda]^{-1}\,\mathcal{J}\phi, S[\Lambda]\eth\xi\big\rangle_\rho \\ &= \big\langle \mathcal{J}\phi, S[\Lambda]^2\eth\xi\big\rangle_\rho \neq \mathfrak{A}^\rho_\eth(\phi, \xi).\end{aligned}$$

This can be corrected by making boosts act on the physical spinors Ψ, $\Psi^\dagger$ (5.51). Namely,

$$\Psi \to S[\Lambda]\Psi = S[\Lambda]\widetilde{\zeta}, \tag{6.8}$$

$$\Psi^\dagger \to \Psi^\dagger S[\Lambda]^\dagger = i(\mathcal{J}\widetilde{\phi})^\dagger S[\Lambda]^\dagger = i\big(S[\Lambda]\mathcal{J}\widetilde{\phi}\big)^\dagger = i\big(\mathcal{J}S[\Lambda]^{-1}\widetilde{\phi}\big)^\dagger. \tag{6.9}$$

Consequently, in order to "boost the fermionic action," instead of ϕ^Λ one should consider

$$\phi^{-\Lambda} := S[\Lambda]^{-1}\phi. \tag{6.10}$$

As a matter of fact, one checks that

$$\begin{aligned}\mathfrak{A}^\rho_{\eth^\Lambda}(\phi^{-\Lambda}, \xi^\Lambda) &= \big\langle \mathcal{J}\, S[\Lambda]^{-1}\phi, \eth_\Lambda S[\Lambda]\xi\big\rangle_\rho \\ &= \big\langle S[\Lambda]\,\mathcal{J}\phi, S[\Lambda]\eth\xi\big\rangle_\rho \\ &= \langle \mathcal{J}\phi, \eth\xi\rangle_\rho = \mathfrak{A}^\rho_\eth(\phi, \xi),\end{aligned} \tag{6.11}$$

and the same holds true for the operator

$$\eth^\Lambda_X := S[\Lambda]\,\eth_X\, S[\Lambda]^{-1} = \eth^\Lambda + \mathbf{X}^\Lambda \quad \text{with } \mathbf{X}^\Lambda := S[\Lambda]\,\mathbf{X}\,S[\Lambda]^{-1}, \tag{6.12}$$

obtained by the action of boosts on the twisted-covariant Dirac operator $\eth_X$. Therefore, the following proposition holds.

Proposition 6.2. *The twisted fermionic action on a Euclidean manifold* (3.34) *is invariant under the boost action*

$$\xi \to \xi^\Lambda, \quad \phi \to \phi^{-\Lambda}, \quad \eth_X \to \eth^\Lambda_X, \tag{6.13}$$

followed by the identification $\phi^\Lambda = \zeta^\Lambda$*; that is,*

$$\mathfrak{A}^\rho_{\eth_X}(\widetilde{\xi}, \widetilde{\xi}) = \mathfrak{A}^\rho_{\eth^\Lambda_X}(\widetilde{\xi}^{-\Lambda}, \widetilde{\xi}^\Lambda). \tag{6.14}$$

Our claim is that the right-hand side of the equation above is the action as seen from a boosted observer. Of course, in order to get the Weyl and Dirac equations, one needs to double the manifold as before, then add a mass matrix. Still, the main features of the boosting are visible on (6.14). In particular, by computing explicitly the bilinear form $\mathfrak{A}^\rho_{\eth^\Lambda_X}$, one sees all the components f_μ of the twisted fluctuation appearing in the action. To this aim, we use the following notations for the boosted spinors.

Definition 6.3. Given $\xi = \binom{c\zeta}{\zeta}$, $\phi = \binom{c\varphi}{\varphi}$ in $\mathcal{H}_\mathcal{R}$, we let $\varphi_{l,r}$, $\zeta_{l,r}$ be the components of

$$\xi^\Lambda = S[\Lambda]\xi = \begin{pmatrix}\Lambda_-\zeta \\ \Lambda_+\zeta\end{pmatrix} =: \begin{pmatrix}\zeta_l \\ \zeta_r\end{pmatrix},$$

$$\mathcal{J}\phi^{-\Lambda} = S[\Lambda]\mathcal{J}\phi = \begin{pmatrix}-\Lambda_-\sigma_2\,\bar{\varphi} \\ \Lambda_+\sigma_2\,\bar{\varphi}\end{pmatrix} =: \begin{pmatrix}\bar{\varphi}_l \\ \bar{\varphi}_r\end{pmatrix}.$$

Proposition 6.4. *Let* $\sigma^\mu_\Lambda := \Lambda_-\sigma^\mu\Lambda_-$ *and* $\tilde{\sigma}^\mu_\Lambda := \Lambda_+\tilde{\sigma}^\mu\Lambda_+$. *Then*

$$\mathfrak{A}^\rho_{\eth^\Lambda_X}(\phi^{-\Lambda},\xi^\Lambda) = -i\int_\mathcal{M} \mathrm{d}\mu\, \overline{\varphi}_l^\dagger\, \tilde{\sigma}^\mu_\Lambda(\partial_\mu + f_\mu)\zeta_l + \overline{\varphi}_r^\dagger \sigma^\mu_\Lambda(\partial_\mu - f_\mu)\zeta_r. \tag{6.15}$$

Proof. Defining

$$\gamma^\mu_\Lambda := S[\Lambda]\gamma^\mu S[\Lambda]^{-1} = \begin{pmatrix} 0 & \Lambda_-\sigma^\mu\Lambda_- \\ \Lambda_+\tilde{\sigma}^\mu\Lambda_+ & 0\end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu_\Lambda \\ \tilde{\sigma}^\mu_\Lambda & 0\end{pmatrix}, \tag{6.16}$$

one has (remembering that ∂_μ and γ^5 commute with $S[\Lambda]$) that

$$\eth^\Lambda\xi^\Lambda = -i\gamma^\mu_\Lambda\partial_\mu\xi^\Lambda = -i\begin{pmatrix}\sigma^\mu_\Lambda\partial_\mu\zeta_r \\ \tilde{\sigma}^\mu_\Lambda\partial_\mu\zeta_l\end{pmatrix},$$

$$\mathbf{X}^\Lambda\xi^\Lambda = -i\gamma^\mu_\Lambda f_\mu\gamma^5\xi^\Lambda = -if_\mu\begin{pmatrix}-\sigma^\mu_\Lambda\zeta_r \\ \tilde{\sigma}^\mu_\lambda\zeta_l\end{pmatrix}.$$

Since $(\mathcal{J}\phi)^\Lambda = \mathcal{J}\phi^{-\Lambda}$, one gets with (3.7)

$$\begin{aligned}
\mathfrak{A}^\rho_{\eth_\Lambda}(\phi^{-\Lambda},\xi^\Lambda) &= \langle(\mathcal{J}\phi)^\Lambda, \mathcal{R}\eth^\Lambda\xi^\Lambda\rangle = -i\int_\mathcal{M}\mathrm{d}\mu\,(\overline{\varphi}_l^\dagger,\overline{\varphi}_r^\dagger)\gamma^0\begin{pmatrix}\sigma^\mu_\Lambda\partial_\mu\zeta_r \\ \tilde{\sigma}^\mu_\Lambda\partial_\mu\zeta_l\end{pmatrix} \\
&= -i\int_\mathcal{M}\mathrm{d}\mu\,(\overline{\varphi}_l^\dagger\tilde{\sigma}^\mu_\Lambda\partial_\mu\zeta_l + \overline{\varphi}_r^\dagger\sigma^\mu_\Lambda\partial_\mu\zeta_r), \\
\mathfrak{A}^\rho_{\mathbf{X}^\Lambda}(\phi^{-\Lambda},\xi^\Lambda) &= \langle(\mathcal{J}\phi)_\Lambda, \mathcal{R}\mathbf{X}^\Lambda\xi^\Lambda\rangle = -i\int_\mathcal{M}\mathrm{d}\mu\,(\overline{\varphi}_l^\dagger,\overline{\varphi}_r^\dagger)\gamma^0\begin{pmatrix}-\sigma^\mu_\Lambda f_\mu\zeta_r \\ \tilde{\sigma}^\mu_\Lambda f_\mu\zeta_l\end{pmatrix} \\
&= -i\int_\mathcal{M}\mathrm{d}\mu\,(\overline{\varphi}_l^\dagger\tilde{\sigma}^\mu_\Lambda f_\mu\zeta_l - \overline{\varphi}_r^\dagger\sigma^\mu_\Lambda f_\mu\zeta_r).
\end{aligned}$$

The results follow summing these two equations. ∎

Remark 6.5. One checks that for $S[\Lambda] = \mathbb{I}$ (no boost), Proposition 6.4 gives back Proposition 3.5. One then has that $\zeta_{l,r} = \zeta$ while $\overline{\varphi}_l^\dagger = -\overline{\varphi}^\dagger\sigma_2$ and $\overline{\varphi}_r = \overline{\varphi}^\dagger\sigma_2$, so by (3.28)

$$\begin{aligned}
\mathfrak{A}^\rho_{\eth^\Lambda_X}(\phi^{-\Lambda},\xi^\Lambda) &= -i\int_\mathcal{M}\mathrm{d}\mu - \overline{\varphi}^\dagger\sigma_2\,\tilde{\sigma}^\mu(\partial_\mu + f_\mu)\zeta + \overline{\varphi}^\dagger\sigma_2\,\sigma^\mu(\partial_\mu - f_\mu)\zeta \\
&= -i\int_\mathcal{M}\mathrm{d}\mu\,\overline{\varphi}^\dagger\sigma_2\big((\sigma^\mu - \tilde{\sigma}^\mu)\partial_\mu - (\tilde{\sigma}^\mu + \sigma^\mu)f_\mu\big)\zeta \\
&= 2\int_\mathcal{M}\mathrm{d}\mu\,\overline{\varphi}^\dagger\sigma_2\left(-\sum_{j=1}^3\sigma_j\partial_j + if_0\right)\zeta.
\end{aligned}$$

The twisted fermionic action (6.14) on a Euclidean manifold, as seen from a boosted observer, is obtained putting $\phi^{-\Lambda} = \zeta^{-\Lambda}$ in (6.15), then turning the entries of the spinors into Graßmann variables. As in Section 3.4, there is not enough spinor degrees of freedom to identify a physically meaningful action. We thus consider the boost of the action (4.36) of the doubled manifold.

6.2. Weyl equations for boosted observers

In agreement with (6.8) and (6.10), we define the action of a boost on $L^2(\mathcal{M}, S) \otimes \mathbb{C}^2$ as $\phi \otimes e + \psi \otimes \bar{e} \to (S[\Lambda]^{-1}\phi) \otimes e + (S[\Lambda]\psi) \otimes \bar{e}$, in such a way that $\eta \in \mathcal{H}_{\mathcal{R}}$ in (4.35) is mapped to

$$\eta^\Lambda = \phi^{-\Lambda} \otimes e + \xi^\Lambda \otimes \bar{e}. \tag{6.17}$$

Proposition 6.6. *The action of a double manifold* (4.36)*, as seen from a boosted observer, is*

$$\begin{aligned}\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}}(\widetilde{\eta}^\Lambda, \widetilde{\eta}^\Lambda) &= 2\,\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}}(\widetilde{\phi}^{-\Lambda}, \widetilde{\xi}^\Lambda)\\ &= -2i\int_{\mathcal{M}} \mathrm{d}\mu\; \widetilde{\varphi}_l^\dagger\, \widetilde{\sigma}^\mu_\Lambda(\partial_\mu + f_\mu)\zeta_l + \widetilde{\varphi}_r^\dagger \sigma^\mu_\Lambda(\partial_\mu - f_\mu)\zeta_r.\end{aligned} \tag{6.18}$$

Proof. Following the analysis below Proposition 6.2, the twisted fermionic action from a boosted observer is $\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}\otimes\mathbb{I}_2}(\widetilde{\eta}^\Lambda, \widetilde{\eta}^\Lambda)$. By a calculation similar to the one of Proposition 4.4, one obtains

$$\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}\otimes\mathbb{I}_2}(\eta^\Lambda, \eta'^\Lambda) = \mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}}(\phi^{-\Lambda}, \xi'^\Lambda) + \mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}}(\xi^\Lambda, \phi'^{-\Lambda}). \tag{6.19}$$

By boost invariance (6.11), the terms in the r.h.s. have the same symmetry as the corresponding expression without Λ, that is, symmetric on graßmanian vectors. Thus, similar to (4.39), one gets $\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}\otimes\mathbb{I}_2}(\widetilde{\eta}^\Lambda, \widetilde{\eta}^\Lambda) = 2\mathfrak{A}^\rho_{\eth^\Lambda_{\mathbf{X}}}(\widetilde{\phi}^{-\Lambda}, \widetilde{\xi}^\Lambda)$. The result follows from Proposition 6.4. ∎

We identify the boosted physical degrees of freedom $\Psi = S[\Lambda]\widetilde{\zeta}$, $\Psi^\dagger = i(\mathcal{J} S[\Lambda]^{-1}\widetilde{\phi})^\dagger$ following (6.8), (6.9). In components (see Definition 6.3), one has that

$$\psi_{l,r} = \widetilde{\zeta}_{l,r}, \quad \psi^\dagger_{l,r} = i\widetilde{\varphi}^\dagger_{l,r}. \tag{6.20}$$

The Lagrangian density in (6.18) then reads

$$\mathcal{L}_\Lambda = -2\big(\psi_l^\dagger \widetilde{\sigma}^\mu_\Lambda(\partial_\mu + f_\mu)\psi_l + \psi_r^\dagger \sigma^\mu_\Lambda(\partial_\mu - f_\mu)\psi_r\big). \tag{6.21}$$

Treating ψ_l, ψ_r, $\psi_l^\dagger$, and $\psi_r^\dagger$ as independent fields, the corresponding equations of motion are

$$\widetilde{\sigma}^\mu_\Lambda(\partial_\mu + f_\mu)\psi_l = 0, \quad \sigma^\mu_\Lambda(\partial_\mu - f_\mu)\psi_r = 0. \tag{6.22}$$

Proposition 6.7. *For a constant twisted fluctuation f_μ, a plane wave solution of the first (resp. the second) equation of* (6.22) *coincides with a plane wave solution of the left- (resp. right-) handed Weyl equation whose (dual) momentum $p^\sharp$ has components $p'_\nu = \Lambda^\mu_\nu p_\mu$ in the boosted frame, where*

$$p_0 = -f_0, \quad p_j = f_j, \quad \textit{resp.} \quad p_0 = f_0, \quad p_j = -f_j. \tag{6.23}$$

Proof. By (C.5), (C.6), a plane wave solution (B.8) of the first equation of (6.22) satisfies

$$0 = \tilde{\sigma}^{\mu}_{\Lambda}(-ip_\mu + f_\mu)\psi_l = \big(\Lambda^0_\nu \tilde{\sigma}^\nu_M(-ip_0 + f_0) - i\Lambda^j_\nu \tilde{\sigma}^\nu_M(-ip_j + f_j)\big)\psi_l, \tag{6.24}$$

$$= -\tilde{\sigma}^\nu_M\big(\Lambda^0_\nu(ip_0 - f_0) + \Lambda^j_\nu(p_j + if_j)\big)\psi_l. \tag{6.25}$$

Similarly, a plane wave solution ψ_r of the second equation of (6.22) satisfies

$$0 = -\sigma^\nu_M\big(\Lambda^0_\nu(ip_0 + f_0) + \Lambda^j_\nu \sigma^\nu_{\mathcal{M}}(p_j - if_j)\big)\psi_r. \tag{6.26}$$

If (6.23) holds, these two equations become

$$0 = -(1+i)\tilde{\sigma}^\nu_M\big(\Lambda^0_\nu p_0 + \Lambda^j_\nu p_j\big)\psi_{0l} = -(1+i)\tilde{\sigma}^\nu_M\, p'_\nu \psi_{0l}, \tag{6.27}$$

$$0 = -(1+i)\sigma^\nu_M(\Lambda^0_\nu p_0 + \Lambda^j_\nu p_j)\psi_{0r} = -(1+i)\sigma^\nu_M\, p'_\nu \psi_{0r}, \tag{6.28}$$

which coincide, up to a constant factor, with the Weyl equations of motion (B.9) for a boosted observer. ∎

Proposition 6.7 is the boosted version of Proposition 4.5: now, the whole field $f_\mu dx^\mu$ (and not only its zeroth component) identifies with the dual $p^\sharp$ of the energy-momentum 4-vector. Nevertheless, the interpretation of the Lagrangian density (6.21) is delicate, because of the sign difference in (6.23):

$$f_0 = -i\partial_0, \quad f_j = i\partial_j \quad \text{versus} \quad f_0 = i\partial_0, \quad f_j = -i\partial_j. \tag{6.29}$$

Substituting the first (resp. second) of these equations in the left- (resp. right-) handed part of (6.21), one obtains

$$2(i-1)\big(\psi_l^\dagger \tilde{\sigma}^\mu_M \partial'_\nu \psi_l + \psi_r^\dagger \sigma^\mu_M \partial'_\mu \psi_r\big) \quad \text{with } \partial'_\nu := \Lambda^\mu_\nu \partial_\mu. \tag{6.30}$$

This agrees with the equations of motion (6.27), (6.28), remembering that $\partial'_\nu = -ip'_\nu$ and the factor -2 that was ignored from (6.21) to (6.22), thus suggesting that $\mathcal{L}_\Lambda$ is the sum – up to a complex factor – of the two Weyl lagrangians $\mathcal{L}^l_{\mathcal{M}}, \mathcal{L}^r_{\mathcal{M}}$ (B.6). The point is that ψ_l, ψ_r come from the action of $\Lambda_\mp$ on the same Weyl spinor φ, and this action leaves the exponential part of the plane wave unaltered. So ψ_l, ψ_r should describe two plane waves with the same momenta, in contradiction with (6.29). We comment on this point in the conclusion.

Remark 6.8. The no-boost limit of the action (6.18) yields back the action (4.36) of the double manifold (along the lines of remark (6.5)). As well for the Lagrangian: identifying $\psi_l, \psi_r \to \zeta$ with ψ in (4.40), $\psi_l^\dagger = i\bar{\varphi}_l^\dagger \to -i\bar{\varphi}^\dagger\sigma_2$ with $\psi^\dagger$, and $\psi_r^\dagger = i\bar{\varphi}_r^\dagger \to i\bar{\varphi}^\dagger\sigma_2$ with $-\psi^\dagger$, then (6.21) becomes $4i(\psi^\dagger(if_0 - \sum_j \sigma_j\partial_j))$, in agreement with (4.41) (the expression with the opposite sign is obtained identifying the no-boost limit of $\psi_r^\dagger$ with ψ, and the one of $\psi_l^\dagger$ with $-\psi$).

6.3. Dirac equation for boosted observers

A boost $S[\Lambda]$ acts on the twisted covariant Dirac operator $\mathcal{D}_Z$ of electrodynamics (5.24) to give

$$\mathcal{D}_Z^\Lambda := S[\Lambda]\mathcal{D}_Z S[\Lambda]^{-1} = \eth^\Lambda \otimes \mathbb{I}_4 + \gamma^5 \otimes D_{\mathcal{F}} + \mathbf{X}^\Lambda \otimes \mathbb{I}' + i\mathbf{Y}^\Lambda \otimes \mathbb{I}'', \tag{6.31}$$

where $\eth^\Lambda$, $\mathbf{X}^\Lambda$ are defined in (6.12); we used $S[\Lambda]\gamma^5 S[\Lambda]^{-1} = \mathbb{I}$ and defined (using notations (6.16))

$$\mathbf{Y}^\Lambda := S[\Lambda]\,\mathbf{Y}\,S[\Lambda]^{-1} = -iS[\Lambda]\gamma^\mu g_\mu \mathbb{I}_4 S[\Lambda]^{-1} = -i\gamma_\Lambda^\mu g_\mu \mathbb{I}_4. \tag{6.32}$$

Similarly to what has been done for the double manifold in (6.17), we make the boost acts on $L^2(\mathcal{M}, S)\otimes\mathbb{C}^4$ in such a way that $\eta \in \mathcal{H}_{\mathcal{R}}$ in (5.31) is mapped to

$$\eta^\Lambda = \phi_1^{-\Lambda}\otimes e_L + \phi_2^{-\Lambda}\otimes e_R + \zeta_1^\Lambda \otimes \overline{e_L} + \zeta_2^\Lambda \otimes \overline{e_R}. \tag{6.33}$$

Proposition 6.9. *The fermionic action from the minimal twist of electrodynamics, as seen from a boosted observer, is the integral*

$$\mathfrak{A}^\rho_{\mathcal{D}_Z}(\widetilde{\eta}^\Lambda, \widetilde{\eta}^\Lambda) = -2\int_{\mathcal{M}} \mathrm{d}\mu\, \mathcal{L}_\Lambda \tag{6.34}$$

of the Lagrangian density

$$\begin{aligned}\mathcal{L}_\Lambda = {} & i\left(\widetilde{\overline{\varphi}}{}_{1l}^\dagger\, \widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\,\widetilde{\zeta}_{1l} + \widetilde{\overline{\varphi}}{}_{1r}^\dagger\, \sigma_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\widetilde{\zeta}_{1r}\right) + d\left(\widetilde{\overline{\varphi}}{}_{2l}^\dagger\widetilde{\zeta}_{1r} - \widetilde{\overline{\varphi}}{}_{2r}^\dagger\widetilde{\zeta}_{1l}\right)\\ & + i\left(\widetilde{\overline{\varphi}}{}_{2l}^\dagger\, \widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\,\widetilde{\zeta}_{2l} + \widetilde{\overline{\varphi}}{}_{2r}^\dagger\, \sigma_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\widetilde{\zeta}_{2r}\right) + \bar{d}\left(\widetilde{\overline{\varphi}}{}_{1l}^\dagger\widetilde{\zeta}_{2r} - \widetilde{\overline{\varphi}}{}_{1r}^\dagger\widetilde{\zeta}_{2l}\right),\end{aligned} \tag{6.35}$$

where $\mathcal{D}_\mu = \partial_\mu - ig_\mu$.

Proof. The computation is similar to the one of Proposition 5.12. One obtains

$$\begin{aligned}\mathfrak{A}^\rho_{\mathcal{D}_Z}(\widetilde{\eta}^\Lambda, \widetilde{\eta}^\Lambda) = {} & 2\mathfrak{A}^\rho_{\eth^\Lambda}(\widetilde{\phi}_1^{-\Lambda}, \widetilde{\xi}_1^\Lambda) + 2\,\mathfrak{A}^\rho_{\eth^\Lambda}(\widetilde{\phi}_2^{-\Lambda}, \widetilde{\xi}_2^\Lambda) + 2\,\mathfrak{A}^\rho_{\mathbf{X}^\Lambda}(\widetilde{\phi}_1^{-\Lambda}, \widetilde{\xi}_1^\Lambda)\\ & - 2\,\mathfrak{A}^\rho_{\mathbf{X}^\Lambda}(\widetilde{\phi}_2^{-\Lambda}, \widetilde{\xi}_2^\Lambda) - 2\mathfrak{A}^\rho_{i\mathbf{Y}^\Lambda}(\widetilde{\phi}_1^{-\Lambda}, \widetilde{\xi}_1^\Lambda) - 2\mathfrak{A}^\rho_{i\mathbf{Y}^\Lambda}(\widetilde{\phi}_2^{-\Lambda}, \widetilde{\xi}_2^\Lambda)\\ & + 2\bar{d}\,\mathfrak{A}^\rho_{\gamma^5}(\widetilde{\phi}_1^{-\Lambda}, \widetilde{\xi}_2^\Lambda) + 2d\,\mathfrak{A}^\rho_{\gamma^5}(\widetilde{\phi}_2^{-\Lambda}, \widetilde{\xi}_1^{-\Lambda}),\end{aligned}$$

where we used that the bilinear forms $\mathfrak{A}^\rho_{\eth^\Lambda}$, $\mathfrak{A}^\rho_{\mathbf{X}^\Lambda}$, $\mathfrak{A}^\rho_{\gamma^5}$, and $\mathfrak{A}^\rho_{i\mathbf{Y}^\Lambda}$ valued on $\widetilde{\phi}_i^{-\Lambda}$, $\widetilde{\zeta}_j^\Lambda$ have the same symmetry properties of the corresponding expressions without Λ (by the invariance (6.11), that holds also for $\mathbf{X}^\Lambda$, γ^5, and $i\mathbf{Y}^\Lambda$). Substituting $\mathfrak{A}^\rho_{\eth^\Lambda}$ and $\mathfrak{A}^\rho_{\mathbf{X}^\Lambda}$ with their explicit form given in the proof of Proposition 6.4 and calculating (with $\mathcal{J}\phi^{-\Lambda} = (\mathcal{J}\phi)^\Lambda$ given in Definition 6.3)

$$\begin{aligned}\mathfrak{A}^\rho_{i\mathbf{Y}^\Lambda}(\phi^{-\Lambda}, \zeta^\Lambda) &= \langle \mathcal{J}\phi^{-\Lambda}, \gamma^0 i\mathbf{Y}^\Lambda \xi^\Lambda\rangle = \begin{pmatrix}\overline{\varphi}_l\\ \overline{\varphi}_r\end{pmatrix}^\dagger \gamma^0 \begin{pmatrix} g_\mu\sigma_\Lambda^\mu\zeta_r\\ g_\mu\widetilde{\sigma}_\Lambda^\mu\zeta_l\end{pmatrix}\\ &= \int_{\mathcal{M}} \mathrm{d}\mu\, g_\mu(\overline{\varphi}_l^\dagger\widetilde{\sigma}_\Lambda^\mu\zeta_l + \overline{\varphi}_r\sigma_\Lambda^\mu\zeta_r),\end{aligned}$$

$$\mathfrak{A}^\rho_{\gamma^5}(\phi^{-\Lambda}, \zeta^\Lambda) = \langle \mathcal{J}\phi^{-\Lambda}, \gamma^5 \xi^\Lambda\rangle = \begin{pmatrix}\overline{\varphi}_l\\ \overline{\varphi}_r\end{pmatrix}^\dagger \gamma^0\gamma^5 \begin{pmatrix}\zeta_l\\ \zeta_r\end{pmatrix} = -\int_{\mathcal{M}} \mathrm{d}\mu(\overline{\varphi}_l^\dagger\zeta_r - \overline{\varphi}_r^\dagger\zeta_l),$$

one obtains the result. ∎

Again, taking the no-boost limit as in Remark 6.5, one checks that (6.34) yields back the fermionic action (5.33) for the minimal twist of the spectral triple of electrodynamics.

Boosting the rule of identification (5.52) in the line of (6.9), one identifies the physical spinors

$$\begin{aligned} \Psi &:= \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} = S[\Lambda]\,\widetilde{\Xi} = \begin{pmatrix} \widetilde{\zeta}_{1l} \\ \widetilde{\zeta}_{2r} \end{pmatrix}, \\ \Psi^\dagger &:= \begin{pmatrix} \psi_l^\dagger \\ \psi_r^\dagger \end{pmatrix} = i\left(S[\Lambda]\mathcal{J}\widetilde{\phi}\right)^\dagger = \begin{pmatrix} i\widetilde{\varphi}_{1l}^\dagger \\ i\widetilde{\varphi}_{2r}^\dagger \end{pmatrix}, \end{aligned} \tag{6.36}$$

$$\begin{aligned} \Psi' &:= \begin{pmatrix} \psi_l' \\ \psi_r' \end{pmatrix} = S[\Lambda]\gamma^0\widetilde{\Xi} = \begin{pmatrix} \widetilde{\zeta}_{2l} \\ \widetilde{\zeta}_{1r} \end{pmatrix}, \\ \Psi'^\dagger &:= \begin{pmatrix} \psi_l'^\dagger \\ \psi_r'^\dagger \end{pmatrix} = -i\left(S[\Lambda]\mathcal{J}\gamma^0\widetilde{\phi}\right)^\dagger = \begin{pmatrix} -i\widetilde{\varphi}_{2l}^\dagger \\ -i\widetilde{\varphi}_{1r}^\dagger, \end{pmatrix}, \end{aligned} \tag{6.37}$$

using Definition 6.3 to write the components. The Lagrangian density (6.35) becomes

$$\begin{aligned} \mathcal{L}_\Lambda &= \psi_l^\dagger\,\widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\,\psi_l + \psi_r^\dagger\,\sigma_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\psi_r + id(\psi_l^\dagger\psi_r + \psi_r^\dagger\psi_l) \\ &\quad - \psi_l'^\dagger\,\widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\,\psi_l' - \psi_r'^\dagger\,\sigma_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\psi_r' - i\bar{d}(\psi_r'^\dagger\psi_l' + \psi_l'^\dagger\psi_r'). \end{aligned} \tag{6.38}$$

Remark 6.10. In the no-boost limit, $\psi_{l,r}$, $\psi_{l,r}^\dagger$ in (6.36) coincide with $\psi_{l,r}$, $\psi_{l,r}^\dagger$ in (5.42), and $\psi_{l,r}'$, $\psi_{l,r}'^\dagger$ in (6.37) with $\psi_{r,l}$, $\psi_{r,l}^\dagger$ in (5.42). This allows to retrieve (5.44) as the no-boost limit of (6.38), imposing $d = im$ and taking into account the factor -2 in (6.34) and 4 in Proposition 5.12. Conversely, $\psi_{l,r}'$, $\psi\dagger_{l,r}'$ in (6.36) and $\psi_{l,r}$, $\psi_{l,r}'^\dagger$ in (6.37) coincide with $\psi_{l,r}'$, $\psi_{l,r}'^\dagger$ and $\psi_{r,l}'$, $\psi_{r,l}'^\dagger$ in (5.42), allowing to retrieve (5.47) as the no-boost limit of (6.38).

Treating all the fields independently, one obtains the two pairs of equations of motion:

$$\widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\psi_l = id\psi_r, \qquad \sigma_\Lambda^\mu(\mathcal{D}_\mu + f_\mu)\psi_r = m\psi_l; \tag{6.39}$$

$$\widetilde{\sigma}_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\psi_l' = -i\bar{d}\psi_r', \qquad \sigma_\Lambda^\mu(\mathcal{D}_\mu - f_\mu)\psi_r' = -i\bar{d}\psi_l'. \tag{6.40}$$

The generalised energy-momentum 4-vector $P := p + g^\flat$ is the sum of the energy momentum p with the musical dual of the 1-form $g = g_\mu dx^\mu$. In practical, this means that

$$\mathcal{D}_\mu\, e^{-ix^\mu p_\mu} = -iP_\mu, \tag{6.41}$$

where P_μ are the components of $P^\flat$. This leads to our final proposition.

Proposition 6.11. *For a constant fluctuation f_μ, a plane wave solution of* (6.39) *(resp.* (6.40)*) coincides with a plane wave solution of the Dirac equation with mass $m = -(1+i)\frac{d}{2}$ (resp. $m = (1+i)\frac{\bar{d}}{2}$), whose (dual) generalised momentum P has components $P_\nu' = \Lambda_\nu^\mu P_\mu$ in the boosted frame, where*

$$P_0 = -f_0, \quad P_j = f_j, \quad \textit{resp.} \quad P_0 = f_0, \quad P_j = -f_j. \tag{6.42}$$

Proof. From (C.5) and (C.6), a plane wave solution (B.3) of (6.39) satisfies

$$\begin{aligned} id\psi_r &= \tilde{\sigma}^{\mu}_{\Lambda}(\mathcal{D}_\mu + f_\mu)\psi_l = \tilde{\sigma}^{\mu}_{\Lambda}(-iP_\mu + f_\mu)\psi_l \\ &= \tilde{\sigma}^{\nu}_{M}\big(\Lambda^0_\nu(-iP_0 + f_0) - i\Lambda^j_\nu(-iP_j + f_j)\big)\psi_l, \end{aligned}$$

and a similar equation with σ^μ, inverting ψ_l and ψ_r. If the first part of (6.42) holds, then these equations are equivalent to $\tilde{\sigma}^{\nu}_{M} P'_\mu \psi_l = \frac{-id}{1+i}\psi_r$ and a similar equation for σ^μ. These coincide with the Dirac equation (B.5), with mass $m = -(1+i)\frac{d}{2}$. Similarly, a plane wave solution of (6.40) satisfies

$$\begin{aligned} -i\bar{d}\psi'_r &= \tilde{\sigma}^{\mu}_{\Lambda}(\mathcal{D}_\mu - f_\mu)\psi'_l = \tilde{\sigma}^{\mu}_{\Lambda}(-iP_\mu - f_\mu)\psi'_l \\ &= \tilde{\sigma}^{\nu}_{M}\big(\Lambda^0_\nu(-iP_0 - f_0) - i\Lambda^j_\nu(-iP_j - f_j)\big)\psi'_l, \end{aligned}$$

which becomes $\tilde{\sigma}^{\nu}_{M} P'_\mu \psi'_l = \frac{i\bar{d}}{1+i}\psi'_r$ if the second part of (6.42) holds. Together with a similar equation for σ^μ, these coincide with the Dirac equations (B.5), with mass $m = (1+i)\frac{\bar{d}}{2}$. ∎

To guarantee a positive mass, one should impose that $d = m(i \pm 1)$ with $m \in \mathbb{R}^+$. Identifying the imaginary/real axis of the complex plane with the space/time directions of two-dimensional Minkowski space, the set of all physically acceptable values of d is the future light cone, while in the non-boosted case, it was the imaginary axis $d = im, m \in \mathbb{R}$.

7. Conclusion and outlook

The twisted fermionic action associated to the minimal twist of a doubled manifold and that of the spectral triple of electrodynamics yield, respectively, the Weyl and the Dirac equations in Lorentzian signature, although one started with a Euclidean manifold. The 1-form field parametrising the twisted fluctuation gets interpreted as an energy-momentum 4-vector. It was known that fluctuations of the geometry generate the bosonic content of the theory (including the Higgs sector). What is new here is that they generate also the energy momentum. In other terms, the dynamics is obtained as a fluctuation of the geometry.

It should be checked that a similar transition from the Riemannian to the pseudo-Riemannian also takes place for the minimal twist of the SM. This will be the subject of future works, as well as the extension of these results to curved Riemannian manifolds.

Some points that deserve to be better understood are the following.

- Is the twisted fermionic action really Lorentzian, since the manifold $\mathcal{M}$ under which one integrates remains Riemannian? Actually, this is not a problem if one takes as domain of integration a local chart (as in quantum field theory: the Wick rotation is usually viewed as a local operation), up to a change of the volume form (see [19] for details). Nevertheless, one may hope that the twist actually changes the metric on the manifold, through Connes distance formula for instance (relations between causal structure and this distance have already been worked out in [26, 27, 33], but without taking into account the twist).

- The twisted fermionic action is invariant under an action of the Lorentz group, and the equations of motions in the boosted frame coincide with those derived from the Weyl and Dirac equations in the boosted frame as well. But the boosted Lagrangians do not agree, because of the difference of sign in the definition of the physical left/right spinors. As stated in the text, this sign difference is not compatible with the initial restriction to $\mathcal{H}_{\mathcal{R}}$. To overcome this difficulty, one may relax this restriction. Whether this still permits to define an antisymmetric bilinear form, that yields a physically meaningful action, will be investigated elsewhere.

In any case, the results presented here suggest an alternative attack to the problem of extending the theory of spectral triples to Lorentzian geometries. That the twist does not fully implement the Wick rotation (it does it only for the Hilbert space but not for the Dirac operator) is not so relevant after all. More than being able to spectrally characterise a pseudo-Riemannian manifold, what matters most for the physics is to obtain an action that makes sense in a Lorentzian context. The present work shows that this happens for the fermionic action.

The spectral action in the twisted context is still an open problem. The interpretation of the 1-form field $f_\mu dx^\mu$ as the energy-momentum 4-vector might be relevant in this context as well.

Contrary to most approaches in the literature (e.g., [1, 35]), we do not obtain a Lorentzian action by implementing a Lorentzian structure on the geometry. The latter somehow “emerges” from the Riemannian one. This actually makes sense, remembering that the regularity condition imposed by Connes and Moscovici (see Remark 2.3) has its origin in Tomita’s modular theory. More precisely, the automorphism ρ that defines a twisted spectral triple should be viewed as the evaluation, at some specific value t, of a one-parameter group of automorphism ρ_t. For the minimal twist of spectral triples, the flip came out as the only automorphism that makes the twisted commutator bounded. It is not yet clear what would be the corresponding one-parameter group of automorphisms. Should it exist, this will indicate that the time evolution in the SM has its origin in the modular group. This is precisely the content of the thermal time hypothesis of Connes and Rovelli [18]. So far, this hypothesis has been applied to algebraic quantum field theory [31, 32], and for general considerations in quantum gravity [34]. Its application to the SM would be a novelty.

A. Gamma matrices in chiral representation

Let $\sigma_{j=1,2,3}$ be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

In a four-dimensional Euclidean space, the Dirac matrices (in chiral representation) are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 := \gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{A.2}$$

where, for $\mu = 0, j$, we define

$$\sigma^\mu := \{\mathbb{I}_2, -i\sigma_j\}, \quad \tilde{\sigma}^\mu := \{\mathbb{I}_2, i\sigma_j\}. \tag{A.3}$$

In Minkowski spacetime with signature $(+,-,-,-)$, the Dirac matrices are

$$\gamma_M^\mu = \begin{pmatrix} 0 & \sigma_M^\mu \\ \bar{\sigma}_M^\mu & 0 \end{pmatrix}, \quad \gamma_M^5 := \gamma_M^1\gamma_M^2\gamma_M^3\gamma_M^0 = -i\gamma^5, \tag{A.4}$$

where, for $\mu = 0, j$, we define

$$\sigma_M^\mu := \{\mathbb{I}_2, \sigma_j\}, \quad \bar{\sigma}_M^\mu := \{\mathbb{I}_2, -\sigma_j\}. \tag{A.5}$$

B. Weyl and Dirac equations

A Dirac spinor $\Psi = \left(\begin{smallmatrix}\psi_l \\ \psi_r\end{smallmatrix}\right) \in L^2(\mathcal{M}, \mathcal{S})$ is the direct sum of two Weyl spinors ψ_l and ψ_r. With our definition of the chiral representation, a left-handed spinor is an eigenspinor of the $+1$-eigenspace $L^2(\mathcal{M}, S)_+$ of the grading operator γ^5, and a right-handed spinor an eigenspinor of the -1 eigenspace $L^2(\mathcal{M}, S)_-$ (in the physics literature, the convention is usually opposite).

The Dirac Lagrangian in Minkowski spacetime is

$$\begin{aligned}
\mathcal{L}_M &= -\bar{\Psi}(\eth_M + m)\Psi \\
&= \begin{pmatrix} \psi_l^\dagger & \psi_r^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & i\sigma_M^\mu\partial_\mu \\ i\tilde{\sigma}_M^\mu\partial_\mu & 0 \end{pmatrix} - m \right] \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} \\
&= i\psi_l^\dagger\tilde{\sigma}_M^\mu\partial_\mu\psi_l + i\psi_r^\dagger\sigma_M^\mu\partial_\mu\psi_r - m\big(\psi_l^\dagger\psi_r + \psi_r^\dagger\psi_l\big),
\end{aligned} \tag{B.1}$$

where $\bar{\Psi} := \Psi^\dagger\gamma^0$ and $\eth = -i\gamma^\mu\partial_\mu$. The equations of motion are derived by a variational principle, treating $\psi_{l/r}$ and their Hermitian conjugates $\psi_{l/r}^\dagger$ as independent variables. In particular, the Euler–Lagrange equations for $\psi_l^\dagger$, $\psi_r^\dagger$ yield Dirac equations (written in components)

$$i\tilde{\sigma}_M^\mu\partial_\mu\psi_l = m\psi_r, \quad i\sigma_M^\mu\partial_\mu\psi_r = m\psi_l. \tag{B.2}$$

By (A.5) one retrieves the familiar form [11, Section 19.77]:

$$i\Big(\partial_0 \mp \sum_j \sigma_j\partial_j\Big)\psi_{l/r} = m\psi_{r/l}.$$

A plane wave solution of (B.2) is

$$\Psi(x^0, x^j) = \Psi_0\, e^{-ip_\mu x^\mu} \quad \text{with } \Psi_0 = \begin{pmatrix} \psi_{0l} \\ \psi_{0r} \end{pmatrix}, \tag{B.3}$$

where $p_\mu := \eta_{\mu\nu} p^\nu$ are the components of the 1-form $p^\sharp$, dual of the energy-momentum 4-vector (p^0, p^j) induced by the Lorentz metric, and Ψ_0 is a constant spinor solution of

$$i\Big(-ip_0 + i\sum_{j=1}^{3}\sigma_j p_j\Big)\psi_{0l} = m\psi_{0r}, \quad i\Big(-ip_0 - i\sum_{j=1}^{3}\sigma_j p_j\Big)\psi_r = m\psi_l. \tag{B.4}$$

The components $\psi_{l/r} = \psi_{0l/r}e^{-ip_\mu x^\mu}$ of the plane wave (B.3) are solutions of

$$\begin{aligned}\tilde{\sigma}_M^\mu p_\mu \psi_l &= \Big(p_0 - \sum_{j=1}^{3}\sigma_j p_j\Big)\psi_l = m\psi_r,\\ \sigma_M^\mu p_\mu \psi_r &= \Big(p_0 + \sum_{j=1}^{3}\sigma_j p_j\Big)\psi_r = m\psi_l.\end{aligned} \tag{B.5}$$

For $m = 0$, the Dirac Lagrangian is the sum of two independent pieces, the Weyl Lagrangians

$$\begin{aligned}\mathcal{L}_M^l &= i\psi_l^\dagger \tilde{\sigma}_M^\mu \partial_\mu \psi_l = i\psi_l^\dagger\Big(\partial_0 - \sum_{j=1}^{3}\sigma_j\partial_j\Big)\psi_l,\\ \mathcal{L}_M^r &= i\psi_r^\dagger \sigma_M^\mu \partial_\mu \psi_r = i\psi_r^\dagger\Big(\partial_0 + \sum_{j=1}^{3}\sigma_j\partial_j\Big)\psi_r,\end{aligned} \tag{B.6}$$

that describe Weyl fermions (massless spin-$\frac{1}{2}$ particle). The corresponding Weyl equations of motion are [11, eq. (19.40), (19.41)]

$$\tilde{\sigma}_M^\mu \partial_\mu \psi_l = \Big(\partial_0 - \sum_{j=1}^{3}\sigma_j\partial_j\Big)\psi_l = 0, \quad \sigma_M^\mu \partial_\mu \psi_r = \Big(\partial_0 + \sum_{j=1}^{3}\sigma_j\partial_j\Big)\psi_r = 0. \tag{B.7}$$

Their plane wave solutions,

$$\psi_l(x^0, x^j) = \psi_{0l}\, e^{-ip_\mu x^\mu}, \quad \psi_r(x^0, x^j) = \psi_{0r}\, e^{-ip_\mu x^\mu}, \tag{B.8}$$

with ψ_{0l}, ψ_{0r} momentum-dependant spinors satisfying (B.4) for $m = 0$, are solutions of

$$\Big(p_0 - \sum_{j=1}^{3}\sigma_j p_j\Big)\psi_{0l} = 0, \quad \Big(p_0 + \sum_{j=1}^{3}\sigma_j p_j\Big)\psi_{0r} = 0. \tag{B.9}$$

C. Spin representation of boosts

The spinor representation of a boost of rapidity $b/2$ in the direction $\mathbf{n}$ is given by

$$S[\Lambda] = \begin{pmatrix}\Lambda_+ & 0\\ 0 & \Lambda_-\end{pmatrix}, \quad \text{where } \Lambda_\pm := \exp(\pm\mathbf{a}\cdot\boldsymbol{\sigma}) \text{ with } \mathbf{a} := \frac{b}{2}\mathbf{n}. \tag{C.1}$$

Collecting the terms with even and odd powers in the expansion of $\exp(\pm\mathbf{a}\cdot\boldsymbol{\sigma})$, one checks that $\Lambda_\pm = \Lambda_1 \pm \Lambda_2$, where $\Lambda_1 := (\cosh|\mathbf{a}|)\mathbb{I}_2$, $\Lambda_2 := (\sinh|\mathbf{a}|)\,\mathbf{n}\cdot\boldsymbol{\sigma}$. Thus Λ_+, Λ_- are both self-adjoint and inverse of one another. Meaning that $S[\Lambda]$ is self-adjoint but not unitary,

$$S[\Lambda]^\dagger = S[\Lambda] \neq S[\Lambda]^{-1}. \tag{C.2}$$

Under such a boost, a Lorentzian spinor and the Lorentzian Dirac operator transform as

$$\psi_M \to S[\Lambda]\psi_M, \quad \eth_M \to S[\Lambda]\,\eth_M\,S[\Lambda]^{-1}. \tag{C.3}$$

By construction, the spin representation of the Lorentz group is such that (see e.g. [11, Section 20.78])

$$(\tilde\sigma_M^\mu)_\Lambda := S[\Lambda]\tilde\sigma_M^\mu S[\Lambda]^{-1} = \Lambda^\mu_\nu\tilde\sigma_M^\nu, \quad (\sigma_M^\mu)_\Lambda = S[\Lambda]\sigma_M^\mu S[\Lambda]^{-1} = \Lambda^\mu_\nu\sigma_M^\nu, \tag{C.4}$$

where $\{\Lambda^\mu_\nu\}$ is the matrix representation of the Lorentz group on Minkowski space. Since $\tilde\sigma^0 = \tilde\sigma_M^0$, $\sigma^0 = \sigma_M^0$ and $\tilde\sigma^j = -i\tilde\sigma_M^j$, $\sigma^j = -i\tilde\sigma_M^j$ for $j = 1,2,3$, one gets

$$\tilde\sigma_\Lambda^0 := S[\Lambda]\,\tilde\sigma^0\,S[\Lambda]^{-1} = \Lambda^0_\nu\,\tilde\sigma_M^\nu, \qquad \sigma_\Lambda^0 := S[\Lambda]\sigma^0 S[\Lambda]^{-1} = \Lambda^0_\nu\sigma_M^\nu, \tag{C.5}$$

$$\tilde\sigma_\Lambda^j := S[\Lambda]\tilde\sigma^j S[\Lambda]^{-1} = -i\Lambda^j_\nu\tilde\sigma_M^\nu, \quad \sigma_\Lambda^j := S[\Lambda]\tilde\sigma^j S[\Lambda]^{-1} = -i\Lambda^j_\nu\sigma_M^j. \tag{C.6}$$

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References

[1] J. W. Barrett, Lorentzian version of the noncommutative geometry of the Standard Model of particle physics. *J. Math. Phys.* **48** (2007), no. 1, Article No. 012303 Zbl 1121.81122 MR 2292605

[2] F. Besnard and N. Bizi, On the definition of spacetimes in noncommutative geometry: Part I. *J. Geom. Phys.* **123** (2018), 292–309 Zbl 1376.83027 MR 3724788

[3] F. Besnard and P. Martinetti, in preparation

[4] T. Brzeziński, N. Ciccoli, L. Dąbrowski, and A. Sitarz, Twisted reality condition for Dirac operators. *Math. Phys. Anal. Geom.* **19** (2016), no. 3, Article No. 16 Zbl 1413.58002 MR 3532813

[5] T. Brzeziński, L. Dąbrowski, and A. Sitarz, On twisted reality conditions. *Lett. Math. Phys.* **109** (2019), no. 3, 643–659 Zbl 1414.58008 MR 3910138

[6] A. Chamseddine and W. D. van Suijlekom, A survey of spectral models of gravity coupled to matter. In *Advances in Noncommutative Geometry*, pp. 1–51, Springer, Cham, 2019 Zbl 1443.58006 MR 4300551

[7] A. H. Chamseddine and A. Connes, Resilience of the spectral standard model. *J. High Energy Phys.* **2012** (2012), no. 9, Article No. 104 Zbl 1397.81412 MR 3044924

[8] A. H. Chamseddine, A. Connes, and M. Marcolli, Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.* **11** (2007), no. 6, 991–1089 Zbl 1140.81022 MR 2368941

[9] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, Beyond the spectral standard model: emergence of Pati–Salam unification. *J. High Energy Phys.* **11** (2013), Article No. 132 Zbl 1410.81032

[10] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, Inner fluctuations in noncommutative geometry without the first order condition. *J. Geom. Phys.* **73** (2013), 222–234 Zbl 1284.58014 MR 3090113

[11] S. Coleman, *Quantum Field Theory*. World Scientific, Hackensack, NJ, 2019 Zbl 1431.81001

[12] A. Connes, *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[13] A. Connes, Noncommutative geometry and reality. *J. Math. Phys.* **36** (1995), no. 11, 6194–6231 Zbl 0871.58008 MR 1355905

[14] A. Connes, Gravity coupled with matter and the foundation of non-commutative geometry. *Comm. Math. Phys.* **182** (1996), no. 1, 155–176 Zbl 0881.58009 MR 1441908

[15] A. Connes, On the spectral characterization of manifolds. *J. Noncommut. Geom.* **7** (2013), no. 1, 1–82 Zbl 1287.58004 MR 3032810

[16] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*. Amer. Math. Soc. Colloq. Publ. 55, American Mathematical Society, Providence, RI, 2008 Zbl 1159.58004 MR 2371808

[17] A. Connes and H. Moscovici, Type III and spectral triples. In *Traces in Number Theory, Geometry and Quantum Fields*, pp. 57–71, Aspects Math. E38, Friedr. Vieweg, Wiesbaden, 2008 Zbl 1159.46041 MR 2427588

[18] A. Connes and C. Rovelli, von Neumann algebra automorphisms and time-thermodynamics relation in generally covariant quantum theories. *Classical Quantum Gravity* **11** (1994), no. 12, 2899–2917 Zbl 0821.46086 MR 1307019

[19] F. D'Andrea, M. A. Kurkov, and F. Lizzi, Wick rotation and fermion doubling in noncommutative geometry. *Phys. Rev. D* **94** (2016), no. 2, Article No. 025030 MR 3669842

[20] A. Devastato, S. Farnsworth, F. Lizzi, and P. Martinetti, Lorentz signature and twisted spectral triples. *J. High Energy Phys.* (2018), no. 3, Article No. 089 Zbl 1388.83534 MR 3798496

[21] A. Devastato, M. Kurkov, and F. Lizzi, Spectral noncommutative geometry standard model and all that. *Internat. J. Modern Phys. A* **34** (2019), no. 19, Article No. 1930010 Zbl 1467.81004 MR 3980723

[22] A. Devastato, F. Lizzi, and P. Martinetti, Grand symmetry, spectral action and the Higgs mass. *J. High Energy Phys.* **2014** (2014), no. 1, Article No. 42

[23] A. Devastato and P. Martinetti, Twisted spectral triple for the standard model and spontaneous breaking of the grand symmetry. *Math. Phys. Anal. Geom.* **20** (2017), no. 1, Article No. 2 Zbl 1413.58003 MR 3589931

[24] N. Franco, Temporal Lorentzian spectral triples. *Rev. Math. Phys.* **26** (2014), no. 8, Article No. 1430007 Zbl 1305.58007 MR 3256858

[25] N. Franco and M. Eckstein, An algebraic formulation of causality for noncommutative geometry. *Classical Quantum Gravity* **30** (2013), no. 13, Article No. 135007 Zbl 1273.83019 MR 3072913
[26] N. Franco and M. Eckstein, Exploring the causal structures of almost commutative geometries. *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Article No. 010 Zbl 1291.58003 MR 3210625
[27] N. Franco and M. Eckstein, Causality in noncommutative two-sheeted space-times. *J. Geom. Phys.* **96** (2015), 42–58 Zbl 1327.58011 MR 3372018
[28] G. Landi and P. Martinetti, On twisting real spectral triples by algebra automorphisms. *Lett. Math. Phys.* **106** (2016), no. 11, 1499–1530 Zbl 1372.58006 MR 3555412
[29] G. Landi and P. Martinetti, Gauge transformations for twisted spectral triples. *Lett. Math. Phys.* **108** (2018), no. 12, 2589–2626 Zbl 1404.58015 MR 3865749
[30] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories. *Phys. Rev. D (3)* **55** (1997), no. 10, 6357–6366 MR 1454002
[31] P. Martinetti, Conformal mapping of Unruh temperature. *Modern Phys. Lett. A* **24** (2009), no. 19, 1473–1483 Zbl 1168.81366 MR 2538067
[32] P. Martinetti and C. Rovelli, Diamond's temperature: Unruh effect for bounded trajectories and thermal time hypothesis. *Classical Quantum Gravity* **20** (2003), no. 22, 4919–4931 Zbl 1170.83390 MR 2018811
[33] V. Moretti, Aspects of noncommutative Lorentzian geometry for globally hyperbolic spacetimes. *Rev. Math. Phys.* **15** (2003), no. 10, 1171–1217 Zbl 1055.46048 MR 2038068
[34] C. Rovelli and M. Smerlak, Thermal time and Tolman–Ehrenfest effect: 'temperature as the speed of time'. *Classical Quantum Gravity* **28** (2011), no. 7, Article No. 075007 Zbl 1213.83123 MR 2777054
[35] K. van den Dungen, Krein spectral triples and the fermionic action. *Math. Phys. Anal. Geom.* **19** (2016), no. 1, Article No. 4 Zbl 1413.53113 MR 3472113
[36] K. van den Dungen, M. Paschke, and A. Rennie, Pseudo–Riemannian spectral triples and the harmonic oscillator. *J. Geom. Phys.* **73** (2013), 37–55 Zbl 1285.53060 MR 3090101
[37] K. van den Dungen and W. D. van Suijlekom, Electrodynamics from noncommutative geometry. *J. Noncommut. Geom.* **7** (2013), no. 2, 433–456 Zbl 1272.58006 MR 3054302
[38] W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*. Math. Phys. Stud., Springer, Dordrecht, 2015 Zbl 1305.81007 MR 3237670

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Pierre Martinetti
Dipartimento di Matematica, Università di Genova; and INFN, Sezione di Genova, Via Dodecaneso, 16146 Genova, Italy; martinetti@dima.unige.it

Devashish Singh
Dipartimento di Matematica, Università di Genova, Via Dodecaneso, 16146 Genova, Italy; devashish@dima.unige.it