J. Noncommut. Geom. 16 (2022), 561–594
DOI 10.4171/JNCG/449

Polarization and deformations of generalized dendriform algebras

Cyrille Ospel, Florin Panaite, and Pol Vanhaecke

Abstract. We generalize three results of M. Aguiar on Loday's dendriform algebras to dendriform algebras associated with algebras satisfying any given set of relations. We adapt the concept of polarization to such algebras, and use it to generalize Aguiar's results on deformations and filtrations of dendriform algebras. We introduce weak Rota–Baxter operators and use them to prove a generalization of another result by Aguiar, which provides an interpretation of the natural relation between infinitesimal bialgebras and pre-Lie algebras in terms of dendriform algebras.

1. Introduction

Dendriform algebras were introduced by J.-L. Loday in [20] as a dichotomized version of associative algebras: if $(A, \prec, \succ)$ is a dendriform algebra, an associative algebra $(A, \star)$ is obtained by setting $a \star b := a \prec b + a \succ b$ for all $a, b \in A$. In [2], M. Aguiar introduced the notion of deformation for a *commutative* dendriform algebra $(A, \prec, \succ)$, where commutativity means that $a \prec b = b \succ a$ for all $a, b \in A$. He shows that such a deformation makes $(A, \succ, \bigcirc)$ into a pre-Poisson algebra, where $\bigcirc$ is constructed from the first order deformation terms of $\prec$ and $\succ$ and establishes a similar result for filtered dendriform algebras; both results are dendriform versions of two well-known results in deformation theory. Even if these results can easily be proven by a direct computation, these computations lack a conceptual understanding, which we will provide in this paper by generalizing these results to arbitrary dendriform algebras.

We define generalized dendriform algebras as follows. Let $\mathcal{C}$ denote the category of all algebras (A, μ) which satisfy a given set of relations $\mathcal{R}_1 = 0, \dots, \mathcal{R}_k = 0$. An algebra $(A, \prec, \succ)$ is said to be a $\mathcal{C}$*-dendriform algebra* if $(A \times A, \boxtimes) \in \mathcal{C}$, where $\boxtimes$ is defined for $(a, x), (b, y) \in A \times A$, by $(a, x) \boxtimes (b, y) := (a \prec b + a \succ b, a \succ y + x \prec b)$. The $\mathcal{C}$-dendriform algebras form a category $\mathcal{C}^{\text{dend}}$ with algebra homomorphisms as morphisms. Generalized dendriform algebras have already been considered from the operadic point of view in [4], but we will not use this formalism since the phenomena and properties which we present are most naturally expressed in terms of the basic algebraic language which we use.

2020 Mathematics Subject Classification. Primary 17A30; Secondary 17B38, 08C05, 53D55.
Keywords. Dendriform algebras, polarization, Rota–Baxter operators, deformations.

The notion of polarization, which was first introduced in [22] for algebras with one operation, is easily generalized to dendriform algebras. We use it to prove the following, above-mentioned result: suppose that $(A[[\nu]], \prec, \succ)$ is a deformation of a commutative algebra $(A, \prec_0, \succ_0)$ in $\mathscr{C}^{\mathrm{dend}}$ and consider the algebra $(A, \succ_0, \bigcirc)$, where $\bigcirc$ is defined for $a, b \in A$ by

$$a \bigcirc b := \frac{a \succ b - b \prec a}{2\nu}\Big|_{\nu=0}.$$

We show that $(A, \succ_0, \bigcirc) \in \underline{\mathscr{C}}^{\mathrm{dend}}_{\mathrm{pol}}$, where $\underline{\mathscr{C}}^{\mathrm{dend}}_{\mathrm{pol}}$ is the category of all polarized dendriform algebras $(A, *, \circ)$ satisfying for each relation $\mathcal{R} = 0$ of $\mathscr{C}^{\mathrm{dend}}_{\mathrm{pol}}$ the relation $\underline{\mathcal{R}} = 0$; here, $\underline{\mathcal{R}}$ stands for the lowest weight part of $\mathcal{R}$, where the weight of a monomial in A is defined as being the number of operations $\circ$ that it contains. For the case in which $(A, \prec_0, \succ_0)$ is a Loday dendriform algebra, we recover Aguiar's result. A similar result holds for filtered commutative algebras in $\mathscr{C}^{\mathrm{dend}}$. Both applications admit also an anticommutative version.

In order to construct (interesting) examples of generalized dendriform algebras, we introduce the notion of a weak Rota–Baxter operator. Given any algebra A, a linear map $\mathfrak{R} : A \to A$ is said to be a *weak Rota–Baxter operator* of A if, for all $a, b \in A$, the element $\mathfrak{R}(a\mathfrak{R}(b) + \mathfrak{R}(a)b) - \mathfrak{R}(a)\mathfrak{R}(b)$ commutes with all elements of A; when it is zero it is a *Rota–Baxter operator*. As an application, we generalize to coboundary ε-bialgebras another result by M. Aguiar [3], which states that the natural functor which associates to any ε-bialgebra (A, μ, Δ) the corresponding pre-Lie algebra $(A, \circ)$, restricted to the category of quasi-triangular ε-bialgebras, admits a natural factorization through the category of dendriform algebras; in our generalization, dendriform algebras are replaced by A_3-dendriform algebras.

The structure of the paper. We introduce in Section 2 the notion of a $\mathscr{C}$-dendriform algebra and show how to obtain the relations in $\mathscr{C}^{\mathrm{dend}}$ from the ones in $\mathscr{C}$. (Weak) Rota–Baxter operators are shown in Section 3 to provide $\mathscr{C}$-dendriform algebras, and applied to ε-bialgebras. The notion of polarization for dendriform algebras is introduced in Section 4. As an application, we give a conceptual proof of the generalization to $\mathscr{C}$-dendriform algebras of Aguiar's results. All results extend to $\mathscr{C}$-tridendriform algebras; throughout the paper, we will indicate these generalizations in some short remarks.

Conventions. All algebraic structures are defined over a commutative ring R in which 2 is invertible and we write $\otimes_R$ for $\otimes$. By "R-algebra" or "algebra," we mean an $(n+1)$-tuple $(A, \mu_1, \dots, \mu_n)$, where A is an R-module and each $\mu_i : A \otimes A \to A$ is a linear map. An *algebra homomorphism* between $(A, \mu_1, \dots, \mu_n)$ and $(A', \mu'_1, \dots, \mu'_n)$ is a linear map $f : A \to A'$ such that $f(\mu_i(a \otimes b)) = \mu'_i(f(a) \otimes f(b))$ for all $a, b \in A$ and $1 \leqslant i \leqslant n$. In the case of an algebra (A, μ) with one product, we usually write ab for $\mu(a \otimes b)$.

2. Dendriform algebras

In this section, we show that the notion of a Loday dendriform algebra naturally generalizes to algebras defined by any finite collection of relations. We show that the relations which hold in a dendriform algebra are easily determined from the ones in the original algebra, when they are multilinear.

2.1. Loday's dendriform algebras

We first recall from [20] the notion of a Loday dendriform algebra.

Definition 2.1. A *Loday dendriform algebra* is an algebra $(A, \prec, \succ)$ satisfying for all $a, b, c \in A$ the following relations:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{2.1}$$

$$(a \succ b) \prec c = a \succ (b \prec c), \tag{2.2}$$

$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c). \tag{2.3}$$

Loday's dendriform algebras can be characterized as follows (see [11]).

Proposition 2.2. *Let $(A, \prec, \succ)$ be an algebra and let $\star$ denote the sum of $\prec$ and $\succ$. Then $(A, \prec, \succ)$ is a Loday dendriform algebra if and only if*

(1) *$(A, \star)$ is an associative algebra;*

(2) *$(A, \succ, \prec)$ is an $(A, \star)$-bimodule.*

In this characterization, the notion of bimodule (over an associative algebra) is the standard one; see the lines following Definition 2.3 below for the more general concept of a bimodule over other types of algebras.

Conditions (1) and (2) can be restated by demanding that $(A \times A, \boxtimes)$ is associative, where the product $\boxtimes$ is defined, for $(a, x), (b, y) \in A \times A$, by

$$(a, x) \boxtimes (b, y) := (a \star b, a \succ y + x \prec b). \tag{2.4}$$

The proof of the equivalence is a direct consequence of the following formulas, valid for all $(a, x), (b, y), (c, z) \in A \times A$:

$$\big((a, x) \boxtimes (b, y)\big) \boxtimes (c, z) = \big((a \star b) \star c, (a \star b) \succ z + (a \succ y) \prec c + (x \prec b) \prec c\big),$$

$$(a, x) \boxtimes \big((b, y) \boxtimes (c, z)\big) = \big(a \star (b \star c), a \succ (b \succ z) + a \succ (y \prec c) + x \prec (b \star c)\big).$$

It follows that a Loday dendriform algebra can equivalently be defined as an algebra $(A, \prec, \succ)$ such that $(A \times A, \boxtimes)$ is associative, where $\boxtimes$ is defined by (2.4). It is this more conceptual definition which we will generalize.

2.2. $\mathscr{C}$-dendriform algebras

Let $\mathscr{R}_1 = 0, \ldots, \mathscr{R}_k = 0$ be given relations and denote by $\mathscr{C}$ the category of all algebras which satisfy these relations, called *the relations of* $\mathscr{C}$. Morphisms in $\mathscr{C}$ are algebra homomorphisms. If (A, μ) is an object of $\mathscr{C}$, we write $(A, \mu) \in \mathscr{C}$.

$\boxtimes$	$\underline{b}_0$	$\underline{b}_1$
$\underline{a}_0$	$\underline{a \star b}_0$	$\underline{a \succ b}_1$
$\underline{a}_1$	$\underline{a \prec b}_1$	$(0,0)$

Table 1. The product $\boxtimes$ for generators of $A \times A$.

Definition 2.3. An algebra $(A, \prec, \succ)$ is said to be a *$\mathcal{C}$-dendriform algebra* if $(A \times A, \boxtimes) \in \mathcal{C}$, where $\boxtimes$ is defined for $(a,x),(b,y) \in A \times A$, by

$$(a,x) \boxtimes (b,y) := (a \prec b + a \succ b, a \succ y + x \prec b). \tag{2.5}$$

Taking $x = y = 0$ in (2.5), it is clear that if $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra, then $(A, \star) \in \mathcal{C}$, where $\star$ denotes the sum of $\prec$ and $\succ$. In the general language of bimodules (see [26]), the property that $(A \times A, \boxtimes)$ belongs to $\mathcal{C}$, where $\boxtimes$ is defined by (2.5), is *by definition* the condition that $(A, \star) \in \mathcal{C}$ and that $(A, \succ, \prec)$ is an $(A, \star)$-bimodule with respect to $\mathcal{C}$.

Remark 2.4. Definition 2.3 admits the following natural generalization: using the notations and under the assumptions of that definition, an algebra $(A, \prec, \succ, \cdot)$ is said to be a *$\mathcal{C}$-tridendriform algebra* if $(A \times A, \boxtimes) \in \mathcal{C}$, where $\boxtimes$ is now defined for $(a,x),(b,y) \in A \times A$, by

$$(a,x) \boxtimes (b,y) := (a \prec b + a \succ b + a \cdot b,\ a \succ y + x \prec b + x \cdot y). \tag{2.6}$$

In the particular case when $a \cdot b = 0$ for all $a, b \in A$, one recovers the above definition of a $\mathcal{C}$-dendriform algebra. Also, taking for $\mathcal{C}$ the category of all associative algebras, one recovers the classical notion of a tridendriform algebra, as first introduced by J.-L. Loday and M. Ronco in [21] (for a proof, see [6] in which our definition of a $\mathcal{C}$-tridendriform algebra appears in the associative case as a characterization of a tridendriform algebra).

2.3. Algebras defined by multilinear relations

The relations which we will consider are multilinear and we will show how for such relations we can easily obtain the corresponding relations which must be satisfied by the corresponding dendriform algebras; we do this for one relation at a time. Our method is based on the fact that, by multilinearity, the condition that $(A \times A, \boxtimes)$ belongs to $\mathcal{C}$ is equivalent to the conditions obtained by demanding that the relations are satisfied for all possible n-tuplets (for an n-linear relation) of elements of $A \times A$, taken from a generating set of $A \times A$. We take this generating set to be the union of $A_0 := A \times \{0\}$ and $A_1 := \{0\} \times A$. We will find it convenient to use for any $a \in A$ the following notation: $\underline{a}_0 := (a, 0)$ and $\underline{a}_1 := (0, a)$; also, when we consider elements $\underline{a}_0 \in A_0$ or $\underline{a}_1 \in A_1$ we implicitly assume that $a \in A$. In this notation, (2.5) is equivalenty described by Table 1, in which a and b stand for arbitrary elements of A. We explain the procedure in the case of a trilinear relation, the case of a bilinear relations being too simple to illustrate how it works; see Remark 2.5 below for the case of an n-linear relation. A general *trilinear relation* is of the form $\mathcal{R} = 0$, where

$$\mathcal{R}(a_1, a_2, a_3) = \sum_{\sigma \in S_3} \lambda_\sigma (a_{\sigma(1)} a_{\sigma(2)}) a_{\sigma(3)} + \sum_{\sigma \in S_3} \lambda'_\sigma a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)}). \tag{2.7}$$

The 12 constants λ_σ and λ'_σ belong to R. The associativity relation is a prime example; we will see many other examples below.

Let $\mathcal{R} = 0$ be a trilinear relation and let us denote by $\mathcal{R}_\boxtimes$ (resp. $\mathcal{R}_\star$) the formula $\mathcal{R}$ in which the product μ is replaced by $\boxtimes$ (resp. $\star$). We show how to obtain the corresponding relations for a $\mathcal{C}$-dendriform algebra.

• If we take three arbitrary elements $\underline{a}_0, \underline{b}_0, \underline{c}_0$ in A_0, then

$$(\underline{a}_0 \boxtimes \underline{b}_0) \boxtimes \underline{c}_0 = \underline{(a \star b) \star c}_0 \quad \text{and} \quad \underline{a}_0 \boxtimes (\underline{b}_0 \boxtimes \underline{c}_0) = \underline{a \star (b \star c)}_0,$$

so that $\mathcal{R}_\boxtimes(\underline{a}_0, \underline{b}_0, \underline{c}_0) = \underline{\mathcal{R}_\star(a, b, c)}_0$, for all $a, b, c \in A$. Therefore, the relation which we find is that $\mathcal{R}_\star = 0$, i.e., that $(A, \star) \in \mathcal{C}$. As we will see in the next item, this relation needs not be stated explicitly, because it follows from the other relations.

• When we take two elements in A_0 and one in A_1, we get from $\mathcal{R}_\boxtimes = 0$ three non-trivial relations which may be linearly dependent. Notice that

$$\begin{aligned}(\underline{a}_0 \boxtimes \underline{b}_0) \boxtimes \underline{c}_1 + (\underline{a}_0 \boxtimes \underline{b}_1) \boxtimes \underline{c}_0 + (\underline{a}_1 \boxtimes \underline{b}_0) \boxtimes \underline{c}_0 \\ = \underline{(a \star b) \succ c + (a \succ b) \prec c + (a \prec b) \prec c}_1 = \underline{(a \star b) \star c}_1,\end{aligned}$$

for any $a, b, c \in A$, and similarly with the opposite parenthesizing,

$$\underline{a}_0 \boxtimes (\underline{b}_0 \boxtimes \underline{c}_1) + \underline{a}_0 \boxtimes (\underline{b}_1 \boxtimes \underline{c}_0) + \underline{a}_1 \boxtimes (\underline{b}_0 \boxtimes \underline{c}_0) = \underline{a \star (b \star c)}_1.$$

If we write $\mathcal{R}$ as in (2.7), then it follows from these two equations that

$$\begin{aligned}&\mathcal{R}_\boxtimes(\underline{a_1}_0, \underline{a_2}_0, \underline{a_3}_1) + \mathcal{R}_\boxtimes(\underline{a_1}_0, \underline{a_2}_1, \underline{a_3}_0) + \mathcal{R}_\boxtimes(\underline{a_1}_1, \underline{a_2}_0, \underline{a_3}_0) \\ &= \sum_{\sigma \in S_3} \lambda_\sigma \underline{(a_{\sigma(1)} \star a_{\sigma(2)}) \star a_{\sigma(3)}}_1 + \sum_{\sigma \in S_3} \lambda_\sigma \underline{(a_{\sigma(1)} \star a_{\sigma(2)}) \star a_{\sigma(3)}}_1 \\ &\quad + \sum_{\sigma \in S_3} \lambda'_\sigma \underline{a_{\sigma(1)} \star (a_{\sigma(2)} \star a_{\sigma(3)})}_1 + \sum_{\sigma \in S_3} \lambda'_\sigma \underline{a_{\sigma(1)} \star (a_{\sigma(2)} \star a_{\sigma(3)})}_1 \\ &= \underline{\mathcal{R}_\star(a_1, a_2, a_3)}_1,\end{aligned} \tag{2.8}$$

and so the sum of the three relations which we just found for $\prec$ and $\succ$ is precisely the corresponding relation for their sum $\star$, as stated above.

• Taking at most one element in A_0 and the other ones in A_1 gives trivial relations, because a triple product in $(A \times A, \boxtimes)$ vanishes as soon as at least two of its factors belong to A_1, as follows at once from the definition of $\boxtimes$.

The upshot is that a trilinear relation $\mathcal{R} = 0$ gives rise to at most three independent relations, which are found by considering $\mathcal{R}_\boxtimes$ for a triplet of elements in $A \times A$, where two of them are arbitrary elements in A_0 and the third one is in A_1. As we will see in the examples below, when $\mathcal{R}$ has some symmetry, only one or two such triplets need to be considered.

Remark 2.5. The above analysis is also valid for n-linear relations, with $n > 3$. First, it follows again from Table 1 that one always gets zero when substituting in any monomial at least two elements from A_1. Also, the relation which is obtained by substituting n elements from A_0 follows from the n relations which are obtained by substituting $n-1$ elements from A_0 and one element from A_1. To show this, consider a monomial $a_1 a_2 \cdots a_n$ in A, with some parenthesizing, and denote for $i = 1, 2, \ldots, n$,

$$X := \underline{a_1}_0 \boxtimes \underline{a_2}_0 \boxtimes \cdots \boxtimes \underline{a_n}_0 = \underline{a_1 \star a_2 \star \cdots \star a_n}_0 \in A_0,$$

$$X_i := \underline{a_1}_0 \boxtimes \underline{a_2}_0 \boxtimes \cdots \boxtimes \underline{a_{i-1}}_0 \boxtimes \underline{a_i}_1 \boxtimes \underline{a_{i+1}}_0 \boxtimes \cdots \boxtimes \underline{a_n}_0 \in A_1,$$

with the same parenthesizing. Defining $a \in A$ by $X = \underline{a}_0$, we show that $\sum_{i=1}^n X_i = \underline{a}_1$. We do this by induction on n, the case of $n = 3$ already being proven above. We can write X (uniquely, as dictated by the parenthesizing) as $X = X' \boxtimes X''$, where

$$X' = \underline{a_1}_0 \boxtimes \underline{a_2}_0 \boxtimes \cdots \boxtimes \underline{a_m}_0, \quad X'' = \underline{a_{m+1}}_0 \boxtimes \underline{a_{m+2}}_0 \boxtimes \cdots \boxtimes \underline{a_n}_0,$$

with $1 \leqslant m < n$, and both X' and X'' come with a parenthesizing inherited from the one of X. We define for $i = 1, \ldots, m$ (resp. for $i = m+1, \ldots, n$) the element X_i' (resp. X_i'') analogously to the definition of X_i above. If we apply the induction hypothesis to X' and X'', we get $\sum_{i=1}^m X_i' = \underline{a'}_1$ and $\sum_{i=m+1}^n X_i'' = \underline{a''}_1$, where $X' = \underline{a'}_0$ and $X'' = \underline{a''}_0$. It follows that

$$\begin{aligned} \sum_{i=1}^n X_i &= \sum_{i=1}^m X_i' \boxtimes X'' + X' \boxtimes \sum_{i=m+1}^n X_i'' = \underline{a'}_1 \boxtimes \underline{a''}_0 + \underline{a'}_0 \boxtimes \underline{a''}_1 \\ &= \underline{a' \prec a''}_1 + \underline{a' \succ a''}_1 = \underline{a' \star a''}_1, \end{aligned}$$

while $X = X' \boxtimes X'' = \underline{a'}_0 \boxtimes \underline{a''}_0 = \underline{a' \star a''}_0$, so that $\sum_{i=1}^n X_i = \underline{a}_1$ where $X = \underline{a}_0$. It proves the announced property for n-linear relations, for all n.

Remark 2.6. For relations which are sums of k-linear relations, with k varying from 1 to n, the above procedure can be adapted, but there is no need to do this since for $k = 1, \ldots, n$ the k-linear part of such a relation $\mathcal{R} = 0$ is itself a relation. To show this, one shows that the leading (n-linear) part is a relation, which follows by substituting successively $a_i = 0$ for $i = 1, \ldots, n$.

Remark 2.7. For $\mathcal{C}$-tridendriform algebras (see Remark 2.4), where $\mathcal{C}$ is defined by multilinear relations, it can similarly be shown that the relation, obtained by substituting in $\mathcal{R}_\boxtimes$ only elements from A_0, is the sum of all $2^n - 1$ relations obtained by substituting in $\mathcal{R}_\boxtimes$ at least one element from A_1 and the other elements from A_0. In general, this is the only dependency between the 2^n relations obtained by the above procedure.

2.4. Examples

We illustrate the above procedure in a few examples.

Example 2.8. We start with the example of a Loday dendriform algebra, recalled above: we show how the relations of Definition 2.1 are obtained from the associativity of $\boxtimes$. First, take $\underline{a}_0, \underline{b}_0$ in A_0 and $\underline{c}_1$ in A_1. Then, by the associativity of $\boxtimes$ and by Table 1,

$$\underline{(a \star b) \succ c}_1 = (\underline{a}_0 \boxtimes \underline{b}_0) \boxtimes \underline{c}_1 = \underline{a}_0 \boxtimes (\underline{b}_0 \boxtimes \underline{c}_1) = \underline{a \succ (b \succ c)}_1,$$

so that $(a \star b) \succ c = a \succ (b \succ c)$, which is (2.3). Taking $\underline{a}_0, \underline{c}_0$ in A_0 and $\underline{b}_1$ in A_1 (resp. $\underline{b}_0, \underline{c}_0$ in A_0 and $\underline{a}_1$ in A_1), one obtains similarly (2.2) and (2.1).

Example 2.9. A *pre-Lie algebra* (A, μ) is an algebra for which the *associator* $(a, b, c) := (ab)c - a(bc)$ is symmetric in its first two variables. Thus, the trilinear relation which defines pre-Lie algebras is given by

$$(ab)c - a(bc) = (ba)c - b(ac). \tag{2.9}$$

Let $\mathcal{C}_{pL}$ denote the category of all pre-Lie algebras. We obtain the relations which any $\mathcal{C}_{pL}$-dendriform algebra $(A, \prec, \succ)$ must satisfy, by substituting in

$$\begin{aligned}\big((a,x) \boxtimes (b,y)\big) \boxtimes (c,z) - (a,x) \boxtimes \big((b,y) \boxtimes (c,z)\big) \\ = \big((b,y) \boxtimes (a,x)\big) \boxtimes (c,z) - (b,y) \boxtimes \big((a,x) \boxtimes (c,z)\big)\end{aligned} \tag{2.10}$$

two elements from A_0 and one from A_1. Substituting $\underline{a}_0$, $\underline{b}_0$, and $\underline{c}_1$ in (2.10), we get, using Table 1,

$$\underline{(a \star b) \succ c}_1 - \underline{a \succ (b \succ c)}_1 = \underline{(b \star a) \succ c}_1 - \underline{b \succ (a \succ c)}_1,$$

which leads to the relation

$$(a \star b) \succ c - a \succ (b \succ c) = (b \star a) \succ c - b \succ (a \succ c). \tag{2.11}$$

Similarly, substituting $\underline{a}_0$, $\underline{b}_1$, and $\underline{c}_0$ in (2.10), we get

$$(a \succ b) \prec c - a \succ (b \prec c) = (b \prec a) \prec c - b \prec (a \star c). \tag{2.12}$$

Since (2.9) is invariant under the transposition which permutes a and b, we have obtained all relations, and so the relations for a $\mathcal{C}_{pL}$-dendriform algebra are given by (2.11) and (2.12). Such dendriform algebras are known as *L-dendriform algebras* (see [7], where they have been introduced).

Example 2.10. The defining relation for an A_3-associative algebra (A, μ) is

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab). \tag{2.13}$$

It can be written in terms of associators in the compact form

$$\sum_{\sigma \in A_3} (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = 0, \tag{2.14}$$

where $a_1, a_2, a_3 \in A$ and where A_3 stands for the alternating group of degree 3. The symmetric form of (2.14) is at the origin of the terminology "A_3" (see [13]); this form is often useful in computations, as we will see below. Since (2.14) is invariant under a cyclic permutation of a_1, a_2, a_3, the corresponding dendriform algebras, which we will call A_3-*dendriform algebras*, need to satisfy only one relation. We obtain it by substituting $\underline{a}_0$, $\underline{b}_0$, and $\underline{c}_1$ for (a_1, x_1), (a_2, x_2), and (a_3, x_3), in the relation

$$\sum_{\sigma\in A_3} \big((a_{\sigma(1)}, x_{\sigma(1)}), (a_{\sigma(2)}, x_{\sigma(2)}), (a_{\sigma(3)}, x_{\sigma(3)})\big)_{\boxtimes} = 0,$$

where $(\cdot,\cdot,\cdot)_{\boxtimes}$ stands for the associator of the product $\boxtimes$. The resulting relation defining A_3-dendriform algebras is given by

$$\begin{aligned} a \succ (b \succ c) - (c \prec a) \prec b + c \prec (a \star b) \\ = (a \star b) \succ c - b \succ (c \prec a) + (b \succ c) \prec a. \end{aligned} \tag{2.15}$$

Notice that, upon defining $a \circ b := a \succ b - b \prec a$ for all $a, b \in A$, the latter relation can be rewritten in the simple form

$$(a \star b) \circ c - b \circ (c \prec a) - a \circ (b \succ c) = 0. \tag{2.16}$$

We determine for this case also the relations of the corresponding tridendriform algebras. To do this, we need to substitute in $\mathcal{R}_{\boxtimes} = 0$ at least one element from A_1 and the other ones from A_0. Notice that, if one substitutes only one element from A_1, one obtains exactly the dendriform relations, with $\star$ standing now for $a \star b := a \prec b + a \succ b + a \cdot b$, so these relations do not have to be computed again. Also, as above, there is only one relation obtained by substituting two elements from A_1 and one from A_0; namely,

$$(a \cdot b) \prec c + (b \prec c) \cdot a + (c \succ a) \cdot b = a \cdot (b \prec c) + b \cdot (c \succ a) + c \succ (a \cdot b). \tag{2.17}$$

A final relation is obtained by substituting three elements from A_1. It is clear that the found relation just says that $(A, \cdot)$ is A_3-associative.

Example 2.11. A *Lie-admissible algebra* (or *LA-algebra*) is an algebra (A, μ) for which the anticommutative product $[\cdot,\cdot]$, defined as the commutator $[a, b] := ab - ba$, is a Lie bracket, i.e., satisfies the Jacobi identity. The trilinear relation which characterizes Lie-admissible algebras is therefore given by

$$\sum_{\sigma\in A_3} \big((a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) - (a_{\sigma(2)}, a_{\sigma(1)}, a_{\sigma(3)})\big) = 0. \tag{2.18}$$

It is invariant under the symmetry group S_3, so that LA-dendriform algebras are defined by a single relation, as in the case of A_3-dendriform algebras. It is obtained in the same way as in that case and is given by

$$\begin{aligned} a \succ (b \succ c - c \prec b) - (b \succ c - c \prec b) \prec a - b \succ (a \succ c - c \prec a) \\ + (a \succ c - c \prec a) \prec b + c \prec (a \star b - b \star a) - (a \star b - b \star a) \succ c = 0, \end{aligned} \tag{2.19}$$

where $\star$ stands again for the sum of $\prec$ and $\succ$. As above, we define $a \circ b := a \succ b - b \prec a$ for all $a, b \in A$ and observe that $a \star b - b \star a = a \circ b - b \circ a$, for all $a, b \in A$. It follows that the relation defining LA-dendriform algebras can be rewritten in the simple form

$$a \circ (b \circ c) - b \circ (a \circ c) - (a \circ b - b \circ a) \circ c = 0. \tag{2.20}$$

It is equivalent to saying that $(A, \circ)$ is a pre-Lie algebra (see Example 2.9).

Example 2.12. An *associative-admissible algebra* (or *AA-algebra*) is defined as an algebra (A, μ) for which the anticommutator $[a, b]^+ := ab + ba$ is associative. AA-algebras are characterized by the trilinear relation

$$(ab + ba)c + c(ab + ba) = a(bc + cb) + (bc + cb)a. \tag{2.21}$$

The relation (2.21) is again S_3-invariant, so AA-dendriform algebras are defined by a single relation. It is most easily obtained from the compact form $[[a, b]^+, c]^+ = [a, [b, c]^+]^+$ of the relation (2.21). Indeed, let us denote by $[\cdot, \cdot]^+_\boxtimes$ the anticommutator of $\boxtimes$, and let $a * b := a \succ b + b \prec a$ for all $a, b \in A$ (not to be confused with $a \star b = a \succ b + a \prec b$). Using the obvious identity $a \star b + b \star a = a * b + b * a$, it is easy to derive from Table 1 that $[\underline{a}_0, \underline{b}_0]^+_\boxtimes = \underline{a * b + b * a}_0$ and that $[\underline{a}_0, \underline{b}_1]^+_\boxtimes = \underline{a * b}_1$, for $a, b \in A$. Substituted in $[[\underline{a}_0, \underline{b}_0]^+_\boxtimes, \underline{c}_1]^+_\boxtimes = [\underline{a}_0, [\underline{b}_0, \underline{c}_0]^+_\boxtimes =]^+_\boxtimes$, we obtain the following relation for AA-dendriform algebras, known as the *Zinbiel* property (see [20]):

$$(a * b + b * a) * c = a * (b * c). \tag{2.22}$$

Example 2.13. Our last example is closely related to Poisson algebras (see Examples 4.3 and 4.9; see also [14] where P-algebras are shown to be A_3-associative). Consider the relation

$$3(ab)c = 3a(bc) + (ac)b + (bc)a - (ba)c - (ca)b. \tag{2.23}$$

We call any algebra satisfying this relation a *P-algebra* and denote the category of all P-algebras by $\mathcal{P}$. The three relations for P-dendriform algebras are given by the following formulas, where the first one is obtained by substituting $\underline{a}_0$, $\underline{b}_0$, and $\underline{c}_1$ for a, b, and c in (2.23), where the product μ has been replaced by $\boxtimes$, and similarly for the other two, where one substitutes $\underline{a}_0, \underline{b}_1, \underline{c}_0$ and $\underline{a}_1, \underline{b}_0, \underline{c}_0$ respectively:

$$\begin{aligned} 3(a \star b) \succ c = 3a \succ (b \succ c) + (a \succ c) \prec b + (b \succ c) \prec a \\ - (b \star a) \succ c - (c \prec a) \prec b, \end{aligned} \tag{2.24}$$

$$\begin{aligned} 3(a \succ b) \prec c = 3a \succ (b \prec c) + (a \star c) \succ b + (b \prec c) \prec a \\ - (b \prec a) \prec c - (c \star a) \succ b, \end{aligned} \tag{2.25}$$

$$\begin{aligned} 3(a \prec b) \prec c = 3a \prec (b \star c) + (a \prec c) \prec b + (b \star c) \succ a \\ - (b \succ a) \prec c - (c \succ a) \prec b. \end{aligned} \tag{2.26}$$

In these formulas, $\star$ stands again for the sum of $\prec$ and $\succ$.

2.5. Commutative and anticommutative dendriform algebras

Many algebras of interest are commutative or anticommutative; i.e., they satisfy the relation $ab = ba$ or $ab = -ba$, besides satisfying some other relations. It follows at once from the defining relations that

(1) associative, pre-Lie, AA, and P-algebras, which are commutative, are precisely commutative associative algebras;

(2) A_3-associative and LA-algebras, which are commutative, are just arbitrary (commutative) algebras; similarly, AA-algebras which are anticommutative are arbitrary (anticommutative) algebras;

(3) A_3-associative, pre-Lie, LA, and P-algebras, which are anticommutative, are precisely Lie algebras;

(4) associative algebras, which are anticommutative, are precisely 2-step nilpotent algebras, i.e., satisfying $(ab)c = a(bc) = 0$ for all $a, b, c \in A$.

It is clear from (2.5) that the corresponding dendriform algebras must satisfy the relation $a \prec b = b \succ a$, respectively, $a \prec b = -b \succ a$.

Definition 2.14. A $\mathscr{C}$-dendriform algebra $(A, \prec, \succ)$ is said to be *commutative* (resp. *anticommutative*) if it satisfies $b \succ a = a \prec b$ (resp. $b \succ a = -a \prec b$) for all $a, b \in A$.

It is then natural to view A as an algebra with only one product, by setting for all $a, b \in A$, $a \times b := a \succ b$.

Example 2.15. We start with (1) above: to obtain the relations of a commutative associative dendriform algebra, we substitute $a \times b$ for $a \succ b$ and for $b \prec a$ in the relations (2.1)–(2.3), to find the relations

$$(a \times b + b \times a) \times c = a \times (b \times c), \quad c \times (a \times b) = a \times (c \times b). \tag{2.27}$$

The first property is the Zinbiel property (see Example 2.12). The second property is known as the *NAP* property; see [19]. Since the Zinbiel property implies the NAP property, commutative associative dendriform algebras are, when written in terms of a single product, the same as Zinbiel algebras.

Example 2.16. For (2) above, arbitrary (anti-) commutative algebras, one only gets the dendriform relation $a \prec b = \pm b \succ a$, with no relation for $\times$.

Example 2.17. For Lie algebras (case (3) above), the quickest way to obtain the relation which $\times$ must satisfy is by substituting $2a \times b$ (or just $a \times b$) for $a \circ b$ in (2.20), so we get the pre-Lie relation (2.9). Thus, Lie dendriform algebras are, when written in terms of a single product, pre-Lie algebras.

Example 2.18. By definition, 2-step nilpotent algebras (case (4) above) satisfy $(ab)c = a(bc) = 0$. Their dendriform algebras satisfy the following six relations:

$$(a \prec b) \succ c = (a \succ b) \prec c = (a \prec b) \prec c = 0,$$
$$c \prec (b \succ a) = c \succ (b \prec a) = c \succ (b \succ a) = 0.$$

It follows that anticommutative associative dendriform algebras are, in terms of a single product, also 2-step nilpotent algebras, as they satisfy the relation $(a \times b) \times c = a \times (b \times c) = 0$.

Example 2.19. The defining relations of a *Jordan algebra* are $(aab)a = (aa)(ba)$ and $ab = ba$ (commutativity). The former relation is not 4-linear but becomes so after linearization and the relations which $\times$ must satisfy can again be easily obtained by the above procedure. Written in terms of a single product, these relations are given in [17, equations (4.1) and (4.2)], where the proposed relations are shown to be equivalent to a bimodule property; the resulting algebras are there called *pre-Jordan* algebras. See [16] for the dendriform algebra associated with a pre-Jordan algebra.

Remark 2.20. Similarly, a tridendriform algebra is said to be *commutative* or *anticommutative* if it satisfies the relations $a \succ b = \pm b \prec a$ and $a \cdot b = \pm b \cdot a$, with the plus sign of course corresponding to the commutative case. Such tridendriform algebras are naturally seen as algebras with two operations "$\times$" and "$\cdot$", upon setting $a \times b := a \succ b$, while keeping "$\cdot$".

Example 2.21. We give an example of an anticommutative tridendriform algebra: a Lie tridendriform algebra. We obtain the relations from the relations of an A_3-tridendriform algebra, given in Example 2.10, by replacing in them $a \succ b$ and $-b \prec a$ by $a \times b$ and using $a \cdot b = -b \cdot a$. After some trivial simplifications, one finds that a Lie tridendriform algebra is a Lie algebra, satisfying the following two relations, obtained from (2.16) and (2.17):

$$(a \cdot b) \times c = a \times (b \times c) - (a \times b) \times c - b \times (a \times c) + (b \times a) \times c,$$
$$c \times (a \cdot b) = (c \times b) \cdot a - (c \times a) \cdot b.$$

Lie tridendriform algebras are known as *Post-Lie* algebras (see [5, 28]).

2.6. Categories of generalized dendriform algebras

Let $\mathcal{C}$ denote, as before, the category of all algebras satisfying a given set of relations $\mathcal{R}_1 = 0, \dots, \mathcal{R}_k = 0$. Clearly, the class of all $\mathcal{C}$-dendriform algebras (over R) also form a category $\mathcal{C}^{\text{dend}}$, with the algebra homomorphisms as morphisms. By the above, we have a (faithful) functor $\mathcal{C}^{\text{dend}} \to \mathcal{C}$ which, on objects $(A, \prec, \succ)$, is defined by $(A, \prec, \succ) \mapsto (A, \star)$, where $\star$ denotes the sum of the products $\prec$ and $\succ$; on morphisms, the functor is just the identity in the sense that it sends the map underlying a morphism to itself.

Let $\mathcal{C}'$ be the category of all algebras verifying another collection of relations $\mathcal{R}'_1 = 0, \dots, \mathcal{R}'_\ell = 0$, where every $\mathcal{R}_i$ is a linear combination of $\mathcal{R}'_1, \dots, \mathcal{R}'_\ell$. Then $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, and $\mathcal{C}'^{\text{dend}}$ is a subcategory of $\mathcal{C}^{\text{dend}}$, since the relations $\mathcal{R}'_i = 0$ can be seen as a subset of the relations $\mathcal{R}_j = 0$. Thus, we have the following commutative diagram of categories:

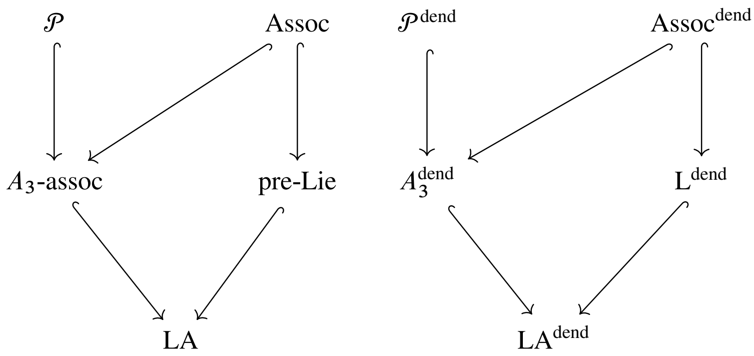
$$
\begin{array}{ccc}
\mathcal{C}', \mu' & \hookrightarrow & \mathcal{C}, \mu \\
\uparrow {\scriptstyle a \prec' b + a \succ' b} & & \uparrow {\scriptstyle a \prec b + a \succ b} \\
\mathcal{C}'^{\mathrm{dend}}, \prec', \succ' & \hookrightarrow & \mathcal{C}^{\mathrm{dend}}, \prec, \succ
\end{array}
$$

As a first application, we denote by $\mathcal{C}_{\mathrm{com}}$ (resp. by $\mathcal{C}_{\mathrm{com}}^{\mathrm{dend}}$) the subcategory of $\mathcal{C}$ (resp. of $\mathcal{C}^{\mathrm{dend}}$) consisting of all commutative algebras in the respective category. Then we have the following commutative diagram of categories:

$$
\begin{array}{ccc}
\mathcal{C}_{\mathrm{com}}, \mu & \hookrightarrow & \mathcal{C}, \mu \\
\uparrow {\scriptstyle a \prec b + a \succ b} & & \uparrow {\scriptstyle a \prec b + a \succ b} \\
\mathcal{C}_{\mathrm{com}}^{\mathrm{dend}}, \prec, \succ & \hookrightarrow & \mathcal{C}^{\mathrm{dend}}, \prec, \succ
\end{array}
$$

Indeed, we can view the commutative algebras in $\mathcal{C}$ as being those which satisfy the extra condition of commutativity, and this relation leads to the condition of commutativity for the corresponding $\mathcal{C}$-dendriform algebras, by the above observation. The same applies, of course, to anticommutative algebras.

As a second application, we show how the above examples of $\mathcal{C}$-dendriform algebras are related. We have the following strict inclusion relations between the original category of algebras on the left; they lead to inclusion relations between their corresponding categories of dendriform algebras on the right:



We have not included AA-algebras and their dendriform algebras, because there are no apparent inclusion relations between the category of AA-algebras and any of the other categories that we considered.

Table 2 shows that the induced inclusions in the rightmost diagram are also strict and that there is no inclusion relation between A_3^{dend} or $\mathcal{P}^{\mathrm{dend}}$ and $\mathrm{L}^{\mathrm{dend}}$. In the table, the algebra $(A, \prec, \succ)$ is a free module of rank at least two and a and b are elements of a basis of A.

$\prec$	$\succ$	$\star$	of type	not of type
$a \prec a = -b$	$a \succ a = a + b$ $b \succ a = b$	$a \star a = a$ $b \star a = b$	A_3-dendri	L-dendri P-dendri
$a \prec b = b$	$b \succ a = b$ $b \succ b = b$	$a \star b = b$ $b \star a = b$ $b \star b = b$	LA-dendri	A_3-dendri L-dendri
—	$b \succ a = a$	$b \star a = a$	L-dendri	A_3-dendri dendri
$a \prec b = -a$	$b \succ a = a$	$a \star b = -a$ $b \star a = a$	P-dendri	L-dendri
$a \prec a = a + b$	—	$a \star a = a + b$	dendri	P-dendri

Table 2. Some examples of generalized dendriform algebras.

The first two columns describe the products $\prec$ and $\succ$ on some of the basis elements; it is understood that all other products between elements of the basis are zero.

3. (Weak) Rota–Baxter operators

In this section, we introduce the notion of a weak Rota–Baxter operator. We show how such operators can be used to construct generalized dendriform algebras and present an application of this. Throughout this section, we denote by $\mathcal{C}$ the category of all algebras satisfying a given collection of multilinear relations $\mathcal{R}_1 = 0, \ldots, \mathcal{R}_k = 0$.

3.1. Dendriform algebras from Rota–Baxter operators

We start with the definition of a Rota–Baxter operator (on an arbitrary algebra), which first appeared in the works of [8, 24, 25]; see [15] for additional information.

Definition 3.1. Let (A, μ) be any algebra, let $\mathfrak{R} : A \to A$ be a linear map, and let $\lambda \in R$. One says that $\mathfrak{R}$ is a *Rota–Baxter operator of weight* λ of A if $\mathfrak{R}$ satisfies the *Rota–Baxter equation*

$$\mathfrak{R}\big(a\mathfrak{R}(b) + \mathfrak{R}(a)b + \lambda ab\big) - \mathfrak{R}(a)\mathfrak{R}(b) = 0, \tag{3.1}$$

for all $a, b, \in A$. When $\lambda = 0$, one simply speaks of a *Rota–Baxter operator*.

We first show that a Rota–Baxter operator on any algebra (A, μ) of $\mathcal{C}$ leads to a $\mathcal{C}$-dendriform algebra $(A, \prec, \succ)$. This was first observed by Aguiar [2] in the associative case.

Proposition 3.2. *Let $\mathfrak{R}$ be a Rota–Baxter operator on an algebra (A, μ) which belongs to $\mathcal{C}$. For $a, b \in A$, let $a \succ b := \mathfrak{R}(a)b$ and $a \prec b := a\mathfrak{R}(b)$. Then $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra.*

Proof. We will give the proof for a trilinear relation $\mathcal{R} = 0$; it is easily generalized to n-linear relations. Recall that $\mathcal{R} = 0$ leads to 3 dendriform relations which are obtained by substituting two elements from A_0 and one element from A_1 in $\mathcal{R}_{\boxtimes} = 0$, where $\boxtimes$ is the product on $A \times A$, defined by (2.4). Recall also that we write $\underline{a}_0$ for $(a, 0)$ and $\underline{a}_1$ for $(0, a)$, where $a \in A$.

We show that this amounts to writing $\mathcal{R}$ for three elements of A, on two of which $\mathfrak{R}$ has been applied, and rewriting the result in terms of the dendriform operations. To show this, we compare the effect of these substitutions on the monomials $(ab)c$ and $a(bc)$, where each time we consider the three possible substitutions. In view of Table 1, the definition of $\prec$ and $\succ$, and (3.1), we get for the first type the correspondence

$$\begin{aligned}
(\underline{a}_1 \boxtimes \underline{b}_0) \boxtimes \underline{c}_0 &= \underline{(a \prec b) \prec c}_1 = \underline{\big(a\mathfrak{R}(b)\big)\mathfrak{R}(c)}_1, \\
(\underline{a}_0 \boxtimes \underline{b}_1) \boxtimes \underline{c}_0 &= \underline{(a \succ b) \prec c}_1 = \underline{\big(\mathfrak{R}(a)b\big)\mathfrak{R}(c)}_1, \\
(\underline{a}_0 \boxtimes \underline{b}_0) \boxtimes \underline{c}_1 &= \underline{(a \star b) \succ c}_1 = \underline{\big(\mathfrak{R}(a)\mathfrak{R}(b)\big)c}_1,
\end{aligned}$$

and similarly for the other type. In the third line, we have used (3.1) with $\lambda = 0$, which says that $\mathfrak{R} : (A, \star) \to (A, \mu)$ is a morphism. ∎

Remark 3.3. Our proof shows that the $\mathscr{C}$-dendriform relations can also formally be obtained from the relations $\mathcal{R}_i = 0$ by formally applying $\mathfrak{R}$ to two of the variables and rewriting the resulting expression in terms of the dendriform operations (using the Rota–Baxter equation). Our proof also explains where the particular form of the Rota–Baxter equation comes from.

As a direct consequence of Proposition 3.2, we have the following result, which is well known in the case of an associative or Lie algebra.

Corollary 3.4. *Let $\mathfrak{R}$ be a Rota–Baxter operator on an algebra (A, μ) in $\mathscr{C}$. For $a, b \in A$, let $a \star b := a\mathfrak{R}(b) + \mathfrak{R}(a)b$. Then $(A, \star)$ also belongs to $\mathscr{C}$.*

Remark 3.5. Proposition 3.2 and its proof are easily adapted to prove case of arbitrary weights: if $\mathfrak{R}$ is a Rota–Baxter operator of weight λ on an algebra (A, μ) which belongs to $\mathscr{C}$, then $(A, \prec, \succ, \cdot)$ is a $\mathscr{C}$-tridendriform algebra, upon defining $a \succ b := \mathfrak{R}(a)b$ and $a \prec b := a\mathfrak{R}(b)$ and $a \cdot b := \lambda ab$, for all $a, b \in A$. In fact, it suffices to change in the proof the meaning of $a \star b$, which should now stand for $a \succ b + a \prec b + a \cdot b$. For associative algebras, this was first observed by Ebrahimi-Fard [12].

Remark 3.6. In the case of Lie algebras, one encounters also the following equation, generalizing the Rota–Baxter equation (of weight 0):

$$\mathfrak{R}\big(a\mathfrak{R}(b) + \mathfrak{R}(a)b\big) = \mathfrak{R}(a)\mathfrak{R}(b) + \nu ab, \tag{3.2}$$

where $\nu \in R$ is a constant. Equation (3.2) is known as the *modified Yang–Baxter equation* and has many applications in the theory of integrable systems (see [27]). The statement and proof of Proposition 3.2, and hence also Corollary 3.4, generalize easily to this case:

in the display in the proof of the proposition, we only need to replace, in line 3, $\mathfrak{R}(a)\mathfrak{R}(b)$ by $\mathfrak{R}(a)\mathfrak{R}(b) + \nu ab$. For the rest the proof is unchanged: these extra terms will disappear because the original product μ satisfies the relation $\mathcal{R} = 0$.

Example 3.7. The prime example of a solution to the modified Yang–Baxter equation is based on the notion of a Lie algebra splitting (see [27]). It naturally generalizes as follows. A *$\mathcal{C}$-algebra splitting* of $(A, \mu) \in \mathcal{C}$ is a module direct sum decomposition $A = A_+ \oplus A_-$ of A, where A_+ and A_- are subalgebras of A. If one denotes by P_+ and P_- projection on A_+ and A_-, then $\mathfrak{R} := P_+ - P_-$ is a solution to (3.2), with $\nu = 1$. The proof is the same as in the classical case.

3.2. Dendriform algebras from weak Rota–Baxter operators

We now introduce the more general notion of a *weak* Rota–Baxter operator. For any algebra (A, μ), we denote by $C(A)$ the submodule[1] of A consisting of those elements which commute with all elements in A.

Definition 3.8. Let $\mathfrak{R} : A \to A$ be a linear map and let $\lambda \in R$. One says that $\mathfrak{R}$ is a *weak Rota–Baxter operator of weight* λ of A if, for all $a, b \in A$,

$$\mathfrak{R}\big(a\mathfrak{R}(b) + \mathfrak{R}(a)b + \lambda ab\big) - \mathfrak{R}(a)\mathfrak{R}(b) \in C(A). \tag{3.3}$$

When $\lambda = 0$, one simply speaks of a *weak Rota–Baxter operator* of A.

We show how Proposition 3.2 can be generalized to weak Rota–Baxter operators, which we do for trilinear relations. A trilinear relation $\mathcal{R} = 0$ has a *commutator form* if it can be written as a linear combination of terms of the form $[ab, c] = (ab)c - c(ab)$, i.e., if $\mathcal{R}$ is of the form

$$\mathcal{R}(a_1, a_2, a_3) = \sum_{\sigma \in S_3} c_\sigma [a_{\sigma(1)} a_{\sigma(2)}, a_{\sigma(3)}], \tag{3.4}$$

for some $c_\sigma \in R$. A set of trilinear relations has a *commutator form* if it is linearly generated by a set of trilinear relations having a commutator form.

Proposition 3.9. *Suppose that the relations $\mathcal{R}_1 = 0, \ldots, \mathcal{R}_k = 0$ are trilinear and have a commutator form. Let $\mathfrak{R}$ be a weak Rota–Baxter operator on an algebra (A, μ) which belongs to $\mathcal{C}$. For $a, b \in A$, define $a \succ b := \mathfrak{R}(a)b$ and $a \prec b := a\mathfrak{R}(b)$. Then $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra.*

Proof. Let $\mathcal{R} = 0$ be a relation which has a commutator form. As in the proof of Proposition 3.2, we need to show that $\prec$ and $\succ$ verify the dendriform relations corresponding to $\mathcal{R}$. We can repeat this proof, except that we need to show how to express the terms of the forms $(\mathfrak{R}(a)\mathfrak{R}(b))c$ and $c(\mathfrak{R}(a)\mathfrak{R}(b))$ in terms of the dendriform operations and that

[1] In general, $C(A)$ is not a subalgebra of A and strictly contains the center $Z(A)$, whose elements are required to have the extra property that any associator containing them vanishes.

by this procedure the same terms are obtained as by substituting in $\mathcal{R}_{\boxtimes}$ two terms from A_0 and one term from A_1. To do this, first observe that (3.3) can (for $\lambda = 0$) be equivalently written as the condition that $[\mathfrak{R}(a \star b), c] = [\mathfrak{R}(a)\mathfrak{R}(b), c]$, where $a \star b = a \succ b + a \prec b = a\mathfrak{R}(b) + \mathfrak{R}(a)b$, leading to the following correspondence:

$$[\underline{a}_0 \boxtimes \underline{b}_0, \underline{c}_1]_{\boxtimes} = \underline{(a \star b) \succ c - c \prec (a \star b)}_1 = \underline{[\mathfrak{R}(a)\mathfrak{R}(b), c]}_1,$$

where $[\cdot,\cdot]_{\boxtimes}$ stands for the commutator of $\boxtimes$. For the two other substitutions, the commutator form is not needed and the claim follows. ∎

The proposition can be applied to A_3-associative algebras and Lie admissible algebras, since (2.13) and (2.18) can be respectively rewritten as

$$[ab, c] + [bc, a] + [ca, b] = 0, \tag{3.5}$$

$$\sum_{\sigma \in A_3} [a_{\sigma(1)}a_{\sigma(2)} - a_{\sigma(2)}a_{\sigma(1)}, a_{\sigma(3)}] = 0. \tag{3.6}$$

However, many relations cannot be written in a commutator form. The associativity relation, $a(bc) = (ab)c$, is a prime example; other examples are the derivation property $a(bc) = (ab)c + b(ac)$, the Zinbiel property $a(bc) = (ab + ba)c$, and the NAP property $a(bc) = b(ac)$, just to mention a few. In such cases, when the relations of $\mathcal{C}$ imply a relation $\mathcal{R} = 0$ which can be written in a commutator form, any dendriform algebra $(A, \prec, \succ)$ obtained by using a weak Rota–Baxter operator on an algebra (A, μ) in $\mathcal{C}$ will satisfy (at least) the $\mathcal{C}$-dendriform relation, derived from $\mathcal{R} = 0$. Moreover, any relation $\mathcal{R} = 0$ which does not involve a product of two of the variables leads to a (single) dendriform relation. We illustrate this in the following example, on which we will elaborate in the following subsection.

Example 3.10. The associativity relation, $a(bc) = (ab)c$, cannot be written in a commutator form. Summing up three instances of this relation, we get $(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab)$, which is the relation of A_3-associativity (3.5), which has a commutator form. Therefore, if $\mathfrak{R}$ is a weak Rota–Baxter operator on an associative algebra (A, μ), then $(A, \prec, \succ)$, with $\prec$ and $\succ$ defined by $a \prec b := a\mathfrak{R}(b)$ and $a \succ b := \mathfrak{R}(a)b$, is a priori not a Loday dendriform algebra but an A_3-dendriform algebra. The associativity relation $a(bc) = (ab)c$ does not contain a product of a and c, so we do not need to use the weak Rota–Baxter equation to rewrite $\mathfrak{R}(a)(b\mathfrak{R}(c)) = (\mathfrak{R}(a)b)\mathfrak{R}(c)$ in terms of the dendriform products. The resulting relation $a \succ (b \prec c) = (a \succ b) \prec c$ of $(A, \prec, \succ)$ is called *inner-associativity*.

It follows that a weak Rota–Baxter operator on an associative algebra leads to an inner-associative A_3-dendriform algebra.

Example 3.11. Let A be a commutative associative algebra. Every linear map $\mathcal{R} : A \to A$ is a weak Rota–Baxter operator since $C(A) = A$, and hence leads to an inner-associative A_3-dendriform algebra. To see that it may not be a Loday dendriform algebra, take $\mathcal{R} = \mathrm{Id}_A$. Then $a \prec b = a \succ b = ab$ and (2.1) cannot be satisfied unless $abc = 0$ for all $a, b, c \in A$.

Remark 3.12. The proof of Proposition 3.9 is easily adapted to prove the following generalization: under the same assumptions on the relations of $\mathcal{C}$, any weak Rota–Baxter operator $\mathfrak{R}$ of weight λ on an algebra $(A, \mu) \in \mathcal{C}$ leads to a $\mathcal{C}$-tridendriform algebra, upon setting $a \succ b := \mathfrak{R}(a)b$, $a \prec b := a\mathfrak{R}(b)$, and $a \cdot b := \lambda ab$, for all $a, b \in A$. Again, it suffices to change in the proof the meaning of $a \star b$, which should now stand for $a \succ b + a \prec b + a \cdot b$.

Remark 3.13. If we denote by $C'(A)$ the set of elements c of A which anticommute with all elements of A, i.e., $ac = -ca$ for all $a \in A$, we can also consider operators $\mathcal{R}$ satisfying (3.3), with $C(A)$ replaced by $C'(A)$. The results of this section, in particular Proposition 3.9, are easily adapted to the case of such operators. An example of a relation having an anticommutator form is the relation (2.21) defining AA-algebras.

3.3. Application: coboundary ε-bialgebras

As an application of weak Rota–Baxter operators, we now generalize a result obtained by M. Aguiar in [3], which we will recall. We first recall the definition of a ε-bialgebra.

Definition 3.14. A *ε-bialgebra* is a triple (A, μ, Δ), where A is an R-module and $\mu : A \otimes A \to A$ and $\Delta : A \to A \otimes A$ are linear maps, such that

(1) μ is associative;

(2) Δ is coassociative;

(3) Δ is a derivation: $\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b$, for all $a, b \in A$.

In item (3), we have used a dot to denote the natural left, resp. right, action of A on $A \otimes A$; later on in this section, it will also be used for the natural left and right actions of A on $A \otimes A \otimes A$.

Let (A, μ, Δ) be a ε-bialgebra and let us write $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ for all $a \in A$ (Sweedler's notation). It is shown in [3] that if one defines $a \circ b := \sum_{(b)} b_{(1)} a b_{(2)}$ for all $a, b \in A$, then $(A, \circ)$ is a pre-Lie algebra. This yields a functor which associates to any ε-bialgebra (A, μ, Δ) the corresponding pre-Lie algebra $(A, \circ)$, and which is identity on morphisms. The fundamental observation of Aguiar is that the restriction of this functor to quasi-triangular ε-bialgebras factors in a natural way through the category of Loday's dendriform algebras, as in the diagram

$$\begin{array}{ccc} \text{QT } \varepsilon\text{-bialg}, \mu, r & \xrightarrow{\; r\cdot a - a \cdot r \;} & \varepsilon\text{-bialg}, \mu, \Delta \\ \Big\downarrow {\scriptstyle \sum_i a u_i b v_i, \sum_i u_i a v_i b} & & \Big\downarrow {\scriptstyle \sum_{(b)} b_{(1)} a b_{(2)}} \\ \text{Assoc}^{\text{dend}}, \prec, \succ & \xrightarrow{\; a \succ b - b \prec a \;} & \text{pre-Lie}, \circ \end{array} \tag{3.7}$$

In order to explain this diagram, we first recall from [3] that a *quasi-triangular ε-bialgebra* is a triple (A, μ, r), where (A, μ) is an associative algebra and $r \in A \otimes A$ is a solution of the *associative Yang–Baxter equation*

$$\text{AYB}(r) := r_{13} r_{12} - r_{12} r_{23} + r_{23} r_{13} = 0.$$

Let (A, μ, r) be a quasi-triangular ε-bialgebra and let $r = \sum_i u_i \otimes v_i$. On the one hand, setting $\Delta_r(a) := r \cdot a - a \cdot r$ for all $a \in A$, we get a ε-bialgebra (A, μ, Δ_r). On the other hand, the map $\mathfrak{R} : A \to A$, defined for all $a \in A$ by $\mathfrak{R}(a) = \sum_i u_i a v_i$, is a Rota–Baxter operator for A, and so, by Proposition 3.2, we get a Loday dendriform algebra $(A, \prec, \succ)$ by setting for all $a, b \in A$

$$a \prec b := \sum_i a u_i b v_i, \quad a \succ b := \sum_i u_i a v_i b. \tag{3.8}$$

The next proposition gives a natural generalization of this construction.

Proposition 3.15 ([1]). *Let (A, μ) be an associative algebra and let $r \in A \otimes A$. Then (A, μ, Δ_r) is a ε-bialgebra if and only if $\mathrm{AYB}(r)$ is invariant; i.e., $a \cdot \mathrm{AYB}(r) = \mathrm{AYB}(r) \cdot a$, for all $a \in A$. One then says that (A, μ, r) is a coboundary ε-bialgebra.*

The natural question arises to generalize Aguiar's construction to coboundary ε-bialgebras, which we do in the following proposition.

Proposition 3.16. *Let $(A, \mu, r = \sum_i u_i \otimes v_i)$ be a coboundary ε-bialgebra.*

(1) *The linear map $\mathfrak{R} : A \to A$, defined for all $a \in A$ by $\mathfrak{R}(a) := \sum_i u_i a v_i$, is a weak Rota–Baxter operator for A.*

(2) *For $a, b \in A$, let $a \succ b := \mathfrak{R}(a)b = \sum_i u_i a v_i b$ and $a \prec b := a\mathfrak{R}(b) = \sum_i a u_i b v_i$. Then $(A, \prec, \succ)$ is an inner-associative A_3-dendriform algebra.*

Proof. In view of Example 3.10, we only need to prove (1). To do this, we show that the linear map $\omega : A \otimes A \to A$, defined for $a, b \in A$ by $\omega(a \otimes b) := \mathfrak{R}(a)\mathfrak{R}(b) - \mathfrak{R}(a\mathfrak{R}(b) + \mathfrak{R}(a)b)$ satisfies $\omega(a \otimes b)c = c\omega(a \otimes b)$ for all $a, b, c \in A$. We do this by relating ω with $\mathrm{AYB}(r)$. Without loss of generality, we may assume that A has a unit 1_A.

$$\begin{aligned}
\mathrm{AYB}(r) &= r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} \\
&= \sum_{i,j}(u_i \otimes 1_A \otimes v_i)(u_j \otimes v_j \otimes 1_A) - \sum_{i,j}(u_i \otimes v_i \otimes 1_A)(1_A \otimes u_j \otimes v_j) \\
&\quad + \sum_{i,j}(1_A \otimes u_i \otimes v_i)(u_j \otimes 1_A \otimes v_j) \\
&= \sum_{i,j}(u_i u_j \otimes v_j \otimes v_i - u_i \otimes v_i u_j \otimes v_j + u_j \otimes u_i \otimes v_i v_j);
\end{aligned}$$

$$\begin{aligned}
\omega(a \otimes b) &= \sum_{i,j} u_i a v_i u_j b v_j - \mathfrak{R}\Big(\sum_i a u_i b v_i + \sum_i u_i a v_i b\Big) \\
&= -\sum_{i,j}(u_j a u_i b v_i v_j + u_j u_i a v_i b v_j - u_i a v_i u_j b v_j) \\
&= -\sum_{i,j}(u_i u_j a v_j b v_i - u_i a v_i u_j b v_j + u_j a u_i b v_i v_j).
\end{aligned}$$

If we compare these two expressions and we write $\mathrm{AYB}(r)$ as $\mathrm{AYB}(r)=\sum_k X_k\otimes Y_k\otimes Z_k$, then we see that $\omega(a\otimes b) = -\sum_k X_k a Y_k b Z_k$. The invariance of $\mathrm{AYB}(r)$, which can be written as $\sum_k cX_k\otimes Y_k\otimes Z_k = \sum_k X_k\otimes Y_k\otimes Z_k c$ for all $c\in A$ therefore yields $\omega(a\otimes b)c = -\sum_k X_k a Y_k b Z_k c = -\sum_k cX_k a Y_k b Z_k = c\omega(a\otimes b)$, as was to be shown. ∎

It leads to the following commutative diagram, generalizing (3.7):

$$\begin{array}{ccc} \text{Cob-}\varepsilon\text{-bialg}, \mu, r & \xrightarrow{\;r\cdot a - a\cdot r\;} & \varepsilon\text{-bialg}, \mu, \Delta \\ \Big\downarrow{\scriptstyle \sum_i a u_i b v_i,\, \sum_i u_i a v_i b} & & \Big\downarrow{\scriptstyle \sum_{(b)} b_{(1)} a b_{(2)}} \\ A_3^{\text{dend}}, \prec, \succ & \xrightarrow{\;a\succ b - b\prec a\;} & \text{pre-Lie}, \circ \end{array}$$

3.4. Curved Rota–Baxter systems

We show in this paragraph that curved Rota–Baxter systems also provide examples of inner-associative A_3-dendriform algebras. We first recall the definition of such systems (see [9]).

Definition 3.17. Let A be an associative algebra endowed with linear maps $\mathfrak{R},\mathfrak{S}: A\to A$ and $\omega: A\otimes A\to A$. The 4-tuple $(A,\mathfrak{R},\mathfrak{S},\omega)$ is called a *curved Rota–Baxter system* if the following conditions are satisfied, for all $a,b\in A$:

$$\mathfrak{R}(a)\mathfrak{R}(b) = \mathfrak{R}\big(\mathfrak{R}(a)b + a\mathfrak{S}(b)\big) + \omega(a\otimes b), \tag{3.9}$$

$$\mathfrak{S}(a)\mathfrak{S}(b) = \mathfrak{S}\big(\mathfrak{R}(a)b + a\mathfrak{S}(b)\big) + \omega(a\otimes b). \tag{3.10}$$

The definition is easily generalized to arbitrary algebras, but not the results which follow; this is why we consider only the case of associative algebras. Notice that weak Rota–Baxter operators on an associative algebra A correspond to curved Rota–Baxter systems $(A,\mathfrak{R},\mathfrak{S},\omega)$ with $\mathfrak{R}=\mathfrak{S}$ and having the property that ω takes values in $Z(A)$, the center of A (which coincides with $C(A)$ because A is associative). Under this correspondence, the following proposition generalizes item (2) of Proposition 3.16.

Proposition 3.18. *Let $(A,\mathfrak{R},\mathfrak{S},\omega)$ be a curved Rota–Baxter system. Define two new products on A by setting $a\succ b := \mathfrak{R}(a)b$ and $a\prec b = a\mathfrak{S}(b)$, for all $a,b\in A$. Then $(A,\prec,\succ)$ is an A_3-dendriform algebra if and only if ω takes values in $Z(A)$. In any case, $(A,\prec,\succ)$ is inner-associative.*

Proof. $(A,\prec,\succ)$ is inner-associative, since for all $a,b,c\in A$,

$$(a\succ b)\prec c = \big(\mathfrak{R}(a)b\big)\prec c = \mathfrak{R}(a)b\mathfrak{S}(c) = a\succ\big(b\mathfrak{S}(c)\big) = a\succ(b\prec c).$$

Using (3.10), we find that

$$\begin{aligned}(a\prec b)\prec c - a\prec(b\prec c + b\succ c) &= a\mathfrak{S}(b)\mathfrak{S}(c) - a\mathfrak{S}\big(b\mathfrak{S}(c)+\mathfrak{R}(b)c\big)\\ &= a\omega(b\otimes c),\end{aligned}$$

and similarly, using (3.9), $b \succ (c \succ a) - (b \prec c + b \succ c) \succ a = \omega(b \otimes c)a$. So, (2.15) is satisfied (i.e., $(A, \prec, \succ)$ is an A_3-dendriform algebra) if and only if $a\omega(b \otimes c) = \omega(b \otimes c)a$, for all $a, b, c \in A$; in turn, this is equivalent to $\omega(b \otimes c) \in Z(A)$, for all $b, c \in A$. ∎

The proof also shows that when $\omega = 0$, the A_3-dendriform algebra which is obtained is a Loday dendriform algebra; this was already observed in [10].

It was proven in [9] that, if $(A, \mathfrak{R}, \mathfrak{S}, \omega)$ is a curved Rota–Baxter system and we define a new product on A by $a \circ b = \mathfrak{R}(a)b - b\mathfrak{S}(a)$, then $(A, \circ)$ is a pre-Lie algebra if and only if $\omega(a \otimes b - b \otimes a) \in Z(A)$, for all $a, b \in A$. In particular, $(A, \circ)$ is a pre-Lie algebra when ω takes values in $Z(A)$. We recover this result as a direct consequence of Example 2.10 and Proposition 3.18.

Example 3.19. Let A be an associative algebra and let $\mathfrak{R}, \mathfrak{S} : A \to A$ be a left (resp. right) Baxter operator; i.e., $\mathfrak{R}(a)\mathfrak{R}(b) = \mathfrak{R}(\mathfrak{R}(a)b)$ and $\mathfrak{S}(a)\mathfrak{S}(b) = \mathfrak{S}(a\mathfrak{S}(b))$, for all $a, b \in A$, satisfying the extra condition that

$$\mathfrak{R}(a)\mathfrak{S}(b) = \mathfrak{R}\big(a\mathfrak{S}(b)\big) = \mathfrak{S}\big(\mathfrak{R}(a)b\big)$$

for all $a, b \in A$. Then $(A, \mathfrak{R}, \mathfrak{S}, \omega)$ is a curved Rota–Baxter system, where $\omega : A \otimes A \to A$ is defined by $\omega(a \otimes b) = -\mathfrak{R}(a)\mathfrak{S}(b)$. If moreover $\mathfrak{R}(a), \mathfrak{S}(a) \in Z(A)$ for all $a \in A$, then ω takes values in $Z(A)$, and hence Proposition 3.18 can be applied to yield an (inner-associative) A_3-dendriform algebra. A particular case of this example already appears in [9], where it is shown that if $r = \sum_i x_i \otimes y_i$ and $s = \sum_j z_j \otimes w_j$ are invariant, then the linear maps $\mathfrak{R}, \mathfrak{S} : A \to A$ and $\omega : A \otimes A \to A$, defined for $a \in A$ by

$$\mathfrak{R}(a) := \sum_i x_i a y_i, \quad \mathfrak{S}(a) := \sum_j z_j a w_j, \quad \omega(a \otimes b) = -\mathfrak{R}(a)\mathfrak{S}(b),$$

make $(A, \mathfrak{R}, \mathfrak{S}, \omega)$ into a curved Rota–Baxter system.

4. Dendriform algebras in polarized form

We now introduce the notion of a dendriform algebra for polarized algebras and relate it to the one of dendriform algebra, introduced in Section 2.

4.1. Polarized algebras

We first define the notion of a polarized algebra. The choice of the terminology *polarized* will become clear in Section 4.4.

Definition 4.1. An algebra $(A, \cdot, [\cdot,\cdot])$ is said to be a *polarized algebra* when "$\cdot$" is commutative and $[\cdot,\cdot]$ is anticommutative.

Example 4.2. If $(A, \cdot)$ is a commutative algebra, any anticommutative product $[\cdot,\cdot]$ on A makes it into a polarized algebra $(A, \cdot, [\cdot,\cdot])$.

Example 4.3. Recall (for example from [18]) that an algebra $(A, \cdot, \{\cdot,\cdot\})$ is a *Poisson algebra* if $(A, \cdot)$ is a commutative associative algebra, $(A, \{\cdot,\cdot\})$ is a Lie algebra, and the two products are compatible in the sense that $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$ for all $a, b, c \in A$. Clearly, every Poisson algebra $(A, \cdot, \{\cdot,\cdot\})$ is a polarized algebra.

4.2. Polarized $\mathscr{C}$-dendriform algebras

In analogy with Definition 2.3, we now define the notion of a dendriform algebra for a polarized algebra. Here, $\mathcal{R}_1 = 0, \dots, \mathcal{R}_k = 0$ are given relations involving the products "$\cdot$" and $[\cdot,\cdot]$ (only). The category of all polarized algebras satisfying these relations is denoted by $\mathscr{C}_{\text{pol}}$. The morphisms in $\mathscr{C}_{\text{pol}}$ are the algebra homomorphisms.

Definition 4.4. An algebra $(A, *, \circ)$ is said to be a *polarized $\mathscr{C}$-dendriform algebra* if $(A \times A, \odot, [\![\cdot,\cdot]\!]) \in \mathscr{C}_{\text{pol}}$, where $\odot$ and $[\![\cdot,\cdot]\!]$ are defined, for (a, x) and (b, y) in $A \times A$, by

$$(a, x) \odot (b, y) := (a * b + b * a, a * y + b * x), \tag{4.1}$$

$$[\![(a, x), (b, y)]\!] := (a \circ b - b \circ a, a \circ y - b \circ x). \tag{4.2}$$

The category of all polarized $\mathscr{C}$-dendriform algebras (over R) is denoted by $\mathscr{C}_{\text{pol}}^{\text{dend}}$. The morphisms in this category are the algebra homomorphisms. Setting $x = y = 0$ in (4.1) and in (4.2), we see that we have again a faithful functor $\mathscr{C}_{\text{pol}}^{\text{dend}} \to \mathscr{C}_{\text{pol}}$, defined on objects by $(A, *, \circ) \mapsto (A, \cdot, [\cdot,\cdot])$, where the two new products on A are defined, for all $a, b \in A$, by

$$a \cdot b := a * b + b * a \quad \text{and} \quad [a, b] := a \circ b - b \circ a. \tag{4.3}$$

Remark 4.5. The above definition of a polarized $\mathscr{C}$-dendriform algebra admits the following natural generalization. An algebra $(A, *, \circ, |, \Box)$ is said to be a *polarized $\mathscr{C}$-tridendriform algebra* if $(A, |, \Box)$ is a polarized algebra and $(A \times A, \odot, [\![\cdot,\cdot]\!]) \in \mathscr{C}_{\text{pol}}$, where $\odot$ and $[\![\cdot,\cdot]\!]$ are defined for (a, x) and (b, y) in $A \times A$, by

$$(a, x) \odot (b, y) := (a * b + b * a + a \mid b, a * y + b * x + x \mid y), \tag{4.4}$$

$$[\![(a, x), (b, y)]\!] := (a \circ b - b \circ a + a \Box b, a \circ y - b \circ x + x \Box y). \tag{4.5}$$

We have a functor from the category $\mathscr{C}_{\text{pol}}^{\text{trid}}$ of all polarized $\mathscr{C}$-tridendriform algebras to $\mathscr{C}_{\text{pol}}$, defined on objects by $(A, *, \circ, |, \Box) \mapsto (A, \cdot, [\cdot,\cdot])$, where

$$a \cdot b := a * b + b * a + a \mid b \quad \text{and} \quad [a, b] := a \circ b - b \circ a + a \Box b,$$

for all $a, b \in A$. It is the identity on morphisms.

4.3. Algebras defined by multilinear relations

In the case of multilinear relations, the relations which every polarized $\mathscr{C}$-dendriform algebra must satisfy can be easily computed, as we show for a trilinear relation $\mathcal{R} = 0$. Thanks

$\odot$	$\underline{b}_0$	$\underline{b}_1$
$\underline{a}_0$	$\underline{a * b + b * a}_0$	$\underline{a * b}_1$
$\underline{a}_1$	$\underline{b * a}_1$	$(0,0)$

$[\![\cdot,\cdot]\!]$	$\underline{b}_0$	$\underline{b}_1$
$\underline{a}_0$	$\underline{a \circ b - b \circ a}_0$	$\underline{a \circ b}_1$
$\underline{a}_1$	$-\underline{b \circ a}_1$	$(0,0)$

Table 3. The products $\odot$ and $[\![\cdot,\cdot]\!]$ for generators of $A \times A$.

to commutativity and anticommutativity, $\mathcal{R}$ is of the form

$$\begin{aligned}\mathcal{R}(a_1,a_2,a_3) = {} & \sum_{\sigma\in A_3} \lambda_\sigma (a_{\sigma(1)} \cdot a_{\sigma(2)}) \cdot a_{\sigma(3)} + \sum_{\sigma\in A_3} \lambda'_\sigma [a_{\sigma(1)}, [a_{\sigma(2)}, a_{\sigma(3)}]] \\ & + \sum_{\sigma\in A_3} \lambda''_\sigma [a_{\sigma(1)}, a_{\sigma(2)}] \cdot a_{\sigma(3)} + \sum_{\sigma\in A_3} \lambda'''_\sigma [a_{\sigma(1)} \cdot a_{\sigma(2)}, a_{\sigma(3)}],\end{aligned}$$

where the 12 constants $\lambda_\sigma, \dots, \lambda'''_\sigma$ belong to R. In Table 3, we exhibit the products $\odot$ and $[\![\cdot,\cdot]\!]$ in terms of a generating set of $A \times A$.

The observations made in the case of algebras with one product are, mutatis mutandis, also valid here; namely, the relations are trivially satisfied when one takes at least two elements in A_1, and the relation which is obtained by taking all elements in A_0 is a consequence of the relations which are obtained by taking two elements in A_0 and taking the other element in A_1. To see the latter claim, it suffices to consider, as in (2.8), the following formulas, which follow easily from Table 3:

$$\begin{aligned}(\underline{a}_0 \odot \underline{b}_0) \odot \underline{c}_1 + (\underline{a}_0 \odot \underline{b}_1) \odot \underline{c}_0 + (\underline{a}_1 \odot \underline{b}_0) \odot \underline{c}_0 &= \underline{(a \cdot b) \cdot c}_0, \\ [\![\underline{a}_0, \underline{b}_0]\!] \odot \underline{c}_1 + [\![\underline{a}_0, \underline{b}_1]\!] \odot \underline{c}_0 + [\![\underline{a}_1, \underline{b}_0]\!] \odot \underline{c}_0 &= \underline{[a,b] \cdot c}_1, \\ [\![\underline{a}_0 \odot \underline{b}_0, \underline{c}_1]\!] + [\![\underline{a}_0 \odot \underline{b}_1, \underline{c}_0]\!] + [\![\underline{a}_1 \odot \underline{b}_0, \underline{c}_0]\!] &= \underline{[a \cdot b, c]}_1, \\ [\![[\![\underline{a}_0, \underline{b}_0]\!], \underline{c}_1]\!] + [\![[\![\underline{a}_0, \underline{b}_1]\!], \underline{c}_0]\!] + [\![[\![\underline{a}_1, \underline{b}_0]\!], \underline{c}_0]\!] &= \underline{[[a,b],c]}_1,\end{aligned}$$

together with the four formulas corresponding to the other parenthesizing. We have used (4.3) to write the above formulas in a compact form.

Example 4.6. We return to the example of a Poisson algebra and show how to obtain the relations in the corresponding dendriform category, which we denote by $\mathcal{P}^{\text{dend}}_{\text{pol}}$. We start with associativity of $\odot$, taking first $\underline{a}_0, \underline{b}_0 \in A_0$ and $\underline{c}_1 \in A_1$, from which we find that

$$\underline{(a * b + b * a) * c}_1 = (\underline{a}_0 \odot \underline{b}_0) \odot \underline{c}_1 = \underline{a}_0 \odot (\underline{b}_0 \odot \underline{c}_1) = \underline{a * (b * c)}_1,$$

so that

$$a * (b * c) = (a * b + b * a) * c, \tag{4.6}$$

for all $a, b, c \in A$, which means that $(A, *)$ is a Zinbiel algebra (see Example 2.15). Similarly, taking $\underline{a}_0, \underline{c}_0 \in A_0$ and $\underline{b}_1 \in A_1$, we find $c * (a * b) = a * (c * b)$ for all $a, b, c \in A$, which means that $(A, *)$ is a NAP algebra. Since every Zinbiel algebra is a

NAP algebra, we do not need to state the NAP condition for $*$. By symmetry we do not need to consider the case of $\underline{b}_0, \underline{c}_0 \in A_0$ and $\underline{a}_1 \in A_1$. Similarly, the derivation property $[a \cdot b, c] = [a, c] \cdot b + a \cdot [b, c]$ is symmetric in a and b, so we get by the above procedure only two equations, which can be written in the following symmetric form:

$$(a * b + b * a) \circ c = a * (b \circ c) + b * (a \circ c), \tag{4.7}$$

$$(a \circ b - b \circ a) * c = a * (b \circ c) - b \circ (a * c). \tag{4.8}$$

Finally, because the Jacobi identity is symmetric in all of its variables, we get only one equation from the Jacobi identity, namely the pre-Lie condition

$$(a \circ b - b \circ a) \circ c = a \circ (b \circ c) - b \circ (a \circ c). \tag{4.9}$$

It follows that (4.6)–(4.9) are the four relations of $\mathcal{P}_{\text{pol}}^{\text{dend}}$. An algebra $(A, *, \circ)$ which satisfies (4.6)–(4.9) (i.e., an algebra in $\mathcal{P}_{\text{pol}}^{\text{dend}}$) is exactly what M. Aguiar in [2] calls a *pre-Poisson algebra*. Thus, our general procedure to obtain $\mathcal{C}_{\text{pol}}^{\text{dend}}$ from $\mathcal{C}_{\text{pol}}$ yields a canonical way to obtain the concept of a pre-Poisson algebra from the concept of a Poisson algebra.

Remark 4.7. The above procedure also applies to polarized $\mathcal{C}$-tridendriform algebras and the comments in Remark 2.7 also apply here.

Example 4.8. We continue Example 4.6 and give the relations which an algebra $(A, *, \circ, |, \square)$ must satisfy in order to be a polarized $\mathcal{P}$-tridendriform algebra. We get the following three equations from associativity, where the first one is obtained using the same substitutions as (4.6), while the two other equations are obtained respectively by substituting in the associativity relation two or three elements from A_0:

$$\begin{aligned} a * (b * c) &= (a * b) * c + (b * a) * c + (a \mid b) * c, \\ a * (b \mid c) &= (a * b) \mid c, \\ a \mid (b \mid c) &= (a \mid b) \mid c. \end{aligned}$$

By symmetry, the Jacobi identity implies that we only get three relations from it, by substituting respectively one, two or three elements from A_0:

$$\begin{aligned} a \circ (b \circ c) - b \circ (a \circ c) &= (a \circ b - b \circ a + a \square b) \circ c, \\ (a \circ b) \square c &= a \circ (b \square c) + (a \circ c) \square b, \\ 0 &= (a \square b) \square c + b \square (c \square a) + c \square (a \square b). \end{aligned}$$

Finally, the derivation property leads to the following five relations:

$$\begin{aligned} a * (b \circ c) + b * (a \circ c) &= (a * b + b * a + a \mid b) \circ c, \\ a * (b \circ c) - b \circ (a * c) &= (a \circ b - b \circ a + a \square b) * c, \\ (a * b) \square c &= a * (b \square c) + b \mid (a \circ c), \\ c \circ (a \mid b) &= a \mid (c \circ b) + b \mid (c \circ a), \\ (a \mid b) \square c &= a \mid (b \square c) + b \mid (a \square c). \end{aligned}$$

These 11 equations are, together with the commutativity and anticommutativity of $|$ and $\square$, precisely the 13 relations [23, equations (48)–(60)] which define the notion of a post-Poisson algebra.

4.4. Polarization

We now show how the two notions of dendriform algebras of Sections 2.2 and 4.2 are related. Following [22], given an algebra (A, μ), two new products "$\cdot$" and $[\cdot,\cdot]$ are defined on A by setting

$$a \cdot b := \frac{1}{2}(ab + ba) \quad \text{and} \quad [a, b] := \frac{1}{2}(ab - ba), \tag{4.10}$$

for all $a, b \in A$. This procedure is called *polarization*. We can easily reconstruct μ from these products, because $ab = a \cdot b + [a, b]$, for all $a, b \in A$; this is what is called *depolarization*. So we have a natural way to associate to each algebra (A, μ) a *polarized* algebra $(A, \cdot, [\cdot, \cdot])$ and vice-versa.

Example 4.9. The P-algebras introduced in Example 2.13 correspond by polarization/depolarization to Poisson algebras; see [22]. This explains why we call the latter P-algebras.

For given relations $\mathcal{R}_1 = 0, \dots, \mathcal{R}_k = 0$ (in one operation), we have constructed four categories $\mathscr{C}$, $\mathscr{C}_{\rm pol}$, $\mathscr{C}^{\rm dend}$, and $\mathscr{C}^{\rm dend}_{\rm pol}$ and three functors, as in the following diagram, which we completed into a square by adding a pair of inverse arrows between $\mathscr{C}^{\rm dend}$ and $\mathscr{C}^{\rm dend}_{\rm pol}$; the commutativity of the diagram is easily established.

$$\begin{array}{ccc} \mathscr{C}, \mu & \underset{a\cdot b+[a,b]}{\overset{(ab+ba)/2,(ab-ba)/2}{\rightleftarrows}} & \mathscr{C}_{\rm pol}, \cdot, [\cdot,\cdot] \\ {\scriptstyle a\prec b+a\succ b}\ \Big\uparrow & & \Big\uparrow\ {\scriptstyle a*b+b*a,\,a\circ b-b\circ a} \\ \mathscr{C}^{\rm dend}, \prec, \succ & \underset{b*a-b\circ a,\,a*b+a\circ b}{\overset{\frac{a\succ b+b\prec a}{2},\,\frac{a\succ b-b\prec a}{2}}{\rightleftarrows}} & \mathscr{C}^{\rm dend}_{\rm pol}, *, \circ \end{array} \tag{4.11}$$

In analogy with the upper arrows, we call the lower arrows *polarization* and *depolarization*. Since polarization and depolarization are inverse operations, the horizontal arrows define functors which are isomorphisms of categories.

Notice that by commutativity of the diagram, a polarized $\mathscr{C}$-dendriform algebra can also be defined as an algebra $(A, *, \circ)$ whose depolarizated algebra $(A, \prec, \succ)$ is a $\mathscr{C}$-dendriform algebra (which justifies the terminology). Indeed, according to the definition and by depolarization, $(A, *, \circ) \in \mathscr{C}^{\rm dend}_{\rm pol}$ if and only if $(A \times A, \bullet) \in \mathscr{C}$, with

$$\begin{aligned} (a, x) \bullet (b, y) &= (a, x) \odot (b, y) + [\![(a, x), (b, y)]\!] \\ &= (b * a - b \circ a + a * b + a \circ b, a * y + a \circ y + b * x - b \circ x) \\ &= (a \prec b + a \succ b, a \succ y + x \prec b). \end{aligned}$$

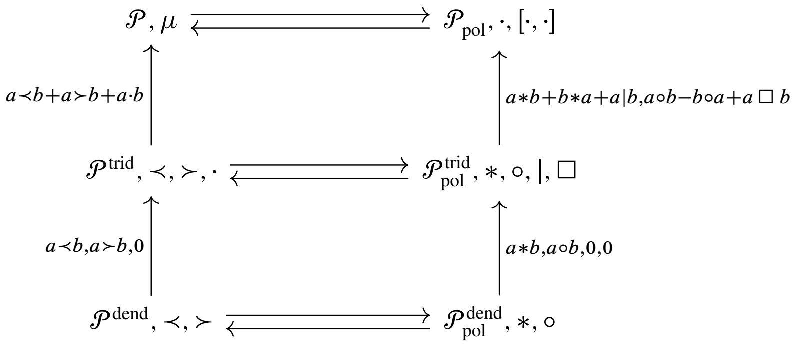
We have obtained exactly the condition that the depolarized form $(A, \prec, \succ)$ of $(A, *, \circ)$ belongs to $\mathscr{C}^{\rm dend}$ (see Definition 2.3), showing our claim.

Remark 4.10. Polarization and depolarization can also be defined for tridendriform and polarized tridendriform algebras, leading, for any category of algebras $\mathscr{C}$ as above, to an isomorphism of the category $\mathscr{C}^{\mathrm{trid}}$ of $\mathscr{C}$-tridendriform algebras and the category $\mathscr{C}^{\mathrm{trid}}_{\mathrm{pol}}$ of polarized $\mathscr{C}$-tridendriform algebras. On objects, the pair of inverse isomorphisms is given by

$$\mathscr{C}^{\mathrm{trid}}, \prec, \succ, \cdot \;\underset{b*a-b\circ a,\, a*b+a\circ b,\, a|b+a\,\square\, b}{\overset{\frac{a\succ b+b\prec a}{2},\, \frac{a\succ b-b\prec a}{2},\, \frac{ab+ba}{2},\, \frac{ab-ba}{2}}{\rightleftarrows}}\; \mathscr{C}^{\mathrm{trid}}_{\mathrm{pol}}, *, \circ, |, \square\,. \tag{4.12}$$

They extend the pair of lower arrows in (4.11) and lead to a commutative diagram, as in (4.11).

Example 4.11. In the case of P-algebras and Poisson algebras, the above results can be summarized in the following commutative diagram, in which the horizontal arrows are given by the horizontal arrows in (4.11) and (4.12):



It was already pointed out by M. Aguiar in [2] that, if $(A, *, \circ) \in \mathscr{P}^{\mathrm{dend}}_{\mathrm{pol}}$, i.e., if it is a pre-Poisson algebra, and we define new operations on A by $a \cdot b = a * b + b * a$ and $\{a, b\} = a \circ b - b \circ a$, for all $a, b \in A$, then $(A, \cdot, \{\cdot, \cdot\})$ is a Poisson algebra. It corresponds to the composition of the two right arrows in the diagram.

4.5. Application I: deformations of dendriform algebras

In [2], M. Aguiar introduced the notion of deformation for a commutative Loday dendriform algebra $(A, \prec, \succ)$ and he showed that such a deformation makes $(A, \times, \bigcirc)$ into a pre-Poisson algebra, where $\times$ stands for $\succ$ and where the product $\bigcirc$ on A is constructed from the first-order deformation terms of the products $\prec$ and $\succ$. In this section, we generalize this result to arbitrary $\mathscr{C}$-dendriform algebras, giving a conceptual proof of Aguiar's result.

As before, $\mathscr{C}$ denotes in this section the category of all R-algebras satisfying a fixed set of relations $\mathscr{R}_1 = 0, \dots, \mathscr{R}_k = 0$. Let ν be an indeterminate and let R^ν denote the ring of formal power series $R[[\nu]]$. More generally, for any R-module A we denote by A^ν the R^ν-module of formal power series in ν with coefficients in A. For a formal power series $X \in A^\nu$, its evaluation at 0, which is the constant term of X, is denoted by X_0.

Definition 4.12. Let $(A, \prec_0, \succ_0)$ be a commutative $\mathcal{C}$-dendriform algebra and denote $a \times b := a \succ_0 b = b \prec_0 a$ for all $a, b \in A$. An R^ν-algebra $(A^\nu, \prec, \succ)$ is said to be a *formal deformation* of $(A, \prec_0, \succ_0)$ if $(A^\nu, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra over R^ν and for any $a, b \in A$,

$$(a \succ b)_0 = a \succ_0 b \quad \text{and} \quad (a \prec b)_0 = a \prec_0 b.$$

We can then define a new product on A by setting, for all $a, b \in A$,

$$a \bigcirc b := \left.\frac{a \succ b - b \prec a}{2\nu}\right|_{\nu=0}. \tag{4.13}$$

The algebra $(A, \times, \bigcirc)$ is called the *infinitesimal algebra* of the deformation.

The question which we study here is to which category $(A, \times, \bigcirc)$ belongs. When $\mathcal{C}$ is the category of associative algebras, the answer is provided by Aguiar [2], who showed that $(A, \times, \bigcirc)$ is a pre-Poisson algebra.

Let M be a monomial which involves the products "$\cdot$" and $[\cdot, \cdot]$ only. We define the *weight* of M as the number of operations $[\cdot, \cdot]$ in M. Similarly, for a monomial $\widetilde{M}$ in the products $*$ and $\circ$, its weight is the number of operations $\circ$ in $\widetilde{M}$. A sum of such monomials is said to be *homogeneous of weight m* if each of its terms has weight m. The lowest weight part of $\mathcal{R}$ is denoted by $\underline{\mathcal{R}}$. Finally, we denote by $\underline{\mathcal{C}}_{\mathrm{pol}}$ (resp. by $\underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}}$) the category of all R-algebras satisfying all relations $\underline{\mathcal{R}} = 0$, where $\mathcal{R}$ runs through the linear space of relations of $\mathcal{C}_{\mathrm{pol}}$ (resp. of $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$).

Proposition 4.13. *Let $(A^\nu, \prec, \succ)$ be a formal deformation of a commutative algebra $(A, \prec_0, \succ_0) \in \mathcal{C}^{\mathrm{dend}}$, with deformation algebra $(A, \times, \bigcirc)$. Then*

$$(A, \times, \bigcirc) \in \underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}}. \tag{4.14}$$

In particular, when the relations of $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$ are generated by weight homogeneous relations, then $(A, \times, \bigcirc) \in \mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$. Also, when the relations of $\mathcal{C}$ are multilinear, $\underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}} = (\underline{\mathcal{C}}_{\mathrm{pol}})^{\mathrm{dend}}$, so that $(A, \times, \bigcirc) \in (\underline{\mathcal{C}}_{\mathrm{pol}})^{\mathrm{dend}}$.

Proof. We prove here only that $(A, \times, \bigcirc) \in \underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}}$, leaving the more technical proof that $\underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}} = (\underline{\mathcal{C}}_{\mathrm{pol}})^{\mathrm{dend}}$ to the end of the section.

Given a formal deformation $(A^\nu, \prec, \succ)$, we can construct by polarization an algebra $(A^\nu, *, \circ)$, which is a polarized dendriform algebra over R^ν. We define new products $*_i$ and $\circ_i$ on A by setting for all $a, b \in A$,

$$\begin{aligned} a * b &= a *_0 b + a *_1 b\nu + a *_2 b\nu^2 + \cdots, \\ a \circ b &= a \circ_0 b + a \circ_1 b\nu + a \circ_2 b\nu^2 + \cdots. \end{aligned} \tag{4.15}$$

Since, by polarization, $a \circ b = (a \succ b - b \prec a)/2$ and $a * b = (a \succ b + b \prec a)/2$ (see (4.11)), we have by commutativity of $(A, \prec, \succ)$ that $a *_0 b = a \times b$ and that $a \circ_0 b = 0$;

also, the definition of $\bigcirc$ implies that $a \circ_1 b = a \bigcirc b$ for all $a, b \in A$. Hence, (4.15) can be rewritten as

$$a * b = a \times b + a *_1 b\nu + a *_2 b\nu^2 + \cdots , \tag{4.16}$$

$$a \circ b = a \bigcirc b\nu + a \circ_2 b\nu^2 + \cdots , \tag{4.17}$$

where the dots stand for terms containing ν^i with $i > 2$. Suppose now that $\mathcal{R} = 0$ is a relation of $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$. Writing $\mathcal{R}$ as $\mathcal{R}_{*,\circ}$ to indicate the products which are involved, we may also consider $\mathcal{R}_{\times,\bigcirc}$. We need to show that $\underline{\mathcal{R}}_{\times,\bigcirc}(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. To do this, consider the relation $\mathcal{R}_{*,\circ}(a_1, \ldots, a_n) = 0$. In view of (4.16) and (4.17),

$$\mathcal{R}_{*,\circ}(a_1, a_2, \ldots, a_n) = \underline{\mathcal{R}}_{\times,\bigcirc}(a_1, a_2, \ldots, a_n)\nu^d + \cdots , \tag{4.18}$$

where d denotes the lowest weight of the terms of $\mathcal{R}$, i.e., the weight of $\underline{\mathcal{R}}$. It follows that $(A, \times, \bigcirc)$ satisfies the relation $\underline{\mathcal{R}}_{\times,\bigcirc} = 0$, as was to be shown. ∎

Example 4.14. Let $\mathcal{C}$ be the category of all associative algebras (over R). Then, by polarization, the following are the relations in $\mathcal{C}_{\mathrm{pol}}$ (see [22]):

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b, \tag{4.19}$$

$$\big[[a, b], c\big] = (b \cdot c) \cdot a - (c \cdot a) \cdot b. \tag{4.20}$$

Recall that (4.20) implies the Jacobi identity, which is weight homogeneous (of weight 2), just like the derivation property (4.19) (of weight 1). Notice that the lowest weight part of (4.20) is $(b \cdot c) \cdot a = (c \cdot a) \cdot b$, which is associativity (since "$\cdot$" commutative). It follows that $\underline{\mathcal{C}}_{\mathrm{pol}}$ is the category of Poisson algebras, hence that $\underline{\mathcal{C}}_{\mathrm{pol}}^{\mathrm{dend}}$ is the category of pre-Poisson algebras. This shows that the infinitesimal algebra of a deformation of a Loday dendriform algebra is a pre-Poisson algebra, as was first shown by Aguiar [2].

Example 4.15. The relations which define Poisson algebras (see Example 4.3) are 3-linear and homogeneous: associativity is of weight 0, the derivation property is of weight 1, and the Jacobi identity is of weight 2. For A_3-associative algebras and LA-algebras in polarized form, the relations are also easily written in a homogeneous form. It follows that the second part of Proposition 4.13 can be applied to these algebras: in each of these cases, the infinitesimal algebra $(A, \times, \bigcirc)$ of the deformation belongs to $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$.

Remark 4.16. Proposition 4.13 is easily adapted to the classical case of formal deformations (A, μ) of commutative algebras $(A, \mu_0) \in \mathcal{C}$. The infinitesimal algebra is then defined as $(A, \mu_0, \diamond)$, where

$$a \diamond b := \frac{\mu(a, b) - \mu(b, a)}{2\nu}\Big|_{\nu=0}.$$

One shows as in the proof of Proposition 4.13 that $(A, \mu_0, \diamond) \in \underline{\mathcal{C}}_{\mathrm{pol}}$. In the case of associative algebras, $\underline{\mathcal{C}}_{\mathrm{pol}}$ is the category of Poisson algebras (see Example 4.14), so we recover the classical result that the infinitesimal algebra of a deformation of an associative algebra is a Poisson algebra.

Remark 4.17. One may also consider more generally deformations of $\mathscr{C}$-tridendriform algebras. Recall that in a commutative $\mathscr{C}$-tridendriform algebra $(A, \prec, \succ, \cdot)$, one also requires the last product to be commutative. The weight of a relation $\mathcal{R} = \mathcal{R}_{*,\circ,|,\Box}$ is now defined such that $*$ and $|$ have weight 0, while $\circ$ and $\Box$ have weight 1. It is clear that all the above results generalize to this case. The infinitesimal algebra has now four operations. For example, when $\mathscr{C}$ is the category of associative algebras, the infinitesimal algebra is a post-Poisson algebra (see Example 4.8).

Remark 4.18. We have considered deformations of commutative dendriform algebras, but everything can be easily adapted to anticommutative dendriform algebras: the roles of $*$ and $\circ$ are exchanged in the sense that one will have now that $*_0 = 0$, that $*_1 = \times$, and $\circ_0 = \bigcirc$, where $(A, \bigcirc)$ is the original anticommutative dendriform algebra (written as an algebra with one operation). As we have seen in Section 2.5, A_3-associative, LA, and P-algebras which are anticommutative are Lie algebras, so there are many natural examples of this case.

To finish this section, we prove that when the relations of $\mathscr{C}$ are multilinear, $\underline{\mathscr{C}}_{\rm pol}^{\rm dend} = (\underline{\mathscr{C}}_{\rm pol})^{\rm dend}$, as stated in Proposition 4.13. The property says that the lowest weight parts of all relations in $\underline{\mathscr{C}}_{\rm pol}^{\rm dend}$ are obtained by dendrifying the lowest weight parts of all relations in $\underline{\mathscr{C}}_{\rm pol}$. Notice that since each dendrification of a monomial of weight k (involving the products "$\cdot$" and $[\cdot,\cdot]$ only) is homogeneous of weight k, one has that all algebras in $(\underline{\mathscr{C}}_{\rm pol})^{\rm dend}$ are also algebras of $\underline{\mathscr{C}}_{\rm pol}^{\rm dend}$. We therefore only need to prove the reciprocal inclusion.

We may restrict ourselves to n-linear relations, for a fixed n, since the dendrification of a k-linear relation is k-linear; i.e., we may suppose that all relations $\mathcal{R}_1, \dots, \mathcal{R}_k$ of $\mathscr{C}_{\rm pol}$, and hence also of $\mathscr{C}_{\rm pol}^{\rm dend}$, are n-linear.

For $0 \leqslant \ell \leqslant n$, consider the free R-modules $\mathcal{M}_\ell$ and $\widetilde{\mathcal{M}}_\ell$, generated by all ℓ-linear monomials M involving only the (commutative and anticommutative) products "$\cdot$" and $[\cdot,\cdot]$, respectively generated by all ℓ-linear monomials $\widetilde{M}$ involving only the products $*$ and $\circ$ in n variables, say $x_1, \dots, x_n$. Their direct sums are denoted by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, respectively. Elements of $\mathcal{M}_\ell$ and $\widetilde{\mathcal{M}}_\ell$ are also said to be of *length* ℓ; notice that the weight of a monomial of length ℓ is between 0 and $\ell - 1$ (included). The modules $\mathcal{M}_\ell$ and $\widetilde{\mathcal{M}}_\ell$ admit natural decompositions

$$\mathcal{M}_\ell = \mathcal{M}_\ell^0 \oplus \cdots \oplus \mathcal{M}_\ell^{\ell-1} \quad \text{and} \quad \widetilde{\mathcal{M}}_\ell = \widetilde{\mathcal{M}}_\ell^0 \oplus \cdots \oplus \widetilde{\mathcal{M}}_\ell^{\ell-1},$$

where $\mathcal{M}_\ell^i \subset \mathcal{M}_\ell$ and $\widetilde{\mathcal{M}}_\ell^i \subset \widetilde{\mathcal{M}}_\ell$ are the submodules generated by the monomials of weight i. Each monomial M of $\mathcal{M}_\ell$ of length at least two can be decomposed as $M = M_1 \cdot M_2$ or $M = [M_1, M_2]$; this decomposition is unique up to the order of the factors.

We describe the process of dendrification of multilinear relations of $\mathscr{C}_{\rm pol}$, introduced and studied in Section 4.3, in terms of the linear maps

$$\varphi_0, \varphi_1, \dots, \varphi_n : \mathcal{M} \to \widetilde{\mathcal{M}},$$

which we define on monomials M, using induction on the length of M:

$$\varphi_0(M) := \begin{cases} x_i & \text{if } M = x_i, \\ \varphi_0(M_1) * \varphi_0(M_2) + \varphi_0(M_2) * \varphi_0(M_1) & \text{if } M = M_1 \cdot M_2, \\ \varphi_0(M_1) \circ \varphi_0(M_2) - \varphi_0(M_2) \circ \varphi_0(M_1) & \text{if } M = [M_1, M_2], \end{cases}$$

and for $p = 1, \ldots, n$ we define

$$\varphi_p(M) := \begin{cases} 0 & \text{if } M \text{ is independent of } x_p, \\ x_p & \text{if } M = x_p, \\ \varphi_0(M_1) * \varphi_p(M_2) & \text{if } M = M_1 \cdot M_2 \text{ and } M_2 \text{ depends on } x_p, \\ \varphi_0(M_1) \circ \varphi_p(M_2) & \text{if } M = [M_1, M_2] \text{ and } M_2 \text{ depends on } x_p. \end{cases}$$

It is clear that these maps are well defined and that they preserve the length and the weight of a monomial. Notice that, by construction, in all terms of $\varphi_p(\mathcal{M})$ the variable x_p is located at the last position. Therefore, the images of the maps $\varphi_1, \ldots, \varphi_n$ are in direct sum.

To see the relation with dendrification, let $\mathcal{R} = 0$ be an n-linear relation of $\mathcal{C}_{\text{pol}}$. Then $\mathcal{R} \in \mathcal{M}$ and for $p = 1, \ldots, n$, the relation $\varphi_p(\mathcal{R}) = 0$ is precisely the relation obtained by substituting in $\mathcal{R}_{\cdot,[\cdot,\cdot]}$ for the pth variable $(0, x_p)$ and for the qth variable $(x_q, 0)$, where $q \neq p$.

Lemma 4.19. *The maps $\varphi_0, \ldots, \varphi_n$ are injective.*

Proof. Let $\widetilde{M}$ be a monomial of $\widetilde{\mathcal{M}}$. We show that there exists a unique monomial $M \in \mathcal{M}$ such that $\widetilde{M}$ is a term of $\varphi_0(M)$; from it the injectivity of φ_0 is clear.

We do this by induction on the length of $\widetilde{M}$. When $\widetilde{M}$ is of length 1, the claim is trivially true, so let us assume that the claim is true for monomials of length strictly less than some $\ell \geqslant 2$. Let $\widetilde{M}$ be a monomial of $\widetilde{\mathcal{M}}$ of length ℓ. We can write $\widetilde{M}$ uniquely as $\widetilde{M} = \widetilde{M}_1 * \widetilde{M}_2$ or $\widetilde{M} = \widetilde{M}_1 \circ \widetilde{M}_2$, up to the order of the factors. By the induction hypothesis there exists a unique couple (M_1, M_2) such that $\widetilde{M}_1$ and $\widetilde{M}_2$ are terms of $\varphi_0(M_1)$ and $\varphi_0(M_2)$, respectively, and hence such that $\widetilde{M}$ is a term of $\varphi_0(M_1) * \varphi_0(M_2)$ or $\varphi_0(M_1) \circ \varphi_0(M_2)$, depending on whether $\widetilde{M} = \widetilde{M}_1 * \widetilde{M}_2$ or $\widetilde{M} = \widetilde{M}_1 \circ \widetilde{M}_2$. It follows that, if we define $M := M_1 \cdot M_2$ or $M := [M_1, M_2]$, depending on whether $\widetilde{M} = \widetilde{M}_1 * \widetilde{M}_2$ or $\widetilde{M} = \widetilde{M}_1 \circ \widetilde{M}_2$, then $\varphi_0(M) = \widetilde{M}$. Since the decomposition of $\widetilde{M}$ is unique up to the order of the factors, M is unique. This shows the claim, and hence the injectivity of φ_0.

In order to show the injectivity of the other maps $\varphi_1, \ldots, \varphi_n$, one proceeds in a similar way: one shows as above that given any monomial $\widetilde{M}$ of $\widetilde{\mathcal{M}}$ there exists a unique monomial M of $\mathcal{M}$ and a unique integer $p \in \{1, \ldots, n\}$ such that $\widetilde{M}$ is a term of $\varphi_p(M)$. ∎

Lemma 4.20. *Let $\mathcal{R}_1, \ldots, \mathcal{R}_k \in \mathcal{M}_n$. For any constants $\lambda_i^p \in R$ ($1 \leqslant i \leqslant k$ and $p = 1, \ldots, n$), not all equal to zero,*

$$\underline{\sum_{i=1}^{k} \sum_{p=1}^{n} \lambda_i^p \varphi_p(\mathcal{R}_i)} = \sum_{p=1}^{n} \varphi_p \left(\underline{\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i} \right). \tag{4.21}$$

Proof. For $i = 1, \dots, k$, let $\mathcal{R}_i = \mathcal{R}_i^0 + \cdots + \mathcal{R}_i^{n-1}$ be the weight decomposition of $\mathcal{R}_i$. By R-linearity of the maps φ_p,

$$\sum_{i=1}^{k} \sum_{p=1}^{n} \lambda_i^p \varphi_p(\mathcal{R}_i) = \sum_{\ell=m}^{n-1} A_\ell, \quad \text{where } A_\ell = \sum_{p=1}^{n} \varphi_p\Big(\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^\ell\Big),$$

and where m is chosen such that $A_0, \dots, A_{m-1} = 0$ and $A_m \neq 0$. Since the maps φ_p are weight-preserving, A_ℓ is homogeneous of weight ℓ, and so A_m is equal to the left-hand side of (4.21). Let $0 \leqslant \ell < m$. Then $\sum_{p=1}^{n} \varphi_p(\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^\ell) = A_\ell = 0$, so that $\varphi_p(\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^\ell)$ for all p, since the images of the maps $\varphi_1, \dots, \varphi_n$ are in direct sum. Since the maps φ_p are injective (Lemma 4.19), this implies that $\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^\ell = 0$ for $\ell = 0, \dots, m-1$. Also, $\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^m \neq 0$ since $A_m \neq 0$. It follows that

$$\underline{\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i} = \underline{\sum_{i=1}^{k} \sum_{\ell=0}^{n-1} \lambda_i^p \mathcal{R}_i^\ell} = \sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i^m,$$

so that A_m is also equal to the right-hand side of (4.21). ■

We use Lemma 4.20 to show that all algebras in $\underline{\mathcal{C}}_{\text{pol}}^{\text{dend}}$ are also algebras of $(\underline{\mathcal{C}}_{\text{pol}})^{\text{dend}}$, so that $\underline{\mathcal{C}}_{\text{pol}}^{\text{dend}} = (\underline{\mathcal{C}}_{\text{pol}})^{\text{dend}}$. Suppose that $\mathcal{R}_1 = 0, \dots, \mathcal{R}_k = 0$ is a basis for the module of all n-linear relations of $\mathcal{C}_{\text{pol}}$. Let $\mathcal{R} = 0$ be a relation of $\underline{\mathcal{C}}_{\text{pol}}^{\text{dend}}$. By definition, $\mathcal{R}$ is the lowest weight part of $\sum_{i=1}^{k} \sum_{p=1}^{n} \lambda_i^p \varphi_p(\mathcal{R}_i)$, for some constants λ_i^p. In view of the lemma, $\mathcal{R}$ is obtained by dendrification of some relations in $\underline{\mathcal{C}}_{\text{pol}}$, namely the p relations $\sum_{i=1}^{k} \lambda_i^p \mathcal{R}_i = 0$, for $p = 1, \dots, n$. This shows that $\mathcal{R} = 0$ is a relation of $(\underline{\mathcal{C}}_{\text{pol}})^{\text{dend}}$.

4.6. Application II: filtered dendriform algebras

As a second application of polarized dendriform algebras, we generalize another result of Aguiar [2], which is itself an analogue for Loday's dendriform algebras of the well-known result which says that the graded algebra associated to an almost commutative filtered associative algebra is a Poisson algebra.

Let $(A, \prec, \succ)$ be an algebra. An (increasing) *filtration* on A is an increasing sequence of subspaces $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ such that

$$A = \bigcup_{i \geqslant 0} A_i \quad \text{and} \quad (A_i \prec A_j + A_i \succ A_j) \subseteq A_{i+j},$$

for all $i, j \geqslant 0$. Then A is called a *filtered algebra*. It is convenient to set $A_i := \{0\}$ for $i < 0$. The associated graded algebra is, as an R-module,

$$\operatorname{gr}(A) := \bigoplus_{i \geqslant 0} \frac{A_i}{A_{i-1}}$$

and inherits two products from $\prec$ and $\succ$, which are still denoted by $\prec$ and $\succ$. They are (well-) defined by setting, for $a \in A_i$ and $b \in A_j$, with $i, j \geqslant 0$,

$$(a + A_{i-1}) \prec (b + A_{j-1}) := (a \prec b + A_{i+j-1}) \in \frac{A_{i+j}}{A_{i+j-1}},$$

and similarly for $\succ$. As in the case of algebras with one operation, A and $\mathrm{gr}(A)$ are canonically isomorphic as R-modules, but not as algebras. It is however clear that any n-linear relation which is satisfied by the original products $\prec$ and $\succ$ will be satisfied by the induced products.

We will be interested in *almost commutative* filtered algebras, which have the property that the associated graded algebra is commutative, i.e., that $a \prec b = b \succ a$ for all $a, b \in \mathrm{gr}(A)$. As before, we then view $\mathrm{gr}(A)$ as an algebra with one operation $\times$ (setting, as usual, $\times$ equal to $\succ$), and $\mathrm{gr}(A)$ can be equipped with another product, defined for $a \in A_i$ and $b \in A_j$, with $i, j \geqslant 0$ by

$$(a + A_{i-1}) \bigcirc (b + A_{j-1}) := (a \succ b - b \prec a + A_{i+j-2}) \in \frac{A_{i+j-1}}{A_{i+j-2}}. \tag{4.22}$$

The question is now again to which category $(\mathrm{gr}(A), \times, \bigcirc)$ belongs. When $\mathscr{C}$ is the category of associative algebras, Aguiar's answer is that $(\mathrm{gr}(A), \times, \bigcirc)$ is a pre-Poisson algebra, as in the case of deformations (see [2]). We will give here the answer for arbitrary algebras; as we will see, the result is very similar to the result which we obtained for deformations (Section 4.5). The definitions and assumptions are the same as in the latter section, except that the relations of $\mathscr{C}$ (and hence of $\mathscr{C}_{\mathrm{pol}}$) are supposed here to be multilinear.

Proposition 4.21. *Suppose that the relations of $\mathscr{C}$ are multilinear. Let $(A = \bigcup_i A_i, \prec, \succ)$ be a commutative filtered algebra in $\mathscr{C}^{\mathrm{dend}}$. On $\mathrm{gr}(A)$, consider the product $\times$, defined for $a, b \in \mathrm{gr}(A)$ by $a \times b := a \succ b$, as well as the product $\bigcirc$, defined by (4.22). Then*

$$\big(\mathrm{gr}(A), \times, \bigcirc\big) \in \underline{\mathscr{C}}_{\mathrm{pol}}^{\mathrm{dend}} = (\underline{\mathscr{C}}_{\mathrm{pol}})^{\mathrm{dend}}.$$

Proof. As in the proof of Proposition 4.13, we use polarization to transform the deformation into an algebra of $\mathscr{C}_{\mathrm{pol}}^{\mathrm{dend}}$. Namely, by polarization, we have a filtered algebra $(A, *, \circ) \in \mathscr{C}_{\mathrm{pol}}^{\mathrm{dend}}$, having the property that

$$A_i * A_j \subset A_{i+j} \quad \text{and} \quad A_i \circ A_j \subset A_{i+j-1}. \tag{4.23}$$

In terms of $*$ and $\circ$, the above definitions of $\times$ and $\bigcirc$ now amount to setting, for $a \in A_i$ and $b \in A_j$,

$$(a + A_{i-1}) \times (b + A_{j-1}) := a * b + A_{i+j-1}, \tag{4.24}$$

$$(a + A_{i-1}) \bigcirc (b + A_{j-1}) := a \circ b + A_{i+j-2}. \tag{4.25}$$

Suppose now that $\mathcal{R} = \mathcal{R}_{*,\circ}$ is an n-linear relation of $\mathcal{C}^{\text{dend}}_{\text{pol}}$ and recall that we denote the lowest weight part of $\mathcal{R}$ by $\underline{\mathcal{R}}$. The weight of $\underline{\mathcal{R}}$ is denoted by d. Let $a_1, a_2, \ldots, a_n \in A$ with $a_i \in A_{j_i}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} \underline{\mathcal{R}}_{\times,\bigcirc}(a_1 + A_{j_1-1}, \ldots, a_n + A_{j_n-1}) \\ &= \underline{\mathcal{R}}_{*,\circ}(a_1, \ldots, a_n) + A_{j_1+\cdots+j_n-d-1} \\ &= \mathcal{R}_{*,\circ}(a_1, \ldots, a_n) + A_{j_1+\cdots+j_n-d-1} \\ &= A_{j_1+\cdots+j_n-d-1}, \end{aligned}$$

where we used in the last step that $(A, *, \circ)$ satisfies $\mathcal{R}$. It follows that $(\text{gr}(A), \times, \bigcirc)$ satisfies the relation $\underline{\mathcal{R}} = 0$. Therefore, $(\text{gr}(A), \times, \bigcirc)$ satisfies all relations of $\underline{\mathcal{C}}^{\text{dend}}_{\text{pol}}$, and so $(\text{gr}(A), \times, \bigcirc) \in \underline{\mathcal{C}}^{\text{dend}}_{\text{pol}}$. ∎

Example 4.22. We return once more to the case where $\mathcal{C}$ is the category of associative algebras. We have already analyzed the relations defining $\mathcal{C}^{\text{dend}}_{\text{pol}}$ in Example 4.14 where we have shown that the lowest weight terms of the relations are the relations which define a pre-Poisson algebra. Hence, we find that if $(A, \prec, \succ)$ is an almost commutative filtered Loday dendriform algebra, then $(\text{gr}(A), \times, \bigcirc)$ is a pre-Poisson algebra. We thereby recover Aguiar's result, cited above.

The strong similarity between our results on filtrations and on deformations is not accidental. Indeed, let $(A^\nu, \prec, \succ)$ be a formal deformation of a commutative algebra $(A, \prec_0, \succ_0) \in \mathcal{C}^{\text{dend}}$, where we assume that the relations which define $\mathcal{C}$ are multilinear. Setting $A^\nu_i := \nu^i A^\nu$ for all $i \in \mathbb{N}$ it is clear that $(A^\nu, \prec, \succ)$ is a filtered $\mathcal{C}$-dendriform algebra. Notice that the filtration is *descending*, so that $\text{gr}(A^\nu)$ is now defined as $\text{gr}(A^\nu) := \bigoplus_{i \geqslant 0} A^\nu_i / A^\nu_{i+1}$, and that $\text{gr}(A^\nu)$ is commutative. Though ascending and descending filtrations (indexed by $\mathbb{N}$) are from many points of view different, it is easily verified that the above results on ascending filtrations hold also for descending filtrations. In particular, $(\text{gr}(A^\nu), \times, \bigcirc) \in \underline{\mathcal{C}}^{\text{dend}}_{\text{pol}}$, as in Proposition 4.21. Under the canonical isomorphisms $A^\nu_i / A^\nu_{i+1} \simeq A$, valid for all $i \in \mathbb{N}$, we get that $(A, \times, \bigcirc) \in \underline{\mathcal{C}}^{\text{dend}}_{\text{pol}}$, where the latter products on A are inherited from the products on $\text{gr}(A)$. It is easily checked that $(A, \times, \bigcirc)$ is the deformation algebra of $(A^\nu, \prec, \succ)$. This shows that under the extra assumption that the relations defining $\mathcal{C}^{\text{dend}}_{\text{pol}}$ are multilinear, Proposition 4.13 is a consequence of Proposition 4.21. It should now be clear that all remarks made in Section 4.5 also apply to almost commutative (or anticommutative) filtered algebras (always under the assumption that the relations defining $\mathcal{C}^{\text{dend}}_{\text{pol}}$ are multilinear).

Acknowledgments. Parts of this paper have been written while the second author was a visiting professor at the University of Poitiers in June 2018.

References

[1] M. Aguiar, Infinitesimal Hopf algebras. In *New Trends in Hopf Algebra Theory (La Falda, 1999)*, pp. 1–29, Contemp. Math. 267, Amer. Math. Soc., Providence, RI, 2000 Zbl 0982.16028 MR 1800704

[2] M. Aguiar, Pre-Poisson algebras. *Lett. Math. Phys.* **54** (2000), no. 4, 263–277 Zbl 1032.17038 MR 1846958

[3] M. Aguiar, Infinitesimal bialgebras, pre-Lie and dendriform algebras. In *Hopf Algebras*, pp. 1–33, Lecture Notes in Pure and Appl. Math. 237, Dekker, New York, 2004 Zbl 1059.16027 MR 2051728

[4] C. Bai, O. Bellier, L. Guo, and X. Ni, Splitting of operations, Manin products, and Rota-Baxter operators. *Int. Math. Res. Not. IMRN* **2013** (2013), no. 3, 485–524 Zbl 1314.18010 MR 3021790

[5] C. Bai, L. Guo, and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras. *Comm. Math. Phys.* **297** (2010), no. 2, 553–596 Zbl 1206.17020 MR 2651910

[6] C. Bai, L. Guo, and X. Ni, $\mathcal{O}$-operators on associative algebras, associative Yang-Baxter equations and dendriform algebras. In *Quantized Algebra and Physics*, pp. 10–51, Nankai Ser. Pure Appl. Math. Theoret. Phys. 8, World Sci. Publ., Hackensack, NJ, 2012 Zbl 1300.16036 MR 2905519

[7] C. Bai, L. Liu, and X. Ni, Some results on L-dendriform algebras. *J. Geom. Phys.* **60** (2010), no. 6-8, 940–950 Zbl 1207.17002 MR 2647294

[8] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math.* **10** (1960), 731–742 Zbl 0095.12705 MR 119224

[9] T. Brzeziński, Curved Rota-Baxter systems. *Bull. Belg. Math. Soc. Simon Stevin* **23** (2016), no. 5, 713–720 Zbl 1404.16032 MR 3593571

[10] T. Brzeziński, Rota-Baxter systems, dendriform algebras and covariant bialgebras. *J. Algebra* **460** (2016), 1–25 Zbl 1376.16039 MR 3510392

[11] P. Cartier, Vinberg algebras, Lie groups and combinatorics. In *Quanta of Maths*, pp. 107–126, Clay Math. Proc. 11, Amer. Math. Soc., Providence, RI, 2010 Zbl 1277.17001 MR 2732048

[12] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation. *Lett. Math. Phys.* **61** (2002), no. 2, 139–147 Zbl 1035.17001 MR 1936573

[13] M. Goze and E. Remm, Lie-admissible algebras and operads. *J. Algebra* **273** (2004), no. 1, 129–152 Zbl 1045.17007 MR 2032454

[14] M. Goze and E. Remm, Poisson algebras in terms of non-associative algebras. *J. Algebra* **320** (2008), no. 1, 294–317 Zbl 1165.17003 MR 2417990

[15] L. Guo, *An Introduction to Rota-Baxter Algebra*. Surv. Mod. Math. 4, International Press, Somerville, MA; Higher Education Press, Beijing, 2012 Zbl 1271.16001 MR 3025028

[16] D. Hou and C. Bai, J-dendriform algebras. *Front. Math. China* **7** (2012), no. 1, 29–49 Zbl 1294.17023 MR 2876897

[17] D. Hou, X. Ni, and C. Bai, Pre-Jordan algebras. *Math. Scand.* **112** (2013), no. 1, 19–48 Zbl 1293.17038 MR 3057597

[18] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson Structures*. Grundlehren Math. Wiss. 347, Springer, Heidelberg, 2013 Zbl 1271.53074 MR 2906391

[19] M. Livernet, A rigidity theorem for pre-Lie algebras. *J. Pure Appl. Algebra* **207** (2006), no. 1, 1–18 Zbl 1134.17001 MR 2244257

[20] J.-L. Loday, Dialgebras. In *Dialgebras and Related Operads*, pp. 7–66, Lecture Notes in Math. 1763, Springer, Berlin, 2001 Zbl 0999.17002 MR 1860994

[21] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes. In *Homotopy Theory: Relations With Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, pp. 369–398, Contemp. Math. 346, Amer. Math. Soc., Providence, RI, 2004 Zbl 1065.18007 MR 2066507

[22] M. Markl and E. Remm, Algebras with one operation including Poisson and other Lie-admissible algebras. *J. Algebra* **299** (2006), no. 1, 171–189 Zbl 1101.18004 MR 2225770

[23] X. Ni and C. Bai, Poisson bialgebras. *J. Math. Phys.* **54** (2013), no. 2, Article No. 023515 Zbl 1290.17019 MR 3076642

[24] G.-C. Rota, Baxter algebras and combinatorial identities. I. *Bull. Amer. Math. Soc.* **75** (1969), 325–329 Zbl 0192.33801 MR 0244070

[25] G.-C. Rota, Baxter algebras and combinatorial identities. II. *Bull. Amer. Math. Soc.* **75** (1969), 330–334 Zbl 0319.05008 MR 0244070

[26] R. D. Schafer, *An Introduction to Nonassociative Algebras. Corrected reprint of the 1966 original*. Dover Publications, New York, 1995 Zbl 0145.25601 MR 1375235

[27] M. A. Semenov-Tyan-Shanskiĭ, What a classical r-matrix is. *Funktsional. Anal. i Prilozhen.* **17** (1983), no. 4, 17–33 Zbl 0535.58031 MR 725413

[28] B. Vallette, Homology of generalized partition posets. *J. Pure Appl. Algebra* **208** (2007), no. 2, 699–725 Zbl 1109.18002 MR 2277706

Received 28 February 2020.

Cyrille Ospel
Laboratoire des Sciences de l'Ingénieur pour l'Environnement (LaSIE) - UMR CNRS 7356, Université de La Rochelle, Av. M. Crépeau, 17042 La Rochelle Cedex 1, France; cospel@univ-lr.fr

Florin Panaite
Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania; florin.panaite@imar.ro

Pol Vanhaecke
Laboratoire de Mathématiques et Applications, Université de Poitiers, Site du Futuroscope, Bât. H3 - TSA 61125, 11 bd Marie et Pierre Curie, 86073 Poitiers Cedex 9, France; pol.vanhaecke@math.univ-poitiers.fr