Publ. RIMS, Kyoto Univ.
18 (1982), 427-438

Semantical Analysis of Constructive PDL

By

Hirokazu NISHIMURA*

§ 1. Introduction

Propositional dynamic logic or PDL is an interesting arena of logical research which was born to modal logic as his father and verification logic in the tradition of Floyd/Hoare as his mother. Several completeness proofs of PDL have been presented and the most recent one is Leivant's [4], where constructive or intuitionistic PDL (simply CPDL) plays an auxiliary role. The main purpose of this paper is to give a semantical analysis of CPDL after the manner of Nishimura [5]. In Section 2 we give a Kripkian semantics to CPDL, with respect to which the semantical completeness of a Gentzen-style system introduced in Section 3 is established in Section 4. A secondary purpose of the paper is to show that the existence of a test program A ? does not make our completeness proof so tedious, contrary to Leivant's remarks.

§ 2. Formal Language and Semantics

There are letters a_i and p_i $(i=0, 1, 2, \cdots)$ for atomic programs and propositions respectively, for which we use $a, b, \cdots$ and $p, q, \cdots$ as syntactic variables. We define the notions of a *formula* and a *program* by simultaneous induction as follows:

(1) Each atomic proposition p is a formula.
(2) If A and B are formulae, so are $A \wedge B$, $A \vee B$, $\neg A$ and $A \supset B$.
(3) If α is a program and A is a formula, then $[\alpha]A$ is a formula.
(4) Each atomic program a is a program.

Received July 8, 1981.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

(5) If α and β are programs, so are

$$\alpha;\beta, \quad \alpha\cup\beta \quad \text{and} \quad \alpha^*.$$

(6) If A is a formula, then A ? is a program.

true is an abbreviation of $p_0 \supset p_0$. We define α^n by induction on n; $\alpha^1=\alpha$ and $\alpha^{n+1}=\alpha^n; \alpha$.

A *sequent* is an ordered pair (Γ, Δ) of finite sets of formulae, which we usually denote by $\Gamma\to\Delta$.

A *structure* is of the form $(S, \leqq, \rho, \pi)$, where

(1) S is a nonempty set;

(2) $\leqq$ is a partial order on S;

(3) ρ is a function assigning to each atomic program a binary relation $\rho(a)$ such that $t\leqq s$ and $(s, s')\in\rho(a)$ imply $(t, s')\in\rho(a)$ for any $s, s', t\in S$;

(4) π is a function assigning a value in $\{0, 1\}$ to each pair (t, p), where $t\in S$ and p is an atomic proposition, such that $\pi(s, p)=1$ and $s\leqq s'$ imply $\pi(s', p)=1$ for any $s, s'\in S$.

ρ and π are extended to all programs and formulae by simultaneous induction as follows:

(1) $\rho(\alpha;\beta)=\rho(\alpha)\circ\rho(\beta)$ (composition).

(2) $\rho(\alpha\cup\beta)=\rho(\alpha)\cup\rho(\beta)$ (union);

(3) $\rho(\alpha^*)=\rho(\mathit{true}\,?)\cup\rho(\alpha)\cup\rho(\alpha^2)\cup\rho(\alpha^3)\cup\cdots$ (iteration).

(4) $\rho(A\,?)=\{(s, t)\in S\times S \mid s\leqq t \text{ and } \pi(t, A)=1\}$.

(5) $\pi(t, A\wedge B)=1$ iff $\pi(t, A)=1$ and $\pi(t, B)=1$.

(6) $\pi(t, A\vee B)=1$ iff $\pi(t, A)=1$ or $\pi(t, B)=1$.

(7) $\pi(t, \neg A)=1$ iff for all $s\in S$, $t\leqq s$ implies $\pi(s, A)=0$.

(8) $\pi(t, A\supset B)=1$ iff for all $s\in S$, $t\leqq s$ and $\pi(s, A)=1$ imply $\pi(s, B)=1$.

(9) $\pi(t, [\alpha]A)=1$ iff for any $s\in S$, $(t, s)\in\rho(\alpha)$ implies $\pi(s, A)=1$.

We can readily see the following proposition.

Proposition 2.1. *For any program* α *and any formula* A, *we have that:*

(1) $t \leqq s$ *and* $(s, s') \in \rho(\alpha)$ *imply* $(t, s') \in \rho(\alpha)$ *for any* $s, s', t \in S$.

(2) $t \leqq s$ *and* $\pi(t, A) = 1$ *imply* $\pi(s, A) = 1$.

Proof. By induction on α or A.

Our syntax is slightly redundant because $A \supset B$ can be regarded as an abbreviation of $[A?]B$ and similarly for $\neg A$. However we do not necessarily prefer to get rid of this redundancy because several subsystems of our syntax (e.g., a test-free variant) are of interest.

A sequent $\Gamma \rightarrow \Delta$ is called *realizable* if for some structure $(S, \leqq, \rho, \pi)$ and some $t \in S$, we have that:

(1) $\pi(t, A) = 1$ for any $A \in \Gamma$.

(2) $\pi(t, B) = 0$ for any $B \in \Delta$.

A sequent $\Gamma \rightarrow \Delta$ which is not realizable is called *valid* (notation: $\models \Gamma \rightarrow \Delta$).

§ 3. Formal System

Our formal system **LJP** for CPDL consists of the following axioms and inference rules:

Axioms: $A \rightarrow A$

Rules:

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \Gamma \rightarrow \Delta, \Sigma} \quad \text{(extension)}$$

$$\frac{\Gamma \rightarrow \Delta, A \qquad A, \Pi \rightarrow \Sigma}{\Gamma, \Pi \rightarrow \Delta, \Sigma} \quad \text{(cut)}$$

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B} \quad (\rightarrow \wedge)$$

$$\left.\begin{array}{c} \dfrac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \\[2ex] \dfrac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \end{array}\right\} \quad (\wedge \rightarrow)$$

$$\left.\begin{array}{c}\dfrac{\Gamma\to\Delta,\ A}{\Gamma\to\Delta,\ A\vee B}\\[2ex]\dfrac{\Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\vee B}\end{array}\right\}\quad(\to\vee)$$

$$\frac{A,\Gamma\to\Delta\qquad B,\Gamma\to\Delta}{A\vee B,\Gamma\to\Delta}\quad(\vee\to)$$

$$\frac{A,\Gamma\to}{\Gamma\to\Delta,\ \neg A}\quad(\to\neg)$$

$$\frac{\Gamma\to\Delta,\ A}{\neg A,\Gamma\to\Delta}\quad(\neg\to)$$

$$\frac{A,\Gamma\to B}{\Gamma\to\Delta,\ A\supset B}\quad(\to\supset)$$

$$\frac{\Gamma\to\Delta,\ A\qquad B,\Pi\to\Sigma}{A\supset B,\Gamma,\Pi\to\Delta,\ \Sigma}\quad(\supset\to)$$

$$\frac{\Gamma\to A}{[\alpha]\Gamma\to[\alpha]A}\quad(\to[\ ])$$

$$\frac{\Gamma\to\Delta,\ [\alpha][\beta]A}{\Gamma\to\Delta,\ [\alpha;\beta]A}\quad(\to[;])$$

$$\frac{[\alpha][\beta]A,\Gamma\to\Delta}{[\alpha;\beta]A,\Gamma\to\Delta}\quad([;]\to)$$

$$\frac{\Gamma\to\Delta,\ [\alpha]A\qquad\Gamma\to\Delta,\ [\beta]A}{\Gamma\to\Delta,\ [\alpha\cup\beta]A}\quad(\to[\cup])$$

$$\left.\begin{array}{c}\dfrac{[\alpha]A,\Gamma\to\Delta}{[\alpha\cup\beta]A,\Gamma\to\Delta}\\[2ex]\dfrac{[\beta]A,\Gamma\to\Delta}{[\alpha\cup\beta]A,\Gamma\to\Delta}\end{array}\right\}\quad([\cup]\to)$$

$$\frac{A\to[\alpha]A}{A\to[\alpha^*]A}\quad(\to[*])$$

$$\left.\begin{array}{c}\dfrac{A,\Gamma\to\Delta}{[\alpha^*]A,\Gamma\to\Delta}\\[2ex]\dfrac{[\alpha][\alpha^*]A,\Gamma\to\Delta}{[\alpha^*]A,\Gamma\to\Delta}\end{array}\right\}\quad([*]\to)$$

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow \Delta, [A?]B} \ (\rightarrow [?])$$

$$\frac{\Gamma \rightarrow \Delta, A \qquad B, \Pi \rightarrow \Sigma}{[A?]B, \Gamma, \Pi \rightarrow \Delta, \Sigma} \ ([?] \rightarrow)$$

A *proof* P (in **LJP**) is a tree of sequents satisfying the following conditions:

(1) The topmost sequents of P are axiom sequents.

(2) Every sequent in P except the lowest one is an upper sequent of an inference rule whose lower sequent is also in P.

A sequent $\Gamma \rightarrow \Delta$ is said to be *provable* (in **LJP**) if there exists a proof whose lowest sequent is $\Gamma \rightarrow \Delta$. If a sequent $\Gamma \rightarrow \Delta$ is provable, then we write $\vdash \Gamma \rightarrow \Delta$ (in **LJP**). A sequent $\Gamma \rightarrow \Delta$ which is not provable is said to be *consistent* (in **LJP**). A sequent $\Gamma \rightarrow \Delta$ is called *intuitionistic* if Δ consists of at most one formula. We denote by **LJP′** the formal system obtained from **LJP** by allowing only intuitionistic sequents.

Proposition 3.1. *For any intuitionistic sequent* $\Gamma \rightarrow \Delta$, $\vdash \Gamma \rightarrow \Delta$ *in* **LJP** *iff* $\vdash \Gamma \rightarrow \Delta$ *in* **LJP′**.

Proof. (1) if part: obvious.

(2) only if part: Prove that for any sequent $\Gamma \rightarrow \Delta$, if $\vdash \Gamma \rightarrow \Delta$ in **LJP**, then $\vdash \Gamma \rightarrow B_1 \vee \cdots \vee B_m$ in **LJP′**, where $\Delta = \{B_1, \cdots, B_m\}$.

Proposition 3.2 (Soundness Theorem of **LJP**). *For any sequent* $\Gamma \rightarrow \Delta$, *if* $\vdash \Gamma \rightarrow \Delta$ *in* **LJP**, *then* $\models \Gamma \rightarrow \Delta$.

Proof. By induction on a proof of $\Gamma \rightarrow \Delta$.

§ 4. Completeness

The main purpose of this section is to establish the following theorem.

Theorem 4.1 (Completeness Theorem for **LJP**). *Any consistent* *sequent* $\Gamma \to \Delta$ *is realizable.*

A finite set Φ of formulae is called *closed* if it satisfies the following conditions:

(1) If $(A \wedge B) \in \Phi$, then $A \in \Phi$ and $B \in \Phi$.
(2) If $(A \vee B) \in \Phi$, then $A \in \Phi$ and $B \in \Phi$.
(3) If $\neg A \in \Phi$, then $A \in \Phi$.
(4) If $(A \supset B) \in \Phi$, then $A \in \Phi$ and $B \in \Phi$.
(5) If $[\alpha] A \in \Phi$, then $A \in \Phi$.
(6) If $[\alpha; \beta] A \in \Phi$, then $[\alpha][\beta] A \in \Phi$.
(7) If $[\alpha \cup \beta] A \in \Phi$, then $[\alpha] A \in \Phi$ and $[\beta] A \in \Phi$.
(8) If $[\alpha^*] A \in \Phi$, then $[\alpha][\alpha^*] A \in \Phi$.
(9) If $[A ?] B \in \Phi$, then $A \in \Phi$ and $B \in \Phi$.

In the rest of this section we fix such a closed set, say, Φ. A sequent $\Gamma \to \Delta$ is called *Φ-saturated* if it satisfies the following conditions:

(1) $\Gamma \to \Delta$ is consistent.
(2) $\Gamma \cup \Delta = \Phi$.

It is easy to see that for any Φ-saturated sequent $\Gamma \to \Delta$, $\Gamma \cap \Delta = \emptyset$.

Lemma 4.2. *Any consistent sequent* $\Gamma \to \Delta$ *can be extended to some consistent sequent* $\tilde{\Gamma} \to \tilde{\Delta}$ *such that* $\Phi \subseteq \tilde{\Gamma} \cup \tilde{\Delta}$.

Corollary 4.3. *Any consistent sequent* $\Gamma \to \Delta$, *where* $\Gamma \cup \Delta \subseteq \Phi$, *can be extended to some Φ-saturated sequent.*

Now we define the *Φ-canonical Structure* $\Omega(\Phi) = (S, \leqq, \rho, \pi)$ *as follows*:

(1) $S = \{\Gamma \to \Delta \mid \Gamma \to \Delta$ is Φ-saturated$\}$.
(2) $(\Gamma_1 \to \Delta_1) \leqq (\Gamma_2 \to \Delta_2)$ iff $\Gamma_1 \subseteq \Gamma_2$.
(3) $\rho(a) = \{(\Gamma_1 \to \Delta_1, \Gamma_2 \to \Delta_2) \in S \times S \mid \{A \mid [a] A \in \Gamma_1\} \subseteq \Gamma_2\}$

for each atomic program a.

(4) $\pi(\Gamma\to\Delta, p)=1$ iff $p\in\Gamma$ for each atomic proposition p.

It is easy to see that $\Omega(\Phi)$ satisfies the conditions of the definition of a structure. The rest of this section is devoted almost completely to the proof of the following theorem, from which Theorem 4.1 follows at once.

Theorem 4.4 (Fundamental Theorem of $\Omega(\Phi)$). *For any formula $A\in\Phi$ and any sequent $\Gamma\to\Delta$ of S, $\pi(\Gamma\to\Delta, A)=1$ if $A\in\Gamma$ and $\pi(\Gamma\to\Delta, A)=0$ if $A\in\Delta$.*

We define a notion of the *test degree* of a program α and a formula A, denoted by $td(\alpha)$ and $td(A)$ respectively, by simultaneous induction as follows:

(1) $td(a)=td(p)=0$ for any atomic program a and atomic proposition p.

(2) $td(A\wedge B)=td(A\vee B)=td(A\supset B)=\max\{td(A), td(B)\}$.

(3) $td(\neg A)=td(A)$.

(4) $td([\alpha]A)=\max\{td(\alpha), td(A)\}$.

(5) $td(\alpha;\beta)=td(\alpha\cup\beta)=\max\{td(\alpha), td(\beta)\}$.

(6) $td(\alpha^*)=td(\alpha)$.

(7) $td(A?)=td(A)+1$.

Our strategy of the proof of Theorem 4.4 is to prove the following theorem by induction on i.

Theorem 4.4 (i). *For any sequent $\Gamma\to\Delta$ of S and any formula $A\in\Phi$ such that $td(A)<i$, $\pi(\Gamma\to\Delta, A)=1$ if $A\in\Gamma$ and $\pi(\Gamma\to\Delta, A)=0$ if $A\in\Delta$.*

It is obvious that Theorem 4.4 (0) holds vacuously. Hence what we have to do is to prove Theorem 4.4 ($i+1$), assuming Theorem 4.4 (i). To do it smoothly, we need several auxiliary notions and lemmas.

We define the notions of the *characteristic formula* $\psi(\Gamma\to\Delta)$ of a sequent $\Gamma\rightarrow\Delta$ and of the *characteristic formula* $\psi(X)$ of a finite set X of sequents as follows:

(1) $\psi(\Gamma\rightarrow\Delta)=A_1\wedge\cdots\wedge A_n$, where $\Gamma=\{A_1,\cdots,A_n\}$.

(2) $\psi(X)=\psi(\Gamma_1\rightarrow\Delta_1)\vee\cdots\vee\psi(\Gamma_k\rightarrow\Gamma_k)$, where

$$X=\{\Gamma_1\rightarrow\Delta_1,\cdots,\Gamma_k\rightarrow\Delta_k\}.$$

For any $Y\subseteq S$ and any program α, the *weakest precondition of* α *with respect to* Y, denoted $wp(\alpha, Y)$, is defined as follows:

$$wp(\alpha,Y)=\{s\in S|\,(s,t)\in\rho(\alpha)\text{ implies }t\in Y\text{ for any }t\in S\}.$$

For any X, $Y\subseteq S$ and any program α, we say that α is *partially correct with respect to precondition X and postcondition Y* (notation: $\{X\}\alpha\{Y\}$) if $X\subseteq wp(\alpha, Y)$

Lemma 4.5 $(i+1)$. *For any* X, $Y\subseteq S$ *and any program* α *such that* $td(\alpha)<i+1$, *if* $\{X\}\alpha\{Y\}$, *then*

$$\vdash\psi(X)\rightarrow[\alpha]\psi(Y).$$

Proof. The proof is carried out by induction on α. Here we deal only with the following three critical cases.

(1) α is an atomic program, say, a:

Let $X=\{\Gamma_j\rightarrow\Delta_j|1\leqq j\leqq n\}$. We assume, for the sake of simplicity, that $n=2$.

Suppose, for the sake of contradiction, that the sequent $\psi(\Gamma_1\rightarrow\Delta_1)\rightarrow[a]\psi(Y)$ is consistent, which implies that the sequent $\Gamma_1\rightarrow[a]\psi(Y)$ is also consistent. So the sequent $\{A|\,[a]A\in\Gamma_1\}\rightarrow\psi(Y)$ is also consistent, for otherwise $\Gamma_1\rightarrow[a]\psi(Y)$ would be provable by rules $(\rightarrow[\ ])$ and (extension). By Lemma 4.2, the sequent $\{A|\,[a]A\in\Gamma_1\}\rightarrow\psi(Y)$ can be extended to some consistent sequent $\tilde{\Gamma}\rightarrow\tilde{\Delta}$ such that $\Phi\subseteq\tilde{\Gamma}\cup\tilde{\Delta}$. Then it is easy to see that $(\Gamma_1\rightarrow\Delta_1, \tilde{\Gamma}\cap\Phi\rightarrow\tilde{\Delta}\cap\Phi)\in\rho(a)$. Since $\{\Gamma_1\rightarrow\Delta_1\}a\{Y\}$ by assumption, $(\tilde{\Gamma}\cap\Phi\rightarrow\tilde{\Delta}\cap\Phi)\in Y$. Hence

$$\vdash\psi(\tilde{\Gamma}\cap\Phi\rightarrow\tilde{\Delta}\cap\Phi)\rightarrow\psi(Y). \tag{A}$$

This implies that

$$\vdash \tilde{\Gamma} \cap \Phi \to \psi(Y). \tag{B}$$

This contradicts the assumption that the sequent $\tilde{\Gamma} \to \tilde{\Delta}$ is consistent and $\psi(Y) \in \tilde{\Delta}$. Thus we can conclude that

$$\vdash \psi(\Gamma_1 \to \Delta_1) \to [a]\psi(Y). \tag{C}$$

A similar argument shows that

$$\vdash \psi(\Gamma_2 \to \Delta_2) \to [a]\psi(Y). \tag{D}$$

By using rule $(\vee \to)$ we can deduce from (C) and (D) that

$$\vdash \psi(\Gamma_1 \to \Delta_1) \vee \psi(\Gamma_2 \to \Delta_2) \to [a]\psi(Y), \tag{E}$$

which was to be proved.

(2) α is of the form A ?:

Let $X = \{\Gamma_j \to \Delta_j | 1 \leqq j \leqq n\}$. We assume, for the sake of simplicity, that $n = 2$. Suppose, for the sake of contradiction, that the sequent $\psi(\Gamma_1 \to \Gamma_1) \to [A\,?]\psi(Y)$ is consistent, which implies that the sequent $\Gamma_1 \to [A\,?]\psi(Y)$ is also consistent. Hence the sequent $A, \Gamma_1 \to \psi(Y)$ is also consistent, for otherwise the sequent $\Gamma_1 \to [A\,?]\psi(Y)$ would be provable by rule $(\to [?])$. By Lemma 4.2, the sequent $A, \Gamma_1 \to \psi(Y)$ can be extended to some consistent sequent $\tilde{\Gamma} \to \tilde{\Delta}$ such that $\Phi \subseteq \tilde{\Gamma} \cup \tilde{\Delta}$. Since $td(A) < i$, $\pi(\tilde{\Gamma} \cap \Phi \to \tilde{\Delta} \cap \tilde{\Phi}, A) = 1$ by Theorem 4.4 (i). Since $\Gamma_1 \subseteq \tilde{\Gamma}$, $(\Gamma_1 \to \Delta_1) \leqq (\tilde{\Gamma} \cap \Phi \to \tilde{\Delta} \cap \Phi)$. Therefore $(\tilde{\Gamma} \cap \Phi \to \tilde{\Delta} \cap \Phi) \in Y$. Hence

$$\vdash \psi(\tilde{\Gamma} \cap \Phi \to \tilde{\Delta} \cap \Phi) \to \psi(Y). \tag{A}$$

This implies that

$$\vdash \tilde{\Gamma} \cap \Phi \to \psi(Y). \tag{B}$$

This contradicts the assumption that the sequent $\tilde{\Gamma} \to \tilde{\Delta}$ is consistent and $\psi(Y) \in \tilde{\Delta}$. Thus we can conclude that

$$\vdash \psi(\Gamma_1 \to \Delta_1) \to [A\,?]\psi(Y). \tag{C}$$

Similarly,

$$\vdash \psi(\Gamma_2 \to \Delta_2) \to [A\,?]\psi(Y). \tag{D}$$

By using rule $(\vee \to)$, we can deduce from (C) and (D) that

$$\psi(\Gamma_1 \to \Gamma_1) \vee \psi(\Gamma_2 \to \Delta_2) \to [A?]\psi(Y), \tag{E}$$

which was to be proved.

(3) α is of the form β^*:

Since $X \subseteq wp(\beta^*, Y)$ by Assumption,

$$\vdash \psi(X) \to \psi(wp(\beta^*, Y)). \tag{A}$$

Since $\{wp(\beta^*, Y)\}\beta\{wp(\beta^*, Y)\}$,

$$\vdash \psi(wp(\beta^*, Y)) \to [\beta]\psi(wp(\beta^*, Y)). \tag{B}$$

Hence by using rule $(\to [*])$, we have that

$$\vdash \psi(wp(\beta^*, Y)) \to [\beta^*]\psi(wp(\beta^*, Y)). \tag{C}$$

Since $\rho(\text{true}\,?) \subseteq \rho(\beta^*)$, $wp(\beta^*, Y) \subseteq Y$.

Hence

$$\vdash \psi(wp(\beta^*, Y)) \to \psi(Y). \tag{D}$$

By using rule $(\to [\ ])$, we can deduce from (D) that

$$[\beta^*]\psi(wp(\beta^*, Y)) \to [\beta^*]\psi(Y). \tag{E}$$

By using rule (cut) twice, we get from (A), (C) and (E) that

$$\vdash \psi(X) \to [\beta^*]\psi(Y). \tag{F}$$

Lemma 4.6 $(i+1)$. *For any formula A any program α and any sequents $\Gamma \to \Delta$ of S such that $td(\alpha) < i+1$, if $[\alpha]A \in \Delta$, then there exists a sequent $\Gamma' \to \Delta'$ of S such that $(\Gamma \to \Delta, \Gamma' \to \Delta') \in \rho(\alpha)$ and $A \in \Delta'$.*

Proof. Let $X = \{(\Pi \to \Sigma) \in S | A \in \Pi\}$. Suppose, for the sake of contradiction, that $\{\Gamma \to \Delta\}\alpha\{X\}$. Then by Lemma 4.5 $(i+1)$

$$\vdash \psi(\Gamma \to \Delta) \to [\alpha]\psi(X). \tag{A}$$

It follows from the definition of X that

$$\vdash \psi(X) \to A. \tag{B}$$

By using rules (cut) and $(\to [\ ])$, we can deduce from (A) and (B) that

$$\vdash \psi(\Gamma \to \Delta) \to [\alpha]A. \tag{C}$$

It follows from (C) that

$$\vdash \Gamma \rightarrow [\alpha] A, \tag{D}$$

which contradicts the assumption that the sequent $\Gamma \rightarrow \Delta$ is consistent and $[\alpha] A \in \Delta$. This completes the proof.

Lemma 4.7 ($i+1$). *For any formula A, any program α and any sequents $\Gamma \rightarrow \Delta$, $\Gamma' \rightarrow \Delta'$ of S such that $td(\alpha) < i+1$ and $(\Gamma \rightarrow \Delta, \Gamma' \rightarrow \Delta') \in \rho(\alpha)$, if $[\alpha] A \in \Gamma$, then $A \in \Gamma'$.*

Proof. Similar to that of Lemma 4.5 ($i+1$).

Now we are ready to complete the proof of Theorem 4.4 ($i+1$).

Proof of Theorem 4.4 ($i+1$). By induction on the construction of a formula $A \in \Phi$. Use Lemmas 4.6 ($i+1$) and 4.7 ($i+1$) in dealing with formulae of the form $[\alpha] A$.

We have completed the proof of Theorem 4.4. By combining Proposition 3.2 and Theorem 4.1, we have

Theorem 4.8. *For any sequent $\Gamma \rightarrow \Delta$,*

$$\vdash \Gamma \rightarrow \Delta \quad \textit{iff} \quad \models \Gamma \rightarrow \Delta$$

The finite model property shown in Theorem 4.4 establishes

Theorem 4.9 (Decidability of LJP). **LJP** *is decidable.*

References

[1] Fitting, M. C., *Intuitionistic logic model theory and forcing*, North-Holland, Amsterdam, 1969.

[2] Gabbay, D. M., *Semantical investigations in Heyting's intuitionistic logic*, D. Reidel, Dordrecht-Boston-London, 1981.

[3] Harel, D., *First-order dynamic logic, Lecture Notes in Computer Science*, **68** Springer, Berlin-Heidelberg-New York, 1979.

[4] Leivant, D., Proof theoretic methodology for propositional dynamic logic, in: Formalization of programming concepts, *Lecture Notes in Computer Science*, **107**

Springer, Berlin-Heidelberg-New York, (1981), 356–373.
[5] Nishimura, H., Sequential methods in propositional dynamic logic, *Acta Informatica*, **12** (1979), 377–400.
[6] Takeuti, G., *Proof theory*, North-Holland, Amsterdam, 1975.